On a Generalization of the Weinstein-Aronszajn Formula and the Infinite Determinant

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1. Introduction

In *Weinstein-Aronszajn's method** of obtaining lower bounds for eigenvalues, it is important to investigate the mutual relation between the spectra of two self-adjoint operators A and A' in the following two cases: (I) $A'-A$ is a degenerate operator; (II) A' is the part of A in a subspace with finite codimension. (The first case is often referred to as *degenerate perturbation.*) In each case the mutual relation is established by the *Weinstein-Aronszajn formula*, which states that the change of the location and the multiplicity of eigenvalues is determined by the location and the order of zeros and poles of a certain meromorphic function called the *Weinstein determinant*. In the present paper the Weinstein-Aronszajn formula for the case (I) (resp. (II)) is provisionally called the *first* (resp. the *second*) W-A *formula*. Heretofore these two cases seem to have been treated separately by analogous methods. So far as the W-A formula is concerned, however, it may be said that the first case is more important than the second. For, on the one hand, the second W-A formula can be deduced from the first and, on the other, the first W-A formula admits of a generalization to the perturbation of closed operator by a wider class of operators than the degenerate operators. (In this paper " operator " always means " linear operator ".)

One of the purposes of the present paper is to prove the first W-A formula in such a generalized form (3.2), by replacing Weinstein determinant by a sort of infinite determinant. Contrary to the original proof, which reduces the problem to the one-dimensional perturbation, we shall prove directly the generalized formula. An essential step of our proof lies in the verification of the formula (3.4), which is done by making use of Dunford's theory of operational calculus. Even in the original case of degenerate perturbation, our method provides a new and, possibly, simple proof of the W-A formula. The second W-A formula is deduced from the first in a slightly generalized form.

Another purpose of the present paper is to make a study of the *infinite determinant* in a Hilbert space in connection with the above mentioned generalization of the W-A formula. The infinite determinant has been treated by

* Aronszajn [1], [2]. See also Fujita [3]. The writer wishes to express his thanks to Dr. H. Fujita, who brought author's attention to this problem.

several authors in various ways (e. g., [4], [8], [9] and [11]). Most of them concern themselves with the Fredholm theory of linear equations for some (not necessarily integral) operators in a Hilbert or a Banach space, and define the infinite determinant of $1-zT$ by a power series of z as the generalized Fredholm determinant. In this paper we shall make a different approach. In 2, confining ourselves to the case of Hilbert space, we shall consider an operator of the form $1-T$ with T belonging to the trace class, and define the infinite determinant of $1-T$ directly (not by a power series) as $\exp\{\mathrm{tr}(\log(1-T))\}$. Some properties of it will be examined also in 2 with special emphasis on its regularity with respect to T. Though our definition is shown to be equivalent to that of Smithies [11], our expression seems more convenient for the application to problems considered.

It should be remarked that Ruston considered in [9] and the subsequent papers the trace class in a Banach space. Nevertheless, it appears that the question is still open whether the trace in a Banach space is a uniquely determined functional on the trace class (see also [4]). Since this situation is quite inconvenient for our purpose, we confine our study to operators in a Hilbert space. But we shall return to this point at the end of the paper.

Here, we shall enumerate those tools and notations which are used throughout the present paper.

We shall consider the problem in a fixed Hilbert space $\mathfrak{H}$. Let A be a densely defined closed operator in $\mathfrak{H}$. We denote by $\rho(A)$, $\sigma(A)$ and $\sigma_p(A)$ the resolvent set, the spectrum and the point spectrum, respectively. Furthermore, we put $R(z;A)=(z-A)^{-1}$, $z\in\rho(A)$. The union of $\rho(A)$ and the set of all isolated singularities of $R(z;A)$ is denoted by $\hat{\rho}(A)$. Let $z\in\hat{\rho}(A)$ and Γ a sufficiently small circle about z. Then the operator $P(z;A)=\dfrac{1}{2\pi i}\displaystyle\int_\Gamma R(\xi;A)d\xi$ is a projection whose range we denote by $\mathfrak{N}(z;A)$.

For each complex number z, the number $\nu(z;A)$ is now defined by

$$\nu(z;A)=\begin{cases}\dim\mathfrak{N}(z;A)\,, & \text{if}\quad z\in\hat{\rho}(A)\,,\\ \infty\,, & \text{if}\quad z\notin\hat{\rho}(A)\,.\end{cases}$$

If $z\in\rho(A)$, then $\nu=0$ and if $z\in\hat{\rho}(A)-\rho(A)$, then ν is the algebraic multiplicity of the eigenvalue z.

