On the First Betti Numbers of Compact Quotient Spaces of Complex Semi-simple Lie Groups by Discrete Subgroups

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Introduction

In his lecture at University of Tokyo, 1961, A. Weil showed, roughly speaking, that any deformation of discrete subgroups Γ of a non-compact simple Lie group G is locally trivial if the rank of G is greater than 1 and the quotient spaces G/Γ are compact. In view of S. Murakami's unpublished result, it seems to the present authors that Weil's result essentially means the triviality of the first cohomology group of the coset space M/Γ with coefficient in the sheaf of germs of Killing vector fields, where M is the quotient space of G by its maximal compact subgroup K. In this article we shall prove that the first Betti numbers of M/Γ and of G/Γ are both zero for complex simple Lie groups of rank greater than 1. The method is Matsushima's, which will be explained in Section 1. With a technique similar to Weil's he proved the vanishing of the first Betti number for the case where G/K is a symmetric bounded domain. Matsushima used some results of A. Borel [1]. What we shall do in the sequel is to obtain the results corresponding to Borel's for our case. In a forthcoming paper we shall treat the same problem more completely parallel to Weil's result for the most general simple Lie groups. The authors are grateful to Professors Y. Matsushima and S. Murakami who gave them a chance to read unpublished manuscripts and suggested them to attack the problem.

1. Matsushima's Method [3]

Consider a homogeneous space $M=G/K$ of a connected semi-simple group G by a compact subgroup K, and assume that M is Riemannian symmetric in the following sense. The corresponding German letters $\mathfrak{g}$ and $\mathfrak{k}$ shall denote the Lie algebras. The vector space $\mathfrak{g}$ admits a direct sum decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ with $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}]\subset\mathfrak{m}$. φ denoting the Killing form of $\mathfrak{g}$, the

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restriction of $-\varphi$ or φ to $\mathfrak{m}$ defines an $(ad\ K)$-invariant metric on $\mathfrak{m}$ according as G is compact or not. By the natural identification of $\mathfrak{m}$ with the tangent space to M at a point left fixed by K, the metric on $\mathfrak{m}$ gives rise to a G-invariant Riemannian metric of M, by which M is Riemannian symmetric.

To formulate Matsushima's theorems, we shall fix some notations. First $\mathfrak{m}\vee\mathfrak{m}$ will denote the symmetric square of $\mathfrak{m}$; $\mathfrak{m}\vee\mathfrak{m}$ is thus the subspace of the tensor product $\mathfrak{m}\otimes\mathfrak{m}$ spanned by $\{x\vee y=(x\otimes y+y\otimes x)/2 \mid x, y\in\mathfrak{m}\}$. Since the metric space $\mathfrak{m}$ is naturally identified with its dual space $\mathfrak{m}^*$, the elements of $\mathfrak{m}\otimes\mathfrak{m}=\mathfrak{m}\otimes\mathfrak{m}^*$ are linear transformations of $\mathfrak{m}$. With this in mind, we define a linear transformation Q' of $\mathfrak{m}\otimes\mathfrak{m}$ by $Q'(x\otimes y)z=-[[y, z], x]$ $(=R(y, z)x$ if R denotes the curvature tensor). Well known properties of R assures us that Q' leaves invariant $\mathfrak{m}\vee\mathfrak{m}$ and the restriction $Q=Q_M$ of Q' to $\mathfrak{m}\vee\mathfrak{m}$ is self-adjoint with respect to the metric $<\,,\,>$ induced by the metric on $\mathfrak{m}$; in particular the eigenvalues of Q are real numbers. It is also obvious that Q commutes with the operation of $ad\,K$ on $\mathfrak{m}\vee\mathfrak{m}$. With Q, we define a quadratic form $H=H_M$ (also denoted by $H_{\mathfrak{g}}$) on $\mathfrak{m}\vee\mathfrak{m}$ by

$$H(\xi)=b<\xi,\xi>+<Q(\xi),\xi>,$$

where b is some constant [3] depending on M; in case G and K are simple, $b=(dim\ \mathfrak{m})/4\ dim\ \mathfrak{k}$. Now the results of Matsushima [3] are:

THEOREM I. *Let G be a non-compact semi-simple Lie group without compact simple factors and Γ be a discrete subgroup of G such that G/Γ is compact. If $H_{\mathfrak{g}(i)}$ is positive definite for each simple factor $\mathfrak{g}(i)$ of the Lie algebra $\mathfrak{g}$, then the first Betti number of G/Γ is equal to zero.*

THEOREM II. *Let G be a semi-simple Lie group without compact simple components, and the maximal compact subgroup K be not semi-simple. Suppose that each simple component of G is not locally isomorphic with the group of all $(n+1)\times(n+1)$ complex unimodular matrices which leave invariant the Hermitian form*

$$z_1\bar{z}_1+\cdots+z_{nn}\bar{z}-z_{n+1}\bar{z}_{n+1} \quad (n\geq 1).$$

If Γ be a discrete subgroup of G with compact quotient space G/Γ, then the first Betti number of G/Γ is equal to zero.

