On the Uniqueness of the Cauchy Problem for Elliptic Systems of Partial Differential Equations

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1. Introduction

Recently, L. Hörmander [2], [3] gave an interesting investigation upon the uniqueness of solutions for the Cauchy problem of elliptic partial differential equations. His argument is based upon the inequalities which he drives from those due to Trèves.

In this paper we shall extend Hörmander's results to some classes of elliptic systems of partial differential equations. Our method is quite similar to that of Hörmander. Firstly we extend the inequalities of Hörmander [3] to a system of elliptic differential operators (Theorem 2). We reduce the system of differential equations to a system of differential equations of the following type;

$$(1)\qquad L_h(x, D)u^j=\sum_{k=1}^{N}\sum_{|\alpha|<m} a_{h,jk,\alpha}(x)D^\alpha u^k \qquad h=1, 2, \cdots,$$

where $\{L_h(x, D)\}$ is a system of differential operators of order m. By virtue of our inequalities we can apply to (1) the arguments as in Hörmander [3], and obtain the uniqueness of solutions for the system of differential equations.

2. Notations and Main Results

Let $\xi=(\xi_1, \cdots, \xi_\nu)$ be a sequence of indeterminates and $\alpha=(\alpha_1, \cdots, \alpha_\nu)$ be a sequence of integers, we write

$$|\alpha|=\alpha_1+\cdots+\alpha_\nu, \qquad |\xi|^2=|\xi_1|^2+\cdots+|\xi_\nu|^2,$$

$$\alpha!=\alpha_1!\cdots\alpha_\nu!, \qquad \xi^\alpha=\xi_1^{\alpha_1}\cdot\xi_2^{\alpha_2}\cdots\xi_\nu^{\alpha_\nu}.$$

Let $\kappa=(\kappa_1, \cdots, \kappa_\nu)$ be another sequence of integers. $\alpha\geqq\kappa$ means that $\alpha_j\geqq\kappa_j$ for every index j, and $\alpha\pm\kappa=(\alpha_1\pm\kappa_1, \cdots, \alpha_\nu\pm\kappa_\nu)$. If $\alpha\geqq\kappa$, then we define

$$\binom{\alpha}{\kappa}=\binom{\alpha_1}{\kappa_1}\cdot\binom{\alpha_2}{\kappa_2}\cdots\binom{\alpha_\nu}{\kappa_\nu}.$$

If $P(\xi)$ is a polynomial of indeterminates $\xi=(\xi_1, \cdots, \xi_\nu)$ with constant coefficients, then we set

$$P^{(\alpha)}(\xi)=\left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\xi_\nu}\right)^{\alpha_\nu}P(\xi_1, \cdots, \xi_\nu),$$

According to definition, if $P(\xi)=\sum a_\kappa \xi^\kappa$, then we have

$$P^{(\alpha)}(\xi)=\sum_{\kappa\geqq\alpha}\frac{\kappa!}{(\kappa-\alpha)!}a_\kappa\xi^{\kappa-\alpha}\ .$$

Let Ω be a domain in ν-dimensional Euclidean space. For a polynomial $P(\xi)=\sum a_\kappa\xi^\kappa$ we define a differential operator $P(D)=\sum a_\kappa D^\kappa$,

$$D^\alpha=i^{-|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1}\cdots\partial x_\nu{}^{\alpha_\nu}}$$

where $x=(x_1,\cdots,x_\nu)$ is a point in Ω and i denotes a square root of -1. If $P(x,\xi)$ is a polynomial of ξ with variable coefficients, that is, $P(x,\xi)=\sum a_\kappa(x)\xi^\kappa$ where $a_\kappa(x)$ is a function of x in Ω, then we employ the following notations.

$$P(x,D)=\sum a_\kappa(x)D^\kappa\ ,$$

$$P^{(\alpha)}(x,D)=\sum_{\kappa\geqq\alpha}\frac{\kappa!}{(\kappa-\alpha)!}a_\kappa(x)D^{\kappa-\alpha}\ ,$$

$$P^{[\alpha]}(x,D)=\sum_{\kappa\geqq0}\{D^\alpha a_\kappa(x)\}D^\kappa\ .$$

Let $\varphi(x)$ be a function of x in Ω. We employ the following inner products and norms.

$$(u,v)_\varphi=\int_\Omega u(x)\overline{v(x)}e^{\varphi(x)}dx\ ,$$

$$\|u\|_\varphi{}^2=(u,u)_\varphi\ ,$$

$$T_m(u)_\varphi{}^2=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\|D^\alpha u\|_\varphi{}^2\ .$$

We state our main results, their proofs will be given in §4. and §5.