Incidentally, we remark that, if $\nu(a;A)$ is finite and positive, a is a pole of order $\mu\leqq\nu(a;A)$ of $R(z;A)$. In such a case, Laurent's expansion of $R(z;A)$ at $z=a$ has the form

$$R(z;A)=\frac{N^{\mu-1}}{(z-a)^\mu}+\cdots+\frac{N}{(z-a)^2}+\frac{P(a;A)}{z-a}+A(z)\,, \tag{1.1}$$

where $A(z)$ is regular at $z=a$ and N is a nilpotent with the range contained in $\mathfrak{N}=\mathfrak{N}(a;A)$. Since $\mathfrak{N}$ is finite-dimensional, it follows from Jordan's canonical form that

$$\mathrm{tr}(N^k)=0\,,\quad k=1,2,\cdots,\quad \text{and}\quad \mathrm{tr}(P(a;A))=\nu(a;A)\,. \tag{1.2}$$

On the other hand, for a function $f(z)$ meromorphic in a domain U of the complex plane, the order of f at a point $a \in U$ is denoted by $\nu(a; f)$. In other words, if $f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$, $a_n \neq 0$, $-\infty < n < \infty$, is Laurent's expansion of $f(z)$ at $z=a$, then $\nu(a; f)=n$. Furthermore, we put $\nu(a; f)=\infty$, if f is identically zero.

Other notations and conventions are as follows. $\boldsymbol{B}$ is the set of all bounded operators on $\mathfrak{H}$ to $\mathfrak{H}$ and $\boldsymbol{T}$ the trace class; $\|T\|_1$ and $\operatorname{tr}(T)$ denote the trace norm and the trace of $T \in \boldsymbol{T}$, respectively. The trace is a bounded functional on $\boldsymbol{T}$ which satisfies

$$\operatorname{tr}(AT) = \operatorname{tr}(TA), \qquad T \in \boldsymbol{T}, \quad A \in \boldsymbol{B}. \tag{1.3}$$

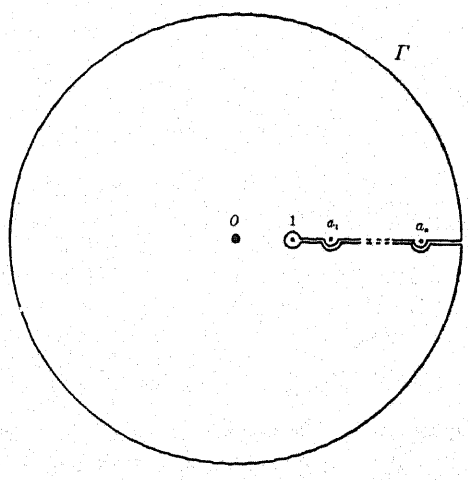
$\boldsymbol{S}$ is the Schmidt class; $\| \ \|$ and $\| \ \|_2$ denote the usual and the Schmidt norm, respectively.

If an operator-valued function $T(z) \in \boldsymbol{T}$ of a complex variable z is continuous (resp. regular) in the sense of $\| \ \|_1$, we say that $T(z)$ is *t-continuous* (resp. *t-regular*). The word "t-convergent" has a similar meaning. (When we say, e. g., that $T(z)$ is continuous, $T(z)$ is continuous in the sense of $\| \ \|$.)

$\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ are the domain and the range of an operator A, respectively; $C(a; r)$ is a circle in the complex plane with center a and radius r. For a square matrix $A=(a_{ij})$ the usual determinant of A is always denoted by $\det(A) = \det(a_{ij})$.

2. Infinite determinant

When $1-T$ is a positive definite symmetric matrix with eigenvalues a_k, we have $\det(1-T) = \Pi(1-a_k) = \exp\{\sum \log(1-a_k)\} = \exp\{\operatorname{tr}(\log(1-T))\}$. In accordance with this formula we shall define in $\mathfrak{H}$ the infinite determinant $\mathrm{d}(1-T)$ of $1-T$, $T \in \boldsymbol{T}$, with the suitable definition of $\log(1-T)$.



Let $T \in \boldsymbol{T}$. For the moment we further assume that $1 \in \rho(T)$ and denote the finite set $\sigma(T) \cap (1, \infty)$ by $\{a_1, a_2, \cdots, a_n\}$, $a_1 < a_2 < \cdots < a_n$. Let Γ be the closed contour shown in the figure. (The radius of the circle about zero is greater than $\|T\|$ and $\sigma(T)$ is entirely inside Γ.) In conformity with Dunford's theory of operational calculus, the operator $\operatorname{Log}(1-T)$ is defined by

$$\operatorname{Log}(1-T) = \frac{1}{2\pi i} \int_{\Gamma} \log(1-\xi) R(\xi; T) d\xi \,, \tag{2.1}$$

where $\log(1-\xi)$ is taken to be real for real $\xi<1$. Since $g(\xi)=\log(1-\xi)/\xi$ is regular on and inside Γ, we have $g(T)\in \boldsymbol{B}$ and hence $\mathrm{Log}(1-T)=Tg(T)\in \boldsymbol{T}$.

Let now $T\in \boldsymbol{T}$ be arbitrary. Then $\mathrm{d}(1-T)$ will be defined by the formula

$$\mathrm{d}(1-T)=\begin{cases} \exp\{\mathrm{tr}(\mathrm{Log}(1-T))\}, & \text{if } 1\in\rho(T), \\ 0, & \text{if } 1\in\sigma(T). \end{cases} \tag{2.2}$$

In the remainder of this section we shall examine some properties of $\mathrm{d}(1-T)$ thus defined. For brevity we shall tacitly assume that $T\in \boldsymbol{T}$.

