公理Ａ系に対するPoissonの法則

平　田　雅　樹

Poisson Law for Axiom A Diffeomorphisms

MASAKI HIRATA

Department of Pure and Applied Sciences
University of Tokyo
Komaba, Meguro-ku, Tokyo 153, Japan

0 Introduction.

Recently, Ya.Sinai studied the distribution of spacings between nearest energy levels of a quantum particle on the two-dimensional compact Riemannian surfaces, and he shows the limiting Poisson distribution for spacings of quasi-classical eigenvalues for the quantum kicked rotator model ([S.I], [S.II]). The essential point of the proof is to reduce the problem to studying some ergodic transformation on $\mathbf{T}^2$. He considers the distribution of the visiting times of the trajectory to a certain horizontal strip, and obtained the limiting Poisson point process as the width of the strip tends to zero. And he points out that the way of appearance of the above Poisson point process is quite different from that in the usual situations in probability theory. This fact is very interesting from the ergodic theoretical view point. Inspired by it, we will consider the following problem.

Let X be a compact metric space, f a continuous map on X, and μ an f-invariant probability measure on X. Fix a point $z \in X$ and take its ϵ-neighborhoods $U_\epsilon(z)$. As a probability measure on $U_\epsilon(z)$, we will take the restriction of μ to $U_\epsilon(z)$, i.e.

$$\mu_\epsilon \equiv \frac{\mu \mid_{U_\epsilon(z)}}{\mu(U_\epsilon(z))}.$$

Denote the k-th return time of a point x from $U_\epsilon(z)$ to $U_\epsilon(z)$ by $T_{\epsilon,f}{}^{(k)}(x)$. Then, we want to know what is the limit distribution of the normalized k-th return times

$$\frac{T_{\epsilon,f}{}^{(k)}}{E_{\mu_\epsilon}(T_{\epsilon,f}{}^{(1)})}$$

as $\epsilon \to 0$.

Next let us introduce a counting measure ($\mathbf{N}^+$-valued Radon measure on $\mathbf{R}^+$), $Y_\epsilon(x)$, defined by

$$Y_\epsilon(x) = \sum_{k=1}^\infty \delta_{c_\epsilon \cdot T_{\epsilon,f}{}^{(k)}(x)},$$

where $c_\epsilon = 1/E_{\mu_\epsilon}(T_{\epsilon,f}{}^{(1)})$ and δ_p is the Dirac δ-measure at $p \in \mathbf{R}^+$. Then, $Y_\epsilon(\cdot)$ is a point process on $\mathbf{R}^+$. We will call it the normalized return time process. And the above problem can be considered as follows: what is the limit of the sequence of the normalized return time processes $\{Y_\epsilon\}_\epsilon$ as $\epsilon \to 0$?

It is expected that the limit distribution of the normalized first return time is the exponential distribution and that the limit distribution of the normalized return time process is the law of Poisson point process if the system (X, f, μ) is "chaotic" in some sense (for example, ergodic, mixing, etc.). Let us say that the Poisson law holds if it is true.

In this paper the author considers the above problem for the typical "chaotic" system, namely, for the Axiom A system, and shows the Poisson law for it.

Let M be a compact C^∞ Riemannian manifold and $f : M \to M$ be an Axiom A diffeomorphism. We denote its non-wandering set by $\Omega = \Omega(f)$ and assume that $f|_\Omega$ is mixing. Take a Lipschitz continuous function $u : \Omega \to \mathbf{R}$ and denote the (unique) Gibbs measure (= the equilibrium state) for u by $\mu = \mu_u$. Fix a point $z \in \Omega$, and take its ϵ-neighborhoods $\{U_\epsilon(z)\}_\epsilon$. The main theorem is the following:

THEOREM. *For $\mu - a.e.$ $z \in \Omega$, the sequence of the normalized return time processes converges to the Poisson point process in finite dimensional distribution: for any disjoint Borel sets $B_1, \cdots, B_n \in \mathcal{B}(\mathbf{R}^+)$, and any non-negative integers $k_1, \cdots, k_n$,*

$$\lim_{\epsilon \to 0} \mu_\epsilon(Y_\epsilon(B_1) = k_1, \cdots, Y_\epsilon(B_n) = k_n) = \prod_{i=1}^n \frac{\ell(B_i)^{k_i}}{k_i!} e^{-\ell(B_i)},$$

where ℓ is the Lebesgue measure.

It should be emphasized that the main theorem holds for $\mu - a.e.$ z, but not for every point.

2

COUNTER-EXAMPLE. *For a periodic point $z \in \Omega$ with period m, the limit distribution of the normalized first return time is the linear combination of the delta-distribution and the exponential distribution. Precisely,*

$$\lim_{\epsilon \to 0} \mu_\epsilon(c_\epsilon T_\epsilon^{(1)} < t) = 1 - \rho_z + \rho_z(1 - e^{-\rho_z t})$$

where $\rho_z = 1 - exp\{u(z) + u(f(z)) + \cdots + u(f^{m-1}(z))\}$.

The main theorem holds only if the eigenvalue of the operator $\tilde{\mathcal{L}}_N$ defined in Section 1 which goes to 1 as $N \to \infty$ is unique, or more precisely, if the number of the eigenvalues of $\tilde{\mathcal{L}}_N$ contained in a small neighborhood of 1 is only one for large N. Otherwise, the limit of the normalized return time process is expected to obey a compound Poisson law.

The proof of the theorem will be given in Section 5. In Section 1, we introduce the singularly perturbed Ruelle- Perron- Frobenius operator which is the main tool to prove the theorem. We study its basic properties in Section 2. The relation between the eigenvalues of that operator and the poles of the Ruelle-Artin-Mazur zeta function is studied in Section 3, which plays the most important role in the proof. In Section 4, we show the Poisson law for symbolic dynamics which is the essential part of the proof of the above theorem. The main theorem can be proved by approximating ϵ-neighborhood by a finite union of cylinder sets associated with a Markov partition of Ω in Section 5. The counter-example will be shown in Section 6.

After the author finished writing the present paper, he found B.Pitskel's paper entitled " Poisson limit law for Markov chains " (*Ergod. Th. and Dynam. Sys.* **11** (1991), 501-513). In that paper, he proved the Poisson law for the recurrence to cylinder sets for a mixing stationary Markov chains with finite state space.

1 Set up.

Let $J = \{1, \cdots, r\}$ be a finite set and $A = (A_{ij})_{i,j=1,\cdots,r}$ be an irreducible $r \times r$ matrix with entries 0 or 1. Define the space Σ_A^+ by

$$\Sigma_A^+ = \{x = \{x_i\}_{i=0}^\infty \in J^{\mathbb{N}}; A_{x_i x_{i+1}} = 1 \quad \text{for all } i \in \mathbb{N}\}.$$

For a fixed $0 < \theta < 1$, we can define the metric $d = d_\theta$ on Σ_A^+ by

$$d_\theta(x, y) = \theta^n \quad \text{if } x_i = y_i \text{ for } i = 0, \cdots, n-1 \quad \text{and } x_n \neq y_n.$$

We denote the shift on Σ_A^+ by σ :

$$(\sigma x)_i = x_{i+1}.$$

Let $\mathcal{F}_\theta(\Sigma_A^+)$ be the totality of real valued Lipschitz continuous functions on Σ_A^+ (with respect to d_θ) and define the norm on $\mathcal{F}_\theta(\Sigma_A^+)$ by

$$\|\|g\|\|_\theta = \|g\|_\infty + \|g\|_\theta$$

where $\|g\|_\infty$ is the supremum norm and $\|g\|_\theta$ is the Lipschitz constant for g:

$$\|g\|_\theta = \sup \left\{ \frac{|g(x) - g(y)|}{d_\theta(x,y)} ; x \neq y \right\}.$$

For $u \in \mathcal{F}_\theta(\Sigma_A^+)$, we define the Ruelle-Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_u : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$ by

$$(1.1) \qquad \mathcal{L}_u f(x) = \sum_{\sigma y = x} e^{u(y)} f(y).$$

We assume that

$$(1.2) \qquad \mathcal{L}_u 1 = 1.$$

If not, we can obtain (1.2) by replacing u by $u' = u + \log h - \log(h \circ \sigma) - P(u)$ where $P(u)$ is the topological pressure for u and h is the eigenfunction of $\mathcal{L}_u$ corresponding to the maximal eigenvalue $e^{P(u)}$. Hence we may assume (1.2) without loss of generality. So we make this assumption throughout the paper.

Let $\mu = \mu_u$ be the Gibbs measure for u. In our situation, the Gibbs measure coincides with the equilibrium state. Hence μ satisfies the following equality

$$(1.3) \qquad P(u) = h_\mu(\sigma) + \int u \, d\mu = 0$$

where $h_\mu(\sigma)$ is the metrical entropy. We remark that $P(u) = 0$ follows from our assumption (1.2) and that $h_\mu(\sigma) > 0$.

4

Now we fix a point $z \in \Sigma_A^+$, and denote cylinder sets by

$$[z]_N \equiv [z_0 z_1 \ldots z_{N-1}] = \{y \in \Sigma_A^+; y_i = z_i, i = 0, \ldots N-1\}, \qquad N = 1, 2, \cdots.$$

Since the measure μ is σ-invariant, we can define a first return time from $[z]_N$ to $[z]_N$, denoted by $T_N(\mathrm{x})$, for $\mu - a.e.$ $x \in [z]_N$ for each N:

$$T_N(x) = \inf\{i \in \mathsf{N}^+; \sigma^i x \in [z]_N\}.$$

We introduce the following singularly perturbed Ruelle-Perron-Frobenius operator $\widetilde{\mathcal{L}}_N : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$:

(1.4)
$$\begin{aligned}
\widetilde{\mathcal{L}}_N f(x) &= \mathcal{L}(1_{[z]_N^c} \cdot f)(x) \\
&= \sum_{\sigma y = x} e^{u(y)} 1_{[z]_N^c}(y) f(y),
\end{aligned}$$

where $[z]_N^c$ denotes the complement of the set $[z]_N$ and $1_{[z]_N^c}$ is its indicator function.

