On Some Compact Transformation Groups on Spheres

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Introduction

Let G be a compact connected Lie transformation group operating on a differentiable manifold S which is homeomorphic to the n-dimensional sphere. We assume that there exists an $(n-1)$-dimensional orbit $G(p)$, $p \in S$, and we shall study the homology groups of the orbits (Theorem 3.5). As an example we shall determine the known homology groups of the Stiefel manifold $V_{N,2}$. The case where the dimension of S is 1 or 2 can be treated very easily, and will be excluded in the sequel. Results in [5] will be frequently used in this paper.

Notations. $n = \dim S > 2$. F_α, $\alpha = 1, 2$, is a singular orbit (See Section 1 for the definition). $f_\alpha = \dim F_\alpha$. $e_\alpha = n-1-f_\alpha$. F is a regular orbit. In a sentence or formula containing the letters α and β, they will always denote 1 or 2 and $\alpha \neq \beta$ (i.e. $\alpha+\beta=3$). Z is the additive group of the integers. Z_2 is the group of order 2. R is the real field.

1. Singular orbits

Throughout this paper we assume the following conditions (A) and (B):

(A) G is a compact connected Lie transformation group of a differentiable manifold S which is a homological sphere, i.e. a compact simply connected manifold with the same homology group as an n-sphere, $2 < n$.

(B) There exists an $(n-1)$-dimensional G-orbit on S.

There exist two-sided and therefore orientable $(n-1)$-dimensional orbits [5], which we shall call *regular orbits.* The other orbits will be said singular. There exists a singular orbit, because otherwise S would admit a fibre bundle structure with a G-orbit as fibre [5], the base space being a one-dimensional manifold by Palais' theorem [7], which is necessarily a circle, and this would lead oviously to a contradiction by observing the fundamental groups. Let F_1 be a singular orbit. Then the set F_2 in S of the farthest points from F_1 is of dimension $< n-1$ or one-sided and so nonorientable. F_2 is thus another singular orbit. There are no other singular orbits [5]. Hence we get

PROPOSITION 1.1. *Under* (A) *and* (B), *there exist exactly two singular orbits.*

All orbits are connected, for G is connected.

2. Homotopy groups of singular oribits

Let f_α denote the dimension of $F_\alpha(\alpha=1,2)$. We put $e_\alpha=n-1-f_\alpha$.

LEMMA 2.1. *If* $r<e_\alpha$, *then the rth homotopy group* $\pi_r(F_\beta)$ *of* F_β *is trivial.*

Proof. We then have $\pi_r(S-F_\alpha)=\pi_r(S)=0$. Since F_β is a deformation retract of $S-F_\alpha$ ([5]), we see that $\pi_r(F_\beta)$ is isomorphic to $\pi_r(S-F_\alpha)$. Lemma 2 is thus proved.

In case $f_\alpha=0$, f_β necessarily vanishes and the other orbits are homeomorphic to the $(n-1)$-sphere [6]. So we assume that

(C) $\quad f_1\cdot f_2\neq 0.$

PROPOSITION 2.2. *Under* (A), (B), *and* (C), *we have the inequality*

$$n-1\leqq f_1+f_2, \ i.e.\ e_\alpha\leqq f_\beta.$$

Proof. Assume the contrary. For any $r\leqq f_\alpha$, we have $f_\beta<n-r-1$. By Lemma 2.1. we find that $\pi_r(F_\alpha)$ is trivial. It follows that f_α equals zero, contrary to (C).

3. The homology groups of the orbits

All regular orbits are homeomorphic to each other ([5]). Let F be one of them. $H_q(X)$ shall denote the q-th homology group over Z of a compact space X, unless otherwise stated.