THEOREM 1. (Hörmander [3]) *Let $P(x,D)$ be a homogeneous differential operator coefficients of which are uniformly bounded and uniformly Lipschitz continuous in Ω. Then there exists a constant C depending only on the dimension ν of Ω, the order m of $P(x,D)$ and the bounds for the coefficients $a_\kappa(x)$ and their first derivatives such that the inequalities*

$$h^{2\beta}\|P^{(\beta)}(x,D)u\|_\varphi{}^2\leqq C\{\|P(x,D)u\|_\varphi{}^2+T_m(u)_\varphi\cdot T_{m-1}(u)_\varphi\} \tag{2}$$

$$\varphi(x)=\sum_{j=1}^{\nu}h_j{}^2(x^j-a^j)^2$$

hold for every function $u\in C_0^\infty(\Omega)$ and for every sequence β of integers.

THEOREM 2. *Let U be a neighbourhood of the origin in R^ν, and let $P_k(x,\xi)$ $(k=1,2,\cdots,n)$ be homogeneous polynomials of degree m whose coefficients are*

Lipschitz continuous in U. *Moreover, we assume that*

(a) *the simultaneous algebraic equations with* ν-*unknowns* $\xi_1, \cdots, \xi_\nu$

$$P_k(0; \xi)=0\,, \qquad k=1, 2, \cdots, n \tag{3}$$

have no real roots except $\xi=0$.

(b) *the simultaneous equations with single unknown* ζ_1

$$P_k(0; \zeta_1, \xi_2, \cdots, \xi_\nu)=0\,, \qquad k=1, 2, \cdots, n \tag{4}$$

have no complex double roots for fixed real $(\xi_2, \cdots, \xi_\nu)\neq 0$.

Then there exist constants M, δ_0, a *and* C *such that the inequalities*

$$(1+\delta^2\tau^2)^{m-|\alpha|-1}\tau^{m-|\alpha|}\|D^\alpha u\|_\varphi{}^2 \leqq C\sum_{k=1}^{n}\|P_k(x, D)u\|_\varphi{}^2 \tag{5}$$

hold for every $u\in C_0^\infty(U_\delta)$ *provided that* $\tau^2\delta>M$ *and* $0<\delta<\delta_0$ *where* $\varphi(x)=\tau^2\varphi_\delta(x)$,

$$\varphi_\delta(x)=(x^1-\delta)^2+\delta\left\{\sum_{j=1}^{\nu}(x^j)^2\right\} \tag{6}$$

and

$$U_\delta=\{x;\ |x^1|<\delta,\ \delta|x^j|<a|x^1-\delta|,\ j=2, 3, \cdots, \nu\}\cap U\,.$$

THEOREM 3. *Let* $\{P_{jk}(x, D)\}$ $j=1, 2, \cdots, N'$, $k=1, 2, \cdots, N$ *be a* $N'\times N$-*matrix of differential operators such that the following hypotheses hold.*

(A) *There exists a sequence of integers* $(s_1, \cdots, s_{N'}: t_1, \cdots, t_N)$ *such that* $P_{jk}(x, D)$ *is of order* s_j+t_k. *We set* $n=\sum_{j=1}^{N}t_j$, $m(\pi)=\sum s_{\pi(k)}$, *where* π *is a one to one mapping from* $\{1, \cdots, N'\}$ *to* $\{1, \cdots, N\}$ *and* $m=\max m(\pi)$. *We may assume* $t_k\geqq 0$.

(B) *The coefficients of* $P_{jk}(x, D)$ *have bounded derivatives up to order* $m+n+t_k$.

