On Specializations of Abelian Varieties

By Shoji KOIZUMI and Goro SHIMURA

Institute of Mathematics, Fuculty of Science, Tokyo University of Education
and
Institute of Mathematics, College of General Education, University of Tokyo

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In the study of specializations of Albanese or Picard varieties as well as in the arithmetic of automorphic functions, there arises a problem which is stated as follows: *Let A and B be abelian varieties defined over a field k with a prime divisor $\mathfrak{p}$. Suppose that there exists a homomorphism of A onto B, defined over k. If A is without defect for $\mathfrak{p}$, then is there an abelian variety which is isomorphic to B over k and is without defect for $\mathfrak{p}$?* Here we say that an abelian variety A is without defect for $\mathfrak{p}$, if the specialization of A with respect to $\mathfrak{p}$ is an abelian variety $\tilde{A}$ and the specialization of the graph of composition-law on A gives that on $\tilde{A}$. The main purpose of the present paper is to solve this problem; an affirmative answer is stated in Theorem 4 (§ 6). Besides this, we shall give, in the first part of the paper, some results which belong to foundations of specialization-theory. We now give a summary of the contents.

Among the fundamental results on abelian varieties, it is known that a rational mapping f of a variety V into an abelian variety A is defined at any simple point on V. In § 1, we shall give a generalization of this result as Theorem 1, which asserts that, if A is without defect for $\mathfrak{p}$ and if the specialization of V with respect to $\mathfrak{p}$ has only one component $\tilde{V}$ of multiplicity 1, then f is defined at any simple point $\bar{a}$ of $\tilde{V}$. Though this result is not needed for the rest of the paper, we have included it in view of future applications. In § 2, we study the expansion of a function f on a variety V, which is defined and finite at a simple point $\bar{a}$ of the specialization $\tilde{V}$ of V, by power-series in local parameters at $\bar{a}$; every coefficient of the power-series is $\mathfrak{p}$-integral and the expansion of the specialization $\tilde{f}$ of f is obtained by the specialization of coefficients of the series for f. This result is used for the specialization-theory of a function-module on V, which is the object of § 3, where we prove that a function-module on V and its specialization are of the same dimension. § 4 is devoted to the study of a problem concerning projective embeddings of an abelian variety without defect for $\mathfrak{p}$ and its field of definition. Problems of this kind are considered for any $\mathfrak{p}$-variety; and one can have a solution generalizing Weil's theory [13], with a certain condition of unramifiedness of $\mathfrak{p}$; in the present paper, however, we have restricted ourselves within the case of abelian varieties. The concept of abstract varieties enables us to construct a group-variety from a pre-group (Weil [10, 11]). In § 5, we shall give a construction of a group $\mathfrak{p}$-variety without defect from a pre-group $\mathfrak{p}$-variety without defect, following the idea of [10]; here is one of the reasons why we have preferred in our treatment abstract and $\mathfrak{p}$-varieties in the sense of [7] to projective varieties. § 6 contains the main theorem, which we have already explained above. In Appendix, it is proved that the specialization ring in the field of functions on V at a simple point $\tilde{a}$ of $\tilde{V}$ is a regular local ring.

Throughout the paper, we shall freely use the terminologies and results of [7].

§ 1. Specialization of rational mappings

Let k be a field with a discrete valuation of rank 1 $\{\mathfrak{o}, \mathfrak{p}, \tilde{k}\}$ where $\mathfrak{o}, \mathfrak{p}$ and $\tilde{k}$ denote respectively the valuation-ring, the maximal ideal of $\mathfrak{o}$ and the residue-field $\mathfrak{o}/\mathfrak{p}$. We shall consider two kinds of algebraic geometry: the one is the geometry under a universal domain $\boldsymbol{K}$ containing k, and the other is the geometry under another universal domain $\bar{\boldsymbol{K}}$ containing $\tilde{k}$. Throughout the paper, by letters with bars such as $\bar{V}, \bar{x}, \bar{\varphi}, \cdots$, we mean geometric objects in the geometry under $\bar{\boldsymbol{K}}$, and by $V, x, \varphi, \cdots$, those under $\boldsymbol{K}$. On the other hand, we shall always denote by $\tilde{V}, \tilde{x}, \tilde{\varphi}, \cdots$, specializations of $V, x, \varphi, \cdots$, with respect to $\mathfrak{o}$, where $V, x, \varphi, \cdots$ may be or may not be defined over k; and we shall write $V \rightarrow \tilde{V}$ ref. $\mathfrak{o}$, etc. If $F(X)$ is a polynomial in $\mathfrak{o}[X]$, we denote by $\tilde{F}(X)$ the polynomial in $\tilde{k}[X]$ obtained from F taking the coefficients of F modulo $\mathfrak{p}$. Let $\mathfrak{a}$ be an ideal of $k[X]$; we shall write

$$\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{o}[X], \quad \tilde{\mathfrak{a}}_0 = \{\tilde{P}(X) \mid P(X) \in \mathfrak{a}_0\}.$$

Let V be a $\mathfrak{p}$-variety[1], x a point of V and $\tilde{x}$ a specialization of x with respect to $\mathfrak{o}$; take affine representatives $x_\alpha, \tilde{x}_\alpha$ of $x, \tilde{x}$ and consider the set of elements $F(x_\alpha)/G(x_\alpha)$ such that $\tilde{G}(\tilde{x}_\alpha) \neq 0$, where $F(X)$ and $G(X)$ are polynomials in $\mathfrak{o}[X]$. This set forms a local ring and is independent of the choice of representatives $x_\alpha, \tilde{x}_\alpha$; we denote this local ring by $[x \rightarrow \tilde{x}; \mathfrak{o}]$.

1) Any affine or projective variety defined over k can be regarded as a $\mathfrak{p}$-variety in a natural way; so we shall identify an affine or a projective variety with the $\mathfrak{p}$-varity in this sense.

Now let us consider specialization of rational mappings. Let V and W be $\mathfrak{p}$-varieties and f a rational mapping of V into W defined over k. Let x be a generic point of V over k and $y=f(x)$. We say that f is *defined at* a point $\bar{a}$ on $\tilde{V}$ if there exists a point $\bar{b}$ on $\tilde{W}$ such that

$$[y\rightarrow\bar{b}\,;\,\mathfrak{o}]\subset[x\rightarrow\bar{a}\,;\,\mathfrak{o}]\,.$$

We can easily verify that $\bar{b}$ is determined only by f and $\bar{a}$; so we write $f(\bar{a})=\bar{b}$. Suppose that V and W are $\mathfrak{p}$-simple. If $\bar{A}$ is a subvariety in $\tilde{V}$, and if f is defined at a point in $\bar{A}$, we say that f is *defined along* $\bar{A}$. If $\bar{x}$ is a generic point of $\tilde{V}$ over $\tilde{k}$, $[x\rightarrow\bar{x}\,;\,\mathfrak{o}]$ is a discrete valuation ring of rank 1 (Prop. 5 of [7]). Hence f is defined along $\tilde{V}$ whenever W is $\mathfrak{p}$-complete. Assuming f to be defined along $\tilde{V}$, put $f(\bar{x})=\bar{y}$ for a generic point $\bar{x}$ of $\tilde{V}$ over $\tilde{k}$. Then we obtain a rational mapping $\tilde{f}$ of $\tilde{V}$ into $\tilde{W}$ defined by $\tilde{f}(\bar{x})=\bar{y}$ with respect to $\tilde{k}$. We call $\tilde{f}$ the *specialization of f with respect to* $\mathfrak{o}$. We see easily that if f is defined at a point $\bar{a}$ of $\tilde{V}$, $\tilde{f}$ is also defined at $\bar{a}$ and $f(\bar{a})=\tilde{f}(\bar{a})$.

Now we begin with the study of behaviour of a rational mapping f of V at a simple point $\bar{a}$ on $\tilde{V}$.

PROPOSITION 1. *Let V be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and f a rational mapping, defined over k, of V into an affine space S, such that f is defined along $\tilde{V}$. Put*

$$\mathcal{S}=\{a\,|\,a\in V, f \textit{ is not defined at } a\}\,,\ \ \bar{\mathcal{S}}=\{\bar{a}\,|\,\bar{a}\in\tilde{V}, f \textit{ is not defined at } \bar{a}\}\,.$$

Then the following assertions hold.

i) *$\mathcal{S}$ (resp. $\bar{\mathcal{S}}$) is a k-normal (resp. $\tilde{k}$-normal) bunch of subvarieties on V (resp. $\tilde{V}$), and $\tilde{\mathcal{S}}\supset\bar{\mathcal{S}}$.*

ii) *A simple point $\bar{a}$ of $\tilde{V}$ is contained in $\bar{\mathcal{S}}$ if and only if $\bar{a}$ is contained in $\tilde{\mathcal{S}}$. In particular, if $\tilde{V}$ is non-singular, we have $\bar{\mathcal{S}}=\tilde{\mathcal{S}}$.*

iii) *There exists a k-normal bunch $\mathcal{T}$ of subvarieties on V such that $\mathcal{T}\supset\mathcal{S}$, $\tilde{\mathcal{T}}\supset\bar{\mathcal{S}}$ and $\mathcal{T}\neq V$.*

Proof. It is not difficult to reduce our proposition to the case where V is an affine variety and f is a numerical function, i.e., a rational mapping of V into the affine 1-space; so we shall deal only with such a case. Let x be a generic point of V over k. Define two ideals in $\mathfrak{o}[X]$ or in $k[X]$ by

$$\mathfrak{a}_1=\{P(X)\,|\,P(X)\in\mathfrak{o}[X], P(x)f(x)\in\mathfrak{o}[x]\}\,,$$

$$\mathfrak{a}_0=\{P(X)\,|\,P(X)\in k[X], P(x)f(x)\in k[x]\}\,.$$

We have then $\mathfrak{a}_0\supset\mathfrak{a}_1$ and $\tilde{\mathfrak{a}}_0\supset\tilde{\mathfrak{a}}_1$. We see easily that

$$\mathcal{S}=\text{the set of zero points of } \mathfrak{a}_0,$$

$$\tilde{\mathcal{S}} = \text{the set of zero points of } \tilde{\mathfrak{a}}_0,$$

$$\bar{\mathcal{S}} = \text{the set of zero points of } \tilde{\mathfrak{a}}_1.$$

This proves the assertion i). Let $\tilde{x}$ be a generic point of $\tilde{V}$ over $\tilde{k}$. Since $\bar{\mathcal{S}}$ does not contain $\tilde{x}$, there exists a polynomial $P(X) \in \mathfrak{a}_1$ such that $\tilde{P}(\tilde{x}) \neq 0$. If we denote by $\mathcal{F}$ the set of points on V where $P(X)$ vanishes, $\mathcal{F}$ satisfies our requirements in iii). The numerical function f defines naturally a rational mapping f_1 of V into the projective 1-space P^1. Let T be the graph of f_1; let S^n and $\tilde{S}^n$ be the ambient spaces for V and $\tilde{V}$. As f is defined along $\tilde{V}$, any component of $\tilde{T}$ is not contained in $\tilde{S}^n \times \overline{\infty}$, so that the intersection-product $\tilde{T} \cdot (\tilde{S}^n \times \overline{\infty})$ is defined; and $\tilde{T} \cap (\tilde{S}^n \times \overline{\infty})$ is a specialization of $T \cap (S^n \times \infty)$ with respect to $\mathfrak{o}$. Let $\bar{a}$ be a simple point on $\tilde{V}$. If $\bar{a} \times \overline{\infty}$ is not contained in $\tilde{T} \cap (\tilde{S}^n \times \overline{\infty})$, every specialization of $f(x)$ over $x \rightarrow \bar{a}$ ref. $\mathfrak{o}$ is finite; since $[x \rightarrow \bar{a}; \mathfrak{o}]$ is integrally closed by Theorem of Appendix, $f(x)$ is contained in $[x \rightarrow \bar{a}; \mathfrak{o}]$, so that f is defined at $\bar{a}$. Therefore, if f is not defined at $\bar{a}$, then $\bar{a} \times \overline{\infty}$ must be contained in $\tilde{T} \cap (\tilde{S}^n \times \overline{\infty})$; and we can find a point a in $T \cap (S \times \infty)$ such that $a \rightarrow \bar{a}$ ref. $\mathfrak{o}$. We see that f is not defined at a since $a \times \infty$ is a specialization of $x \times f(x)$ over k. Hence we have $a \in \mathcal{S}$, so that $\bar{a} \in \tilde{\mathcal{S}}$. This proves ii).

