Geometrical Approach to Topological Sigma Model

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Contents

1 **Introduction** **3**

2 **Topological Sigma Model (A-model)** **11**

2.1 Action 11

2.1.1 The Ghost Number anomaly and BRST observables 16

2.2 Evaluation of Path Integral 17

2.3 Topological Sigma Model coupled to Topological Gravity. 18

2.4 Reduction to an Integral of Forms on Moduli Spaces 21

3 **Geometrical Calculation** **27**

3.1 Moduli Space of the Pure Matter Theory 28

3.2 Schubert Calculus 31

3.3 Gravitational Moduli Space of CP^1 40

3.3.1 Trees 40

3.3.2 From trees to moduli spaces 41

3.3.3 Homology of moduli spaces 43

3.4 Stable Map 44

3.5 Torus Action Method 46

3.5.1 Introduction of the Torus Action and the Bott Residue Formula 46

3.5.2 Construction of Fixed Point Set 48

3.5.3 Determination of the contribution from Normal and Vector bundles 49

3.5.4 Some Explicit Calculation of Amplitudes 55

3.5.5 Construction of Generating Function 59

4 **Operator Product Approach** **66**

4.1 Pure Matter Case 67

4.1.1 Strategy for Determination of Quantum Cohomology Ring of M_N^k 67

4.1.2 Reformulation as One Variable Polynomial Algebra 69

4.2 Gravitational Case 83

4.2.1 Meaning of the Correlation Function 83

4.2.2 Set up of the calculation 85

4.2.3 The Calculations 87

4.2.4 Appendix of Section 4.2:Derivation of Initial Conditions and Some Direct Counting of Amplitudes 110

5 **Mirror Symmetry** **112**

5.1 Construction of M_N^* 112

5.2 B-model 115

5.3 The Observables 117

5.4 n-point Correlation Function 117

5.5 Kodaira-Spencer equation 119

5.6 B-model on M_N^* 121

5.7 Construction of Holomorphic $(N-2,0)$ form Ω 122

5.8 Calculation of $\langle\prod_{i=1}^{N-2}\mathcal{O}_B(z_i)\rangle$ 125

5.9 The mirror map and the translation into A-model 127

6 **Conclusion** **131**

Chapter 1

Introduction

Topological Field Theory was first constructed by Edward Witten to answer Atiyah's question : "Is there any physical field theory that corresponds to Donaldson theory in mathematics ?"
We don't know details of Donaldson theory, but it is well-known that moduli space of anti-self dual instanton in $SU(2)$ gauge theory plays an important role in the proof of main theorems of this theory. Moduli spaces come out naturally in field theory through path-integral which integrates all the field configurations with weight $\exp(-\text{Action})$. Roughly speaking, the question at the beginning arises from these two facts. In ordinary field theory, moduli space is non-dynamical degree of freedom which should be integrated after integrating out all the dynamical degrees of freedom. In other words, it is continuous family of equation of motion which should be treated as background. So, at first sight, it seems unnatural to seek for the theory that picks up non-dynamical degrees of freedom.

But with success in "Supersymmetry and Morse Theory" that explained the Hodge theory and Morse theory on finite dimensional real manifold M in terms of $N = 1$ super symmetric sigma model on M and its deformation via Morse function, Witten thought the key is super symmetry (BRST-symmetry), or fermionic degrees of freedom that kill corresponding dynamical bosonic degrees of freedom. Then in case of $SU(2)$ Yang-Mills theory, he introduced entire ghost field ψ_μ and anti-ghost field $\chi_{\mu\nu}$ which kill entire bosonic gauge field A_μ and many other fields which are needed to make BRST-transformation closed, and wrote out Lagrangian of Topological Yang-Mills theory on 4 dimensional manifold M.

$$\begin{aligned} L = \int_M d^4x\sqrt{g}Tr[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi D_\mu D^\mu\lambda - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\cdot\chi^{\mu\nu} \\ -\frac{i}{8}\phi[\chi_{\mu\nu},\chi^{\mu\nu}] - \frac{i}{2}\lambda[\psi_\mu,\psi^\mu] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\lambda]^2] \end{aligned} \quad (1.0.1)$$

The solution of bosonic equation of motion of this theory is anti self-dual instanton of $SU(2)$ Yang-Mills theory. And Lagrangian L has BRST-symmetry, i.e., $\{Q, L\} = 0$.

Moreover it can be written as $L = \{Q, S\}$ modulo equations of motion. Then moduli space of instanton reveals itself under the following logic. Under the condition $L = \{Q, S\}$, we can take weak coupling limit of the theory because path integral is invariant under the variation of coupling constant. In this limit, contributions from dynamical modes are integrated by Gaussian integration around the instanton solution and equal $\frac{det'(D^\dagger)det'(D)}{det'(D^\dagger D)} = 1$ where numerator comes from fermionic modes and denominator comes from bosonic modes and $D, D^\dagger$ are differential operators obtained from quadratic term of Lagrangian expanded around instanton configurations. Then there remains integration over moduli space of instantons as the bosonic degrees of freedom. We denote this moduli space as $\mathcal{M}$. Correspondingly, fermionic degrees of freedom arise as the zero-modes of differential operator D, the number of which equals the dimension of moduli space. These zero-modes can be regarded as differential form on $\mathcal{M}$.

Next, consider the observables of this theory. Since this theory has BRST-symmetry, observables $\mathcal{O}$ has to be BRST-closed, i.e., $\{Q, \mathcal{O}\} = 0$. Then Witten daringly but correctly assumed that this condition is equal to the condition $d_\mathcal{M}\mathcal{O} = 0$ if we regard $\mathcal{O}$ as the form on $\mathcal{M}$. And correlation function $\langle\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_k\rangle$ reduces to intersection number on moduli space $\mathcal{M}$, $\sharp(PD_\mathcal{M}(\mathcal{O}_1)\cap PD_\mathcal{M}(\mathcal{O}_2)\cap\cdots PD_\mathcal{M}(\mathcal{O}_k))$ where $PD_M(w)$ represents Poincare dual of $w \in H^*(M)$.

In this way, this theory turns into intersection theory of moduli space $\mathcal{M}$ of $SU(2)$ anti-self dual instanton on M and Donaldson invariants are reconstructed as correlation functions of topological Yang-Mills theory.[9]

After that, he constructed various topological field theory. We write out four of these.

1. Topological Yang-Mills theory on four dimensional manifold.
2. Chern-Simons gauge theory on 3-dimensional manifold.
3. Topological Sigma Model from Riemann surface Σ to Kähler manifold M.
4. Two dimensional Topological Gravity.

When these theory are constructed, people ought to have had the impression :"Of course, they are well-formed, but how to treat these theories or to compute correlation functions ?"

We think break-through of this problem was occurred in the following order $4 \to 3 \to 1$. (As for 2, we don't know much about it and don't mention it).

Topological quantum gravity was identified with intersection theory on moduli space $\mathcal{M}_{g,n}$ of complex structure of genus g Riemann surface $\Sigma_{g,n}$ with n-punctures by Witten. Observables of this theory are given as Mumford-Morita classes $c_1^n(T_{z_i}\Sigma_{g,n})$.[30]

Using this identification, he derived some recursion relation between correlation functions of this theory and reproduce the dispersion-less limit result of matrix model.

Complete treatment was done by Kontsevich, who constructed matrix integral representation of generating function of all correlation functions by identifying $\mathcal{M}_{g,n}$ with the moduli space of ribbon graphs. This work also revealed the relation between topological gravity and KP-hierarchy.[31]

Break-through on 3 occurred in quite non-trivial way. It came out from compactification of the heterotic string theory on Calabi-Yau manifolds of complex dimension 3. Then at energies small compared to Planck scale, an effective four-dimensional super gravity theory whose component fields correspond to the parameters that describe the possible deformations of Calabi-Yau manifold emerges. These massless fields correspond to the parameters that take one vacuum into nearby equivalent one. The terms that make up the effective Lagrangian of the low energy were said to have topological significance, one of which is the Yukawa coupling, the cubic term of the effective Lagrangian.

For Calabi-Yau manifolds, the deformation parameters (massless fields) are factorized into two moduli spaces, one of which is the Kähler moduli space that deforms the size of Calabi-Yau manifolds, the other of which is the Complex structure moduli space that changes the shape of Calabi-Yau manifolds. Then the above Yukawa coupling broke up into two pieces,i.e., the coupling of Kähler moduli space and the coupling of complex structure moduli space. The coupling of Kähler moduli space turn out to receive instanton corrections but the coupling of complex moduli space is exact at classical level.

Candelas et al proposed Mirror Symmetry between two Calabi-Yau manifold M and M^*, i.e., string theory compactified on M and the one compactified on M^* are isomorphic to each other under exchange of Kähler moduli space and complex structure moduli space. To give more concrete foundation of this isomorphism, they calculated Kähler Yukawa coupling on string theory compactified on Calabi-Yau manifold of Fermat type in CP^4 (we denote it as M_5), by using the result of complex structure Yukawa coupling compactified on its mirror manifold M_5^* and mirror map that relates deformation parameter of complex structure of M_5^* and that of Kähler structure of M_5. Their result tells us that Yukawa coupling arising from Kähler class deformation indeed has instanton corrections which come from holomorphic maps from CP^1 (string world sheet) to M_5. By evaluating contributions from one instanton solution, one can count the number of instantons from CP^1 to M_5 from their result.[15]

Then reinterpretation of this result emerged.

In [28], Eguchi and Yang, inspired by the statement of Witten that topological Yang-Mills theory can be regarded as twisted $N=2$ super symmetric Yang-Mills theory, proposed that a class of topological field theories are constructed from twist-

ing $N=2$ super conformal field theory. Witten pointed out in [8] that $N=2$ super symmetric sigma model becomes conformal invariant when target space is Calabi-Yau manifold and that there are two ways of twisting (we denote them as A-twist and B-twist).

Then mirror symmetry can be reinterpreted as isomorphism between A-twisted topological sigma model (A-model) on M and B-twisted topological sigma model (B-model) on M^*. He also pointed out in [30] that the above Yukawa coupling is equivalent to three point function $\langle\mathcal{O}_e(z_1)\mathcal{O}_e(z_2)\mathcal{O}_e(z_3)\rangle$ of A-model on M_5 where $\mathcal{O}_e$ is the BRST-closed operator induced from Kähler form of M_5.

Another flow occurred from Batyrev who gave systematic way of construction of mirror pair of Calabi-Yau manifolds. He suggested that two toric variety P_Δ and P_{Δ^*} that are constructed from two reflexive polyhedra Δ and Δ^* dual to each other are ambient spaces of mirror pair of Calabi-Yau manifolds M_Δ and M_{Δ^*}. This construction tells us that mirror pair of Calabi-Yau manifolds exist in arbitrary dimension. This was supported by the work of Nagura and Sugiyama who generalized the result of P.Candelas et al to the cases of torus and K3 surface.[2],[37] They generalized the mirror symmetry as the symmetry of topological sigma model of Calabi-Yau manifold of dimension lower than three and concluded that corresponding correlation functions (in our words, $\langle\mathcal{O}_e(z)\rangle$ for torus and $\langle\mathcal{O}_e(z_1)\mathcal{O}_e(z_2)\rangle$ for K3-surface) have no instanton corrections. Then Nagura and myself analyzed topological sigma model (A-model) on Calabi-Yau hypersurface M_N in CP^{N-1} and (B-model) on M_N^*(mirror manifold of M_N). Assuming that the result of Candelas et al is the one of topological sigma model on M_5, we generalized the treatment of [15] and computed $(N-2)$-point function

$$\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)(t)\rangle=\sum_{d=0}^{\infty}\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d e^{-dt} \tag{1.0.2}$$

where d represents degree of holomorphic maps and $\mathcal{O}_e$ is BRST-closed observable induced from Kähler form $e_{M_N}\in H^*(M_N)$. Then we find that for $N\geq 5$ case, these correlation functions have instanton corrections whose values are integers. Of course, at the same time, Morrison et al. treated the same model and compute some correlation functions, they did not explain clearly the meaning of instanton corrections.