LEMMA 1.1.

$$\mu(\{x \in [z]_N; T_N(x) = i\}) = \int \widetilde{\mathcal{L}}_N^{i-1}(\mathcal{L}(1_{[z]_N}))(x) 1_{[z]_N}(x) \mu(dx).$$

PROOF: Recall that

(1.5)
$$\int \mathcal{L} f \cdot g \, d\mu = \int f \cdot (g \circ \sigma) d\mu$$

holds for $f, g \in \mathcal{F}_\theta(\Sigma_A^+)$. Using this fact, we can immediately obtain the above lemma. ∎

2 Basic properties of $\widetilde{\mathcal{L}}_N$.

The properties of spectrum of analytically perturbed Ruelle operator is well known by the results of Ruelle and Pollicott ([R.I],[P.]). But we can not apply their results directly to $\widetilde{\mathcal{L}}_N$ because it is a singularly perturbed one. So, in this section, we will study some basic properties of $\widetilde{\mathcal{L}}_N$.

LEMMA 2.1(LASOTA-YORKE TYPE INEQUALITY). *For each* $N \in \mathbb{N}^+$, *there exists a constant* c_N, *which depends only on* N, *such that the following inequality holds for any* $h \in \mathcal{F}_\theta(\Sigma_A^+)$ *and any* $p \in \mathbb{N}$:

$$(2.1) \qquad \|\widetilde{\mathcal{L}}_N^p h\|_\theta \le \theta^p \|h\|_\theta + c_N \|h\|_\infty.$$

PROOF: By the definition of $\widetilde{\mathcal{L}}_N$, we can write

$$\widetilde{\mathcal{L}}_N^p h(x) = \sum_{a_1,\ldots,a_p} e^{S_p u(a_1 \ldots a_p x)} 1_{[z]_N^c}(a_1 \ldots a_p x) \ldots 1_{[z]_N^c}(a_p x) h(a_1 \ldots a_p x)$$

where $S_p u(\cdot) = \sum_{j=0}^{p-1} u(\sigma^j \cdot)$ and the summation is taken over all words $a_1 \cdots a_p$ such that $a_1 \cdots a_p x \in \Sigma_A^+$.
Put

$$I_n^p(h) = \sup_{d(x,y) \le \theta^n} |\widetilde{\mathcal{L}}_N^p h(x) - \widetilde{\mathcal{L}}_N^p h(y)|.$$

For $n \ge N - 1$,

$$I_n^p(h) = \sup_{d(x,y) \le \theta^n} \Big| \sum_{a_1,\ldots,a_p} e^{S_p u(a_1 \ldots a_p x)} 1_{[z]_N^c}(a_1 \ldots a_p x) \cdots 1_{[z]_N^c}(a_p x) h(a_1 \cdots a_p x)$$

$$- \sum_{a_1,\ldots,a_p} e^{S_p u(a_1 \ldots a_p x)} 1_{[z]_N^c}(a_1 \ldots a_p y) \cdots 1_{[z]_N^c}(a_p y) h(a_1 \cdots a_p y) \Big|$$

$$\le \sup_{d(x,y) \le \theta^n} \sum_{a_1,\ldots,a_p} |e^{S_p u(a_1 \cdots a_p x)} h(a_1 \cdots a_p x) - e^{S_p u(a_1 \cdots a_p y)} h(a_1 \cdots a_p y)|$$

$$\le \sup_{d(x,y) \le \theta^n} \sum_{a_1,\ldots,a_p} e^{S_p u(a_1 \ldots a_p x)} |h(a_1 \cdots a_p x) - h(a_1 \cdots a_p y)|$$

$$+ \sup_{d(x,y) \le \theta^n} \sum_{a_1,\ldots,a_p} e^{S_p u(a_1 \ldots a_p y)} |e^{S_p u(a_1 \cdots a_p x) - S_p u(a_1 \cdots a_p y)} - 1| \cdot \|h\|_\infty$$

$$\le \|h\|_\theta \cdot \theta^{n+p} + C' e^{C' \cdot \theta^{n+1}} \|h\|_\infty \cdot \theta^{n+1} \qquad \text{with } C' = \frac{\|u\|_\theta}{1-\theta},$$

where we use the assumption (1.2): $\mathcal{L}_u 1 = 1$. Hence,

$$(2.2) \qquad \theta^{-n} \cdot I_n^p(h) \le \|h\|_\theta \cdot \theta^p + C \|h\|_\infty$$

where C is a constant depending only on u and θ.

6

For $n < N - 1$, the following inequality is trivial,

$$I_n^p(h) \leq 2\|h\|_\infty.$$

So we get

(2.3) $$\theta^{-n} I_n^p(h) \leq C_N' \|h\|_\infty$$

with $C_N' = 2\theta^{1-N}$.

By (2.2) and (2.3), we get

$$\|\widetilde{\mathcal{L}}_N^p h\|_\theta = \sup_{n \geq 0} I_n^p(h)$$

$$\leq \theta^p \cdot \|h\|_\theta + c_N \cdot \|h\|_\infty.$$

where $c_N = \max\{C, C_N'\}$. ∎

By Lemma 2.1, we can estimate the upper bound of the essential spectral radius of $\widetilde{\mathcal{L}}_N$.

LEMMA 2.2. *The essential spectral radius of $\widetilde{\mathcal{L}}_N$ is not greater than θ.*

PROOF: We can prove this lemma by the technique used by Pollicott ([P.]). Define a compact linear operator $E_p : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$ by

$$E_p h(\cdot) = \sum_{[a_1 \cdots a_p]} \frac{\int_{[a_1 \cdots a_p]} h \, d\mu}{\mu([a_1 \cdots a_p])} 1_{[a_1 \cdots a_p]}(\cdot)$$

where the summation is taken over all cylinder sets of length p.

By simple calculation, we can get the following inequalities:

$$\|h - E_p h\|_\infty \leq \|h\|_\theta \cdot \theta^p$$

and

$$\|h - E_p h\|_\theta \leq \|h\|_\theta.$$

Put $\widetilde{K}_{N,p} = \widetilde{\mathcal{L}}_N^p E_p$. The operator $\widetilde{K}_{N,p}$ is also compact because it is a composition of a compact operator E_p and a bounded operator $\widetilde{\mathcal{L}}_N^p$. By Lemma 2.1, we get

$$\|(\widetilde{\mathcal{L}}_N^p - \widetilde{K}_{N,p})h\|_\theta = \|\widetilde{\mathcal{L}}_N^p(h - E_p h)\|_\theta$$

$$\leq \theta^p \cdot \|h - E_p h\|_\theta + c_N \|h - E_p h\|_\infty$$

$$\leq \theta^p \cdot (1 + c_N)\|h\|_\theta.$$

7

And

$$\|(\widetilde{\mathcal{L}}_N^p - \widetilde{K}_{N,p})h\|_\infty = \|\widetilde{\mathcal{L}}_N^p(h - E_p h)\|_\infty$$
$$\leq \|h - E_p h\|_\infty$$
$$\leq \|h\|_\theta \cdot \theta^p.$$

Therefore,

$$||| \ \widetilde{\mathcal{L}}_N^p - \widetilde{K}_{N,p} \ |||_\theta \leq (c_N + 2)\theta^p.$$

·Using the above inequality, we can see that the essential spectral radius of $\widetilde{\mathcal{L}}_N$ is not greater than θ by Nussbaum's essential spectral radius formula ([N.]):

$$\sup\{|\lambda|; \ \lambda \ \text{is in the essential spectrum of} \ \ \widetilde{\mathcal{L}}_N\} = \lim_{p \to \infty} \|\widetilde{\mathcal{L}}_N^p\|_c^{1/p}$$

where

$$\|\widetilde{\mathcal{L}}_N^p\|_c = \inf\{||| \widetilde{\mathcal{L}}_N^p - K |||_\theta \, ; K : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+) \ \ \text{is a compact operator}\}.$$

REMARK: From the definition of $\widetilde{\mathcal{L}}_N$ and the assumption (1.2): $\mathcal{L}_u 1 = 1$, it follows that the spectral radius of $\widetilde{\mathcal{L}}_N$ is not greater than that of $\mathcal{L}_u$ which is equal to 1. So, by Lemma 2.2, the spectra of $\widetilde{\mathcal{L}}_N$ in the annulus $\{t \in \mathbf{C}; \theta < |t| \leq 1\}$ consists only of isolated eigenvalues of finite mutiplicity. We will denote them by $\{\lambda_N^{(j)}\}_j$. Similarly, we will denote by $\{\lambda^{(j)}\}_j$ the isolated eigenvalues of $\mathcal{L}_u$ in the annulus $\{t \in \mathbf{C}; \theta < |t| \leq 1\}$.

3. The Zeta-function associated with $\widetilde{\mathcal{L}}_N$.

The Ruelle-Artin-Mazur zeta function $\zeta(t)$ is defined as follows:

$$\zeta(t) = \exp\{\sum_{p=1}^\infty \frac{t^p}{p} \sum_{x \in Fix_p \sigma} e^{S_p u(x)}\}$$

(3.1)
$$= \exp\{\sum_{p=1}^\infty \frac{t^p}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)\}$$

where $\dot{a}_1 \cdots \dot{a}_p$ is a periodic point $a \in \Sigma_A^+$ such that $a_{kp+i} = a_i$ for any $k \in \mathbf{N}$. It is well known that the poles of $\zeta(t)$ are corresponding to the eigenvalues of $\mathcal{L}$.

8

PROPOSITION 3.1 (RUELLE [R.I]). *Let $\lambda^{(j)}$ be the eigenvalue of $\mathcal{L}$ in the annulus $\{t \in \mathbf{C}; \theta < |t| \leq 1\}$ of multiplicity m_j. Then $\frac{1}{\lambda^{(j)}}$ is the pole of $\zeta(t)$ in $\{t \in \mathbf{C}; 1 \leq |t| < \theta^{-1}\}$ of the same multiplicity m_j, and vice versa.*

Now, we define a formal power series $\tilde{\zeta}_N(t)$ as follows:

$$\tilde{\zeta}_N(t) = \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{x \in Fix_p \sigma} e^{S_p u(x)} \prod_{j=0}^{p-1} 1_{[z]_N^c}(\sigma^j x)\}$$

(3.2)
$$= \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)\}.$$

We call it the zeta function associated with $\widetilde{\mathcal{L}}_N$. Then, we can show the same correspondence as in Proposition 3.1 between the eigenvalues of $\widetilde{\mathcal{L}}_N$ and the poles of $\tilde{\zeta}_N(t)$.

PROPOSITION 3.2. *Let $\lambda_N^{(j)}$ be the eigenvalue of $\widetilde{\mathcal{L}}_N$ in $\{t \in \mathbf{C}; \theta < |t| \leq 1\}$ of multiplicity m_j. Then, $\frac{1}{\lambda_N^{(j)}}$ is the pole of $\tilde{\zeta}_N(t)$ in $\{t \in \mathbf{C}; 1 \leq |t| < \theta^{-1}\}$ of the same multiplicity m_j, and vice versa.*

PROOF: This proposition can be proved by almost the same technique as in the proof of Theorem A.1 in [R.II], and so we will only sketch the outline of the proof.

