On a Generalization of Hilbert's Theory of Ramification

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Introduction

The purpose of this note is to generalize the ramification theory of Hilbert ([5]; completed by Herbrand [3]) of finite normal extensions of discretely valued fields. The theory will be extended here to some type of infinite normal extensions over the ground fields with valuations not necessarily discrete, which we shall call (H_1)-extensions over semi-discretely valued fields. In § 1 we shall give a summary of Hilbert-Herbrand's theory on ramification as well as its generalization to the infinite case. § 2 contains the definition and the propositions on (H)-extensions over discretely valued fields. In § 3 we shall give the definition of semi-discretely valued fields and then discuss the ramification theory of normal extensions (especially (H_1)-extensions) over these fields. As an application of our theory we shall consider in § 4 abelian extensions of finite algebraic number fields and give a new proof for 'norm residue theorem' of Hasse ([2]). In this last section our considerations are connected with a recent investigation of Tamagawa ([7]) on Artin conductor of Weil's L-function ([8]).

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Notations:—Let k_0 be a field with a discrete valuation, $\mathfrak{o}_0$ and $\mathfrak{p}_0$ be its valuation ring and prime ideal, respectively. We shall assume that the residue class field $\mathfrak{k}_0=\mathfrak{o}_0/\mathfrak{p}_0$ be perfect. Let $\tilde{k}_0$ be a maximal separable extension of k_0, and $\tilde{\mathfrak{p}}_0$ be some fixed extension of $\mathfrak{p}_0$ to $\tilde{k}_0$. In the following we shall exclusively consider intermediate fields k of k_0 and $\tilde{k}_0$: $k_0\subseteq k\subseteq\tilde{k}_0$, with valuations $\mathfrak{p}$ induced there by $\tilde{\mathfrak{p}}_0$: $\mathfrak{p}=\tilde{\mathfrak{p}}_0\frown k$, $\mathfrak{o}=\tilde{\mathfrak{o}}_0\frown k$. The residue class field $\mathfrak{k}=\mathfrak{o}/\mathfrak{p}$ of k can be considered as an intermediate field of $\mathfrak{k}_0$ and $\tilde{\mathfrak{k}}_0=\tilde{\mathfrak{o}}_0/\tilde{\mathfrak{p}}_0$ and so it is always perfect. If $\mathfrak{p}$ is discrete, we shall say simply that k is discrete. If k, k' are discrete, the composite field kk' and subfields of k are discrete, too.

Let K/k be a (finite or infinite) normal extension. We shall denote its galois group considered as a compact topological group by $\mathfrak{g}=\mathfrak{g}(K/k)$. As is known (Herbrand [4], II), we can define as usual the splitting group $\mathfrak{g}_Z$ and the inertia group $\mathfrak{g}_T$ of the valuation of K which is determined uniquely by the above conventions. They are closed subgroups of $\mathfrak{g}$ and $\mathfrak{g}_T$ is a normal subgroup of $\mathfrak{g}_Z$. The factor group $\mathfrak{g}_Z/\mathfrak{g}_T$ is isomorphic to the galois group of the residue class extension $\mathfrak{K}/\mathfrak{k}$, which is also a normal extension. If $K\supseteq k'\supseteq k$, $\mathfrak{g}'=\mathfrak{g}(K/k')$, we have

$$\mathfrak{g}'_Z=\mathfrak{g}_Z\frown\mathfrak{g}', \qquad \mathfrak{g}'_T=\mathfrak{g}_T\frown\mathfrak{g}',$$

and if k'/k is also normal, $\bar{\mathfrak{g}}=\mathfrak{g}(k'/k)$, we have

$$\bar{\mathfrak{g}}=\mathfrak{g}_Z\mathfrak{g}'/\mathfrak{g}', \qquad \bar{\mathfrak{g}}_T=\mathfrak{g}_T\mathfrak{g}'/\mathfrak{g}'.$$

§ 1. Summary of known results.[1]

Let k be a discrete field, K/k its finite normal extension with galois group $\mathfrak{g}=\mathfrak{g}(K/k)$. Let $\mathfrak{O}$, $\mathfrak{P}$ be the valuation ring and its prime ideal in K. For $\sigma\in\mathfrak{g}_T$ we define the *ramification function* (or *v-function*) $v(\sigma)=v_{K/k}(\sigma)$ by

(1) $$v(\sigma)=\mathrm{Max}\{i;\ \alpha^\sigma\equiv\alpha \pmod{\mathfrak{P}^{i+1}} \text{ for all } \alpha\in\mathfrak{O}\}.$$

The values $v(\sigma)$ ($\sigma\in\mathfrak{g}_T$) are called the *ramification numbers* (or *v-values*). The v-function satisfies the following conditions:

(i) $v(\sigma)$ ($\sigma\neq1$) are non-negative integers and $v(1)=\infty$,

(ii) $v(\sigma)$ is continuous at $\sigma\neq1$ and if $\lim_{n\to\infty}\sigma_n=1$ then $\lim_{n\to\infty}v(\sigma_n)=\infty$,

1) Cf. Hilbert [5] and Herbrand [3], or Artin [1]. For the treatment of infinite case, see Herbrand [4] II, and Kawada [6].

(iii) $v(\sigma\tau) \geqslant \mathrm{Min}\{v(\sigma), v(\tau)\}$, where the equality sign holds when $v(\sigma) \neq v(\tau)$,

(iv) $v(\tau^{-1}\sigma\tau)=v(\sigma)$ for $\tau \in \mathfrak{g}_Z$.

These imply of course the possibility of definition of ramification groups. Namely we define the *ramification groups* (or *v-groups*) by

$$\mathfrak{g}_{V(v)}=\{\sigma;\ \sigma \in \mathfrak{g}_T,\ v(\sigma) \geqslant v\}, \tag{2}$$

for any non-negative integer (or real number) v. They are normal subgroups of $\mathfrak{g}_Z$. v_1 being the first ramification number, i. e. that which is positive and the smallest, we call $\mathfrak{g}_V=\mathfrak{g}_{V(v_1)}$ the first ramification group. It reduces to the unity group or is an p-group according as the characteristic of the residue class field is 0 or $p \neq 0$.