Our study in the sequel will be based on Theorem I. In short we have to show that H is positive definite, or what amounts to the same thing that the minimum eigenvalue of Q is greater than $-b$. In our case where G is complex simple and K is its maximal compact subgroup, there corresponds to the symmetric space $M=G/K$ canonically the compact symmetric space $K=(K\times K)/K$; in particular $\dim\mathfrak{m}=\dim\mathfrak{k}$. We then have $Q_M=-Q_K$ by the obvious identification. We thus have to show that the maximum eigenvalue of Q_K is smaller than $b=(\dim\mathfrak{m})/(4\dim\mathfrak{k})=1/4$.

To conclude this section, let us note that for a compact irreducible symmetric space M the trace of Q equals $(\dim M)/4$, in view of the fact that the Ricci tensor is (the metric tensor)/2 for such a space. The same fact also implies that $-1/2$ is the eigenvalue of Q with an eigen-vector $g^* \in \mathfrak{m} \vee \mathfrak{m}$, the element dual to the metric tensor g.

2. The Case of Group Manifolds

Under the assumption that K is a compact simple group and M is the quotient manifold of $G=K\times K$ by the diagonal $\{(k, k) \mid k \in K\} \cong K$, we shall explain how to find the maximum eigenvalue of Q. If one makes correspond to $k \in \mathfrak{k}$ the element $(k, -k)/\sqrt{2}$ in $\mathfrak{m}=\{(k, -k) | k \in \mathfrak{k}\}$, one obtains an isometric mapping π of $\mathfrak{k}$ with the usual metric defined by the Killing form of K onto $\mathfrak{m}$ with the metric mentioned in Section 1. Thus the two inner products in $\mathfrak{k}$ and in $\mathfrak{m}$ can be denoted by the same symbol $\langle\ ,\ \rangle$. This extends to bilinear forms on the complexifications of $\mathfrak{k}$ and $\mathfrak{m}$. The operation of $\mathfrak{k}$ on $\mathfrak{m}$ is given by the adjoint representation *ad* of the complexified Lie algebra $\mathfrak{k}^c$ on $\mathfrak{m}^c$. The linear transformation Q extends to $\mathfrak{m}^c \vee \mathfrak{m}^c$ and commutes with the representation $\Lambda \vee \Lambda = \mathrm{ad} \vee \mathrm{ad}$ induced on $\mathfrak{m}^c \vee \mathfrak{m}^c$. $\Lambda \vee \Lambda$ may be decomposed into the sum of irreducible representations; $\Lambda \vee \Lambda = \sum_p \Lambda_{(p)}$. The representation space of each $\Lambda_{(p)}$ is contained in the eigen-space of Q corresponding to an eigenvalue, say κ_p. We write $m_\lambda(\mu)$ for the multiplicity of a weight μ in a representation λ of $\mathfrak{k}^c$, and use the same notation for an irreducible representation of $\mathfrak{k}$ as for its highest weight. We agree that $\Lambda_{(p)} \geq \Lambda_{(p+1)}$ for all p with respect to some usual ordering. To simplify the description we assume that $\Lambda_{(p)} \neq \Lambda_{(p+1)}$ for all p, a fact to be shown later case by case. If W_p is the subspace of $m^c \vee m^c$ spanned by the weight vectors of the weight $\Lambda_{(p)}$, each W_p is Q-invariant and we have $\dim W = \sum_{q \leq p} m_q(\Lambda_{(p)})$ where m_q is m_λ with $\lambda = \Lambda_{(q)}$, together with

$$(2.1) \qquad (Tr\ Q)_p = \sum_{q \leq p} m_q(\Lambda_{(p)})\kappa_q,$$

$(Tr\ Q)_p$ being the trace of Q restricted to W_p. Each $m_q(\Lambda_{(p)})$ other than $m_p(\Lambda_{(p)})=1$ is given by the

Lemma 3.1 (Freudenthal [2]).

$$(2.2) \qquad m_\lambda(\mu)(\langle \lambda+\delta,\ \lambda+\delta\rangle - \langle \mu+\delta,\ \mu+\delta\rangle) = 2\sum_\alpha \sum_{k=1}^{\infty} m_\lambda(\mu+k\alpha)\langle \mu+k\alpha,\ \alpha\rangle,$$

where δ is the sum of the fundamental weights of $\mathfrak{k}$ and the first sum in the right hand side ranges over all positive roots and the second over all integers >0.