(C) *Let* $L_\pi(x, \xi)$ *be the determinant of the matrix* $(P_{\pi(j)k}(x, \xi))$, *and let* $L_\pi{}^0(x, \xi)$ *be the* $\{n+m(\pi)\}$-*homogeneous part of* $L_\pi(x, \xi)$. *Then, polynomials* $\{L_\pi{}^0(x, \xi)\}$ *have properties (a) and (b) in Theorem 2.*

Let $u(x)$ *be a solution of a system of differential equations*

$$\sum_k P_{jk}(x, D)u^k=0\,. \tag{7}$$

such that u *is identically zero in the intersection of a neighbourhood of the origin and the domain*

$$x^1<(x^2)^2+\cdots+(x^\nu)^2\,. \tag{8}$$

Then $u=0$ *in a neighbourhood of the origin.*

3. Trèves' inequalities

In this section we shall prove Trèves' inequalities. Our proof are slightly different from that of Trèves and the author hopes that his proof might be a little more natural than that of Trèves.

We shall begin with some algebraic lemmas. Let $\mathfrak{A}$ be a algebra over complex number field C with unite I. We set $[A_1, B_1]=A_1B_1-B_1A_1$ where A_1, B_1 are elements of $\mathfrak{A}$. Let $P(\xi)$ be a polynomial of $\xi=(\xi_1, \cdots, \xi_\nu)$ and $A=(A_1, \cdots, A_\nu)$ be a vector of elements in $\mathfrak{A}$. Then we denote by $P(A)$ an element of $\mathfrak{A}$ obtained by substituting A for ξ in $P(\xi)$.

The following two lemmas are easily checked by mathematical induction.

LEMMA 1. Suppose that $[A_1, B_1]=g_1 I$. Then

$$A_1{}^l B_1{}^m=\sum_{m,l\geqq k\geqq 0}\frac{g_1{}^k}{k!}\frac{l!}{(l-k)!}\frac{m!}{(m-k)!}B_1{}^{m-k}A_1{}^{l-k} \tag{9}$$

LEMMA 2. Suppose that $[A_j, B_k]=\delta_{jk}g_j I$ $(j=1, \cdots, \nu)$, $[A_j, A_k]=[B_j, B_k]=0$. Then

$$A^\alpha B^\beta=\sum_{\alpha,\beta\geqq\kappa\geqq 0}\frac{g^\kappa}{\kappa!}\frac{\alpha!}{(\alpha-\kappa)!}\frac{\beta!}{(\beta-\kappa)!}B^{\beta-\kappa}A^{\alpha-\kappa} \tag{10}$$

holds, where $A=(A_1, \cdots, A_\nu)$ and $B=(B_1, \cdots, B_\nu)$.

LEMMA 3. If A and B have same properties as in Lemma 2, then

$$P(A)Q(B)=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}Q^{(\alpha)}(B)P^{(\alpha)}(A) \tag{11}$$

holds, where $P(\xi)$ and $Q(B)$ are arbitrary polynomials.

Proof. Suppose that $P(\xi)=\sum a_\lambda\xi^\lambda$ and $Q(\xi)=\sum b_\mu\xi^\mu$. Then we have

$$\begin{aligned}P(A)Q(B)&=\sum_\lambda a_\lambda\sum_\mu b_\mu A^\lambda B^\mu\\&=\sum_\lambda\sum_\mu\sum_{\alpha\leqq\lambda,\mu}a_\lambda b_\mu\frac{g^\alpha}{\alpha!}\frac{\lambda!}{(\lambda-\alpha)!}\frac{\mu!}{(\mu-\alpha)!}B^{\mu-\alpha}A^{\lambda-\alpha}\\&=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}\Big(\sum_{\mu\geqq\alpha}\frac{\mu!}{(\mu-\alpha)!}b_\mu B^{\mu-\alpha}\Big)\Big(\sum_{\lambda\geqq\alpha}\frac{\lambda!}{(\lambda-\alpha)!}a_\lambda A^{\lambda-\alpha}\Big)\\&=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}Q^{(\alpha)}(B)P^{(\alpha)}(A)\ .\end{aligned}$$

Now let H be a pre-Hilbert space whose elements are infinitely differentiable functions with compact carriers in R^ν and whose inner product is $(u, v)_\varphi$ where $\varphi(x)$ is a continuous function of x in R^ν.

We denote by $\tilde{D}_j$ the formal adjoint of $D_j=i^{-1}\partial/\partial x^j$ in H.

Let $P(\xi)=\sum a_\lambda\xi^\lambda$ be a polynomial. Then the formal adjoint of $P(D)$ is $\bar{P}(\tilde{D})$ where $\bar{P}(\xi)=\sum\bar{a}_\lambda\xi^\lambda$.