Following Hasse [2] we associate with the extension K/k a real-valued function $v=\varphi(u)=\varphi_{K/k}(u)$ of non-negative real variable u such that $\varphi(0)=0$ and

$$D^-\varphi(u)=[\mathfrak{g}_T:\mathfrak{g}_{V(v)}], \tag{3}$$

where $v=\varphi(u)$ and $D^-\varphi(u)$ denotes the left derivative of $\varphi(u)$. The *Hasse's function* is continuous, convex and strictly increasing, and its inverse function $u=\varphi^{-1}(v)$ is given by

$$\varphi^{-1}(v)=(1/[\mathfrak{g}_T:1]) \sum_{\sigma \in \mathfrak{g}_T} \mathrm{Min}\{v, v(\sigma)\}. \tag{4}$$

We have $\varphi(u) \geqslant u$ for any u and the equality sign holds for all u if and only if K/k is unramified. For convenience we shall set in the following

$$\varphi(\infty)=\infty, \qquad \varphi^{-1}(\infty)=\infty. \tag{5}$$

By means of Hasse's function we can define another sort of ramification function (*u-function*) $u(\sigma)=u_{K/k}(\sigma)$ by

$$v(\sigma)=\varphi \circ u(\sigma) \qquad \text{for } \sigma \in \mathfrak{g}_T. \tag{6}$$

It satisfies also the conditions (ii), (iii), (iv), but now (i) should be replaced by the following

(i)′ $u(\sigma)$ $(\sigma \neq 1)$ are non-negative real numbers and $u(1)=\infty$.

Definitions of *u-values* and *u-groups* are given in a similar way as above. u-values (v-values) are nothing other than the u-coordinates (v-coordinates) of the vertices of the graph of the Hasse's function. (The origin is counted in the vertices if and only if $D^+\varphi(0) \neq 1$, i. e. $\mathfrak{g}_T \neq \mathfrak{g}_V$.)

Now let $K \supseteq k' \supseteq k$, $\mathfrak{g}'=\mathfrak{g}(K/k')$. Then we have

$$v_{K/k'}(\sigma)=v_{K/k}(\sigma) \qquad \text{for } \sigma \in \mathfrak{g}'_T, \tag{7}$$

and, if k'/k is normal, $\bar{\mathfrak{g}}=\mathfrak{g}(k'/k)=\mathfrak{g}/\mathfrak{g}'$,

(8) $$\varphi_{K/k'} \circ v_{k'/k}(\bar{\sigma}) = \mathrm{Max}\{v_{K/k}(\sigma\tau)\,;\, \tau \in \mathfrak{g}'_T\},$$

where $\bar{\sigma} \in \bar{\mathfrak{g}}_T$ denotes the coset of $\sigma \in \mathfrak{g}_T$ mod $\mathfrak{g}'$. (Hilbert-Herbrand's theorem) It follows immediately that the corresponding Hasse's functions satisfy the following important formula:

(9) $$\varphi_{K/k}(u) = \varphi_{K/k'} \circ \varphi_{k'/k}(u).$$

By means of this formula we can readily extend the definition of Hasse's function $\varphi_{k'/k}$ uniquely for non-normal extensions k'/k. Then the Hilbert-Herbrand's theorem can be formulated in terms of u-functions as follows:

(7)′ $$u_{K/k'}(\sigma) = \varphi_{k'/k} \circ u_{K/k}(\sigma) \quad \text{for } \sigma \in \mathfrak{g}'_T,$$

(8)′ $$u_{k'/k}(\bar{\sigma}) = \mathrm{Max}\{u_{K/k}(\sigma\tau)\,;\, \tau \in \mathfrak{g}'_T\} \quad \text{for } \bar{\sigma} \in \bar{\mathfrak{g}}_T.$$

We shall now extend these considerations to the case where K/k is an infinite normal extension. Let $\{k_\lambda\}$ be the set of all finite normal extension fields of k which are contained in K. The set of indices $\Lambda = \{\lambda\}$ is considered as a directed system, the order being defined such that $\lambda > \mu$ is equivalent to $k_\lambda \supset k_\mu$. Let $\mathfrak{g} = \mathfrak{g}(K/k)$, $\mathfrak{g}_\lambda = \mathfrak{g}(k_\lambda/k)$ and denote by σ_λ the element of $\mathfrak{g}_\lambda$ induced by $\sigma \in \mathfrak{g}$. It follows from (8)′ that if $\lambda > \mu$

$$u_{k_\lambda/k}(\sigma_\lambda) \leqslant u_{k_\mu/k}(\sigma_\mu)$$

and the set of all u-values of k_λ/k contains that of k_μ/k. Hence we can define $u_{K/k}(\sigma)$ as a limit of $u_{k_\lambda/k}(\sigma_\lambda)$ $(\lambda \in \Lambda)$ as follows:

(10) $$u_{K/k}(\sigma) = \lim_{\lambda\to\infty} u_{k_\lambda/k}(\sigma_\lambda).$$

It satisfies (i)′, (iii), (iv), so that we can define two kinds of u-groups as follows:

(11) $$\mathfrak{g}_{U(u)} = \{\sigma\,;\, \sigma \in \mathfrak{g}_T,\ u_{K/k}(\sigma) \geqslant u\},$$

$$\mathfrak{g}_{U(u+0)} = \{\sigma\,;\, \sigma \in \mathfrak{g}_T,\ u_{K/k}(\sigma) > u\}.$$

These are normal subgroups of $\mathfrak{g}_Z$ and $\mathfrak{g}_{U(u)}$ are closed subgroups. $\mathfrak{g}_{U(u)}$ and the closures of $\mathfrak{g}_{U(u+0)}$ are limit groups of those corresponding for k_λ/k $(\lambda \in \Lambda)$. Therefore the closure of $\mathfrak{g}_{U(+0)}$, which is called the first ramification group, reduces to the unity group or is a topological p-group according as the characteristic of the residue class field is 0 or $p \neq 0$. It follows also that the set of all u-values of K/k contains the union of those of k_λ/k $(\lambda \in \Lambda)$, and conversely the former is contained in the left closure of the latter, i.e. in the set of all limit values of decreasing sequences contained in the latter. $\mathfrak{g}_{U(u+0)}$ coincides with $\mathfrak{g}_{U(u)}$ if and only if u is not an u-value of K/k, while the closure of $\mathfrak{g}_{U(u+0)}$ coincides with $\mathfrak{g}_{U(u)}$ if and only if u is not an u-value of k_λ/k for any $\lambda \in \Lambda$. Furthermore, the Hilbert-Herbrand's theorem (7)$'$, (8)$'$ remains true in this case also (k' is assumed to be discrete in (7$_1'$)).[2) 2a)]

