On the Embedding of Topological Rings into Quotient Rings II

By Mikihiko ENDO

Institute of Mathematics, Colledge of General Education, University of Tokyo
(Received February 21, 1964)

In the previous paper [1] we have developed a general theory of embedding of topological rings. In this paper, we shall give some examples of embedding and, in example 3, we shall prove that the relation $\sim$ is not open in the ring given by Gould [2] as he expected. For the proof, the concept of continuous (right) transitivity plays an important rôle.

In the following discussion we assume that the readers are acquainted with the results of the previous paper [1].

1. A typical example of open embeddability is valuation rings of a division ring. Let D be a division ring and v be a non-archimedian exponential valuation of D. We shall exclude the trivial valuation, i.e. the valuation v such that $v(x)=0$ for all non-zero elements x of D. The set R of elements in D with non-negative values with respect to v is the valuation ring of v. Further, the set P of elements with positive values forms the maximal (two-sided) ideal of R. Then, as is well known, R is a Q-ring with continuous quasi-inverses and the radical of R is P. If we define V_α as the set of elements $x \in R$ such that $v(x)$ is greater than α (α: a positive number), then the set $\{V_\alpha: \alpha>0\}$ forms a fundamental system of neighbourhoods of 0 in R. Further, for any open neighbourhood V of 0 and for any non-zero element x of R, xV and Vx are also open neighbourhoods of 0. Then all the condition of theorem 7 in [1] hold so that R is openly embeddable into the quotient ring S of R with denominator system $H=R^*$ $(=R\backslash\{0\})$. It is clear that the topological division ring S is algebraically and topologically isomorphic to the original division ring D with the topology induced by v.

The completion $\bar{R}$ of R with respect to v (the completion of topological ring is always possible: see for example [6] pp. 47–49) is also a Q-ring with the continuous quasi-inverse and the conditions of open embeddability are satisfied. It is easily seen that the quotient ring $\bar{S}$ of $\bar{R}$ with the denominator system $\bar{R}^*$ is topologically isomorphic to the completion $\bar{D}$ of D.

A similar investigation also holds when D has a generalized valuation v, that is, the value group of v is not necessarily a subgroup of the real numbers but an arbitrary totally ordered group.

2. Let Z be the ring of rational integers. Then, the set of non-zero ideals of Z may be taken as a fundamental system of neighbourhoods of 0 in Z. It is easily seen that the relation $\sim$ is open with respect to this topology (cf. Cor. of lemma 1 [1]). Since Z is a commutative ring, $Z^* \times Z$ is continuously right transitive (lemma 4. 1). [1]). Therefore we can endow the quotient field Q (the rational number field) with a topology (theorem 2. [1]). The topology of this field is not locally bounded. Further, this topological field Q is dense in the locally direct sum of Q_p with respect to Z_p, where Q_p is the completion of the rational number field with respect to the p-adic valuation and Z_p is the subring of p-adic integers (p runs over all the prime integers). However, it is discrete in the ring of valuation vectors of the rational number field (Kaplansky [5], Iwasawa [3]).

3. Let K be a non-discrete locally bounded topological division ring and σ be a topological automorphism of K. And let $R=K[X]$ be the set of polinomials $p(X)=a_0+a_1X+\cdots+a_nX^n$ of an indeterminate X with K as left coefficient domain. Multiplication in R is defined by $Xa=a^{\sigma}X$ and by other known rules. It is easily shown that R is a principal ideal domain (cf. Jacobson [4] pp. 29–30). We endow R with a topology as follows:

Let $\mathfrak{B}=\{V_\alpha;\ \alpha\in A$ (an index set)$\}$ be the complete system of neighbourhoods of 0 in K and let N be the set of natural numbers. Further let A^N be the set of mappings of N into A. For $\nu\in A^N$, we define

$$W_\nu=\{p(X)=\sum a_iX^i\in R;\ a_i\in V_{\nu(i)},\ i=1,2,\cdots\},$$

and take $\mathfrak{W}=\{W_\nu;\ \nu\in A^N\}$ as a fundamental system of neighbourhoods of 0 in R. Then R becomes a topological ring with respect to this neighbourhood system. In order to see this fact, we need to show the following conditions:

1) For all W_ν in $\mathfrak{W}$, 0 is in W_ν,

2) for any W_μ, W_ν in $\mathfrak{W}$, there exists a W_λ in $\mathfrak{W}$ such that $W_\lambda\subseteqq W_\mu\cap W_\nu$,

3) for any W_ν, there exists a W_μ satisfying $W_\mu-W_\mu\subseteqq W_\nu$,

4) for any $p_0(X)$ in R and for any W_ν in $\mathfrak{W}$, there exists a W_μ such that $p_0(X)W_\mu\subseteqq W_\nu$ and $W_\mu p_0(X)\subseteqq W_\nu$,

5) for any W_ν, there exists a W_μ satisfying $W_\mu W_\mu\subseteqq W_\nu$,

6) for any $p_0(X)\neq 0$, there exists a W_ν such that $p_0(X)$ is not in W_ν,

7) every W_ν contains non-zero element.

