On Spectral Theory of Completely Continuous and Some Other Closed Operators

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Introduction

N. Dunford[1)] and A. E. Taylor[2)] respectively have developed spectral theories for linear and closed operators in a Banach space. If we apply those theories to completely continuous operators on a Banach space, we are able to obtain Riesz-Schauder's theory in a more refined form.

It is not a main purpose of this paper to discuss completely continuous operators themselves but to consider closed operators satisfying weaker conditions and to deduce more general results than those of Riesz-Schauder's. In our discussions, Taylor's spectral theory will play the most important role.

Throughout this paper, we denote by X and X^* a complex Banach space and its conjugate space respectively, and the elements of X by x, y, z etc., and those of X^* by f, g, h etc. By linear operators we mean those of bounded ones and those without this assumptions are called additive operators. They are denoted by T and their adjoints by T^* when they exist. We shall consider exclusively the case where the domain $\mathfrak{D}(T)$ and the range $\mathfrak{R}(T)$ of a given operator T are included in the same space, and the identity operator is denoted by I. We mean by nullmanifold of T the set of all elements x such that $x\in\mathfrak{D}(T)$ and $Tx=0$. By $\mathfrak{D}_n(T)(n\geqq 1)$ we mean the set of all $x\in X$ such that $x, Tx,\dots, T^nx\in\mathfrak{D}(T)$. We write $\mathfrak{D}^0(T)=X$ and $\mathfrak{D}_\infty(T)=\bigcap_{n=1}^{\infty}\mathfrak{D}_n(T)$. For any complex number λ and positive integer $n\geqq 1$, $\mathfrak{N}_n[\lambda]$ and $\overline{\mathfrak{N}}_n[\lambda]$ denote the nullmanifolds of operators $(\lambda I-T)^n$(with domain $\mathfrak{D}_n(T)$) and $(\lambda I^*-T^*)^n$(with domain $\mathfrak{D}_n(T^*)$) respectively, and $\mathfrak{N}_n[\lambda]$ and $\overline{\mathfrak{N}}_n[\lambda]$ the

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1) N. Dunford, Trans. Amer. Math. Soc. 54, 185-217 (1943).

2) A. E. Taylor, Acta Math. 84, 189-224 (1951).

ranges of operators $(\lambda I-T)^n$(with domain as above) and $(\lambda I^*-T^*)^n$(with domain as above) respectively. We write simply $\mathfrak{N}_n[0]=\mathfrak{N}_n$, etc. For the details of other definitions and notations such as *spectra*[3] $\sigma(T)$ of a closed operator T, the *resolvent* set[3] $\rho(T)$ of T, projection operator[4] $E_\sigma[T]$ or E_σ corresponding to a *spectral set*[4] σ and its range $X_\sigma[T]$ or X_σ etc., see Taylor[2]. For any subset M of X, we shall denote by $M^\perp$ the set of all linear functionals which vanish identically on M, and similarly for any subset $\overline{M}$ of the conjugate space X^* of X, we shall denote by $\overline{M}^\perp$ the set of all elements x in X such that $f(x)=0$ for any $f\in M$. Let us agree to the following: A scalar multiple of any linear functional f is such that $(\lambda f)(x)=\bar{\lambda}f(x)$ for any $x\in X$ and any complex number λ, in order to adjust it to the inner products in complex Hilbert spaces. Hence the adjoint operator of λT is $\bar{\lambda}T^*$. For any set σ of complex numbers we denote by σ^* its symmetry with respect to the real line.

§ 1. Schauder's theorem for closed operators densely defined on a Banach space

We begin this section with discussion of the spectra of adjoint operators. The proofs of the Lemmas 1.1 and 1.2 below may be obtained by modifying those given in J. Schwarz[5] where the space is assumed to be reflexive.

LEMMA 1.1. *Let T be an additive operator densely defined on X. Then,*

(i) *if T^* has a densely defined inverse, then T also has a densely defined inverse and $(T^*)^{-1}=(T^{-1})^*$;*

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3) Spectra $\sigma(T)$ of a closed operator T is the set of all complex numbers λ such that $\lambda I-T$ has not a linear inverse. The resolvent set $\rho(T)$ of T is its complement.

