On the Minimum Eigenvalues of the Laplacians in Riemannian Manifolds

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In a Riemannian manifold M the Laplacian $\Delta=d\delta+\delta d$ operates on the space H of the differential forms on M. When M is compact and orientable, H is the direct sum of the finite-dimensional eigen-spaces of Δ. This decomposition is compatible with the decomposition of H by the degree of the forms. The eigenvalues are non-negative. In this paper we shall find the minimum positive eigenvalues of Δ for the degrees 0 and 1 in a class of Riemann manifolds which includes irreducible symmetric spaces, and their geometric meaning will also be clarified. Some relations between eigenspaces and transformation groups will be discussed in a forthcoming paper.

1. Eigenspace decomposition

In a compact orientable manifold M with a fixed Riemannian metric, inner product $(\ ,\ )_M$ is defined on the space of the tensor fields on M. We call $\Delta=d\delta+\delta d$ the Laplacian of M, where δ denotes the adjoint operator of the exterior differential d; $(d\alpha, \beta)_M=(\alpha, \delta\beta)_M$ for differential forms α and β. Since $-\Delta$ is strongly elliptic, the space of all forms is decomposed into the sum finite-dimensional eigenspaces of the Laplacian.

Let α be a form satisfying $\Delta\alpha=c\alpha$, where c is an eigenvalue. We are not interested in harmonic forms and assume $c\neq 0$ (hence $0<c$). α is then uniquely written as $\alpha=d\beta+\delta\gamma$ with β and γ subject to $\Delta\beta=c\beta$ and $\Delta\gamma=c\gamma$. In fact one has only to put $\beta=\delta\alpha/c$ and $\gamma=d\alpha/c$. Since $d\lambda=\delta\mu$ implies $d\lambda=0$ in general, the uniqueness is proved, while β and γ are not unique. Thus we have

PROPOSITION 1. *Let B_c^p [resp. C_c^p] be the space of all derived [coderived] p-forms belonging to the non-zero eigenvalue c of the Laplacian on a compact orientable Riemann manifold M. Then $B_c^p+C_c^p$ is the totality of the p-forms α with $\Delta\alpha=c\alpha$. $d: C_c^p \to B_c^{p+1}$ [resp. $\delta: B_c^p \to C_c^{p-1}$] is bijective. The star-operation maps B_c^p [resp. C_c^p] onto C_c^{n-p} [resp. B_c^{n-p}], $n=\dim M$.*

EXAMPLE. If M is moreover an Einstein space with scalar curvature R and c denotes $2R/n$, then a vector field u (which is identified with the dual 1-form by the Riemannian metric) satisfying $\Delta u=cu$ is called geodesic. $A_c^1=B_c^1+C_c^1$ is the totality of geodesic vector field [7]. C_c^1 is the space of all Killing vector fields [5]. If M is furthermore Kählerian, then A_c^1 is the Lie algebra of the (holomorphic) automorphism group of M and the almost complex structure is a bijective map of B_c^1 onto C_c^1 [4]. When M is irreducible symmetric, in order that A_c^1 different from C_c^1 it is necessary and sufficient that M is Kählerian or the group manifold of G_2 [8].

2. The least eigenvalues of Δ for 1-forms

THEOREM 1. *In a compact orientable Einstein space M with non-vanishing scalar curvature R, the Laplacian restricted to the coderived 1-forms has no eigenvalues smaller than $2R/n$, $n=dim\ M$.*

The proof is accomplished by modifying that of Theorem 2.16 in [5]. Summing up the formula (2.75) times (-1) and the formula (2.77) times $(1-t)$, $0<t<2$, both in [5], one gets

$$(\Delta u-tRu/n\,,\ u)_M-(v,\, v)_M/(1+a^2)=0\,,$$

for any 1-form u, where v is the tensor $\nabla_j u_i - a\nabla_i u_j$ and a is the real number satisfying $1-t=2a/(1+a^2)$ and $-1<a<1$. Thus $\Delta u=tRu/n$ implies $v=0$, hence $\nabla u=0$ and so $u=0$ because of $R\neq 0$.

REMARK. $2R/n$ is an eigenvalue of Δ operating on coderived 1-forms if and only if the isometry group of M is not discrete [5]. The least eigenvalue of Δ for derived 1-forms coincides with that of Δ for functions.

