On Certain Pairs of Mappings of Modular Lattices

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Introduction. Recently M. Hukuhara[1] has treated the theory of endomorphisms of vector spaces in view of applications to Analysis. The purpose of this paper is to show that some of his results can be regarded as theorems on modular lattices. It is a well-known fact, that subspaces of the vector space form a modular lattice; also normal subsystems of certain algebraic systems[2] (e.g. groups) form modular lattices; thus our results generalize theorems of Hukuhara, and our proofs are also simpler than his. The author wishes to express his gratitude to Prof. M. Hukuhara for his kind criticism.

1. All the lattices which we shall consider in the following are supposed to be modular and have the least element 0 and the largest element 1; we shall no more refer to these conditions explicitly.
Let L and L' be such lattices. Then the set $Hom\{L, L'\}$ is defined as the set of all the pairs of mappings $\Phi=\{\varphi, \varphi'; \varphi: L\rightarrow L', \varphi': L'\rightarrow L\}$ which satisfy following axioms:

$$\text{I.}\quad \varphi(\alpha\cup\beta)=\varphi(\alpha)\cup\varphi(\beta)\ ,\qquad \text{I}'.\quad \varphi'(\alpha'\cap\beta')=\varphi'(\alpha')\cap\varphi'(\beta')\ ,$$
$$\text{II.}\quad \varphi'\cdot\varphi(\alpha)=\alpha\cup\varphi'(0')\ ,\qquad \text{II}'.\quad \varphi\cdot\varphi'(\alpha')=\alpha'\cap\varphi(1)\ ,$$

where $\alpha, \beta\in L$, $\alpha', \beta'\in L'$, and $0'$, $1'$ denote the least and the largest elements of L' to distinguish them with $0, 1\in L$.

THEOREM 1. *If $\Phi=\{\varphi, \varphi'\}$ is an element of $Hom\{L, L'\}$, then we have*

(1) $\varphi(\alpha\cap\beta)\leqq\varphi(\alpha)\cap\varphi(\beta)$, (1)$'$ $\varphi'(\alpha'\cup\beta')\geqq\varphi'(\alpha')\cup\varphi'(\beta')$,
(2) $\varphi(0)=0'$, (2)$'$ $\varphi'(1')=1$.

Proofs are evident, because φ and φ' preserve the semi-order of lattices, as is easily seen.

THEOREM 2. (Transitivity) *If $\Phi=\{\varphi, \varphi'\}\in Hom\{L, L'\}$ and $\Psi=\{\varphi, \varphi'\}$ $\in Hom\{L', L''\}$ then we have $\Psi\cdot\Phi=\{\psi\cdot\varphi, \varphi'\cdot\psi'\}\in Hom\{L, L''\}$,*

1) M. Hukuhara: Théorie des endomorphismes de l'espace vectoriel, Jour. Fac. Sci., Univ, Tokyo, I, 7, 129-192 (1954).

2) Cf. G. Birknoff: Lattice theory, Second edition, New York, 1948.

Proof: It suffices to prove that $\Psi\cdot\Phi$ satisfies I and II, then I′ and II′ follows automatically because of the duality. It is clear from the definition of *Hom* $\{L, L'\}$ that $\Psi\cdot\Phi$ satisfies I. On the other hand, we have

$$\begin{aligned}&\varphi'\cdot\psi'\cdot\psi\cdot\varphi(\alpha)=\varphi'(\varphi(\alpha)\cup\psi'(0''))=\varphi'((\varphi(\alpha)\cup\psi'(0''))\cap\varphi(1))\\&=\varphi'(\varphi(\alpha)\cup(\psi'(0'')\cap\varphi(1)))=\varphi'(\varphi(\alpha)\cup\varphi\cdot\varphi'\cdot\psi'(0''))=\varphi'\cdot\varphi(\alpha\cup\varphi'\cdot\psi'(0''))\\&=\alpha\cup\varphi'\cdot\psi'(0'')\cup\varphi'(0')=\alpha\cap\varphi'\cdot\psi'(0''),\end{aligned}$$

which shows that $\Psi\cdot\Phi$ satisfies II ($0''$ denotes the least element of L'').