As we have from (9)

$$\varphi^{-1}_{k_\lambda/k}(v) \leqslant \varphi^{-1}_{k_\mu/k}(v), \qquad \varphi_{k_\lambda/k}(u) \geqslant \varphi_{k_\mu/k}(u)$$

for $\lambda > \mu$, we can define the Hasse's function of K/k by

$$\varphi^{-1}_{K/k}(v) = \lim_{\lambda\to\infty} \varphi^{-1}_{k_\lambda/k}(v), \qquad \varphi_{K/k}(u) = \lim_{\lambda\to\infty} \varphi_{k_\lambda/k}(u), \tag{12}$$

in which latter case the function value $\varphi(u)$ ($u<\infty$) may take ∞. Let us put

$$u_\infty = \varphi^{-1}(\infty) = \operatorname{Sup}\{\varphi^{-1}(v)\,;\, 0 \leqslant v < \infty\}. \tag{13}$$

Then we can prove easily that $\varphi(u) < \infty$ for $u < u_\infty$ and $\varphi(u) = \infty$ for $u > u_\infty$, and that $v = \varphi(u)$ $(0 \leqslant u \leqslant u_\infty)$ is continuous, convex and strictly increasing. (In case $\varphi(u_\infty) = \infty$, the left continuity of φ at u_∞ means $\lim_{u\to u_\infty-0} \varphi(u) = \infty$. Here again we put $\varphi(\infty) = \infty$.) We can also prove that

$$D^-\varphi(u) = [\mathfrak{g}_T : \mathfrak{g}_{U(u)}], \tag{14}$$

$$D^+\varphi(u) = [\mathfrak{g}_T : \text{closure of } \mathfrak{g}_{U(u+0)}],$$

where $D^+\varphi(u)$ denotes the right derivative of $\varphi(u)$ and we put $D^-\varphi(u) = \infty$ for $u > u_\infty$, $D^+\varphi(n) = \infty$ for $u \geqslant u_\infty$. It follows that the set of u-values of K/k such that $u < u_\infty$ is either finite or countably infinite and that any such u-value is an u-value of k_λ/k for some $\lambda \in \Lambda$. Assume it be finite; then if $u_\infty = \infty$ we have $[\mathfrak{g}_T : 1] < \infty$, which case is essentially the same as the case of a finite normal extension, and if $u_\infty < \infty$ we have $\varphi(u_\infty) < \infty$ and u_∞ is also an u-value of k_λ/k for some $\lambda \in \Lambda$. Assume, on the contrary, the set be countably infinite, then it makes a convergent (or divergent) sequence with limit u_∞ and $\varphi(u_\infty) = \infty$.

Now we define the v-function of K/k by (6). The conditions (iii), (iv) are satisfied again but it should be noted that $v(\sigma)$ $(\sigma \neq 1)$ may be ∞ and $v(\sigma)$ is continuous only at σ such that $v(\sigma) < \infty$. Any v-value of K/k is an v-value of k_λ/k for some $\lambda \in \Lambda$. The u-values ($\leqslant u_\infty$) and the v-values (sometimes except ∞) are in a one-to-one correspondence and give the u-coordinates and v-coordinates of the vertices of the graph of the Hasse's function, respec-

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2) This generalization by means of 'u' is due to Kawada, who called $\mathfrak{g}_{U(u)}$ and the closure of $\mathfrak{g}_{U(u+0)}$ the ramification groups of the first and the second kind, respectively. The detailed treatments on these groups will be found in his paper [6].

2a) In (7)$'$ the Hasse's function $\varphi_{k'/k}$ is a generalized one. But its meaning is clear, since it is easily reduced to the case $[k':k] < \infty$.

tively. $\mathfrak{g}_{V(\infty)}$ coincides with the closure of $\mathfrak{g}_{U(u_\infty+0)}$ or $\mathfrak{g}_{U(u_\infty)}$ according as the set of all v-values is finite and $u_\infty < \infty$ or not. Except for the case of $[\mathfrak{g}_T:1] < \infty$ it is not an open subgroup of $\mathfrak{g}_T$, while all the other v-groups (or the corresponding u-groups with $u \leqslant u_\infty$) are all open subgroups of $\mathfrak{g}_T$. We have also (3) and the following generalization of (4):

$$\varphi_{K/k}^{-1}(v) = \int_{\mathfrak{g}_T} \mathrm{Min}\{v, v(\sigma)\} d\sigma \,,^{3)} \tag{15}$$

Finally, it can be easily seen that (7), (8) and (9) are still valid (k': discrete).

§ 2. (H)-extensions of discretely valued fields.

Let k be a discrete field, K/k its normal extension and $\mathfrak{g}=\mathfrak{g}(K/k)$. In the preceding section we have remarked that $\mathfrak{g}_{V(\infty)}$ does not reduce in general to the unity group and $u_\infty = \varphi^{-1}(\infty)$ is not always equal to ∞. These difficulties show that the consideration of the v-function alone is not sufficient for the general theory of ramification. Hence we shall study here the range of infinite extensions where the ramifications are determined completely only by the v-functions.

DEFINITION. *We shall call K/k an (H)-extension if $\mathfrak{g}_{V(\infty)}$ is equal to the unity group. It is called an (H_1)-extension if $\varphi_{K/k}^{-1}(\infty)=\infty$ and an (H_2)-extension if not.*

Thus K/k is an (H_1)-extension if and only if $\varphi_{K/k}(u) < \infty$ for all $0 \leqslant u < \infty$, and is an (H)-extension if and only if $\varphi_{K/k}(u) < \infty$ for all u-values ($<\infty$) of K/k. If we use the u-functions, the above definition is equivalent to the following

DEFINITION. *K/k is called an (H)-extension if $u_{K/k}(\sigma)$ satisfies the first part of* (ii). *It is called an (H_1)-extension if $u_{K/k}(\sigma)$ satisfies also the second part of* (ii) *and an (H_2)-extension if not.*

For instance, all finite normal extensions are (H_1)-extensions. The v-function for an (H)-extension satisfies the conditions (i), (iii), (iv) and the first part of (ii); of which the second part is satisfied if and only if the v-groups (including $\mathfrak{g}_{V(\infty)}$ when $[\mathfrak{g}_T:1] < \infty$) make a system of neighbourhoods of the unity in $\mathfrak{g}_T$. This last is surely the case for an (H_1)-extension.