Let S_{j,γ_j} be the bases of generalized eigenspace of $\widetilde{\mathcal{L}}_N$ corresponding to $\lambda_N^{(j)}$ and σ_{j,γ_j} be the dual bases of the dual operator $\widetilde{\mathcal{L}}_N^*$ so that $\sigma_{j,\gamma_j}(S_{j,\gamma_j}) = 1$.

We define compact operators $\overline{E}_p$ and $\overline{K}_{N,p}$ as follows:

(3.3)
$$\overline{E}_p h(\cdot) = \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} h(\dot{a}_1 \cdots \dot{a}_p) \cdot 1_{[a_1 \cdots a_p]}(\cdot)$$

and

(3.4)
$$\overline{K}_{N,p} = \widetilde{\mathcal{L}}_N^p \overline{E}_p.$$

Then, it is easy to see that the following two inequalities (3.5), (3.6) hold:

(3.5)
$$\|(\widetilde{\mathcal{L}}_N^p - \overline{K}_{N,p})h\|_\theta \leq \theta^p(1 + c_N)\|h\|_\theta$$

9

where c_N is a constant depending only on N, and

(3.6) $$\|(\widetilde{\mathcal{L}}_N^p - \overline{K}_{N,p})h\|_\infty \le \theta^p \cdot \|h\|_\theta.$$

Now,

$$\sum_j m_j (\lambda_N^{(j)})^p = \sum_j (\lambda_N^{(j)})^p \sum_{\gamma_j} \sigma_{j,\gamma_j}(S_{j,\gamma_j})$$

$$= \sum_{j,\gamma_j} \sigma_{j,\gamma_j}(\widetilde{\mathcal{L}}_N^p S_{j,\gamma_j})$$

$$= \sum_{j,\gamma_j} \sigma_{j,\gamma_j}((\widetilde{\mathcal{L}}_N^p - \overline{K}_{N,p})S_{j,\gamma_j}) + \sum_{j,\gamma_j} \sigma_{j,\gamma_j}(\overline{K}_{N,p}S_{j,\gamma_j})$$

$$= (I) + (II)$$

where (I) is the first term and (II) is the second term of the right hand side.

From the inequality (3.5) (3.6), it is easy to see that

$$(I) \le const \cdot \theta^p.$$

Next, we will estimate (II).

$$(II) = \sum_{j,\gamma_j} \sigma_{j,\gamma_j}(\widetilde{\mathcal{L}}_N^p \overline{E}_p S_{j,\gamma_j})$$

$$= \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \sum_{j,\gamma_j} S_{j,\gamma_j}(\dot{a}_1 \cdots \dot{a}_p) \sigma_{j,\gamma_j}(\widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]})$$

$$= \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \sum_j \mathcal{P}_j(\widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]})(\dot{a}_1 \cdots \dot{a}_p)$$

$$= \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p) - \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \mathcal{P}\widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)$$

where $\mathcal{P}_j$ is the projection to the generalized eigenspace corresponding to $\lambda_N^{(j)}$ and $\mathcal{P}$ is the projection corresponding to the part of the spectrum contained in the disc $\{t \in \mathbf{C}; |t| \le \theta\}$.

10

The second term of (II) is bounded by $const \cdot \theta'^p$ for $\theta' > \theta$ and therefore,

$$|\sum_j m_j(\lambda_N^{(j)})^p - \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)| \le const \cdot \theta'^p.$$

Consequently, we can see

$$\tilde{\zeta}_N(t) \cdot \prod_j (1 - \lambda_N^{(j)} t)^{m_j}$$

$$= \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} (\sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p) - \sum_j m_j(\lambda_N^{(j)})^p)\}$$

converges for $t \in \mathbb{C}$ such that $|t| \cdot \theta' < 1$. ∎

THEOREM 3.3. Let us denote the convergence radius of $\tilde{\zeta}_N(t)$ by $\tilde{t}_N$. Then,.

$$\lim_{N \to \infty} \tilde{t}_N = 1 \qquad for \quad \mu - a.e. \ z.$$

We remark that the convergence radius of $\zeta(t)$ equals to 1. So, the above theorem implies that the convergence radius of $\tilde{\zeta}_N(t)$ converges to that of $\zeta(t)$. In order to prove this theorem, we need two lemmas, Lemma 3.4 and Lemma 3.5 below. The proof of the theorem will be given thereafter.

In preparation, let us introduce some notations. Put

$$C_p = \frac{1}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)$$

and

$$C_p^{(N)} = \frac{1}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p).$$

Then, of course,

$$\zeta(t) = \exp \sum_{p=1}^{\infty} C_p t^p$$

11

and

$$\tilde{\zeta}_N(t) = \exp \sum_{p=1}^{\infty} C_p^{(N)} t^p.$$

We put

$$D_p^{(N)} = \begin{cases} C_p - C_p^{(N)}, & p \leq N, \\ 0 & p > N \end{cases}$$

and

$$E_p^{(N)} = \begin{cases} C_p & p \leq N, \\ C_p^{(N)} & p > N. \end{cases}$$

We define $\tilde{\zeta}_N^{(D)}(t)$ and $\tilde{\zeta}_N^{(E)}(t)$ as follows:

$$\tilde{\zeta}_N^{(D)}(t) = \exp \sum_{p=1}^{\infty} D_p^{(N)} t^p$$

and

$$\tilde{\zeta}_N^{(E)}(t) = \exp \sum_{p=1}^{\infty} E_p^{(N)} t^p.$$

Then,

$$\tilde{\zeta}_N(t) \cdot \tilde{\zeta}_N^{(D)}(t) = \tilde{\zeta}_N^{(E)}(t),$$

because $C_p^{(N)} + D_p^{(N)} = E_p^{(N)}$.

LEMMA 3.4. For $\mu - a.e.\ z$,

$$\lim_{N \to \infty} E_p^{(N)} = C_p \qquad \text{uniformly in } p.$$

PROOF:

i) For $p \leq N$, it is trivial by the definition of $E_p^{(N)}$.
ii) For $p > N$,

$$C_p - E_p^{(N)} = C_p - C_p^{(N)}$$

$$= \frac{1}{p} \sum_{a_1 \cdots a_p}^{*} e^{S_p u(\hat{a}_1 \cdots \hat{a}_p)}$$

12

where $\displaystyle\sum_{a_1\cdots a_p}{}^{*}$ means the summation taken over all words $a_1\cdots a_p$ such that

$.a_{j+1}\cdots a_p\dot a_1\cdots\dot a_j\in[z]_N$ for some $j\in\{0,\cdots,p-1\}$. Hence,

$$C_p-E_p^{(N)}\leq\sum_{\substack{a_1,\cdots,a_p\\a_1\cdots a_p\in[z]_N}}e^{S_p u(\dot a_1\cdots\dot a_p)}$$

$$=\sum_{\substack{b_1,\cdots,b_{p-N}\\A_{z_{N-1}b_1}=A_{b_{p-N}z_0}=1}}e^{S_p u(\dot z_0\cdots z_{N-1}b_1\cdots\dot b_{p-N})}.$$

Note

$$S_p u(\dot z_0\cdots z_{N-1}b_1\cdots\dot b_{p-N})\leq S_N u(z)+S_{p-N}u(b_1\cdots b_{p-N}z)+K$$

with $K=\frac{2\|u\|_\theta}{1-\theta}$. Therefore,

$$C_p-C_p^{(N)}\leq e^{S_N u(z)}e^K\sum_{\substack{b_1\cdots b_{p-N}\\A_{z_{N-1}b_1}=A_{b_{p-N}z_0}=1}}e^{S_{p-N}u(b_1\cdots b_{p-N}z)}$$

$$\leq e^{S_N u(z)}e^K\mathcal{L}^{p-N}1(z)$$

$$=e^{S_N u(z)}e^K,$$

where we use the assumption $(1.2){:}\mathcal{L}1=1$. By the ergodic theorem,

$$\lim_{N\to\infty}\frac{1}{N}S_N u(z)=\int u\,d\mu\qquad\mu-a.e.\ z,$$

and by (1.3),

$$\int u\,d\mu=-h_\mu(\sigma)<0.$$

Therefore,

$$\lim_{N\to\infty}S_N u(z)=-\infty\qquad\text{for}\quad\mu-a.e.\ z.$$

Hence we can see

$$\lim_{N\to\infty}|C_p-C_p^{(N)}|=0\qquad\text{uniformly in}\quad p,\qquad\text{for}\quad\mu-a.e.\ z.$$

From i) and ii), we obtain

$$\lim_{N\to\infty}|C_p-E_p^{(N)}|=0\qquad\text{uniformly in}\quad p,\qquad\text{for}\quad\mu-a.e.\ z.$$

∎

13

LEMMA 3.5. *The convergence radius of $\tilde{\zeta}_\infty^{(D)}(t)$ is greater than 1.*

PROOF: Recall that $\tilde{\zeta}_N^{(D)}(t) = \exp \sum_{p=1}^{N}(C_p - C_p^{(N)})t^p$.
For $p \leq N$,

$$C_p - C_p^{(N)} = \frac{1}{p}\sum_{a_1 \cdots a_p}^{*} e^{S_p u(\dot{a}_1 \cdots \dot{a}_p)}$$

$$\leq e^{K'} e^{S_p u(z)}$$

$$= e^{K'}(e^{\frac{1}{p}S_p u(z)})^p,$$

with $K' = \frac{\|u\|_\theta}{1-\theta}$. Therefore, the convergence radius of $\tilde{\zeta}_\infty^{(D)}(t)$ is not less than $\exp\{-\lim_{p\to\infty}\frac{1}{p}S_p u(z)\}$.