If $\varphi(x)$ is differentiable in x^j then we have, by partial integration,

$$\tilde{D}_j=D_j+\frac{1}{i}\frac{\partial\varphi}{\partial x^j}\ .$$

If $\varphi(x)$ is twice differentiable, it follows from this formula that

$$[\tilde{D}_j, D_k]=\frac{\partial^2\varphi}{\partial x^j\partial x^k}\,.$$

Hence, the assumptions in Lemma 2 with $A_j=\tilde{D}_j$, $B_j=D_j$ hold if and only if

$$\frac{\partial^2\varphi}{(\partial x^j)^2}=g_j \qquad \frac{\partial^2\varphi}{\partial x^j\partial x^k}=0$$

where g_j $(j=1, 2, \cdots, \nu)$ are constants. Integrating this system of partial differential equations, we have

$$\varphi(x)=\frac{1}{2}\sum_{j=1}^{\nu} g_j(x^j-a^j)^2+c$$

where a_j $(j=1, 2, \cdots, \nu)$ and c are constants. A constant multiplier in the inner product of H is irrelevant, so we may assume that $c=0$.

We obtain from Lemma 3 that

$$\begin{aligned}(Q(D)u, P(D)v)_\varphi&=(\bar{P}(\tilde{D})\cdot Q(D)u, v)_\varphi\\ &=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}(Q^{(\alpha)}(D)\bar{P}^{(\alpha)}(\tilde{D})u, v)_\varphi\\ &=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}(\bar{P}^{(\alpha)}(\tilde{D})u, \bar{Q}^{(\alpha)}(\tilde{D})v)_\varphi\end{aligned}$$

holds for $u, v\in H$. Setting $v=u$, $Q(\xi)=P(\xi)$, we have

$$\|P(D)u\|_\varphi^2=\sum_{\alpha\geqq 0}\frac{g^\alpha}{\alpha!}\|\bar{P}^{(\alpha)}(\tilde{D})u\|_\varphi^2\,. \tag{12}$$

Applying this formula to $P^{(\alpha)}(D)$, we have

$$\|P^{(\alpha)}(D)u\|_\varphi^2=\frac{\alpha!}{g^\alpha}\sum_\kappa\binom{\alpha+\kappa}{\kappa}\frac{g^{\alpha+\kappa}}{(\alpha+\kappa)!}\|\bar{P}^{(\alpha+\kappa)}(\tilde{D})u\|_\varphi^2\,.$$

Remark that if $|\alpha|$ is greater than the degree of $P(\xi)$, then $P^{(\alpha)}(\xi)=0$ and that $\binom{\alpha}{\kappa}\leqq 2^{|\alpha|}$.

Setting $g_j=2h_j^2$, we have the following:

Lemma 4. (Trèves [1])

$$\|P(D)u\|_\varphi^2\geqq\frac{2^{|\alpha|}h^{2\alpha}}{2^m\alpha!}\|P^{(\alpha)}(D)u\|_\varphi^2 \tag{13}$$

holds for every $u\in C_0^\infty(R^\nu)$ where $\varphi(x)=\sum h_j^2(x^j-a^j)^2$ and a^j, h_j are constants.

In the following we denote by $\varphi(x)$ the function of x defined above.

By virtue of (12), we have

$$\|D_j u\|_\varphi^2=\|\tilde{D}_j u\|_\varphi^2+2h_j^2\|u\|_\varphi^2\geqq 2h_j^2\|u\|_\varphi^2\,,$$

Hence, we have

$$T_1(u)_\varphi^2=\sum \|D_j u\|_\varphi^2\geqq 2|h|^2\|u\|_\varphi^2$$

where $|h|^2=h_1^2+\cdots+h_\nu^2$, so that

$$\begin{aligned}T_{m+1}(u)&=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_j \|D_j D^\alpha u\|_\varphi^2\geqq \sum_{|\alpha|=m}2\frac{m!}{\alpha!}|h|^2\|D^\alpha u\|_\varphi^2\\&=2|h|^2 T_m(u)_\varphi^2\,.\end{aligned}$$

Repeated application of this formula gives the following lemma.

LEMMA 5.

$$T(u)_\varphi^2\geqq 2^{m-k}|h|^{2(m-k)}T_k(u)_\varphi^2 \qquad (k\leqq m) \tag{14}$$

holds for every $u\in C_0^\infty(R^\nu)$.