PROPOSITION 1. *Let k be a discrete field and K/k its normal extension.*

3) The integrations extended over the inertia group $\mathfrak{g}_T$ are always assumed to be normalized, i.e. they are based on the Haar measure of $\mathfrak{g}_T$ such that $\int_{\mathfrak{g}_T} d\sigma = 1$. It is also noted that the v-function $v(\sigma)$ is a measurable function on $\mathfrak{g}_T$.

Let k' be an intermediate field, which is also discrete, and $\mathfrak{g}=\mathfrak{g}(K/k)$, $\mathfrak{g}'=\mathfrak{g}(K/k')$. *Then we have*

(i) $\mathfrak{g}_Z'=\mathfrak{g}_Z \frown \mathfrak{g}'$, $\quad \mathfrak{g}_T'=\mathfrak{g}_T \frown \mathfrak{g}'$.

(ii) *If K/k is an (H)-extension then K/k' is also an (H)-extension. K/k is an (H_1)-extension if and only if K/k' is an (H_1)-extension.*

(iii) $v_{K/k'}(\sigma)=v_{K/k}(\sigma)$ *for* $\sigma \in \mathfrak{g}_T'$.

Proof. We know already (i) and (iii), whence follows immediately the first half of (ii). To prove the rest part of (ii), we first assume that K/k is completely ramified. Then we have $[k':k]<\infty$. As we have by (9)

$$\varphi_{K/k}^{-1}=\varphi_{k'/k}^{-1}\circ\varphi_{K/k'}^{-1}$$

and as $\varphi_{k'/k}^{-1}(\infty)=\infty$, it follows that $\varphi_{K/k}^{-1}(\infty)=\infty$ is equivalent to $\varphi_{K/k'}^{-1}(\infty)=\infty$. This proves our statement in this special case. Now in the general case, let k_T be the inertia field of K/k. Then by (i) k_Tk' is that of K/k' and we have obviously

$$\varphi_{K/k}^{-1}=\varphi_{K/k_T}^{-1}, \qquad \varphi_{K/k'}^{-1}=\varphi_{K/k_Tk'}^{-1}.$$

Hence we obtain the same result as in the above case, q. e. d.

Remark. If K/k is an (H_2)-extension then K/k' is also an (H_2)-extension. The converse of this, however, is not always true. When k'/k is normal, it is true if and only if $\varphi_{K/k'}(v(k'/k))<\infty$, where $v(k'/k)$ denotes the last v-value of k'/k, i. e. that which is $<\infty$ and the largest. This follows from the fact that the set of all u-values of K/k is the union of that of k'/k and the set of all v-values of K/k'.

Proposition 2. *Let k be a discrete field and K/k its normal extension. Let K' be an intermediate field, which is normal over k, and* $\mathfrak{g}=\mathfrak{g}(K/k)$, $\mathfrak{g}'=\mathfrak{g}(K/K')$, $\bar{\mathfrak{g}}=\mathfrak{g}(K'/k)=\mathfrak{g}/\mathfrak{g}'$. *Then we have*

(i) $\bar{\mathfrak{g}}_Z=\mathfrak{g}_Z\mathfrak{g}'/\mathfrak{g}'$, $\bar{\mathfrak{g}}_T=\mathfrak{g}_T\mathfrak{g}'/\mathfrak{g}'$.

(ii) *If K/k is an (H)-extension K'/k is also an (H)-extension. If K/k is an (H_1)-extension K'/k is also an (H_1)extension.*

(iii) *If K/k is an (H)-extension, we have*

$$v_{K'/k}(\bar{\sigma})=\int_{\mathfrak{g}_T'} v_{K/k}(\sigma\tau)\,d\tau \qquad \text{for } \bar{\mathfrak{g}}_T \ni \bar{\sigma} \neq 1. \tag{16}$$

If K/k is an (H_1)-extension, this formula is valid also for $\bar{\sigma}=1$ in the sense that

$$v_{K'/k}(1)=\infty=\int_{\mathfrak{g}_T'} v(\tau)\,d\tau. \tag{17}$$

For the proof we use the following lemma.

LEMMA 1. *Under the same assumption as in Prop. 2, we have*

$$\varphi_{K'/k}\circ\varphi_{K/k}^{-1}(v)=\int_{\mathfrak{g}'_T}\mathrm{Min}\{v, v(\tau)\}d\tau, \tag{18}$$

$$\varphi_{K'/k}\circ\varphi_{K/k}^{-1}(\infty)=\int_{\mathfrak{g}'_T}v(\tau)\,d\tau. \tag{19}$$

Proof. Let us put

$$\psi_1(v)=\varphi_{K'/k}\circ\varphi_{K/k}^{-1}(v),\qquad \psi_2(v)=\int_{\mathfrak{g}'_T}\mathrm{Min}\{v, v(\tau)\}d\tau.$$

They are both continuous and strictly increasing functions of real variable $0\leqslant v<\infty$ and it is obvious that

$$\psi_1(0)=0=\psi_2(0).$$

Therefore to prove $\psi_1(v)=\psi_2(v)$ it is sufficient to show $D^-\psi_1(v)=D^-\psi_2(v)$. We have by (3), (14) and Hilbert-Herbrand's theorem (8)$'$

$$\begin{aligned} D^-\psi_1(v)&=D^-\varphi_{K'/k}(\varphi_{K/k}^{-1}(v))\cdot D^-\varphi_{K/k}^{-1}(v)\\ &=[\bar{\mathfrak{g}}_T:\bar{\mathfrak{g}}_{U(u)}]/[\mathfrak{g}_T:\mathfrak{g}_{V(v)}]\\ &=[\mathfrak{g}_T\mathfrak{g}'/\mathfrak{g}':\mathfrak{g}_{U(u)}\mathfrak{g}'/\mathfrak{g}']/[\mathfrak{g}_T:\mathfrak{g}_{V(v)}]\\ &=[\mathfrak{g}_T\mathfrak{g}':\mathfrak{g}_{V(v)}\mathfrak{g}']/[\mathfrak{g}_T:\mathfrak{g}_{V(v)}]\\ &=[\mathfrak{g}_T\frown\mathfrak{g}':\mathfrak{g}_{V(v)}\frown\mathfrak{g}']^{-1}, \end{aligned}$$

where $u=\varphi_{K/k}^{-1}(v)$ and so $\mathfrak{g}_{U(u)}=\mathfrak{g}_{V(v)}$. On the other hand, it is clear that

$$D^-\psi_2(v)=[\mathfrak{g}'_T:\mathfrak{g}_{V(v)}\frown\mathfrak{g}']^{-1}.$$

Thus (18) is proved; (19) follows immediately from (18) by the limiting process $v\to\infty$.