And as we have seen in the proof of the previous lemma,

$$-\lim_{p\to\infty}\frac{1}{p}S_p u(z) = h_\mu(\sigma) > 0 \quad for \quad \mu - a.e. \ z.$$

Hence the convergence radius of $\tilde{\zeta}_\infty^{(D)}(t)$ is greater than 1. ∎

PROOF OF THEOREM 3.3:
Recall that $\tilde{\zeta}_N(t) \cdot \tilde{\zeta}_N^{(D)}(t) = \tilde{\zeta}_N^{(E)}(t)$. By Lemma 3.4, we can see that the convergence radius of $\tilde{\zeta}_N^{(E)}(t)$ goes to 1 as $N \to \infty$, which is the convergence radius of $\zeta(t)$, for $\mu - a.e. \ z$. But the convergence radius of $\tilde{\zeta}_\infty^{(D)}(t)$ is greater than 1 by Lemma 3.5. Therefore,

$$\lim_{N\to\infty}\tilde{t}_N = 1 \quad for \quad \mu - a.e. \ z.$$

∎

COROLLARY 3.6. *Denote by $\tilde{\lambda}_N$ the eigenvalue of $\widetilde{\mathcal{L}}_N$ of maximal modulus. Then,*

$$\lim_{N\to\infty}\tilde{\lambda}_N = 1 \quad for \quad \mu - a.e. \ z.$$

PROOF: By Proposition 3.2, $\tilde{\lambda}_N = 1/\tilde{t}_N$. Hence, by Theorem 3.3, $\tilde{\lambda}_N$ goes to 1 as $N \to \infty$. ∎

REMARK: We can easily check by Lemma 3.4 and Corollary 3.6 that the convergence radius of $(1 - \tilde{\lambda}_N t) \cdot \tilde{\zeta}_N^{(E)}(t)$ goes to that of $(1 - t) \cdot \zeta(t)$ as $N \to \infty$. Since 1 is the

14

simple pole of $\zeta(t)$, the modulus of the eigenvalues of $\widetilde{\mathcal{L}}_N$ execpt $\tilde{\lambda}_N$ do not go to 1 as $N \to \infty$. Precisely, there exists a number $0 < q < 1$ such that for any $N \in \mathbb{N}$,

$$\sup\{|\lambda|; \ \lambda \in \mathrm{Spec}(\widetilde{\mathcal{L}}_N) \setminus \tilde{\lambda}_N\} < q$$

where $\mathrm{Spec}(\widetilde{\mathcal{L}}_N)$ is the spectrum of $\widetilde{\mathcal{L}}_N$.

4. Poisson law for Symbolic Dynamics.

In this section, we will show the Poisson law for symbolic dynamics $(\Sigma_A^+, \sigma, \mu)$.

We fix a point $z \in \Sigma_A^+$, and take a cylinder set $[z]_N$ as a neighborhood of z. On $[z]_N$, we define a probability measure μ_N as the restriction of the equilibrium state μ to $[z]_N$, i.e.,

$$\mu_N = \frac{\mu|_{[z]_N}}{\mu([z]_N)}.$$

In order to study the limit distribution of the normalized first return time $\epsilon_N T_N$, where $\epsilon_N = 1/E_{\mu_N}(T_N)$, as $N \to \infty$, we consider its Laplace transform $\phi_N(\alpha)$:

$$\phi_N(\alpha) \equiv \mu_N(e^{-\alpha \epsilon_N T_N})$$
$$= \int e^{-\alpha \epsilon_N T_N(x)} \mu_N(dx).$$

Before we compute the limit of $\phi_N(\alpha)$ as $N \to \infty$, we prepare several lemmas.

LEMMA 4.1. The operator $\widetilde{\mathcal{L}}_N : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$ can be decomposed as follows:

$$(4.1) \qquad\qquad \widetilde{\mathcal{L}}_N = \tilde{\lambda}_N \tilde{E}_N + \tilde{\Psi}_N$$

where $\tilde{E}_N$ is the projection to the eigenspace corresponding to the eigenvalue $\tilde{\lambda}_N$ of maximal modulus, and $\tilde{\Psi}_N$ is a bounded linear operator such that

$$\tilde{E}_N \tilde{\Psi}_N = \tilde{\Psi}_N \tilde{E}_N = 0.$$

REMARK: The eigenfunction of $\widetilde{\mathcal{L}}_N$ corresponding to the maximal eigenvalue $\tilde{\lambda}_N$ is positive.

15

PROOF: By Lemma 2.1 (Lasota-Yorke type inequality), it is easy to see that the operator $\tilde{\lambda}_N^{-1} \cdot \widetilde{\mathcal{L}}_N : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$ satisfies the conditions of the Ionescu-Tulcia-Marinescu theorem([I.T.M.]). Then, it is a direct conclusion of that theorem that $\tilde{\lambda}_N^{-1} \cdot \widetilde{\mathcal{L}}_N$ can be decomposed as follows:

$$\tilde{\lambda}_N^{-1} \cdot \widetilde{\mathcal{L}}_N = \tilde{E}_N + \tilde{\Psi}'_N$$

where $\tilde{E}_N$ is the projection to the eigenspace corresponding to the eigenvalue 1, and $\tilde{\Psi}'_N$ is a bounded operator satisfying $\tilde{E}_N \tilde{\Psi}'_N = \tilde{\Psi}'_N \tilde{E}_N = 0$. Put $\tilde{\Psi}_N = \tilde{\lambda}_N \tilde{\Psi}'_N$ and we obtain the above lemma. ∎

From Lemma 4.1, we obtain the following decomposition:

$$(4.2) \qquad \widetilde{\mathcal{L}}_N^i = \tilde{\lambda}_N^i \tilde{E}_N + \tilde{\Psi}_N^i.$$

Then, by Lemma 1.1, we get for $i \geq 2$,

$$\mu_N(T_N = i) = \int \widetilde{\mathcal{L}}_N^{i-1}(\mathcal{L}1_{[z]_N})d\mu_N$$

$$(4.3) \qquad = \tilde{\lambda}_N^{i-1} \int \tilde{E}_N(\mathcal{L}1_{[z]_N})d\mu_N + \int \tilde{\Psi}_N^{i-1}(\mathcal{L}1_{[z]_N})d\mu_N.$$

The following result is well-known as a part of Ambrose-Kakutani's theorem ([PE.]), but we will give a proof by using the operator $\widetilde{\mathcal{L}}_N$ for the completeness of the paper.

LEMMA 4.2.

$$\epsilon_N = \frac{1}{E_{\mu_N}(T_N)} = \mu([z]_N)$$

PROOF:

$$E_{\mu_N}(T_N) = \mu_N(T_N = 1) + \sum_{i=2}^{\infty} i\mu_N(T_N = i)$$

$$= \int \mathcal{L}(1_{[z]_N})d\mu_N + \sum_{i=2}^{\infty} i \int \widetilde{\mathcal{L}}_N^{i-1}(\mathcal{L}1_{[z]_N})d\mu_N$$

$$= 1 + \sum_{i=1}^{\infty} \int \widetilde{\mathcal{L}}_N^i 1 d\mu_N,$$

16

where we used the fact that

$$(4.4) \qquad \mathcal{L}1_{[z]_N} = 1 - \widetilde{\mathcal{L}}_N 1.$$

But, using the property (1.5), we can see

$$\mu([z]_N^c) = \sum_{i=1}^{\infty} \int 1_{[z]_N^c} \cdot (1_{[z]_N^c} \circ \sigma) \cdots (1_{[z]_N^c} \circ \sigma^{i-1}) \cdot (1_{[z]_N} \circ \sigma^i) d\mu$$

$$= \sum_{i=1}^{\infty} \int \widetilde{\mathcal{L}}_N^i 1 \cdot 1_{[z]_N} d\mu.$$

Then,

$$\sum_{i=1}^{\infty} \int \widetilde{\mathcal{L}}_N^i 1 d\mu_N = \frac{\mu([z]_N^c)}{\mu([z]_N)}.$$

Therefore,

$$E_{\mu_N}(T_N) = 1 + \frac{\mu([z]_N^c)}{\mu([z]_N)} = \frac{1}{\mu([z]_N)}.$$

∎

LEMMA 4.3. *For $\mu - a.e.$ z, there exist a positive integer N_0 and a constant H and a number q, $0 < q < 1$, such that for any $N > N_o$*

$$\|\tilde{\Psi}_N^p 1\|_{\infty} < H \cdot q^p \qquad \text{for any } p \in \mathbb{N}.$$

PROOF: The resolvent operator of $\widetilde{\mathcal{L}}_N$, say $\tilde{R}_N$, can be formally expressed as follows:

$$\tilde{R}_N(t) = (tI - \widetilde{\mathcal{L}}_N)^{-1}$$

$$= \sum_{p=0}^{\infty} \frac{1}{t^{p+1}} \widetilde{\mathcal{L}}_N^p.$$

We have already seen that $\tilde{\lambda}_N \to 1$ as $N \to \infty$. But as we remarked below Corollary 3.6, the eigenvalues of $\widetilde{\mathcal{L}}_N$ of the second maximal modulus do not go to 1 as $N \to \infty$. Therefore, we can choose $0 < q < 1$ such that for N large enough,

$$\sup\{|\lambda|; \ \lambda \in \operatorname{Spec}(\widetilde{\mathcal{L}}_N) \setminus \tilde{\lambda}_N\} < q < |\tilde{\lambda}_N|$$

17

and

$$\sup\{|\lambda|;\ \lambda \in \mathrm{Spec}(\mathcal{L}) \setminus 1\} < q$$

where $\mathrm{Spec}(\widetilde{\mathcal{L}}_N)$ is the spectrum of $\widetilde{\mathcal{L}}_N$ and $\mathrm{Spec}(\mathcal{L})$ is that of $\mathcal{L}$.