REMARK. 1) By the above discussion, we see that every simple component of $\mathcal{S}, \bar{\mathcal{S}}, \tilde{\mathcal{S}}$ is of codimension 1.

2) The assumption that the image of f is embedded in an affine variety is not necessary for i) and iii); but the assertion ii) requires the assumption.

3) In the proof of ii), we have only needed that $[x \rightarrow \bar{a}; \mathfrak{o}]$ is integrally closed. Since $[x \rightarrow \bar{a}; \mathfrak{o}]$ is integrally closed if $\bar{a}$ is $\tilde{k}$-normal, (Hironaka [2]), we can replace, in the assertion ii), the simplicity of $\bar{a}$ by the $\tilde{k}$-normality of $\bar{a}$.

4) By ii) and 3), we know that, $\tilde{V}$ being a $\tilde{k}$-normal affine variety, we have $\tilde{f}(\tilde{x}) \in \tilde{k}[\tilde{x}]$ if $f(x) \in k[x]$ is finite on V. This means that the defining ideal of $\tilde{V}$ is the specialization of that of V with respect to $\mathfrak{o}$. When V and $\tilde{V}$ are projective varieties, the defining ideal of $\tilde{V}$ is the intersection of the specialization of the defining ideal of V and an irrelevant ideal in $\tilde{k}[X]$. This is a natural way to the equality of arithmetic genera of V and $\tilde{V}$ (Igusa [3]).

Let G be a group (resp. an abelian) variety defined over k, having a structure of a $\mathfrak{p}$-variety. Then the notion of *group* (resp. *abelian*) *$\mathfrak{p}$-variety*, denoted by the same letter G, is defined by the combination of the structure of group (resp. abelian) variety and the structure of $\mathfrak{p}$-variety.

DEFINITION 1. Let G be a group $\mathfrak{p}$-variety; let $\varphi: G \times G \rightarrow G$ and $\psi: G \rightarrow G$ be respectively the group-composition function and the rational mapping which corresponds a point of G to its inverse. We say that G is a *group* $\mathfrak{p}$-*variety without defect* if the following conditions are satisfied:

1) *G is $\mathfrak{p}$-simple, i.e., $\tilde{G}$ is a variety;*
2) *φ is everywhere defined on $\tilde{G}\times\tilde{G}$;*
3) *ψ is everywhere defined on $\tilde{G}$.*

Moreover, *if G is $\mathfrak{p}$-complete*, we say that G is an *abelian $\mathfrak{p}$-variety without defect.*

From the definition we can easily see that if G is a group (resp. an abelian) $\mathfrak{p}$-variety without defect, $\tilde{G}$ is considered in a natural way to be a group (resp. an abelian) variety defined over $\tilde{k}$ and the specialization of φ and ψ give the corresponding mappings of $\tilde{G}$.

Let G and G' be group $\mathfrak{p}$-varieties, both without defect, and λ a homomorphism of G into G' defined over k. If λ is defined along $\tilde{G}$, λ is everywhere defined on $\tilde{G}$ and $\tilde{\lambda}$ induces a homomorphism $\tilde{G}\to\tilde{G}'$. We shall say that G and G' are isomorphic to each other (with respect to the structure of group $\mathfrak{p}$-varieties) if there exists a surjective isomorphism λ of G onto G' such that λ is defined along $\tilde{G}$ and that $\tilde{\lambda}$ is also an isomorphism of $\tilde{G}$ onto $\tilde{G}'$. We note that if both G and G' are abelian $\mathfrak{p}$-varieties without defect, any group-isomorphism between G and G' in a usual sense is always an isomorphism between abelian $\mathfrak{p}$-varieties G and G'.

PROPOSITION 2. *Let f be a rational mapping, defined over k, of a $\mathfrak{p}$-simple $\mathfrak{p}$-variety V into a group $\mathfrak{p}$-variety G without defect, and let F be a rational mapping of $V\times V$ into G defined by $F(x,y)=f(x)f(y)^{-1}$. Let $\bar{a}$ be a simple point of $\tilde{V}$. Then f is defined at $\bar{a}$ if and only if F is defined at $(\bar{a},\bar{a})$ and $F(\bar{a},\bar{a})=\bar{e}$, where $\bar{e}$ is the identity element of $\tilde{G}$.*

We omit the proof because it is easy and is quite similar to the discussion in n° 15 of Weil [10]; we shall make use of the idea given there in the following treatment.

Notations being as in Prop. 2, let G_α be an affine representative of G such that the corresponding representative $\tilde{G}_\alpha$ of $\tilde{G}$ has the representative of $\bar{e}$. Let $\bar{a}$ be a simple point of $\tilde{V}$. If we denote by F_α the rational mapping $V\times V\to G_\alpha$ induced by F, F is defined at $(\bar{a},\bar{a})$ and $F(\bar{a},\bar{a})=\bar{e}$ if and only if every coordinate-function of F_α is defined at $(\bar{a},\bar{a})$. Suppose that F_α is not defined at $(\bar{a},\bar{a})$. By ii) of Prop. 1, there exists a point (a_1,a_2) on $V\times V$ such that F_α is not defined at (a_1,a_2) and $(a_1,a_2)\to(\bar{a},\bar{a})$ ref. $\mathfrak{o}$. By Remark 1) below Prop. 1, there exists a simple subvariety X of $V\times V$, of codimension 1, containing (a_1,a_2), where F_α is not defined. Let $\tilde{X}$ be a specialization of X over $(a_1,a_2)\to(\bar{a},\bar{a})$ ref. $\mathfrak{o}$. Let Δ and $\bar{\Delta}$ denote respectively the diagonals on $V\times V$ and on $\tilde{V}\times\tilde{V}$. Then the support of $\tilde{X}$ does not contain $\bar{\Delta}$ because F_α is defined along $\bar{\Delta}$. Hence, both the intersection-products $X\cdot\Delta$ and $\tilde{X}\cdot\bar{\Delta}$ are defined and

$(\bar{a}, \bar{a})$ is contained in a component of $\tilde{X}\cdot\bar{\Delta}$; so there exists a point (a, a) in $X \cap \Delta$ such that $(a, a) \rightarrow (\bar{a}, \bar{a})$ ref. $\mathfrak{o}$. As $(\bar{a}, \bar{a})$ is simple on $V\times V$, (a, a) is simple on $V\times V$. F_α is not defined at (a, a) since (a, a) is contained in X. Thus we have shown that if f is not defined at a simple point $\bar{a}$ of $\bar{V}$, there exists a simple point a of V where f is not defined. In view of the results in $n°$ 15 of Weil [10], we have:

THEOREM 1. *Let A be an abelian $\mathfrak{p}$-variety and f a rational mapping, defined over k, of a $\mathfrak{p}$-simple $\mathfrak{p}$-variety V into A. Suppose that A is without defect as group $\mathfrak{p}$-variety and that f is defined along $\tilde{V}$. Then f is defined at any simple point on $\tilde{V}$.*

REMARK. 1) A is not necessarily without defect as abelian $\mathfrak{p}$-variety even if it is without defect as group $\mathfrak{p}$-variety.

2) f is defined along $\tilde{V}$ whenever A is without defect as abelian $\mathfrak{p}$-variety.

§ 2. Specialization-ring at a simple point

Let V be a $\mathfrak{p}$-variety, x a generic point of V over k and $\bar{a}$ a point of $\tilde{V}$ which is simple on V. We can prove that $[x\rightarrow\bar{a}\,;\mathfrak{o}]$ is a regular local ring (Theorem in Appendix); it is not so easy, however, to prove this result. If $\bar{a}$ is $\bar{k}$-rational, the situation becomes easier and we can consider expansion by power-series in local parameters; so in this section we shall concern ourselves only with such a case. First we give a definition of local parameters at $\bar{a}$, generalizing a definition in Koizumi [4]. Let n be the dimension of V. We say that a set of n rational functions $\{\tau_1, \cdots, \tau_n\}$ on V is a *set of local parameters* on V at $\bar{a}$, if the following conditions are satisfied:

i) *the τ_i are defined and finite at $\bar{a}$.*

ii) *Let $V_\alpha, x_\alpha, \bar{a}_\alpha$ be respectively representatives of $V, x, \bar{a}$ and S^N the ambient space for V_α. Then there exists a set of N polynomials $F_i(X_1, \cdots, X_N, T_1, \cdots, T_n)$ in $\mathfrak{o}[X, T]$ such that $F_i(x, \tau(x))=0$ for $1\leqq i\leqq N$ and*

$$\det\left(\frac{\partial\bar{F}_i}{\partial X_j}(\bar{a}, \tau(\bar{a}))\right)\neq 0\,.$$

We say that $\{\tau_1, \cdots, \tau_n\}$ is defined over k if the τ_i are all defined over k. We can verify that the condition ii) is independent of the choice of representatives $V_\alpha, x_\alpha, \bar{a}_\alpha$. The existence of a set of local parameters on V at $\bar{a}$ is a direct consequence of the definition of simple point[2].

PROPOSITION 3. *With the same notations and assumptions as above, if $\bar{a}$ is $\bar{k}$-rational, the specialization-ring $\mathfrak{R}=[x\rightarrow\bar{a}\,;\mathfrak{o}]$ is a regular local ring of dimen-*

2) For a more detailed treatment, see [8].

sion $n+1$. *More precisely, if* $\{\tau_1, \cdots, \tau_n\}$ *is a set of local parameters on* V *at* $\bar{a}$, *defined over* k, *such that* $\tau_i(\bar{a})=0$ *for* $i=1, \cdots, n$, *and* π *is a generator of the maximal ideal* $\mathfrak{p}$ *of* $\mathfrak{o}$, *then the maximal ideal* $\mathfrak{M}$ *of* $\mathfrak{R}$ *is generated by* $\{\pi, \tau_1(x), \cdots, \tau_n(x)\}$.

Proof. It is almost obvious that $[x \to \bar{a}; \mathfrak{o}]$ is a local ring of dimension $\geqq n+1$ and that there exists a set of local parameters $\{\tau_1, \cdots, \tau_n\}$ at $\bar{a}$, defined over k, such that $\tau_i(\bar{a})=0$ for $i=1, \cdots, n$. Hence our proposition is proved if we show that $\mathfrak{M}$ is generated by $\{\pi, \tau_1(x), \cdots, \tau_n(x)\}$. We may assume that V is an affine variety in the affine space S^N and that $\bar{a}$ is the origin $\bar{0}$. Then we have clearly $\mathfrak{M}=\mathfrak{R}x_1+\cdots+\mathfrak{R}x_N+\mathfrak{R}\pi$, where the x_i are the coordinates of x; so we have only to prove that every x_i is contained in $\mathfrak{R}t_1+\cdots+\mathfrak{R}t_n+\mathfrak{R}\pi$, where $t_i=\tau_i(x)$. Let the $F_i(X, T)$ be polynomials in the condition ii). If we express $F_i(X, T)$ in the form

$$F_i(X, T)=F_{i1}(X, T)X_1+\cdots+F_{iN}(X, T)X_N+F_{i0}(T),$$

where $F_{ij}(X, T) \in \mathfrak{o}[X, T]$ for $1 \leqq i \leqq N$, $0 \leqq j \leqq N$, then we have

$$\frac{\partial \tilde{F}_i}{\partial X_j}(\bar{0}, \bar{0})=\tilde{F}_{ij}(\bar{0}, \bar{0}),$$

and hence $\det(\tilde{F}_{ij}(\bar{0}, \bar{0})) \neq 0$. As we have $\tilde{F}_i(\bar{0}, \bar{0})=0$, $F_{i0}(T)$ is expressed in the form

$$F_{i0}(T)=\sum_{\nu=1}^{n} T_\nu F_{i0}^{(\nu)}(T)+\pi\alpha_i,$$

where $F_{i0}^{(\nu)}(T) \in \mathfrak{o}[T]$ and $\alpha_i \in \mathfrak{o}$. We have then

$$F_{i1}(x, t)x_1+\cdots+F_{iN}(x, t)x_N=-\sum_{\nu=1}^{n} F_{i0}^{(\nu)}(t)t_\nu-\pi\alpha_i \qquad (1 \leqq i \leqq N).$$

Solving these equations with respect to $x_1, \cdots, x_N$, we see that the x_i are contained in $\mathfrak{R}t_1+\mathfrak{R}t_2+\cdots+\mathfrak{R}t_n+\mathfrak{R}\pi$; this completes our proof.