LEMMA 3.1. *There exists an onto isomorphism*

$$\lambda_q: H_q(F)\rightarrow H_q(F_1)+H_q(F_2) \quad for \quad 0<q<n-1.$$

Proof. The subspace $S-F$ has two connected components. Let X_α be the closure of the component which contains F_α. Then we have $S=X_1\cup X_2$ and $F=X_1\cap X_2$. The Mayer-Vietoris sequence

$$\cdots\rightarrow H_{q+1}(S)\rightarrow H_q(F)\rightarrow H_q(H_1)+H_q(X_2)\rightarrow H_q(S)\rightarrow\cdots$$

is exact [1]. By the assumption we have $H_{q+1}(S)=H_q(S)=0$ for $0<q<n-1$. Since F_α is a deformation retract of X_α ([5]), we conclude the lemma from the above sequence.

REMARK. We have $f_\alpha\leqq n-2$. This inequality is due to Montgomery, Samelson and Yang [4]. But in our case the proof is simple. In fact the above Mayer-Vietoris sequence is exact for the homology groups over any coefficient group A. Putting $A=Z_2$, we get $0\rightarrow Z_2\rightarrow Z_2\rightarrow H_{n-1}(F_1)+H_{n-1}(F_2)\rightarrow 0$. Hence $H_{n-1}(F_\alpha)=0$, and so f_α cannot be $n-1$.

LEMMA 3.2. *When F_α is simply connected, we have*

$$H_{q-e_\alpha}(F_\alpha)=H_q(F_\beta) \quad \textit{for} \quad 0<q<n-2.$$

Proof. F is an e_α-sphere bundle over F_α ([5]). The Gysin sequence [8] reads

$$\cdots \longrightarrow H_{q+1}(F_\alpha) \longrightarrow H_{q-e_\alpha}(F_\alpha) \xrightarrow{\mu_{\alpha q}} H_q(F) \xrightarrow{\lambda_{q\alpha}'} H_q(F_\alpha) \longrightarrow \cdots.$$

$\lambda_{q\alpha}'$ is induced from the projection of F onto F_α. The composition $\lambda_{q\alpha}'' : H_q(F) \to H_q(F_\alpha)$ of λ_q and the canonical homomorphism of $H_q(F_1)+H_q(F_2)$ onto $H_q(F_\alpha)$ is induced from the injection of F into X_α and the deformation of X_α onto F_α. Hence $\lambda_{q\alpha}'$ coincides with $\lambda_{q\alpha}''$ for $0<q<n-1$, as is easily seen from the definition [5] of the deformation. Thus $\lambda_{q\alpha}'$ is surjective. Since $\lambda_{q+1,\alpha}'$ is surjective, for $0<q+1<n-1$, we find that $\mu_{\alpha q}$ is injective. Furthermore there exists an isomorphism $\nu_{\alpha q}$ of $H_q(F_\alpha)$ into $H_q(F)$ such that the composition $\lambda_{q\alpha}'\nu_{\alpha q}$ is the identity. Therefore we get an isomorphism

$$H_q(F)=H_q(F_\alpha)+H_{q-e_\alpha}(F_\alpha) \quad \text{for} \quad 0<q<n-2.$$

Combined with Lemma 3.1, this gives the Lemma 3.2.

LEMMA 3.3. *If both F_1 and F_2 are simply connected, then there exists the isomorphism*

$$H_q(F_\alpha)=H_{q-e_1-e_2}(F_\alpha) \quad \textit{for} \quad e_\beta<q<n-2.$$

Proof. From Lemma 3.2, follows

$$H_q(F_\alpha)=H_{q-e_\beta}(F_\beta)=H_{q-e_\beta-e_\alpha}(F_\alpha).$$

LEMMA 3.4. *When F_1 and F_2 are orientable, we have the isomorphisms in lemmas 3.2 and 3.3 for the homology groups over R.*

This fact can be proved by means of Alexander's duality theorem [3]. The details are omitted.