Proof of Prop. 2. (i) is known already. Assume that K/k be an (H)-extension and let $\bar{\mathfrak{g}}_T\ni\bar{\sigma}\neq 1$. Then $v_{K/k}(\sigma\tau)=\varphi_{K/k}\circ u_{K/k}(\sigma\tau)$ $(\tau\in\mathfrak{g}'_T)$ are $<\infty$ and we have $u_{K/k}(\sigma\tau)=\varphi_{K/k}^{-1}\circ v_{K/k}(\sigma\tau)$. Since the function $\varphi_{K/k}^{-1}(v)$ is strictly increasing for v-values $(<\infty)$, we have by (8)$'$

$$\begin{aligned} v_{K'/k}(\bar{\sigma})=\varphi_{K'/k}\circ u_{K'/k}(\bar{\sigma})&=\varphi_{K'/k}\,(\mathrm{Max}\{u_{K/k}(\sigma\tau)\,;\,\tau\in\mathfrak{g}'_T\})\\ &=\varphi_{K'/k}\circ\varphi_{K/k}^{-1}\,(\mathrm{Max}\{v_{K/k}(\sigma\tau)\,;\,\tau\in\mathfrak{g}'_T\}). \end{aligned}$$

Choosing the representative σ of the coset $\bar{\sigma}$ such that $v_{K/k}(\sigma)=\mathrm{Max}\{v_{K/k}(\sigma\tau)\,;\,\tau\in\mathfrak{g}'_T\}$, we have $\mathrm{Min}\{v_{K/k}(\sigma), v_{K/k}(\tau)\}=v_{K/k}(\sigma\tau)$. This together with the above lemma proves the first half of (iii). It follows, in particular, $v_{K'/k}(\bar{\sigma})<\infty$ for

$\bar{\mathfrak{g}}_T \ni \bar{\sigma} \neq 1$ and thus the first half of (ii). If K/k is an (H_1)-extension, i. e. $\varphi_{K/k}^{-1}(\infty)=\infty$, we have also by the lemma

$$\int_{\mathfrak{g}'_T} v(\tau)\, d\tau = \varphi_{K'/k} \circ \varphi_{K/k}^{-1}(\infty) = \varphi_{K'/k}(\infty) = \infty\,,$$

and as we have $\varphi_{K/k}^{-1}(v) \leqslant \varphi_{K'/k}^{-1}(v)$, $\varphi_{K'/k}^{-1}(\infty)=\infty$. The remainder of the proposition is proved.

The following lemma gives a partial converse of the proposition.

LEMMA 2. *Let k be a discrete field, $K \supseteq K' \supseteq k$, K/k normal, K/K' finite and K'/k an (H_1)-extension. Then K/k is also an (H_1)-extension.*

Proof. Let $\mathfrak{g}, \mathfrak{g}', \bar{\mathfrak{g}}$ be as above. We have by the Hilbert-Herbrand's theorem (8)$'$

$$\begin{aligned} D^-\varphi_{K/k}(u) &= [\mathfrak{g}_T : \mathfrak{g}_{U(u)}] = [\mathfrak{g}_T\mathfrak{g}' : \mathfrak{g}_{U(u)}\mathfrak{g}'] \cdot [\mathfrak{g}_T \frown \mathfrak{g}' : \mathfrak{g}_{U(u)} \frown \mathfrak{g}'] \\ &\leqslant [\bar{\mathfrak{g}}_T : \bar{\mathfrak{g}}_{U(u)}] \cdot [\mathfrak{g}' : 1] \\ &= D^-\varphi_{K'/k}(u) \cdot [K : K'] < \infty \end{aligned}$$

for any $0 \leqslant u < \infty$, which proves the lemma.

Remark. This lemma does not hold for an (H_2)-extension. In fact if we replace the words (H_1) by (H_2) the lemma is true if and only if none of the u-values of K/k is greater than all of the u-values of K'/k. When $K=K'(\alpha)$ and $\varphi_{K'/k}(u(k(\alpha)/k)) < \infty$, $u(k(\alpha)/k)$ being the last u-value of $k(\alpha)/k$, it can be proved that this condition is fulfilled.

PROPOSITION 3. *Let k, k' be discrete fields.*

(i) *If K/k be an (H_1)-extension then $K'=Kk'$ is an (H_1)-extension field of kk'.*

(ii) *If $K' \supseteq K$, K/k be a mormal extension and K'/k' an (H_1)-extension, then K/k is an (H_1)-extension.*

Proof. By Prop. 1, 2 we can assume without loss of generality that $K'=Kk'$, $k=K \frown k'$. Furthermore, as in the proof of Prop. 1, it is sufficient to consider the completely ramified case. Hence we suppose that K'/k is completely ramified. Then $[k' : k] < \infty$, $[k^* : k] < \infty$, where k^* is the smallest normal extension field of k containing k'. By Prop. 2 and Lem. 2 K/k and K'/k' are (H_1)-extensions if and only if Kk^*/k and Kk^*/k' are (H_1)-extensions, respectively. But by Prop. 1 Kk^*/k is an (H_1)-extension if and only if Kk^*/k' is an (H_1)-extension, q. e. d.

§ 3. (H_1)-extensions of semi-discretely valued fields.

In this section we shall extend the above considerations to the case where the ground field K is not discrete but an intermediate field of an (H_1)-extension of a discrete field k.

DEFINITION. *We call K semi-discrete if there exist two fields k, Ω^* such that $k \subseteq K \subseteq \Omega^*$, k is discrete and Ω^*/k is an (H_1)-extension.*

It follows from Prop. 3 that if K is semi-discrete and k' discrete then Kk' is also semi-discrete. But when K and K' are semi-discrete KK' and subfields of K are not always semi-discrete.

DEFINITION. *Let K be semi-discrete, Ω/K a normal extension. Ω/K is called an (H_1)-extension if there exist two fields k, Ω^* such that $k \subseteq K \subseteq \Omega \subseteq \Omega^*$, k is discrete and Ω^*/k an (H_1)-extension.*

Clearly this gives a generalization of the definition of an (H_1)-extension of a discrete field.