LEMMA 2.1. *Let Γ_0 be a closed contour such that* i) *Γ_0 is contained in $\rho(T)$,* ii) *Γ_0 is entirely inside* $C(0, 1)$ *and* iii) 0 *is inside Γ_0; let $\{a_1, \cdots, a_n\}$ be the set of all points in $\sigma(T)$ which are outside Γ_0 and put $\nu_k=\nu(a_k; T)$. Then we have*

$$\mathrm{d}(1-T)=\left\{\prod_{k=1}^{n}(1-a_k)^{\nu_k}\right\}\exp\left\{\mathrm{tr}\left(\frac{1}{2\pi i}\int_{\Gamma_0}\log(1-\xi)R(\xi;\, T)\,d\xi\right)\right\}. \tag{2.3}$$

(*We agree that, when $\{a_k\}$ is empty, the factor* Π *is replaced by* 1.)

Proof. If $1\in\sigma(T)$, (2.3) is obvious. If $1\in\rho(T)$, we replace Γ in (2.1) by the union of Γ_0 and sufficiently small circles Γ_k about a_k and then substitute for $R(\xi;\, T)$ under $\int_{\Gamma_k}$ Laurent's expansion (1.1) of R at a_k. By virtue of (2.2) and (1.2) we now obtain (2.3) at once.

Remark. It may seem somewhat artificial that we used in (2.2) the operator $\mathrm{Log}(1-T)$, which should be called the "principal value" of $\log(1-T)$. More naturally, Γ in (2.1) should be the union of Γ_k, $k=0, 1, \cdots, n$, defined above and on each Γ_k, $k=1, \cdots, n$, $\log(1-\xi)$ should be taken to be on an arbitrary branch depending on k. (On Γ_0 the same branch as before must be taken in order that the operator $\log(1-T)$ should belong to $\boldsymbol{T}$.) Nevertheless, we can see as above that $\exp\{\mathrm{tr}(\log(1-T))\}$ also satisfies (2.3) and hence coincides with $\mathrm{d}(1-T)$.

THEOREM 2.1. *Let $T(z)\in \boldsymbol{T}$ be t-regular* (*resp. t-continuous*) *in a domain U. Then the function* $\mathrm{d}(1-T(z))$ *is regular* (*resp. continuous*) *in U.*

Proof. Let $T(z)$ be t-regular in U. Let a be an arbitrary point of U and Γ_0 the contour given in Lemma 2.1 with T replaced by $T(a)$. Then there exists an open disque V containing a such that $\Gamma_0\subset\rho(T(z))$ for each $z\in V$. Then (2.3) holds true with T replaced by $T(z)$, $z\in V$, Γ_0 being independent of z. On the other hand, if $z'\to z\in V$, $(z'-z)^{-1}\{R(\xi;\, T(z'))-R(\xi;\, T(z))\}$ is t-convergent to $R(\xi;\, T(z))T'(z)R(\xi;\, T(z))$ uniformly on Γ_0. Hence, it follows that $\int_{\Gamma_0}\log(1-\xi)R(\xi;\, T(z))d\xi$ is t-regular in V. This implies that the factor $\exp\{\ \}$ of (2.3) is regular in V, because the trace is a bounded functional on $\boldsymbol{T}$. According to the theory of regular perturbation of eigenvalues (Kato [5]), however, the factor Π in (2.3) is also regular in V. Thus $\mathrm{d}(1-T(z))$ is regular in V and consequently in U. The other part of the theorem can be proved in a similar way.

From the above theorem it follows that $d(z)=\mathrm{d}(1-zT)$, is an entire function of z. Actually, $d(z)$ coincides with the Fredholm determinant introduced by Smithies [11]. To show this, we assume for the moment that $|z|<\|T\|^{-1}$. Since $\sigma(zT)$ is then entirely inside a circle $C=C(0, r)$, $0<r<1$, $d(z)$ is given by (2.1) and (2.2) with T and Γ replaced by zT and C, respectively. On C, however, we can expand $\log(1-\xi)$ in a power series $\log(1-\xi)=-\sum_{n=1}^{\infty}\frac{1}{n}\xi^{n}$ and perform the integration along C term by term. Hence, noting the relation $(2\pi i)^{-1}\int_{C}\xi^{n}R(\xi;\ T)d\xi=T^{n}$, we have $\log(1-zT)=-\sum_{n=1}^{\infty}\frac{1}{n}z^{n}T^{n}$, the series being convergent in the sence of $\|\ \|$. Since this series is also t-convergent by virtue of $\|T^{n}\|_{1}\leqq\|T\|_{1}\|T\|^{n-1}$ and $|z|<\|T\|^{-1}$, it follows that

$$
d(z)=\exp\left\{-\sum_{n=1}^{\infty}\frac{1}{n}\tau_{n}z^{n}\right\}, \qquad |z|<\|T\|^{-1}, \tag{2.4}
$$

where $\tau_{n}=\mathrm{tr}\,(T^{n})$. By differentiating both sides of this formula, we see that the coefficients d_{n} of the expansion $d(z)=\sum_{n=1}^{\infty}d_{n}z^{n}$ are determined by the recursion formulas

$$
\begin{cases} d_{0}=1\ , \\ d_{n}=-\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\tau_{k}d_{n-k}\ , \qquad n\geqq 1\ . \end{cases} \tag{2.5}
$$