In addition to the assumptions in Prop. 3, assume that *there exists a* k*-rational point* a *on* V *such that* $a \to \bar{a}$ ref. $\mathfrak{o}$. We can choose a set of local parameters $\tau_1, \cdots, \tau_n$ at $\bar{a}$ in such a way that $\tau_i(a)=0$ for $1 \leqq i \leqq n$. Put $\mathfrak{R}_1=[x \to a; k]$ and call $\mathfrak{M}_1$ the maximal ideal of $\mathfrak{R}_1$. Then, since $\tau_1, \cdots, \tau_n$ is a set of local parameters on V at a, we have $\mathfrak{M}_1=\mathfrak{R}_1t_1+\cdots+\mathfrak{R}_1t_n$. Put $\mathfrak{t}=\mathfrak{R}t_1+\cdots+\mathfrak{R}t_n$. Then, by the same argument as the above proof, we see that $x_1-a_1, \cdots, x_N-a_N$ are contained in $\mathfrak{t}$; and the ring of quotients of $\mathfrak{R}$ with respect to the prime ideal $\mathfrak{t}$, coincides with $\mathfrak{R}_1$. Hence the residue class ring $\mathfrak{R}/\mathfrak{t}$ is canonically isomorphic to a subring of $\mathfrak{R}_1/\mathfrak{M}_1$; the ring $\mathfrak{R}_1/\mathfrak{M}_1$ admits k as a complete set of representatives, and $\mathfrak{o}$ is contained in a complete set of representatives for $\mathfrak{R}/\mathfrak{t}$. Since $\mathfrak{R}/\mathfrak{t}$ is not a field and $\mathfrak{o}$ is a maximal ring in k, we conclude that $\mathfrak{o}$ *is a complete set of representatives of* $\mathfrak{R}/\mathfrak{t}$.

PROPOSITION 4. *Notations and assumptions being as above, let z be an element of $\mathfrak{R}$. Then there exist homogeneous forms $H_i(T_1, \cdots, T_n)$ of degree $i=0, 1, 2, \cdots$ in $\mathfrak{o}[T]$ such that*

$$z \equiv H_0 + H_1(t) + \cdots + H_\rho(t) \qquad (\mathrm{mod.}\ \mathfrak{t}^{\rho+1}).$$

Such forms $H_i(T)$ are uniquely determined by z.

Proof. The existence of $H_i(T)$ is due to that $\mathfrak{o}$ is a complete set of representatives of $\mathfrak{R}/\mathfrak{t}$ and the uniqueness follows from the fact that we have in $\mathfrak{R}_1$,

$$z \equiv H_0 + H_1(t) + \cdots + H_\rho(t) \qquad (\mathrm{mod.}\ \mathfrak{M}_1{}^{\rho+1}).$$

§ 3. Specialization of function-modules[3]

Let V be a variety defined over k. We call a subset L of the field of rational functions on V a *function-module* on V if L is a vector space of finite dimension over the field of constant functions on V[4]. L is said to be *defined over* k if L has a base over K consisting of the functions defined over k. If L is a function-module defined over k, we denote by L_k the subset of L composed of the elements in L defined over k. Let $\{f_1, \cdots, f_m\}$ be a base of L such that the f_i are defined over k. Then we see that $L_k = kf_1 + \cdots + kf_m$. Now assume that V has a structure of $\mathfrak{p}$-simple $\mathfrak{p}$-variety. L being a function-module on V defined over k, we denote by $\tilde{L}_k$ the set of rational functions $\tilde{f}$ on $\tilde{V}$ such that there exists an element f of L_k having $\tilde{f}$ as a specialization with respect to $\mathfrak{o}$. Then $\tilde{L}_k$ is clearly a vector space over $\tilde{k}$, and $\dim_{\tilde{k}}\tilde{L}_k$ is not greater than $\dim_k L_k$; in fact, the specializations $\tilde{f}_1, \cdots, \tilde{f}_r$ of functions $f_1, \cdots, f_r$ can not be linearly independent over $\tilde{k}$ unless the f_i are so over k. We obtain, more precisely,

PROPOSITION 5. $\dim_{\tilde{k}}\tilde{L}_k = \dim_k L_k$.

Proof. Denote by $\mathcal{B}$ the totality of bases of L_k over k, consisting of functions defined and finite along $\tilde{V}$. Let $a \in V$ and $\bar{a} \in \tilde{V}$ be points, both simple on V, such that $a \rightarrow \bar{a}$ ref. $\mathfrak{o}$ and that every function in L_k is defined and finite at $\bar{a}$. Let $\{k', \mathfrak{o}', \mathfrak{p}'\}$ be a prolongation of $\{k, \mathfrak{o}, \mathfrak{p}\}$ such that $a \rightarrow \bar{a}$ ref. $\mathfrak{o}'$ and that a is rational over k'. Take a set of local parameters $\{\tau_1, \cdots, \tau_n\}$ on V at $\bar{a}$, defined over k', such that $\tau_i(a)=0$ for $1 \leqq i \leqq n$. For every $\{f\} = \{f_1, \cdots, f_m\}$ in $\mathcal{B}$, by virtue of Prop. 4, we get an expansion of f_ν by power-series:

$$f_\nu = \sum_{(i)} f_{(i)}{}^{(\nu)} \tau_1{}^{i_1} \cdots \tau_n{}^{i_n}, \qquad (1 \leqq \nu \leqq m),$$

3) Another approach to the same subject will be found in [8].

4) The field of constant functions on V may be identified with the universal domain K.

where the $f_{(i)}^{(\nu)}$ are elements of $\mathfrak{o}'$. Since $f_1, \cdots, f_m$ are linearly independent over k', there exists a set of m indices among $\{(i_1, \cdots, i_n)\}$, briefly denoted by $1, 2, \cdots, m$, such that

$$\begin{vmatrix} f_1^{(1)} & f_2^{(1)} & \cdots & f_m^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_m^{(2)} \\ \cdots & & \cdots & \\ f_1^{(m)} & f_2^{(m)} & \cdots & f_m^{(m)} \end{vmatrix} \neq 0 .$$

If $\{g\} = \{g_1, \cdots, g_m\}$ is another element of $\mathscr{B}$, there exists a non-singular matrix $M_{(g,f)}$ of degree m with coefficients in k such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} = M_{(g,f)} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} .$$

We observe that

$$\det(g_j^{(\nu)}) = \det M_{(g,f)}(f_j^{(\nu)}) \neq 0 ,$$

where $g_j^{(\nu)}$ is the corresponding coefficient of the expansion for g_j. Let v' be the normalized exponential valuation of k' defined by $\mathfrak{p}'$. Put $\mu(f_1, \cdots, f_m) = v'(\det(f_j^{(\nu)}))$. Then we have $\mu(f_1, \cdots, f_m) \geqq 0$ because every $f_j^{(\nu)}$ is contained in $\mathfrak{o}'$. Now put

$$\mu_0 = \operatorname*{Min}_{\{f\} \in \mathscr{B}} \mu(f_1, \cdots, f_m) ;$$

and let $\{h_1, \cdots, h_m\}$ be an element of $\mathscr{B}$ such that

$$\mu_0 = \mu(h_1, \cdots, h_m) .$$

Then, $\tilde{h}_1, \cdots, \tilde{h}_m$ are linearly independent over $\tilde{k}$. In fact, if there can be found m elements $c_1, \cdots, c_m$ in $\mathfrak{o}$, not all non-units of $\mathfrak{o}$, such that $\tilde{c}_1\tilde{h}_1 + \cdots + \tilde{c}_m\tilde{h}_m = 0$, then $g = \pi^{-1}(c_1h_1 + \cdots + c_mh_m)$ is defined and finite along $\tilde{V}$, where π is a prime element of $\mathfrak{o}$. If $\tilde{c}_1 \neq 0$, $\{g, h_2, \cdots, h_m\}$ is contained in $\mathscr{B}$ and we have

$$\mu(h_1, \cdots, h_m) > \mu(g, h_2, \cdots, h_m) .$$

This contradicts the definition of $(h_1, \cdots, h_m)$; so our proposition is proved.

We shall say that a rational function f on V, defined over k, is $\mathfrak{p}$-*finite* if f is defined and finite along $\tilde{V}$; $\mathfrak{p}$-finite functions $f_1, \cdots, f_r$ are said to be *linearly* $\mathfrak{p}$-*independent* if $\tilde{f}_1, \cdots, \tilde{f}_r$ are linearly independent over $\tilde{k}$. L and L_k being as above, let $\{f_1, \cdots, f_m\}$ be a base of L_k, consisting of $\mathfrak{p}$-finite functions. Then, $f_1, \cdots, f_m$ are linearly $\mathfrak{p}$-independent if and only if for every $\mathfrak{p}$-finite function $h = c_1f_1 + \cdots + c_mf_m \in L_k$ with $c_i \in k$, we have $c_i \in \mathfrak{o}$. On the other hand, when $f_1, \cdots, f_m$ are linearly $\mathfrak{p}$-independent, a function $h = c_1f_1 + \cdots + c_mf_m$ with $c_i \in k$ is $\mathfrak{p}$-finite if and only if $c_i \in \mathfrak{o}$ for every i. Now we shall define the specialization of a function-module. Let L be as before a function-module on a $\mathfrak{p}$-simple $\mathfrak{p}$-variety V, defined over k. We denote by $\tilde{L}$ the set of functions $\bar{f}$ on $\tilde{V}$ such that there exists an element f of L having $\bar{f}$ as a specialization with respect to $\mathfrak{o}$; we call $\tilde{L}$ the *specialization of* L with respect to $\mathfrak{o}$. We see that $\tilde{L}$ is a function-module on $\tilde{V}$, defined over $\tilde{k}$, whose dimension is equal to that of L.

We shall apply our result to linear systems on a variety. Let V be a $\mathfrak{p}$-simple $\mathfrak{p}$-complete $\mathfrak{p}$-variety and X a k-rational divisor on V. If both V and $\tilde{V}$ are non-singular in codimension 1, we can define two function-modules $L(X)$ on V and $\bar{L}(\tilde{X})$ on $\tilde{V}$, defined over k and $\tilde{k}$, respectively, such that

$$L(X)=\{f\,|\,f \text{ is a function on } V, (f)\succ -X\},$$
$$\bar{L}(\tilde{X})=\{\bar{f}\,|\,\bar{f} \text{ is a function on } \tilde{V}, (\bar{f})\succ -\tilde{X}\}.$$

We denote by $\tilde{L}(X)$ the specialization of $L(X)$ with respect to $\mathfrak{o}$. In view of Theorem 20 of [7], we see that $\tilde{L}(X)$ is contained in $\bar{L}(\tilde{X})$. If we denote by $l(X)$ (resp. $\bar{l}(\tilde{X})$) the dimension of $L(X)$ (resp. $\bar{L}(\tilde{X})$), the above discussion leads us to the inequality $l(X)\leqq \bar{l}(\tilde{X})$.