Then what remains to show is that our result is really a correlation function of A-model on M_N, in other words, the number of holomorphic maps f from CP^1 to M_N which satisfy the following condition.

$$f(z_j)\in PD_{M_N}(e_{M_N})\quad (j=1,2,\cdots,N-2) \tag{1.0.3}$$

The reason why correlation functions are determined from (1.0.3) will be explained in Section 2. First idea came from discussion with Dr.Okai. With him, we found the simple statement in the famous textbook of algebraic geometry [19].

$$PD(e_{M_N})=PD(e_{CP^{N-1}})\cap M_N \tag{1.0.4}$$

Then condition (1.0.3) decomposes into the following two conditions on holomorphic map f from CP^1 to CP^{N-1}.

$$\begin{aligned} f(CP^1) &\subset M_N \\ f(z_i) &\in PD_{CP^{N-1}}(e_{CP^{N-1}}) \end{aligned} \qquad (1.0.5)$$

With this idea and the following fact that holomorphic map f from CP^1 of degree d is described by polynomial map,

$$f:(s:t)\mapsto (\sum_{j=1}^{d} a_j^1 s^j t^{d-j}:\cdots:\sum_{j=1}^{d} a_j^N s^j t^{d-j}) \qquad (1.0.6)$$

we roughly evaluated $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d$ as N^{dN+1}. This result is different from our result from mirror symmetry, but reproduces the top term of N-expansion of $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle$. Then we thought "the condition (1.0.5) is fundamentally right, but we have to know more about moduli space $\mathcal{M}_{0,d}^{CP^{N-1}}$ of holomorphic maps from CP^1 to CP^{N-1} of degree d". We also concluded that differences between exact results and N^{Nd+1} come from boundary part of space of polynomial maps that consists of maps superficially of degree d but truly of lower degree by projective equivalence. In degree 1 case, with the help of Prof.Oguiso and Dr.Hori, we exactly reproduced $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_1$ using the condition (1.0.5) and the fact that $\mathcal{M}_{0,1}^{CP^{N-1}}/SL(2,C) = Gr(2,N)$. This result was the first step for explicit statement that mirror symmetry is the symmetry between A-model on M_N and B-model on M_N^*.

In the same year, many works on topological sigma model (A-model) appeared mainly from Kontsevich. These works focused on A-model coupled to gravity. In [21], Kontsevich and Manin proposed that A-model on Fano variety (complex manifold with positive first Chern class or positive Ricci curvature) can be solved by using Dijkgraaf-Witten-Verlinde-Verlinde equation (later we abbreviate it as DWVV equation) or associativity of operator algebra of BRST-closed observables. With this statement, Dr.Y.Sun and myself solved topological sigma models coupled to gravity on CP^3, CP^4, and $Gr(2,4)$ and found that in $Gr(2,4)$ case, symmetry in classical cohomology ring are conserved in correlation functions of BRST-closed observables induced from elements of $H^*(Gr(2,4))$. This work suggested that the associativity condition is powerful in treating topological sigma model coupled to gravity on Fano variety.

The reason why mathematicians prefer A-model coupled to gravity to pure A-model lies in the fact that (gravitational) moduli space of complex structure of CP^1 with n-punctures $\mathcal{M}_{0,n}$ is compactified by stable curves and analyzed completely in terms of tree graphs. Geometrical proof of DWVV equation was given by Kontsevich and Manin in [21] using the result on homology of $\mathcal{M}_{0,n}$ by Keel [36] and splitting axiom. With this concept, Kontsevich proceed further to the notion of stable maps, which compactify moduli space of holomorphic maps from CP^1 with n-punctures to Kähler manifold M that corresponds to moduli space associated with A-model

coupled to gravity from CP^1 to M. With these set up, he performed exact calculation of correlation functions on M_5 and CP^2 by use of Bott-residue formula in [1] (we call this method torus action method).

We thought that by combining the condition (1.0.5) and torus action method (to be more precise, some subtle changes occur because we couple gravity to the model), we can calculate correlation correlation functions of A-model on M_N coupled to gravity. This speculation turned out to be right and we reached integral representation of generating function of correlation functions of this model. Using the fact that 3-point functions of pure A-model and the ones coupled with gravity coincide (notified by Prof. Y. Yamada) and that fusion rule holds in pure A-model (pointed out by Witten in [30]), we reproduced $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d$ up to degree 3 and suggested that we can reproduce them to arbitrary degree d. Thus we gave practical proof of string tree level mirror symmetry in the sense of symmetry between A-model on M_N and B-model on M_N^*.

Topological sigma model (pure matter theory) on Fano variety (esp. for CP^N and Grassmannian) was studied from another point of view, by Vafa and Intriligator. They argued that classical cohomology ring of Grassmann variety (including CP^N) is described as polynomial ring divided by the ideal generated by the derivative dW of Landau-Ginzuburg super potential W. And they suggested perturbation of W by the elements corresponding to Kähler form gives quantum cohomology ring which is equivalent to pure matter theory on Grassmannian. Because resulting correlation functions given as the residue of perturbed super potential are non-zero only if topological selection rule of pure matter theory is satisfied. Of course, they are integers. But their argument was merely a conjecture.[5]

Geometrical proof of this was given by Bertram [4] who constructed the compactified matter moduli space of holomorphic maps from CP^1 to Grassmanians and evaluate three point functions of pure matter theory with this moduli space. And using the fact that fusion rule which reduces correlation functions into products of three point functions holds in pure matter theory, he reproduced the result of Vafa and Intriligator.

This geometrical reproduction tells us that if we can evaluate three point functions of pure matter theory, we can solve pure matter theory on any target space.

With this idea, we analyzed pure A-model on degree k hypersurface in CP^{N-1} (we denote it as M_N^k) to seek for the reason why N^{dN+1} gives the top term of $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d$. Tracing the same logic which gives $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d \simeq N^{dN+1}$, we roughly evaluated $\langle\prod_{j=1}^{N-2+(N-k)d}\mathcal{O}_e(z_j)\rangle_d \simeq k^{kd+1}$ for pure A-model on M_N^k . Then we used the idea which reproduced $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_d$ in $M_N^N = M_N$ case, i.e., fusion rule that decomposes any correlation functions into sum of products of three point functions. More explicitly, we assumed pure matter theory on M_N^k is constructed by two relations,

$$\begin{aligned} \mathcal{O}_{e^l} \cdot \mathcal{O}_{e^j} &= \langle \mathcal{O}_{e^l}\mathcal{O}_{e^j}\mathcal{O}_{e^l}\rangle \frac{1}{k}\delta_{l+m,N-2}\mathcal{O}_{e^m} & (1.0.7) \\ \langle \mathcal{O}_{e^l}\mathcal{O}_{e^j} *\rangle &= \langle \mathcal{O}_{e^l}\mathcal{O}_{e^j}\mathcal{O}_{e^l}\rangle \frac{1}{k}\delta_{l+m,N-2}\langle \mathcal{O}_{e^m} *\rangle & (1.0.8) \end{aligned}$$

where $\mathcal{O}_{e^j}$ denotes BRST-closed operator induced from $e^j_{M^k_N} \in H^*(M^k_N)$. We evaluated all the three point functions in need by torus action method and found the following relations hold if k is no more than $N-2$.

$$(\mathcal{O}_e)^{N-1} = k^k e^{-t} (\mathcal{O}_e)^{k-1} \tag{1.0.9}$$

This is the natural generalization of the well-known result of CP^{N-2} model,

$$(\mathcal{O}_e)^{N-1} = e^{-t} \tag{1.0.10}$$

and it corresponds to $\partial_X W(X)$ if we set $X := \mathcal{O}_e$. (1.0.9) tells us that if the condition $k \leq N-2$ is satisfied, $\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_j)\rangle_d = k^{kd+1}e^{-dt}$. The exactness of the rough evaluation can be explained from the dimensional counting of boundary parts of polynomial maps mentioned before. In this way, we showed that (1.0.5) is fundamentally right and the first speculation about difference between N^{dN+1} and exact result of $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)\rangle_d$ is adequate. Of course, we limited the Hilbert space of this model to the space spanned by $\mathcal{O}_{e^j}$, so generalization of this discussion to full Hilbert space is expected. But we think this is the first step to generalization of the discussion of Vafa and Intriligator to pure matter theory on arbitrary Kähler manifolds.

This thesis consists of these works. Chapters are ordered from geometric formulation to field theoretic formulation.

In Chapter 2, we review topological sigma model (A-model) and show the strategy to treat topological sigma model on M^k_N.

In Chapter 3, we perform geometrical calculations of correlation functions on M^k_N. In Section 3.1, we introduce pure matter moduli space of holomorphic maps from CP^1 to CP^{N-1} and under the strategy of Chapter 2, we derive $\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_j)\rangle_d \simeq k^{kd+1}e^{-dt}$ and discuss the limitation of this evaluation. In Section 3.2, we perform exact calculation of $\langle \prod_{j=1}^{N-2+(N-k)} \mathcal{O}_e(z_j)\rangle_1$ using Schubert calculus of $H^*(Gr(2,4))$. In section 3.3, we review mathematical theory of gravitational moduli space of CP^1 with punctures to prepare for the notion of stable map. In Section 3.4, we introduce stable map and using the results of Section 3.3, we give geometrical proof of DWVV equation. In Section 3.5, we introduce torus action method and perform some explicit calculation of correlation functions of A-model coupled to gravity on M_N and evaluate $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)\rangle_d$ of pure A-model on M_N up to degree 3. Finally, we give integral representation of generating function of A-model coupled to gravity on M^k_N.

Chapter 4 is devoted to topics in connection with operator algebra of BRST-closed observables. In Section 4.1, we construct quantum cohomology ring of pure A-model on M^k_N and derive the formula (1.0.9). In Section 4.2, we solve A-model coupled to gravity on CP^3, CP^4 and $Gr(2,4)$ using DWVV equation derived from associativity of operator product algebra.

In Chapter 5, we treat mirror symmetry between A-model on M_N and B-model on M^*_N, both of which are Calabi-Yau manifolds. In section 5.1, we construct mirror pair of M_N and M^*_N using the result of Batyrev, Hosono et al. Section 5.2 and 5.3 are given for review of B-model. In section 5.4 and 5.5, we introduce Kodaira-Spencer equation and give the formalism for calculating the correlation function of B-model. From Section 5.6 to Section 5.8, we apply the above formalism to B-model on M^*_N and calculate $\langle \prod_{j=1}^{N-2} \mathcal{O}_B(z_j)(x)\rangle$ where x is deformation parameter of complex structure of M^*_N. It corresponds to $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)(t)\rangle$ of A-model on M_N. In section 5.9, we construct mirror map which translates x into coupling constant (Kähler deformation parameter) t of A-model on M_N and give the N-expansion form of $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)(t)\rangle = \sum_{d=0}^{\infty} \langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)\rangle_d e^{-dt}$. Finally we write out some numerical results and see the coincidence with the result of Section 3.5.

Chapter 2

Topological Sigma Model (A-model)

2.1 Action

The A-model is obtained by *twisting* a N=2 super symmetric non-linear sigma model defined on a Kähler manifold. N=2 super symmetric non-linear sigma model is defined as follows. Let M be a n-dimensional Kähler manifold and ϕ^i be a holomorphic coordinate on M $(i = 1, \cdots n)$(and $\phi^{\bar i}$ be a anti-holomorphic coordinate), Σ be a Riemann surface, which, in this thesis, is restricted to genus zero, and z be a holomorphic coordinate on Σ.

