On E. Hille's Theorem

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1. Let G be a locally compact abelian group and $\hat{G}$ the dual group of G. The elements of G and $\hat{G}$ are denoted by $x, y,$ ...... and $\hat{x}, \hat{y}$ ...... respectively; $(x, \hat{x})$ will mean the value of the character $\hat{x}$ at x.

$L^1(G)$, $C(G)$ denote as usual the set of all integrable functions on G with respect to Haar measure, and the set of all continuous functions on G, respectively, τ_a the translation by $a \in G$, i.e. $\tau_a f(x) = f(xa^{-1})$. For $f, g \in L^1(G)$, $f * g$ means the convolution product of f and g. We shall denote by $\mathfrak{E}[L^1(G)]$, $\mathfrak{E}[C(G)]$ the set of all bounded linear transformations on $L^1(G)$, $C(G)$ to itself. The subset of $\mathfrak{E}[L^1(G)]$, $\mathfrak{E}[C(G)]$ consisting of all elements which commute with every translation τ_a, $a \in G$, will be denoted by $A[L^1(G)]$, $A[C(G)]$.

E. Hille[1] has proved that the following propositions hold in the cases where G is either the additive group of real numbers or the toral groups.

(a) For every element T of $A[L^1(G)]$ there exists uniquely a function $\mu(\hat{x})$, called the factor function of T, which is continuous in $\hat{x} \in \hat{G}$, such that:

$$\widehat{T \cdot f}(\hat{x}) = \mu(\hat{x}) \hat{f}(\hat{x}).$$

(b) There exists a bounded Radon measure on G, $\mathfrak{M}(\hat{x})$, such that:

$$\hat{\mathfrak{M}}(\hat{x}) = \mu(\hat{x}) \quad \text{and} \quad T \cdot f = f * \mathfrak{M} \left(= \int \overline{f(xy^{-1})} d\mathfrak{M}(y) \right)$$

(c) Let $\mathfrak{S} = \{T(\xi), \xi > 0\}$ be a semi-group in $A[L^1(G)]$, i.e. a set of elements of $A[L^1(G)]$ defined for a positive real parameter ξ, satisfying

$$T(\xi_1 + \xi_2) = T(\xi_1) T(\xi_2).$$

If we denote by $\mu(\hat{x}, \xi)$ the factor function of $T(\xi)$, then we have

$$\mu(\hat{x}, \xi_1 + \xi_2) = \mu(\hat{x}, \xi_1) \mu(\hat{x}, \xi_2),$$

and the properties of $T(\xi)$ such as being weakly continuous in ξ, possessing infinitesimal generator, have their counter-parts in properties of $\mu(\hat{x}, \xi)$. (c.f. Theorem 4. below).

1) E. Hille, *Functional analysis and semi-groups*, p. 368 (1946).

Though it is not certain that all these hold for any locally compact abelian group G, we could establish the following results.

(a′) The proposition (a) of Hille hold for every locally compact abelian group. (Theorem 1 below).

(b′) The proposition (b) of Hille holds for every discrete abelian group. It holds also for compact abelian groups, if $A[L^1(G)]$ is replaced by $A[C(G)]$. (Theorem 2.3. below).

(c′) The proposition (c) of Hille holds for every locally compact abelian group. (Theorem 4 below).

Our proofs are simpler than that of Hille in that we do not need the classical Zygmund's theorem[2]. The author wishes to present here her hearty thanks to Prof. S. Iyanaga and Prof. K. Yosida for their kind criticisms.

2. First we shall consider a single linear bounded transformation $T\in\mathfrak{E}[L^1(G)]$ as given, and define that linear transformation $\hat{T}$ on $\widehat{L^1(G)}$ to itself by:

$$\hat{T}\cdot\hat{f}=\widehat{T\cdot f}.$$

Then we have:

Lemma. $T\tau_a=\tau_a T \Longleftrightarrow \hat{T}(a^{-1},\hat{x})=(a^{-1},\hat{x})\hat{T}$ *for all* $\hat{x}\in\hat{G}$.

