On Fibred Riemann Manifolds

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We consider a Riemann manifold M with a differentiable map p of M onto a manifold L with constant rank=dim L; the Riemannian metric on M shall be bundle-like (in Reinhart's sense) so that p induces a Riemannian metric on L. The study was preceded by Y. Mutô [1] and K. Yano [7] and others. If M is complete, M becomes a fibred space in Ehresmann's sense and admits a connection (Section 5). When the fibres are totally geodesic, M admits a structure of differentiable fibre bundles. But the existence of a bundle-like metric hardly influence the topology (Section 1). The fibres are totally geodesic in important examples (Section 6). When an isometry group N leaves the fibres invariant and acts transitively on them, M becomes a fibre bundle over L with the structure group $N(K)/K$ where $N(K)$ is the isotropy subgroup K (Section 7). From this fact it follows that if N is semisimple and N/K is a symmetric space, then the connection is unique for all N-invariant bundle-like metrics. All these are derived from the fundamental formulas analogous to Weingarten's formula for submanifolds (Section 3). It is to be noted that the tensor analogous to the second fundamental form is skew-symmetric.

1. The postulates (P. 1–2)

The following postulate will be preserved throughout the paper.

(P. 1) *p is a differentiable map of a connected paracompact differentiable manifold M onto a differentiable manifold L, with costant rank, "differentiable" meaning C^∞-differentiable.*

The rank of p then equals the dimension n of L ([5]). The inverse image $p^{-1}(x)$ of any point x of L will be called *a fibre*, though M is not necessarily a fibred space even in the sense of Serre. The fibres are closed submanifolds of M and of constant dimension $m-n$, $m=\dim M$. Let M' denote the disjoint union of all fibres. M' is a regularly imbedded submanifold of M. A connected component of M' is contained in a fibre as a connected component. A vector tangent to M is called *vertical* if it is tangent to M'. A vertical (vector) field assigns a vertical vector to each point of M by definition.

Now we shall prove the existence of a Riemannian metric g on M such that

(P. 2) *For any two vectors X and Y both normal to M' with the same dp-image, the length $\|X\|_g$ of X equals $\|Y\|_g$.*

Let h and $\hat{g}$ be arbitrary Riemannian metrics on M and L respectively. If X is vertical, $\|X\|_g$ shall be the length of X with respect to h. If Y is normal to M' with respect to h, then $\|Y\|_g$ shall be the length of $dp(Y)$ with respect to $\hat{g}$. A Riemannian metric g on M is clearly defined in this way.

Under the hypothesis (P. 2), a vector normal to M' shall be called *horizontal*. Then g induces a Riemannian metric $\hat{g}$ on L such that, restricted to horizontal vectors, dp is an isometry.

Example 1. Let M be a differentiable fibre bundle over L with projection p. Since L is paracompact, there exists a connection of M. Let h and $\hat{g}$ be Riemannian metrics on M and L respectively. The length of a vertical vector is defined by means of h as above. The horizontal vectors shall be horizontal with respect to the connection. Thus a Riemannian metric g is defined and satisfies (P. 2).

Example 2. Assume that a connected Riemann manifold M admits a connected transitive isometry group G and that the isotropy subgroup H of G at a point of M is compact. This is the case when G is the maximal connected isometry group. Provided that G has a closed normal subgroup N, the orbit space $L=M/N$, a space whose points are N-orbits $N(x)$, $x\in M$, is naturally a differentiable manifold. In fact the subgroup NH is closed in G, and L is identified with the homogeneous space G/NH. Let p be the canonical projection of $M=G/H$ onto L. (P. 1) is clearly satisfied. M is a differentiable fibre bundle over L with each N-orbit as fibre and associated with the principal bundle $G\to G/NH$. N leaves each fibre invariant and acts transitively on it. It follows that the Riemannian metric on M has the property (P. 2).

