Non-Representability of Real General Linear Groups in Higher Dimensional Lorentz Groups

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(Received Dec. 1, 1951)

Let F be any field of characteristic 0 or $p(\neq 2)$. Then the general linear group $GL(n,F)$ of degree n over F (i.e. the group of all non-singular linear transformations of n variables over F) is isomorphic with a subgroup of the group $O(n,n,F)$ of all linear transformations which leave invariant the quadratic form of $2n$ variables $x_1, x_2, \cdots, x_{2n}$:

$$2(x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n}).$$

In fact, for a linear transformation in $GL(n,F)$

$$z_i' = \sum_{j=1}^{n} a_{ij} z_j \qquad (i=1,2,\cdots,n)$$

let us correspond the linear transformation

$$\begin{cases} x_i' = \sum_{j=1}^{n} a_{ij} x_j \\ x_{n+i}' = \sum_{j=1}^{n} \alpha_{ij} x_{n+j} \qquad (i=1,2,\cdots,n), \end{cases}$$

where (α_{ij}) is the transposed matrix of the inverse matrix of (a_{ij}). Then we have a linear transformation in $O(n,n,F)$ and this correspondence is an isomorphism from $GL(n,F)$ into $O(n,n,F)$.[1]

The above correspondence is also an isomorphism from $GL(n,F)$ into the symplectic group $Sp(2n,F)$ of degree $2n$ over F, i.e. the group of all linear transformations which leave invariant the non-degenerated skew-symmetric bilinear form of $(x_1,\cdots,x_{2n}; y_1,\cdots,y_{2n})$: $(x_1 y_{n+1} - x_{n+1} y_1) + \cdots + (x_n y_{2n} - x_{2n} y_n)$. Then the following problem may arise.

"*Let $O(m,n,F)$ be the group of all linear transformations which leave invariant the quadratic form of $x_1, \cdots, x_m, \cdots, x_{m+n}$:*

$$x_1^2 + \cdots + x_m^2 - (x_{m+1}^2 + \cdots + x_{m+n}^2).$$

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1) This remark is due to Mr. A. Hattori. The author wishes to express here his best thanks to him.

When two integres l, n are given, does there exist an integer m such that $GL(l, F)$ is isomorphic to a subgroup of $O(m, n, F)$?" We cannot yet solve this problem. In this note we shall give the following weaker result.

THEOREM. *Let R be the field of all real numbers. If $n \geqq 3$, then for any integer $m>0$ there exists no one-to-one continuous homomorphism from $GL(n, R)$ into $O(m, 1, R)$.*

REMARK. It will be shown that under the assumption of the theorem, even locally isomorphic continuous homomorphism cannot exist from $GL(n, R)$ into $O(m, 1, R)$

Let $T(n, R)$ be the subgroup of $GL(n, R)$ consisting of all the linear transformations with the following type

$$z_i' = a_{ii} z_i + a_{i,i+1} z_{i+1} + \cdots + a_{in} z_n \qquad (i=1, 2, \cdots, n).$$

Then it will be shown that if $n \geqq 3$, there is no locally isomorphic continuous homomorphism from $T(n, R)$ into $O(m, 1, R)$ for any integer $m>0$.

To prove our theorem, we shall study the Lie algebra $\mathfrak{o}(m, 1, R)$ of $O(m, 1, R)$. In the following we shall omit R for brevity, and denote the linear transformations by matrices. Hence in the following we use the following notations.

$GL(n, R)$=the group of all non-singular matrices of degree n over R,

$$O(m, n) = \{A\,;\, A \in GL(m+n, R),\ {}^tAI_{m,n}A = I_{m,n}\},$$

where ${}^tA$ means the transposed matrix of A, and $I_{m,n}$ means

$$I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}, \quad I_m = \text{the unit matrix of degree } m.$$

The Lie algebra of $O(m, n)$ is realized by the matrix Lie algebra $\mathfrak{o}(m, n)$: $\mathfrak{o}(m, n) = \{A\,;\, A$ is a real matrix of degree $m+n$ such that ${}^tAI_{m,n} + I_{m,n}A = 0\}$

$$= \{A\,;\, A = \begin{pmatrix} X & Y \\ {}^tY & Z \end{pmatrix}, \quad {}^tX + X = 0, \quad {}^tZ + Z = 0,$$

where X is real and of degree m, Z is real and of degree n, and Y is an arbitrary real matrix with m rows and n columns.$\}$

$O(m, n)$ may be disconnected, so we write by $O^+(m, n)$ the connected component of the unity in $O(m, n)$.