4) Extended spectra $\sigma_e(T)$ is the set $\sigma(T)$ when T is linear, and otherwise the set $\sigma(T)$ together with ∞.

Spectral sets are defined to be the subsets of $\sigma_e(T)$ both open and closed in $\sigma_e(T)$. To any spectral set σ of T, there corresponds a projection (i.e., linear and idempotent) operator $E_\sigma[T]$(or E_σ). If we put $\sigma'=\sigma_e(T)-\sigma$, σ' is also a spectral set of T and we have $E_\sigma[T]+E_{\sigma'}[T]=I$ and $E_\sigma[T]E_{\sigma'}[T]=0$.

If we denote by $X_\sigma[T]$(or X_σ) the range of $E_\sigma[T]$, and by $\sigma_e(T, X_\sigma)$ the extended spectra of T considered as an operator on X_σ, we have

(i) $E_\sigma(\mathfrak{D}(T))\subset\mathfrak{D}(T)$; (ii) $T(\mathfrak{D}(T)\cap X_\sigma)\subset X_\sigma$; (iii) $\sigma=\sigma_e(T, X_\sigma)$ etc.

If the spectral set σ is bounded we have further conclusions

(iv) $X_\sigma\subset\mathfrak{D}_\infty(T)$; (v) as an operator on X_σ, T is linear.

If λ is an isolated point of the spectra, one point set $\{\lambda\}$ constitutes a spectral set. We write $E_\lambda[T]$, $X_\lambda[T]$ instead of $E_{\{\lambda\}}[T]$, $X_{\{\lambda\}}[T]$.

5) J. Schwarz, Paci. J. Math. 4, 415-458 (1954).

(ii) *if T has a densely defined inverse then T^* also has an inverse and* $(T^*)^{-1}=(T^{-1})^*$.

LEMMA 1.2. *Let T be an additive closed operator densely defined on X. Then,*

(i) *T and T^* have both bounded inverse if either does, and* $(T^*)^{-1}=(T^{-1})^*$;

(ii) *if B is a linear operator,* $(T+B)^*=T^*+B^*$;

(iii) *we have* $\sigma(T^*)=(\sigma(T))^*$.

For any two subsets A, B of X, we denote by $A\oplus B$ the set of all elements of the form $x+y$, where $x\in A$ and $y\in B$.

LEMMA 1.3. *Let A, B and C be three subspaces such that $A\supset B$. Then we have* $A\cap(B\oplus C)=B\oplus(A\cap C)$.

LEMMA 1.4. *Let T be a linear operator from a Banach space X into itself with a closed range. Then we have* $\mathfrak{N}_1=\overline{\mathfrak{N}}_1{}^{+}$ *and* $\overline{\mathfrak{N}}_1=\mathfrak{N}_1{}^{+}$.

PROOF. If we define f on the range of T by $f(Tx)=g(x)$, f is one-valued, additive and continuous by the hypothesis (see L. M. Grave[6]). If we extend the domain of f from $\mathfrak{R}(T)$ over the whole space X, we shall obtain the lemma.

LEMMA 1.5. *Let $\mathfrak{D}(T)$ be dense in X. Then σ is a spectral set of T if and only if σ^* is a spectral set of T^* and we have* $(E_\sigma[T])^*=E_{\sigma^*}[T^*]$.

The second part is obtained by comparing their approximation sums[7].