Remark. In this example 2, G carries fibres onto fibres. Conversely, for a Riemann homogeneous manifold $M=G/H$ with the map p satisfying (P. 1), the elements of G which leave all fibres invariant constitute a closed normal subgroup N. But N is not transitive on the fibre in general.

Example 3. Let G be a connected isometry group of a connected Riemann manifold M. Assume that all G-orbits are regular in Palais' sense [3]. Then the orbit space $M/G=\{G(x)|x\in M\}$ is a differentiable manifold ([3]). (P. 1) is clearly satisfied. p denoting the projection of M onto $L=M/G$, we define the length of a vector X tangent to L by the length of a horizontal vector Y with $dp(Y)=X$. (P. 2) is satisfied, as is easily seen.

2. The maps B and C

Let $V(M; M')$ be the totality of the vector fields u on M with $dp\circ u$ constant on each fibre. For any differentiable manifold A, we denote by $V(A)$ the totality of the vector fields on A. $V(A)$ is a Lie algebra. Assigning to u the vector field $dp\circ u\circ p^{-1}$ on L, we obtain a homomorphism B' of the vector space $V(M; M')$ into $V(L)$.

PROPOSITION 2.1. *$V(M; M')$ is a subalgebra of $V(M)$, and B' is a homomorphism of the Lie algebra $V(M; M')$ onto $V(L)$, whose kernel is the totality $V'(M)$*

of all vertical vector fields.

Proof. A vector field u on M belongs to $V(M; M')$ if and only if the function uf with $f=\hat{f}\circ p$ is constant on each fibre for any function $\hat{f}$ on L. Since $(B'u)\hat{f}=uf$ for each u in $V(M; M')$ and each function $\hat{f}$ on L, we have $u(vf)$, hence $[u, v]f$, is constant on each fibre for any u and v in $V(M; M')$, and moreover $[B'u, B'v]\hat{f}=u(vf)p^{-1}-v(uf)p^{-1}=B'[u, v]\hat{f}$. Hence $V(M; M')$ is a subalgebra, and B' is a homomorphism of a Lie algebra. $V'(M)$ is obviously the kernel of B'. (Hence $V'(M)$ is an ideal of $V(M; M')$.)

Let g be a Riemannian metric on M having the property (P. 2). By definition B is an isomorphism of the vector space $V(L)$ into $V(M; M')$ which assigns to $\hat{u}$ in $V(L)$ the horizontal vector field u with $dp\circ u=\hat{u}$. Then $B'B$ is clearly an identity; in particular B' is onto.

Remark. The proposition 2.1 gives the meaning of (1.14) in [7].

Proposition 2.2. *If each fibre is connected, then $V(M; M')$ is characterized as the normalizer of $V'(M)$ in $V(M)$.* (See [7]).

Proof. Let u and v be in $V(M; M')$ and $V'(M)$ respectively. For a function f on M with $f=fp^{-1}p$, we have $[u, v]f=uvf-vuf=-vuf$. Since $uf=(uf)p^{-1}p$, we obtain $[u, v]f=0$, whence $[u, v]$ belongs to $V'(M)$. Conversely let u belong to the normalizer of $V'(M)$ in $V(M)$ and v be in $V'(M)$. Then we have $-vuf=0$, which implies that u belongs to $V(M; M')$ on account of the connectedness of the fibres.

Now any vertical field can be regarded as a vector field on M'. Thus we get an isomorphism C' of the Lie algebra $V'(M)$ into $V(M')$. The inverse of C' defined on the C'-image $V'(M')$ of $V'(M)$ will be denoted by C. Thus we have the exact sequence: $0\longrightarrow V'(M')\xrightarrow{C} V(M; M')\xrightarrow{B'} V(L)\longrightarrow 0$. C' composed with the restriction to a fibre F is onto $V(F)$. We shall confound Bu (resp. Cv) with u (resp. v) when no confusion is to fear.