LEMMA. 1. *If $A \in \mathfrak{o}(m, n)$, then*

$$A^q \in \mathfrak{o}(m, n) \quad \textit{for an odd integer } q,$$

$$A^q=\begin{pmatrix} S & Q \\ -{}^tQ & T \end{pmatrix} \qquad {}^tS=S,\ {}^tT=T \textit{ for an even integer } q.$$

PROOF. Define the mapping $A \to A^*$ as follows:

$$A^*=\begin{pmatrix} {}^tX & -{}^tZ \\ -{}^tY & {}^tW \end{pmatrix} \quad \text{for} \quad A=\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Then for any two matrices A, B of degree $m+n$, we have $(AB)^*=B^*A^*$. because $A^*=I_{m,n}\,{}^tA\,I_{m,n}$. Now let $A \in \mathfrak{o}(m,n)$, then $A^*=-A$. Hence we have

$$(A^q)^*=-A^q \quad \text{for an odd integer } q,$$

$$(A^q)^*=\ A^q \quad \text{for an even integer } q.$$

From these follows easily our conclusion, Q. E. D.

LEMMA. 2. i) *If* $A \in \mathfrak{o}(m,1)$ *and* $A^2=0$, *then* $A=0$.
ii) *If A is a nilpotent matrix in* $\mathfrak{o}(m,1)$, *then* $A^3=0$.

PROOF. i) Let $A=\begin{pmatrix} X & Y \\ {}^tY & Z \end{pmatrix}$, then ${}^tZ+Z=0$ and Z is of degree 1. Hence $Z=0$. Since $A^2=0$, we have

$$X^2+Y{}^tY=0, \qquad XY=0, \qquad {}^tYY=0.$$

Hence we have $Y=0$ and $X^2=0$. As X is skew-symmetric, we have at last $X=0$.

ii) Let $A^k=0$. Take an integer l such that $k \leqq 2(2l+1)$. Then $(A^{2l+1})^2=0$, where $A^{2l+1} \in \mathfrak{o}(m,1)$ by Lemma 1. Hence $A^{2l+1}=0$ by i). If $l>1$, then $2l+1 < 2(2l'+1)$, $(l'=l-1)$. Hence $A^{2l'+1}=0$. Proceeding this method, we have $A^3=0$, Q. E. D.

REMARK. Lemma 2, ii) can be generalized as follows: ii)' *If A is a nilpotent matrix in* $\mathfrak{o}(m,n)$, *then* $A^{2\nu+1}=0$, *where* $\nu=\mathrm{Min}(m,n)$. (For the proof see the appendix.)

Now, as $O^+(m,1)$ is a linear Lie group, it is known[2] that there are two closed Lie subgroups K and H such that

$$O^+(m,1)=KH=HK, \qquad H \frown K=1.$$

K: a maximal compact subgroup.
H: a solvable subgroup which is homeomorphic to an Euclidean space.

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2) K. Iwasawa, On some types of topological groups, Annals of Math., 50 (1949), 507-558, Lemma 3.11, cf. also M. Gotô, Faithful representations of Lie groups, II. Nagoya Math. J., 1 (1950), 91-107, Th. 7, Cor. 1.

An example of such a decomposition is given for $m>1$ as follows: Put

$$\mathfrak{K}=\{A\,;A\in\mathfrak{o}(m,1).\quad A=\begin{pmatrix} S & 0 \\ 0 & 0\end{pmatrix}.\quad {}^tS+S=0\,,\ S \text{ is of degree } m\}$$

$$\mathfrak{R}=\{A\,;A\in\mathfrak{o}(m,1),\quad A=\begin{pmatrix} 0, & -a_1, & -a_2,\cdots, & -a_{m-1}, & c \\ a_1 & & & & -a_1 \\ a_2 & & 0 & & -a_2 \\ \vdots & & & & \vdots \\ a_{m-1} & & & & -a_{m-1} \\ c, & -a_1, & -a_2,\cdots, & -a_{m-1}, & 0 \end{pmatrix}\}$$

Then $\mathfrak{K}$ and $\mathfrak{R}$ are Lie subalgebras of $\mathfrak{o}(m,1)$. Let $\mathfrak{K}$ and $\mathfrak{R}$ generate the Lie subgroups K and H respectively. Then K is a compact subgroup and is isomorphic to the connected component $SO(m)$ of the real orthogonal group of degree m.