PROPOSITION 4. *Let K be semi-discrete, Ω/K an (H_1)-extension. If $K \subseteq K' \subseteq \Omega$ then K' is semi-discrete and Ω/K' an (H_1)-extension. If moreover K'/K be normal then K'/K is an (H_1)-extension.*

PROPOSITION 5. *Let k be discrete, K semi-discrete. If K/k is a normal extension then it is an (H_1)-extension.*

Prop. 4 is an immediate consequence of the definitions. Prop. 5 follows from the definitions and Prop. 3 (ii).

PROPOSITION 6. *Let K be semi-discrete, Ω/K a finite normal extension. Then Ω/K is an (H_1)-extension.*

Proof. Let k, Ω^* be such that $k \subseteq K \subseteq \Omega^*$, k is discrete and Ω^*/k an (H_1)-extension. Let $\Omega = K(\alpha)$. Then k^* being the smallest normal extension field of k containing $k(\alpha)$, we have $k \subseteq K \subseteq \Omega \subseteq \Omega^* k^*$ and $\Omega^* k^*/k$ is an (H_1)-extension by Lem. 2, q. e. d.

Now we shall define v-function and Hasse's function of an (H_1)-extension Ω/K. Let $k \subseteq K \subseteq \Omega \subseteq \Omega^*$, k be discrete and Ω^*/k an (H_1)-extension. Put $\mathfrak{G} = \mathfrak{g}(\Omega^*/k)$, $\mathfrak{g}^* = \mathfrak{g}(\Omega^*/K)$, $\mathfrak{n} = \mathfrak{g}(\Omega^*/\Omega)$ and $\mathfrak{g} = \mathfrak{g}(\Omega/K) = \mathfrak{g}^*/\mathfrak{n}$.

Assume first that $\Omega = \Omega^*$, i. e. $\mathfrak{g} = \mathfrak{g}^*$. In this case we define $v_{\Omega/K}$ for $\sigma \in \mathfrak{g}_T = \mathfrak{G}_T \frown \mathfrak{g}$ as follows:

$$v_{\Omega/K}(\sigma) = v_{\Omega^*/k}(\sigma) \,. \tag{20}$$

It is obvious by Prop. 1 that this definition is independent of the choice of k. It is also obvious that $v_{\Omega/K}(\sigma)$ satisfies the conditions (i)—(iv) of v-functions. Defining Hasse's function of Ω/K by

$$\varphi^{-1}_{\Omega/K}(v) = \int_{\mathfrak{g}_T} \mathrm{Min}\{v, v_{\Omega/K}(\sigma)\} d\sigma \,, \tag{21}$$

we have by Lem. 1

$$\varphi^{-1}_{\Omega/K}(v) = \varphi_{K/k} \circ \varphi^{-1}_{\Omega/k}(v) \tag{22}$$

and by (17)

(23) $$\varphi^{-1}_{\Omega/K}(\infty)=\mathrm{Sup}\{\varphi^{-1}_{\Omega/K}(v)\,;\ 0\leqslant v<\infty\}=\infty\,.$$

Now in the general case we have $\mathfrak{g}_T=\mathfrak{g}_T^*\mathfrak{n}/\mathfrak{n}$, $\mathfrak{g}_T^*=\mathfrak{G}_T\frown\mathfrak{g}^*$. Denoting a representative of $\sigma\in\mathfrak{g}_T$ by $\sigma^*\in\mathfrak{g}_T^*$, we define v-function by

(24) $$v_{\Omega/K}(\sigma)=\int_{\mathfrak{n}_T}v_{\Omega^*/K}(\sigma^*\tau^*)\,d\tau^*\,.$$

The independence of this definition of the choice of k, Ω^* is proved as follows. Denote for a time by $v[k,\Omega^*]$ the function defined above and by $v[k',\Omega^{*\prime}]$ that defined similarly for the other choice k', $\Omega^{*\prime}$. If $k\subseteq k'$, $\Omega^*=\Omega^{*\prime}$ we have $v[k,\Omega^*]=v[k',\Omega^{*\prime}]$ by Prop. 1. If $k=k'$, $\Omega^*\supseteq\Omega^{*\prime}$, let $\mathfrak{z}=\mathfrak{g}(\Omega^*/\Omega^{*\prime})$, $\mathfrak{G}'=\mathfrak{G}/\mathfrak{z}$, $\mathfrak{g}^{*\prime}=\mathfrak{g}^*/\mathfrak{z}$, $\mathfrak{n}'=\mathfrak{n}/\mathfrak{z}$ and denote by $\sigma^{*\prime}\in\mathfrak{g}_T^{*\prime}$ the coset of $\sigma^*\in\mathfrak{g}_T^*$ mod $\mathfrak{z}$. Making use of Prop. 2 and a Weil's formula[4] we have

$$\begin{aligned}v[k,\Omega^*](\sigma)&=\int_{\mathfrak{n}_T}v_{\Omega^*/k}(\sigma^*\tau^*)\,d\tau^*\\&=\int_{\mathfrak{n}'_T}d\tau^{*\prime}\int_{\mathfrak{z}_T}v_{\Omega^*/k}(\sigma^*\tau^*\rho^*)\,d\rho^*\\&=\int_{\mathfrak{n}'_T}v_{\Omega^{*\prime}/k}(\sigma^{*\prime}\tau^{*\prime})\,d\tau^{*\prime}\\&=v[k,\Omega^{*\prime}](\sigma)\,.\end{aligned}$$

Thus $v[k,\Omega^*]=v[k',\Omega^{*\prime}]$. In the general case we have by what we have proved $v[k,\Omega^*]=v[kk',\Omega^*]=v[kk',\Omega^*\frown\Omega^{*\prime}]$. Similarly we have $v[k',\Omega^{*\prime}]=v[kk',\Omega^*\frown\Omega^{*\prime}]$ and so $v[k,\Omega^*]=v[k',\Omega^{*\prime}]$, which proves our assertion. We have by the arguments used in the proof of Prop. 2

(25) $$v_{\Omega/K}(\sigma)=\varphi^{-1}_{\Omega^*/\Omega}(\mathrm{Max}\{v_{\Omega^*/K}(\sigma^*\tau^*)\,;\ \tau^*\in\mathfrak{n}_T\})\,.$$

This implies that the Hilbert-Herbrand's theorem holds for Ω^*/K and Ω/K, by which we can conclude that $v_{\Omega/K}(\sigma)$ satisfies also the conditions (i)—(iv). On the other hand, if we define the Hasse's function by (21) we have

(26) $$\varphi^{-1}_{\Omega/K}(v)=\varphi^{-1}_{\Omega^*/K}\circ\varphi_{\Omega^*/\Omega}(v)$$

and (23).