LEMMA 1.6. *Let $X=M\oplus N$ be a direct decomposition of X into its two closed subspaces M and N, and E be a projection onto M. Then, we can define one-to-one, linear (hence bicontinuous, too) correspondence from $M^*=(EX)^*$ onto E^*X^*. If $\|E\|=1$, this correspondence is isometric.*

PROOF. For any given $f\in M^*$, let us define a functional $\dot{f}$ on X as follows:

$$\dot{f}(x)=\begin{cases} f(x), & \text{if } x\in M \\ 0, & \text{if } x\in N \end{cases}.$$

Then $\dot{f}$ is not only linear by Lorch[8], but also belongs to E^*X^*. Moreover, we have

$$\|f\|=\sup_{\substack{\|x\|\leq 1\\ x\in M}}|f(x)|\leq\|\dot{f}\|=\sup_{\substack{\|x\|\leq 1\\ x\in X}}|\dot{f}(x)|\leq\|E\|\,\|f\|.$$

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6) L.M. Graves, Trans. Amer. Math. Soc. 7, 141-149 (1955) (See corollary of Lemma 5).

7) $E_\sigma[T]$ is defined to be an integral of the resolvent of T along a certain kind of closed curves(see Taylor[2]).

8) E.R. Lorch, Trans. Amer. Math. Soc. 45, 217-334 (1939).

Thus, the mapping $f \longleftrightarrow \dot{f}$ satisfies the conditions stated in the lemma.

THEOREM 1.1. *Let T be a densely defined closed operator from a Banach space X into itself, and σ be a bounded spectral set of T. If for some positive integer n and some complex number λ in σ, $R_n[\lambda]$ is closed, then we have*

(i) $\mathfrak{R}_n[\lambda]=\overline{\mathfrak{N}}_n[\bar{\lambda}]^{\perp}$, *i. e., the equation $y=(\lambda I-T)^n x$ has a solution x, if and only if y is orthogonal to the nullmanifold of $(\bar{\lambda}I^*-T^*)^n$, and*

(ii) $\overline{\mathfrak{R}}_n[\bar{\lambda}]=\mathfrak{N}_n[\lambda]^{\perp}$, *i. e., the equation $g=(\bar{\lambda}I^*-T^*)^n f$ has a solution f if and only if g is orthogonal to the nullmanifold of $(\lambda I-T)^n$.*

PROOF. (i) At first, we shall show that $(\lambda I-T)^n X_\sigma[T]$ is closed. Put $\sigma'=\sigma_e(T)-\sigma$. As σ is bounded, we have, by Taylor[2], $X_\sigma[T]\subset\mathfrak{D}_\infty(T)$ and by Lemma 1.3

$$\mathfrak{D}_n(T)=\mathfrak{D}_n(T)\cap(X_\sigma[T]\oplus X_{\sigma'}[T])$$
$$=X_\sigma[T]\oplus(\mathfrak{D}_n(T)\cap X_\sigma[T]).$$

As $\lambda\,\bar{\in}\,\sigma'$, $(\lambda I-T)^n$ gives one-to-one correspondence from $X_{\sigma'}[T]\cap\mathfrak{D}_n(T)$ onto $X_{\sigma'}[T]$, and hence

$$(\lambda I-T)^n\mathfrak{D}_n(T)$$
$$=(\lambda I-T)^n X_\sigma[T]\oplus(\lambda I-T)^n(\mathfrak{D}_n(T)\cap X_{\sigma'}[T]).$$
$$=(\lambda I-T)^n X_\sigma[T]\oplus X_\sigma[T].$$

If a sequence $\{x_i\}$ from $(\lambda I-T)^n X_\sigma[T]$ converges to x, then $x\in(\lambda I-T)^n\mathfrak{D}^n(T)$. Hence x is decomposed uniquely as follows:

$$x=x'+x'',\ x'\in(\lambda I-T)^n X_\sigma[T] \text{ and } x''\in X_\sigma[T].$$

Since $x'=E_\sigma[T]x=\lim_{i\to\infty}E_\sigma[T]x_i=\lim_{i\to\infty}x_i$, $\{x_i\}$ converges also to x', and $x=x'\in(\lambda I-T)^n X_\sigma[T]$. This shows that $(\lambda I-T)^n X_\sigma[T]$ is closed