§ 4. Projective embedding of an abelian $\mathfrak{p}$-variety without defect

Let A be an abelian $\mathfrak{p}$-variety without defect. If X_1 is a positive k-rational and non-degenerate divisor on A, the specialization $\tilde{X}_1$ of X_1 with respect to $\mathfrak{o}$ is also non-degenerate on $\tilde{A}$; and by a result of Nishi [5], we have $l(X_1)=\bar{l}(\tilde{X}_1)$. From Weil [12], we know that for a sufficiently large integer s, the divisors $sX_1=X$ and $s\tilde{X}_1=\tilde{X}$ are ample on A and $\tilde{A}$ respectively; and we have $l(X)=\bar{l}(\tilde{X})$. Hence, if $\{f_0,f_1,\cdots,f_m\}$ is a linearly $\mathfrak{p}$-independent base of $L(X)$, then $\{\tilde{f}_0,\tilde{f}_1,\cdots\tilde{f}_m\}$ is a base of $\tilde{L}(X)=\bar{L}(\tilde{X})$. Take a generic point x of A over k and consider the locus A_1 of $(f_0(x),\cdots,f_m(x))$ over k in the projective space P^m; then we obtain a birational mapping τ of A onto A_1 defined by $\tau(x)=(f_0(x),\cdots,f_m(x))$ with respect to k. We can put into A_1 a structure of abelian variety so that τ is an isomorphism of A onto A_1. As A_1 is a projective variety, A_1 is naturally endowed with a structure of $\mathfrak{p}$-variety. We shall now prove that A_1 is an abelian $\mathfrak{p}$-variety without defect. Since $\tilde{X}$ is ample on $\tilde{A}$, and since $\{\tilde{f}_0,\cdots,\tilde{f}_m\}$ is a base of $\bar{L}(\tilde{X})$, for any point $\bar{a}$ of $\tilde{A}$, there is a function among $\tilde{f}_0,\cdots,\tilde{f}_m$, say $\tilde{f}_0$, such that $\bar{a}$ is not contained in the support of $(\tilde{f}_0)+\tilde{X}$. Then we see easily that the functions f_i/f_0 are all defined and finite at $\bar{a}$; so τ is defined at $\bar{a}$. Call T the graph of τ; then, as τ is everywhere defined on $\tilde{A}$, the specialization of T with respect to $\mathfrak{o}$ is a variety and coincides with the locus of $\bar{\tau}$. Let $\tilde{A}_1$ be the image of $\tilde{A}$ by $\bar{\tau}$. Recalling that $\tilde{X}$ is ample on $\tilde{A}$ and $\{\tilde{f}_0,\cdots,\tilde{f}_m\}$ is a base of $\bar{L}(\tilde{X})$, we see that

$\tilde{\tau}$ gives a birational biregular mapping of $\tilde{A}$ onto $\bar{A}_1$. It follows from this that A_1 is an abelian $\mathfrak{p}$-variety without defect and $\tilde{A}_1 = \bar{A}_1$. This solves the problem about a projective embedding of an abelian $\mathfrak{p}$-variety without defect. We shall now consider the problem concerning the field of definition for A_1.

THEOREM 2. *Let A be an abelian variety defined over k; suppose the following conditions are satisfied.*

D1) *There are a prolongation $\{k', \mathfrak{o}', \mathfrak{p}'\}$ of $\{k, \mathfrak{o}, \mathfrak{p}\}$, an abelian $\mathfrak{p}'$-variety A^* without defect, defined over k', and an isomorphism θ of A onto A^*, defined over k'.*

D2) *There are a $\mathfrak{p}$-simple $\mathfrak{p}$-variety V and a surjective rational mapping φ of V to A, both defined over k, such that the specialization $\widetilde{\theta\circ\varphi}$ of $\theta\circ\varphi$ with respect to $\mathfrak{o}'$ induces a surjective rational mapping of $\tilde{V}$ to $\tilde{A}^*$.*

Then there exists a projective abelian variety A_1, defined over k, without defect with respect to a natural structure of $\mathfrak{p}$-variety, which is k-isomorphic to A.

Proof. Let X be a positive k-rational divisor on A such that the divisor $X^* = \theta(X)$ on A^* and its specialization $\tilde{X}^*$ with respect to $\mathfrak{o}'$ on $\tilde{A}^*$ are both ample on A^* and $\tilde{A}^*$, respectively. Then $L(X^*)$ is a function-module on A^*, defined over k', whose specialization is $\bar{L}(\tilde{X})$. Now consider a function-module L^0 on V composed of all functions g of the form $g = f^*\circ\theta\circ\varphi$ for some $f^* \in L(X^*)$; then, L^0 coincides with the set of all functions $f\circ\varphi$ for $f \in L(X)$, because θ is an isomorphism; this implies that L^0 is defined over k. Let $\{g_0, \cdots, g_m\}$ be a linearly $\mathfrak{p}$-independent base of $L_k{}^0$; for every i, there exists a function f_i^* in $L_{k'}(X^*)$ such that $g_i = f_i^*\circ\theta\circ\varphi$. Since $\widetilde{\theta\circ\varphi}$ is surjective, we have $\tilde{g}_i = \tilde{f}_i^*\circ(\widetilde{\theta\circ\varphi})$. As $\tilde{g}_0, \cdots, \tilde{g}_m$ are linearly independent over the constant field, we see that $\tilde{f}_0^*, \cdots, \tilde{f}_m^*$ are also linearly independent over the constant field. It follows that $\{f_0^*, \cdots, f_m^*\}$ is a linearly $\mathfrak{p}'$-independent base of $L(X^*)$. Using this base $\{f_0^*, \cdots, f_m^*\}$, we obtain a projective abelian variety A_1 defined over k', without defect as an abelian $\mathfrak{p}'$-variety. On the other hand, we see that $\{f_0^*\circ\theta, \cdots, f_m^*\circ\theta\}$ is a base of $L_k(X)$. Hence A_1 is a projective embedding of A defined over k. This concludes that A_1 satisfies our requirements.

REMARK. The assumption D2) in Theorem 2 is indispensable. For instance: for a prime number $p > 2$, the elliptic curve $y^2 = 4x^3 - px - p$ defined over the field $\boldsymbol{Q}$ of rational numbers does not admit any model, without defect for p, defined over $\boldsymbol{Q}$, while it is birationally equivalent to the elliptic curve $y^2 = 4x^3 - p^{1/3}x - 1$, which is without defect for the prime divisor $(p^{1/3})$.

§ 5. Construction of group $\mathfrak{p}$-varieties without defect

In this section we shall translate a part of n° 32–33 of Weil [10] to the case of $\mathfrak{p}$-varieties. We first restate the definition of pre-group given in Weil [11]. A *pre-group* is a pair (V,f) of two geometric objects such that:

P 1) *V is a variety;*

P 2) *f is a rational mapping of $V\times V$ into V which defines a normal law of composition on V, namely, if x, y, z are independent generic points of V over a common field k of definition for V and f, we have* i) $k(x,y)=k(x,f(x,y))=k(y,f(x,y))$ *and* ii) $f(x,f(y,z))=f(f(x,y),z)$.

In these circumstances we say that k is a field of definition for a pre-group (V,f) or that (V,f) is defined over k. Two pre-groups (V,f) and (V',f') are said to be isomorphic to each other if there exists a birational correspondence between V and V' by which f corresponds to f'.

Let (V,f) be a pre-group. If V has a structure of $\mathfrak{p}$-variety, the pair (V,f) of the $\mathfrak{p}$-variety V and f is called a *pre-group* $\mathfrak{p}$*-variety*. In particular, when V is an affine variety with the natural structure of $\mathfrak{p}$-variety, (V,f) is called an *affine pre-group* $\mathfrak{p}$*-variety.*

DEFINITION 2. *A pre-group $\mathfrak{p}$-variety (V,f) is said to be without defect if the following two conditions are satisfied.*

PD 1) *V is $\mathfrak{p}$-simple;*

PD 2) *f is defined along $\tilde{V}\times\tilde{V}$ and $\tilde{f}$ induces a normal law of composition on $\tilde{V}$.*

Hence, if (V,f) is without defect, $(\tilde{V},\tilde{f})$ is a pre-group defined over $\tilde{k}$.

Two pre-group $\mathfrak{p}$-varieties (V,f) and (V_1,f_1), both without defect, are said to be *(k-)isomorphic* to each other if there exists a birational mapping σ, defined over k, of V onto V_1 such that:

PDI 1) *f_1 is the translation of f by σ;*

PDI 2) *σ is defined along $\tilde{V}$ and $\tilde{\sigma}$ induces a birational mapping of $\tilde{V}$ onto $\tilde{V}_1$.*

From the conditions PDI 1-2), we can see that $\tilde{f}_1$ is the translation of $\tilde{f}$ by $\tilde{\sigma}$, so that $(\tilde{V},\tilde{f})$ is isomorphic to $(\tilde{V}_1,\tilde{f}_1)$. It is not difficult to show that, for any pre-group $\mathfrak{p}$-variety (V,f) without defect, there is an affine pre-group $\mathfrak{p}$-variety without defect, isomorphic to (V,f). The purpose in this section is to prove that for any pre-group $\mathfrak{p}$-variety (V,f) without defect, we can construct a group $\mathfrak{p}$-variety without defect, which is isomorphic to (V,f). Since the method is quite similar to Weil's construction of group variety in [10], we shall only state two preliminary propositions and the main theorem without proof. Let (V,f) be a pre-group defined over k and x,y two independent generic points of V over k. Put $z=f(x,y)$. We obtain then two rational mappings φ and ψ: $V\times V\rightarrow V$, defined by

$$y=\varphi(x,z), \quad x=\psi(z,y)$$

with respect to k.

PROPOSITION 6. *Let (V, f) be a pre-group $\mathfrak{p}$-variety without defect. Then there exist a prolongation $\{k', \mathfrak{o}', \mathfrak{p}'\}$ of $\{k, \mathfrak{o}, \mathfrak{p}\}$ and a frontier $\mathfrak{F}$ on V, normally algebraic over k', having the following properties:*

i) *for any point a of $V-\mathfrak{F}$ and any generic point x of V over $k'(a)$, $f(x, a)$ and $\psi(x, a)$ are defined; $\varphi(x, f(x, a))$ is defined and equal to a; and $\psi(f(x, a), a)$ and $f(\psi(x, a), a)$ are defined and equal to x;*

ii) *for any point $\bar{a}$ of $\tilde{V}-\mathfrak{F}$ and any generic point $\bar{x}$ of $\tilde{V}$ over $\tilde{k}'(\bar{a})$, all assertions, replaced x, a by $\bar{x}, \bar{a}$, in* i) *are true.*

PROPOSITION 7. *Let (V, f) be a pre-group $\mathfrak{p}$-variety without defect; let $\{k', \mathfrak{o}', \mathfrak{p}'\}$ and $\mathfrak{F}$ be a prolongation of $\{k, \mathfrak{o}, \mathfrak{p}\}$ and a frontier on V, having the properties in Prop.* 6. *Let x be a generic point of V over k' and $\bar{x}$ a generic point of $\tilde{V}$ over $\tilde{k}$; put $\mathfrak{o}_1=[x \rightarrow \bar{x}; \mathfrak{o}']$. Let y be a generic point of V over $k'(x)$ and T_x the locus of $y \times f(x, y)$ on $V \times V$ over $k'(x)$; and let $\tilde{T}_x$ be the specialization of T_x with respect to $\mathfrak{o}_1$. Then, T_x is a birational correspondence of V onto itself, and:*

i) *if (a, b) is a point of T_x, such that both a and b are points in $V-\mathfrak{F}$, the points a and b are regularly corresponding points of V by T_x;*

ii) *if $(\bar{a}, \bar{b})$ is a point of the support of $\tilde{T}_x$, such that both $\bar{a}$ and $\bar{b}$ are points in $\tilde{V}-\tilde{\mathfrak{F}}$, $\bar{a}$ and $\bar{b}$ are regularly corresponding points of $\tilde{V}$ by T_x.*

The above two propositions are translations of the Lemmas 6, 7 in [10, p. 52–53]. After these, we are now in a position of stating the construction theorem.