3. Relation to the isometry group

From now on, M will be assumed to be a compact homogeneous Riemann manifold G/H, G compact. An isometry g of M carries a form α to $(\delta g)^{-1}\alpha$. G thus operates on the space of forms, and defines *the regular representation* T. $T(G)$ leaves B_c^p's and C_c^p's invariant. Let x be a point left fixed by H. If U is an irreducible representation space of T, then U is the direct sum of subspaces V and W such that V and W are invariant under $T(H)$ and V consists of forms which vanishes at x. W is not trivial and $T(H)$ operates on W as on some invariant space of Grassmann algebra of cotangent space at x. The converse is true. Namely, given an orthogonal representation T' of G operating on a finite dimensional vector space U' such that the operation of $T'(G)$ on some non-trivial invariant subspace W' with $T'(G)W'=U'$ is equivalent to operations of H on some subspace (identified with W') of Grassmann algebra of cotangent space at x, and given an element α' of W', we define a form α on M by $\alpha(g\cdot x)=(\delta g)PT'(g)^{-1}\alpha'$, $g\in G$, where P is the orthogonal projection of U' onto W'. Then $\{T(g)\alpha | g\in G\}$ spans a finite dimensional $T(G)$-invariant space. This representation is equivalent to T'. Such a representation will be said *of class* $(1, p)$, if W' consists of elements with degree p.

EXAMPLE 1. The adjoint representation of G occurs on C_c^1, $c=2R/n$.

EXAMPLE 2.. ([1]). Consider the case $p=0$. G operates on the space

$F(M)$ on M by T. Here we allow the functions are complex-valued. If M is irreducible symmetric, then each irreducible representation of class (1,0) of G occurs just once on $F(M)$ operated by $T(G)$. To be "of class (1,0)" means that H leaves invariant some non-zero vector. Each irreducible space of functions corresponds in a one-to-one way to a simultaneous eigenspace of all G-invariant differential operators, one of which is the Laplacian Δ.

4. The least eigenvalue of Δ for functions

LEMMA 1. *Let M be a compact homogeneous Riemannian space such that G is compact and the linear isotropy group is irreducible. If a subspace $\boldsymbol{F}$ ($N=\dim \boldsymbol{F}>1$) of the space $\boldsymbol{F}(M)$ of the functions is invariant by $T(G)$ and irreducible, then there exists an isometric immersion ϕ of M into the euclidean space $\boldsymbol{R}^N$; ϕ is at the same time an equivariant map of the transformation group.* [1].

PROOF. Let $(f_\lambda)=(f_1, \ldots, f_N)$ be an orthonormal base of $\boldsymbol{F}$ with respect to the $T(G)$-invariant inner product and ϕ be the map: $M\rightarrow \boldsymbol{R}^N$ given by $\phi(x)=(f_1(x), \ldots, f_N(x))$. We define a function L on the tangent bundle of M by $L(X)=\Sigma_\nu(Xf_\nu)^2$. L is G-invariant. Restricted to a tangent space, L is a positive semi-definite function. It is definite; in fact $L^{-1}(\{0\})$ is G-invariant and its intersection with a tangent space is a vector subspace invariant under the isotropy subgroup, hence it consists of zero vectors, otherwise it would be the tangent bundle and would consist of constant functions, contrary to the assumption $1<N$. Hence L gives rise to a G-invariant Riemannian metric on M. It is induced by ϕ from the usual metric in $\boldsymbol{R}^N$. Since the linear isotropy group is irreducible, the metric coincides with the given one up to a constant factor. Multiplying ϕ by a constant, we therefore obtain an isometric immersion.

REMARK. Under the assumption concerning G and H of Lemma 1, assume that c is a non-zero eigenvalue of the Laplacian. Then M can be isometrically immersed into (an irreducible part of) C_c^0 with G-invariant euclidean metric.

LEMMA 2. *Let ϕ be an isometric immersion of M into an irreducible part U of C_c^0, $c\neq 0$. $\phi(M)$ is contained in a hypersphere. Let r be its radius. Then we have $n=cr^2$, $n=\dim M$.*

PROOF. If $(f_1, \ldots, f_N)$ denotes an orthonormal basis of U, then we have $r^2=\Sigma(f_\nu)^2=\text{const}$. It follows $0=\Delta r^2=-2\Sigma(df_\nu, df_\nu)+2\Sigma f_\nu \Delta f_\nu=-2n+2\Sigma f_\nu(cf_\nu)=-2n+2cr^2$, where (df_ν, df_ν) is the square of the length of df_ν.

Let ϕ be an isometric immersion of M into the euclidean space $\boldsymbol{R}^N\subset C_c^0$, given as in the remark. We identify a small neighborhood V of an arbitrary point x with the submanifold $\phi(V)$. Let ∇ and ∇' denote covariant differentiations with respect to the given metrics on V and $\boldsymbol{R}^N$ respectively. Then for vector fields X and Y on V.

$$\Omega(X, Y)=\nabla'_X Y-\nabla_X Y$$

is the normal component of $\nabla'_X Y$, Ω being the second fundamental form by definition. Now we take an orthonormal basis in R^N such that the normal space to V at x is given by $x^1=\ldots.=x^n=0$. Put $\phi(y)=(f_1(y), \ldots., f_n(y))$, $y\in V$, $\nabla f_\nu=cf_\nu$. Then we have

$$\Omega^\mu_{ij}=\nabla_j\nabla_i f_\mu\,, \quad \mu>n\,,$$

where $\Omega(\partial_i, \partial_j)=(0, \ldots.0, \Omega^{n+1}_{ij}, \ldots., \Omega^\mu_{ij}, \ldots.)$. By the equations of Gauss ([2] p. 163),

