A Proof of Tannaka Duality Theorem

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1. The purpose of this paper is to give a simple proof of the well-known Tannaka duality theorem[1)].

Let $G \ni a, b \ldots, x, \ldots$ be a compact group, $C(G)$ the ring of all continuous functions on G. The Fourier polynomials on G, i.e. the elements $f(x)$ of $C(G)$ expressible in the form:

$$f(x)=\sum_{a,i,j} c_{ij}^{(a)} d_{ij}^{(a)}(x) \qquad \text{(finite sum)}$$

where $c_{ij}^{(a)}$'s are complex numbers and $d_{ij}^{(a)}(x)$'s are the components of some continuous irreducible representation $x \rightarrow D^{(a)}(x)=(d_{ij}^{(a)}(x))$ of G by matrices of a finite degree, form a subring $R(G)$ of $C(G)$.

For an element a in G, left (right) translation $L_a(R_a)$ is defined as a linear operator on $C(G)$ as follows:

$$(L_a f)(x)=f(a^{-1}x), \quad (R_a f)(x)=f(xa) \quad (x \in G,\ f \in C(G)).$$

Let T be a linear mapping from $R(G)$ into $R(G)$. We shall call T *left-admissible* if $TL_a=L_aT$ for every $a \in G$.

Now we shall prove the following theorem, which we shall show afterwards to be equivalent with Tannaka duality theorem:

THEOREM. *Let T be a left-admissible linear operator of $R(G)$ such that (a) $T1=1$, (b) $\overline{Tf}=T\bar{f}$, (c) $T(fg)=Tf\cdot Tg$ (for every f, g in $R(G)$). Then T coincides with a right translation R_a for some $a \in G$.*

In the following proof, an essential point is to deduce the continuity of T with respect to the uniform norm from the given algebraic conditions on T. It is to be noticed that this deduction constituted one of the difficulties in the proofs hitherto given of Tannaka duality theorem.

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1) T. Tannaka, Über den Dualitätssatz der nichtkommutativen topologischen Gruppen, *Tohoku Math. J.*, 45, 1-12 (1938).

K. Yoshida, On the duality theorem of non-commutative compact groups, *Proc. Imp. Acad. Tokyo*, 19, 181-183 (1943).

PROOF. First, we have

(1) $\int_G (Tf)(x)dx = \int_G f(x)dx$,

for every $f(x)$ in $R(G)$ where dx is the Haar measure on G such that $\int_G dx=1$. In fact, (1) is valid for every component $d_{ij}(x)$ of an irreducible representation D which is not equivalent to the unit respresentation. To prove this, let $(TD)(x)$ mean the matrix $((Td_{ij})(x))$ for $D(x)=(d_{ij}(x))$, then we have by $(L_{a^{-1}}D)(x)=D(ax)=D(a)\cdot D(x)$, $(TL_{a^{-1}}D)(x)=D(a)\cdot(TD)(x)$. Moreover, by $(L_{a^{-1}}TD)(x)=(TD)(ax)$, we have $(TD)(ax)=D(a)\cdot(TD)(x)$ for every $a, x\in G$. Putting $x=e$ and $a=x$, we have (e is the unit element of G)

$$(TD)(x)=D(x)\cdot(TD)(e), \text{ for every } x\in G.$$

Thus, $(Td_{ij})(x)$ is a linear combination of $d_{ij}(x)$'s and $\int_G d_{ij}(x)dx=0$ by orthogonality relations. Hence both sides of (1) become zero. Moreover (1) is valid for $f=1$ by (a). Thus (1) is proved.

Then we have for $p=1,2,\ldots$,

(2) $\|Tf\|_{2p}=\|f\|_{2p}$, for every $f(x)$ in $R(G)$.

In fact, $|Tf|^{2p}=(Tf\cdot\overline{Tf})^p=(Tf\,T\bar{f})^p=(T(f\bar{f}))^p=T\big((f\bar{f})^p\big)$ by virtue of (b), (c). Then, we have (2) by (1). Now, since $\|f\|_\infty=\lim_{p\to\infty}\|f\|_p$ ($\|f\|_\infty=\operatorname{Max}_{x\in G}|f(x)|$), we have

(3) $\|Tf\|_\infty=\|f\|_\infty$, for every $f(x)$ in $R(G)$.

Thus, *T is continuous with respect to the uniform norm.* Now by Peter-Weyl's theorem, $R(G)$ is dense in $C(G)$ with respect to the uniform norm, and T can be extended uniquely to a continuous linear operator $\tilde{T}$ of $C(G)$. We have obviously $\tilde{T}L_a=L_a\tilde{T}$ for every $a\in G$.

Now, let us show that T is a *one-to-one* linear mapping from $R(G)$ *onto* $R(G)$. To show this, it is clearly sufficient to show that $(TD)(e)$ is a non-singular matrix for every representation $D(x)$. Now, the contragredient representation ${}^tD^{-1}(x)=(d'_{ij}(x))$ of $D(x)$ satisfies ${}^tD^{-1}\cdot{}^tD=I$, so we have $T{}^tD^{-1}\cdot{}^t(TD)=I$ by (a) and (c), then $(T{}^tD^{-1})(e)\cdot{}^t((TD)(e))=I$. Thus $(TD)(e)$ is non-singular.