The Lagrangian is

$$L = 2t\int_\Sigma d^2z\Big(\frac{1}{2}g_{i\bar j}(\partial_z\phi^i\partial_{\bar z}\phi^{\bar j}+\partial_{\bar z}\phi^i\partial_z\phi^{\bar j})+i\psi_-^{\bar j}D_{\bar z}\psi_-^i g_{i\bar j}+i\psi_+^{\bar j}D_z\psi_+^i g_{i\bar j}+R_{i\bar i j\bar j}\psi_+^i\psi_+^{\bar i}\psi_-^j\psi_-^{\bar j}\Big) \tag{2.1.1}$$

where $\phi^i(z)$ is a map from Σ to M (these are the main dynamical variables in this model). ψ fields are fermionic degrees of freedom. We put world sheet spin quantum number +1 to $\bar z$ and −1 to z. For ψ fields, + (resp.−) means spin quantum number $\frac{1}{2}$ (resp.$-\frac{1}{2}$). D_z, $D_{\bar z}$ represent covariant derivatives with respect to pull-back of tangent bundle on M and to world sheet spin. Explicitly, they are

$$\begin{aligned} D_{\bar z}\psi_-^i &= \partial_{\bar z}\psi_-^i + \partial_{\bar z}\phi^j\Gamma^i_{jk}\psi_-^k \\ D_z\psi_+^i &= \partial_z\psi_+^i + \partial_z\phi^j\Gamma^i_{jk}\psi_+^k. \end{aligned} \tag{2.1.2}$$

Note that since M and Σ are Kähler, remaining connections on tangent bundle are Γ^i_{jk}, $\Gamma^{\bar i}_{\bar j\bar k}$, Γ^z_{zz} and $\Gamma^{\bar z}_{\bar z\bar z}$. They are written in terms of Kähler metric $g_{i\bar j}, g_{z,\bar z}$ as follows.

$$\begin{aligned} \Gamma^i_{jk} = g^{i\bar l}\partial_j g_{k\bar l}, &\quad \Gamma^{\bar i}_{\bar j\bar k} = g^{\bar i l}\partial_{\bar j} g_{\bar k l} \\ \Gamma^z_{zz} = g^{z\bar z'}\partial_z g_{z\bar z'}, &\quad \Gamma^{\bar z}_{\bar z\bar z} = g^{z'\bar z}\partial_{\bar z} g_{z'\bar z} \end{aligned} \tag{2.1.3}$$

This Lagrangian possesses N=2 super symmetry. In terms of fermionic parameter $\alpha_-, \tilde\alpha_-$, and $\alpha_+, \tilde\alpha_+$, the super transformation laws are given as follows.

$$\begin{aligned} \delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \\ \delta\phi^{\bar i} &= i\tilde\alpha_-\psi_+^{\bar i} + i\tilde\alpha_+\psi_-^{\bar i} \\ \delta\psi_+^i &= -\tilde\alpha_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jm}\psi_+^m \\ \delta\psi_+^{\bar i} &= -\alpha_-\partial_z\phi^{\bar i} - i\tilde\alpha_+\psi_-^{\bar j}\Gamma^{\bar i}_{\bar j\bar m}\psi_+^{\bar m} \\ \delta\psi_-^i &= -\tilde\alpha_+\partial_{\bar z}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jm}\psi_-^m \\ \delta\psi_-^{\bar i} &= -\alpha_+\partial_{\bar z}\phi^{\bar i} - i\tilde\alpha_-\psi_+^{\bar j}\Gamma^{\bar i}_{\bar j\bar m}\psi_-^{\bar m} \end{aligned} \tag{2.1.4}$$

A-model is obtained by twisting the fermionic degrees of freedom. In this case we subtract half of $U(1)$ charge (we put −1 to i and 1 to $\bar i$) from fermionic world sheet spin quantum number.

$$\begin{aligned} \psi_+^i &\to \psi_z^i \\ \psi_+^{\bar i} &\to \chi^{\bar i} \\ \psi_-^i &\to \chi^i \\ \psi_-^{\bar i} &\to \psi_{\bar z}^{\bar i} \end{aligned} \tag{2.1.5}$$

Then we have the A-model Lagrangian.

$$L = 2t\int_\Sigma d^2z\Big(\frac{1}{2}g_{i\bar j}(\partial_z\phi^i\partial_{\bar z}\phi^{\bar j}+\partial_{\bar z}\phi^i\partial_z\phi^{\bar j}) + i\psi_{\bar z}^{\bar j}D_z\chi^i g_{i\bar j} + i\psi_z^i D_{\bar z}\chi^{\bar j} g_{i\bar j} - R_{i\bar i j\bar j}\psi_z^i\psi_{\bar z}^{\bar i}\chi^j\chi^{\bar j}\Big) \tag{2.1.6}$$

(2.1.6) is invariant under the infinitesimal BRST - transformation obtained from (2.1.4) by twisting fermionic degrees of freedom and setting $\alpha_+ = \tilde{\alpha}_- = \alpha$ and $\tilde{\alpha}_+ = \alpha_- = 0$ (Note that remaining fermionic infinitesimal variables become spin 0 under twisting).

$$\begin{aligned} \delta\phi^i &= i\alpha\chi^i \\ \delta\phi^{\bar i} &= i\alpha\chi^{\bar i} \\ \delta\chi^i &= \delta\chi^{\bar i} = 0 \\ \delta\psi_{\bar z}^{\bar i} &= -\alpha\partial_{\bar z}\phi^{\bar i} - i\alpha\chi^{\bar j}\Gamma^{\bar i}_{\bar j\bar m}\psi_{\bar z}^{\bar m} \\ \delta\psi_z^i &= -\alpha\partial_z\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi_z^m \end{aligned} \tag{2.1.7}$$

This invariance allows us to consider only BRST-invariant observables. We define BRST operator Q by $\delta V = -i\alpha\{Q, V\}$ for any field V. Of course, $Q^2 = 0$.

Moreover, we can rewrite the Lagrangian (2.1.6) using the ψ equation of motion,

$$\begin{aligned} i\psi_{\bar z}^{\bar i}D_z\chi^j g_{j\bar i} &= R_{i\bar i j\bar j}\psi_z^i\psi_{\bar z}^{\bar i}\chi^j\chi^{\bar j} \\ i\psi_z^i D_{\bar z}\chi^{\bar j} g_{i\bar j} &= R_{i\bar i j\bar j}\psi_z^i\psi_{\bar z}^{\bar i}\chi^j\chi^{\bar j} \end{aligned} \tag{2.1.8}$$

as follows,

$$L = it\int_{\Sigma} d^2z\{Q,V\} + t\int_{\Sigma}\Phi^*(e) \tag{2.1.9}$$

where

$$\begin{aligned} V &= g_{i\bar{j}}(\psi_z^{\bar{j}}\partial_{\bar{z}}\phi^i + \partial_z\phi^{\bar{j}}\psi_{\bar{z}}^i) \\ \{Q,V\} &= -2ig_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}} + 2g_{i\bar{j}}\psi_{\bar{z}}^i D_z\chi^{\bar{j}} + 2g_{i\bar{j}}\psi_z^{\bar{j}} D_{\bar{z}}\chi^i - 2R_{i\bar{i}j\bar{j}}\psi_{\bar{z}}^i\psi_z^{\bar{i}}\chi^j\chi^{\bar{j}} \end{aligned} \tag{2.1.10}$$

and

$$\int_{\Sigma}\Phi^*(e) = \int_{\Sigma}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}g_{i\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}g_{i\bar{j}}). \tag{2.1.11}$$

(2.1.11) is the integral of the pull-back of the Kähler form e of M, and it depends only on the intersection number between $\Phi_*(\Sigma)$ and $PD(e)$ ($PD(e)$ denotes the Poincare Dual of e), which equals to the degree of Φ. By an appropriate normalization of $g_{i\bar{j}}$, we have

$$\int_{\Sigma}\Phi^*(e) = d \tag{2.1.12}$$

where d is the degree (or winding number) of ϕ. Next, we consider the correlation function of BRST-invariant observables $\{\mathcal{O}_i\}$, i.e.

$$\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle = \int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi e^{-L}\prod_{i=1}^{k}\mathcal{O}_i. \tag{2.1.13}$$

We have seen $\int_{\Sigma}\Phi^*(e) = d$ and we decompose the space of maps ϕ into different topological sectors $\{B_d\}$ in each of which $\deg(\Phi)$ is a fixed integer.

We can rewrite (2.1.12) as follows.

$$\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle = \int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi e^{-L}\prod_{i=1}^{k}\mathcal{O}_i = \sum_{d=0}^{\infty}e^{-dt}\int_{B_d}\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi e^{-it\int_{\Sigma}d^2z\,\{Q,V\}}\prod_{i=1}^{k}\mathcal{O}_i \tag{2.1.14}$$

And we set

$$\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle_d = \int_{B_d}\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi e^{-it\int_{\Sigma}d^2z\,\{Q,V\}}\prod_{i=1}^{k}\mathcal{O}_i \tag{2.1.15}$$

We can easily see that $\int_{\Sigma}d^2z\{Q,V\} = \{Q,\int_{\Sigma}d^2zV\}$, i.e., Lagrangian is BRST exact except for topological terms. Then by taking infinitesimal variation of coupling constants, we have insertion of $\delta t\{Q,\int_{\Sigma}d^2zV\}$. It follows from this and $\{Q,\mathcal{O}_i\} = 0$ that $\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle_d$ doesn't depend on the coupling constant t and we can take weak coupling limit $t \to \infty$ in evaluating the path integral.

In this limit, the saddle point approximation of the path integral becomes exact. Saddle points of the Lagrangian are evaluated from (2.1.10) as follows.

$$\partial_{\bar{z}}\phi^i = 0 \quad \partial_z\phi^{\bar{i}} = 0 \quad \chi^i = \chi^{\bar{i}} = \psi_{\bar{z}}^i = \psi_z^{\bar{i}} = 0 \tag{2.1.16}$$

We have to note one important things. The saddle point equation shown in (2.1.16) has moduli. Dimension of moduli space can be counted by infinitesimal variation of ϕ.

$$\phi = \phi_0 + \delta\phi \tag{2.1.17}$$

Note that $\delta\phi$ takes value in tangent bundle on M. Since $\partial_{\bar{z}}\phi_0^i = \partial_z\phi_0^{\bar{i}} = 0$, we have the following equation for $\delta\phi$.

$$D_{\bar{z}}\delta\phi^i = D_z\delta\phi^{\bar{i}} = 0 \tag{2.1.18}$$

The number of linear independent solutions is the dimension of moduli space. In this thesis, we will treat the genus 0 world sheet, and we denote the moduli space in this case as $\mathcal{M}_{0,d}^M$.

We write out the fermionic equation of motion in the saddle point. Solutions of these equations are zero modes of fermionic degrees of freedom.

$$\begin{aligned} D_{\bar{z}}\chi^i &= D_z\chi^{\bar{i}} = 0 \\ D_{\bar{z}}\psi_z^{\bar{i}} &= D_z\psi_{\bar{z}}^i = 0 \end{aligned} \tag{2.1.19}$$

We can see from (2.1.19) that χ equation of motion is the same as the bosonic moduli equation. This tells us that χ-zero mode can be regarded as the basis of tangent space of $\mathcal{M}_{0,d}^M$. Of course, $\dim(\mathcal{M}_{0,d}^M) = \{\sharp \text{ of } \chi\text{−zero modes}\}$ (In this thesis, we use $\sharp$ to represent the number of elements of a set). With these discussions, we separate the integration into the integration of saddle point moduli and the integration of variations around the saddle points,

$$\begin{aligned} \langle\prod_{i=1}^{k}\mathcal{O}_i\rangle_d &= \int_{B_d}\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi e^{-it\int_{\Sigma}d^2z\,\{Q,V\}}\prod_{i=1}^{k}\mathcal{O}_i \\ &= \int_{\mathcal{M}_{0,d}^M}\mathcal{D}\mathcal{M}\int\mathcal{D}\psi_0\mathcal{D}\chi_0\int\mathcal{D}\phi'\mathcal{D}\psi'\mathcal{D}\chi' e^{-itL_{quad.}}\prod_{i=1}^{k}\mathcal{O}_i \end{aligned} \tag{2.1.20}$$

where

$$L_{quad.} = -2iD_z\phi_i'D_{\bar{z}}\phi'^{i} + 2\psi_{\bar{z}}'^{i}D_z\chi_i' + 2\psi_{z,i}'D_{\bar{z}}\chi'^{i} \tag{2.1.21}$$

ϕ', ψ'andχ' represent oscillation modes perpendicular to zero modes. In (2.1.21), we assumed that there are no ψ-zero modes, which results in no insertion of $R_{i\bar{j}k\bar{l}}\chi^{0i}\chi^{0\bar{j}}\psi_{\bar{z}}^{0k}\psi_z^{0\bar{l}}$ because there are no measure for ψ^0. Then integration of ϕ', ψ' and χ' results in

$$\frac{det(D_z')det(D_{\bar{z}}')}{det(D_z'D_{\bar{z}}')} = 1, \tag{2.1.22}$$

where $D_z', D_{\bar{z}}'$ represent operators that act on functional space except for zero modes.