Let $\mathfrak{N}=[\mathfrak{R},\mathfrak{R}]$, then a matrix A in $\mathfrak{R}$ belongs to $\mathfrak{N}$ if and only if its $(1,m+1)$-component is 0. We can easily verify that $[\mathfrak{N},\mathfrak{N}]=0$ and if $A\in\mathfrak{N}$ then $A^3=0$. Thus $\mathfrak{R}$ is a solvable Lie subalgebra. Now let us show that H is closed and homeomorphic to an Euclidean space. To prove this, it is sufficient to show that for every matrix $A(\neq 0)$ in $\mathfrak{R}$, $\{\exp tA\,;\ -\infty<t<\infty\}$ is not bounded[3].

Let $A\in\mathfrak{R}$ and $\{\exp tA\,;\ -\infty<t<\infty\}$ be bounded. Then A is a semi-simple matrix (i.e., all elementary divisors are linear) and all proper values of A are purely imaginaly. Hence

$$tr(A^2)\leqq 0.$$

On the other hand, if

$$A=\begin{pmatrix} 0, & -a_1,\cdots, & -a_{m-1}, & c \\ a_1 & & & -a_1 \\ \vdots & & 0 & \vdots \\ a_{m-1} & & & -a_{m-1} \\ c, & -a_1,\cdots, & -a_{m-1}, & 0 \end{pmatrix}$$

then

$$tr(A^2)=2c^2\geqq 0$$

Hence we have $c=0$ and $A\in\mathfrak{N}$. Then $A^3=0$, hence we have $A=0$. Now as is seen easily,

$$\mathfrak{o}(m,1)=\mathfrak{K}+\mathfrak{R}\,,\qquad \mathfrak{K}\frown\mathfrak{R}=0\,.$$

3) M. Gotô, Faithful representations of Lie groups, I, Mathematica Japonicae, 1 (1949), 1-13. Lem. 6.

As K is compact, it follows that[4)]

$$O^+(m,1)=KH=HK, \qquad H\frown K=1$$

and K is a maximal compact subgroup of $O^+(m,1)$.

Now let us call a linear Lie algebra (a Lie algebra of matrices) consisting of only nilpotent matrices an *n-algebra.*

Then we can determine all n-algebras in $\mathfrak{o}(m,1)$ by the following

LEMMA 3. *Let* $\mathfrak{n}$ *be an n-algebra in* $\mathfrak{o}(m,1)$, $(m\geqq 2)$ *then there exists a matrix P in K such that*

$$P^{-1}\cdot\mathfrak{n}\cdot P\subset\mathfrak{N}.$$

PROOF. As is known, n-algebra is a nilpotent Lie algebra[5)]. Hence there exists a matrix $C\neq 0$ in the center of $\mathfrak{n}$. Then we have $C^2\neq 0$, $C^3=0$ by Lemma 2.
Put

$$C=\begin{pmatrix} S & X \\ {}^tX & 0\end{pmatrix}, \qquad C^2=\begin{pmatrix} T & Y \\ -{}^tY & P\end{pmatrix}$$

where $P=(p)$ is a matrix of degree 1 and

$$T=S^2+X{}^tX, \qquad Y=SX, \qquad P={}^tXX>0.$$

$C^4=0$ implies that

$$(1) \qquad T^2=Y{}^tY, \qquad TY+YP=0, \qquad {}^tYY=P^2.$$

Now K-conjugates of C are given by

$$C_1=\begin{pmatrix} {}^tU & 0 \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} S & X \\ {}^tX & 0\end{pmatrix}\cdot\begin{pmatrix} U & 0 \\ 0 & 1\end{pmatrix}=\begin{pmatrix} {}^tUSU & {}^tUX \\ {}^tXU & 0\end{pmatrix},$$

where $U\in SO(m)$. If C is replaced by C_1, then

$$C_1^2=\begin{pmatrix} {}^tUTU & {}^tUY \\ -{}^tYU & P\end{pmatrix}$$

Since $m\geqq 2$, there exists a matrix U in $SO(m)$ such that

$${}^tUY=\begin{pmatrix} x \\ 0 \\ \vdots \\ 0\end{pmatrix} \qquad (x>0)$$

4) cf. K. Iwasawa, loc. cit., see the latter part of the proof of Lemma 3.11.