4. Trèves' inequalities in the case of variable coefficients

Firstly we state

LEMMA 6.

$$\|(D,\tilde{D})^\alpha u\|_\varphi^2\leqq\|D^\alpha u\|_\varphi^2\leqq T_{|\alpha|}(u)_\varphi^2 \tag{15}$$

holds for every $u\in C_0^\infty(R^\nu)$, where $(D,\tilde{D})^\alpha=(D_1,\tilde{D}_1)^{\alpha_1}(D_2,\tilde{D}_2)^{\alpha_2}\cdots(D_\nu,\tilde{D}_\nu)^{\alpha_\nu}$, and $(D_j,\tilde{D}_j)^{\alpha_j}$ is any product of the form

$$D_j^{a_1}\tilde{D}_j^{b_1}D_j^{a_2}\tilde{D}_j^{b_2}\cdots D_j^{a_p}\tilde{D}_j^{b_p};\ a_1+b_1+a_2+b_2+\cdots+a_p+b_p=\alpha_j\,.$$

This lemma is easily checked by using the inequality

$$\|D_j u\|_\varphi^2=\|\tilde{D}_j u\|_\varphi^2+2h_j^2\|u\|_\varphi^2\geqq\|\tilde{D}_j u\|_\varphi^2\,.$$

Let $a(x)$ be a function of x in Ω which has bounded first derivatives. Denote $a^{(j)}=D_j a(x)$. We denote by $\mathfrak{A}_{m,n}=\mathfrak{A}_{n,m}$ $(m\leqq n)$ the C-module generated by operators of the form

$$g^\gamma(D,\tilde{D})^\alpha a^{(j)}(x)(D,\tilde{D})^\beta;$$

$$2|\gamma|+|\alpha|+|\beta|\leqq m+n,\ |\gamma|\leqq m,\ n+|\beta|\geqq|\alpha|+m,\ n+|\alpha|\geqq|\beta|+m$$

and denote by $\mathfrak{A}_{m,+}$ $(\mathfrak{A}_{m,-})$ the C-module generated by operators of the form

$$g^\gamma(D,\tilde{D})^\alpha a^{(j)}(x)(D,\tilde{D})^\beta;\ 2|\gamma|+|\alpha|+|\beta|\leqq 2m-1,\ |\alpha|=|\beta|+1\quad(|\alpha|=|\beta|-1)\,.$$

By virtue of definition, we have

(i) $\mathfrak{A}_{m-1,n-1}\subset\mathfrak{A}_{m,n}$; $\mathfrak{A}_{m-1,\pm}\subset\mathfrak{A}_{m,\pm}$,

(ii) $\tilde{D}_j\mathfrak{A}_{m,n}$, $\mathfrak{A}_{m,n}D_k\subset\mathfrak{A}_{m,n+1}$ $(n\leqq m)$,

(iii) $\mathfrak{A}_{m,+}D_k$, $\tilde{D}_j\mathfrak{A}_{m,-}\supset\mathfrak{A}_{m,m}$,

(iv) $\tilde{D}_j\mathfrak{A}_{m,m}\subset\mathfrak{A}_{m+1,+}$, $\mathfrak{A}_{m,m}D_k\subset\mathfrak{A}_{m+1,-}$,

(v) $\tilde{D}_j\mathfrak{A}_{m,n}D_k\subset\mathfrak{A}_{m+1,n+1}$, $\tilde{D}_j\mathfrak{A}_{m,\pm}D_k\subset\mathfrak{A}_{m+1,\pm}$.

We set

$$\{\tilde{D}^\lambda aD^\mu\}=\sum_{0<\kappa\leqq\lambda,\mu}\frac{g}{\kappa!}\frac{\lambda!}{(\lambda-\kappa)!}\frac{\mu!}{(\mu-\kappa)!}D^{\mu-\kappa}a\tilde{D}^{\lambda-\kappa}.$$