As σ is bounded, T is linear on $X_\sigma[T]$, and the adjoint operator of its restriction to $X_\sigma[T]$ can be considered as the restriction of T^* to $X_{\sigma^*}[T^*]$, since by lemma 1.5 and 1.6 $X_{\sigma^*}[T^*]$ is considered to be the adjoint space of $X_\sigma[T]$. Under these preliminaries, we will give the proof of (i). Suppose $y\in\overline{\mathfrak{N}}_n[\lambda]$ and let f be an arbitrary element of $\overline{\mathfrak{N}}_n[\bar{\lambda}]$. Then cleraly $f\in X_{\sigma^*}[T^*]$ and

$$f(E_\sigma[T]y)=(E_{\sigma^*}[T^*]f)(y)=f(y)=0.$$

Thus, from Lemma 2.4, we can find an element $z\in X_\sigma[T]$ such that $E_\sigma[T]y=(\lambda I-T)^n z$. On the other hand, by the fact that $\lambda\,\bar{\in}\,\sigma'$, we can determine a uniquely defined element z' in $X_{\sigma'}[T]\cap\mathfrak{D}_n(T)$ such that $E_{\sigma'}[T]y=(\lambda I-T)^n z'$. Hence we have

$$y=E_\sigma[T]y+E_{\sigma'}[T]y=(\lambda I-T)^n(z+z').$$

This shows that $\overline{\mathfrak{N}_n[\bar{\lambda}]^\perp}\subset\mathfrak{N}_n[\lambda]$. As the converse inclusion is obvious, we have obtained $\overline{\mathfrak{N}_n[\bar{\lambda}]^\perp}=\mathfrak{N}_n[\lambda]$.

The second part of the theorem may be proved similarly.

REMARK. Let $K(x, y)(0\leqq x\leqq 1, 0\leqq y\leqq 1)$ be a complex valued continuous function of both variables. Let $C[0, 1]$ be a Banach space composed of all complex valued continuous functions on the closed interval $[0, 1]$ and with uniform norm. Then the operator K defined on $C[0, 1]$ by the equation:

$$(Kf)(x)=\int_0^1 K(x, y)f(y)dy$$

is completely continuous, and its adjoint K^* is given by

$$(K^*\mu)(y)=\int_0^1 K(x, y)d\mu(x),\ \text{where}\ \mu\in(C[0, 1])^*.$$

But, the transposed equation appearing in Fredholm's theory is

$$f(x)=\phi(x)-\int_0^1 K(x, y)f(x)dx\ .$$

For the compromise we proceed as follows: Let V be a completely continuous operator from a Banach space X into itself. We denote by $|||f|||$ the norm of $f\in X^*$ as a linear functional on X. Let $\overline{X}$ be a Banach space with the following properties:

(i) $\overline{X}$ is an algebraic subspace of X^*, invariant under V^* and total for X;

(ii) $\overline{X}$ is a Banach space by means of its own norm, which we denote by $\|\ \ \|$;

(iii) We have $|||f|||\leqq\|f\|$ for every $f\in\overline{X}$;

(iv) The restriction $\overline{V}$ of V^* to $\overline{X}$ is completely continuous.

It is clear that such is the case for $X=C[0, 1]$, $\overline{X}=$ the totality of $\mu\in C[0, 1]^*$ with density $f(x)\in C[0, 1]$ and $\|\mu\|=\int_0^1|f(x)|dx$, $\overline{V}$ being equal to the above K. Under these assumptions, it is clear that $\sigma(V)$ and $\sigma(\overline{V})$ completely correspond to each other by taking complex conjugate, and for any non-zero spectrum λ of V, index of Riesz of V is equal to that of V. Finally, the projection $E_{\bar{\lambda}}[\overline{V}]$ associated with any non-zero spectrum λ of V is the restriction of $E_\lambda[V^*]=(E_\lambda[V])^*$ to X and the dimension of its range $X_{\bar{\lambda}}[\overline{V}]$ is finite and equal to that of $X_\lambda[V]$ by the totality of $\overline{X}$.