THEOREM 3.5. *Under the hypotheses (A), (B) and (C), assume that $f_1, f_2<n-2$. Then the Poincaré polynomials with respect to any field of F and F_α are of the forms;*

$$P(F_\alpha;t)=(1+t^{e_\beta})Q(t),$$

$$P(F;t)=(1+t^{e_1})(1+t^{e_2})Q(t),$$

where $Q(t)=1+t^e+t^{2e}+\cdots+t^{me}$, $e=e_1+e_2$, $(m+1)e=n-1$,
or else $e_1=e_2$ and

$$P(F_\alpha;t)=1+t^{e'}+t^e+\cdots+t^{ke}=(1-t^{(2k+1)e'})/(1-t^{e'}),$$

$$P(F;t)=P(F_\alpha;t)(1+t^{e'}),$$

where $e'=e_\alpha$ and $ke+e'=(2k+1)e'=n-1$.

Proof. On account of Lemma 2.1, the spaces F_1 and F_2 are simply connected. By Lemma 3.2 we obtain

$$H_q(F_\alpha)=0 \quad \text{for} \quad 0<q<e_\beta ,$$

and

$$H_{e_\beta}(F_\alpha)=Z .$$

It follows from Lemma 3.2 that $H_q(F_\alpha)=0$ for $e_\beta<q<e$. On the other hand we have $H_{f_\alpha-q}(F_\alpha)=H_q(F_\alpha)$. Therefore Lemma 3.3 gives the above forms of $P(F_\alpha; t)$. Finally one finds that of $P(F; t)$ by virtue of Lemma 3.1.

REMARK. Under the hypotheses (A), (B) and (C), $H(F_2)$ can be determined if F_2 is simply connected and $H(F_1)$ is known. (See Example 4 in Section 4).

$H(F)$ and $H(F_\alpha)$ are determined by the dimensions of F and F_α.

COROLLARY 3.6. *Preserve the assumptions of Theorem* 3.5. *If e_α is odd, then the Euler characteristics $\chi(F)$ and $\chi(F_\alpha)$ of F and F_α vanish, and $\chi(F_\beta)=0$ or $1+(-1)^m$ according as e_β is odd or not; or else $e_1=e_2$, $\chi(F)=0$ and $\chi(F_1)=\chi(F_2)=1$. If both e_1 and e_2 are even, then we have $\chi(F_1)=\chi(F_2)=2(m+1)=2(n-1)/e$ and $\chi(F)=2\chi(F_\alpha)$; or else $e_1=e_2$, $\chi(F_\alpha)=(n-1)/e$ and $\chi(F)=2\chi(F_\alpha)$.*

PROPOSITION 3.7. *Under the assumption of (A) and (B), we have*

$$\chi(F_1)+\chi(F_2)=\chi(F)+1+(-1)^n .$$

Proof. This follows from Lemma 3.1, orientability of F and the remark below Lemma 3.1.

THEOREM 3.8. *Under the assumptions (A) and (B), assume that $f_1=1$. Then F_1, F_2 and F are a 1-sphere, an $(n-2)$-sphere and the direct product of these two spheres respectively.*

Proof. Since F is an orientable $(n-2)$-sphere bundle over a 1-sphere F_1, the well known classification theorem yields that F is a product bundle. It remains to show that F_2 is an $(n-2)$-sphere in case $3<n$. Then f_2 is $n-2$ and F_2 is simply connected by Lemma 2.1. Lemma 3.1 gives that F_2 is a homological sphere. On the other hand it is a homogeneous space. Hence it is a sphere [6].

REMARK. It can be proved that more generally, under the assumptions (A), (B) and $f_1+f_2=n-1$, F_α is a sphere and F is the direct product $F_1\times F_2$. (See Example 1 in Section 4.)

THEOREM 3.9. *Under the hypotheses (A), (B) and (C), if F_1 and F_2 are orientable, the analogous conclusion as Theorem* 3.6 *is valid if the Poincaré polynomials are considered to be over the real field R.*

This follows from Lemma 3.4.