Proof. Let f be any element of $L^1(G)$. Taking Fourier transform, we have $(\widehat{T\tau_a f(x)})(\hat{x})=\widehat{Tf(xa^{-1})}(\hat{x})=\hat{T}\int_G f(xa^{-1})\overline{(x,\hat{x})}dx=\hat{T}\int_G f(x)\overline{(ax,\hat{x})}dx$

$=\hat{T}(a^{-1},\hat{x})\hat{f}(\hat{x}).$

On the other hand we have

$(\widehat{\tau_a Tf(x)})(\hat{x})=(\widehat{Tf})(xa^{-1}(\hat{x})=\int_G (Tf)(xa^{-1})\overline{(x,\hat{x})}\,dx=(a^{-1},\hat{x})\,\hat{T}\hat{f}(\hat{x}).$

Theorem 1. *There exists a continuous function* $\mu(\hat{x})$ *on* G *for* $T\in\mathfrak{E}[L^1(G)]$ *with* $\widehat{Tf}=\mu\hat{f}$, *if and only if* $T\in A[L^1(G)]$, *i.e.* $T\tau_a=\tau_a T$ *for all* $a\in G$. *Moreover if* $T\in A[L^1(G)]$, *then the function* $\mu(\hat{x})$ *is uniquely determined.*

Proof. Assume $T\tau_a=\tau_a T$ for all $a\in G$, then we show the existence of μ. First we have the following relation for arbitrary f, g in $L(G)$:

$$T(f*g)=(Tf)*g=f*(Tg).$$

In fact, we have $f*g(x)=\int_G f(xy^{-1})\,g(y)\,dy=\int_G \tau_y f(x)\,g(y)\,dy,$

and so, $T(f*g(x))=\int_G T\tau_y f(x)\,g(y)\,dy=\int_G \tau_y\,Tf(x)\,g(y)\,dy=(Tf)*g(x),$

As G is an abelian group, $T(f*g)=f*(Tg)$.

Hence taking the Fourier transforms of both sides of the above equation, we obtain:

$$(1)\qquad (\hat{T}\hat{f})\hat{g}=\hat{f}(\hat{T}\hat{g})$$

2) A. Zygmund, Trigonometrical series, p. 332 Warszawa (1935).

By H. Cartan's theorem[3], there exists for arbitrary compact set $\hat{K}$ in $\hat{G}$, and a compact neighborhood $\hat{V}$ of $\hat{K}$, a function f in $L^1(G)$ such that

$$\hat{f}(\hat{x})=\begin{cases}1 & \text{for } \hat{x}\in\hat{K}\\ 0 & \text{for } \hat{x}\in\hat{V}^c\end{cases}$$

($\hat{V}^c$ means the complementary set of $\hat{V}$ in $\hat{G}$).

Therefore we may define μ as $\hat{Tf}/\hat{f}=\mu$.
It is obvious that μ is continuous and independent of f from (1). The uniqueness of μ is also clear from the above. Conversely, if there exists such a function μ, then we have $\hat{Tf}=\mu\hat{f}$, so that $\hat{T}(a^{-1},\hat{x})\hat{f}=(a^{-1},\hat{x})\hat{Tf}$. So we have by lemma $T\tau_a f=\tau_a Tf$, and the proof is completed.

Theorem 2. *Let G be a discrete group and $T\in A[L^1(G)]$, then we have*

$$\mu(\hat{x})=\sum_{g\in G}a_g\overline{(g,\hat{x})}$$

where $\alpha=(a_g)$ is an element of $L^1(G)$; i.e.

$$\sum_{g\in G}|a_g|<\infty.$$

And G has a bounded Radon measure $\mathfrak{M}$ such that $\hat{\mathfrak{M}}=\mu$.