On the other hand, the Fredholm determinant $\hat{d}(z)$ of $T\in \boldsymbol{T}$ is defined by a power series $\hat{d}(z)=\sum_{n=0}^{\infty}\hat{d}_{n}z^{n}$ and it is shown that $\hat{d}_{n}$ satisfy the same recursion formulas (2.5) as d_{n} ([11], (3.7.3)). Hence we have $d(z)=\hat{d}(z)$.

Here it should be noted that, if we define $\mathrm{d}(1-T)$ as the value of the power series $\sum_{n=0}^{\infty}d_{n}z^{n}$ at $z=1$, we can also prove Theorem 2.1 using an estimation of $|d_{n}|$ and the whole argument of this paper would be carried out without referring to the theory of operational calculus. Nevertheless, we prefer the form (2.2) for our purpose, because it not only exhibits the regularity of $\mathrm{d}(1-T(z))$ in a clearer way, but also reduces the calculation in **3** to a simpler form.

We shall next examine the relations between $\mathrm{d}(1-T)$ and the usual determinant of the part of $\mathrm{d}(1-T)$ in a finite-dimensional subspace. We begin with a preliminary lemma.

LEMMA 2.2. *Let* P_{n} *and* Q_{n}, $n=1, 2, \cdots$, *belong to* $\boldsymbol{B}$ *and let* $\text{s-lim}\,P_{n}=P$ *and* $\text{s-lim}\,Q_{n}=Q$. *Then* $\|P_{n}AQ_{n}^{*}-PAQ^{*}\|_{2}\rightarrow 0$ *for each* $A\in\boldsymbol{S}$ *and* $\|P_{n}AQ_{n}^{*}-PAQ^{*}\|_{1}\rightarrow 0$ *for each* $A\in\boldsymbol{T}$.

Proof. Since $\|P_{n}AQ_{n}^{*}-PAQ^{*}\|_{k}\leqq\|P_{n}\|\|(Q_{n}-Q)A^{*}\|_{k}+\|(P_{n}-P)A\|_{k}\|Q^{*}\|$,

$k=1, 2$, we may assume that $P=0$ and $Q_n=Q=1$. Let now $A \in \boldsymbol{S}$ and let $\{\varphi_k\}$ be a c. o. n. s. (complete orthonormal set) of the orthogonal complement $\mathfrak{M}$ of the null space of A. (Note that $\mathfrak{M}$ is separable.) Then we have

$$\|P_n A\|_2{}^2 = \sum_{k=1}^{\infty} \|P_n A\varphi_k\|^2 \leqq \sum_{k=1}^{N} \|P_n A\varphi_k\|^2 + M \sum_{k=N+1}^{\infty} \|A\varphi_k\|^2 ,$$

where $M=\sup \|P_n\|$ and N is an arbitrary positive integer. From this $\|P_n A\|_2 \to 0$ follows by a standard argument. If $A \in \boldsymbol{T}$, A is expressible in the form $A=S_1S_2$, $S_k \in \boldsymbol{S}$. Hence we obtain $\|P_n A\|_1 \leqq \|P_n S_1\|_2 \|S_2\|_2 \to 0$.

LEMMA 2.3. i) *Let P be a (not necessarily orthogonal) projection such that $\mathfrak{R}(P) \supset \mathfrak{R}(T)$. Then, if we put $T'=PTP$, we have* $\mathrm{d}(1-T)=\mathrm{d}(1-T')$.

ii) *Let T be of rank r, $0<r<\infty$, and $\{\varphi_1, \cdots, \varphi_r\}$ a base of $\mathfrak{R}(T)$. Then, if we put $t_{ij}=(T\varphi_j, \varphi_i)$, $1 \leqq i, j \leqq r$, we have* $\mathrm{d}(1-T)=\det(\delta_{ij}-t_{ij})$.

Proof. i) Since $PT=P$, we have $\mathrm{tr}(T'^k)=\mathrm{tr}(T^k)$ in virtue of $P^2=P$ and (1.3). Hence, it follows from (2.4) that $\mathrm{d}(1-zT)=\mathrm{d}(1-zT')$, if $|z|<\|T\|^{-1} \leqq \|T'\|^{-1}$. Then the result follows by analytic continuation. ii) By virtue of i) we may assume that $\mathfrak{H}$ itself is r-dimensional and $\mathfrak{R}(T)=\mathfrak{H}$. Then $\nu(z, T)$ is also finite at $z=0$. Hence in the same way as we got (2.3) we obtain

$$\mathrm{d}(1-T) = \prod_{k=1}^{p} (1-a_k)^{\nu_k} , \tag{2.6}$$

where $\{a_1, \cdots, a_p\}$ is the set of all eigenvalues of T and ν_k the algebraic multiplicity of a_k. From this the result follows.