(1) The scalar curvature $R=\Sigma_\mu((\Omega_i^{\mu i})^2-\Omega^\mu_{ik}\Omega^{\mu,ik})$

$$\leqslant\Sigma_\mu(\Sigma_i\Omega_i^{\mu i})^2-\Sigma_{\mu,i}(\Omega^\mu_{ii}\Omega^{\mu,ii}).$$

(2) $\Sigma(\Omega_i^{\mu i})^2=\Sigma(\Delta f_\mu)^2=c^2\Sigma(f_\mu)^2=c^2r^2$.

Let X be the unit tangent vector field along a geodesic γ in V. Then we have

$$\nabla'_X X=\nabla_x X+\Omega(X, X)=\Omega(X, X).$$

The length of $\nabla'_X X$ is the principal curvature of $\alpha\circ\gamma$. It is not smaller than $1/r$, if $\phi(U)$ is contained in the hypersphere with radius r in R^N. Hence we have

(3) $\Sigma\Omega^\mu_{ii}\Omega^{\mu,ii}\leqslant n/r^2$.

From (1), (2) and (3) it follows that

$$R\leqslant c^2r^2-n/r^2=(n-1)c\,.$$

We have proved

THEOREM 2. *For a compact homogeneous Riemannian manifold $M=G/H$ with irreducible isotropy group H, any non-zero eigenvalue of the Laplacian Δ operating on functions is not smaller than $R/(n-1)$, where R is the scalar curvature and n is the dimension of M.*

REMARK. If $c=R/(n-1)$ is an eigenvalue of Δ, then $B_c^1+C^1_{2R/n}$ is the Lie algebra of the conformal transformation group of the Einstein space. Therefore M is isometric to the sphere (=the elliptic space) [6].

5. Symmetric spaces

Let M be a compact irreducible symmetric space G/H. G is a compact connected semisimple Lie group. M is given the Riemannian metric induced naturally from the Killing form of G. Thus, p denoting the projection of G onto G/H, $\Delta'(f\circ p)=(\Delta f)\circ p$ for any function f on M where Δ' is the Laplacian on G. The eigenvalues of Δ operating on the functions are then those of the Casimir operators of the corresponding representations of G, and $2R/n=1$ (See a forthcoming paper for details).

To the irreducible representations of class (1, 0) of G correspond linear combinations with non-negative integers as coefficients of p dominant weights $\mu_1, \ldots, \mu_p$, p=rank M, (E. Cartan [1]). The generators have recently been determined by M. Sugiura [3].

Using his results and Freudenthal's formula concerning the eigenvalues of Casimir operators, we can calculate the eigenvalues of Δ for functions. They are tabulated in the following for classical groups G. $r(\mu_i)$ denotes the eigenvalue corresponding to the representation μ_i.

Symmetric space	$r(\mu_i)$, $1 \leqq i \leqq p$	Min $r(\mu_i)$
$AI = SU(m)/SO(m)$	$i(m-i)(m+2)/m^2$	$(m-1)(m+2)/m^2$
$AII = SU(2m)/Sp(m)$	$i(m-1)(2m+1)/(2m^2)$	$(m-1)(2m+1)/(2m^2)$
$AIII$, $IV = U(m)/U(p) \times U(m-p)$	$i(m-i+1)/m$	1
BI, $II = SO(2m+1)/SO(p) \times SO(2m+1-p)$	$i(2m+2-i)/(2m-1)$, $i<p$; $i(2m+1-i)/2(2m-1)$, if $i=p<m$; $m(m+1)/4(2m-1)$, if $i=p=m$	
$CI = Sp(m)/U(m)$	$i(2m+3-i)/2(m+1)$	1
$CII = Sp(m)/Sp(p) \times Sp(m-p)$	$i(m+1-i)/(m+1)$	$m/(m+1)$
DI, $II = SO(2m)/SO(p) \times SO(2m-p)$	$i(2m+1-i)/2(m-1)$, $i<p$, $i<m-1$; $p(2m-p)/4(m-1)$, $i=p<m-1$; $m(2m-1)/8(m-1)$, $i=p=m-1$; $m^2/4(m-1)$, $i \geqq m-1<p$	
$DIII = SO(2m)/U(m)$ $3<m$	$i(m-i)/(m-1)$, $i<p$; $(m+1)/4$, $i=p$, $m=odd$; $m^2/4(m-1)$, $i=p$, $m=even$	1

REMARK. For a Kählerian space (i.e. for a space whose isotropy subgroup has a non-discrete center), the least eigenvalue is $1=2R/n$. This fact is compactible with the example in Section 1.

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