Since the coefficients of $\{D^\lambda aD^\mu\}$ are equal to those of $[\tilde{D}^\lambda, D^\mu]$, we obtain

$$\{\tilde{D}^\lambda aD^\mu\}=[D_j, D^\mu]a\tilde{D}^{\lambda*}+\{\tilde{D}^{\lambda*}a[\tilde{D}_j, D^\mu]\}+\{\tilde{D}^{\lambda*}aD^\mu\}\tilde{D}_j\,, \tag{16}$$

$$\{\tilde{D}^\lambda aD^\mu\}=D^{\mu*}a[\tilde{D}^\lambda, D_k]+\{[\tilde{D}^\lambda, D_k]aD^{\mu*}\}+D_k\{\tilde{D}^\lambda aD^{\mu*}\}\,, \tag{17}$$

where $\lambda^*=(\lambda_1, \cdots, \lambda_j-1, \cdots, \lambda_\nu)$ and $\mu^*=(\mu_1, \cdots, \mu_k-1, \cdots, \mu_\nu)$.
By virtue of (i)-(v), (16) and (17), we can prove

LEMMA 7.

$$\tilde{D}^\lambda aD^\mu-D^\mu a\tilde{D}^\lambda\equiv\{\tilde{D}^\lambda aD^\mu\} \bmod. \mathfrak{A}_{|\lambda|,|\mu|+1}(\mathfrak{A}_{|\mu|,\pm}) \tag{18}$$

holds if $|\lambda|<|\mu|$ $(|\lambda|=|\mu|)$.

Proof. Consider the identities

$$\begin{aligned}&\tilde{D}^\lambda aD^\mu-D^\mu a\tilde{D}^\lambda\\&\quad=D^{\mu*}a[\tilde{D}^\lambda, D_k]+([\tilde{D}^\lambda, D_k]aD^{\mu*}-D^{\mu*}a[\tilde{D}^\lambda, D_k])\\&\qquad+D_k(\tilde{D}^\lambda aD^{\mu*}-D^{\mu*}a\tilde{D}^\lambda)+\tilde{D}^\lambda(aD_k-D_ka)D^{\mu*}\\&\quad=[\tilde{D}_j, D^\mu]a\tilde{D}^{\lambda*}+(\tilde{D}^{\lambda*}a[\tilde{D}_j, D^\mu]-[D_j, D^\mu]a\tilde{D}^{\lambda*})\\&\qquad+(\tilde{D}^{\lambda*}aD^\mu-D^\mu a\tilde{D}^{\lambda*})\tilde{D}_j+\tilde{D}^{\lambda*}(\tilde{D}_ja-a\tilde{D}_j)D^\mu\,.\end{aligned}$$

Then, it is obvious that this lemma is proved inductively.
By virtue of Lemma 5, Lemma 6 and the Schwarz' inequality, we have

LEMMA 8. If A is an operator in $\mathfrak{A}_{m,\pm}$, then there exists a constant C depending on A such that

$$|(Au, v)|\leqq C\{T_m(u)_\varphi T_{m-1}(v)_\varphi+T_{m-1}(u)_\varphi\cdot T_m(v)_\varphi\} \tag{19}$$

holds for every u, v in $C_0^\infty(\Omega)$.
Using above two lemmas, we can prove

LEMMA 9. If $P(x, D)$, $Q(x, D)$ are homogeneous differential operators of order m whose coefficients have bounded first derivatives in Ω, then

$$(20)\qquad (P(x, D)u, Q(x, D)v)_\varphi = \sum_\alpha \frac{2^{|\alpha|}\cdot h^\alpha}{\alpha!}(\bar{Q}^{(\alpha)}(x, \tilde{D})u, \bar{P}^{(\alpha)}(x, \tilde{D})v)_\varphi + \text{remainder}$$

such that

$$|\text{remainder}| \leqq C\{T_m(u)_\varphi T_{m-1}(v)_\varphi + T_{m-1}(u)_\varphi T_m(v)_\varphi\}$$

holds for every $u \in C_0^\infty(\Omega)$ where C is a constant depending only on P, Q and the dimention ν of Ω.

Now take $P(x, D)=Q(x, D)$, $u=v$ in (20). Then we have

$$(21)\qquad \|P(x, D)u\|_\varphi^2 = \sum_\alpha \frac{2^{|\alpha|}h^{2\alpha}}{\alpha!}\|\bar{P}^{(\alpha)}(x, D)u\|_\varphi^2 + \text{remainder}$$

where $|\text{remainder}| \leqq CT_m(u)_\varphi T_{m-1}(u)_\varphi$.