Let Γ_q be a circle of radius q centered at the origin. Then, we can write

$$\tilde{\Psi}_N^p = \frac{1}{2\pi i} \int_{\Gamma_q} t^p \tilde{R}_N(t) dt.$$

·Recall the compact operator $\overline{K}_{N,p}$ defined by (3.4) in the proof of Proposition 3.2. For that operator, we have already seen the following inequality,

$$(3.6) \qquad \|(\widetilde{\mathcal{L}}_N^p - \overline{K}_{N,p})h\|_\infty \leq \theta^p \cdot \|h\|_\theta.$$

Then, for any $x \in \Sigma_A^+$,

$$|\overline{K}_{N,p}h(x)| = |\sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1}=1}} h(\dot{a}_1 \cdots \dot{a}_p)\widetilde{\mathcal{L}}_N^p 1_{[a_1 \cdots a_p]}(x)|$$

$$\leq \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1}=1}} |h(\dot{a}_1 \cdots \dot{a}_p)|\mathcal{L}^p 1_{[a_1 \cdots a_p]}(x)$$

$$\leq \|h\|_\infty \cdot e^{\frac{\theta}{1-\theta}\cdot\|u\|_\theta} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1}=1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p).$$

Therefore, using (3.6), we can see

$$\|\tilde{R}_N(t)h\|_\infty = \|\frac{1}{t}I + \sum_{p=1}^\infty \frac{1}{t^{p+1}}(\widetilde{\mathcal{L}}_N^p - \overline{K}_{N,p})h + \sum_{p=1}^\infty \frac{1}{t^{p+1}}\overline{K}_{N,p}h\|_\infty$$

$$\leq \|h\|_\theta \cdot \sum_{p=0}^\infty \frac{\theta^p}{|t|^{p+1}} + \|h\|_\infty e^{\frac{\theta}{1-\theta}\|u\|_\theta} \sum_{p=1}^\infty \frac{1}{|t|^{p+1}} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1}=1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p).$$

Hence we obtain

$$\|\tilde{R}_N(t)1\|_\infty \leq e^{\frac{\theta}{1-\theta}\|u\|_\theta} \sum_{p=1}^\infty \frac{1}{|t|^{p+1}} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1}=1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p).$$

18

We remark that $1/q$ is not a pole of

$$\zeta(t) = \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \mathcal{L}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)\},$$

because we choose q so that it is not an eigenvalue of $\mathcal{L}$. Hence, as $\zeta(1/q)$ converges, there exists a constant C such that

$$\|\tilde{R}_N(q)1\|_\infty \le C.$$

Then, for N large enough,

$$\|\tilde{\Psi}_N^p 1\|_\infty \le \|\tilde{R}_N(q)1\|_\infty \cdot |\frac{1}{2\pi i} \int_{C_q} t^p dt|$$

$$\le H \cdot q^p$$

where H is some constant which is independent of N. ∎

LEMMA 4.4. For $\mu - a.e.\ z$,

$$\lim_{N \to \infty} \frac{\int \tilde{E}_N(\mathcal{L}1_{[z]_N}) d\mu_N}{1 - \tilde{\lambda}_N} = \lim_{N \to \infty} \int \tilde{E}_N 1 d\mu_N = 1.$$

PROOF: For the simplicity, we put

$$[\tilde{E}_N] = \int \tilde{E}_N(\mathcal{L}1_{[z]_N}) d\mu_N.$$

Then, by using $\mathcal{L}1_{[z]_N} = 1 - \widetilde{\mathcal{L}}_N 1$,

$$[\tilde{E}_N] = (1 - \tilde{\lambda}_N) \int \tilde{E}_N 1 d\mu_N.$$

Since $\mu(\{z \in \Sigma_A^+;\ z \text{ is periodic}\}) = 0$, we may assume that z is not a periodic point. For an integer $p > 0$, put

$$N_p = p + \max_{1 \le i \le p} \frac{\log(d_\theta(z, \sigma^i z))}{\log \theta}.$$

19

Then, for any $x \in [z]_{N_p}$ and any words $a_1 \cdots a_p$,

$$a_j \cdots a_p x \notin [z]_{N_p} \qquad j = 1, \cdots, p.$$

Therefore, for any $N > N_p$ and any $x \in [z]_N$, we can see

$$1_{[z]_N^c}(a_1 \cdots a_p x) \cdots 1_{[z]_N^c}(a_p x) = 1.$$

Recall that

$$\widetilde{\mathcal{L}}_N^p 1(x) = \sum_{a_1 \cdots a_p} e^{S_p u(a_1 \cdots a_p x)} 1_{[z]_N^c}(a_1 \cdots a_p x) \cdots 1_{[z]_N^c}(a_p x).$$

Hence, for $N > N_p$,

$$1_{[z]_N}(x) \cdot \widetilde{\mathcal{L}}_N^p 1(x) = 1_{[z]_N}(x) \cdot \sum_{a_1 \cdots a_p} e^{S_p u(a_1 \cdots a_p x)}$$

$$= 1_{[z]_N}(x) \cdot \mathcal{L}^p 1(x)$$

$$= 1_{[z]_N}(x).$$

Therefore, $\int \widetilde{\mathcal{L}}_N^p 1 d\mu_N = 1$ for $N > N_p$.

Now, we use the decomposition $\widetilde{\mathcal{L}}_N^p = \tilde{\lambda}_N^p \tilde{E}_N + \tilde{\Psi}_N^p$ for any p and $N > N_p$. Since $\int \tilde{E}_N 1 d\mu_N > 0$ and $0 < \tilde{\lambda}_N < 1$, we can obtain

$$|1 - \int \tilde{E}_N 1 d\mu_N| \le |1 - \tilde{\lambda}_N^p \int \tilde{E}_N 1 d\mu_N|$$

$$= |\int \tilde{\Psi}_N^p 1 d\mu_N|$$

$$\le \|\tilde{\Psi}_N^p 1\|_\infty.$$

Hence, by Lemma 4.3,

$$\lim_{N \to \infty} \frac{[\tilde{E}_N]}{1 - \tilde{\lambda}_N} = \lim_{N \to \infty} \int \tilde{E}_N 1 d\mu_N = 1.$$

LEMMA 4.5. For $\mu - a.e.$ z,

$$\lim_{N \to \infty} \frac{\int \tilde{E}_N(\mathcal{L}1_{[z]_N}) d\mu_N}{\epsilon_N} = 1.$$

PROOF: Put $[\tilde{E}_N] = \int \tilde{E}_N(\mathcal{L}1_{[z]_N}) d\mu_N$, and $[\tilde{\Psi}_N^i] = \int \tilde{\Psi}_N^i(\mathcal{L}1_{[z]_N}) d\mu_N$. Then,

$$E_{\mu_N}(T_N) = \int \mathcal{L}1_{[z]_N} d\mu_N + \sum_{i=1}^{\infty}(i+1)\{\bar{\lambda}_N^i[\tilde{E}_N] + [\tilde{\Psi}_N^i]\}$$

$$= \frac{[\tilde{E}_N]}{(1-\bar{\lambda}_N)^2} + 1 - \frac{[\tilde{E}_N]}{1-\bar{\lambda}_N} + \sum_{i=1}^{\infty}\int \tilde{\Psi}_N^i 1 d\mu_N$$

where we use $[\tilde{\Psi}_N^i] = \int \tilde{\Psi}_N^i(\mathcal{L}1_{[z]_N}) d\mu_N = \int \tilde{\Psi}_N^i 1 d\mu_N - \int \tilde{\Psi}_N^{i+1} 1 d\mu_N$. Hence

$$\frac{[\tilde{E}_N]}{\epsilon_N} = [\tilde{E}_N] \cdot E_{\mu_N}(T_N)$$

$$= (\frac{[\tilde{E}_N]}{1-\bar{\lambda}_N})^2 + [\tilde{E}_N] \cdot (1 - \frac{[\tilde{E}_N]}{1-\bar{\lambda}_N} + \sum_{i=1}^{\infty}\int \tilde{\Psi}_N^i 1 d\mu_N).$$

By Lemma 4.3 and Lemma 4.4, for $\mu - a.e.$ z,

$$\lim_{N \to \infty} \frac{[\tilde{E}_N]}{\epsilon_N} = 1.$$

∎

LEMMA 4.6. For $\mu - a.e.$ z,

$$\lim_{N \to \infty} \epsilon_N \sum_{i=1}^{\infty}\int \tilde{\Psi}_N^i 1 d\mu_N = 0.$$

PROOF: This lemma is obvious from Lemma 4.2 and Lemma 4.3. ∎

Using the above lemmas, we can prove the following result.

21

THEOREM 4.7. *For $\mu - a.e.\ z$, the limit distribution of $\epsilon_N T_N$ as $N \to \infty$ exists and it is the exponential distribution with parameter 1, where $\epsilon_N = 1/E_{\mu_N}(T_N)$.*

PROOF: In order to prove the theorem, we will consider the limit of $\phi_N(\alpha)$.

$$\phi_N(\alpha) = \int e^{-\alpha \epsilon_N T_N} d\mu_N$$

$$= \sum_{i=1}^{\infty} e^{-\alpha \epsilon_N i} \mu_N(T_N = i)$$

$$= e^{-\alpha \epsilon_N} \{ \int \mathcal{L} 1_{[z]_N} d\mu_N + \sum_{i=1}^{\infty} e^{-\alpha \epsilon_N i} \tilde{\lambda}_N^i [\tilde{E}_N] + \sum_{i=1}^{\infty} e^{-\alpha \epsilon_N i} [\tilde{\Psi}_N^i] \}$$

$$= e^{-\alpha \epsilon_N} \{ 1 - \tilde{\lambda}_N \int \tilde{E}_N 1 d\mu_N + \frac{\tilde{\lambda}_N}{e^{\alpha \epsilon_N} - \tilde{\lambda}_N} \cdot [\tilde{E}_N] + \sum_{i=1}^{\infty} (e^{-\alpha \epsilon_N i} - 1)[\tilde{\Psi}_N^i] \}.$$

where we used (4.3) and (4.4).

By Lemma 4.4, we can see that for $\mu - a.e.\ z$,

$$\lim_{N \to \infty} (1 - \tilde{\lambda}_N \int \tilde{E}_N 1 d\mu_N) = 0.$$

The following equality is obtained by Lemma 4.4 and Lemma 4.5,

$$\lim_{N \to \infty} \frac{\tilde{\lambda}_N}{e^{\alpha \epsilon_N} - \tilde{\lambda}_N} \cdot [\tilde{E}_N] = \frac{1}{1 + \alpha}.$$

And we can see by Lemma 4.6,

$$|\sum_{i=1}^{\infty} (e^{-\alpha \epsilon_N i} - 1)[\tilde{\Psi}_N^i]| \leq \alpha \epsilon_N \sum_{i=1}^{\infty} i [\tilde{\Psi}_N^i]$$

$$= \alpha \epsilon_N \sum_{i=1}^{\infty} \int \tilde{\Psi}_N^i 1 d\mu_N$$

$$\to 0.$$

Therefore, for $\mu - a.e.\ z$,

$$\lim_{N \to \infty} \phi_N(\alpha) = \frac{1}{1 + \alpha}.$$

22

This implies that the limit distribution of $\epsilon_N T_N$ is the exponential distribution with parameter 1. ∎

Next we will study the k-th return times $T_N^{(k)}$:

$$T_N^{(k)}(x) \equiv \sum_{j=0}^{k-1} T_N(\sigma^{T_N^{(j)}(x)} x), \qquad k = 1, 2, \cdots, $$

where $T_N^{(0)}(x) \equiv 0$.