Next, we discuss the case of ψ-zero modes integration. We decompose ϕ, χ and ψ into zero mode part and oscillation mode part.

$$\begin{aligned} \phi &= \phi^0 + \phi' \\ \chi &= \chi^0 + \chi' \\ \psi &= \psi^0 + \psi' \end{aligned} \tag{2.1.23}$$

Then we have quadratic part of Lagrangian by carefully treating the pararell transport of χ and ψ caused by ϕ' perturbation like

$$\begin{aligned} \chi^{i} &= \chi^{i} - \phi'^{k}\Gamma^{i}_{kj}\chi^{j} \\ \psi^{i}_{\bar{z}} &= \psi^{i}_{\bar{z}} - \phi'^{k}\Gamma^{i}_{kj}\psi^{j}_{\bar{z}}. \end{aligned} \tag{2.1.24}$$

And $L_{quad.}$ is given as follows,

$$L_{quad.} = 2t(D_{z}\phi'^{i}D_{\bar{z}}\phi'_{i} + i\psi'_{zi}D_{\bar{z}}\chi'^{i} + i\psi'^{i}_{\bar{z}}D_{z}\chi'_{i} + i\phi'^{i}\varphi_{\bar{z}zi} - i\phi'^{\bar{i}}\bar{\varphi}_{z\bar{z}\bar{i}} - R_{i\bar{j}k\bar{l}}\psi^{i0}_{\bar{z}}\psi^{\bar{j}0}_{z}\chi^{k0}\chi^{\bar{l}0}) \tag{2.1.25}$$

where

$$\begin{aligned} \varphi_{\bar{z}zi} &= \chi^{j0}\partial_{\bar{z}}\phi^{m0}R_{jmi\bar{l}}\psi^{\bar{l}0}_{z} \\ \bar{\varphi}_{z\bar{z}\bar{i}} &= \psi^{k0}_{\bar{z}}\chi^{\bar{j}0}\partial_{z}\phi^{m0}R_{\bar{j}m\bar{i}k}. \end{aligned} \tag{2.1.26}$$

In deriving this, we used the following equations.

$$R^{i}_{jl\bar{m}} = \partial_{\bar{m}}\Gamma^{i}_{jl}, \;\; R^{\bar{i}}_{\bar{j}\bar{l}m} = \partial_{m}\Gamma^{\bar{i}}_{\bar{j}\bar{l}} \tag{2.1.27}$$

For later convenience, we rescale ψ into $\frac{1}{\sqrt{t}}\psi$. Then $L_{quad.}$ changes into,

$$\begin{aligned} L_{quad.} &= -2t\phi'^{i}D_{z}D_{\bar{z}}\phi'_{i} + 2\sqrt{t}i\psi'_{zi}D_{\bar{z}}\chi'^{i} + 2\sqrt{t}i\psi'^{i}_{\bar{z}}D_{z}\chi'_{i} \\ &\quad +2\sqrt{t}i\phi'^{i}\varphi_{\bar{z}zi} - 2\sqrt{t}i\phi'^{\bar{i}}\bar{\varphi}_{z\bar{z}\bar{i}} - 2R_{i\bar{j}k\bar{l}}\psi^{i0}_{\bar{z}}\psi^{\bar{j}0}_{z}\chi^{k0}\chi^{\bar{l}0}. \end{aligned} \tag{2.1.28}$$

We introduce Green's operator $G = \frac{1}{D'_{z}D'_{\bar{z}}}$ to rewrite bosonic part of $L_{quad.}$. Then bosonic part turns into,

$$\begin{aligned} & -2t\phi'^{i}D_{z}D_{\bar{z}}\phi'_{i} + 2\sqrt{t}i\phi'^{i}\varphi_{\bar{z}zi} - 2\sqrt{t}i\phi'_{i}\bar{\varphi}^{i}_{z\bar{z}} \\ =& -2t(\phi'^{i} + \cdots)D_{z}D_{\bar{z}}(\phi'_{i} + \cdots) + 2\bar{\varphi}^{i}_{z\bar{z}}G(\varphi_{\bar{z}zi}). \end{aligned} \tag{2.1.29}$$

Integrating out oscillation modes results in cancelation of bosonic and fermionic determinant like no ψ-zero modes case. Then we finally get effective Lagrangian for this case.

$$L_{eff.} = 2\bar{\varphi}^{i}_{z\bar{z}}G(\varphi_{\bar{z}zi}) - 2R_{i\bar{j}k\bar{l}}\psi^{i0}_{\bar{z}}\psi^{\bar{j}0}_{z}\chi^{k0}\chi^{\bar{l}0} \tag{2.1.30}$$

Of course, in no ψ-zero modes case, we have $L_{eff.} = 0$. Then we have

$$\begin{aligned} \langle\prod_{i=1}^{k}\mathcal{O}_{i}\rangle_{d} &= \int_{\mathcal{M}^{M}_{0,d}}\mathcal{DM}\int\mathcal{D}\psi_{0}\mathcal{D}\chi_{0}\exp(-L_{eff.})\prod_{i=1}^{k}\mathcal{O}_{i} \\ &= \int_{\mathcal{M}^{M}_{0,d}}\mathcal{DM}\int\mathcal{D}\chi_{0}(\chi(\nu))\prod_{i=1}^{k}\mathcal{O}_{i} \end{aligned} \tag{2.1.31}$$

where we define Euler class $\chi(\nu)$ as differential form on $\mathcal{M}^{M}_{0,d}$ obtained from integrating out ψ-zero modes of $L_{eff.}$. If we regard $\mathcal{O}_{i}$ as closed forms on $\mathcal{M}^{M}_{0,d}$, we can rewrite (2.1.31) as follows.

$$\langle\prod_{i=1}^{k}\mathcal{O}_{i}\rangle_{d} = \int_{\mathcal{M}^{M}_{0,d}}\chi(\nu)\wedge(\bigwedge_{i=1}^{k}\mathcal{O}_{i}) \tag{2.1.32}$$

2.1.1 The Ghost Number anomaly and BRST observables

In the previous subsection, we come to the conclusion that path integral (2.1.12) is reduced to an integral over $\mathcal{M}^{M}_{0,d}$ weighted by one loop determinants of the non zero modes (which turns out to be 1). But as we saw in (2.1.19), there are fermion zero modes which are given as the solution of $D_{\bar{z}}\chi^{i} = D_{z}\chi^{\bar{i}} = 0$ and $D_{\bar{z}}\psi^{\bar{i}}_{z} = D_{z}\psi^{i}_{\bar{z}} = 0$. Let a_{d} (resp.b_{d}) be the number of χ (resp.ψ) zero modes. We can see from Riemann-Roch Theorem (we treat the matter more explicitly later),

$$w_{d} = a_{d} - b_{d} = 2(\dim(M) + dc_{1}(T'M)). \tag{2.1.33}$$

(In this thesis, we denote $T'M$ (resp.$T''M$) as the holomorphic (resp. anti-holomorphic) part of tangent bundle on M.) The factor 2 comes from left-right symmetry of this model. We will omit this factor in later discussion. The existence of Fermion zero modes is understood as Ghost number anomaly, because Lagrangian (2.1.6) classically conserves the ghost number. In path integration, these zero modes appear only in the integration measure except in $\prod_{i=1}^{k}\mathcal{O}_{i}$, and the correlation function $\langle\prod_{i=1}^{k}\mathcal{O}_{i}\rangle_{d}$ vanishes unless the sum of the ghost number of $\mathcal{O}_{i}$ is equal to w_{d}.

w_{d} is usually called "virtual dimension " of $\mathcal{M}^{M}_{0,d}$. In generic case $b_{d} = 0$ and $\dim(\mathcal{M}^{M}_{0,d}) = a_{d} = w_{d}$ holds.

BRST cohomology classes of the A-model are constructed from the de Rham cohomology classes $H^{*}(M)$ of the manifold M. Let $W = W_{I_{1}I_{2}\cdots I_{n}}(\phi)d\phi^{I_{1}}\wedge\cdots\wedge d\phi^{I_{n}}$ be an n form on M. Then we define a corresponding local operator of the A-Model,

$$\mathcal{O}_{W}(z) = W_{I_{1}\cdots I_{n}}\chi^{I_{1}}\cdots\chi^{I_{n}}(z) \tag{2.1.34}$$

From (2.1.7) we have

$$\{Q, \mathcal{O}_{W}\} = -\mathcal{O}_{dW} \tag{2.1.35}$$

which shows that if $W \in H^{*}(M)$, $\mathcal{O}_{W}(P)$ is BRST-closed. Note that if we limit our interest to analytic class ,i.e., $H^{i,i}(M)$ and define $\dim(W) = i$ for $W \in H^{i,i}(M)$, ghost number of $\mathcal{O}_{W}$ is equal to i. (We ignore left-right multiplicity.)

We can construct non-local operators $\mathcal{O}^{(i)}_{W}, (i = 1, 2)$ from the following recursion relation,

$$\begin{aligned} d_{\Sigma}\mathcal{O}_{W} &= i\{Q, \mathcal{O}^{(1)}_{W}\} \\ d_{\Sigma}\mathcal{O}^{(1)}_{W} &= i\{Q, \mathcal{O}^{(2)}_{W}\} \end{aligned} \tag{2.1.36}$$

where

$$\begin{aligned} \mathcal{O}^{(1)}_{W} &= inW_{I_{1}I_{2}\cdots I_{n}}d_{\Sigma}\phi^{I_{1}}\chi^{I_{2}}\cdots\chi^{I_{n}} \\ \mathcal{O}^{(2)}_{W} &= -\frac{n(n-1)}{2}W_{I_{1}I_{2}\cdots I_{n}}d_{\Sigma}\phi^{I_{1}}\wedge d_{\Sigma}\phi^{I_{2}}\chi^{I_{3}}\cdots\chi^{I_{n}} \end{aligned} \tag{2.1.37}$$

, and d_Σ represents the external differential operator on world sheet. (2.1.36) shows $\int_{C_i} \mathcal{O}_W^{(i)}$ (where C_i denotes non-trivial i-dimensional cycle on Σ.) is BRST-closed. When world sheet is genus 0, these non-local operator has little meaning, so we concentrate on $\mathcal{O}_W$. But with these set up, one can show one important fact, i.e., correlation functions $\langle \prod_{j=1}^k \mathcal{O}_{W_j}(z_j) \rangle$ does not depend on the position of insertion point z_j.

Let us evaluate the difference between $\langle \mathcal{O}_{W_1}(z_1) \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle$ and $\langle \mathcal{O}_{W_1}(z_1') \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle$.

$$\begin{aligned} &\langle \mathcal{O}_{W_1}(z_1) \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle - \langle \mathcal{O}_{W_1}(z_1') \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle \\ &= \langle \int_{z_1'}^{z_1} d_\Sigma \mathcal{O}_{W_1}(z) \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle \\ &= \langle \int_{z_1'}^{z_1} i\{Q, \mathcal{O}_{W_1}^{(1)}\} \prod_{j=2}^k \mathcal{O}_{W_j}(z_j) \rangle \\ &= 0 \end{aligned} \tag{2.1.38}$$

From this, one can see that correlation functions are just the numbers.