3. Fundamental formulas

Hereafter we shall always assume (P. 1–2). We shall prove the following fundamental formulas (3.1–4) which are analogous to Weingarten's formula for a submanifold of a Riemann manifold.

Any vector field u on M is the sum of a horizontal vector field h and a vertical vector field v. The unique decomposition $u=h+v$ will be called *canonical*. u is inducible if and only if h belongs to the B-image of $V(L)$.

For any u and v in $V(L)$, consider the vecter field

$$j(u, v)=\nabla_{Bu}Bv-B\nabla_{u}v,$$

where the covariant differentiation ∇ in the first term is with respect to g on M, while ∇ in the second is with respect to the induced $\hat{g}$ on L from g. For a function $\hat{f}$ on L and $f=\hat{f}\circ p$, we have $j(\hat{f}u, v)=f\cdot j(u, v)$ and $j(u, \hat{f}v)=\nabla_{Bu}B(\hat{f}v)$

$-B\nabla_u(\hat{f}v)=\nabla_{Bu}(fBv)-B\nabla_u(\hat{f}v)=((Bu)f)Bv+f\nabla_{Bu}Bv-B((u\hat{f})v+\hat{f}\nabla_u v)=f\cdot j(u, v)$, because $((Bu)f)Bv=((u\hat{f})\circ p)Bv=B((u\hat{f})v)$. Thus there exists uniquely a (1, 2)-type tensor field J on M such that $J(u, v)=j(B'u, B'v)$ for horizontal vector fields on M and $J(u, v)=0$ if at least one of u and v is vertical.

$$\nabla_{Bu}Bv=B\nabla_u v+J(u, v) \tag{3.1}$$

is the canonical decomposition of $\nabla_{Bu}Bv$. $J(u, v)$ is vertical, because of

$$(Bu)g(Bv, Bw)=u\hat{g}(v, w)\circ p,\ \ g([Bu, Bv], Bw)=g(B'[Bu, Bv], B'Bw)=\hat{g}([u, v], w)\circ p$$

and

$$2g(\nabla_u v, w)=ug(v, w)+vg(w, u)-wg(u, v)+g([u, v], w)+g([w, u], v)-g([v, w], u)\ .$$

Let

$$\nabla_{Ca}Bu=h(a, u)+v(a, u)\ ,\qquad a\in V'(M')\ ,\quad u\in V(L)\ ,$$

be the canonical decomposition. Since $\nabla_{Ca}(\hat{f}p)=0$ for any function $\hat{f}$ on L, there exist uniquely (1, 2)-type tensor fields H and V with $H(a, v)=h(a, v)$, $V(a, v)=v(a, v)$ for a vertical a and horizontal v, and with $H(a, v)=V(a, v)=0$ for a horizontal a or a vertical v. Thus

$$\nabla_{Ca}Bv=H(a, v)+V(a, v)\ , \tag{3.2}$$

for a vertical a and a horizontal v. Since $V'(M)=\operatorname{Ker} B'$ is an ideal of $V(M; M')$ we find that

$$\nabla_{Bu}Cb=H(b, u)+(V(b, u)+[b, u]) \tag{3.3}$$

is the canonical decomposition. Analogously we obtain the canonical decomposition

$$\nabla_{Ca}Cb=K(a, b)+C\nabla_a b\ . \tag{3.4}$$

The tensor fields J, H, V and K may be considered to be generalized "second fundamental forms".

J (resp. K) is intimately related to H (resp. V) with the formulas

$$g(H(a, u), v)=-g(a, J(u, v))\ , \tag{3.5}$$

$$g(V(b, v), a)=-g(K(b, a), v)\ . \tag{3.6}$$

These are proved as follows. The identity $g(a, v)=0$ gives (3.5) and (3.6) if it is covariantly differentiated with respect to a horizontal u and a vertical b respectively with (3.1–4) applied.