LEMMA 4.8. For each $k \geq 1$, $\epsilon_N(T_N^{(k+1)} - T_N^{(k)})$ has the same distribution as $\epsilon_N T_N$. Therefore, for $\mu - a.e.$ z, the limit distribution of $\epsilon_N(T_N^{(k+1)} - T_N^{(k)})$ is the exponential distribution.

PROOF: The measure μ_N on $[z]_N$ is an invariant measure of the induced transformation of the shift σ to $[z]_N$, i.e. $\sigma^{T_N(\cdot)}(\cdot) : [z]_N \to [z]_N$. And therefore the following equality holds:

$$\mu(\{x \in [z]_N; T_N^{(k+1)}(x) - T_N^{(k)}(x) = i\}) = \mu(\{x \in [z]_N; T_N(x) = i\}).$$

This shows Lemma 4.8. ∎

We remark that the limit distributions of $\epsilon_N(T_N^{(k+1)} - T_N^{(k)})$ are mutualy independent because $(\Sigma_A^+, \sigma, \mu)$ is weakly Bernoulli. (See [B.])

Hence we obtain the following proposition.

PROPOSITION 4.9. For $\mu - a.e.$ z,

$$\lim_{N \to \infty} \mu_N(\epsilon_N T_N^{(k)} \leq t \text{ and } \epsilon_N T_N^{(k+1)} > t) = \frac{t^k}{k!} e^{-t}.$$

Here, let us define a point process on $\mathbf{R}^+$, say $Y_N(\cdot)$, as follows:

$$Y_N(\cdot) = \sum_{k=1}^{\infty} \delta_{\epsilon_N T_N^{(k)}(\cdot)}$$

where δ_p is the Dirac δ measure at $p \in \mathbf{R}^+$. We will call it the normalized return time process.

Then, the above proposition implies the following .

23

THEOREM 4.10. *For $\mu-a.e.$ $z \in \Sigma_A^+$, the sequence of the normalized return time processes $\{Y_N\}_N$ converges to the Poisson point process as $N \to \infty$ in finite dimensional distribution, i.e. for any disjoint Borel sets $B_1, \cdots, B_n \in \mathcal{B}(\mathbf{R}^+)$ and any non-negative integers $k_1, \cdots, k_n$,*

$$\lim_{N \to \infty} \mu_N(Y_N(B_1) = k_1, \cdots, Y_N(B_n) = k_n) = \prod_{i=1}^{n} \frac{\ell(B_i)^{k_i}}{k_i!} e^{-\ell(B_i)},$$

where ℓ is the Lebesgue measure.

REMARK: By using the technique of Section 5, we can see that the above theorem holds not only for the normalized return time processes associated with cylinder sets $\{[z]_N\}_N$ but also for that associated with open neighborhoods $\{U_\epsilon(z)\}_\epsilon$ such that $U_\epsilon(z) \to \{z\}$ as $\epsilon \to 0$.

Now, we can easily extend Theorem 4.10 to two-sided symbolic dynamics. Put

$$\Sigma_A = \{x = \{x_i\}_{i=-\infty}^{\infty} \in J^{\mathbf{Z}}; A_{x_i x_{i+1}} = 1, \text{ for all } i \in \mathbf{Z}\},$$

and define the metric $\hat{d}$ on Σ_A by

$$\hat{d}(x, y) = \theta^n \quad \text{if} \quad x_i = y_i \quad \text{for } i = -(n-1), \cdots, n-1,$$
$$\text{and if} \quad x_{-n} \neq y_{-n} \quad \text{or} \quad x_n \neq y_n.$$

Let $\mathcal{F}_\theta(\Sigma_A)$ be the totality of real valued Lipschitz contiuous functions on Σ_A. Take $u \in \mathcal{F}_\theta(\Sigma_A)$, and let $\hat{\mu}_u$ be the (unique) equilibrium state for u. But, one can find u' which depends on only $\{x_i\}_{i=0}^{\infty}$ and is homologous to u. Indeed, put

$$u_0(x) = \min_y u(y)$$

and

$$u_n(x) = \min\{u(y) - u_{n-1}(x) \; ; \; x_i = y_i \quad \text{for } i = -n, \cdots, n\},$$

then, $u = u' + w - w \circ \sigma$ where $u' = \sum_{n=0}^{\infty} u_n \circ \sigma^n$ and $w = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} u_n \circ \sigma^k$. We remark that $u' \in \mathcal{F}_\theta(\Sigma_A^+)$ and $\hat{\mu}_u = \hat{\mu}_{u'}$. Therefore, we may assume that u depends on only $\{x_i\}_{i=0}^{\infty}$.

Fix a point $z \in \Sigma_A$, and denote

$$_{-a}[z]_b = \{x \in \Sigma_A; \; x_i = z_i \quad \text{for } i = -a, \cdots, b\},$$

24

and

$$[z]_{(N)} = \begin{cases} -m[z]_m & \text{if} \quad N = 2m \quad m \geq 0 \\ -m+1[z]_m & \text{if} \quad N = 2m-1 \quad m \geq 1. \end{cases}$$

For a point $x \in [z]_{(N)}$, we denote its first return time to $[z]_{(N)}$ by $T_{(N)}(x)$, and the k-th return time by $T_{(N)}^{(k)}(x)$. As $\hat{\mu}_u$ is σ-invariant,

$$\hat{\mu}_u(\{x \in [z]_{(N)}; T_{(N)}(x) = i\}) = \hat{\mu}_u(1_{[\hat{z}]_N} \cdot (1_{[\hat{z}]_N^c} \circ \sigma) \cdots (1_{[\hat{z}]_N^c} \circ \sigma^{i-1}) \cdot (1_{[\hat{z}]_N} \circ \sigma^i))$$

$$= \int \widetilde{\mathcal{L}}_{[\hat{z}]_N}^{i-1}(\mathcal{L}1_{[\hat{z}]_N}) \cdot 1_{[\hat{z}]_N} d\hat{\mu}_u$$

$$= \int \widetilde{\mathcal{L}}_{[\hat{z}]_N}^{i-1}(\mathcal{L}1_{[\hat{z}]_N}) \cdot 1_{[\hat{z}]_N} d\mu_u,$$

where $[\hat{z}]_N =_0 [\sigma^{-[\frac{N}{2}]}z]_N$, $\widetilde{\mathcal{L}}_{[\hat{z}]_N}f \equiv \mathcal{L}(1_{[\hat{z}]_N^c} \cdot f)$, and μ_u is the equilibrium state on Σ_A^+ corresponding to u.

Then, the problem is reduced to the case of $(\Sigma_A^+, \sigma, \mu_u)$. It is easy to see that Lemmas, Propositions and Theorems in Section 1, 2 also hold for $\widetilde{\mathcal{L}}_{[\hat{z}]_N}$ instead of $\widetilde{\mathcal{L}}_N$. Define a probability measure $\hat{\mu}_{(N)}$ on $[z]_{(N)}$ by

$$\hat{\mu}_{(N)} = \frac{\hat{\mu}_u \mid_{[z]_{(N)}}}{\hat{\mu}_u([z]_{(N)})}.$$

Denote the normalized return time process on $\mathbf{R}^+$ by $Y_{(N)}$:

$$Y_{(N)}(\cdot) = \sum_{k=1}^{\infty} \delta_{\hat{\epsilon}_N T_{(N)}^{(k)}(\cdot)}$$

where $\hat{\epsilon}_N = 1/E_{\hat{\mu}_{(N)}}(T_{(N)})$.

Then, Lemmas and Theorems in the previous section hold for this system. Hence, we can obtain the following assertion.

PROPOSITION 4.11. For $\hat{\mu}_u - a.e.$ $z \in \Sigma_A$,

$$\lim_{N \to \infty} \hat{\mu}_{(N)}(\hat{\epsilon}_N \cdot T_{(N)}^{(k)} \leq t \text{ and } \hat{\epsilon}_N \cdot T_{(N)}^{(k+1)} > t) = \frac{t^k}{k!}e^{-t}.$$

25

And the sequence of the normalized return time processes $\{Y_{(N)}\}_N$ converges to the Poisson point process as $N \to \infty$ in finite dimensional distribution.

5. Poisson law for Axiom A system.

Now, we will think about the Poisson law for Axiom A system. Before going to prove the main theorem, we will recall some basic facts of Axiom A diffeomorphisms (see [B.] about the details).

Let $f : M \to M$ be a Axiom A diffeomorphism of a compact C^∞ Riemannian manifold M, and $\Omega = \Omega(f)$ be its non-wandering set. By considering the spectral decomposition of Ω, if necessary, we will assume that $f \mid_\Omega$ is topologically mixing. Then a Markov partition of Ω, say $\mathcal{R} = \{R_1, \cdots, R_r\}$, can be constructed. Write $\partial^s \mathcal{R} = \bigcup_j \partial^s R_j$ and $\partial^u \mathcal{R} = \bigcup_j \partial^u R_j$ where $\partial^s R_j$ is the stable boundary of R_j and $\partial^u R_j$ is the unstable one. The structure matrix $A = (A_{ij})_{i,j=1,\cdots,r}$ is defined by

$$A_{ij} = \begin{cases} 1 & \text{if} \quad int\, R_i \cap f^{-1}(\, int\, R_j\,) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a continuous surjection $\pi : \Sigma_A \to \Omega$ which satisfies $\pi \circ \sigma = f \circ \pi$. Furtheremore, $\pi : \pi^{-1}(\tilde{\Omega}) \to \tilde{\Omega}$ is one to one, where $\tilde{\Omega}$ is a residual set defined by $\tilde{\Omega} = \Omega \setminus \bigcup_{j \in \mathbf{Z}} f^j(\partial^s \mathcal{R} \cup \partial^u \mathcal{R})$.

Let $u : \Omega \to \mathbf{R}$ be a Lipschitz continuous function and μ_u be the unique equilibrium state for u. We remark that $\mu_u(\Omega \setminus \tilde{\Omega}) = 0$.

Now, fix $z \in \tilde{\Omega}$ and put $\bar{z} = \pi^{-1}(z) = \{z_i\}_{i=-\infty}^\infty \in \Sigma_A$. Let $u^* = u \circ \pi$ and μ_{u^*} be the equilibrium state for u^* on Σ_A. Then $\mu_u = \pi^* \mu_{u^*}$. For the fixed $z \in \tilde{\Omega}$, denote its ϵ-neighborhood by $U_\epsilon(z)$, and define a probability measure μ_ϵ on U_ϵ by

$$\mu_\epsilon = \frac{\mu_u\big|_{U_\epsilon(z)}}{\mu_u(U_\epsilon(z))}.$$

Denote by $T_{\epsilon,f}$ the first return time from $U_\epsilon(z)$ to $U_\epsilon(z)$. Put $c_\epsilon = 1/E_{\mu_\epsilon}(T_{\epsilon,f})$. Remark that $c_\epsilon = \mu_u(U_\epsilon(z))$. Then the following theorem holds.