THEOREM 3. *Let (V, f) be a pre-group $\mathfrak{p}$-variety without defect. Then there exist a prolongation $\{K, \mathfrak{O}, \mathfrak{P}\}$ of $\{k, \mathfrak{o}, \mathfrak{p}\}$ and a group $\mathfrak{P}$-variety G, defined over K, without defect, which is K-isomorphic to (V, f). G is uniquely determined by (V, f) up to an isomorphism.*

We shall only give an outline of the proof. At first we may assume that (V, f) is an affine pre-group $\mathfrak{p}$-variety because we can always find an affine model isomorphic to (V, f). $\{k', \mathfrak{o}', \mathfrak{p}'\}$ and $\mathfrak{F}$ having the same meanings as in Prop. 6, let $x, t_1, \cdots, t_N$ be independent generic points of V over k' and $\bar{t}_1, \cdots, \bar{t}_N$ be independent generic points of $\tilde{V}$ over $\tilde{k}$, for a sufficiently large N. Put, for $1 \leqq \alpha \leqq N$, $G_\alpha = V$, $\mathfrak{F}_\alpha = \mathfrak{F}$ and $x_\alpha = f(t_\alpha, x)$. We define $\{K, \mathfrak{O}, \mathfrak{P}\}$ by

$$K = k'(t_1, \cdots, t_N), \qquad \mathfrak{O} = [(t_1, \cdots, t_N) \rightarrow (\bar{t}_1, \cdots, \bar{t}_N); \mathfrak{o}'].$$

Then, $\mathfrak{O}$ is a discrete valuation ring of rank 1 and $\{K, \mathfrak{O}, \mathfrak{P}\}$ is a prolongation of $\{k, \mathfrak{o}, \mathfrak{p}\}$. If we denote by $T_{\beta\alpha}$, the locus of (x_α, x_β) on $G_\alpha \times G_\beta$ with respect to K, we know that $[G_\alpha, \mathfrak{F}_\alpha, \tilde{\mathfrak{F}}_\alpha, T_{\beta\alpha}]$ defines a $\mathfrak{P}$-variety, and that this is just the one which we want to construct. The uniqueness is a consequence of the fact that any two group $\mathfrak{p}$-varieties, both without defect, which are isomorphic to each other with respect to the structure of pre-group $\mathfrak{p}$-variety, are also isomorphic with respect to the structure of group $\mathfrak{p}$-variety.

COROLLARY. *(V,f) and G being as in Theorem 3, suppose that G is an abelian $\mathfrak{P}$-variety without defect. Then there exists a projective abelian $\mathfrak{p}$-variety A, defined over k, without defect, k-isomorphic to (V,f).*

Proof. By Weil [11], we know that there exists an abelian variety, defined over k, which is derived from (V,f) by a birational transformation defined over k. Applying then Theorem 2 to the present case we obtain our result.

§ 6. Specialization of homomorphic images of an abelian $\mathfrak{p}$-variety without defect

In this section we want to prove:

THEOREM 4. *Let A be an abelian $\mathfrak{p}$-variety without defect, and let λ be a surjective homomorphism of A onto another abelian variety B, where both λ and B are defined over k. Then there exists an abelian $\mathfrak{p}$-variety B_1 without defect, which is birationally equivalent to B over k.*

Let A_1 and A_2 be abelian varieties and λ a homomorphism of A_1 onto A_2. We shall call λ a *solid homomorphism*, if, for a common field k_1 of definition for A_1, A_2 and λ, and for a generic point x of A_1 over k_1, $k_1(x)$ is a regular extension of $k_1(\lambda(x))$. Now any surjective homomorphism λ of A onto B is decomposed into two surjective homomorphisms, $\lambda_1 : A \to B^*$, $\lambda_2 : B^* \to B$ and $\lambda = \lambda_2 \circ \lambda_1$, where B^* is an abelian variety, λ_1 is an isogeny and λ_2 is a solid homomorphism. Hence, if the problem concerning the common field of definition for B_1 and the birational correspondence between B and B_1, is left out of consideration, it is sufficient to prove Theorem 4 in two special cases where 1) λ is an isogeny or 2) λ is a solid homomorphism. If this is done, the theorem would be a consequence of Theorem 2.

At first we shall make some preliminary considerations. X and u being a cycle on an abelian variety A and a point on A, we denote by X_u the transform of X by the translation $x \to x+u$ on A. Now let A be an abelian variety embedded in a projective space and C an abelian subvariety of A, both defined over k. Then $\{C_u \mid u \in A\}$ forms an algebraic system of positive cycles on A. Let B and Γ be the Chow variety associated with the algebraic system $\{C_u\}$ and the corresponding graph of $\{C_u\}$ to B. Then B is a model of the factor group variety of A by C, Γ is the graph of the natural homomorphism of A onto B; and both B and Γ are defined over k (Chow [1]). Moreover, if A is an abelian $\mathfrak{p}$-variety without defect, we can easily see that the specialization $\tilde{C}$ of C with respect to $\mathfrak{p}$ is a multiple of an abelian subvariety $\bar{C}$ of $\tilde{A}$, i.e., $\tilde{C}=s\bar{C}$ for a positive integer s. Let $\bar{B}$ and $\bar{\Gamma}$ (resp. $\bar{B}'$ and $\bar{\Gamma}'$) be the Chow variety associated with the algebraic system $\{\tilde{C}_{\bar{u}}|\bar{u}\in A\}$ (resp. $\{\bar{C}_{\bar{u}}|\bar{u}\in A\}$) and the corresponding graph of $\{\tilde{C}_{\bar{u}}\}$ to $\bar{B}$ (resp. $\{\bar{C}_{\bar{u}}\}$ to $\bar{B}'$). Denote by p the characteristic of $\tilde{k}$. In the following discussion, the case $p\neq 0$ is essential; in fact, Theorem 4 itself is rather trivial if p is equal to 0. The following proposition is concerned with the case $p\neq 0$; it is also true, however, in the case $p=0$, if we put 1 in place of any exponent of p.

Proposition 8. *With the above notations, if p is positive, we have*

i) $\tilde{\Gamma}=\bar{\Gamma}$;

ii) *there exists an integer e such that $s=p^e$, i.e., $\tilde{C}=p^e\bar{C}$;*

iii) *$\bar{B}$ is an abelian variety and is the image of $\bar{B}'$ by the rational mapping $\mathbf{p}^e$, where $\mathbf{p}$ is a rational mapping defined by*

$$\mathbf{p}(x_0, \cdots, x_n)=(x_0{}^p, \cdots, x_n{}^p)$$

for every point $(x_0, \cdots, x_n)$ in the projective space;

iv) *$\tilde{B}=p^{(m-1)e}\bar{B}$, where m is the dimension of B, and the specialization of the graph of the composition-law in B is a multiple of that of $\bar{B}$.*

Proof. The assertion i) and the fact that the support of $\tilde{B}$ coincides with $\bar{B}$, are directly derived from our definition. Put $\tilde{B}=r\bar{B}$, $s=s'p^e$, where s' is prime to p, $\deg\bar{B}=\bar{b}$, $\deg\bar{C}=\bar{c}$, $\deg B=\deg\tilde{B}=b$ and $\deg C=\deg\tilde{C}=c$. Then we have $b=r\bar{b}$ and $c=s\bar{c}$. Let $P, P', \bar{P}, \bar{P}'$ be respectively the ambient projective spaces for $A, B, \tilde{A}, \tilde{B}$; and let $L, M, \bar{L}, \bar{M}$ be respectively independent generic linear varieties of dual dimension to $C, B, \tilde{C}, \tilde{B}$ in $P, P', \bar{P}, \bar{P}'$. Then $(\tilde{\Gamma}, \bar{L}, \bar{M})$ is a specialization of (Γ, L, M) with respect to $\mathfrak{o}$; and the intersections $\Gamma\cdot(L\times M)$ in $P\times P'$ and $\tilde{\Gamma}\cdot(\bar{L}\times\bar{M})$ in $\bar{P}\times\bar{P}'$ are defined. Comparing the degrees of both 0-cycles $\Gamma\cdot(L\times M)$ and $\tilde{\Gamma}\cdot(\bar{L}\times\bar{M})$, we have the equality $bc=sr\bar{b}\bar{c}=s'rp^e\bar{b}\bar{c}=p^{me}\bar{b}\bar{c}$, and hence $s'=1$, $r=p^{(m-1)e}$. The remaining part of the proposition can be shown immediately.

In order to apply Theorem 3 to the proof of Theorem 4, we shall first prove the existence of a pre-group $\mathfrak{p}$-variety without defect, k-isomorphic to B.

Proposition 9. *Besides the assumptions in Theorem 4, suppose that λ is an isogeny or a solid homomorphism. Then there exists a pre-group $\mathfrak{p}$-variety without defect, which is k-isomorphic to B.*

Proof. On account of Theorem 2, we may assume that A is a projective variety. In the following, under the titles I) or S), λ will be considered an isogeny or a solid homomorphism. We first fix a generic point x of A over k and a generic point $\tilde{x}$ of $\tilde{A}$ over $\tilde{k}$.

i) I) There is a positive integer n and an isogeny $\lambda' : B \rightarrow A$ such that $n\delta_A = \lambda' \circ \lambda$, where δ_A is the identity mapping of A onto itself. Put $\lambda(x) = y$, $nx = z$ and $n\tilde{x} = \tilde{z}$.

S) Denote by C the kernel of λ and by B^* the canonical model of the factor group variety of A by C, defined above Prop. 8. B^* is an abelian variety, defined over k, embedded in a projective space; and there is a k-isomorphism $\kappa : B \rightarrow B^*$. Applying Prop. 8 to our case, we have

$\tilde{C} = p^e \bar{C}$, where $\bar{C}$ is an abelian subvariety of A,

$\tilde{B}^* = p^{(m-1)e} \bar{B}^*$, where $\bar{B}^*$ is an abelian variety and $m = \dim B$.

We can easily see that the specialization $\widetilde{\kappa \circ \lambda}$ of $\kappa \circ \lambda$ is a homomorphism $\tilde{A} \rightarrow \bar{B}^*$. Put $\lambda(x) = y$, $\kappa(y) = z$ and $\widetilde{\kappa \circ \lambda}(\tilde{x}) = \tilde{z}$.

ii) If we denote by $\mathfrak{o}^*$ and $\mathfrak{p}^*$ the specialization ring $[x \rightarrow \tilde{x}; \mathfrak{o}]$ and the maximal ideal of $\mathfrak{o}^*$, $\mathfrak{o}^*$ is a discrete valuation ring of rank 1 in $k(x)$ and $\{k(x), \mathfrak{o}^*, \mathfrak{p}^*\}$ is a prolongation of $\{k, \mathfrak{o}, \mathfrak{p}\}$. Put furthermore $\mathfrak{o}_1 = \mathfrak{o}^* \cap k(y)$; and denote by $\mathfrak{p}_1$ the maximal ideal of $\mathfrak{o}_1$. Then, $\mathfrak{o}_1$ is a discrete valuation ring of rank 1 in $k(y)$; and $\{k(x), \mathfrak{o}^*, \mathfrak{p}^*\}$ is a prolongation of $\{k(y), \mathfrak{o}_1, \mathfrak{p}_1\}$.

ASSERTION (I). *$\mathfrak{o}_1$ is the integral closure of the specialization ring $[z \rightarrow \tilde{z}; \mathfrak{o}]$ in $k(y)$.*

Proof of (I). Since $\mathfrak{o}_1$ is integrally closed in $k(y)$ and contains $[z \rightarrow \tilde{z}; \mathfrak{o}]$, $\mathfrak{o}_1$ contains the integral closure of $[z \rightarrow \tilde{z}; \mathfrak{o}]$ in $k(y)$. Now we shall prove that every element t in $\mathfrak{o}_1$ is integral with respect to $[z \rightarrow \tilde{z}; \mathfrak{o}]$, namely, that ∞ is not a specialization of t over $z \rightarrow \tilde{z}$ with respect to $\mathfrak{o}$. Let $\bar{t}$ be a specialization of t over $z \rightarrow \tilde{z}$ with respect to $\mathfrak{o}$; let $\tilde{x}'$ be an isolated specialization of x over the specialization $(z, t) \rightarrow (\tilde{z}, \bar{t})$ ref. $\mathfrak{o}$. Then, by Th. 6 of [7], we have

$$\dim_{\tilde{k}(\tilde{z})}(\tilde{x}') \geqq \dim_{\tilde{k}(\tilde{z}, \bar{t})}(\tilde{x}') \geqq \dim_{k(z, t)}(x) = \dim_{k(z)}(x).$$

On the other hand, we have $k(x) \supset k(z)$, $\tilde{k}(\tilde{x}') \supset \tilde{k}(\tilde{z})$ and $\dim_k(z) = \dim_{\tilde{k}}(\tilde{z})$. Therefore we must have $\dim_{\tilde{k}}(\tilde{x}') \geqq \dim_k(x)$. This shows that $\tilde{x}'$ is a generic point of $\tilde{A}$ over $\tilde{k}$. Hence $[x \rightarrow \tilde{x}'; \mathfrak{o}]$ coincides with $\mathfrak{o}^*$ which contains t; so $\bar{t}$ is not ∞; this proves the assertion.

iii) I) Let A_0 be an affine representative of A and z^0 the representative of z on A_0.