THEOREM 2.2. *Let $\mathfrak{H}'$ be a separable closed subspace of $\mathfrak{H}$ containing $\mathfrak{R}(T)$, $\{\varphi_k\}$ a c. o. n. s. of $\mathfrak{H}'$ and $t_{ij}=(T\varphi_j, \varphi_i)$. Furthermore, let T_n and 1_n be the square matrix of degree n defined by $T_n=(t_{ij})$, $1 \leqq i, j \leqq n$, and $1_n=(\delta_{ij})$. Then we have*

$$\mathrm{d}(1-T) = \lim_{n\to\infty} \det(1_n - T_n) . \tag{2.7}$$

Proof. Let P and P_n, $n=1, 2, \cdots$, be the orthogonal projections on $\mathfrak{H}'$ and the subspace determined by $\{\varphi_1, \cdots, \varphi_n\}$, respectively. Then by Lemma 2.2 we have $\|P_n T P_n - PTP\|_1 \to 0$, $n\to\infty$. By referring to Theorem 2.1 and Lemma 2.3, we therefore obtain $\lim \det(1_n - T_n) = \lim \mathrm{d}(1-P_nTP_n) = \mathrm{d}(1-PTP) = \mathrm{d}(1-T)$.

COROLLARY. i) $\mathrm{d}(1-T^*)=\overline{\mathrm{d}(1-T)}$. ii) *If $1-T=(1-T_1)(1-T_2)$, we have*

$$\mathrm{d}(1-T) = \mathrm{d}(1-T_1)\,\mathrm{d}(1-T_2) . \tag{2.8}$$

Proof. We first prove ii). Let $\mathfrak{R}$ be the closed subspace determined by $\mathfrak{R}(T_1) \cup \mathfrak{R}(T_2)$. Since $T_k \in \boldsymbol{T}$, $k=1, 2$, implies that $\mathfrak{R}(T_k)$ is separable, $\mathfrak{R}$ is also separable. Let now P_n be constructed as above with $\mathfrak{R}'$ replaced by $\mathfrak{R}$. Then, by Lemma 2.3 and the formula (2.8) for the usual determinants we have $\mathrm{d}((1-P_nT_1P_n)(1-P_nT_2P_n)) = \mathrm{d}(1-P_nT_1P_n)\,\mathrm{d}(1-P_nT_2P_n)$. Therefore, by letting $n\to\infty$ on both sides and referring to Lemma 2.2 and Theorems 2.1 and 2.2, we obtain (2.8). i) is proved in a similar way.

As we saw above, the t-regularity of $T(z)$ implies the regularity of $d(z)=\mathrm{d}\,(1-T(z))$. If $T(z)$ has a pole at $z=a$, however, $d(z)$ may have an essential singularity at a. Nevertheless, if a pole-like singularity of $T(z)$ occurs in a fixed finite-dimensional subspace, the possible singularity of d is pole-like. Namely, we have the following

THEOREM 2.3. *Let U be a domain in the complex plane and let $a\in U$. Let $T_1(z)$ and $T_2(z)$ be t-regular in U and put*

$$T(z)=\frac{T_1(z)}{(z-a)^p}+T_2(z)\,,\qquad z\in U-\{a\}\,,$$

where p is a positive integer. Then, if $\mathfrak{R}(T_1)$ is contained in a r-dimensional $(0\leqq r<\infty)$ subspace $\mathfrak{R}$ independent of $z\in U$, $d(z)=\mathrm{d}\,(1-T(z))$ is meromorphic in U and we have $\nu(a;\,d)\geqq -pr$.

To prove this theorem we need the following estimate of $|\mathrm{d}\,(1-T)|$.

LEMMA 2.4.* For each $T\in \boldsymbol{T}$ we have

$$\mathrm{d}\,(1-T)\leqq \exp\,(\|T\|_1)\,. \tag{2.9}$$

Proof. We first assume that T is of finite rank. Then (2.6) implies $|\mathrm{d}\,(1-T)|\leqq \prod_{k=1}^{p}(1+|a_k|)^{\nu_k}\leqq \exp\,(\sum_{k=1}^{p}\nu_k|a_k|)$. Let now $\{b_1,\cdots,b_q\}$ be the set of all eigenvalues of $|T|=(T^*T)^{1/2}$ and ρ_k the multiplicity of b_k. Then it is known that (see Weyl [12])

$$\sum_{k=1}^{p}\nu_k|a_k|\leqq \sum_{k=1}^{q}\rho_k b_k=\|T\|_1\,.$$

Hence we see that (2.9) holds for each T of finite rank. The general result follows from this by virtue of (2.7) and the relation $\|T_n\|_1\leqq\|T\|_1$.