THEOREM 5.1. *For* $\mu_u - a.e.$ $z \in \Omega$, *the limit distribution of the first return time* $c_\epsilon T_{\epsilon,f}$ *as* $\epsilon \to 0$ *exists and it is the exponential distribution with parameter 1.*

The proof of the above theorem will be given after several lemmas.

Define for $\epsilon > 0$ sufficiently small

$$N^+(\epsilon) = \max\{i \in \mathbb{N}^+; U_\epsilon(z) \subset \pi(_0[\tilde{z}]_i)\}$$

and

$$N^-(\epsilon) = \max\{i \in \mathbb{N}^+; U_\epsilon(z) \subset \pi(_{-i}[\tilde{z}]_0)\}.$$

For simplicity, we put $U_\epsilon = \pi^{-1}(U_\epsilon(z))$ and $\mu = \mu_{u^*}$. We can choose a set $V_\epsilon \subset \Sigma_A$ which is a finite union of cylinder sets such that

$$U_\epsilon \subset V_\epsilon \subset \, _{-N^-(\epsilon)}[\tilde{z}]_{N^+(\epsilon)}$$

and

$$\mu(V_\epsilon \setminus U_\epsilon) \leq \mu(V_\epsilon)^3.$$

And we can choose $W_\epsilon \subset \Sigma_A$ which is a finite union of cylinder sets such that

$$W_\epsilon \subset U_\epsilon$$

and

$$\mu(U_\epsilon \setminus W_\epsilon) \leq \mu(W_\epsilon)^3.$$

For an open set $V \subset \Sigma_A$, we denote the first return time from V to V by T_V:

$$T_V(x) = \inf\{i \in \mathbb{N}^+; \sigma^i x \in V\}.$$

Remark that $T_V(x)$ is finite for $\mu - a.e.$ x by Poincaré's recurrence theorem. And we define a probability measure μ_V on V by

$$\mu_V = \frac{\mu|_V}{\mu(V)}.$$

LEMMA 5.2.

$$\mu_{V_\epsilon}(T_{V_\epsilon} = i) - 2\mu(V_\epsilon)^2 \leq \mu_{U_\epsilon}(T_{U_\epsilon} = i)$$
$$\leq \mu_{W_\epsilon}(T_{W_\epsilon} = i) + 2\mu(W_\epsilon)^2.$$

PROOF: Denote the first hitting time from U_ϵ to $V_\epsilon \setminus U_\epsilon$ by $\tilde{T}_{U_\epsilon}$, i.e., for $\mu - a.e.$ $x \in U_\epsilon$,

$$\tilde{T}_{U_\epsilon}(x) = \inf\{i \in \mathbb{N}^+; \sigma^i x \in V_\epsilon \setminus U_\epsilon\}.$$

27

Similarly, we denote the first hitting time from $V_\epsilon \setminus U_\epsilon$ to U_ϵ by $\tilde{T}_{V_\epsilon \setminus U_\epsilon}$. Then the following inequality is obvious:

$$
\begin{aligned}
\mu(x \in V_\epsilon; T_{V_\epsilon}(x) = i) &\leq \mu(x \in U_\epsilon; T_{U_\epsilon}(x) = i) + \mu(x \in U_\epsilon; \tilde{T}_{U_\epsilon}(x) = i) \\
&\quad + \mu(x \in V_\epsilon \setminus U_\epsilon; T_{V_\epsilon \setminus U_\epsilon}(x) = i) + \mu(x \in V_\epsilon \setminus U_\epsilon; \tilde{T}_{V_\epsilon \setminus U_\epsilon}(x) = i) \\
&= (1) + (2) + (3) + (4).
\end{aligned}
$$

But it is evident that

$$(3) + (4) \leq \mu(V_\epsilon \setminus U_\epsilon) \leq \mu(V_\epsilon)^3$$

and

$$(2) \leq \mu(V_\epsilon \setminus U_\epsilon) \leq \mu(V_\epsilon)^3.$$

Therefore,

$$\mu_{V_\epsilon}(T_{V_\epsilon} = i) \leq \frac{(1)}{\mu(V_\epsilon)} + 2\mu(V_\epsilon)^2$$

$$\leq \mu_{U_\epsilon}(T_{U_\epsilon} = i) + 2\mu(V_\epsilon)^2.$$

The second inequality can be proved in a simillar manner. ∎

LEMMA 5.3. *For any positive number $\alpha > 0$,*

$$\lim_{\epsilon \to 0} \mu(V_\epsilon)^2 \frac{e^{-\alpha\mu(U_\epsilon)}}{1 - e^{-\alpha\mu(U_\epsilon)}} = 0$$

and

$$\lim_{\epsilon \to 0} \mu(W_\epsilon)^2 \frac{e^{-\alpha\mu(U_\epsilon)}}{1 - e^{-\alpha\mu(U_\epsilon)}} = 0.$$

PROOF: By the way of the choice of V_ϵ and W_ϵ, it is obvious that

$$0 \leq \mu(V_\epsilon) - \mu(U_\epsilon) \leq \mu(V_\epsilon)^3$$

and

$$0 \leq \mu(U_\epsilon) - \mu(W_\epsilon) \leq \mu(W_\epsilon)^3.$$

Hence $\mu(U_\epsilon)/\mu(V_\epsilon)$ and $\mu(U_\epsilon)/\mu(W_\epsilon)$ go to 1 as $\epsilon \to 0$. Therefore we can obtain the above lemma by simple calculation. ∎

LEMMA 5.4. For $\mu_u - a.e.$ z,

$$(5.1) \qquad \lim_{\epsilon \to 0} \sum_{i=1}^{\infty} e^{-\alpha \mu(V_\epsilon) i} \mu_{V_\epsilon}(T_{V_\epsilon} = i) = \frac{1}{1 + \alpha}$$

and

$$(5.2) \qquad \lim_{\epsilon \to 0} \sum_{i=1}^{\infty} e^{-\alpha \mu(W_\epsilon) i} \mu_{W_\epsilon}(T_{W_\epsilon} = i) = \frac{1}{1 + \alpha}.$$

PROOF: As we have seen in the previous section, it is enough to prove the lemma in the case of one-sided symbolic dynamics. Therefore we may assume that V_ϵ, $W_\epsilon \subset [\bar{z}]_{N+(\epsilon)} \subset \Sigma_A^+$.

We will only prove (5.1), because (5.2) can be proved in a simillar manner.

Define a singular perturbed Perron-Frobenius operator $\tilde{\mathcal{L}}_{V_\epsilon} : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$ by

$$\tilde{\mathcal{L}}_{V_\epsilon} f(x) = \mathcal{L}(1_{V_\epsilon^c} \cdot f)(x).$$

As the set V_ϵ is a finite union of cylinder sets, we can obtain the following Lasota-Yorke type inequality by a similar calculation as in the proof of Lemma 2.1 ;

$$(5.3) \qquad \|\tilde{\mathcal{L}}_{V_\epsilon}^p f\|_\theta \le \theta^p \|f\|_\theta + c_{V_\epsilon} \|f\|_\infty$$

where c_{V_ϵ} is a constant depending only on V_ϵ.

Then we can see the essential spectral radius of $\tilde{\mathcal{L}}_{V_\epsilon}$ is not greater than θ . (cf. Lemma 2.2)

Define the zeta function associated with $\tilde{\mathcal{L}}_{V_\epsilon}$ as follows :

$$\tilde{\zeta}_{V_\epsilon}(t) = \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{x \in Fix_p \sigma} e^{S_p u(x)} \prod_{j=0}^{p-1} 1_{V_\epsilon^c}(\sigma^j x)\}$$

$$= \exp\{\sum_{p=1}^{\infty} \frac{t^p}{p} \sum_{\substack{a_1 \cdots a_p \\ A_{a_p a_1} = 1}} \tilde{\mathcal{L}}_{V_\epsilon}^p 1_{[a_1 \cdots a_p]}(\dot{a}_1 \cdots \dot{a}_p)\}.$$

Then the same correspondance as Proposition 3.2 holds between the eigenvalues of $\tilde{\mathcal{L}}_{V_\epsilon}$ and the poles of $\tilde{\zeta}_{V_\epsilon}(t)$. In particular, the eigenvalue of $\tilde{\mathcal{L}}_{V_\epsilon}$ of maximal modulus,

29

say $\tilde{\lambda}_{V_\epsilon}$, is equal to the inverse of the convergence radius of $\tilde{\zeta}_{V_\epsilon}(t)$, say $\tilde{t}_{V_\epsilon}$. As $V_\epsilon \subset [\bar{z}]_{N+(\epsilon)}$,

$$\sum_{x \in Fix_p\sigma} e^{S_p u(x)} \prod_{j=0}^{p-1} 1_{[\bar{z}]^c_{N+(\epsilon)}}(\sigma^j x) \le \sum_{x \in Fix_p\sigma} e^{S_p u(x)} \prod_{j=0}^{p-1} 1_{V_\epsilon^c}(\sigma^j x).$$

Hence $\tilde{t}_{V_\epsilon} \le \tilde{t}_{N+(\epsilon)}$, i.e., $\tilde{\lambda}_{N+(\epsilon)} \le \tilde{\lambda}_{V_\epsilon}$. Consequently, as $\lim_{\epsilon \to 0} \tilde{\lambda}_{N+(\epsilon)} = 1$, we get

$$\lim_{\epsilon \to 0} \tilde{\lambda}_{V_\epsilon} = 1.$$

By the Lasota-Yorke type inequality (5.3) and the Ionescu-Tulcia-Marinescu theorem, we can obtain the decomposition:

$$\tilde{\mathcal{L}}_{V_\epsilon} = \tilde{\lambda}_{V_\epsilon} \tilde{E}_{V_\epsilon} + \tilde{\Psi}_{V_\epsilon}$$

where the oprator $\tilde{E}_{V_\epsilon}$ is the projection to the eigenspace corresponding to the eigenvalue $\tilde{\lambda}_{V_\epsilon}$ and the oprator $\tilde{\Psi}_{V_\epsilon}$ is a bounded linear operator such that $\tilde{E}_{V_\epsilon} \tilde{\Psi}_{V_\epsilon} = \tilde{\Psi}_{V_\epsilon} \tilde{E}_{V_\epsilon} = 0$. Then the same assertions as Lemma 4.3, 4.4, 4.5, 4.6 remain valid for $\tilde{E}_{V_\epsilon}$ and $\tilde{\Psi}_{V_\epsilon}$ as $\epsilon \to 0$. Therefore we obtain (5.1) by a similar calculation as in the proof of Theorem 4.7.