S) Let B_0^* be an affine representative of B^* and z^0 the representative of z on B_0^*.

I, S) Denote by K and $\tilde{K}$ the field $k(y)$ and the residue field $\mathfrak{o}_1/\mathfrak{p}_1$.

ASSERTION (II). *There exists a set $(t) = (t_1, \cdots, t_s)$ of quantities in K such that every t_i is integral over $\mathfrak{o}[z^0]$ and that $k(z^0, t) = K$, $\tilde{k}(\tilde{z}^0, \tilde{t}) = \tilde{K}$, where tilde means the specialization with respect to $\mathfrak{o}_1$.*

Proof of (II). From the assertion (I) we know that there is a set $(u)=(u_1,\cdots,u_s)$ of quantities in K such that every u_i is integral with respect to $[z^0\to\tilde{z}^0;\mathfrak{o}]$ and that $\tilde{k}(\tilde{z}^0,\tilde{u})=\tilde{K}, k(z^0,u)=K$. For each u_i, we can find a polynomial $f_i(U)$ in $k(z^0)[U]$ such that

$$f_i(u_i)=0,\qquad f_i(U)=U^M+\frac{P_1(z^0)}{Q(z^0)}U^{M-1}+\cdots+\frac{P_M(z^0)}{Q(z^0)},$$

where the P_j and Q are polynomials with coefficients in $\mathfrak{o}$ and $\tilde{Q}(\tilde{z}^0)\neq 0$. If we put $t_i=Q(z^0)u_i$, (t) is a set of quantities which we wanted to find.

iv) Let V be the locus of (t,z^0) over k in an affine space. Since V is birationally equivalent to B, V itself can be considered to be a pre-group (V,f) in a natural way. We shall now prove that (V,f) is a pre-group $\mathfrak{p}$-variety without defect.

Assertion (III) *The support of* $\tilde{V}$ *is a variety.*

Proof of (III). Let $(t)\to(\bar{a})$ be a specialization over any finite specialization $(z^0)\to(\bar{b})$ ref. $\mathfrak{o}$. Since the quantities t_i are integral over $\mathfrak{o}[z^0]$, (a) is finite and we have $\dim_{\tilde{k}}(\bar{a},\bar{b})=\dim_{\tilde{k}}(\bar{b})$. In particular this implies that the support of $\tilde{V}$ is not empty. Let $(\bar{t},\bar{z}^0)$ and $(\bar{t}_1,\bar{z}^0{}_1)$ be generic points of any two (same or different) components of $\tilde{V}$ over the algebraic closure of $\tilde{k}$. In order to conclude our assertion it is sufficient to show that $(\bar{t}_1,\bar{z}^0{}_1)$ is a generic specialization of $(\bar{t},\bar{z}^0)$ over $\tilde{k}$ and that $\tilde{k}(\bar{t},\bar{z}^0)$ is a regular extension of $\tilde{k}$. From the fact that $\dim_{\tilde{k}}(\bar{t},\bar{z}^0)=\dim_{\tilde{k}}(\bar{z}^0)$, $\dim_{\tilde{k}}(\bar{t}_1,\bar{z}^0{}_1)=\dim_{\tilde{k}}(\bar{z}^0{}_1)$, we know that both $\dim_{\tilde{k}}(\bar{z}^0)$ and $\dim_{\tilde{k}}(\bar{z}^0{}_1)$ are equal to $m=\dim B$. Consider isolated specializations $\bar{x}$ and $\bar{x}_1$ of x respectively over $(t,z^0)\to(\bar{t},\bar{z}^0)$ ref. $\mathfrak{o}$ and $(t,z^0)\to(\bar{t}_1,\bar{z}^0{}_1)$ ref. $\mathfrak{o}$. We see then, as in the proof of (I), that both $\bar{x}$ and $\bar{x}_1$ are generic on $\tilde{A}$ over $\tilde{k}$. From this it follows that every coordinate of the point (t,z^0) is contained in the specialization-ring $[x\to\bar{x};\mathfrak{o}]=[x\to\bar{x}_1;\mathfrak{o}]$; so we have $\tilde{k}(\bar{x})\supset\tilde{k}(\bar{t},\bar{z}^0)$; this shows that $\tilde{k}(\bar{t},\bar{z}^0)$ is regular over $\tilde{k}$. At the same time, we observe that $(\bar{t}_1,\bar{z}^0{}_1)$ is a generic specialization of $(\bar{t},\bar{z}^0)$ over $\tilde{k}$; so the assertion is proved.

Assertion (IV) V *is* $\mathfrak{p}$*-simple.*

Proof of (IV). Using the notations (t,z^0), $(\bar{t},\bar{z}^0)$ in the above proof, it is sufficient, on account of Theorem 12 of [7], to show the equalities:

I) $[\tilde{k}(\bar{t},\bar{z}^0):\tilde{k}(\bar{z}^0)]=[k(t,z^0):k(z^0)]$,

S) $[\tilde{k}(\bar{t},\bar{z}^0):\tilde{k}(\bar{z}^0)]=p^{(m-1)e}$.

I) By Assertion (II) and by a property of specialization, we have

$$[k(x):k(t,z^0)]\geqq[\tilde{k}(\bar{x}):\tilde{k}(\bar{t},\bar{z}^0)],\qquad [k(t,z^0):k(z^0)]\geqq[\tilde{k}(\bar{t},\bar{z}^0):\tilde{k}(\bar{z}^0)].$$

On the other hand, we have $[k(x):k(z^0)]=\nu(n\delta_A)=\nu(n\delta_{\tilde{A}})=[\tilde{k}(\bar{x}):\tilde{k}(\bar{z}^0)]$. Hence we have the above equality.

S) From the fact that the specialization $\tilde{B}^*$ of B^* is equal to $p^{(m-1)e}\bar{B}^*$ follows the inequality $[\tilde{k}(\bar{t},\bar{z}^0):\tilde{k}(\bar{z}^0)]\leqq p^{(m-1)e}$. On the other hand since the specialization $(\tilde{C})_{\bar{x}}=p^e(\bar{C})_{\bar{x}}$ of C_x is rational over $\tilde{k}(\bar{t},\bar{z}^0)$, the opposite inequality must hold.

ASSERTION (V) *(V,f) is a pre-group $\mathfrak{p}$-variety without defect.*

Proof of (V). We shall only prove the case S) as the other case will be obtained by substituting B^* for A in the following proof.

Let $(t_1, z^0{}_1)\times(t_2, z^0{}_2)\times(t_3, z^0{}_3)$ be a generic point of the graph of the composition-law in V over k. Then, $z^0{}_1\times z^0{}_2\times z^0{}_3$ is a generic point of the graph of the composition-law in $B_0{}^*$ over k. We know that $\bar{B}_0{}^*$ is a pre-group defined over the algebraic closure $\tilde{k}_c$ of $\tilde{k}$, and $\tilde{B}_0{}^*=p^{(m-1)e}\bar{B}_0{}^*$. Let $\bar{z}^0{}_1\times\bar{z}^0{}_2\times\bar{z}^0{}_3$ be a generic point of the graph of the composition-law in $\bar{B}_0{}^*$ over $\tilde{k}_c$; then $(\bar{z}^0{}_1,\bar{z}^0{}_2,\bar{z}^0{}_3)$ is a specialization of $(z^0{}_1,z^0{}_2,z^0{}_3)$ with respect to $\mathfrak{o}$. If we extend this to a specialization

$$((t_1, z^0{}_1), (t_2, z^0{}_2), (t_3, z^0{}_3))\rightarrow((\bar{t}_1, \bar{z}^0{}_1), (\bar{t}_2,\bar{z}^0{}_2), (\bar{t}_3, \bar{z}^0{}_3)) \text{ ref. } \mathfrak{o},$$

then, for every i, $\bar{t}_i$ is finite and algebraic over $\tilde{k}(\bar{z}^0{}_i)$. It is not difficult to see that the locus of $((\bar{t}_1, \bar{z}^0{}_1), (\bar{t}_2, \bar{z}^0{}_2), (\bar{t}_3,\bar{z}^0{}_3))$ over $\tilde{k}_c$ is a simple component of the specialization of the composition-law on V, and that it defines a normal law of composition on $\tilde{V}$; so the assertion is proved.

Proof of Theorem 4. If λ is an isogeny or a solid homomorphism, we can obtain, by Prop. 9, a pre-group $\mathfrak{p}$-variety (V,f) without defect, which is k-isomorphic to B. Then, by Theorem 3, there exist a prolongation $\{K, \mathfrak{O}, \mathfrak{P}\}$ of $\{k, \mathfrak{o}, \mathfrak{p}\}$ and a group $\mathfrak{P}$-variety B_1, without defect, which is K-isomorphic to (V,f). By the uniqueness of group variety isomorphic to a given pre-group, B_1 is isomorphic to B; so there exists a homomorphism λ_1 of A onto B_1. Since both A and B_1 are without defect as group $\mathfrak{P}$-variety, λ_1 is everywhere defined on $\tilde{A}$. It follows from this and the fact that A is $\mathfrak{p}$-complete, that B_1 is $\mathfrak{P}$-complete; so B_1 is an abelian $\mathfrak{P}$-variety without defect. In the general case, we decompose λ into two homomorphisms, one of which is an isogeny and the other is a solid homomorphism. Applying our result in special cases to these two homomorphisms, we can find a prolongation $\{k', \mathfrak{o}', \mathfrak{p}'\}$ of $\{k, \mathfrak{o}, \mathfrak{p}\}$ and an abelian $\mathfrak{p}$-variety B', without defect, isomorphic to B. Now apply Theorem 2 to the present case, considering $\{B, B', A\}$ to be $\{A, A^*, V\}$ in that theorem; then we obtain a projective abelian $\mathfrak{p}$-variety, which is without defect and birationally equivalent to B over k. Thus Theorem 4 is completely proved.

REMARK. Let B be an abelian subvariety of an abelian $\mathfrak{p}$-variety A; and suppose that A is without defect. Then the specialization $\tilde{B}$ of B is a multiple of an abelian subvariety $\bar{B}$ of $\tilde{A}$: we have $\tilde{B}=p^e\bar{B}$, where p is the characteristic of $\tilde{k}$ (we put 1 in place of p^e if $p=0$). The multipilicity p^e is independent of the choice of models of A, which are of course assumed to be without defect. On the other hand, since there is a homomorphism of A onto B, by Theorem 4, we know that B is birationally equivalent to an abelian $\mathfrak{p}$-variety without defect. Thus a question arises whether B itself is always an abelian variety without defect. The following example will show that this is not so, namely, B is not necessarily without defect.

Example. Let E be an elliptic curve, embedded in a projective space, over a field of characteristic 0, such that the specialization $\tilde{E}$ of E with respect to $\mathfrak{p}$ is an elliptic curve over $\tilde{k}$, having no point of order p, where p is the characteristic of $\tilde{k}$. Assume that every point t on E of order p is rational over k. Let T_1 and T_2 be two distinct subgroups of E, of order p; and let E_i be the canonical model (by the Chow variety) of the factor group variety of E by T_i; and denote by λ_i the natural homomorphism of E onto E_i. Then, E_1 and E_2 are abelian $\mathfrak{p}$-varieties without defect; and they have the same specialization $\tilde{E}^{(p)}$. Take a point x generic on E over k, and put $x_1 = \lambda_1(x)$, $x_2 = \lambda_2(x)$. Call E_0 the locus of $x_1 \times x_2$ over k on $E_1 \times E_2$. We see then that the specialization of E_0 is $p\tilde{\Delta}^{(p)}$, where $\tilde{\Delta}^{(p)}$ is the diagonal on $\tilde{E}^{(p)} \times \tilde{E}^{(p)}$, so that E_0 is not without defect, while $E_1 \times E_2$ is without defect.