2.2 Evaluation of Path Integral

Now we discuss how we can evaluate $\langle \prod_{i=1}^k \mathcal{O}_i \rangle_d$. We take $\mathcal{O}_i$ to be $\mathcal{O}_{W_i}$ which is induced from $W_i \in H^*(M)$. By adding appropriate exact forms we can make W_i into the differential form which has delta function support on $PD(W_i)$. Then $\mathcal{O}_{W_i}(P_i)$ is non zero only if

$$\phi(z_i) \in PD(W_i) \tag{2.2.39}$$

Then integration over $\mathcal{M}_{0,d}^M$ is restricted to $\tilde{\mathcal{M}}_{0,d}^M$, which consists of $\Phi \in \mathcal{M}_{0,d}^M$ satisfying (2.2.39). In evaluating $\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle_d$, (2.2.39) imposes $\sum_{i=1}^k \dim(W_i)$ conditions, so $\dim(\tilde{\mathcal{M}}_{0,d}^M) = \dim(\mathcal{M}_{0,d}^M) - \sum_{i=1}^k \dim(W_i) = w_d + b_d - \sum_{i=1}^k \dim(W_i)$. But from the fact that ghost number of $\mathcal{O}_{W_i}$ equals to $\dim(W_i)$ (contribution from χ) and anomaly cancelation condition, we have $\dim(\tilde{\mathcal{M}}_{0,d}^M) = b_d$. In generic case where $b_d = 0$, $\tilde{\mathcal{M}}_{0,d}^M$ turns into finite set of points. Then we perform an one loop integral over each of these points. The result is a ratio of boson and fermion determinants, which cancel each other. Then contributions to $\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle_d$ in the generic case equals to the number of instantons which satisfies (2.2.39), i.e.,

$$\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle_{generic} = \sharp \tilde{\mathcal{M}}_{0,d}^M. \tag{2.2.40}$$

When $\dim(\tilde{\mathcal{M}}_{0,d}^M) = b_d \geq 1$, there are b_d ψ zero modes which we can regard as the fiber of the vector bundle ν on $\tilde{\mathcal{M}}_{0,d}^M$. In this case, contributions to $\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle_d$ are known as the integration of Euler class $\chi(\nu)$ on $\tilde{\mathcal{M}}_{0,d}^M$. If we consider ν as a 0-dimensional vector bundle on a point in the generic case, we can apply the same logic there. We denote each component of $\tilde{\mathcal{M}}_{0,d}^M$ as $\tilde{\mathcal{M}}_{0,d,m}^M$ and obtain

$$\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle_d = \sum_m \delta_{(\sum_{i=1}^k dim(W_i), w_d)} \int_{\tilde{\mathcal{M}}_{0,d,m}^M} \chi(\nu) \tag{2.2.41}$$

Hence from (2.1.14)

$$\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle = \sum_{d=0}^{\infty} \sum_{m=1}^{m_d} \delta_{(\sum_{i=1}^k dim(W_i), w_d)} e^{-dt} \int_{\tilde{\mathcal{M}}_{0,d,m}^M} \chi(\nu) \tag{2.2.42}$$

In algebraic geometry, generic instantons of degree d correspond to irreducible maps of degree d, which can not be written as the combination of maps from CP^1 to CP^1 of degree $j > 1$ and instantons of degree d with non-zero ψ zero mode correspond to reducible maps which are combination of degree j multiple cover maps from CP^1 to CP^1 and irreducible maps of degree d/j (Of course, j must divide d and we represent this condition as $j|d$). We will discuss it in section 2.4.

Let $\tilde{\mathcal{M}}_{0,d,j,m}^M$ be the m-th connected component of moduli spaces which are j-th multiple cover of d/j-th irreducible instantons, and $\nu_{j,m}$ be vector bundle of ψ zero modes on $\tilde{\mathcal{M}}_{0,d,j,m}^M$. Then we have from (2.2.42),

$$\langle \prod_{i=1}^k \mathcal{O}_{W_i} \rangle = \sum_{d=0}^{\infty} \sum_{j|d} \sum_{m=1}^{m_{d,j}} \delta_{(\sum_{i=1}^k dim(W_i), w_d)} e^{-dt} \int_{\tilde{\mathcal{M}}_{0,d,j,m}^M} \chi(\nu_{j,m}) \tag{2.2.43}$$

2.3 Topological Sigma Model coupled to Topological Gravity.

Lagrangian of topological sigma model coupled to topological gravity is written as sum of gravity part and matter part.

$$L = L_{gravity} + L_{matter} \tag{2.3.44}$$

We write out the matter part of Lagrangian.

$$\begin{aligned} L_{matter} \;=\; & \int_\Sigma d^2 z (g_{i\bar{j}} \partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i + i\psi_{zi}(D_{\bar{z}} \chi^i + \chi_{\bar{z}}^z \partial_z \phi^i) + i\psi_{\bar{z}\bar{i}}(D_z \chi^{\bar{i}} + \chi_z^{\bar{z}} \partial_{\bar{z}} \phi^{\bar{i}}) \\ & - R_{i\bar{j}k\bar{l}} \psi_{\bar{z}}^i \psi_z^{\bar{j}} \chi^k \chi^{\bar{l}} - \chi_{\bar{z}}^z \chi_z^{\bar{z}} \psi_z^i \psi_{\bar{z}i} + i\psi_{\bar{z}}^i \psi_{zi}(D_z \gamma^z + D_{\bar{z}} \gamma^{\bar{z}}) \\ & + i\psi_{zi} \gamma^z D_z \psi_{\bar{z}}^i + i\psi_{\bar{z}}^i \gamma^{\bar{z}} D_{\bar{z}} \psi_{zi}) \end{aligned} \tag{2.3.45}$$

This system has additional degrees of freedom that are complex structure of Riemann surface $J^{z}_{\bar{z}}$, corresponding ghosts $\chi^{z}_{\bar{z}}$ and reparameterization ghosts (b, c) and (β, γ). This Lagrangian also has BRST-symmetry. BRST-transformation law is given as follows.

$$\begin{aligned}
\delta J^{z}_{\bar{z}} &= 2\alpha\chi^{z}_{\bar{z}} + \cdots \quad \delta J^{\bar{z}}_{z} = 2\alpha\chi^{\bar{z}}_{z} + \cdots \\
\delta\chi^{z}_{\bar{z}} &= \cdots \quad \delta\chi^{\bar{z}}_{z} = \cdots \\
\delta\phi^{i} &= i\alpha\chi^{i} \quad \delta\phi^{\bar{i}} = i\alpha\chi^{\bar{i}} \\
\delta\chi^{i} &= \cdots \quad \delta\chi^{\bar{i}} = \cdots \\
\delta\psi^{\bar{i}}_{z} &= -\alpha\partial_{z}\phi^{\bar{i}} - i\alpha\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_{z} + \cdots \quad \delta\psi^{i}_{\bar{z}} = -\alpha\partial_{\bar{z}}\phi^{i} - i\alpha\chi^{j}\Gamma^{i}_{jm}\psi^{m}_{\bar{z}} + \cdots
\end{aligned} \tag{2.3.46}$$

$\cdots$ means the terms involving reparameterization ghosts. For later use, we write out explicit BRST-transformation law for χ.

$$\begin{aligned}
\delta\chi^{i} &= \alpha\gamma^{z}\partial_{z}\phi^{i} + \alpha\gamma^{\bar{z}}\partial_{\bar{z}}\phi^{i} \\
\delta\chi^{\bar{i}} &= \alpha\gamma^{z}\partial_{z}\phi^{\bar{i}} + \alpha\gamma^{\bar{z}}\partial_{\bar{z}}\phi^{\bar{i}}
\end{aligned} \tag{2.3.47}$$

Then Lagrangian is written as BRST-exact form modulo equation motion and remaining degrees of freedom are moduli degrees, which is the same as pure matter case.

In this section, we just notify some differences from pure matter theory. First, remaining moduli space include moduli space of complex structure of Σ. We denote this moduli space as $\mathcal{M}^{M}_{g,d,grav.}$ and it can be realized as follows.

$$\mathcal{M}^{M}_{g,d,grav.} = \{(J, f) | f : \Sigma_{g,J} \overset{hol.}{\mapsto} M \text{of degree d}\} \tag{2.3.48}$$

$\Sigma_{g,J}$ is the Riemann surface of genus g with complex structure J. Using local coordinates, the condition of f to be holomorphic map is written as

$$\frac{1}{2}dx^{\mu}(\delta^{\nu}_{\mu} + iJ^{\nu}_{\mu})\partial_{\nu}f^{i} = 0. \tag{2.3.49}$$

Then taking the variation of (2.3.49) results in

$$D_{\bar{z}}\delta f^{i} + \frac{i}{2}\delta J^{z}_{\bar{z}}\partial_{z}f^{i} = 0 \tag{2.3.50}$$

where we used local holomorphic coordinates z. Then zero modes of δf part is counted in the same way as pure matter case. So with respect to dimension of moduli space, we simply add the dimension of complex structure of Riemann surface. This dimension can be counted by counting the dimension of $H^{1}(\Sigma, T'\Sigma)$ or equivalently, number of holomorphic section of $\phi_{zz} \in H^{0}(\Sigma, K^{\otimes 2}_{\Sigma})$ where

$$\begin{aligned}
\phi_{zz} &:= g_{z\bar{z}}\bar{v}^{\bar{z}}_{z} \\
\bar{v}^{\bar{z}}_{z} &= v^{z}_{\bar{z}} \in H^{1}(\Sigma, T'\Sigma).
\end{aligned} \tag{2.3.51}$$

Then $\dim(H^{0}(\Sigma, K^{\otimes 2}_{\Sigma}))$ is counted by using Riemann-Roch Theorem,

$$\dim(H^{0}(\Sigma, K^{\otimes 2}_{\Sigma})) - \dim(H^{1}(\Sigma, K^{\otimes 2}_{\Sigma})) = (2g-2)\cdot 2 - (g-1) = 3g-3 \tag{2.3.52}$$

where we used the fact that $deg(K_{\Sigma}) = 2g-2$. $\phi_{\bar{z}\bar{z}'\bar{z}'} \in H^{1}(\Sigma, K^{\otimes 2}_{\Sigma})$ is mapped into $v^{\bar{z}} = g^{z\bar{z}}\phi_{z\bar{z}'\bar{z}'}g^{z'\bar{z}'}$ which corresponds to conformal Killing vector of Σ. Then we turn to genus zero case which is the main subject of this thesis. In this case, we have $\dim(H^{1}(\Sigma, K^{\otimes 2}_{\Sigma})) = 3$ which tells us that CP^{1} has no non-trivial complex moduli but three conformal Killing vectors. Since these Killing vectors are integrated into global action of $SL(2, C)$ of CP^{1}, we conclude that

$$\mathcal{M}^{M}_{0,d,grav.} = \mathcal{M}^{M}_{0,d}/SL(2, C) \tag{2.3.53}$$

Next we turn to BRST-closed observables of this model. This is important also for the notion of moduli space of this theory. $\mathcal{O}_{W}$ is not an BRST-closed operator because of (2.3.47). In this case we have to introduce gravitational dressing of $\mathcal{O}_{W}$ defined as follows.

$$\mathcal{O}_{W}(z) \mapsto \tilde{\mathcal{O}}_{W} := \mathcal{O}_{W}(z)c^{z}(z)c^{\bar{z}}(z)\delta(\gamma^{z}(z))\delta(\gamma^{\bar{z}}(z)) \tag{2.3.54}$$

Then $\tilde{\mathcal{O}}_{W}$ becomes BRST-closed operator again because of insertion of delta functions of reparameterization ghosts. This dressing has geometrical meaning. Since insertion of delta functions of reparameterization ghosts corresponds to taking diffeomorphism gauge group as diffeomorphisms fixing operator insertion points, we have additional moduli degree of freedom, i.e., position of operator insertion points. Then we define $\mathcal{M}^{M}_{0,d,k}$ as gravitational moduli space with k-operator insertions.

$$\mathcal{M}^{M}_{0,d,k} = \{(z_1, z_2, \cdots, z_k),\ \mathcal{M}^{M}_{0,d}\}/SL(2, C) \tag{2.3.55}$$

Then $SL(2, C)$ action is naturally defined as follows.

$$\begin{aligned}
u \circ \{(z_1, \cdots, z_k), f \in \mathcal{M}^{M}_{0,d}\} &:= \{(u(z_1), \cdots, u(z_k)), (u^{-1*})f\} \\
u \in SL(2, C) &
\end{aligned} \tag{2.3.56}$$

Finally we write out the representation of correlation functions as an integral of moduli space.

$$\langle \prod_{j=1}^{k} \tilde{\mathcal{O}}_{W_j} \rangle = \int_{\mathcal{M}^{M}_{0,d,k}} \chi(\nu) \bigwedge_{j=1}^{k} \tilde{\mathcal{O}}_{W_j} \tag{2.3.57}$$

We regard $\mathcal{O}_{W_j}$ as a form on $\mathcal{M}^{M}_{0,d,k}$. Euler class $\chi(\nu)$ is evaluated fundamentally the same way as pure matter case. See for details [33]. Later we use $\mathcal{O}_{W}$ for $\tilde{\mathcal{O}}_{W}$ in case of theory coupled with gravity.