A root α of $\mathfrak{k}$ is a non-zero element of a fixed maximal abelian subalgebra $\mathfrak{h}$ such that there exists a non-zero element e_α of $\mathfrak{k}^c$ with $[h, e_\alpha] = i\langle h, \alpha\rangle e_\alpha$ for all $h \in \mathfrak{h}$, $i^2=-1$. e_α's are so chosen that $e_{-\alpha}$ is the complex conjugate of e_α with respect to $\mathfrak{k}$ and we have

$$(2.3) \qquad [e_\alpha, e_{-\alpha}] = i\alpha.$$

This implies

$$(2.4) \qquad \langle e_\alpha, e_{-\alpha}\rangle = 1.$$

Let R be the set of all roots. For $\alpha, \beta \in R$, a real number $N_{\alpha,\beta}$ satisfying

$[e_\alpha, e_\beta]=N_{\alpha,\beta}\, e_{\alpha+\beta}$ exists uniquely if $\alpha+\beta\in R$ and $N_{\alpha,\beta}=0$ otherwise by definition. Then we readily get

(2.5) $$N_{-\alpha,-\beta}=N_{\alpha,\beta}\,.$$

Let $(F_a)_{a\in A}$, A=an index set, be a base of $\mathfrak{m}^c$ and σ be an involutive mapping of A onto A with the property that $\langle F_a,F_b\rangle=1$ for $b=\sigma(a)$, and$=0$ for $b\neq\sigma(a)$. $Q(F_a\vee F_b)$ is uniquely written in the form $\sum_{u,v} a_{uv}F_u\vee F_v$, $a_{uv}=a_{vu}\in \boldsymbol{C}$. $Q(F_a\vee F_b)\in\mathfrak{m}^c\otimes\mathfrak{m}^c$ is a linear transformation of $\mathfrak{m}^c$ which sends F_c to

$$Q(F_a\vee F_b)F_c=\Sigma_{u,v}a_{uv}(\langle F_c,F_v\rangle F_u+\langle F_u,F_c\rangle F_v)/2=\Sigma_u a_{u,\sigma(c)}F_u\,.$$

Thus we have

$$\langle Q(F_a\vee F_b)F_c,F_d\rangle=\textstyle\sum_u a_{u,\sigma(c)}\langle F_u,F_d\rangle=a_{\sigma(d),\,\sigma(c)}=a_{\sigma(c),\,\sigma(d)}\,.$$

Hence the definition of Q gives

$$2a_{\sigma(c),\,\sigma(d)}=\langle Q(F_a\otimes F_b+F_b\otimes F_a)\,F_c,\,F_d\rangle=-\langle[[F_b,F_c],F_a]+[[F_a,F_c],F_b],F_d\rangle\,.$$

Since the relation $\langle[[F_b,F_c],F_a],F_d\rangle=-\langle[F_b,F_c],\,[F_a,F_d]\rangle$ follows from compactness of G, we conclude

$$a_{\sigma(c),\,\sigma(d)}=-\tfrac{1}{2}(\langle[F_b,F_c],\,[F_a,F_d]\rangle+\langle[F_a,F_c],\,[F_b,F_d]\rangle)\,.$$

We rewrite this formula in terms of $f_a=\pi^{-1}(F_a)$ (π naturally extended to a map of $\mathfrak{k}^c$), obtaining

(3.6) $$a_{c,d}=\tfrac{1}{4}(\langle[f_a,f_{\sigma(c)}],\,[f_{\sigma(d)},f_b]\rangle+\langle[f_a,f_{\sigma(d)}],\,[f_{\sigma(c)},f_b]\rangle)\,,$$

where we have used the property $\sigma\circ\sigma$=identity. *The coefficient of* $f_a\vee f_b$ *in* $Q(F_a\vee F_b)$ is by definition a_{ab} if $a=b$ and $a_{ab}+a_{ba}$ if $a\neq b$.

Now an orthonormal base $(h_j)_{1\leq j\leq \mathrm{rank}\,\mathfrak{k}}$ and the root vectors e_α form a base of $\mathfrak{k}^c$. Putting $H_j=\pi(h_j)$, $E_\alpha=\pi(e_\alpha)$, $\sigma(j)=j$, and $\sigma(\alpha)=-\alpha$, we obtain a base $(\ldots, H_j,\ldots, E_\alpha,\ldots)$ satisfying the above mentioned property. By the preceding definitions and the formulas (2.3) to (2.6), it is quite easy to establish the

LEMMA. 2.2. *Given roots* $\alpha,\beta\in R$, *we have*

(1) *The coefficient of* $e_\alpha\vee e_\beta$ *in* $Q(E_\alpha\vee E_\beta)$ *is* $\frac{1}{2}(\langle\alpha,\beta\rangle+N_{\alpha,-\beta}^2)$,