COROLLARY 1. *Any spectral set σ of T such that $E_\sigma[T]$ is completely continuous (or equivalently $X_\sigma[T]$ is finite dimensional), is a finite set not containing ∞, and any λ in σ is a pole of the resolvent of T, the order of which we denote by ν. Then we have $\mathfrak{N}_n[\lambda]=\overline{\mathfrak{N}}_n[\bar{\lambda}]^{\perp}$ and $\overline{\mathfrak{N}}_n[\bar{\lambda}]=\mathfrak{N}_n[\lambda]^{\perp}$ for each n such that $1\leqq n\leqq \nu$.*

COROLLARY 2. *Riesz-Schauder's theory holds also for any regular*[9] *operator densely defined on a (not necessarily reflexive) Banach space and with range in the same space.*

COROLLARY 3. *Riesz-Schauder's theory holds also for any linear operator from a Banach space into itself with the following properties (such is the case, for example, for the linear operator with a completely continuous power):*

(i) *$\sigma(T)$ is a finite or countable set having at most one cluster point 0;*

(ii) *Every projection $E_\lambda(\lambda\neq 0$ and $\lambda\in\sigma(T))$ is completely continuous.*

THEOREM 1.2. *Let T be a densely defined closed operator from a Banach space into itself and σ be a bounded spectral set of T. Then for any $\lambda\in\sigma$ we have*

(i) $\mathfrak{N}_n[\lambda]^{\perp}=\overline{\mathfrak{N}}_n[\bar{\lambda}]$, $n=1, 2, 3,\dots$;

(ii) $\overline{\mathfrak{N}}_n[\lambda]^{\perp}=\mathfrak{N}_n[\lambda]$, $n=1, 2, 3,\dots$.

The proof of this theorem may be obtained from Theorem 1.1.

§ 2. On an isolated point of the spectra

If λ is an isolated point of $\sigma(T)$, we can expand $R_\xi(T)=(\xi I-T)^{-1}$ into the Laurent series in the neighbourhood of $\xi=\lambda$;

$$R_\xi(T)=\sum_{n=0}^{\infty}(\xi-\lambda)^n A_n+\sum_{n=1}^{\infty}(\xi-\lambda)^{-n}B_n.$$

The coefficients A_n, B_n satisfy various relations (see Taylor[2]), and in particular,

$$(T-\lambda I)A_0=B_1-I,$$
$$B_{n+1}=(T-\lambda I)B_n=(T-\lambda I)^n E_\lambda,\ n=1, 2,\dots.$$

THEOREM 2.1. *Let T be a closed operator from a Banach space X into itself, λ an isolated point of the spectra of T and σ' its complement with respect to $\sigma_e(T)$. Then,*

9) When the resolvent set of T is not empty and one of its resolvenls is completely continuous, T is called a *regular* closed operator (cf. J. Schwarz[5]).

(i) *X_λ is identical to the set of all elements $x \in \mathfrak{D}_\infty(T)$ such that*

$$\lim_{n\to\infty} \sqrt[n]{\|(T-\lambda I)^n x\|}=0; \tag{2.1}$$

(ii) *if λ is not a proper value of T and $\bigcap_{n=1}^{\infty} \mathfrak{R}_n[\lambda]$ is closed, then $X_{\sigma'} = \bigcap_{n=1}^{\infty} \mathfrak{R}_n[\lambda]$.*

PROOF. (i) That any element x in X_λ satisfies the relation (3.1) is a result of the fact that the radius of convergence of the series of the negative powers in the Laurent expansion of $R_\xi(T)$ in the neighbourhood of $\xi=\lambda$ is infinite. On the other hand, if $x(\in \mathfrak{D}_\infty(T))$ satisfies the relation (2.1), then, from Taylor[2] and the form of A_0, it follows that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{\|E_{\sigma'}x\|} &= \lim_{n\to\infty} \sqrt[n]{\|A_0^n(T-\lambda I)^n x\|}\\ &\leqq \lim_{n\to\infty} \sqrt[n]{\|A_0^n\|}\cdot \sqrt[n]{\|(T-\lambda I)^n x\|}\\ &\to 0 \qquad as\ n\to\infty\end{aligned}$$

Hence we get $E_{\sigma'}x=0$ and $x\in X_\lambda$.