2. If $L=L'$, *Hom*$\{L, L'\}$ is denoted by *Endo*$\{L\}$, to which we shall restrict our considerations hereafter. Let us put $\sigma_n(\Phi)=\varphi^n(1)$ and $\sigma^n(\Phi)=\varphi'^n(0)$ for $n=0, 1, 2, \ldots\ldots$[3] Then we can define two functions $\mu(\Phi)$. $\nu(\Phi)$ on *Endo*$\{L\}$ by

$$\mu(\Phi)=inf\{n\,;\ \sigma_n(\Phi)=\sigma_{n+1}(\Phi)\}\leqq+\infty,$$
$$\nu(\Phi)=inf\{n\,;\ \sigma^n(\Phi)=\sigma^{n+1}(\Phi)\}\leqq+\infty,$$

For a fixed $\Phi\in$ *Endo*$\{L\}$, $\sigma_n(\Phi)$, $\sigma^n(\Phi)$, $\mu(\Phi)$ and $\nu(\Phi)$ will be simply written as σ_n, σ^n, μ and ν, respectively, if no confusion is to fear.

THEOREM 3.[4] *For $\Phi\in$ Endo$\{L\}$ we have*

(1) *if $\sigma^n=\sigma^{n+1}$, then $\sigma^n\cap\sigma_n=0$,* (1)′ *if $\sigma_n=\sigma_{n+1}$, then $\sigma^n\cup\sigma_n=1$,*
(2) *if $\sigma^n\cap\sigma_n=0$, then $\sigma^n=\sigma^{n+1}$,* (2)′ *if $\sigma^n\cup\sigma_n=1$, then $\sigma_n=\sigma_{n+1}$,*
(3) *if $\sigma^n\cap\sigma_n=0$, then $\sigma_n=\sigma_{n+1}$ or $\mu=\infty$,*
(3)′ *if $\sigma^n\cup\sigma_n=1$, then $\sigma^n=\sigma^{n+1}$ or $\nu=\infty$,*
(4) *if $\nu<\infty$ and $\sigma^n\cap\sigma_n=0$, then $\sigma^n\cup\sigma_n=1$,*
(4)′ *if $\mu<\infty$ and $\sigma^n\cup\sigma_n=1$, then $\sigma^n\cap\sigma_n=0$,*
(5) *if μ, $\nu<\infty$, then the relations $\sigma_n=\sigma_{n+1}$, $\sigma^n=\sigma^{n+1}$, $\sigma^n\cap\sigma_n=0$ and $\sigma^n\cup\sigma_n=1$ are equivalent with each other and therefore $\mu=\nu$.*

Proof is evident in case $n=0$. On the other hand, it is sufficient to prove the theorem in case $n=1$, in order to obtain it for $n\geqq1$, because of the fact that $\Phi^n=\{\varphi^n, \varphi'^n\}\in$ *Endo*$\{L\}$, and $\sigma_n=\sigma_{2n}$ ($\sigma^n=\sigma^{2n}$) is equivalent to $\sigma_n=\sigma^{n+1}$ ($\sigma^n=\sigma^{n+1}$), as is easily seen. As the duality dominates over our theory, we have only to prove (1), (2), (3), (4) and (5) in case $n=1$.[5]

Proof of (1): We haue $\varphi(\sigma^1)=\varphi(\sigma^2)$ from $\sigma^1=\sigma^2$. By II′ it takes the form: $\sigma^1\cap\sigma_1=0$, which proves (1) in case $n=1$.

Proof of (2): From $\sigma^1\cap\sigma_1=0$, we have $\varphi'(\sigma^1\cap\sigma_1)=\sigma^1$, the left member of which becomes $\varphi'(\sigma^1\cap\sigma_1)=\varphi'(\sigma^1)\cap\varphi'(\sigma_1)=\sigma^2\cap(1\cup\sigma^1)=\sigma^2$. Then we have

3) For $n=0$ we put $\varphi^0(\alpha)=\alpha$, $\varphi'^0(\alpha)=\alpha$.
4) Cf. M. Hukuhara (loc. cit.), Chap. II, I.
5) The same reason allows us to abridge our proofs in the rest of the paper.

$\sigma^1 = \sigma^2$.