5) cf. for example, C. Chevalley, Algebraic Lie algebras, Annals of Math., 48 (1947) 91–100, II, Th. 1.

Thus, if necessary, replacing C by its K-conjugate, we can assume that

$$Y=\begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (x>0).$$

Then, by (1) we have

$$T=\begin{pmatrix} -p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad x=p, \quad C^2=\begin{pmatrix} -p, & 0, & \cdots & 0, & p \\ 0, & 0, & \cdots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots & 0, & 0 \\ -p, & 0, & \cdots & 0, & p \end{pmatrix}$$

Let A be any matrix in $\mathfrak{n}$. As C is in the center of $\mathfrak{n}$ we have $C^2A=CAC$ Then as in Lemma 1, we have $(CAC)^*=-CAC$, hence

$$CAC \in \mathfrak{o}(m,1).$$

On the other hand $(CAC)^2=C^4A^2=0$, hence by Lemma 2,

$$C^2A=0$$

Put

$$A=\begin{pmatrix} 0 & -{}^tZ & \alpha \\ Z & S_0 & W \\ \alpha & {}^tW & 0 \end{pmatrix} \qquad (S_0 \text{ is of degree } m-1)$$

then $C^2A=0$ implies that

$$\alpha=0, \qquad W=-Z.$$

Since A is nilpotnet, we have

$$0=tr(A^2)=tr(S_0^2)$$

As S_0 is real skew-symmetric, we have $S_0=0$. Thus $A \in \mathfrak{N}$, Q. E. D.

COROLLARY. *Every n-algebra* $\mathfrak{n}$ *in* $\mathfrak{o}(m,1)$ $(m \geqq 2)$ *is abelian and for every* A, B, C *in* $\mathfrak{n}$ *we have* $ABC=0$.

Any two maximal n-algebras in $\mathfrak{o}(m,1)$ *are K-conjugate with each other.* (They are all $m-1$-dimensional.)

PROOF. We have only to show that for every A, B, C in $\mathfrak{N}$, $ABC=0$. This is verified easily if we remark that

$$AB=\begin{pmatrix} -x, & 0,\cdots,0, & x \\ 0 & & 0 \\ \vdots & 0 & \vdots \\ 0 & & 0 \\ -x, & 0,\cdots,0, & x \end{pmatrix} \qquad \text{Q.E.D.}$$

PROOF OF THE THEOREM. Suppose that there exists a locally isomorphic continuous homomorphism f from $GL(n, R)$ $(n \geqq 3)$ into some $O(m, 1)$. Then $m \geqq 2$. Now f induces a one-to-one homomorphism $\tilde{f}$ from the Lie algebra $\mathfrak{gl}(n, R)$ of $GL(n, R)$ into $\mathfrak{o}(m, 1)$. ($\mathfrak{gl}(n, R)$ is the matrix Lie algebra consisting of all real matrices of degree n).

Let $\mathfrak{r}_0$ be the Lie subalgebra of $\mathfrak{gl}(n, R)$ such that

$$\mathfrak{r}_0=\{A\,;\ A \in \mathfrak{gl}(n, R), \qquad A=\begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots \\ 0 & & a_{nn} \end{pmatrix}$$

Then $\mathfrak{r}_0$ is solvable, and $\mathfrak{n}_0=[\mathfrak{r}_0, \mathfrak{r}_0]$ is not abelian since $n \geqq 3$. On the other hand, $\mathfrak{r}=\tilde{f}(\mathfrak{r}_0)$ is solvable, hence as is known,[6] $\mathfrak{n}=[\mathfrak{r}, \mathfrak{r}]$ is an n-algebra in $\mathfrak{o}(m, 1)$. Hence by the Corollary of Lemma 3, $\mathfrak{n}$ is abelian. This contradicts to $\mathfrak{n}_0 \cong \mathfrak{n}$, Q.E.D.