(2) *that of* $h_j\vee e_\alpha$ *in* $Q(H_j\vee E_\alpha)$ *is* $\frac{1}{2}\langle\alpha, h_j\rangle^2$, *and*

(3) *that of* $h_j\vee h_k$ *in* $Q(H_j\vee H_k)$ *vanishes.*

$(Tr\ Q)_p$ in (2.1) can be calculated with Lemma 2.2 on account of the fact that W_p is spanned by some of $E_\alpha\vee E_\beta$, $H_j\vee E_\beta$, and $H_j\vee H_k$; if $\Lambda_{(p)}$ is neither a root nor the zero for instance, W_p is spanned by $\{E_\alpha\vee E_\beta\,|\,\alpha,\beta\in R,\ \alpha+\beta\in\Lambda_{(p)}\}$.

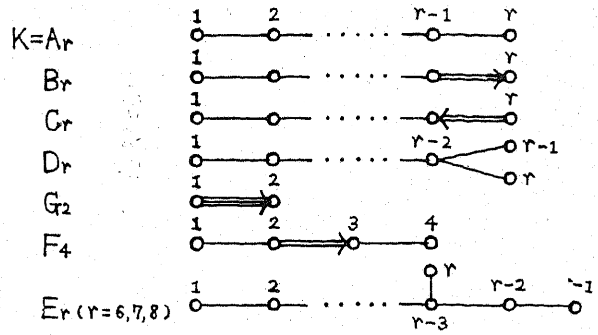
Thus it should be our task to find

1) each irreducible component $\Lambda_{(p)}$ of $\Lambda=ad\vee ad$,
2) the number in which $\Lambda_{(p)}$ is written as the sum of two roots,
3) $(Tr\ Q)_p$ as above,
4) the multiplicity $m_q(\Lambda_{(p)})$ in (2.1) by means of lemma 2.1 and
5) each eigenvalue κ_q of Q with repeated use of (2.1).

We shall also take advantage of the remark at the end of Section 1. As a matter of fact some of those steps could be replaced by additional observations so that the whole proof would considerably be shortened, though we shall rather choose simple description to show our tedious and easy procedure.

3. The Decomposition of the Representation Λ

This section will be devoted to the survey of 1), 2) and 4) in the program at the end of Section 2. Let $(\alpha_1, \ldots, \alpha_r)$, $r=\operatorname{rank} K$, be a system of simple roots. The inner products $\langle\alpha_j, \alpha_k\rangle$ are given by the diagrams:



The j-th fundamental wight Λ_j is a linear combination $\sum_k u_{jk}\alpha_k$ with $2\langle\Lambda_j, \alpha_k\rangle=\delta_{jk}\langle\alpha_k, \alpha_k\rangle$. The dominant forms (=the linear combinations of Λ_j's with non-negative integers as coefficients) are canonically identified with the equivalence classes of the irreducible complex linear representations of $\mathfrak{k}^c$.

As an example, we consider the case $K=B_r$. First we assume $r>4$. The adjoint representation is then Λ_2. Knowing the values $u_{jk}=\operatorname{Min}\{j, k\}\div(1+\delta_{jr})$ for B_r, the dominant forms which are linear combinations of α_k's with integral coefficients and which are not higher than $2\Lambda_2$ are readily enumerated. In the case of B_r they are all weights of $\Lambda=\Lambda_2\vee\Lambda_2$; in fact they are sums of two roots: $2\Lambda_2=\Lambda_2+\Lambda_2$, $\Lambda_2=\Lambda_1+(\Lambda_1-\alpha_1)$, $2\Lambda_1=\Lambda_1+\Lambda_1=\alpha_1+\Lambda_2$, $\Lambda_1+\Lambda_3=\Lambda_2+(\Lambda_2-\alpha_2)$, $\Lambda_4=\Lambda_2+(\Lambda_2-\alpha_1-\alpha_2-\alpha_3)$, $\Lambda_3=\Lambda_2+(\Lambda_1-\alpha_1-\alpha_2)$, $\Lambda_1=\alpha_1+(\Lambda_1-\alpha_1)$, and $0=\alpha_1+(-\alpha_1)$. There are no other dominant forms which are weights of Λ. Their multiplicities in Λ are the number by which they are expressed as the sums of two roots or the zeros, and are shown in Table I; a number N in Tables I to VIII is the multiplicity $m_\lambda(\mu)$ of the weight μ on the left end of the same row in the representation λ at the top of the same column as N, which is calculated with Lemma 2.1.

From Table I, we conclude that Λ is the sum of the irreducible representations $2\Lambda_2$, $2\Lambda_1$, Λ_4 and 0; $\Lambda=2\Lambda_2+2\Lambda_1+\Lambda_4+0$.