2.4 Reduction to an Integral of Forms on Moduli Spaces

In this thesis, we treat the topological sigma model (A-Model) on degree k hypersurface M_N^k in CP^{N-1} ($k \leq N$). This manifold is realized as the zero-locus of section of kH (H is hyperplane bundle of CP^{N-1} and kH represents k-times tensor product of hyperplane bundle). Note that when $N = k$, kH is equivalent to $-K_{CP^{N-1}}$ and M_N^N turns into Calabi-Yau manifold (K_M represents the canonical line bundle of M which is equivalent to $A(M, \wedge^{\dim_M} T'^{*}M)$). We can take homogeneous polynomial of degree k as the defining equation of M_N^k. For example,

$$M_N^k := \{(X_1 : X_2 : \cdots : X_N) \in CP^{N-1} | X_1^k + X_2^k + \cdots + X_N^k = 0\} \qquad (2.4.58)$$

Observables of this model can be constructed from elements of $w \in H^*(M_N^k)$ which we denote as $\mathcal{O}_w$, and in the following discussion we consider the observables which are induced from the sub-ring of $H^*(M_N^k, C)$ generated by Kähler form e of M_N^k (we denote it as $H_e^*(M_N^k, C)$). One of the reason why we take this sub-ring is that we can obtain it directly from $H^*(CP^{N-1}, C)$ and Poincare dual of its elements are analytic submanifold of M_N^k. More explicitly, elements of $H_e^*(M_N^k, C)$ are given as e^j ($j = 1, 2, \cdots, N-2$) and Poincare dual of e^j is the intersection of the zero locus of the section of $H^0(CP^{N-1}, \mathcal{O}(j \cdot H))$ and M_N^k. So in the following discussion we treat the observables *

$$\mathcal{O}_1, \mathcal{O}_e, \mathcal{O}_{e^2}, \cdots, \mathcal{O}_{e^{N-2}} \qquad (2.4.59)$$

Then the fact that Lagrangian of the topological sigma model is BRST-exact allows us to take the weak coupling limit and correlation functions of this model reduces to the integral of closed forms corresponding to the BRST closed observables on moduli spaces of holomorphic maps f from Riemann surface Σ_g to target space M_N^k (we focus our attention to the case of $g = 0$, i.e, CP^1). When the target space is a hypersurface of simple projective space CP^{N-1}, we can classify moduli spaces by the degree $d = \sharp(f(CP^1) \cap PD(e))$ and we denote the moduli space of degree d as $\mathcal{M}_{0,d}^{M_N^k}$. Dimension of $\mathcal{M}_{0,d}^{M_N^k}$ which counts the number of χ-zero modes is evaluated by the Riemann-Roch Theorem as follows,

$$\begin{aligned}
\dim(\mathcal{M}_{0,d}^{M_N^k}) &:= \dim(H^0(CP^1, f^*(T'M_N^k))) \\
&= \dim(M_N^k) + deg(f) \cdot c_1(T'M_N^k) + \dim(H^1(CP^1, f^*(T'M_N^k))) \\
&= \dim(M_N^k) + d(N-k) + \dim(H^1(CP^1, f^*(T'M_N^k))) \\
&= N - 2 + d(N-k) + \dim(H^1(CP^1, f^*(T'M_N^k))) \qquad (2.4.60)
\end{aligned}$$

*When coupled to gravity, $\mathcal{O}_1$ corresponds to puncture operator P, but in the small phase space, P insertion is suppressed except for constant map sector because of puncture equation. And in Calabi-Yau case, as we know from the later discussion of topological selection rule, ghost number of inserted operator must be less than $N-3$. So it suffices to consider only $N-4$ elements $\mathcal{O}_e, \mathcal{O}_{e^2}, \cdots, \mathcal{O}_{e^{N-4}}$ in this case.

where we used the fact that $c_1(T'M_N^k) = N - k$. This can be derived as follows. Since $T'M_N^k$ and $T'^{*}M_N^k$ are dual to each other, we have the identity,

$$c_1(T'M_N^k) = -c_1(T'^{*}M_N^k) = -c_1(K_{M_N^k}). \qquad (2.4.61)$$

Then from the adjunction formula,

$$K_{M_N^k} = K_{CP^{N-1}} \otimes kH \qquad (2.4.62)$$

we have

$$c_1(K_{M_N^k}) = c_1(K_{CP^{N-1}}) + c_1(kH) = -N + k. \qquad (2.4.63)$$

Then $c_1(T'M_N^k) = N - k$ follows.

First, we consider the generic case where $\dim(H^1(CP^1, f^*(T'M_N^k))) = 0$. From the argument of previous section, we can heuristically represent correlation functions,

$$\begin{aligned}
&\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2) \cdots \mathcal{O}_{e^{j_m}}(z_m)\rangle_{d,generic} \\
&= \int_{\mathcal{M}_{0,d}^{M_N^k}} \alpha(\mathcal{O}_{e^{j_1}}) \wedge \alpha(\mathcal{O}_{e^{j_2}}) \wedge \cdots \wedge \alpha(\mathcal{O}_{e^{j_m}}) \qquad (2.4.64)
\end{aligned}$$

where $\alpha(\mathcal{O}_{e^j})$ is the closed form on $\mathcal{M}_{0,d}^{M_N^k}$ induced from $\mathcal{O}_{e^j}$. Since the form degree of $\alpha(\mathcal{O}_{e^j})$ equals the ghost number of $\mathcal{O}_{e^j} (= \dim(e^j) = j)$, correlation functions are nonzero only if the following conditions are satisfied.

$$\begin{aligned}
\dim(\mathcal{M}_{0,d}^{M_N^k}) &= \sum_{i=1}^{m} j_i \\
\Longleftrightarrow N - 2 + (N-k)d &= \sum_{i=1}^{m} j_i \qquad (2.4.65)
\end{aligned}$$

If we take e^j as the forms which has the delta function support on $PD(e^j)$, then from (2.1.34), $\alpha(\mathcal{O}_{e^j})$ can be interpreted as the constraint condition on f,

$$\alpha(\mathcal{O}_{e^{j_i}}) \leftrightarrow f(z_i) \in PD(e^{j_i}) \qquad (2.4.66)$$

(2.4.66) imposes $(\dim(e^j) - 1) + 1 = j$ independent conditions on $\mathcal{M}_{0,d}^{M_N^k}$ ($(\dim(e^j) - 1)$ corresponds to the degree of freedom which makes $f(CP^1) \cap PD(e^j) \neq \emptyset$ and 1 to the one which sends $f(z_i)$ into $PD(e^j)$). And from (2.4.65), what remains is the discrete point set of holomorphic maps f which satisfy (2.4.66) for all i. Then we have

$$\begin{aligned}
&\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2) \cdots \mathcal{O}_{e^{j_m}}(z_m)\rangle_{d,generic} \\
&= \sharp\{f : CP^1 \overset{hol.}{\to} M_N^k \text{ of degree } d | f(z_i) \in PD(e^{j_i})\} \qquad (2.4.67)
\end{aligned}$$

Now, let us consider non-generic case. In this case, $\dim(H^1(CP^1, f^*(T'M_N^k))) > 0$ and moduli space have additional $\dim(H^1(CP^1, f^*(T'M_N^k)))$ degrees of freedom.

We can see these degrees of freedom correspond to multiple cover maps by the following argument. A multiple cover map f can be decomposed into the form $f = \tilde{f} \circ \varphi$ where $\tilde{f}$ is irreducible map from CP^1 to M_N^k and φ represents the map from CP^1 to CP^1 of degree $n \geq 2$. Then let us count $\dim(\mathcal{M}_{0,d}^{M_N^k})$ by taking the (holomorphic) variation of $\tilde{f} \circ \varphi$.

$$\begin{aligned} &\delta(\tilde{f} \circ \varphi) \\ &= \delta\tilde{f} \circ \varphi + \tilde{f} \circ \delta\varphi \end{aligned} \tag{2.4.68}$$

$\delta\tilde{f}$ corresponds to the generic degrees of freedom, i.e.,

$$\dim(\delta\tilde{f} \circ \varphi) = N - 2 + (N-k)\frac{d}{n} \tag{2.4.69}$$

$\tilde{f} \circ \delta\varphi$ counts the deformation of the multiple cover map which can be realized using the section $\varphi^z \in H^0(CP^1, \varphi^*(T'CP^1))$ as $\varphi^z \partial_z \tilde{f}$. We can count these by using Riemann-Roch,

$$\begin{aligned} \dim(\tilde{f} \circ \delta\varphi) &= \dim(H^0(CP^1, \varphi^*(T'CP^1))) - 3 \\ &= 1 + deg(\varphi) \cdot c_1(T'CP^1) - 3 \\ &= 2n - 2 \end{aligned} \tag{2.4.70}$$

In (2.4.70) we subtract the double counted $SL(2,C)$ which comes from the indetermination of the decomposition of f, i.e.,

$$\begin{aligned} f &= \tilde{f} \circ \varphi \\ &= \tilde{f} \circ u \circ u^{-1} \circ \varphi \qquad u \in SL(2,C) \end{aligned} \tag{2.4.71}$$

After all when k equals N, generic degrees of freedom doesn't depend on d. And we find additional $2n-2$ χ zero modes in this case (When k is less than N, we can see that contribution of multiple cover maps don't exceed the generic dimension of $\mathcal{M}_{0,d}^{M}$ and we can conclude that $\dim(H^1(CP^1, f^*(T'M_N^k))) = 0$). But we can also construct $2n-2$ $\psi_{\bar{z}}^i$ which comes from $H^1(CP^1, f^*(T'M_N))$. By the Kodaira-Serre duality, the following equation holds.

$$\begin{aligned} \dim(H^1(CP^1, f^*(T'M_N^k)) &= \dim(H^0(CP^1, K \otimes f^*(T'^*M_N^k))) \\ (&= \dim(H^0(CP^1, \bar{K} \otimes f^*(T''^*M_N^k)))) \end{aligned} \tag{2.4.72}$$

and

$$\psi_{\bar{z}}^i = g^{i\bar{j}}\bar{\psi}_{\bar{z}\bar{j}} \qquad (\bar{\psi}_{\bar{z}\bar{j}} \in H^0(CP^1, \bar{K} \otimes f^*(T''^{*\prime}M_N^k))) \tag{2.4.73}$$