Proof. As $\hat{G}$ is compact, there exists an element α_0 in $L^1(G)$ such that $\hat{\alpha}_0(\hat{x})\equiv 1$. Therefore $\hat{T\alpha_0}=\mu$, hence we have

$$\mu(\hat{x})=\hat{T\alpha_0}(\hat{x})=\sum_{g\in G}a_g(g,\hat{x}),\quad T\alpha_0=\alpha\in L^1(G),$$

and this α may be considered as a bounded Radon measure $\mathfrak{M}$.

Theorem 3. *Let G be a compact group and $T\in A[C(G)]$. Then there exists a bounded Radon measure $\mathfrak{M}$ on G such that $Tf=f*\mathfrak{M}$, and this measure $\mathfrak{M}$ is uniquely determined.*

Proof. a). $C(G)$ may be considered as a Banach space with the uniform norm, i.e. $\|f\|_\infty=\sup|f(x)|$, and as $C(G)\subset L^1(G)$ in this case, T has by theorem 1 the factor function μ: $\widehat{Tf}=\mu\hat{f}$.
μ is not only continuous, but also bounded, because $(x,\hat{x})=f_0(x)$ is an element of $C(G)$ and the Fourier transform of $f_0(x)$ vanishes except at the point $\hat{x}$.

Thus we have $|\mu(\hat{x})|\leqq\|\widehat{Tf_0}\|_\infty\leqq\|Tf_0\|_\infty\leqq K\|f_0\|_\infty=K<\infty$, where $K=\|T\|_\infty$.

b). Construction of $\mathfrak{M}$.

Put $S(x,\hat{x})=\overline{\mu(\hat{x})}$ then S maps $\hat{G}\,(\subset C(G))$ in the complex number field. We shall show that a necessary and sufficient condition for S to be extended linearly and continuously all over $C(G)$, is that μ has the property:

$$g(x)=\sum_{i=1}^{p}c_i\overline{(x,\hat{x}_i)}\Longrightarrow\left|\sum_{i=1}^{p}c_i\,\mu(\hat{x}_i)\right|\leqq K\|g\|_\infty\quad\ldots\ldots\ldots\ldots(*)$$

3) cf. R. Godement, Théorèmes taubériens et théorie spectrale. *Ann. Sci. École Norm. Sup.* (3) 64, 119-138 (1947).

In fact $h(x)=g(x^{-1})=\sum_{i=1}^{p} c_i\,(x, \hat{x}_i)$ and $Th(x)=k(x)$, then we have

$$\hat{k}(\hat{x})=\hat{T}\hat{h}(\hat{x})=\mu(\hat{x})\,\hat{h}(\hat{x})=\sum_{i=1}^{p}\mu(\hat{x}_i)\,c_i\,\delta_{\hat{x},\hat{x}_i}.$$

Let us consider the function $k'(x)$ defined by

$$k'(x)=\sum_{i=1}^{p} c_i\,\mu(\hat{x}_i)\,(x, \hat{x}_i),$$

then

$$\hat{k}'(\hat{x})=\sum_{i=1}^{p} c_i\,\mu(\hat{x}_i)\,\delta_{\hat{x},\hat{x}_i}.$$

By the uniqueness of the Fourier transform, we have $k(x)=k'(x)$. Therefore $\left|\sum_{i=1}^{p} c_i\,\mu(\hat{x}_i)\right|=|k(e)|\leqq\|k\|_\infty\leqq K\|h\|_\infty=K\cdot\|g\|_\infty$.
Thus the necessity of the property ($*$) is proved. We shall show now the sufficiency. If $g(x)=\sum_{i=1}^{p} c_i(x, \hat{x}_i)$, then we shall define $S(g(x))$ as $\sum_{i=1}^{p} c_i\,\mu(\hat{x}_i)$.
It may happen that $g(x)=\sum_{i=1}^{p} c_i(x, \hat{x}_i)=\sum_{i=1}^{p} c_i'(x, \hat{x}_i)$, then we have $\sum_{i=1}^{p}(c_i-c'_i)(x, \hat{x}_i)=0$, and $\sum(c_i-\hat{c}_i)\,\mu(\hat{x}_i)=0$ by($*$) so that $\sum c_i\,\mu(\hat{x}_i)=\sum c'_i\,\mu(\hat{x}_i)$. Thus $S(g(x))$ is uniquely determined for such $g(x)$. Now let $f(x)$ be any element of $C(G)$. There exists a sequence $\{g_n\}$ approximating $f(x)$ uniformly such that each g_n is a finite linear combination of characters of G.