Proof of Theorem 1. From (21) and the corresponding formula obtained by replacing P by $P^{(\alpha)}$, it follows that

$$(22)\qquad h^{2\alpha}\|P^{(\alpha)}(x, D)u\|_\varphi^2 \leqq C\{\|P(x, D)u\|_\varphi^2 + T_m(u)_\varphi T_{m-1}(u)_\varphi + h^{2\alpha}T_{m-|\alpha|}(u)T_{m-|\alpha|-1}(u)_\varphi\} .$$

Using Lemma 5 we complete the proof of Theorem 1.

Another proof of (21) can be found in Hörmander [2].

The proof of Theorem 2 is quite similar to that of Theorem 4 in Hörmander [3]. Therefore we omit it.

5. Uniqueness theorem for a system

Let $P(x, D)$ be a $N' \times N$-matrix of differential operators, that is, $P(x, D) = P_{jk}(x, D)$, $j=1, \cdots, N'$; $k=1, \cdots, N$, where $P_{jk}(x, D)=\sum a_{jk;\,\alpha}(x)D^\alpha$. We rewrite this operator as $P(x, D)=\sum a_\alpha(x)D^\alpha$, where $a_\alpha(x)=(a_{jk;\alpha})$, $j=1, \cdots, N'$; $k=1, \cdots, N$.

Let $Q(x, D)=\sum b_\beta(x)D^\beta$ be another such operator. We define

$$(23)\qquad P(x, D)\circ Q(x, D) = \sum_{\alpha,\beta} a_\alpha(x)b_\beta(x)D^{\alpha+\beta}$$

After simple calculations, we easily see the following:

LEMMA 10. If the coefficients of $Q(x, D)$ are differentiable up to the order of $P(x, D)$, then

$$(24)\qquad P(x, D)\cdot Q(x, D) = \sum_{\kappa \geqq 0} \frac{1}{\kappa!} P^{(\kappa)}(x, D)\circ Q^{[\kappa]}(x, D)$$

holds.

Remark that the order of the κ-th term in the right hand is equal to the order of the left hand minus $|\kappa|$. This fact plays an essential role in the following.

LEMMA 11. Let Ω be a bounded domain in R^ν, $P(x, D)$ be a matrix of differential operators such that the hypotheses (A) and (B) in Theorem 3 are satisfied. Let u be a solution of a system of equations (7), which is $m+n$ times differentiable.

Then we have

$$L_\pi^0(x, D)D^\beta u^j=\sum_{k=1}^N \sum_{|\alpha|>m+n+t_k} c_{\pi,\beta;jk;\alpha}(x)D^\alpha u^k, \tag{25}$$

where $c_{\pi,\beta;jk;\alpha}(x)$ is a linear combination of the coefficients of $P(x, D)$ and its derivatives, and β is an arbitrary vector of non negative integers such that $|\beta|=t_j+m-m(\pi)$.

Proof. Let $Q_{jk}(x, \xi)$ be the (k, j) cofactor of the determinant of the matrix $P(x, \xi)=(P_{\pi(j)k})$, $j, k=1, \cdots, N$. Then we have

$$Q(x, D)\circ P_\pi(x, D)=\begin{pmatrix} L_\pi(x, D) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & L_\pi(x, D)\end{pmatrix}.$$

Accoording to Lemma 10 we have

$$Q\circ P_\pi-Q\cdot P_\pi=-\sum_{|\alpha|\geq 1}\frac{1}{\alpha!}Q^{(\alpha)}\circ P^{[\alpha]}.$$

The (j, k) component of the κ-th operator in the right hand is at most of order $n+m(\pi)-t_j+t_k-|\alpha|$. It follows from this that

$$[D^\beta(Q\circ P_\pi-Q\cdot P_\pi)u]^j=\sum_{k=1}^N \sum_{|\alpha|<m+n+t_k} c^{(1)}_{\pi,\beta;jk;\alpha}(x)D^\alpha u^k. \tag{26}$$

In the same way, we obtain

$$D^\beta L_\pi(x, D)u^j-L_\pi(x, D)D^\beta u^j=\sum_{|\alpha|<m+n+t_j} c^{(2)}_{\pi,\beta;j;\alpha}(x)D^\alpha u^j. \tag{27}$$

Now that $L_\pi(x, D)u^j=L_\pi^0(x, D)u^j+$the lower order terms, we have

$$L_\pi^0(x, D)D^\beta u^j-L_\pi(x, D)D^\beta u^j=\sum_{|\alpha|<m+n+t_j} c^{(3)}_{\pi,\beta;j;\alpha}(x)Du^j. \tag{28}$$