The vertical fields forming a subalgebra $V'(M)$, K is symmetric;

$$K(a, b)=K(b, a)\ . \tag{3.7}$$

J is, however, skew-symmetric;

$$J(u, v)=-J(v, u)\ . \tag{3.8}$$

Proof. For a vertical vector field b and a horizontal vector field u, we get $0=\nabla_b g(u, u)=2g(\nabla_b u, u)=-2g(b, J(u, u))$, which implies $J(u, u)=0$.

Remark. From the fundamental formulas, one can deduce formulas which are analogous to the Gauss-Codazzi equation for submanifolds; for instance,

$$(3.9)\quad \begin{aligned} R(Bu, Bv)Bw = &\{H(J(v, w), u) - H(J(u, w), u) - 2H(J(u, v), \mathrm{w}) \\ &+ BR(u, v), w\} + \{V(J(v, w), u) - V(J(u, w), v) \\ &- V(J(u, w), v) - 2V(J(u, v), w) - J([u, v], w)\}, \end{aligned}$$

where the first four terms in $\{\cdots\}$ at the right hand side are horizontal and the rest are vertical, while R is the curvature tensor; $R(X, Y)Z = (\nabla_X, \nabla_Y)Z - \nabla_{(X,Y)}Z$, for any vector fields X, Y and Z.

4. Geodesics

A piecewise differentiable curve γ in M is called a *lift* of the curve $p\gamma$ in L when each tangent vector to it is horizontal.

PROPOSITION 4.1. *The lifts of geodesics in L are geodesics.*

The proof is immediate by $J(u, u) = 0$ (See (3.8)). This was proved by Mutô [1] and by Yano [6] for Example 3.

Assigning to each point x in M the horizontal vectors at x, we obtain a distribution, which we call the *horizontal distribution.* It is integrable if and only if B is a homomorphism of a Lie algebra. Then the integral manifolds are called the *horizontal submanifolds.*

PROPOSITION 4.2. *If B is a homomorphism of a Lie algebra, then the integral submanifolds are totally geodesic.*

In fact J in (3.1) is then symmetric, and therefore J vanishes by (3.8).

Example 2′. Let G be a connected Lie group with the radical group R and the maximal semisimple group S. Assume that g is any G-left-invariant Riemannian metric such that R is orthogonal to S at the neutral element of G. Then S is totally geodesic with respect to g.

Proof. Put $G = M$ and $S = G/R = L$. Then S is imbedded as a horizontal submanifold (Proposition 4.2).

Since M and L are connected Riemann manifolds, they are metric spaces. Let d and $\hat{d}$ denote the distance functions. One can define another metric d' on L by $d(\hat{x}, \hat{y}) = d(p^{-1}(\hat{x}), p^{-1}(\hat{y})) = \inf\{d(x, y) | p(x) = \hat{x}), p(y) = \hat{y}\}$.

PROPOSITION 4.3. *We have* 1) $d'(\hat{x}, \hat{y}) \geqq \hat{d}(\hat{x}, \hat{y})$ *for any two point $\hat{x}$ and $\hat{y}$* in L, 2) *the equality holds in* 1) *if $\hat{x}$ and $\hat{y}$ are sufficiently near, and* 3) *the same holds if M is complete.*

This follows from Proposition 4.1. (The equality does not holds in general if M is not complete but the fibres are complete.)

5. Complete M

PROPOSITION 5.1. *If M is complete, M is a differentiable fibred space in*

Ehresmann's sence with the projection p. The horizontal distribution defines an (infinitesimal) connection.

This follows from the Lemmas 5.2–3 below.

LEMMA 5.2. *Assume that M is complete. Then for any piecewise differentiable curve γ on L starting at $\hat{x}$ in L, there exists a lift of γ starting at an arbitrary point x with $p(x)=\hat{x}$.*

Remark. Without the condition that M is complete, the conclusions in Proposition 5.1 and Lemma 5.2 hold good if all fibres are compact but not in general if all fibres are complete.