These considerations show that the v-function and the Hasse's function thus defined for an (H_1)-extension Ω/K of a semi-discrete field K possess the same properties as those of an (H_1)-extension of a discrete field. Since, in particular, all finite normal extensions of a semi-discrete field are (H_1)-extensions, we

4) See Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1939, p. 45.

can define by a similar method as in § 1 the ramification function (u-function) for any normal extension of a semi-discrete field. In this way, replacing the words 'discrete' by 'semi-discrete', almost all the results in § 1 can be transferred literally to our generalized case. Above all, the following generalizations of the Hilbert-Herbrand's theorem are obtained without any difficulties.

THEOREM 1. *Let K be a semi-discrete field and Ω/K its (H_1)-extension. If K' is an intermediate field of Ω/K, then K' is also semi-discrete, Ω/K' an (H_1)-extension and we have*

$$v_{\Omega/K'}(\sigma)=v_{\Omega/K}(\sigma) \qquad \text{for } \sigma\in\mathfrak{g}_T(\Omega/K'). \tag{27}$$

If moreover K'/K be normal, then K'/K is also an (H_1)-extension and we have

$$\begin{aligned} v_{K'/K}(\bar\sigma)&=\int_{\mathfrak{g}_T(\Omega/K')} v_{\Omega/K}(\sigma\tau)\,d\tau \\ &=\varphi^{-1}_{\Omega/K'}(\mathrm{Max}\{v_{\Omega/K}(\sigma\tau)\,;\ \tau\in\mathfrak{g}_T(\Omega/K')\}) \\ &\qquad\qquad \text{for } \bar\sigma\in\mathfrak{g}_T(K'/K), \end{aligned} \tag{28}$$

where σ is a representative of $\bar\sigma$ in $\mathfrak{g}_T(\Omega/K)$.

THEOREM 2. *Let K be a semi-discrete field and Ω/K its normal extension. If K' is an intermediate field of Ω/K, which is contained in some (H_1)-extension of K and thus semi-discrete, then we have*

$$u_{\Omega/K'}(\sigma)=\varphi_{K'/K}\circ u_{\Omega/K}(\sigma) \qquad \text{for } \sigma\in\mathfrak{g}_T(\Omega/K'). \tag{29}$$

On the other hand, if K' is an intermediate field of Ω/K, which is normal over K, we have

$$u_{K'/K}(\bar\sigma)=\mathrm{Max}\{u_{\Omega/K}(\sigma\tau)\,;\ \tau\in\mathfrak{g}_T(\Omega/K')\} \text{ for } \bar\sigma\in\mathfrak{g}_T(K'/K). \tag{30}$$

In (29) $\varphi_{K'/K}(u)$ is the Hasse's function defined for non-normal extension K'/K. Ω' being an (H_1)-extension field of K containing K', it is defined by the following formula

$$\varphi_{\Omega/K}(u)=\varphi_{\Omega/K'}\circ\varphi_{K'/K}(u) \quad \text{for} \quad 0\leqslant u<\infty, \tag{31}$$

which surely holds in case K'/K is normal.

It might seem more natural to extend the definition of (H_1)-extensions so that Ω/K is an (H_1)-extension if and only if $\varphi^{-1}_{\Omega/K}(\infty)=\infty$. Then the notion of semi-discreteness should be extended by a similar method as above. Thus this same process would be repeated again and again. But on the way of this process we must need Prop. 1, 2, 3 in § 2, in which the words 'discrete' are now replaced by 'semi-discrete'. Unfortunately we could not prove them (except Prop. 2) in general and for the proofs some further assumption seems to be unavoidable.

We conclude this section with some remarks on (H_2)-extensions. Namely, let us consider whether it is possible or not to replace the words (H_1) by (H_2) in the above considerations. Some of the results will remain true, but naturally with suitable modifications. On the other hand, as is seen from the remarks in § 2, Prop. 5, 6 do not hold in this case. Hence the whole theory, including the Hilbert-Herbrand's theorem, would become a complicated one. It should be noted that the case treated in Herbrand [4] I will be contained in this case and give its special (and the simplest) example.

§ 4. The case of algebraic number fields.

We shall consider in this section the case where k_0 is the field of rational numbers R, or its p-adic completion R_p. When we want to apply the above theory a fundamental fact is contained in the following

THEOREM 3.[5] *Let k be a finite algebraic number field (or its $\mathfrak{p}$-adic completion) and K/k its abelian extension. Then K/k is an (H_1)-extension with respect to every discrete valuation $\mathfrak{p}$ in k.*

As the theorem is easily reduced to the local case, it is sufficient to prove it for an $\mathfrak{p}$-adic field $k=k_{\mathfrak{p}}$. Assume namely $k_0=R_p$, $[k:k_0]<\infty$, $\Omega=A_k$ (the greatest abelian extension of k) and let us prove that Ω/k is an (H_1)-extension. Then the theorem will follow from Prop. 2. Let q be the absolute norm of $\mathfrak{p}$, k^* the multiplicative group of k, $U_{\mathfrak{p}}$ the group of units of k and

$$U_{\mathfrak{p}}^{(i)}=\{\alpha\,;\ \alpha\in U_{\mathfrak{p}},\ \alpha\equiv 1 \pmod{\mathfrak{p}^i}\}\,. \tag{32}$$

Let further $\mathfrak{G}=\mathfrak{g}(\Omega/k)$ and $G=G_k$ be the Weil's group of k. The inertia group G_T of G is identified with $\mathfrak{G}_T$.[6]

The local class field theory yields a cononical isomorphism

$$G\cong k^*,\qquad G_T=\mathfrak{G}_T\cong U_{\mathfrak{p}}, \tag{33}$$

and thus a one-to-one correspondence between the finite abelian extension fields k' of k and the closed subgroups $N_{k'}$ of k^* with finite indeces. Then by the 'conductor theorem' we have $N_{k'}\supseteq U_{\mathfrak{p}}^{(i)}$ if and only if $u(k'/k)<i$, $u(k'/k)$ being the last u-value of k'/k. It follows that the canonical isomorphism (33) induces the following isomorphism

$$\mathfrak{G}_{U(u)}\cong U_{\mathfrak{p}}^{(i)}\quad\text{for}\quad i-1<u\leqq i\,. \tag{34}$$

Hence we have for the Hasse's function $\varphi_k(u)=\varphi_{\Omega/k}(u)$

$$D^-\varphi_k(u)=[U_{\mathfrak{p}}:U_{\mathfrak{p}}^{(i)}]=q^{i-1}(q-1)\quad\text{for}\quad i-1<u\leqq i \tag{35}$$

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5) This theorem is due to Tamagawa [7].