REMARK. The conjugateness of maximal n-algebras in a linear Lie algebra will perhaps hold for l-algebraic Lie algebras. Yet we cannot show this.

Appendix

PROOF OF LEMMA 2, ii)'. We can assume that $n \leqq m$, because $I_{m,n}$ and $I_{n,m}$ are cogredient to each other in complex number field.

Let A be a nilpotent matrix $(\neq 0)$ in $\mathfrak{o}(m, n)$ and let

$$A^{l-1} \neq 0, \qquad A^l=0. \qquad (l \geqq 2)$$

Case I. l: odd.

$O(m, n)$ contains a subgroup K consisting of matrices

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad U, V \text{ are real orthogonal matrices of degree } m, n \text{ respectively.}$$

Put

$$A=\begin{pmatrix} X & Y \\ {}^tY & Z \end{pmatrix} \qquad A^{l-1}=\begin{pmatrix} S & Q \\ -{}^tQ & T \end{pmatrix} \qquad {}^tS=S\,, \quad {}^tT=T$$

6) Cf. for example C. Chevalley, loc. cit., Cor. of Th. 3.

Then, if necessary, replacing A by its K-conjugate, we may assume that S and T are diagonal. Since $A^{2(l-1)}=0$, we have

$$S^2=Q{}^tQ\,, \quad T^2={}^tQQ\,, \quad SQ+QT=0\,.$$

Hence Q, S and T have the same rank. Let it be r. Then

$$S=\begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad T=\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \tau_1 & & \\ & & & & \ddots & \\ & & & & & \tau_r \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & Q_0 \\ 0 & 0 \end{pmatrix}.$$

Where Q_0 is a non-singular matrix of degree r, and

$$(\sigma_i)^2=Q_0{}^tQ_0\,, \quad (\tau_i)^2={}^tQ_0\,Q_0\,, \quad Q_0^{-1}(\sigma_i)Q_0=-(\tau_i)\,.$$

Hence $(\sigma_1, \cdots, \sigma_r)$ and $-(\tau_1, \cdots, \tau_r)$ coincide up to the order. So if necessary, by replacing A by its K-conjugate and by preserving S and T in diagonal forms, we may assume further that

$$\sigma_i+\tau_i=0 \qquad (i=1, 2, \cdots, r)$$

Then Q_0 is commutative with (σ_i). Now we may assume that

$$(\sigma_i)=\begin{pmatrix} \sigma_1 I_{n_1} & & & \\ & \sigma_2 I_{n_2} & & \\ & & \ddots & \\ & & & \sigma_s I_{n_s} \end{pmatrix} \quad (\sigma_i \neq \sigma_j, \ 1 \leqq i \neq j \leqq s)$$

Then

$$Q_0=\begin{pmatrix} \sigma_1 Q_1 & & & \\ & \sigma_2 Q_2 & & \\ & & \ddots & \\ & & & \sigma_s Q_s \end{pmatrix}$$

Q_i is orthogonal $(i=1, 2, \cdots, s)$.
Put

$$\widetilde{Q}=\begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_s \end{pmatrix}.$$

Then is $\widetilde{Q}$ orthogonal and we have

$$A^{l-1}=\begin{pmatrix} (\sigma_i) & 0 & (\sigma_i)\widetilde{Q} \\ 0 & 0 & 0 \\ (-\sigma_i){}^t\widetilde{Q} & 0 & (-\sigma_i) \end{pmatrix}=\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & {}^t\widetilde{Q} \end{pmatrix}\begin{pmatrix} (\sigma_i) & 0 & (\sigma_i) \\ 0 & 0 & 0 \\ (-\sigma_i) & 0 & (-\sigma_i) \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \widetilde{Q} \end{pmatrix}$$

Replacing A by K-conjugate we may assume

$$A^{l-1}=\begin{pmatrix} (\sigma_i) & 0 & (\sigma_i) \\ 0 & 0 & 0 \\ (-\sigma_i) & 0 & (-\sigma_i) \end{pmatrix}$$

Then $A^{l-1}A=0$ implies that

$$A=\left(\begin{array}{c|cc|c} L & -{}^tY & 0 & L \\ \hline Y & & & Y \\ & & X & \\ 0 & & & 0 \\ \hline -L & {}^tY & 0 & -L \end{array}\right)$$

Where L is skew-symmetric and of degree r, Y is a matrix with $(m-r)$-rows and r-columns, X is in $\mathfrak{o}(m-r, n-r)$.