LEMMA 5.3. *If the conclusion of Lemma 5.2 is valid, then M is a fibred space.*

Proof. By Lemma 5.2, all fibres are diffeomorphic to each other. Let F be one of them. Let U_x, $x\in L$, be the neighborhood of x on which the normal coordinate system is valid. There exists a diffeomorphism ρ_x of $F\times U_x$ onto $p^{-1}(U_x)$ such that $\rho_x(f, y)$ is the end point of the lift from $\rho_x(f, x)$ of the geodesic joining x to y in U_x. Given two points x and z, there exists a map α of $U_x\cap U_z$ into the diffeomorphism group of F such that we have $\rho_x(f, y)=\rho_z(\alpha(y)f, y)$ for any y in $U_x\cap U_z$.

Remark on the universal covering. Let p_M (resp. p_L) be the projection of the universal covering $\tilde{M}$ (resp. $\tilde{L}$) onto M (resp. L). Then there exists a map $\tilde{p}$ of $\tilde{M}$ into $\tilde{L}$ with $pp_M=p_L\tilde{p}$. When M is complete, $\tilde{p}$ is onto and a fibre of $\tilde{M}$ is mapped by p_M onto a connected component of a fibre of M. If moreover the horizontal distribution is integrable, then $\tilde{M}$ is the direct product of the manifolds $\tilde{F}$ (a fibre) and $\tilde{L}$.

Remark. It is probable that to a conjugate point along a geodesic γ on L corresponds a focal point of the lift of γ.

6. Transformations of the metric

The Lie derivative of the metric tensor g with respect to a vertical vector field Ca is given by

$$(\mathfrak{L}_{Ca}g)(Bv, Bv)=0\ , \tag{6.1}$$

$$(\mathfrak{L}_{Ca}g)(Bv, Cb)=-g([Ca, Bv], Cb)\ , \quad \text{and} \tag{6.2}$$

$$(\mathfrak{L}_{Ca}g)(Cb, Cb)i=(\mathfrak{L}_a g')(b, b) \tag{6.3}$$

for vector fields a and b on M', where g' is the metric tensor induced on M' and i is the injection of M' into M.

Proof. Since a Lie derivative is a derivation of the algebra of all tensor fields and commutes with the contraction, we have

$$\begin{aligned}(\mathfrak{L}_X g)(Y, Z)&=\mathfrak{L}_X(g(Y, Z))-g(\mathfrak{L}_X Y, Z)-g(Y, \mathfrak{L}_X Z)\\ &=\mathfrak{L}_X g(Y, Z)-g([X, Y], Z)-g(Y, [X, Z])\ .\end{aligned}$$

(6.1) follows from

$$(\mathfrak{L}_{Ca}g)(Bv, Bv)=\mathfrak{L}_{Ca}g(Bv, Bv)-2g([Ca, Bv], Bv)=0-0$$

by (P. 2) and Proposition 2.1. (6.2) follows from

$$(\mathfrak{L}_{Ca}g)(Bv, Cb)=\mathfrak{L}_{Ca}g(Bv, Cb)-g([Ca, Bv], Cb)-g(Bv, [Ca, Cb])$$
$$=0-g([Ca, Bv], Cb)+0$$

owing to $g(Bv, Cb)=0$ and Proposition 2.1. (6.3) is trivial.

These formulas (6.1–3) give

Proposition 6.1. *A vertical vector field Ca on M is a Killing vector field if and only if it commutes with any horizontal vector fields and a is a Killing vector field. A vertical Killing vector field vanishes on M if it vanishes on a fibre.*

The Lie derivative of g with respect to a horizontal vector field is given by the following:

(6.4) $$(\mathfrak{L}_{Bu}g)(Bv, Bv)=(\mathfrak{L}_u\hat{g})(v, v)p\ ,$$

(6.5) $$(\mathfrak{L}_{Bu}g)(Bv, Ca)=-2g(J(u, v), Ca), \quad \text{and}$$

(6.6) $$(\mathfrak{L}_{Bu}g)(Ca, Ca)=-2g(u, K(a, a))\ ,$$

where $\hat{g}$ is the induced metric on L from g.