In this case, as we have said in section 2.1, by integrating ψ zero-modes first, we have the Euler class $\chi(\nu)$ where $\nu \simeq H^1(CP^1, \varphi^*(T'M_N^k))$. This leads us to

$$\begin{aligned} &\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2)\cdots\mathcal{O}_{e^{j_m}}(z_m)\rangle_d \\ &= \int_{\mathcal{M}_{0,d}^{M_N^k}} \alpha(\mathcal{O}_{e^{j_1}}(z_1)) \wedge \cdots \wedge \alpha(\mathcal{O}_{e^{j_m}}(z_m)) \quad (k < N) \\ &= \int_{\mathcal{M}_{0,d}^{M_N^k}} \chi(\nu) \wedge \alpha(\mathcal{O}_{e^{j_1}}(z_1)) \wedge \cdots \wedge \alpha(\mathcal{O}_{e^{j_m}}(z_m)) \quad (k = N) \end{aligned} \tag{2.4.74}$$

We can refine (2.4.74) by using the argument which leads us to (2.4.66) and define the evaluation map,

$$\varphi_i : \mathcal{M}_{0,d}^{M_N^k} \to M_N^k : f \mapsto f(z_i) \tag{2.4.75}$$

We have

$$\begin{aligned} &\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2)\cdots\mathcal{O}_{e^{j_m}}(z_m)\rangle_d \\ &= \int_{\mathcal{M}_{0,d}^{M_N^k}} \varphi_1^*(e^{j_1}) \wedge \cdots \wedge \varphi_m^*(e^{j_m}) \quad (k < N) \\ &= \int_{\mathcal{M}_{0,d}^{M_N^k}} \chi(\nu) \wedge \varphi_1^*(e^{j_1}) \wedge \cdots \wedge \varphi_m^*(e^{j_m}) \quad (k = N) \end{aligned} \tag{2.4.76}$$

In $k = N$ case, we can relate the non-generic part of the correlation functions to the ones of lower degree, because in such case f decomposes into $f = \tilde{f} \circ \varphi$ where $\deg(\varphi) = n$ and $\deg(\tilde{f}) = d/n < d$. But good results are given only in the case of $k = 3$, which was derived by Greene, Aspinwall, Morrison and Plesser [7] [34]. Of course, if we use the fusion rule that holds in the matter theory, we can reduce the correlation functions into the product of three point functions and formally distinguish the non-generic part from the generic ones. But geometrical meaning is still not clear.

Then we slightly change our point of view. Since M_N^k is a hypersurface in CP^{N-1}, we can see $\mathcal{M}_{0,d}^{M_N^k}$ as a submanifold of $\mathcal{M}_{0,d}^{CP^{N-1}}$ which consists of maps satisfying the following condition.

$$\begin{aligned} f : CP^1 &\to CP^{N-1} \\ f(CP^1) &\subset M_N^k \end{aligned} \tag{2.4.77}$$

If we can realize the condition (2.4.77) as the closed forms (which we hypothetically denote as $c_d(M_N^k)$) on $\mathcal{M}_{0,d}^{CP^{N-1}}$, we have an alternate representation for the correlation functions as follows,

$$\begin{aligned} &\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2)\cdots\mathcal{O}_{e^{j_m}}(z_m)\rangle_{d,alt} \\ &= \int_{\mathcal{M}_{0,d}^{CP^{N-1}}} c_d(M_N^k) \wedge \tilde{\varphi}_1^*(e^{j_1}) \wedge \cdots \wedge \tilde{\varphi}_m^*(e^{j_m}) \\ &\tilde{\varphi}_i : \mathcal{M}_{0,d}^{CP^{N-1}} \to CP^{N-1} : f \mapsto f(z_i) \end{aligned} \tag{2.4.78}$$

Note that e represents the Kähler class of CP^{N-1}. In (2.4.78), we can drop off the Euler class $\chi(\nu)$. This is because

$$\dim(H^1(CP^1, f^*(T'CP^{N-1}))) = 0. \tag{2.4.79}$$

Dimension of moduli space does not jump in this case. Then naturally arises the question about the relation between (2.4.76) and (2.4.78). However we want to proceed further with the formula (2.4.78).

Then we want to use the torus action method invented by Kontsevich in Section 3.5, which enables us to compute correlation functions of topological sigma model coupled to (topological) gravity. And we couple gravity to the topological sigma model. Roughly speaking, we add to the moduli space "puncture" degrees of freedom which correspond to the insertion points of external operators. So for m-point correlation function, dimension of moduli space (we denote it as $\mathcal{M}_{0,d,m}^{CP^{N-1}}$) increases by $m-3$. -3 corresponds to dividing by automorphism of CP^1, i.e., $SL(2,C)$ which is induced by c-ghost zero-modes. And topological selection rule (2.4.65) is changed into

$$\begin{aligned} N-2+(N-k)d+m-3 &= \sum_{i=1}^{m} j_i \\ \Longleftrightarrow \quad N-5+(N-k)d &= \sum_{i=1}^{m}(j_i-1) \end{aligned} \tag{2.4.80}$$

$\mathcal{M}_{0,d,m}^{CP^{N-1}}$ can be generically represented as follows,

$$\mathcal{M}_{0,d,m}^{CP^{N-1}} \simeq \{(z_1, z_2, \cdots, z_m), f\}/SL(2,C) \qquad f \in \mathcal{M}_{0,d}^{CP^{N-1}} \tag{2.4.81}$$

where $u \in SL(2,C)$ acts

$$u \circ \{(z_1, z_2, \cdots, z_m), f\} = \{(u(z_1), \cdots, u(z_m)), (u^{-1})^* \circ f\} \tag{2.4.82}$$

This action of $SL(2,C)$ is compatible with the "evaluation map",

$$\begin{aligned} \phi_i : \mathcal{M}_{0,d,m}^{CP^{N-1}} &\mapsto CP^{N-1} \\ \{(z_1, z_2, \cdots, z_m), f\}/SL(2,C) &\mapsto f(z_i) \end{aligned} \tag{2.4.83}$$

because $(u^{-1})^* f(u(z_i)) = f(z_i)$. In (2.4.81), $(z_1, \cdots, z_m)$ are considered as distinct points, but to compactify the moduli space, we have to add boundary parts which describe coincidence of these points. We will discuss it in section 3.4 and 3.5.

Then the integral representation of amplitudes (2.4.78) turns into

$$\begin{aligned} &\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2)\cdots\mathcal{O}_{e^{j_k}}(z_m)\rangle_{d,alt,grav.} \\ &= \int_{\mathcal{M}_{0,d,m}^{CP^{N-1}}} c_d(M_N^k) \wedge \phi_1^*(c_1^{j_1}(H)) \wedge \cdots \wedge \phi_k^*(c_1^{j_m}(H)) \end{aligned} \tag{2.4.84}$$

where we used the fact that e corresponds to the first Chern class of hyperplane bundle H. Then we have to find the realization of $c_d(M_N^k)$. We can roughly do it as follows. First consider the coordinate representation of $\mathcal{M}_{0,d}^{CP^{N-1}}$,

$$\begin{aligned} f : CP^1 &\mapsto CP^{N-1} \\ f : (s:t) &\mapsto (\sum_{i=0}^{d} a_1^i s^{d-i} t^i : \cdots : \sum_{i=0}^{d} a_N^i s^{d-i} t^i) \end{aligned} \tag{2.4.85}$$

where (a_j^i)'s are the coordinates of $\mathcal{M}_{0,d}^{CP^{N-1}}$. Then the condition imposed by $c_d(M_N^k)$ is equal to

$$\begin{aligned} & f(s:t) \in M_N^k \quad \text{for all}(s,t) \\ \Longleftrightarrow \quad & (\sum_{i=0}^{d} a_1^i s^{d-i} t^i)^k + \cdots + (\sum_{i=0}^{d} a_N^i s^{d-i} t^i)^k = 0 \quad \text{for all}(s,t) \\ \Longleftrightarrow \quad & f^m(a_j^i) = 0 \qquad (m = 0, 1, \cdots, kd) \end{aligned} \tag{2.4.86}$$

where $f^m(a_j^i)$'s are the coefficient polynomials of $s^m t^{kd-m}$ of the l.h.s of the second line of (2.4.86). This imposes $kd+1$ condition on $\mathcal{M}_{0,d}^{CP^{N-1}}$. We can describe this condition mathematically in terms of moduli space $\mathcal{M}_{0,d,m}^{CP^{N-1}}$. Let π_j be a forgetful map $\pi_j : \mathcal{M}_{0,d,j}^{CP^{N-1}} \to \mathcal{M}_{0,d,j-1}^{CP^{N-1}}$ which "forget" the existence of one of the punctures. Then for $j=1$, the fiber of π_1 is CP^1. And consider the sheaf $\phi_1^*(kH)$ on $\mathcal{M}_{0,d,1}^{CP^{N-1}}$ where kH corresponds to defining polynomial of M_N^k and $H^0(\mathcal{M}_{0,d,1}^{CP^{N-1}}, \phi_1^*(kH))$ to the second line of (2.4.86) modulo $SL(2,C)$ equivalence. Then consider direct image sheaf $R_{\pi_1}^0(\phi_1^*(kH))$ (we denote it as $\mathcal{E}_{kd+1}$). It locally equals $H^0(CP^1, f^*\mathcal{O}(kH))$ and has rank $(kd+1)$. We can translate the operation in going from the second line of (2.4.86) to the third one into the evaluation of the zero locus of the section of $\mathcal{E}_{kd+1}$. This condition is equivalent to the insertion of top Chern class $c_T(\mathcal{E}_{kd+1})$ by Gauss-Bonnet Theorem.

Considering the map,

$$\tilde{\pi}_m := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : \mathcal{M}_{0,d,m}^{CP^{N-1}} \mapsto \mathcal{M}_{0,d,0}^{CP^{N-1}} \tag{2.4.87}$$

We have

$$c_d(M_N^k) = c_T(\tilde{\pi}_m^*(\mathcal{E}_{kd+1})) \tag{2.4.88}$$

Finally the representation (2.4.84) turns into

$$\begin{aligned} &\langle \mathcal{O}_{e^{j_1}}(z_1)\mathcal{O}_{e^{j_2}}(z_2)\cdots\mathcal{O}_{e^{j_m}}(z_m)\rangle_{d,alt,grav.} \\ &= \int_{\mathcal{M}_{0,d,m}^{CP^{N-1}}} c_T(\tilde{\pi}_m^*(\mathcal{E}_{kd+1})) \wedge \phi_1^*(c_1^{j_1}(H)) \wedge \cdots \wedge \phi_k^*(c_1^{j_m}(H)) \end{aligned} \tag{2.4.89}$$

We will use this formula in explicit calculation of amplitudes with the aid of torus action method in section 3.5.

Chapter 3

Geometrical Calculation

In this chapter, we perform geometrical calculation of correlation functions of topological sigma model on M_N^k from the point of view of the formula (2.4.89). In section 3.1, we approximately evaluate $\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_j) \rangle$ using compactification of $\mathcal{M}_{0,d}^{CP^{N-1}}$ and discuss its limitation of this method. In section 3.2, we exactly evaluate $\langle \prod_{j=1}^{N-2+N-k} \mathcal{O}_e(z_j) \rangle$ by using the fact that $\mathcal{M}_{0,1}^{CP^{N-1}}/SL(2,C) = Gr(2,N)$. This calculation is an example of classical method for enumeration of instantons of algebraic manifolds. But the application in this thesis is rather exceptional because we treat the case where operator insertion points are fixed or we treat pure matter theory. We think this is the first treatment of direct geometrical calculation of pure matter theory. Section 3.3, 3.4 and 3.5 are devoted to the geometrical calculation of correlation functions of general degree instantons. For general degree, the calculation of amplitudes from the point of view of (2.4.89) is difficult. But in case of theory coupled with gravity, the notion of stable map and the developments in topological gravity enable us to calculate them through torus action method . It is application of Bott residue formula (a variation of fixed point theorem for complex manifold) to the integral on $\mathcal{M}_{0,d,m}^{CP^{N-1}}$. In section 3.3 and 3.4 we introduce the notion of stable curve and stable map which compactify $\mathcal{M}_{0,d,m}^{M}$. In section 3.5, we review the torus action method and by using the formula (2.4.89), we perform some explicit calculations of amplitudes of the theory coupled with gravity on M_N^N ,i.e., Calabi-Yau manifold in CP^{N-1}. We also compute $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j) \rangle_{matter}$ by use of the fact that three point functions of pure matter theory and the theory coupled with gravity coincide and fusion rules that hold in matter theory. We finally construct the integral representation of generating function of amplitudes for the theory coupled with gravity on M_N^k. Relation of pure matter theory and the theory coupled with gravity are pursued further in section 4.1. We argue that this is the first application of torus action method to the general hypersurfaces in CP^{N-1}.