Remark. This decomposition may be interpreted geometrically as follows. B_r is the Lie algebra of the orthogonal group $O(n)$, $n=2r+1$. Hence B_r consists of skew-symmetric matrices of degree n. B_r is the space of 2-forms over an n-dimensional vector space C^n. The representation space of Λ is thus

Table I

B_r, $r>4$, $ad=\Lambda_2$

	$\Lambda_2\vee\Lambda_2$	$2\Lambda_2$	$2\Lambda_1$	Λ_4	0
$2\Lambda_2$	1	1	0	0	0
$\Lambda_1+\Lambda_3$	1	1	0	0	0
$\Lambda_1+\Lambda_2$	1	1	0	0	0
$2\Lambda_1$	r	$r-1$	1	0	0
Λ_4	3	2	0	1	0
Λ_3	3	2	0	1	0
Λ_2	$3r-3$	$2r-2$	1	$r-2$	0
Λ_1	$3r-2$	$2r-2$	1	$r-1$	0
0	$r^2+\frac{1}{2}r(r+1)$	r^2-1	r	$\frac{1}{2}r(r-1)$	1

Table II

A_r, $r>2$, $ad=\Lambda_1+\Lambda_r$

	$(\Lambda_1+\Lambda_r)\vee(\Lambda_1+\Lambda_r)$	$2(\Lambda_1+\Lambda_r)$	$\Lambda_2+\Lambda_{r-1}$	$\Lambda_1+\Lambda_r$	0
$2(\Lambda_1+\Lambda_r)$	1	1	0	0	0
$2\Lambda_1+\Lambda_{r-1}$	1	1	0	0	0
$\Lambda_2+2\Lambda_r$	1	1	0	0	0
$\Lambda_2+\Lambda_{r-1}$	2	1	1	0	0
$\Lambda_1+\Lambda_r$	$2r-1$	r	$r-2$	1	0
0	$r(r+1)$	$\frac{1}{2}r(r+1)$	$\frac{1}{2}(r+1)(r-2)$	r	1

For the case $r=2$, the rows and the columns of $\Lambda_2+\Lambda_{r-1}$ must be omitted.

the symmetric square V of this space.

The space F of 4-forms over C^n is contained in V and invariant under $O(n)$. This corresponds to Λ_4. Another invariant space is the one consisting of the tensors in V which satisfies "the Bianchi identity" or the space of the "Riemannian curvature tensors." It is not hard to see that this space, say B, is the orthogonal complement of F is V. If one corresponds the "Ricci tensor" to a curvature tensors, one obtains a homomorphism of representations. Thus B contains the irreducible invariant subspaces corresponding to $2\Lambda_1$ and 0. (The tensors corresponding to 0 are the curvature tensors of Riemannian spaces of constant curvature.) The orthogonal complement of the space of $2\Lambda_1$ in B is the space of curvature tensors of Einstein spaces. It is the sum $2\Lambda_2+0$. $2\Lambda_2$ corresponds to the space of curvature tensors with vanishing Ricci tensors or the space of Weyl's conformal curvature tensors. D_r admits a parallel treatment.

For B_4, Λ_4 in Table I must be replaced by $2\Lambda_4$, but no other changes are necessary. For B_3, the space of 4-forms are equivalent to the space of

3-forms and Λ_4 must be replaced by $2\Lambda_3$, and Λ_3 must be omitted. For B_2, the adjoint representation is $2\Lambda_2$. The space of 4-forms is equivalent to the space of 1-forms, and Λ_4 must be replaced by Λ_1. Some other trivial changes are necessary. For B_1, we have $ad=2\Lambda_1$, and Weyl's conformal curvature tensors vanish. The space of 4-forms has no sense. Since $B_1=A_1$, the result is also obtained with Clebsch-Gordan's formula.

For other groups we only show the results by tables.

Table III

C_r, $r>3$, $ad=2\Lambda_1$

	$2\Lambda_1\vee 2\Lambda_1$	$4\Lambda_1$	$2\Lambda_2$	Λ_2	0
$4\Lambda_1$	1	1	0	0	0
$2\Lambda_1+\Lambda_2$	1	1	0	0	0
$2\Lambda_2$	2	1	1	0	0
$\Lambda_1+\Lambda_3$	2	1	1	0	0
$2\Lambda_1$	$2r-1$	r	$r-1$	0	0
Λ_4	3	1	2	0	0
Λ_2	$3r-2$	r	$2r-3$	1	0
0	$r^2+\frac{1}{2}r(r+1)$	$\frac{1}{2}r(r+1)$	$r(r-1)$	$r-1$	1

In case $r=3$, the row of Λ_4 must be omitted.