Then we define $S(f(x))=\lim_{n\to\infty} S(g_n(x))$.
It is obvious from above that we have $|Sg|\leqq K\cdot\|g\|_\infty$, S is continuous and $S(f(x))$ is independent of the approximating sequence of $f(x)$.
Therefore S defines a bounded Radon measure. Denoting it by $\mathfrak{M}$ we have

$$\mu(\hat{x})=S\overline{(x, \hat{x})}=\int_G\overline{(x, \hat{x})}\,d\mathfrak{M}(x)=\hat{\mathfrak{M}}(\hat{x}),$$

hence

$$\hat{\mathfrak{M}}=\mu, \quad \text{and} \quad Tf=f*\mathfrak{M}.$$

c). Uniqueness of $\mathfrak{M}$.

As we have shown above, we have $\hat{\mathfrak{M}}=\mu$. Hence by the uniqueness of μ and of the Fourier transform, $\mathfrak{M}$ is also uniquely determined.

Corollary. *Let G be a compact abelian group, μ a continuous function on G. Then there exists a bounded linear transformation T of $C(G)$ to itself, with μ as a factor function, if and only if μ has the following property: there exists a positive real number K for any complex numbers $c_1, c_2, \ldots\ldots c_p$, and for any $\hat{x}_1, \hat{x}_2, \ldots\ldots, \hat{x}_p$ in $\hat{G}$ such that*

$$\left|\sum_{i=1}^{p} c_i\,\mu(\hat{x}_i)\right|\leqq K\cdot\sup_{x\in G}\left|\sum_{i=1}^{p} c_i\overline{(x, \hat{x}_i)}\right|.$$

3. Now we shall consider a one-parameter semi-group of bounded linear transformations on a locally compact abelian group.

Theorem 4. *Let G be a locally compact abelian group and $\mathfrak{S}=\{T(\xi), \xi>0\}$ be a semi-group in $A[L^1(G)]$. Then the factor functions $\mu(\hat{x}, \xi)$ satisfy the following equation: $\mu(\hat{x}, \xi_1+\xi_2)=\mu(\hat{x}, \xi_1)\,\mu(\hat{x}, \xi_2)$.*

If $T(\xi)$ is weakly measurable, $\mu(\hat{x}, \xi)$ is L-measurable in ξ for any fixed $\hat{x}$.

Moreover there exist two disjoint sets J_0 and J_1 of $\hat{G}$ whose union is $\hat{G}$, such that $\mu(\hat{x}, \xi)=0$ or $\mu(\hat{x}, \xi)=exp(-\xi\lambda(\hat{x}))$ according as $x \in J_0$ or $\hat{x} \in J_1$. A necessary condition for $\mathfrak{S}$ to have an infinitesimal generator A with dense domain is: $J_0=\phi$.

If $f \in \mathfrak{D}[A]$, then $\widehat{Af}(\hat{x})=-\lambda(\hat{x})\hat{f}(\hat{x})$, and $\lambda(\hat{x})$ is continuous.

Proof. We have for any $f \in L^1(G)$

$$\widehat{T(\xi_1+\xi_2)f}(\hat{x}) = \mu(\hat{x}, \xi_1+\xi_2)\hat{f}(\hat{x}).$$

On the other hand

$$\widehat{T(\xi_1+\xi_2)f}(\hat{x}) = \widehat{T(\xi_1)\,T(\xi_2)f}(\hat{x}) = \widehat{T(\xi_1)}\;\widehat{T(\xi_2)f}(\hat{x})$$
$$= \widehat{T(\xi_1)}\,\mu(\hat{x}, \xi_2)\hat{f}(\hat{x}) = \mu(\hat{x}, \xi_1)\,\mu(\hat{x}, \xi_2)\hat{f}(\hat{x}).$$