PROOF OF THEOREM 5.1: We will denote by $\phi_\epsilon(\alpha)$ the Laplace transform of the normalized first return time $c_\epsilon T_{\epsilon, f}$:

$$\phi_\epsilon(\alpha) = \int e^{-\alpha c_\epsilon T_{\epsilon, f}} d\mu_\epsilon$$

$$= \sum_{i=1}^{\infty} e^{-\alpha c_\epsilon i} \mu_\epsilon(T_{\epsilon, f} = i)$$

$$= \sum_{i=1}^{\infty} e^{-\alpha c_\epsilon \mu(U_\epsilon)} \mu_{U_\epsilon}(T_{U_\epsilon} = i)$$

and consider the limit of $\phi_\epsilon(\alpha)$ as $\epsilon \to 0$. By Lemma 5.2 and the inequality $\mu(W_\epsilon) \le \mu(U_\epsilon) \le \mu(V_\epsilon)$,

$$\sum_{i=1}^{\infty} e^{-\alpha \mu(V_\epsilon)} \mu_{V_\epsilon}(T_{V_\epsilon} = i) - 2\mu(V_\epsilon) \sum_{i=1}^{\infty} e^{-\alpha \mu(U_\epsilon)i}$$

$$\le \phi_\epsilon(\alpha)$$

$$\le \sum_{i=1}^{\infty} e^{-\alpha \mu(W_\epsilon)} \mu_{W_\epsilon}(T_{W_\epsilon} = i) + 2\mu(W_\epsilon)^2 \sum_{i=1}^{\infty} e^{-\alpha \mu(U_\epsilon)i}.$$

By Lemma 5.3 and Lemma 5.4, we get

$$\lim_{\epsilon \to 0} \phi_\epsilon(\alpha) = \frac{1}{1+\alpha}.$$

This implies Theorem 5.1. ∎

We denote the k-th return time by $T_{\epsilon,f}{}^{(k)}$:

$$T_{\epsilon,f}{}^{(k)}(x) = \sum_{j=0}^{k-1} T_{\epsilon,f}(f^{T_{\epsilon,f}{}^{(j)}(x)}x) \qquad k = 1, 2, \cdots$$

where $T_{\epsilon,f}{}^{(0)} \equiv 0$. And we define the normalized return time process Y_ϵ as follows :

$$Y_\epsilon(\cdot) = \sum_{k=1}^{\infty} \delta_{c_\epsilon T_{\epsilon,f}{}^{(k)}}(\cdot).$$

As (Ω, μ_u, f) is mixing, the same statements as Lemma 4.8 and Proposition 4.9 hold for the normalized k-th return times $c_\epsilon T_{\epsilon,f}{}^{(k)}$. Therefore we obtain the main theorem.

THEOREM 5.5. *For* $\mu_u - a.e.$ $z \in \Omega$, *the sequence of the normalized return time processes* $\{Y_\epsilon\}_\epsilon$ *converges to the Poisson point process as* $\epsilon \to 0$ *in finite dimensional distribution, i.e. for any disjoint Borel sets* $B_1, \cdots, B_n \in \mathcal{B}(\mathbf{R}^+)$ *and any non-negative integers* $k_1, \cdots, k_n$,

$$\lim_{\epsilon \to 0} \mu_\epsilon(Y_\epsilon(B_1) = k_1, \cdots, Y_\epsilon(B_n) = k_n) = \prod_{i=1}^{n} \frac{\ell(B_i)^{k_i}}{k_i!} e^{-\ell(B_i)}$$

where ℓ *is the Lebesgue measure.*

6. Counter-example.

As is mentioned in the introduction, Theorem 5.5 does not hold for every point $z \in \Omega$. In this section, we will consider the typical counter-example, the case where a point $z \in \Omega$ is periodic. As we have seen in the above sections, the problem can be essentially reduced to the case of one-sided symbolic dynamics, and so we will only consider the system $(\Sigma_A^+, \sigma, \mu)$. The essential point is that Theorem 4.7 fails whenever a point $z \in \Sigma_A^+$ is periodic. Indeed, the limit distribution of the normalized first return

31

time $\epsilon_N T_N$ as $N \to \infty$ exists but is not an exponential distribution, but it turns out to be a linear combination of the delta-distribution and an exponential distribution.

We assume that a point $z \in \Sigma_A^+$ is a periodic point with period m. By the same technique of the proof of Lemma 3.4 and Lemma 3.5, we can see that the equality in Lemma 3.4 and the statement of Lemma 3.5 remain valid even if a point z is periodic. Hence the eigenvalue of $\widetilde{\mathcal{L}}_N$ of the maximal moduls, $\tilde{\lambda}_N$, converges to 1 as $N \to \infty$ and the statement of Lemma 4.3 holds. But in this case, Lemma 4.4 does fail, i.e., $\int \tilde{E}_N 1 \mu_N$ does not converge to 1. Indeed, we can obtain the following.

LEMMA 6.1. For a periodic point $z \in \Sigma_A^+$ with period m,

$$\lim_{N \to \infty} \frac{\int \tilde{E}_N(\mathcal{L} 1_{[z]_N}) d\mu_N}{1 - \tilde{\lambda}_N} = \lim_{N \to \infty} \int \tilde{E}_N 1 d\mu_N = 1 - e^{S_m u(z)}.$$

PROOF: Recall that

$$\widetilde{\mathcal{L}}_N^p 1(x) = \sum_{a_1, \ldots, a_p} e^{S_p u(a_1 \ldots a_p x)} 1_{[z]_N^c}(a_1 \ldots a_p x) \ldots 1_{[z]_N^c}(a_p x).$$

Fix $k \in \mathbb{N}$ arbitrary and put $p = km$. For N large enough,

$$1_{[z]_N}(x) - 1_{[z]_N}(x) \cdot \widetilde{\mathcal{L}}_N^p 1(x) = 1_{[z]_N}(x) \cdot \sum_{a_1 \cdots a_{m(k-1)}} e^{S_p u(a_1 \cdots a_{m(k-1)} z_0 \cdots z_{m-1} x)}$$

$$= 1_{[z]_N}(x) \cdot \mathcal{L}^{p-m} e^{S_m u(z_0 \cdots z_{m-1} x)}.$$

Then,

$$\left| 1 - \int \widetilde{\mathcal{L}}_N^p 1 d\mu_N - e^{S_m u(z)} \right| \le \int \mathcal{L}^{p-m} |e^{S_m u(z_0 \cdots z_{m-1} x)} - e^{S_m u(z)}| d\mu_N$$

$$\le e^{S_m u(z)} (e^{\theta^{N+m} \|u\|_\theta / 1 - \theta} - 1)$$

$$\to 0 \qquad (N \to \infty).$$

Therefore, for any $p = km$,

$$\lim_{N \to \infty} \int \widetilde{\mathcal{L}}_N^p 1 d\mu_N = 1 - e^{S_m u(z)}.$$

32

On the other hand, by Lemma 4.3, the following inequality holds for large N,

$$\left| \int \widetilde{\mathcal{L}}_N^p 1 d\mu_N - \bar{\lambda}_N^p \int \bar{E}_N 1 d\mu_N \right| = \left| \int \bar{\Psi}_N^p 1 d\mu_N \right|$$
$$\leq \|\bar{\Psi}_N^p 1\|_\infty$$
$$\leq H q^p.$$

We fixed $p = km$ arbitrary and $\tilde{\lambda}_N$ goes to 1, therefore,

$$\lim_{N \to \infty} \int \bar{E}_N 1 d\mu_N = 1 - e^{S_m u(z)}.$$

∎

THEOREM 6.2. *For a periodic point $z \in \Sigma_A^+$ with period m, the limit distribution of the normalized first return time $\epsilon_N T_N$ as $N \to \infty$ exists and it is the linear combination of the delta-distribution and the exponential distribution. Precisely,*

$$\lim_{N \to \infty} \mu_N(\epsilon_N T_N < t) = 1 - \rho_z + \rho_z(1 - e^{-\rho_z t})$$

where $\rho_z = 1 - e^{S_m u(z)}$.

PROOF: By Lemma 6.1 and a similar calculation as in the proof of Lemma 4.5 , we obtain

$$\lim_{N \to \infty} \frac{\int \bar{E}_N(\mathcal{L} 1_{[z]_N}) d\mu_N}{\epsilon_N} = \rho_z^2.$$

Then the limit of the Laplace transform of the normalized first return time , $\phi_N(\alpha)$, is given by

$$\lim_{N \to \infty} \phi_N(\alpha) = 1 - \rho_z + \rho_z \cdot \frac{\rho_z}{\alpha + \rho_z}.$$

This proves the above theorem. ∎

<div align="center">REFERENCES</div>

[**B.**] R. Bowen, "*Equilibrium states and the ergodic theory of Anosov diffeomorphisms,*" Lec. Note. in Math. vol.470, Springer, 1975.

[I.T.M.] C. T. Ionescu-Tulcea, G. Marinescu, Théorie ergodique pour des classes d'operations non complètement continues, *Ann. of Math.* **52** (1950), 140–147.

[M.] T. Morita, Random iteration of one-dimensional transformations, *Osaka J. Math.* **22** (1985), 489–518.

[N.] R. D. Nussbaum, The radius of the essential spectrum, *Duke Math. J.* **37** (1970), 473–478.

[PE.] K. Petersen, "*Ergodic theory*," Cambridge University Press, 1983.

[P.] M. Pollicott, Meromorphic extensions of generalized zeta function, *Invent. math.* **85** (1986), 147–164.

[R.I] D. Ruelle, "*Thermodynamic formalism*," Encyclopedia of Mathematics and its Applications, vol.5, Addison-Wesley, 1978.

[R.II] D. Ruelle, The thermodynamic formalism for expanding maps, *Comm. Math. Phys.* **125** (1989), 239–262.

[S.I] Ya. G. Sinai, Some mathematical problems in the theory of quantum chaos, *Physica A* **163** (1990), 197–204.

[S.II] Ya. G. Sinai, Mathematical problems in the theory of quantum chaos, preprint.