Table IV

D_r, $r>3$, $ad=\Lambda_2$

	$\Lambda_2\vee\Lambda_2$	$2\Lambda_2$	$2\Lambda_1$	Λ_4	0
$2\Lambda_2$	1	1	0	0	0
$\Lambda_1+\Lambda_3$	1	1	0	0	0
$2\Lambda_1$	$r-1$	$r-2$	1	0	0
Λ_4	3	2	0	1	0
Λ_2	$3r-4$				0
0	$r(r-1)+\frac{1}{2}r(r+1)$				1

Table V

E_6 $ad=\Lambda_6$

	$\Lambda_6\vee\Lambda_6$	$2\Lambda_6$	$\Lambda_1+\Lambda_5$	0
$2\Lambda_6$	1	1	0	0
Λ_3	1	1	0	0
$\Lambda_1+\Lambda_5$	4	3	1	0
Λ_6	16			0
0	57			1

Table VI

E_7 $ad=\Lambda_6$

	$\Lambda_6\vee\Lambda_6$	$2\Lambda_6$	Λ_2	0
$2\Lambda_6$	1	1	0	0
Λ_5	1	1	0	0
Λ_2	5	4	1	0
Λ_6	23			0
0	91			1

Table VII

E_8 $ad=\Lambda_1$

	$\Lambda_1\vee\Lambda_1$	$2\Lambda_1$	Λ_7	0
$2\Lambda_1$	1	1	0	0
Λ_2	1	1	0	0
Λ_7	7	6	1	0
Λ_1	36			0
0	156			1

Table VIII

F_4 $ad=\Lambda_1$

	$\Lambda_1\vee\Lambda_1$	$2\Lambda_1$	$2\Lambda_4$	0
$2\Lambda_1$	1	1	0	0
Λ_2	1	1	0	0
$\Lambda_1+\Lambda_4$	1	1	0	0
$2\Lambda_4$	4	3	1	0
Λ_3	4			0
Λ_1	11			0
Λ_4	14			0
0	34			1

Table IX

G_2 $ad=\Lambda_1$

	$\Lambda_1\vee\Lambda_1$	$2\Lambda_1$	$2\Lambda_2$	0
$2\Lambda_1$	1	1	0	0
$\Lambda_1+\Lambda_2$	1	1	0	0
$3\Lambda_2$	1	1	0	0
$2\Lambda_2$	3	2	1	0
Λ_1	4	3	1	0
Λ_2	5	3	2	0
0	9	5	3	1

These tables give the irreducible decomposition:

For $K=A_r$, $r>2$, $\Lambda=ad\vee ad=(\Lambda_1+\Lambda_r)\vee(\Lambda_1+\Lambda_r)$
$=2(\Lambda_1+\Lambda_r)+(\Lambda_2+\Lambda_{r-1})+(\Lambda_1+\Lambda_r)+0$,

A_2, $2(\Lambda_1+\Lambda_2)+(\Lambda_1+\Lambda_2)+0$.

A_1, $2\Lambda_1\vee 2\Lambda_1=4\Lambda_1+0$,

B_r, $r>4$, $\Lambda_2\vee\Lambda_2=2\Lambda_2+2\Lambda_1+\Lambda_4+0$,

B_r, $r=3, 4$, $2\Lambda_2+2\Lambda_1+2\Lambda_r+0$,

B_2, $(2\Lambda_2)\vee(2\Lambda_2)=4\Lambda_2+2\Lambda_1+\Lambda_1+0$,

C_r, $r>2$, $(2\Lambda_1)\vee(2\Lambda_1)=4\Lambda_1+2\Lambda_2+\Lambda_2+0$,

D_r, $r>5$, $\Lambda_2\vee\Lambda_2=2\Lambda_2+2\Lambda_1+\Lambda_4+0$,

D_r, $r=4, 5$, $2\Lambda_2+2\Lambda_1+(\Lambda_{r-1}+\Lambda_r)+0$,

E_6, $\Lambda_6\vee\Lambda_6=2\Lambda_6+(\Lambda_1+\Lambda_5)+0$,

E_7, $\Lambda_6\vee\Lambda_6=2\Lambda_6+\Lambda_2+0$,

E_8, $\Lambda_1\vee\Lambda_1=2\Lambda_1+\Lambda_7+0$,

F_4, $\Lambda_1\vee\Lambda_1=2\Lambda_1+2\Lambda_4+0$,

G_2, $\Lambda_1\vee\Lambda_1=2\Lambda_1+2\Lambda_2+0$.