Therefore we have $\mu(\hat{x}, \xi_1+\xi_2) = \mu(\hat{x}, \xi_1)\,\mu(\hat{x}, \xi_2)$ ....(i)

Let $T(\xi)$ be weakly measurable, then $\widehat{T(\xi)f}=\mu(\hat{x}, \xi)\hat{f}(\hat{x})$ is L-measurable in ξ for all $f \in L^1(G)$ and fixed $\hat{x}$ from the definition. From the functional equation (i), we have either $\mu(\hat{x}, \xi)\equiv 0$ or $\exp(-\xi\lambda(\hat{x}))$ for fixed $\hat{x}$. Let $J_0=\{\hat{x}\,;\ \mu(\hat{x}, \xi)\equiv 0$ for all $\xi>0\}$.

Since $\mu(\hat{x}, \xi)$ is continuous in $\hat{x}$ for any $\xi>0$, J_0 is a closed set in $\hat{G}$ and $J_1=\hat{G}-J_0$ is open. In order to obtain a necessary condition for $\mathfrak{S}$ to have an infinitesimal generator A, let us prove first the following proposition.

If $T(\xi)\xrightarrow[(\xi\to 0)]{}1$ (strongly), then $\mu(\hat{x}, \xi)f(\hat{x})\xrightarrow[(\xi\to 0)]{}\hat{f}(\hat{x})$ (uniformly) for all $f \in L^1(G)$.

Indeed, we have for $f \in L^1(G)$

$$\|\hat{f}\|_\infty = \sup_{\hat{x}\in\hat{G}}|\hat{f}(\hat{x})| \leqq \int_G |f(x)|\,dx = \|f\|_1.$$

Therefore $T(\xi)\longrightarrow 1$ (strongly) $\Longleftrightarrow T(\xi)f\longrightarrow f$ $(f \in L^1(G))$

$\Longleftrightarrow \|T(\xi)f-f\|_1\to 0 \Rightarrow \|\widehat{T(\xi)f}-\hat{f}\|_\infty\to 0 \Longleftrightarrow \mu(\hat{x}, \xi)\hat{f}(\hat{x})\to\hat{f}(\hat{x})$ (uniformly).

Let A be the infinitesimal generator of $\mathfrak{S}$: i.e.

$$\lim_{\xi\to 0}\frac{1}{\xi}[T(\xi)f-1\cdot f]=Af.$$

If $J_0\neq\phi$, then there exists $\hat{x}\in J_0$ such that $\mu(\hat{x}, \xi)\equiv 0$.

Therefore by H. Cartan's theorem there exists $f \in L^1(G)$, for $\hat{K}\subset J_0$: $\hat{f}(\hat{x})=0$ on a neighborhood of $\hat{K}$, $\hat{f}(\hat{x})\neq 0$ in $\hat{K}$, and $\mu(\hat{x}, \xi)\hat{f}(\hat{x}) \not\to \hat{f}(\hat{x})$ (uniformly), which is a contradiction. Hence, if A has a dense domain, we must have $J^0=\phi$. Let $f \in \mathfrak{D}[A]$, then we have

$$Af(\hat{x})=\lim_{\xi\to 0}\frac{1}{\xi}[\widehat{T(\zeta)f}(\hat{x})-\hat{f}(\hat{x})] = \lim_{\xi\to 1}\frac{1}{\xi}[\exp(-\xi\lambda(\hat{x})-1]\hat{f}(\hat{x})$$
$$=-\lambda(\hat{x})\hat{f}(\hat{x}).$$

From this equation we can show easily the continuity of $\lambda(\hat{x})$.

Remark. If G is a compact abelian group, and $\mathfrak{S}$ is a semi-group in $A[C(G)]$ then the conclusions of theorem 4 can be easily reformulated as the facts on Radon measures $\mathfrak{M}(\xi)$.