Proof. We deduce (6.4) from $g([Bu, Bv], Bv)=g(B[u, v], Bv)$ obtained from (3.1). Proposition 2.1 and (3.1) give (6.5). Finally (6.6) follows from (3.2) and (3.6) combined with the well known formula: $(\mathfrak{L}_Xg)(Y, Y)=2g(\nabla_YX, Y)$.

If a is a vertical Killing vector field and h is a Killing vector field such that $h(x)$, $x\in M$, is horizontal, then we have

(6.7) $$g(K(a, a), h)=g(a, [a, h]) \text{ at } x\ .$$

Proof. Since h is a Killing field, we have $2g([h, a], a)=\mathfrak{L}_hg(a, a)$. There exists a horizontal vector field u with $u(x)=h(x)$. At the point x, $\mathfrak{L}_hg(a, a)$ equals $\mathfrak{L}_ug(a, a)=\nabla_ug(a, a)=2g(\nabla_ua, a)=2g([u, a]+V(a, u), a)$ (See (3.3)). a being a vertical Killing field, $[u, a]$ vanishes by (6.2). Now (6.7) follows from the above and (3.6).

Proposition 6.2. 1) *A Killing vector field k on M naturally gives rise to a Killing field $B'k$ on L if it is inducible* ([7]). 2) *Conversely for a Killing vector field u on K, the vector field $Bu+Ca$ is a Killing field if a satisfies the differential equations*

$$[Ca, Bv]=2J(u, v) \quad \textit{for any } v \textit{ in } V(L),$$
$$\mathfrak{L}_ag'(b, b)=2g(u, K(b, b)) \quad \textit{for any } b \textit{ in } V'(M').$$

Easy to prove by the above formula (6.1–6).

Corollary 6.3. *When a connected isometry group G is transitive on L, the following two conditions are equivalent*:

1) *The Lie algebra $\mathfrak{g}(M; M')$ consisting of all Killing vector fields in $V(M; M')$ contains the B-image of the Lie algebra $\mathfrak{g}(L)$ of G, or in other words, for any killing field u in $\mathfrak{g}(L)$, the vector field Bu on M is a killing vector field.*

2) *M is a fibre bundle with fibre $F=p^{-1}(o)$, $o\in L$, and associated with the principal bundle $G\to G/H$, where H is the isotropy subgroup of G at o. Moreover M is locally the Riemann product of F and L in the sense that L is covered with open subsets $\{U_\alpha\}$ such that $p^{-1}(U_\alpha)$ is isometric to the Riemann product $F\times U_\alpha$.*

PROPOSITION 6.4. *Assume that $p:M\to N$ defines a structure of a fibred space and the horizontal distribution defines an infinitesimal connection. Then in order that parallel displacements give rise to isometries of fibres it is necessary and sufficient that all the fibres are totally geodesic.* (See [1]).

This ia an immediate consequence of (6.6).

COROLLARY 6.5. *Under the hypotheses of the preceding proposition, in order that M is locally the Riemann product of F and L, it is necessary and sufficient that the connection defined with the horizontal distribution is locally flat and the parallel displacements preserve the metric of fibres.*

COROLLARY 6.6. *Under the hypotheses of Proposition 6.4, assume that the fibres are totally geodesic. A Killing vector field a'' on a fibre F can be extended to a vertical Killing vector field a on M if and only if a'' is invariant under the holonomy group of the connection defined with the horizontal distribution; if a exists, it is unique.*

PROPOSITION 6.7. *In order that J and K vanish it is necessary and sufficient that every geodesics on M are mapped to geodesics on L by p.*