4. The Eigenvalues of Q

The aim of this section is to accomplish the tasks 3) and 5) in the program at the end of Section 2. We take B_r, $r>4$, as an example again. The heights of the irreducible components $2\Lambda_2$, Λ_4, $2\Lambda_1$ and 0 of Λ are in this order. Obviously we have dim $W_1=1$ and, applying (2.1) and Lemma 2.2, we find

$$\tfrac{1}{2}\langle \Lambda_2, \Lambda_2\rangle=(Tr\ Q)_1=\kappa_1 .$$

We saw $\dim W_2=m_\Lambda\ (\Lambda_4)=3$ in Table I; in fact Λ_4 is written as the sum of two roots α, β in three ways: $\alpha=\Lambda_2$, $\Lambda_2-\alpha_2$, and $\Lambda_2-(\alpha_1+\alpha_2)$. Since the latter two are obtained from Λ_2 by reflections with respect to α_2 and $\alpha_1+\alpha_2$, which leave Λ_4 fixed, $(\langle\alpha, \beta\rangle+N_{\alpha,-\beta}^2)$ takes the same value for the three unordered pairs (α, β) with $\alpha+\beta=\Lambda_4$. If in general $\alpha+\beta$ is not a root and $\langle\alpha, \beta\rangle$ vanishes, then we have $N_{\alpha,-\beta}=0$. Therefore, applying (2.1) and lemma 2.2, we get $\quad 2\kappa_1+\kappa_2=(Tr\ Q)_2=0$, or $\kappa_2=-2\kappa_1=-(\Lambda_2, \Lambda_2)$.

Table I reads $m_1(2\Lambda_1)=1$, $m_2(2\Lambda_1)=0$ and $m_3(2\Lambda_1)=1$; in fact $2\Lambda_1=\alpha+(2\Lambda_1-\alpha)$, where $\alpha=\alpha_1+\alpha_2+\cdots+\alpha_j$, with $1\leq j<r$, and $2\Lambda_1=\Lambda_1+\Lambda_1$. For the first $(r-1)$ pairs, $\langle\alpha, 2\Lambda_1-\alpha\rangle=\langle\alpha_1, 2\Lambda_1-\alpha_1\rangle=0$, hence $N_{\alpha,-\beta}=0$. For the last one (Λ_1, Λ_1), we have $N_{\alpha,-\beta}=0$ also. Therefore we obtain

$$(r-1)\kappa_1+\kappa_3=(Tr\ Q)_3=\tfrac{1}{2}\langle\Lambda_1, \Lambda_1\rangle, \quad \text{whence}$$
$$\kappa_3=\tfrac{1}{2}\langle\Lambda_1, \Lambda_1\rangle-(r-1)\kappa_1=-(2r-3)\langle\Lambda_1, \Lambda_1\rangle/2 .$$

κ_3 can also be evaluated with known values of $\kappa_4=-\frac{1}{2}$ and $Tr\, Q$ after one determined κ_1, κ_2 and the degrees of the irreducible components of Λ. Finally the lengths of the roots are calculated with Freudental's formula on the eigenvalues of the Casimir operators since that of the adjoint representation is the identity. The tables IX to XVI are completed in this way. The first columns will be the irreducible components of Λ, the second the eigenvalues of Q on the spaces of the irreducible representations on their left, and the third the multiplicities of the eigenvalues, or in other words, the degrees of the representations.

Table X

$\boldsymbol{B_r}$, $r>1$

	κ_p	multiplicity
$2\Lambda_2$	$1/2(2r-1)$	$(r-1)(r+1)(2r+1)(2r+3)/3$
Λ_4	$-1/(2r-1)$	$r(r-1)(2r-1)(2r+1)/6$
$2\Lambda_1$	$-(2r-3)/4(2r-1)$	$r(2r+3)$
0	$-\frac{1}{2}$	1

For the case $2\leq r\leq 4$, some modifications in the first column are necessary, which are similar to the ones indicated as regards Table I.

For $r=1$, the eigenvalues are $\frac{1}{4}$ and $-\frac{1}{2}$.

Table XI

$\boldsymbol{A_r}$, $r>1$

$2(\Lambda_1+\Lambda_r)$	$1/2(r+1)$	$\frac{1}{4}r(r+1)^2(r+4)$
$\Lambda_2+\Lambda_{r-1}$	$-1/2(r+1)$	$\frac{1}{4}(r+1)^2(r^2-4)$
$\Lambda_1+\Lambda_r$	$-\frac{1}{4}$	$(r+1)^2-1$
0	$-\frac{1}{2}$	1

For the case $r=2$, the second row of $\Lambda_2+\Lambda_{r-1}$ must be omitted.