Appendix

First we recall some terminologies and elementary results on local rings. We shall call a commutative ring $\mathfrak{R}$ with an identity element a *local ring* if $\mathfrak{R}$ is Noetherian and has a unique maximal ideal. A local ring having no zero-divisor is called a *local domain.* Let $\mathfrak{R}$ be a local ring and $\mathfrak{m}$ the maximal ideal of $\mathfrak{R}$. We call a set of generators $\{u_1, \cdots, u_r\}$ of $\mathfrak{m}$ a *minimal base* of $\mathfrak{m}$ if no proper subset of $\{u_1, \cdots, u_r\}$ generates $\mathfrak{m}$. A set of elements $\{u_1, \cdots, u_r\}$ in $\mathfrak{R}$ is a minimal base of $\mathfrak{m}$ if and only if $\{u_1, \cdots, u_r\}$ gives a base of the module $\mathfrak{m}/\mathfrak{m}^2$ over $\mathfrak{R}/\mathfrak{m}$. Hence the number of elements in any minimal base of $\mathfrak{m}$ is determined by $\mathfrak{R}$. A local ring $\mathfrak{R}$ is said to be *regular* if this number is equal to the dimension of $\mathfrak{R}$. If $\mathfrak{R}$ is regular, every minimal base $\{u_1, \cdots, u_r\}$ of $\mathfrak{m}$ satisfies the following condition:

(R) *If $F(X_1, \cdots, X_r)$ is a homogeneous polynomial in $(X_1, \cdots, X_r)$ of degree ν with coefficients in $\mathfrak{R}$ and if*

$$F(u_1, \cdots, u_r) \in \mathfrak{m}^{\nu+1},$$

then every coefficient of F is contained in $\mathfrak{m}$.

Conversely, if a set of generators $\{u_1, \cdots, u_r\}$ of the maximal ideal $\mathfrak{m}$ of a local ring $\mathfrak{R}$ satisfies this condition, then $\mathfrak{R}$ is regular and $\{u_1, \cdots, u_r\}$ is a minimal base of $\mathfrak{m}$. Every regular local ring has no zero-divisor and is integrally closed. If $\mathfrak{R}$ is a local ring and $\mathfrak{m}$ is the maximal ideal of $\mathfrak{R}$, then the powers of $\mathfrak{m}$ define a topology on $\mathfrak{R}$. With respect to this topology, $\mathfrak{R}$ has a completion $\mathfrak{R}^*$, which is a local ring containing $\mathfrak{R}$ as subring and subspace, and in which $\mathfrak{R}$ is dense. $\mathfrak{R}$ is regular if and only if its completion $\mathfrak{R}^*$ is regular; and every minimal base of $\mathfrak{m}$ gives a minimal base of the maximal ideal $\mathfrak{m}^*$ of $\mathfrak{R}^*$.

Our main purpose is to prove the following theorem.

THEOREM. *Notations being as in* § 1, *let* V *be a* $\mathfrak{p}$-*variety and* x *a generic point of* V *over* k. *If a point* $\bar{a}$ *of* $\tilde{V}$ *is simple on* V, *then the local ring* $[x \rightarrow \bar{a}\,; \mathfrak{o}]$ *is regular.*

To prove this we need several lemmas. First we generalize the concept of specialization (cf. Northcott [6], Shimura [7]). Let $\mathfrak{R}$ be a local domain and $\mathfrak{m}$ the maximal ideal; let K be the quotient field of $\mathfrak{R}$ and $\tilde{K}$ the residue-field $\mathfrak{R}/\mathfrak{m}$. Let $(x_1, \cdots, x_n)$ be a set of n elements in an extension field of K and $(\xi_1, \cdots, \xi_n)$ a set of n elements in an extension field of $\tilde{K}$. We say that (ξ) is a *specialization of* (x) *over* $\mathfrak{R}$, if the natural homomorphism of $\mathfrak{R}$ onto $\tilde{K} = \mathfrak{R}/\mathfrak{m}$ can be extended to a homomorphism of $\mathfrak{R}[x]$ onto $\tilde{K}[\xi]$ which maps (x) on (ξ). For any polynomial $F(X)$ with coefficients in $\mathfrak{R}$, we denote by $\tilde{F}(X)$ the polynomial with coefficients in $\tilde{K}$ obtained from F considering the coefficients of F modulo $\mathfrak{m}$. (ξ) being a specialization of (x) over $\mathfrak{R}$, we denote by

$$[(x) \rightarrow (\xi)\,; \mathfrak{R}]$$

the set of elements $F(x)/G(x)$ such that $\tilde{G}(\xi) \neq 0$, where $F(X)$ and $G(X)$ are polynomials in $\mathfrak{R}[X]$. This set is also a local domain.

LEMMA 1. *Let* $\mathfrak{R}$ *be a regular local ring and* K *the quotient field of* $\mathfrak{R}$. *Let* a *be an element in an algebraic extension of* K *and* α *a specialization of* a *over* $\mathfrak{R}$. *If there exists a polynomial* $F(X)$ *in* $\mathfrak{R}[X]$ *such that* $F(a) = 0$ *and* $\tilde{F}'(\alpha) \neq 0$, *then* $[a \rightarrow \alpha\,; \mathfrak{R}]$ *is a regular local ring, where* F' *denotes the derivative of* F.

Proof. Put $\mathfrak{S} = [a \rightarrow \alpha\,; \mathfrak{R}]$. Let $\mathfrak{m}$ and $\mathfrak{M}$ denote respectively the maximal ideals of $\mathfrak{R}$ and $\mathfrak{S}$; and let $\tilde{K}$ be the residue field $\mathfrak{R}/\mathfrak{m}$. Let $\overline{F}_0(X) = 0$ be an irreducible equation for α over $\tilde{K}$. As we have $\tilde{F}(\alpha) = 0$, $\tilde{F}(X)$ is divisible by $\overline{F}_0(X)$; so there exists a polynomial $F_1(X)$ in $\mathfrak{R}[X]$ such that $\tilde{F}(X) = \overline{F}_0(X)\tilde{F}_1(X)$. By the assumption $\tilde{F}'(\alpha) \neq 0$, we must have $\tilde{F}_1(\alpha) \neq 0$. Let b be an element of $\mathfrak{M}$; then we can find two polynomials $P(X)$ and $Q(X)$ in $\mathfrak{R}[X]$, such that $b = P(a)/Q(a)$, $\tilde{P}(\alpha) = 0$, $\tilde{Q}(\alpha) \neq 0$. There exists a polynomial $G(X)$ in $\mathfrak{R}[X]$ such that $\tilde{P}(X) = \overline{F}_0(X)\tilde{G}(X)$. Let $\{u_1, \cdots, u_r\}$ be a minimal base of $\mathfrak{m}$. Since we have $\tilde{P}(X)\tilde{F}_1(X) = \tilde{F}(X)\tilde{G}(X)$, there exists r polynomials $H_1(X), \cdots, H_r(X)$ in $\mathfrak{R}[X]$ such that

$$P(X)F_1(X)=F(X)G(X)+\sum_{i=1}^{r}u_iH_i(X).$$

We have then $b=[\sum_i u_iH_i(a)]/[Q(a)F_1(a)]$. We observe $\tilde{Q}(\alpha)\tilde{F}_1(\alpha)\neq 0$, so that r elements $H_i(a)/[Q(a)F_1(a)]$ are all contained in $\mathfrak{S}$. Hence b is contained in $\mathfrak{S}u_1+\cdots+\mathfrak{S}u_r$. This shows that $\mathfrak{M}$ is generated by $\{u_1,\cdots,u_r\}$. Let $\mathfrak{R}^*$ be the completion of $\mathfrak{R}$ and K^* the quotient field of $\mathfrak{R}^*$. Then, by Theorem 1 of Northcott [6], there exists an isomorphism of $K(a)$ into the algebraic closure of K^*, such that, if a' is the image of a, α is a specialization of a' over $\mathfrak{R}^*$. For our purpose, we may put $a=a'$, so that α is a specialization of a over $\mathfrak{R}^*$. We can easily verify that α is a proper specialization of a over $\mathfrak{R}^*$, in the sense of [6], [7]. Then, by Theorem 3 of [6], a is integral over $\mathfrak{R}^*$. Hence we can find an irreducible polynomial $M(X)$ in $\mathfrak{R}^*[X]$ with the leading coefficient 1 such that $M(a)=0$. Now we shall show that $\{u_1,\cdots,u_r\}$ has the property (R) for $\mathfrak{M}$. Let $\sum c_{(i)}X_1^{i_1}\cdots X_r^{i_r}$ be a homogeneous polynomial of degree ν with $c_{(i)}$ in $\mathfrak{S}$ such that

$$\sum_{(i)}c_{(i)}u_1^{i_1}\cdots u_r^{i_r}\in\mathfrak{M}^{\nu+1}.$$

Then there exists a homogeneous polynomial $\sum_{(j)}d_{(j)}X_1^{j_1}\cdots X_r^{j_r}$ of degree $\nu+1$ with $d_{(j)}$ in $\mathfrak{S}$ such that

$$\sum_{(i)}c_{(i)}u_1^{i_1}\cdots u_r^{i_r}=\sum_{(j)}d_{(j)}u_1^{j_1}\cdots u_r^{j_r}.$$

We can find a polynomial $\Phi(X)$ in $\mathfrak{R}[X]$ such that $\tilde{\Phi}(\alpha)\neq 0$ and the elements $\Phi(a)c_{(i)}$, $\Phi(a)d_{(j)}$ are contained in $\mathfrak{R}[a]$. Since a satisfies the equation $M(X)=0$ with the leading coefficient 1, there exist elements $c_{(i)\mu}$, $d_{(j)\mu}$ in $\mathfrak{R}^*$ such that

$$\Phi(a)c_{(i)}=\sum_{\mu=0}^{s-1}c_{(i)\mu}a^{\mu},\quad \Phi(a)d_{(j)}=\sum_{\mu=0}^{s-1}d_{(j)\mu}a^{\mu},$$

where s is the degree of $M(X)$. We have then

$$\sum_{\mu=0}^{s-1}[\sum_{(i)}c_{(i)\mu}u_1^{i_1}\cdots u_r^{i_r}-\sum_{(j)}d_{(j)\mu}u_1^{j_1}\cdots u_r^{j_r}]a^{\mu}=0,$$

so that we get, for every μ,

$$\sum_{(i)}c_{(i)\mu}u_1^{i_1}\cdots u_r^{i_r}=\sum_{(j)}d_{(j)\mu}u_1^{j_1}\cdots u_r^{j_r}\in(\mathfrak{m}^*)^{\nu+1}.$$

As $\{u_1,\cdots,u_r\}$ satisfies the condition (R) for $\mathfrak{m}^*$, we have $c_{(i)\mu}\in\mathfrak{m}^*$, so that the $c_{(i)}$ are contained in the maximal ideal of $[a\to\alpha;\mathfrak{R}^*]$. Since the $c_{(i)}$ are elements of $\mathfrak{S}$, we have $c_{(i)}\in\mathfrak{M}$ for every (i). Thus we have shown that $\{u_1,\cdots,u_r\}$ satisfies the condition (R) for $\mathfrak{M}$. This proves our lemma.

LEMMA 2. *Let $\mathfrak{R}$ be a local domain and $\mathfrak{m}$ the maximal ideal of $\mathfrak{R}$. Let K be the quotient field of $\mathfrak{R}$ and $\tilde{K}$ the residue-field $\mathfrak{R}/\mathfrak{m}$. Let t be a variable over*

K and τ a variable over $\tilde{K}$. If the local domain $[t\to\tau;\mathfrak{R}]$ is regular, so is $\mathfrak{R}$.