Proof of Theorem 2.3. Since it is easily seen that $\mathfrak{R}(T(z))$ is contained in a separable closed subspace independent of z, we may assume that $\mathfrak{H}$ itself is separable. Let now $\{P_n\}$ be an increasing sequence of finite-dimensional orthogonal projections such that $P_1\mathfrak{H}=\mathfrak{R}$ and $\text{s-lim}\,P_n=1$. Furthermore we put $d_n(z)=\mathrm{d}\,(1-P_nT(z)P_n)$. By Lemma 2.3 $d_n(z)$ is equal to the determinant of a matrix representing $1-T(z)$ in $P_n\mathfrak{H}$. We choose a base $\{\varphi_k\}$ of $P_n\mathfrak{H}$ in such a way that $\{\varphi_1,\cdots,\varphi_r\}$ determines $\mathfrak{R}$ and construct the above mentioned matrix $(d_{ij}(z))$ with respect to this base. Then by the assumption of the theorem $d_{ij}(z)$ is meromorphic in U and

$$\nu(a;\,d_{ij}(z))\geqq\begin{cases}-p\,, & \text{if } 1\leqq i\leqq r\,,\\ 0\,, & \text{if } r<i\,.\end{cases}$$

Since $d_n(z)=\det\,(d_{ij}(z))$, this implies $\nu(a;\,d_n)\geqq -pr$. The general result is obtained by the limiting procedure based upon the fact that $|d_n(z)|$ is uniformly bounded on a closed contour contained in $U-\{a\}$ (Lemma 2.4) and $d_n(z)\to d(z)$.

* The writer is indebted to Prof. T. Kato for this lemma.

3. Generalization of the Weinstein-Aronszajn formula

Let A be a densely defined colsed operator in $\mathfrak{H}$ with non-empty resolvent set and B an operator which satisfies the following condition:

$$(3.1)\qquad \begin{cases} \mathfrak{D}(B)\supset\mathfrak{D}(A) & \text{and} \\ T(z)=BR(z;\ A)\in \boldsymbol{T} & \text{for some (or equivalently for all) } z\in\rho(A)\,. \end{cases}$$

Since $BR(z;\ A)$ is then completely continuous, $A'=A+B$ is a closed operator with the domain $\mathfrak{D}(A')=\mathfrak{D}(A)$ (Kato [6], Theorem 2a). Furthermore, the function $\mathrm{d}\,(1-T(z))$, $z\in\rho(A)$, is denoted in this section by $w(z)$, namely

$$w(z)=\mathrm{d}\,(1-BR(z;\ A))\,,$$

for the reason that it corresponds to the Weinstein determinant in the case (I).

THEOREM 3.1. *Let A, B and $w(z)$ be as above. We further assume that $R(z;\ A)$ is meromorphic in a domain U of the complex plane and $\nu(z;\ A)$ is finite for all $z\in U$. Then $w(z)$ is meromorphic in U and we have*

$$(3.2)\qquad \nu(z;\ w)=\nu(z;\ A')-\nu(z;\ A)\,,\qquad z\in U\,,\quad A'=A+B\,.$$

(*Both sides of* (3.2) *may be* $+\infty$.)

Before proceeding to the proof, we shall make some remarks related to this theorem.

Remark 1. Special cases. If B is of finite rank, that is, $B=\sum_{k=1}^{r} c_k(\cdot,\varphi_k)\psi_k$, $(\varphi_i,\varphi_j)=(\psi_i,\psi_j)=\delta_{ij}$, then we have by Lemma 2.3 $w(z)=\det(\delta_{ij}-c_i(R(z;\ A)\psi_j,\varphi_i))$. If in particular B is self-adjoint $(\varphi_i=\psi_i)$, $w(z)$ coincides with the Weinstein determinant defined for the case (I) and (3.2) reduces itself to the first W-A formula.

In the next place, if $A=0$ and $B\in\boldsymbol{T}$, (3.2) implies a well-known relation between the eigenvalues of B and the zeros of the Fredholm determinant.

Remark 2. Returning to the general case, we see that the nature of the spectrum of A' in U is classified into two different types according as $w(z)$ is identically zero or not. If $w(z)\equiv0$, (3.2) implies $\nu(z;\ A')=\infty$ everywhere in U. Hence $U\subset\sigma(A')$. If, on the contrary, $w(z)\not\equiv0$, then Theorem 3.1 shows that U is contained in $\rho(A')$ except for an at most denumerable number of eigenvalues of A' whose algebraic multiplicity are finite and whose accumulation point (if any) lies on the boundary of U. Here we remark that this classification was obtained by Kato [6] by a different method under weaker conditions than (3.1).

Remark 3. In referrence to the above remark it is desirable to have some criteria which ensure $w(z)\not\equiv0$ without explicit calculation of w. $w(z)\not\equiv0$ is equivalent to the following condition:

(C_1) $U\cap\rho(A')$ is not empty.

We shall enumerate some sufficient conditions (C_2)-(C_4) which imply (C_1). The proof is not difficult and will be omitted.