Table XII

$C_r,\ r>2$

$4\Lambda_1$	$1/2(r+1)$	$r(r+1)(2r+1)(2r+3)$
$2\Lambda_2$	$-1/4(r+1)$	$r(r-1)(2r-1)(2r+3)/3$
Λ_2	$-(r+2)/4(r+1)$	$(2r+1)(r-1)$
0	$-\frac{1}{2}$	1

Table XIII

$D_r,\ r>3$

$2\Lambda_2$	$1/4(r-1)$	$r(r+1)(2r-3)(2r+1)/3$
Λ_4	$-1/2(r-1)$	$r(r-1)(2r-1)(2r-3)/6$
$2\Lambda_1$	$-(r-2)/4(r-1)$	$2r^2+r-1$
0	$-\frac{1}{2}$	1

Table XIV

E_6

$2\Lambda_6$	1/24	2430
$\Lambda_1+\Lambda_5$	−/8	650
0	$-\frac{1}{2}$	1

Table XV

E_7

$2\Lambda_6$	1/36	7371
Λ_2	−1/9	1539
0	$-\frac{1}{2}$	1

Table XVI

E_8

$2\Lambda_1$	1/60	27000
Λ_7	−1/10	3875
0	$-\frac{1}{2}$	1

Table XVII

E_4

$2\Lambda_1$	1/18	1053
$2\Lambda_4$	−5/36	324
0	$-\frac{1}{2}$	1

Table XVIII

G_2

$2\Lambda_1$	1/8	77
$2\Lambda_2$	−5/24	27
0	$-\frac{1}{2}$	1

5. Conclusions

Using the results in the preceeding sections, we shall prove our main theorems.

THEOREM 1. *Let G be a complex semi-simple Lie group, and Γ be a discrete subgroup of G. If no simple components of G are of rank* 1 *and the factor space G/Γ is compact, the first Betti number of G/Γ equals zero and the group $\Gamma/[\Gamma, \Gamma]$ is finite, where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ.*

Proof. The tables in Section 4 tell us that the linear operator Q for the symmetric space G/K, K a maximal compact subgroup, has the minimum eigenvalue strictly greater than $-\frac{1}{4}$, provided that G is simple and has rank greater than 1. Thus Theorem I applies, and we find the first part true. Let $\tilde{G}$ denote the universal covering group of G, p the canonical projection of $\tilde{G}$ onto G with kernel Δ. The inverse image Γ of $\tilde{\Gamma}$ under p is then a discrete subgroup of $\tilde{G}$ containing Δ. $\tilde{G}/\tilde{\Gamma}$ is diffeomorphic to G/Γ. Furthermore $\tilde{\Gamma}$ is isomorphic with the fundamental group of $\tilde{G}/\tilde{\Gamma} \approx G/\Gamma$. Since the first Betti number of G/Γ is zero, $\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}]$ is a finite group. On the other hand p carries $\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}]$ onto $\Gamma/[\Gamma, \Gamma]$, which is thus finite.

REMARK. A complex manifold is called *complex parallisable* if the tangent bundle is holomorphically trivial. H.C. Wang [4] proved that any compact connected complex parallisable manifold is non-kählerian in general and is holomorphically equivalent to the coset space of a complex Lie group by a discrete subgroup. Theorem 1 shows that if this complex Lie group satisfies the hypotheses of theorem 1, the first Betti number of the corresponding compact complex parallisable manifold is zero.

THEOREM 2. *Assume that M is a simply connected symmetric Riemannian space whose isometry group $I(M)$ is complex semi-simple. Let Γ be a properly discontinuous group of isometries of M such that the quotient space M/Γ is compact. If the rank of each simple component of $I(M)$ is larger than* 1 *and Γ contains no element of finite order, then the first Betti number of M/Γ is equal to zero.*

Proof. The proof given here is due to S. Murakami essentially. Let K' and K respectively be the isotropy subgroups of $I(M)$ and of G, the identity component of $I(M)$. Then the index $[K' : K]$ of K in K' is finite, and so is the index $[I(M) : G]$. Denoting $\Gamma \cap G$ by Γ_0, we have $[\Gamma : \Gamma_0] \leq [I(M) : G]$, and consequently $[\Gamma : \Gamma_0]$ is finite. Since the group Γ without element of finite order operates on M properly discontinuously, the quotient space M/Γ is a manifold, and Γ is the fundamental group of M/Γ. Also M/Γ_0 is a finite convering of M/Γ, and M/Γ_0 is thus compact. Thus G/Γ_0 is compact by compactness of K. By theorem 1, $\Gamma_0/[\Gamma_0, \Gamma_0]$ is a finite group, and so is $\Gamma/[\Gamma, \Gamma]$. Therefore the first Betti number of M/Γ is zero.

References

[1] Borel, A., *Ann. of Math.*, **71**, 508-521 (1960).
[2] Freudenthal, H., *Indagationes*, **16**, 369-376 (1954).
[3] Matsushima, Y., *Ann. of Math.*, **75**, 312-330 (1962).
[4] Wang, H.C., *Proc. Amer. Math. Soc.*, **5**, 771-776 (1954).

Added in proof.

The proofs of Theorems 1 and 2 for the bounded symmetric domains are found in [3] which appeared in Annals of Mathematics after the preparation of the present paper.