4. Examples

Example 1. Let us assume that a connected subgroup H_α, $\alpha=1, 2$, of the orthogonal group $O(f_\alpha+1)$ is transitive on the f_α-sphere. Denote by G the direct product of H_1 and H_2. Then G operates on an n-sphere with $n-1=f_1+f_2$ through the direct sum of the natural representations of H_1 and H_2, and has an $(n-1)$-dimensional orbit F. Then the singular orbit F_α is an f_α-sphere, and F is the direct product of these two singular orbits.

Professor H. C. Wang suggested the author to investigate the following three examples.

Example 2. Let G be the adjoint group of the compact exceptional simple Lie group G_2, G operating naturally on the Lie algebra of itself as an orthogonal group. Hence G operates on a 13-sphere, admitting 12-dimensional orbits. The regular orbit is a simply connected homogeneous space G_2/T_2, T_2 being a 2-torus. The singular orbits are both simply connected and homeomorphic to the homogeneous space $G_2/(T_1\times A_1)$. Thus we have

$$P(F_\alpha;t)=(1+t^2)(1+t^4+t^8),\ \text{and}\ P(F;t)=(1+t^2)P(F_\alpha;t),\ \text{over any field.}$$

In particular F (A. Borel: Ann. of Math. 57, 115-207, 1953) and F_α are without torsion.

Example 3. Let G be a compact simply connected exceptional simple Lie group F_4 which operates irreducibly on a 26-dimensional vector space over R. (See [2], especially paragraphs 4.11, 4.12 and 5.1). The regular orbits are the homogeneous space F_4/D_4, and each singular orbit is F_4/B_4. We have

$$P(F_\alpha;t)=1+t^8+t^{16},\ P(F;t)=(1+t^8)(1+t^8+t^{16}),\ \text{over any field.}$$

Example 4. Let G be the direct product of $SO(2)$ and $SO(N)$, $3<N$. G operates on the $2N$-dimensional real vector space through the Kronecker product of the natural representations of $SO(2)$ and $SO(N)$. The unit $(2N-1)$-sphere S is left invariant by G. G admits $(2N-2)$-dimensional orbits on S. Let H and H_α be the isotropy subgroups of G at points on F and F_α respectively. Assume $f_1\leqq f_2$. Then we have $H_1=SO(N-1)$ or $SO(N-1)Z_2$ according as N is even or odd. Hence F_1 is clearly an $(N-1)$-sphere bundle over a 1-sphere, which is trivial or not according as the parity of N. If N is odd, it follows that $H_0(F_1)=Z$, $H_{N-1}(F_1)=Z_2$ and $H_q(F_1)=0$ for other q's. Since $H_2=SO(2)\times SO(N-2)$, F_2 is the Stiefel manifold $V_{N,2}=SO(N)/SO(N-2)$ whose dimension is $2N-3$. F_2 is simply connected by Lemma 2.1 or as is well known. Now Lemma 3.2 applies to $F_\alpha=F_2$, and one can easily determine the homology groups of $F_2=V_{N,2}$ ([9]) and F.

When N is even, $P(F_2;t)=P(V_{N,2};t)=(1+t^{N-2})(1+t^{N-1})$;

$$P(F;t)=(1+t)P(F_2;t)=(1+t^{N-2})P(F_1;t),$$

each over any field.

When N is odd, $H_0(F_2)=H_{2N-3}(F_2)=Z, H_{N-2}(F_2)=Z_2$,
$H_q(F_2)=0$ $(q\neq 0, 2N-3, N-2)$;
$H_0(F)=H_1(F)=H_{2N-3}(F)=H_{2N-2}(F)=Z$,
$H_{N-2}(F)=Z_2+Z_2, H_{N-1}(F)=H_N(F)=Z_2$,
and $H_i(F)=0$ for other i's.

Since we have $H^2(F_2)=0$, the 1-sphere bundle F over F_2 is trivial.

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