(C_2) $\|BR(z;\ A)\|<1$ holds for a $z\in U\cap\rho(A)$.
(C_3) There exists a sequence $\{z_n\}$ in $U\cap\rho(A)$ such that $|z_n|\to\infty$ and $|z_n-z_1|\|R(z_n;\ A)\|$ is bounded.
(C_4) $A\in \boldsymbol{B}$ and there exists a $z\in U$ such that $|z|\geqq\|A\|$.

We now proceed to the proof of Theorem 3.1. We first prove that $w(z)$ is meromorphic in U.

Let $\{a_k\}$ be the set of all poles of $R(z;\ A)$ in U and put $U'=U-\{a_k\}$. Then, as in the proof of Theorem 2.1, we see that, $T(z)$ is t-regular in U'. Hence we have only to prove that each a_k is not an essential singularity of w. By (1.1) we see that, in a neighbourhood V of a_k, $T(z)$ is expressible in the form

$$T(z)=\frac{T_1(z)}{(z-a_k)^{\mu}}+T_2(z)\ ,$$

where $T_1(z)$ and $T_2(z)$ are regular in V and $\mathfrak{R}(T_1(z))\subset B\mathfrak{N}(a_k;\ A)$. Since $B\mathfrak{N}(a_k;\ A)$ is finite-dimensional and independent of z, Theorem 2.3 is applicable with the desired result.

In order to prove (3.2) we need the following lemma, which is partly given in Kleinecke [7].

Lemma 3.1. *Let $z\in U'$. Then $z\in\sigma_p(A')$ or $z\in\rho(A')$ according as $w(z)=0$ or $w(z)\neq0$.*

Proof. (2.2) implies that $1\in\sigma_p(T(z))$ or $1\in\rho(T(z))$ according as $w(z)=0$ or $w(z)\neq0$. Hence, the result follows directly from the relation $1-T(z)=(z-A')\cdot R(z;\ A)$.

Let now $w(z)\equiv0$ in U. Then from Lemma 3.1 it follows that $U'\subset\sigma_p(A')$ and hence $U\subset\sigma(A')$. Namely $\nu(z;\ A')=\infty$ for each $z\in U$. Therefore (3.2) holds in the sense $\infty=\infty$.

Next let $w(z)\not\equiv0$ and put $U''=\{z\in U'|w(z)\neq0\}$. Then $U''\subset\rho(A)\cap\rho(A')$ and for each $z\in U''$ we have

$$R(z;\ A')-R(z;\ A)=R(z;\ A')BR(z;\ A)\in \boldsymbol{T} \tag{3.3}$$

by (3.1). Now the following lemma is fundamental to the proof of (3.2).

Lemma 3.2. *For each $z\in U''$ we have*

$$f(z)\equiv\frac{w'(z)}{w(z)}=\{\mathrm{tr}\,(R(z;\ A')-R(z;\ A))\}\ . \tag{3.4}$$

Proof. Let $a\in U''$ be arbitrarily fixed and Γ a fixed contour defined as in (2.1) with T replaced by $T(a)$. (Note that $1\in\rho(T(a))$ by the definition of U''.) Since $T(z)$ is t-regular at a, there exists an open disque $V\subset U''$ containing a such that $\Gamma\subset\rho(T(z)))$ and $1\in\rho(T(z))$ for each $z\in V$. Then we see, as in the proof of Theorem 2.1, that the integral $(2\pi i)^{-1}\int_\Gamma \log(1-\xi)R(\xi;\ T(z))d\xi$ is significant for each $z\in V$ and t-regular in V with respect to z. We temporarily denote this integral by $\log(1-T(z))$. Though it may be different from $\mathrm{Log}(1-T(z))$, we have $w(z)=\exp\{\mathrm{tr}(\log(1-T(z)))\}$ as is stated in the remark after Lemma 2.1. Hence it follows that

$$f(z)=\frac{d}{dz}\,\mathrm{tr}\,(\log(1-T(z)))$$
$$=\frac{1}{2\pi i}\frac{d}{dz}\,\mathrm{tr}\left(\int_\Gamma \log(1-\xi)R(\xi;\,T(z))d\xi\right), \qquad z\in V.$$

Since $\log(1-T(z))$ is t-regular, the differentiation can be performed before calculating the trace. Furthermore, since $\Gamma\subset\rho(T(z))$, we can carry out the differentiation under the integral sign with the result (we write T instead of $T(z)$)

$$f(z)=-\frac{1}{2\pi i}\,\mathrm{tr}\left(\int_\Gamma \log(1-\xi)R(\xi;\,T)BR(z;\,A)^2R(\xi;\,T)d\xi\right).$$

Now, the order of the two operations, the integration and the calculation of the trace, can be freely inverted, as long as the integrand is t-continuous along Γ. Therefore, using (1.3) we obtain by means of partial integration

$$f(z)=-\frac{1}{2\pi i}\,\mathrm{tr}\left(\int_\Gamma \log(1-\xi)R(\xi;\,T)^2d\xi\cdot BR(z;\,A)^2\right)$$
$$=\frac{1}{2\pi i}\,\mathrm{tr}\left(\int_\Gamma \frac{1}{1-\xi}R(\xi;\,T)d\xi\cdot BR(z;\,A)^2\right).$$

Since $\sigma(T)$ is entirely inside Γ, the integral on the right-hand side multiplied by $(2\pi i)^{-1}$ is equal to $(1-T)^{-1}=(z-A)R(z;\,A')$. Hence, noting (3.3) we have

$$f(z)=\mathrm{tr}\,((z-A)R(z;\,A')BR(z;\,A)^2)=\mathrm{tr}\,(R(z;\,A')BR(z;\,A))$$
$$=\mathrm{tr}\,(R(z;\,A')-R(z;\,A)), \qquad z\in V.$$

Since V is a neighbourhood of an arbitrarily fixed $a\in U''$, this proves (3.4).