Proof of (3): Assume that $\infty > \mu > 1$, then $\sigma_\mu = \sigma_{\mu+1}$. Operating φ' to the both members of the last relation, we have $\varphi'(\sigma_\mu) = \varphi'(\sigma_{\mu+1})$, from which follows $\sigma_{\mu-1} \cup \sigma^1 = \sigma_\mu \cup \sigma^1$. Then we get $(\sigma_{\mu-1} \cup \sigma^1) \cap \sigma_1 = (\sigma_\mu \cup \sigma^1) \cap \sigma_1$, from which it follows that $\sigma_{\mu-1} = \sigma_\mu$. This relation contradicts with the definition of μ. Therefore, μ must :orcibly be $\leqq 1$ or $= \infty$.

Thus, the proof of the theorem is completed.

THEOREM 4. *If $\Phi = \{\varphi, \varphi'\} \in Endo\{L\}$ is such that $\Phi = \Phi^2$,* i.e.

$$\text{A.} \quad \varphi(\alpha) = \varphi^2(\alpha), \qquad \text{A}'. \quad \varphi'(\alpha) = \varphi'^2(\alpha)$$

for every $\alpha \in L$, then Φ is of the form:

$$(1) \quad \varphi(\alpha) = (\alpha \cup \sigma^1) \cap \sigma_1, \qquad (1)' \quad \varphi'(\alpha) = (\alpha \cap \sigma_1) \cup \sigma^1$$

and $\mu = \nu \leqq 1$. Especially, if σ_1 or σ^1 is a central element of L, then

$$(2) \quad \varphi(\alpha) = \alpha \cap \sigma_1, \qquad (2)' \quad \varphi'(\alpha) = \alpha \cup \sigma^1.$$

Proof:. From A$'$ we have $\varphi' \cdot \varphi(\alpha) = \varphi'^2 \cdot \varphi(\alpha)$, which is rewritten as $\alpha \cup \sigma^1 = \varphi'(\alpha \cup \sigma^1)$. Operating φ to the both members of the last relation, we finally obtain (1). The rest of the theorem is evident from the definition.

3. Now, let us define the *commutability* of $\Phi = \{\varphi, \varphi'\}$ and $\Psi = \{\psi, \psi'\} \in Endo\{L\}$ by the following conditions B, B$'$:

$$\text{B.} \quad \varphi \cdot \psi(\alpha) = \psi \cdot \varphi(\alpha), \qquad \text{B}'. \quad \varphi' \cdot \psi'(\alpha) = \psi' \varphi'(\alpha).$$

THEOREM 5. *If $\Phi = \{\varphi, \varphi'\}$, $\Psi = \{\psi, \psi'\}$ are commutable with each other and $\mu(\Phi) = \nu(\Phi) = h$, $\mu(\Psi) = \nu(\Psi) = k$ are both finite, then*

$$
\begin{array}{llll}
(1) & \varphi^m \cdot \psi^n(1) \leqq \sigma_m \cap \tau_n, & (1)' & \varphi'^m \cdot \psi'^n(0) \geqq \sigma^m \cup \tau^n, \\
(2) & \varphi^m \cdot \psi^{-n}(0) \leqq \sigma_m \cap \tau^n, & (2)' & \varphi'^m \cdot \psi^n(1) \geqq \sigma^m \cup \tau_n, \\
(3) & \sigma^m = (\sigma^m \cap \tau^k) \cup (\sigma^m \cap \tau_k), & (3)' & \sigma_m = (\sigma_m \cup \tau^k) \cap (\sigma_m \cup \tau_k), \\
(4) & \sigma^m = (\sigma^m \cup \tau^k) \cap (\sigma^m \cup \tau_k), & (4)' & \sigma_m = (\sigma_m \cap \tau^k) \cup (\sigma_m \cap \tau_k), \\
(5) & \varphi^h \cdot \psi^k(1) = \sigma_h \cap \tau_k, & (5)' & \varphi'^h \cdot \psi'^k(0) = \sigma^h \cup \tau^k, \\
(6) & \varphi^h \cdot \psi'^k(0) = \sigma_h \cap \tau^k, & (6)' & \varphi'^h \cdot \psi^k(1) = \sigma^h \cup \tau_k, \\
(7) & \multicolumn{3}{l}{\mu(\Psi \cdot \Phi) = \nu(\cdot \Psi \Phi) = \mu(\Phi \cdot \Psi) = \nu(\Phi \cdot \Psi) \leqq sup\{h, k\}}
\end{array}
$$

where σ_m, σ^m, τ_n and τ^n are $\varphi^m(1)$, $\varphi'^m(0)$, $\psi^n(1)$ and $\psi'^n(0)$, respectively.