Hence T has the inverse T^{-1} which is also clearly left-admissible and satisfies the conditions (a), (b), (c). Consequently T^{-1} can be extended to a continuous linear operator $\tilde{T}^{-1}$ of $C(G)$, which is the inverse of $\tilde{T}$. Since T is an automorphism of the ring $R(G)$, $\tilde{T}$ is also an automorphism of the ring $C(G)$. Now, to complete our proof it is sufficient to show:

Let $\tilde{T}$ be an automorphism of the ring $C(G)$ such that $\tilde{T}L_a=L_a\tilde{T}$ for every $a\in G$. Then $\tilde{T}$ is a right translation.

Let us now prove this. For $a \in G$, the subset $\mathfrak{J}_a=\{f \in C(G);\ f(a)=0\}$ is a maximal ideal of $C(G)$. Conversely, as is well-known, every maximal ideal of $C(G)$ is of this form. Now for any $f \in C(G)$, $a \in G$ we have $f(x)-f(a) \in \mathfrak{J}_a$. Hence, putting $g=\tilde{T}f$, we have $g(x)-f(a) \in \tilde{T}(\mathfrak{J}_a)$. Since we have $\mathfrak{J}_a=L_a(\mathfrak{J}_e)$ (e: the unit element of G), $g(x)-f(a) \in \tilde{T}L_a(\mathfrak{J}_e)=L_a\tilde{T}(\mathfrak{J}_e)$. Now, $\tilde{T}$ being an automorphism of $C(G)$, $\tilde{T}(\mathfrak{J}_e)$ is a maximal ideal of $C(G)$. So there is an element $b \in G$ such that $\tilde{T}(\mathfrak{J}_e)=\mathfrak{J}_b$. Then $g(x)-f(a) \in L_a(\mathfrak{J}_b)=\mathfrak{J}_{ab}$, hence we have $g(ab)=f(a)$ or $(\tilde{T}f)(ab)=f(a)$ for any $a \in G$. In other words we have $\tilde{T}f(x)=f(xb^{-1})$ for any $x \in G$, i.e., $\tilde{T}=R_{b^{-1}}$, q.e.d.

2. Now we shall show that our theorem means indeed the classical Tannaka duality theorem. For a representation $D(x)=(d_{ij}(x))$ of G and a left-admissible operator T we shall denote the constant matrix $(TD)(e)$ by $\mu_D(T)$ or simply by μ_D. We note the following equality:

$$(TD)(x)=D(x)\cdot\mu_D. \quad \text{(for every } x \text{ in } G).$$

Thus T is determined by $\{\mu_D\}$. We have easily

(4) $\mu_{D_1 \dot{+} D_2}=\mu_{D_1} \dot{+} \mu_{D_2}$ ($\dot{+}$ means the direct sum)

(5) $\mu_{PDP^{-1}}=P\mu_D P^{-1}$ (P: any constant non-singular matrix).

Conversely let $D \to \mu_D$ be a mapping which associates for every representation D of G a constant matrix μ_D with the same degree as D satisfying (4), (5). Then we have:

There exists one (and only one) left-admissible operator T such that $\mu_D=(TD)(e)$ for every representation D of G.

Indeed, let $\{D^{(\alpha)}(x)\}$ be a complete set of mutually inequivalent, continuous, irreducible representations of G. Then $\{d_{ij}^{(\alpha)}(x)\}$'s form a base of $R(G)$ as a vector space over the complex number field.

Then we may define a linear operator T of $R(G)$ by $TD^{(\alpha)}=D^{(\alpha)}\mu_{D^{(\alpha)}}$ for everyre presentative irreducible representation $D^{(\alpha)}$. Then since every reducible representation D is expressible in the form $D=P(D^{(\alpha)}\dot{+}\dots\dot{+}D^{(\gamma)})P^{-1}$, we have $TD=D\cdot\mu_D$. Then it is easily seen that T is left-admissible and $(TD)(e)=\mu_D$.

Thus we can formulate the properties of a left-admissible operator T by means of the properties of $\mu_D=\mu_D(T)$. In particular we have immediately the following:

(6) $T1=1$ if and only if $\mu_E=I_1$ (E is the unit representation of degree 1 and I_1 is the unit matrix of degree 1).

(7) $\overline{Tf}=T\bar{f}$ (for every f in $R(G)$) if and only if $\mu_{\bar{D}}=\bar{\mu}_D$ (for every representation D).

(8) $T(fg)=Tf\cdot Tg$ (for every f, g in $R(G)$) if and only if $\mu_{D_1\otimes D_2}=\mu_{D_1}\otimes\mu_{D_2}$ (for every representation D_1, D_2), where $\otimes$ means the Kronecker product.

(9) $T=R_a$ if and only if $\mu_D=D(a)$ for every representation D.

Now, the classical Tannaka duality theorem is expressed in terms of μ_D. It asserts that μ_D satisfying the conditions (4)~(5) for μ_D is in fact $D(a)$ for some $a\in G$. Translating this in terms of T by the above results, we see that the classical theorem is equivalent with ours.