Let now $z\in U$ be arbitrary. Then there exists a circle C about z such that i) $C\subset U$ and ii) there exists no points of $U-U''$ on and inside C except for, possibly, z. Since $w(z)$ is meromorphic in U, it then follows that $(2\pi i)^{-1}\int_C f(z')dz'=\nu(z;\,w)$. Therefore, by integrating both sides of (3.4) along C and noting (1.2) we obtain (3.2).

Thus Theorem 3.1 is completely proved.

We shall next deduce the second W-A formula in a slightly generalized form.

THEOREM 3.2. *Let A and U be as in Theorem 3.1. Let $\mathfrak{N}$ be a r-dimensional $(0\leqq r<\infty)$ subspace of $\mathfrak{D}(A)$, P the orthogonal projection onto $\mathfrak{M}=\mathfrak{H}\ominus\mathfrak{N}$ and $A'=PAP$. Then A' is a closed operator with the domain $\mathfrak{D}(A')=\mathfrak{D}(A)$. Let $\{\psi_k|k=1,\cdots,r\}$ be a c.o.n.s. of $\mathfrak{N}$ and put*

$$\hat{w}(z)=\det(d_{ij}(z)), \qquad d_{ij}(z)=(R(z;\,A)\psi_j,\,\psi_i), \qquad z\in U.$$

Then we have

(3.5) $$\nu(z;\hat{w})=\nu(z;A')-\nu(z;A)\,,\qquad z\in U\,,\quad z\neq 0\,.$$

Remark. When A is self-adjoint $\hat{w}(z)$ coincides with the Weinstein determinant defined for the case (II) and (3.5) reduces itself to the second W-A formula.

Proof of Theorem 3.2. It is easily verified that

(3.6) $$A'=A+B\,,\qquad B=-PAQ-QA\,.$$

Since Q is of finite rank and $A(A-z)^{-1}\in \boldsymbol{B}$, $z\in\rho(A)$, it follows that $B(A-z)^{-1}$ is of finite rank and hence belongs to $\boldsymbol{T}$. Therefore the first assertion of the theorem follows as before. Moreover, Theorem 3.1 is applicable to this problem. Now, by virtue of (3.6) and the corollary to Theorem 2.2 we obtain for each $z\in U'$, $z\neq 0$, (U' is the same as before)

(3.7) $$w(z)=\mathrm{d}\,(1-BR(z;A))=\mathrm{d}(1+(1/z)PAQ)\cdot\mathrm{d}\,(1-Q+zQ(z-A)^{-1})\,.$$

Since $\mathrm{tr}\,((PAQ)^k)=0$, $k=1,2,\cdots$, a simple consideration using (2.4) shows that $\mathrm{d}\,(1+(1/z)PAQ)=1$. Hence, by applying Theorem 2.2 to the second factor on the right-hand side of (3.7) we get $w(z)=z^m\hat{w}(z)$, $z\in U$, $z\neq 0$. From this and (3.2), (3.5) follows at once.

Remark. In the above deduction of (3.5) from (3.2) the condition (3.1) of Theorem 3.1 is essentially utilized, because in general B defined by (3.6) is not of finite rank. If A is self-adjoint (or more generally if $\mathfrak{N}\subset\mathfrak{D}(A^*)$), however, it follows easily that QA and hence B are of finite rank. Then the above proof is performable without any change and without any reference to the generalized formula (3.2). We can therefore say that the original second W-A formula can be deduced from the original first W-A formula.

Finally, we remark that most of the arguments in this paper can be applied to the case of a Banach space $\mathfrak{X}$, if we replace the trace class $\boldsymbol{T}$ by the class $\boldsymbol{F}$ of all degenerate operators on $\mathfrak{X}$ to $\mathfrak{X}$. The trace norm and the trace of $T\in\boldsymbol{F}$ are taken in the sense of Schatten (Schatten [10], Ruston [9]). In particular, we can also prove a theorem similar to Theorem 3.1. In the proof the incompleteness of $\boldsymbol{F}$ provides no difficulty. The only point to be noted is that in the proof of Theorem 2.3 we used essentially the unitary character of $\mathfrak{H}$. In the present case of degenerate perturbation, however, we need Theorem 2.3 only $T_2(z)=0$. In this case the theorem is trivial and requires no such proof as before. Further details of slight changes in the arguments need not be stated.

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