3) means the continuity of addition and subtraction, 4) and 5) mean that of multiplication, 6) is the Hausdorff property, and 7) means that the topology of R is not discrete. The properties 1), 2), 3) and 6) are easily proved. 7) is also clear from the fact that K is not discrete. 4) can be proved as follows:

For a neighbourhood V of 0 in K, we denote by $\frac{1}{n}V$ a neighbourhood W of 0 in K such that $\underbrace{W+W+\cdots+W}_{n\text{ times}}\subseteqq V$. Let $p_0(X)=a_0+a_1X+\cdots+a_kX^k$. We can choose a $\lambda\in A^N$ such that

$$V_{\lambda(n)}\subseteqq\bigcap_i(a_i^{-1})^{\sigma^{-i}}\left(\frac{1}{n+i+1}V_{\nu(n+i+1)}\right)^{\sigma^{-i}},$$

where a_i runs over all non-zero coeffitients of $p_0(X)$. Then $p_0(X)W_\lambda\subseteqq W_\nu$. Similarly we can take a $\kappa\in A^N$ satisfying $W_\kappa p_0(X)\subseteqq W_\nu$. Take a W_μ such that $W_\mu \subseteqq W_\kappa \cap W_\lambda$, then W_μ satisfies the condition 4).

To show 5), we define a $\mu \in A^N$ by induction. Let $\mu(0)$ be an index such as $V_{\mu(0)} V_{\mu(0)} \subseteqq V_{\nu(0)}$. Further, when $\mu(0), \cdots, \mu(n-1)$ are determined, we choose $\mu(n)$ satisfying

$$V_{\mu(n)} V_{\mu(i)}^{\sigma^n} \subseteqq \frac{1}{n+i} V_{\nu(n+i)} \quad \text{and} \quad V_{\mu(i)} V_{\mu(n)}^{\sigma^i} \subseteqq \frac{1}{n+i} V_{\nu(n+i)}$$

for $i=0, 1, \cdots, n-1$ (here the locally boundedness of K is used). Then this $\mu \in A^N$ satisfies the condition 5).

Therefore R becomes a topological ring. But, in general, the relation $\sim$ is not open in R. For example, let K be a commutative field and σ be the identity mapping of K. Then R is a commutative ring so that $R^* \times R$ is continuously right transitive (lemma 4. 1). [1]). In this case, the relation $\sim$ is not open. Assume, on the contrary, that the relation $\sim$ is open in R. Let $a=x=X$ and $a'=x'=1$, then $a'x=x'a=1X=X$. By proposition 5 [1], to arbitrary neighbourhoods U_1 and U_2 of 1, there exists suitable neighbourhoods V_1 and V_2 of X such that, if $a_1 \in V_1$ and $x_1 \in V_2$, we can choose $a_1' \in U_1$ and $x_1' \in U_2$ satisfying $a_1'x_1=x_1'a_1$. Let $V_2=X+W_\mu$, $a_1=X$ and $x_1=X+c$, where c is a non-zero element of $V_{\mu(0)}$. Then any common multiple of a_1 and x_1 is necessarily a multiple of $X(X+c)$, so that x_1' is a multiple of X. Consequently, if we select ν satisfying $-1 \notin V_{\nu(0)}$ and let $U_2=1+W_\nu$, then $x_1' \in U_2$, a contradiction. Therefore the relation $\sim$ is not open.

Hence, by our theory the ring R is not embeddable in a quotient ring. Now we change the topology of R so that the relation $\sim$ becomes open. Let $\mathfrak{U}=\{p(X)W_\nu;\ p(X)\neq 0, \in R,\ W_\nu \in \mathfrak{W}\}$ be a fundamental system of neighbourhoods of 0 in R. For this system, the conditions 1), 3), 4), 6) and 7) are obviously satisfied. 2) can be described as follows. Given $p(X)W_\mu$ and $q(X)W_\nu$, there exists a right common multiple $r(X)=p(X)q'(X)=q(X)p'(X)$ of $p(X)$ and $q(X)$, since R is a principal ideal domain. Let $W_{\mu'}$ and $W_{\nu'}$ be such that $q'(X)W_{\mu'} \subseteqq W_\mu$ and $p'(X)W_{\nu'} \subseteqq W_\nu$, and take W_λ as $W_\lambda \subseteqq W_{\mu'} \cap W_{\nu'}$, then $r(X)W_\lambda=p(X)q'(X)W_\lambda=q(X)p'(X)W_\lambda \subseteqq p(X)W_\mu \cap q(X)W_\nu$. The proof of 5): Given $p(X)W_\nu \in \mathfrak{U}$, we take $W_{\nu'}$ as $W_{\nu'}W_{\nu'} \subseteqq W_\nu$ and further take W_μ satisfying $p(X)W_\mu \subseteqq W_{\nu'}$ and $W_\mu \subseteqq W_{\nu'}$, then

$$p(X)W_\mu p(X)W_\mu \subseteqq p(X)W_\mu W_{\nu'} \subseteqq p(X)W_{\nu'}W_{\nu'} \subseteqq p(X)W_\nu .$$

Now it is clear that R becomes a topological ring. By corollary of lemma 1 [1], the relation $\sim$ is open. By lemma 4. b) and the theorem 2 [1], $R^* \times R$ is continuously right transitive if and only if R can be weakly embeddable into S. For example, R is weakly embeddable in S if R is commutative. Especially, if K is a field with non-trivial valuation and $\sigma=1$, R is weakly embeddable in S. In this case, by slight modification of the method of Gould [2], we can prove that S is not locally bounded.

References

[1] Endo, M., On the embedding of topological rings into quotient rings, *J. Fac. Sci. Univ. Tokyo, Sec. I*, **10**, 196–214 (1963).

[2] Gould, G. G., Locally unbounded topological fields and box topologies on products of vector spaces, *J. London Math. Soc.*, **36**, 273-281 (1961).
[3] Iwasawa, K., On the rings of valuation vectors, *Ann. of Math.*, **57** (2), 331-356 (1953).
[4] Jacobson, N., The theory of rings, *Amer. Math. Soc. Mathematical Surveys*, No. II (1943).
[5] Kaplansky, I., Locally compact rings II, *Amer. J. of Math.*, **73**, 20-24 (1951).
[6] Van der Waerden, *Algebra* II, 4 Aufl.