6) For the construction and the properties of the Weil's group, see Weil [8].

and thus

$$\varphi_k(i)=q^i-1. \tag{36}$$

Our theorem is therefore proved.

Now let K/k a finite normal extension, $K=K_{\mathfrak{P}}$. Let $\mathfrak{G}=\mathfrak{g}(A_K/k)$, $\mathfrak{G}'=\mathfrak{g}(A_K/K)$, $\mathfrak{g}=\mathfrak{g}(K/k)$ and $G=G_{K,k}$ be the Weil's group. Then we can consider G as an extension group of $G_K=K^*$, the multiplicutive group of K, by $\mathfrak{g}$:

$$G\supseteq K^*, \qquad G/K^*\cong\mathfrak{g}. \tag{37}$$

G_T being the inertia group of G, we have

$$G_T\frown K^*=U_{\mathfrak{P}}, \qquad G_T/U_{\mathfrak{P}}\cong\mathfrak{g}_T. \tag{38}$$

By Prop. 1 A_K/k is an (H_1)-extension. We have by the Hilbert-Herbrand's theorem ((8) or (28)) for $\varphi_K(i-1)<j\leqslant\varphi_K(i)$

$$\mathfrak{G}'_{U(i)}=\mathfrak{G}'_{V(j)}=\mathfrak{G}_{V(j)}\frown\mathfrak{G}', \qquad \mathfrak{g}_{V(i)}=\mathfrak{G}_{V(j)}\mathfrak{G}'/\mathfrak{G}'. \tag{39}$$

This implies that $\mathfrak{G}_{V(j)}$ for $\varphi_K(i-1)<j\leqslant\varphi_K(i)$ are all equal. Denoting by G_{V_i} the corresponding subgroup of G_T, we have

$$G_{V_i}\frown K^*=U_{\mathfrak{P}}^{(i)}, \qquad G_{V_i}/U_{\mathfrak{P}}^{(i)}\cong\mathfrak{g}_{V(i)}. \tag{40}$$

If we put for $s\in G_T=\mathfrak{G}_T$

$$w_{K,k}(s)=\mathrm{Max}\{i\,;\, s\in G_{V_i}\}, \tag{41}$$

we have

$$v_{A_K/k}(s)=\varphi_K\circ w_{K,k}(s), \qquad w_{K,k}(s)=\varphi_{K/k}\circ u_{A_K/k}(s). \tag{42}$$

The following theorem gives an analogy of the Hilbert-Herbrand's theorem for the Weil's groups.

THEOREM 4.[7] *Let k be an $\mathfrak{p}$-adic number field and K/k its finite normal extension. Let $k\subseteq k'\subseteq K$, $G=G_{K,k}$, $G'=G_{K,k'}$. Then we have*

$$G'\subseteq G,\ G'_T=G_T\frown G' \tag{43}$$

and

$$w_{K,k'}(s)=w_{K,k}(s) \text{ for } s\in G'_T. \tag{44}$$

If moreover k'/k is normal, $\overline{G}=G_{k',k}$, we have

$$\overline{G}=G/G'^c,\ \overline{G}_T=G_TG'^c/G'^c, \tag{45}$$

where G'^c denotes the closure of the commutator subgroup of G', and

$$\varphi_{K/k'}\circ w_{k',k}(\bar{s})=\mathrm{Max}\{w_{K,k}(st)\,;\, t\in G_T\frown G'^c\}, \tag{46}$$

7) This theorem is due to Tamagawa [7]. His original proof is based on the norm residue theorem of Hasse.

where $\bar{s} \in \overline{G}_T$ *denotes the coset of* $s \in G_T$ *mod* G'^c.

Proof. (43) and (45) are known ([8]). (44) follows immediately from (27) (or (7)) and (42). (46) follows from (28), (31) and (42) as follows.[8)]

$$\varphi_{A_K/A_{k'}} \circ v_{A_{k'}/k}(\bar{s}) = \mathrm{Max}\{v_{A_K/k}(st)\,;\ t \in \mathfrak{g}_T(A_K/A_{k'})\}\,.$$

Hence

$$\varphi_K \circ \varphi_{K/k'} \circ w_{k',k}(\bar{s}) = \varphi_{A_K/A_{k'}} \circ \varphi_{k'} \circ w_{k',k}(\bar{s})$$
$$= \mathrm{Max}\{\varphi_K \circ w_{K,k}(st)\,;\ t \in G_T \frown G'^c\}\,.$$

As the function φ_K is strictly increasing, we have (46), q. e. d.

It follows from (46) that for $\varphi_{K/k'}(i-1) < j \leqslant \varphi_{K'k'}(i)$

$$\overline{G}_{V_i} = G_{V_j}\, G'^c/G'^c\,, \tag{46$'$}$$

and, in particular, if $k'=k$, we have for $\varphi_{K/k}(i-1) < j \leqslant \varphi_{K/k}(i)$

$$U_{\mathfrak{p}}^{(i)} = G_{V_j}\, G^c/G^c\,. \tag{47}$$

Since this identification is given by the transference (Verlagerung), we have by (40)

$$N_{K/k}\, U_{\mathfrak{P}}^{(j)} \subseteq U_{\mathfrak{p}}^{(i)}\,,\ [U_{\mathfrak{p}}^{(i)} : N_{K/k}\, U_{\mathfrak{P}}^{(j)}] \leqslant [\mathfrak{g}_{V^{(j)}} : 1]\,. \tag{48}$$

This is of course equivalent to the 'norm residue theorem' of Hasse ([2]).

In this way the 'norm residue theorem' for arbitrary finite normal extensions is proved as a corollary of Th. 5. But it should be noted that for the proof of the latter we have assumed the former theorem in its abelian case, i. e. the 'conductor theorem' of the class field theory.

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8) It can be proved also from (8)$'$, (9) and (42), and thus for the proofs of Th. 3, 4 we need not use the results of § 3. This remark is due to Kawada [6].