Proof of (1): $\sigma_m \cap \tau_n = \varphi^m \cdot \varphi'^m(\tau_n) = \varphi^m \cdot \varphi'^m \cdot \psi^n(1) \geqq \varphi^m \cdot \varphi'^m \cdot \psi^n \cdot \varphi^m(1) = \varphi^m \cdot \varphi'^m \cdot \varphi^m \cdot \psi^n(1) = \varphi^m \cdot \psi^n(1)$.

(2) can be proved in a similar manner as (1).

Proof of (3): We have $(\sigma^m \cap \tau^k) \cup (\sigma^m \cap \tau_k) = \sigma^m \cap (\tau^k \cup (\sigma^m \cap \tau_k))$ by the modular low. On the other hand, $\tau_k \cup (\sigma^m \cap \tau_k) = \psi'^k \cdot \psi^k \cdot \psi^k \psi'^k \varphi'^m(0)$
$= \psi'^k \cdot \psi^{2k} \cdot \varphi'^m(\tau^k) = \psi'^k \cdot \psi^{2k} \cdot \varphi'^m(\tau^{2k}) = \psi'^k \cdot \psi^{2k} \cdot \varphi'^m \psi'^{2k}(0) = \psi'^k \cdot \psi^{2k} \cdot \psi'^{2k} \cdot \varphi'^m(0)$

$=\psi'^k(\sigma^m \wedge \tau_{2k}) = \psi'^k(\sigma^m \wedge \tau_k) = \psi'^k \cdot \psi^k \cdot \psi'^k(\sigma^m) = \psi'^k \cdot \varphi'^m(0) = \varphi'^m(\tau^k) \geqq \varphi'^m(0)$
$=\sigma^m$.

Then we have (3).

Proof of (4): Evidently, $(\sigma^m \vee \tau^k) \wedge (\sigma^m \vee \perp_k) = ((\sigma^m \vee \tau^k) \wedge \tau_k) \vee \sigma^m$. On the other hand, $(\sigma^m \vee \tau^k) \wedge \tau_k = \psi^k \cdot \psi'^k \cdot \psi'^k \cdot \psi^k \cdot \varphi'^m(0) \leq \psi^k \cdot \psi'^{2k} \cdot \psi^k \cdot \varphi'^m \cdot \psi'^k(0)$
$= \psi^k \cdot \psi'^{2k} \cdot \psi^k \psi'^k \cdot \varphi'^m(0) = \psi^k \cdot \psi'^{2k} \varphi'^m(0) = \psi^k \varphi'^m(\tau^{2k}) = \psi^k \cdot \varphi'^m(\tau^k) = \psi^k \cdot \psi'^k \cdot \varphi'^m(0)$
$= \sigma^m \wedge \tau_k$.

It is evident that the reversed inequality holds. Thus (4) is obtained.

Proof of (5): $\sigma_h \wedge \tau_k = \varphi^h(1) \wedge \tau_k = \varphi^h(\tau^k \vee \tau_k) \wedge \tau_k = (\varphi^h \cdot \psi'^k(0) \vee \varphi^h \cdot \psi^k(1))$
$\wedge \tau_k \leq (\varphi^h \cdot \psi'^k(\sigma^h) \vee \varphi^h \cdot \psi^k(1)) \wedge \tau_k = (\varphi^h \cdot \varphi'^h \cdot \psi'^k(0) \vee \varphi^h \cdot \psi^k(1)) \wedge \tau_k \leq (\psi'^k(0)$
$\vee \varphi^h \cdot \psi^k(1)) \wedge \tau_k = \varphi^h \cdot \psi^k(1) \vee (\tau^k \wedge \tau_k) = \varphi^h \cdot \psi^k(1)$.
From (1) we have the reversed inequality and (5) is proved.

Proof of (6): $\sigma_h \wedge \tau^k = \varphi^h \cdot \varphi'^h \cdot \psi'^k(0) = \varphi^h \cdot \varphi'^h \cdot \psi^{k\prime} \cdot \varphi^{2h}(0) = \varphi^h \cdot \psi'^k \cdot \varphi'^h \cdot \varphi^{2h}(0)$
$= \varphi^h \cdot \psi'^k(0)$. This proves (6).

(7) is clear from (5).