By a simple calculation we have

$$A^q=\begin{pmatrix} K & W & K \\ Z & X^q & Z \\ -K & -W & -K \end{pmatrix}$$

Hence X is nilpotent.

Now to prove by the induction, let us suppose that ii)$'$ holds for $n'<n$. Then since $n-r<n$, we have

$$X^{2(n-r)+1}=0$$

Then

$$A^{2(n-r)+1}=\begin{pmatrix} K & W & K \\ Z & 0 & Z \\ -K & -W & -K \end{pmatrix}$$

Comparing the both sides of $A^{2(n-r)+1}\cdot A=A\cdot A^{2(n-r)+1}$, we have

$$A^{2(n-r)+1}A=\begin{pmatrix} F & 0 & F \\ 0 & 0 & 0 \\ -F & 0 & -F \end{pmatrix}$$

Then a simple calculation shows that

$$A^{2(n-r)+1+2}=0$$

Hence (using $r\geqq 1$)

$$A^{2n+1}=0, \qquad \text{Q. E. D.}$$

Case II. l: even

Put

$$A^{l-1}=\begin{pmatrix} S & Q \\ {}^tQ & T \end{pmatrix} \qquad {}^tS+S=0, \qquad {}^tT+T=0.$$

Replacing A by its K-conjugate, we may assume that

$$S=\begin{pmatrix} \sigma_1 J_1 & & & & & \\ & \sigma_2 J_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_s J_s & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}, \quad \sigma_i^2 \neq \sigma_j^2 \quad (1 \leqq i \neq j \leqq s)$$

where

$$J_i=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \quad (i=1,2,\cdots,s)$$

and J_i is of degree $2p_i$[7] $(i=1,2,\cdots,s)$.

$$T=\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \tau_1 J_1' & \\ & & & & \ddots \\ & & & & & \tau_t J_s' \end{pmatrix}, \quad \tau_i^2 \neq \tau_j^2 (1 \leqq i \neq j \leqq t)$$

$$J_i'=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}$$

and J_i' is of degree $2q_i (i=1,2,\cdots,t)$. As in case I, we have $s=t$, and we may assume

$$Q=\begin{pmatrix} 0 & Q_0 \\ 0 & 0 \end{pmatrix} \qquad \sigma_i+\tau_i=0, \quad J_i=J_i' \quad (i=1,2,\cdots,s)$$

Then Q_0 is commutative with $\begin{pmatrix} \sigma_1 J_1 & & \\ & \ddots & \\ & & \sigma_s J_s \end{pmatrix}^2$, hence

$$Q_0=\begin{pmatrix} \sigma_1 Q_1 & & \\ & \ddots & \\ & & \sigma_s Q_s \end{pmatrix}$$

Q_i is a orthogonal matrix $(i=1,2,\cdots,s)$, so we have

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7) Notice that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$

$$A^{l-1}=\begin{pmatrix} (\sigma) & 0 & Q_0 \\ 0 & 0 & 0 \\ {}^tQ_0 & 0 & -(\sigma) \end{pmatrix}, \quad \text{where} \quad (\sigma)=\begin{pmatrix} \sigma_1 J_1 & & \\ & \ddots & \\ & & \sigma_s J_s \end{pmatrix}$$

Put $Q_0=(\sigma)\cdot\widetilde{Q}$, then $\widetilde{Q}$ is orthogonal and commutative with (σ). Transform A by

$$\begin{pmatrix} I_{2s} & 0 & 0 \\ 0 & I_{m+1-4s} & 0 \\ 0 & 0 & \widetilde{Q} \end{pmatrix}$$

and apply the similar discussion to A, then we shall arrive at the analogous result to case I, Q. E. D.