Proof. The necessity is obvious. Assume that for any geodesic γ on M the curve $p\gamma$ is a geodesic on L. If at a point x the geodesic γ is tangent to the fibre F containing x, then $p\gamma$ has the tangent vector vanishing at $p(x)$. It follows that γ is contained in F. Hence F is totally geodesic; i. e. $K=0$. Take any short geodesic γ on M. There exists on M an inducible vector field $u=Bv+Ca$ which gives rise to tangent vectors to γ, i. e. $u\circ\gamma=d\gamma$. Along γ and $p\gamma$ we have $\nabla_u u=0$ and $\nabla_v v=0$ respectively. From $K=0$ (hence $V=0$), we deduce

$$0=\nabla_u u=2H(a, v)+2V(a, v)+[Bv, Ca]+C\nabla_a a\ .$$

Hence the horizontal part $2H(a, v)$ vanishes, whence $J=0$.

PROPOSITION 6.8. *If every fibres are complete and totally geodesic, then the map $p:M\to L$ gives a fibre-bundle structure on M, with an isometry group of a fibre as the structure group.*

In fact any piecewise differentiable curve on L starting at an arbitrary point x in L admits a lift starting at any point in $p^{-1}(x)$, as is easily seen by completeness assumption and propositon 6.4. For the proof it is sufficient to note that, given a map λ of a differentiable manifold U into an isometry group G of a Riemann manifold F, if the map $(x, y)\in U\times F\to\lambda(g)(y)\in F$ is differentiable, then λ is differentiable.

Example 1'. Let M be the tangent bundle of a connected Riemann manifold L, and p be the projection. (P. 1) is satisfied. Consider a connection Γ of the bundle M. Γ gives rise to a Riemannian metric on M satisfying (P. 2) (Example

1). Assume that Γ is without torsion. Then in order that each fibre is totally geodesic it is necessary and sufficient that the affine homogeneous connection is metric, i. e. it is the Levi-Civita connection. (See [4]).

Example 3'. Assume that L is a Riemann homogeneous space. Let G be the maximal connected isometry group of L, and H be the isotropy subgroup at a point o in L. H is compact. Put $M=G$. p denoting the projection of M onto L, one finds (P. 1) satisfied. G operates on M to the left. At the same time we make H operate on M to the right, so that $G'=G\times H$ operates on M. Since H is compact, there exists a G'-invariant Riemannian metric which satisfies (P. 2) and the metric induced on L coincides with the original one. (See Example 1 and note that operation of H on the tangent space to L at o can be regarded as the operation of *ad* H on an orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in the Lie algebra $\mathfrak{g}$ of G with respect to an arbitrary metric of $\mathfrak{g}$ which is invariant under *ad* H.) Now *all fibres are totally geodesic in* M. This fact follows from (6.7), since every elements in G commutes with the elements of H operating on M to the right and H is transitive on each fibre. The restricted homogeneous holonomy group of the Levi-Civita connection on L can be determined (see (3.9)) by $BR(u, v)w=R(Bu, Bv)Bw-H(J(v, w), u)+H(J(u, w), v)+2H(J(u, v), w)$, if the curvature tensor of the metric on M is known and J (and so H) can be calculated with the first formula in Proposition 6.2.

7. Isometry groups transitive on fibres

Proposition 7.1 *Let F be a homogeneous space N/K of a Lie group N which operates on F effectively. Then the centralizer C of N in the group of all diffeomorphisms of F is a Lie group and isomorphic to $N(K)/K$, where $N(K)$ is the normalizer of K in N.*

Proof. To each element g in $N(K)$ we assign a transformation $\alpha(g)$ of F defined by $\alpha(g)(h(0))=hg^{-1}(0)$, where 0 is a point left fixed by K and h is an arbitrary element h in N; $\alpha(g)(h(0))$ depends on $h(0)$ but not on h. Let C' be the totality of all $\alpha(g)$'s; $C'=\alpha(N(K))$. C' is a transformation group of F and α is a homomorphism. Clearly C' is contained in C. Moreover $C'=C$. In fact for any λ in C there exists an element g in N such that $g^{-1}(0)=\lambda(0)$ and it is not hard to see that g belongs to $N(K)$ and $\alpha(g)=\lambda$. Finally the kernel of α is K.