Before giving the proof of the second part of the theorem let us prove the following lemma.

LEMMA 2.1. *Let T be a linear operator from a Banach space X into itself, such that its spectra $\sigma(T)$ consist of 0 alone and 0 is not a proper value of T. If $\bigcap_{n=1}^{\infty} \mathfrak{R}_n$ is a closed subspace, then it reduces to one elemeut 0.*

Proof. Put $\mathfrak{R}=\bigcap_{n=1}^{\infty} \mathfrak{R}_n$. From the assumption it is clear that T gives a one-to-one correspondence from $\mathfrak{R}$ onto itself. And so, if we denote by $\sigma(T, \mathfrak{R})$ the spectra of T as an operator of $\mathfrak{R}$, 0 is not in $\sigma(T, \mathfrak{R})$. Let T_0 be a restriction of T to $\mathfrak{R}$. Then from the definition of the norms of operators and the assumption,

$$\lim_{n\to\infty} \sqrt[n]{\|(T_0)^n\|} \leqq \lim_{n\to\infty} \sqrt[n]{\|T^n\|}=0.$$

Thus, $\sigma(T, \mathfrak{R})=\sigma(T_0)\subset \{0\}$ and hence $\sigma(T, \mathfrak{R})$ is empty, and so $\mathfrak{R}$ reduces to one element 0.

The proof of the second part ot the theorem. As

$$\begin{aligned}\bigcap_{n=1}^{\infty} \mathfrak{R}_n[\lambda] &= \bigcap_{n=1}^{\infty}\{(\lambda I-T)^n X_\lambda \oplus X_{\sigma'}\}\\ &=\{\bigcap_{n=1}^{\infty}(\lambda I-T)^n X_\lambda\} \oplus X_{\sigma'},\end{aligned}$$

it follows that $\bigcap_{n=1}^{\infty}(\lambda I-T)^n X_\lambda$ is closed. From lemmas 2.1, it is easily seen that $\bigcap_{n=1}^{\infty}(\lambda I-T)^n X_\lambda$ reduces to one element 0. This implies $X_{\sigma'}=\bigcap_{n=1}^{\infty}\mathfrak{N}_n[\lambda]$.

COROLLARY. *If T is an unbounded closed operator from a Banach space into itself aud $\sigma(T)$ is bounded, then it follows that*

(i) *X_∞ is identical to the set of all elements x such that*

$$\lim_{n\to\infty} \sqrt[n]{\|(T-\alpha I)^n x\|}=0 \quad \text{for any } \alpha\in\rho(T);$$

(ii) *if $\mathfrak{D}_\infty(T)$ is closed, then $X_{\sigma(T)}=\mathfrak{D}_\infty(T)$.*

§ 3. Decomposition of the space

Let T be a linear operator from a Banach space X into itself such that its spectra $\sigma(T)$ is a finite or countable set with at most only one cluster point 0. From the form of the Laurent expansion of $R_\xi(T)$ in the neighbourhood of $\xi=\lambda$, we can easily see that $(\xi I-T)^{-1}x$ can be extended regularly to $\xi=\lambda$ if and only if $E_\lambda x=0$.

THEOREM 3.1. *Let T be a linear operator as above. We denote by $X^{(0)}$ the set of all elements x such that $\lim_{n\to\infty}\sqrt[n]{\|T^n x\|}=0$. Then $X^{(0)}$ is identical to the set of all elements x such that for any non-zero spectrum λ of T, $E_\lambda x=0$. Moreover, $X^{(0)}$ is a closed subspace invariant under T, and the spectra $\sigma(T, X^{(0)})$ of T as an operator on X consist of at most 0 only. More generally, for any finite set $\{\mu_1, \dots, \mu_n\}$ of non-zero spectra of T, $X^{(0)}\oplus X\mu_1\oplus\dots\oplus X\mu_n$ is a closed subspace invariant under T, and the spectra $\sigma(T, X^{(0)}\oplus X\mu_1\oplus\dots\oplus X\mu_n)$ of T as an operator on $X^{(0)}\oplus X\mu_1\oplus\dots\oplus X\mu_n$ consists of 0 and $\mu_i(1\leqq i\leqq n)$ or of μ_i $(1\leqq i\leqq n)$ according as $X^{(0)}$ contains non-zero elements or only one element 0.*