The formulae $(Q\circ P_\pi u)^j=L_\pi(x, D)u^j$, (26), (27) and (28) imply that

$$\begin{aligned} L_\pi^0(x, D)D^\beta u^j &=\{L_\pi^0(x, D)D^\beta u^j-L_\pi(x, D)D^\beta u^j\} \\ &\quad -\{D^\beta L_\pi(x, D)u^j-L_\pi(x, D)D^\beta u^j\} \\ &\quad +\{D^\beta L_\pi(x, D)u^j-D^\beta(Q\cdot P_\pi u)^j\} \\ &=\sum_{k=1}^N \sum_{|\alpha|<m+n+t_k} c_{\pi,\beta;jk;\alpha}(x)D^\alpha u^k. \end{aligned}$$

Proof of Theorem 3. By virtue of the hypothesis (C), the hypotheses of Theorem 2 hold for a system $\{L_\pi^0(x, D)D^\beta\}_{\pi,|\beta|=m-m(\pi)+t_j}$. Hence, we can apply

Theorem 2. We may assume that M, and C do not depend on $j=1, 2, \cdots, N$. and that $a\leqq\delta/2$.

It is easily seen that there exists a constant ρ smaller than $\varphi_\delta(0)=\delta^2$ such that $M_\rho=\left\{x;\ x^1\geqq\sum_{j=2}^{\nu}(x^j)^2,\ \varphi_\delta(x)\geqq\rho\right\}$ is contained in U_δ. Moreover there exists a neighbourhood U_1 of the origin such that $U_\delta\supset\bar{U}_1\supset U_1\supset M_\rho$ where $\bar{U}_1$ is the closure of U_1. Set $U_2=U_1\cap\{x;\ \varphi_\delta(x)>\rho\}$. Then U_2 is a neighbourhood of the origin.

Let $\chi(x)$ be a C^∞-function which is identically equal to 1 on $\bar{U}_1$ and has compact carrier in U_δ. Set $v=\chi u$. Applying Theorem 2 for v^* and $\varphi(x)=\tau^2\varphi_\delta(x)$ we have

$$\sum_{k=1}^{N}\sum_{l<m+n+t_k}T_l(u^k)_{\varphi,U_2}^2\leqq C\tau^{-2}\sum_{k=1}^{N}\sum_{\pi}\sum_{|\beta|=t_k+m-m(\pi)}\|L_\pi^0D^\beta v^k\|_{\varphi,U_2}^2$$
$$+C\tau^{-2}\sum_{k=1}^{N}\sum_{\pi}\sum_{|\beta|=t_k+m-m(\pi)}\|L_\pi^0D^\beta v^k\|_{\varphi,U_\delta-U_2}^2$$

where $\|\ \ \|_{\varphi,U}$ means L_2-norm on U with the weight function $e^{\varphi(x)}$. The fact $u=v$ on U_2 and (25) imply that

$$(1-C'\tau^{-2})\sum_{k=1}^{N}\sum_{l>m+n+t_k}T_l(u^k)_{\varphi,U_2}^2\leqq C\tau^{-2}\sum_{k=1}^{N}\sum_{\pi}\sum_{|\beta|=t_k+m-m(\pi)}\|L_\pi^0D^\beta v^k\|_{\varphi,U_\delta-U_2}^2\,.$$

Take $\tau>1/\sqrt{2C'}$. Remarking that $\inf\{\varphi(x);\ x\in U_2\cap\operatorname{car}u\}$ is equal to $\sup\{\varphi(x);\ x\in(U_\delta-U_2)\cap\operatorname{car}v\}$, we get

$$\sum_{k=1}^{N}\sum_{|\alpha|<m+n+t_k}\frac{|\alpha|!}{\alpha!}\int_{U_2}|D^\alpha u^k|^2dx \tag{29}$$
$$\leqq 2C\tau^{-2}\sum_{k=1}^{N}\sum_{|\beta|=t_k+m-m(\pi)}\int_{U_\delta-U_2}|L_\pi^0(x, D)D^\beta v^k|^2dx\,.$$

When τ tends to infinity, the right hand tends to zero, so that u is identically equal to zero on U_2. This completes the proof of Theorem 3.

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* The differentiability of solutions can be proved. See Hörmander [4], Komatsu [9].

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