Remark. $N(K)/K$ can be regarded as a subset of F. For points x and y in $N(K)/K$, the following (7.1–2) are evident;

(7.1) An element g in N belongs to $N(K)$ if and only if gx belongs to $N(K)/K$,

(7.2) There exists a unique coset gK such that $gx=y$.

Remark. The proposition 7.1 considerably shortens the proof of Theorem 2 in [2].

Lemma 7.2. *Assume that*

(7.3) *A connected isometry group N of M leaves invariant all fibres and is transitive on them.*

Then any piecewise differentiable curve γ on L admits a lift. Every two lifts *of γ are carried to each other by some transformations in N.*

Proof. Obvious.

THEOREM 7.3. *Under the assumption* (P. 3), *the totality P of the fixed points of the isotropy subgroup K of N at a point* 0, *$P=\{x\in M|K(x)=x\}$, is a differentiable principal bundle over L with structure group $C\cong N(K)/K$. Moreover M is a differentiable fibre bundle with fibre $F=p^{-1}p(0)$ associated with P, where $N(K)$ is the normalizer of K in N.*

Proof. P is a closed submanifold of M, for K is an isometry group of a Riemann manifold M. $N(K)/K=F\cap P$ is a closed submanifold and it is a Lie group. Any horizontal curve intersecting with $N(K)/K$ is contained in P by virtue of Proposition 6.1. Conversely an arbitrary point of P is on some of such curves by the same proposition and Lemma 7.2. P satisfies (P. 1) and (P. 2) with M replaced by P. Now P is a differentiable principal C-bundle over L with the right operation of C: $x\alpha(g)=g^{-1}(x)$ for $x\in P$ and $g\in N(K)$. Now we define a map μ of $P\times F$ into M by $\mu(x, y)=g(x)$ where g is any element in N with $g(0)=y$. μ is onto and can be regarded as the canonical projection of $P\times F$ onto the factor manifold defined by the equivalence relation $(gh, f)\equiv(g, hf)$ where h is an arbitrary element of $N(K)$ (see (7.2)).

PROPOSITION 7.4. *If $N(K)/K$ in the preceding theorem is discrete, then the horizontal distribution is integrable* ($J=0$) *and independent of the N-invariant Riemannian metrics on M having the property* (P. 2).

Proof. P is then an integral manifold of the horizontal distribution, which is thus integrable. Let g'' be another metric mentioned in the theorem. Then P which was defined by means of K and was independent of the metric, gives integral manifolds of the distribution with respect to g''.

Remark. $N(K)/K$ is discrete if N is semisimple and $F=N/K$ is a symmetric space. More precisely $N(K)/K$ is discrete if and only if K does not leave fixed any nonzero tangent vector F at 0.

The metric g gives rise to Riemannian metrics g' on M' and $\hat{g}$ on L in a natural way. g' and $\hat{g}$ fixed, a Riemannian metric g_1 is not unique even if g_1 is under an isometry group G leaving g invariant. If G is abelian and simply transitive on M, there exists, however, a fibre-preserving transformation ϕ of M which carries g to g_1. In general, if there exists a fibre-preserving transformation A of M which carries g to g_1, then A is contained in the normalizer of the isometry group of M in the diffeomorphism group of M. Hence A induces an automorphism of G.

Added in proof. The author has found that Proposition 5.1 and Proposition 6.8 were proved by Robert Hermann in his recent paper: A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle, Proc. Amer. Math. Soc. 11, 236–242 (1960).

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