Proof. Put $\mathfrak{S}=[t\to\tau;\mathfrak{R}]$; let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{S}$. We can easily verify that $\mathfrak{M}=\mathfrak{S}\mathfrak{m}$ and $\mathfrak{m}=\mathfrak{M}\cap\mathfrak{R}$. Let $\{u_1,\cdots,u_r\}$ be a minimal base of $\mathfrak{m}$. We shall prove that $\{u_1,\cdots,u_r\}$ gives a base of $\mathfrak{M}/\mathfrak{M}^2$ over $\mathfrak{S}/\mathfrak{M}$. Suppose that $\sum a_iu_i\in\mathfrak{M}^2$ for r elements a_i of $\mathfrak{S}$. Then we can find a polynomial $f(t)$ in $\mathfrak{R}[t]$ such that $\tilde{f}(\tau)\neq 0, f(t)a_i\in\mathfrak{R}[t]$ and

$$f(t)\sum_i a_iu_i=\sum_{i,j}g_{ij}(t)u_iu_j\,,$$

where the $g_{ij}(t)$ are elements of $\mathfrak{R}[t]$. Put

$$f(t)a_i=\sum_\nu a_{i\nu}t^\nu,\quad g_{ij}(t)=\sum_\nu b_{ij\nu}t^\nu\,,$$

where the $a_{i\nu}$ and the $b_{ij\nu}$ are elements of $\mathfrak{R}$. Then we have $\sum_i a_{i\nu}u_i=\sum_{i,j}b_{ij\nu}u_iu_j$ for every ν; since $\{u_1,\cdots,u_r\}$ is a base of $\mathfrak{m}/\mathfrak{m}^2$ over $\mathfrak{R}/\mathfrak{m}$, we have $a_{i\nu}\in\mathfrak{m}$, so that the a_i are contained in $\mathfrak{M}$. Thus we have proved that $\{u_1,\cdots,u_r\}$ is a minimal base of $\mathfrak{M}$, so that $\{u_1,\cdots,u_r\}$ satisfies the condition (R) for the ideal $\mathfrak{M}$. Then it is obvious that $\{u_1,\cdots,u_r\}$ satisfies (R) for the ideal $\mathfrak{m}$. This proves our lemma.

LEMMA 3. *Notations being as in Lemma 2, let α be an element which is algebraic over $\tilde{K}$. If $\mathfrak{R}$ is regular, so is $[t\to\alpha;\mathfrak{R}]$.*

Proof. Put $\mathfrak{S}=[t\to\alpha;\mathfrak{R}]$; let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{S}$. We can find a polynomial $F(X)$ in $\mathfrak{R}[X]$, such that $\tilde{F}(X)=0$ is an irreducible equation for α over $\tilde{K}$; we may assume that F and $\tilde{F}$ have the same degree d. Let $\{u_1,\cdots,u_r\}$ be a minimal base of $\mathfrak{m}$; put $u_0=F(t)$. We shall prove that $\{u_0,u_1,\cdots,u_r\}$ satisfies the condition (R) for $\mathfrak{M}$. Let x be an element of $\mathfrak{M}$; then we can find two polynomials $P(t)$, $Q(t)$ in $\mathfrak{R}[t]$ such that $x=P(t)/Q(t)$, $\tilde{P}(\alpha)=0$ and $\tilde{Q}(\alpha)\neq 0$. There exists a polynomial $G(t)$ in $\mathfrak{R}[t]$ such that $\tilde{P}(X)=\tilde{F}(X)\tilde{G}(X)$. We see that $P(t)-F(t)G(t)$ is contained in $\mathfrak{m}[t]$. This shows that $u_0,u_1,\cdots,u_r$ generate $\mathfrak{M}$. Let $\sum a_{(i)}X_0^{i_0}\cdots X_r^{i_r}$ be a homogeneous polynomial of degree ν with a_i in $\mathfrak{S}$ such that

$$\sum_{(i)}a_{(i)}u_0^{i_0}u_1^{i_1}\cdots u_r^{i_r}\in\mathfrak{M}^{\nu+1}\,.$$

Then there exists a homogeneous polynomial $\sum b_{(j)}X_0^{j_0}X_1^{j_1}\cdots X_r^{j_r}$ of degree $\nu+1$ with $b_{(j)}$ in $\mathfrak{S}$ such that

$$\sum_{(i)}a_{(i)}u_0^{i_0}u_1^{i_1}\cdots u_r^{i_r}=\sum_{(j)}b_{(j)}u_0^{j_0}u_1^{j_1}\cdots u_r^{j_r}\,.$$

We can find a polynomial $f(t)$ in $\mathfrak{R}[t]$ such that $\tilde{f}(\alpha)\neq 0$ and the elements $f(t)a_{(i)}$, $f(t)b_{(j)}$ are contained in $\mathfrak{R}[t]$. Since $u_0=F(t)$ is a polynomial in t of degree d, there exist polynomials $a_{(i)\mu}(t)$, $b_{(i)\mu}(t)$ in $\mathfrak{R}[t]$, all of degree less than d, such that

$$f(t)a_{(i)}=\sum_{\mu=0}^{m}a_{(i)\mu}(t)u_0{}^{\mu},\quad f(t)b_{(j)}=\sum_{\mu=0}^{m}b_{(j)\mu}(t)u_0{}^{\mu}.$$

We have then

$$\sum_{(i)}a_{(i)0}u_0{}^{i_0}\cdots u_r{}^{i_r}=\sum_{\mu=0}^{m}\sum_{(j)}b_{(j)\mu}u_0{}^{j_0+\mu}u_1{}^{j_1}\cdots u_r{}^{j_r}-\sum_{\mu=1}^{m}\sum_{(i)}a_{(i)\mu}u_0{}^{i_0+\mu}u_1{}^{i_1}\cdots u_r{}^{i_r}.$$

We can rewrite this equation in the form

$$\sum_{\lambda=0}^{\nu}u_0{}^{\lambda}\Phi_\lambda(t,u_1,\cdots,u_r)=\sum_{\lambda=0}^{n}u_0{}^{\lambda}\Psi_\lambda(t,u_1,\cdots,u_r),$$

where Φ_λ is a polynomial with coefficients in $\mathfrak{R}$ of degree $<d$ in t and homogeneous in $(u_1,\cdots,u_r)$ of degree $\nu-\lambda$ and Ψ_λ is a polynomial with coefficients in $\mathfrak{R}$ of degree $<d$ in t and of degree $\geqq\nu-\lambda+1$ in $(u_1,\cdots,u_r)$. We have then, for every λ,

$$\Phi_\lambda(t,u_1,\cdots,u_r)=\Psi_\lambda(t,u_1,\cdots,u_r).$$

We see that the right hand side is contained in $\mathfrak{m}^{\nu-\lambda+1}[t]$. Since $\{u_1,\cdots,u_r\}$ satisfies (R) for $\mathfrak{m}$, the coefficients of the polynomial $\Phi_\lambda(T,U_1,\cdots,U_r)$ are contained in $\mathfrak{m}$. It follows from this that the $a_{(i)}$ are contained in $\mathfrak{M}$. Hence $\{u_0,u_1,\cdots,u_r\}$ satisfies (R) for $\mathfrak{M}$.

LEMMA 4. *The notations* $\mathfrak{o}$, $\mathfrak{p}$, k, $\tilde{k}$ *being as in* §1, *let* $t_1,\cdots,t_n$ *be* n *independent variables over* k *and* $\bar{a}_1,\cdots,\bar{a}_n$ *be* n *elements in an extension of* $\tilde{k}$. *Then*, $[(t)\to(\bar{a});\mathfrak{o}]$ *is a regular local ring.*

Proof. Let s be the dimension of $(\bar{a})$ over $\tilde{k}$; if s is not 0, we may, after reordering the $\bar{a}_i$ if necessary, assume that $\bar{a}_1,\cdots,\bar{a}_s$ are independent variables over $\tilde{k}$ and $(\bar{a})$ is algebraic over $\tilde{k}(\bar{a}_1,\cdots,\bar{a}_s)$. Put $\mathfrak{o}'=[(t_1,\cdots,t_s)\to(\bar{a}_1,\cdots,\bar{a}_s);\mathfrak{o}]$. Then $\mathfrak{o}'$ is a discrete valuation ring of rank 1; and we have

$$[(t)\to(\bar{a});\mathfrak{o}]=[(t_{s+1},\cdots,t_n)\to(\bar{a}_{s+1},\cdots,\bar{a}_n);\mathfrak{o}'].$$

Hence by lemma 3, $[(t)\to(\bar{a});\mathfrak{o}]$ is a regular local ring.

We shall now prove Theorem 1. We may assume that V is an affine variety. Let r and n be respectively the dimensions of V and the ambient space for V. Let t_{ij}, for $0\leqq i\leqq r$, $1\leqq j\leqq n$, be $(r+1)n$ independent variables over $k(x)$; and let $\bar{t}_{ij}$, for $0\leqq i\leqq r$, $1\leqq j\leqq n$, be $(r+1)n$ independent variables over $\tilde{k}(\bar{a})$. Put $y_i=\sum_j t_{ij}x_j$, $\bar{b}_i=\sum_j\bar{t}_{ij}\bar{a}_i$ and

$$\mathfrak{o}'=[(t_{ij})\to(\bar{t}_{ij});\mathfrak{o}],$$
$$\mathfrak{R}=[(y_1,\cdots,y_r)\to(\bar{b}_1,\cdots,\bar{b}_r);\mathfrak{o}'],$$
$$\mathfrak{S}=[y_0\to\bar{b}_0;\mathfrak{R}].$$

Then, $\mathfrak{o}'$ is a discrete valuation ring. Since $y_1,\cdots,y_r$ are independent variables over $k(t_{ij})$, $\mathfrak{R}$ is a regular local ring by virtue of Lemma 4. By the proof of

Theorem 15 of [7], $\bar{b}_0$ is a proper specialization of y_0 over $\mathfrak{R}$ of multiplicity 1. Let

$$F(Y_0, Y_1, \cdots, Y_r) = 0$$

be an irreducible equation for $(y_0, y_1, \cdots, y_r)$ over $k(t_{ij})$. We may assume that all coefficients of F are contained in $\mathfrak{o}'$ and at least one of them is equal to 1. Then, we have $\tilde{F}(Y_0, \bar{b}_1, \cdots, \bar{b}_r) \neq 0$; for otherwise, $\bar{c}$ being a variable over $\tilde{k}(\bar{t}_{ij}, \bar{b}_i)$, $\bar{c}$ would be a specialization of y_0 over $\mathfrak{R}$; this contradicts the fact that $\bar{b}_0$ is a proper specialization of y_0 over $\mathfrak{R}$. Since $\bar{b}_0$ is of multiplicity 1, $\bar{b}_0$ is a simple root of the equation

$$\tilde{F}(Y_0, \bar{b}_1, \cdots, \bar{b}_r) = 0;$$

so we have $\partial\tilde{F}/\partial Y_0(\bar{b}_0, \bar{b}_1, \cdots, \bar{b}_r) \neq 0$. By Lemma 1, this proves that $\mathfrak{S}$ is a regular local ring; in particular, $\mathfrak{S}$ is integrally closed in its quotient field $k(t, y)$. By Proposition 16 of [7], we have $k(t, x) = k(t, y)$ and x is finite over $\mathfrak{S}$. Since $\mathfrak{S}$ is integrally closed, the coordinates of x must be contained in $\mathfrak{S}$. This shows

$$\mathfrak{S} = [x \rightarrow a; \mathfrak{o}'].$$

Put $\mathfrak{T} = [x \rightarrow \bar{a}; \mathfrak{o}]$; then it is easy to see $\mathfrak{S} = [(t_{ij}) \rightarrow (\bar{t}_{ij}); \mathfrak{T}]$. Hence, by Lemma 2, $\mathfrak{T}$ is a regular local ring; so our theorem is proved.

We profit by this opportunity to revise some points in [7].

(1) The proof of Proposition 17 is omitted by reason of that it is a translation of Proposition 19 of [9] Chap. V. The first part which asserts $\partial\bar{F}/\partial Z(\eta, \zeta) \neq 0$ is proved in fact in the same way as in Weil's book. It is hardly possible, however, to prove the remaining part by the same argument as in [9]. The above Theorem 1, or Lemma 1, with their proofs, will supply this gap.

(2) p. 150, the lowest line. "Obviously, (η) is" should be read "Obviously, (ξ) is".

(3) p. 151, the first line. (τ_{ij}, τ_i) should be read $(\delta_{ij}, \varepsilon_i)$.

(4) p. 155. Corollary of Theorem 10 should be as follows:

COROLLARY. *Let V be a variety defined over k and $\mathfrak{B}$ a component of $\bar{V}$. If $\mathfrak{B}$ is simple on V, then we have $\mu(V, \mathfrak{B}) = 1$ and $[\mathfrak{B} : \kappa]_i = 1$.*

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