PROOF. The first part of the theorem is derived from the equivalence of the following conditions:

(i) $x\in X^{(0)}$;

(ii) The Laurent series $\sum_{n=0}^{\infty}\rho^{-n-1}T^n x$, which is equal to $(\rho I-T)^{-1}x$ for any ρ such that $|\rho|>\|T\|$, is regular in the domain given by an inequality $0<|\rho|\leqq\infty$;

(iii) $E_\lambda x=0$ for any non-zero spectrum λ of T. Clearly, $X^{(0)}$ is a closed subspace invariant under T.

Next, let λ be any complex number different from 0. Then for any $x\in X^{(0)}$, $y=\sum\lambda^{n-1}T^n x$ exists and is contained in $X^{(0)}$ and clearly we have $x=(\lambda I-T)y$.

On the other hand, if for some $x(\neq 0)$ in $X^{(0)}$ we had an equa ity $(\lambda I-T)x=0$, then λ would be a proper value of T and x a proper vector belonging to it and so we should have the contradiction, $x\in X_\lambda$ and $x=E_\lambda x=0$.

Thus, $\lambda I-T$ gives one-to-one correspondence from $X^{(0)}$ onto itself, and we have obtained $\sigma(T, X^{(0)})\subseteq \{0\}$.

Let σ be any finite set consisting of non-zero spectra μ_i $(i=1, 2, \dots, N)$ of T. σ is a spectral set of T, and $X^{(0)}\oplus X\mu_1\oplus \dots \oplus X\mu_n=X^{(0)}\oplus X_\sigma$. Let $\{x_i\}$ be a sequence from $X^{(0)}\oplus X_\sigma$ converging to x. For every i, we can decompose x_i uniquely as follows: $x_i=y_i+E_\sigma x_i$, $y_i\in X^{(0)}$.
Since $\{E_\sigma x_i\}$ tends to $E_\sigma x$, $\{y_i\}$ forms a Cauchy sequence. and so $\{y_i\}$ converges to some element y, which belongs to $X^{(0)}$. Thus we have obtained $x=y+E_\sigma x\in X^{(0)}\oplus X_\sigma$, which shows that $X^{(0)}\oplus X_\sigma$ is closed. (Q. E. D.)

Similar results hold also for any (not necessarily densely defined) closed operator T from X into itself, whose set of spectra is a finite or a countable set without finite cluster points.

From the above theorem, it seems that X is decomposable into the direct sum of its subspaces $X^{(0)}$, $X\lambda_i,\dots$ where $\{\lambda_i\}$ denotes the total spectra of T different from 0. But the situation is not so if the direct decomposition of X is meant as above, viz. that every x in X can be expressed as the sum of a strongly convergent series $x=\sum_{i=0}^{\infty} x_i$, where $x_0\in X^{(0)}$ and $x_i\in X_i$ for $i=1, 2,\dots$. In order that such decomposition holds good we must assume a certain kind of uniform boundedness of $E\lambda_i$'s. For instance, if X is reflexive and if the norms of all finite sums of $E\lambda_i$'s are bounded, then the series $\sum_{i=1}^{\infty} E\lambda_i$ is strongly convergent and $E_0=I-\sum_{i=1}^{\infty} E\lambda_i$ is a projection onto $X^{(0)}$. In such case X is decomposed in the above sense, and we have obtained an example of spectral operators of N. Dunford[10]: T is decomposable into the sum of uniformly convergent series $T=\sum_{i=1}^{\infty} E\lambda_i+N$, were N is a generalized nilpotent commuting with all $E\lambda_i$.

10) N. Dunford, Paci. J. Math. 4, 321-354 (1954).