3.1 Moduli Space of the Pure Matter Theory

In this section, we take our first step of geometrical calculation of correlation function of A-model (pure matter theory) on M_N^k. We have mentioned the strategy in previous section. Since M_N^k is a hypersurface in CP^{N-1}, moduli space $\mathcal{M}_{0,d}^{M_N^k}$ is realized as submanifold of $\mathcal{M}_{0,d}^{CP^{N-1}}$. The condition of $f \in \mathcal{M}_{0,d}^{CP^{N-1}}$ to be a point of $\mathcal{M}_{0,d}^{M_N^k}$ is,

$$f(CP^1) \subset M_N^k. \tag{3.1.1}$$

Then if we have appropriate realization of $\mathcal{M}_{0,d}^{CP^{N-1}}$ and good description of (3.1.1), we can calculate correlation functions. We will calculate simple, but non-trivial correlation function $\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_i) \rangle_d$ by simple compactification of $\mathcal{M}_{0,d}^{CP^{N-1}}$, and discuss limitation of this generic argument. It is based on the following assumption.

Assumption. Any holomorphic map f from CP^1 to CP^{N-1} of degree d is described by polynomial map of degree d.

$$f : (s:t) \mapsto (\sum_{j=0}^{d} a_j^1 s^j t^{d-j} : \sum_{j=0}^{d} a_j^2 s^j t^{d-j} : \cdots : \sum_{j=0}^{d} a_j^N s^j t^{d-j}) \tag{3.1.2}$$

We can regard $\mathcal{M}_{0,d}^{CP^{N-1}}$ as $CP^{(d+1)N-1}$ if we ignore the boundary parts of positive codimension by the following identification.

$$\begin{aligned}\Phi : f = (\sum_{j=0}^{d} a_j^1 s^j t^{d-j} : \sum_{j=0}^{d} a_j^2 s^j t^{d-j} : \cdots : \sum_{j=0}^{d} a_j^N s^j t^{d-j}) \in \mathcal{M}_{0,d}^{CP^{N-1}} \\ \mapsto (a_0^1 : a_1^1 : \cdots : a_d^1 : a_0^2 : \cdots : a_d^N) \in CP^{(d+1)N-1}\end{aligned} \tag{3.1.3}$$

Boundary parts will be discussed later.

Then we can realize the condition (3.1.1) as the constraint on $CP^{(d+1)N-1}$,

$$\begin{aligned} & f(CP^1) \subset M_N^k \\ \Longleftrightarrow \quad & f(s:t) \in M_N^k \quad \text{for all } (s:t) \\ \Longleftrightarrow \quad & \textstyle\sum_{i=1}^{N} (\sum_{j=0}^{d} a_j^i s^j t^{d-j})^k = 0 \quad \text{for all } (s:t) \\ \Longleftrightarrow \quad & \textstyle\sum_{m=0}^{kd} g^m(a_j^i) s^m t^{dk-m} = 0 \quad \text{for all } (s:t) \\ \Longleftrightarrow \quad & g^m(a_j^i) = 0 \ \ (\text{for } m = 0, 1, \cdots, kd) \end{aligned} \tag{3.1.4}$$

where $g^m(a_j^i)$ are homogeneous polynomials of degree k. Of course, the condition (3.1.4) is imposed only for the elements of $\Phi(\mathcal{M}_{0,d}^{CP^{N-1}}) \subset CP^{(d+1)N-1}$ and has no meaning on $CP^{(d+1)N-1} - \Phi(\mathcal{M}_{0,d}^{CP^{N-1}})$. But let us assume the condition (3.1.4) is extended to the whole $CP^{(d+1)N-1}$. Then we can regard (3.1.4) as $kd+1$ homogeneous polynomial constraint of degree k. This constraint is equivalent to insertion of $(k\bar{e})^{kd+1}$ where $\bar{e}$ is the Kähler form of $CP^{(d+1)N-1}$. We have the following generic result.

$$\mathcal{M}_{0,d}^{M_N^k} \simeq PD_{CP^{(d+1)N-1}}((k\bar{e})^{kd+1}) \tag{3.1.5}$$

where $PD_M(w)$ denotes the Poincare dual of the closed form $w \in H^*(M)$. From (3.1.5), we can see $\dim(\mathcal{M}_{0,d}^{M_N^k}) = \dim(PD_{CP^{(d+1)N-1}}) = N(d+1) - 1 - (kd+1) = (N-k)d + N - 2$ which is consistent with the result of Riemann-Roch Theorem.

Next, we evaluate $\langle\prod_{j=1}^{N-2+(N-k)d}\mathcal{O}_e(z_i)\rangle_d$ using this generic argument. Combining (2.4.78) and (3.1.5), we have the following formula.

$$\begin{aligned}\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_i)\rangle_d &= \int_{\mathcal{M}_{0,d}^{CP^{N-1}}} c_d(M_N^k) \bigwedge_{j=1}^{(N-k)d+N-2} \bar{\varphi}_i^*(e) \\ &= \int_{CP^{(d+1)N-1}} (k\bar{e})^{kd+1} \bigwedge_{j=1}^{(N-k)d+N-2} \bar{\varphi}_i^*(e)\end{aligned} \tag{3.1.6}$$

where e denotes Kähler form of CP^{N-1} and $\bar{\varphi}_i$ is the evaluation map,

$$\bar{\varphi}_i : f \in \mathcal{M}_{0,d}^{CP^{N-1}} \mapsto f(s_i : t_i) \in CP^{N-1}. \tag{3.1.7}$$

Since $PD_{CP^{N-1}}(e)$ is hyperplane in CP^{N-1}, it is realized as zero locus of linear equation.

$$PD_{CP^{N-1}}(e) = \{(X_1 : X_2 : \cdots : X_N) \in CP^{N-1} | \sum_{i=1}^{N} \gamma_i X_i = 0\} \tag{3.1.8}$$

Then by taking Poincare dual, $\bar{\varphi}_i^*(e)$ corresponds to the condition that $f(s_i : t_i)$ should be in $PD_{CP^{N-1}}(e)$, i.e.,

$$\sum_{l=1}^{N}\sum_{j=0}^{d} a_j^l \gamma_l s_i^j t_i^{d-j} = 0 \tag{3.1.9}$$

In treating pure matter theory, $(s_i : t_i)$ is kept fixed and γ_l's are constant. And we can regard (3.1.9) as linear relation on $CP^{(d+1)N-1}$ in our generic treatment. Then again taking Poincare dual, we conclude $\bar{\varphi}_i^*(e) = \bar{e}$. Finally from (3.1.6), we evaluate $\langle\prod_{j=1}^{N-2+(N-k)d}\mathcal{O}_e(z_i)\rangle_d$.

$$\begin{aligned}\langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_i)\rangle_d &\simeq \int_{CP^{(d+1)N-1}} (k\bar{e})^{kd+1} \wedge \bar{e}^{(N-k)d+(N-2)} \\ &= k^{kd+1}\end{aligned} \tag{3.1.10}$$

Now we discuss the limitation of this generic argument. Boundary part $CP^{(d+1)N-1} - \Phi(\mathcal{M}_{0,d}^{CP^{N-1}})$ is described by the polynomial maps superficially of degree d but really of lower degree by projective equivalence. Let us consider the following map (from

now on, we omit the operation Φ which turns the coefficients (a_i^j) into the variables of $CP^{(d+1)N-1}$).

$$\begin{aligned}&\eta_{d-m} : CP^{(m+1)N-1} \times CP^{k-m} \mapsto CP^{(k+1)N-1} \\ &\quad (d_j^i)/C^* \ \ (i = 1, 2, \cdots, N \ \ j = 0, 1, \cdots, m) \times (f_0 : f_1 : \cdots : f_{d-m}) \mapsto \\ &\quad ((\sum_{j=0}^{m} d_j^1 s^j t^{m-j})(\sum_{i=0}^{d-m} f_i s^i t^{d-m-i}) : \cdots : (\sum_{j=0}^{m} d_j^N s^j t^{m-j})(\sum_{i=0}^{d-m} f_i s^i t^{d-m-i})) \\ &\quad \simeq ((\sum_{j=0}^{m} d_j^1 s^j t^{m-j}) : (\sum_{j=0}^{m} d_j^2 s^j t^{m-j}) : \cdots : (\sum_{j=0}^{m} d_j^N s^j t^{m-j}))\end{aligned} \tag{3.1.11}$$

Or, if we take Fundamental Theorem of Algebra into account, we can use following map sequence instead of η_m's.

$$\begin{aligned}&CP^{N(d+1)-1} \xleftarrow{\tilde{\eta}_1} \\ &\quad CP^{N(d)-1} \times CP^1 \xleftarrow{\tilde{\eta}_2} CP^{N(d-1)-1} \times (CP^1)^2 \xleftarrow{\tilde{\eta}_3} \\ &\quad \cdots \\ &\quad CP^{N(d-k+1)-1} \times (CP^1)^k \xleftarrow{\tilde{\eta}_{k+1}} CP^{N(d-k)-1} \times (CP^1)^{k+1} \xleftarrow{\tilde{\eta}_{k+2}} \\ &\quad \cdots \\ &\quad CP^{2N-1} \times (CP^1)^{d-1} \xleftarrow{\tilde{\eta}_d} CP^{N-1} \times (CP^1)^d\end{aligned} \tag{3.1.12}$$

where

$$\begin{aligned}\tilde{\eta}_j : &CP^{N(d-j+1)-1} \times (CP^1)^j \mapsto CP^{N(d-j+2)-1} \times (CP^1)^{j-1} \\ &((A_{d-j}^1(s,t), \cdots, A_{d-j}^N(s,t)), (a^1 s + b^1 t), \cdots, (a^j s + b^j t)) \\ &\mapsto ((A_{d-j}^1(s,t)(a^1 s + b^1 t), \cdots, A_{d-j}^N(s,t)(a^1 s + b^1 t)) \\ &, (a^2 s + b^2 t), \cdots, (a^j s + b^j t)) \\ &A_d^i(s,t) := \sum_{j=0}^{d} a_j^i s^j t^{d-j}\end{aligned} \tag{3.1.13}$$

The image of the map η_{d-m} corresponds to, by projective equivalence, the space of maps of degree m in $CP^{(d+1)N-1}$, so obviously it belongs to $CP^{(d+1)N-1} - \mathcal{M}_{0,d}^{CP^{N-1}}$. Naive counting of $\dim(Im(\eta_{d-m}))$ concludes that it equals to $((m+1)N-1) + (d-m) = (m+1)N + d - m - 1$, while the condition for $f \in Im(\eta_i)$ to be a holomorphic map from CP^1 to M_N^k only reduces the dimension of $CP^{(m+1)N-1}$ from $((m+1)N-1)$ to $N-2+(N-k)m$. Then in translating the condition (3.1.4) into $(k\bar{e})^{kd+1}$, we make mistakes in dimensional counting on $Im(\eta_{d-m})$. These mistakes becomes relevant if contributions from $Im(\eta_{d-m})$ exceeds $\mathcal{M}_{0,d}^{M_N^k}$ with respect to dimensional counting, i.e.,

$$\begin{aligned}&(N-k)m + (N-2) + d - m \geq (N-k)d + (N-2) \\ &\iff \ (N-k-1)(d-m) \leq 0\end{aligned} \tag{3.1.14}$$

From (3.1.14), if $k \geq N-1$, we cannot believe the approximation (3.1.10), but in $k \leq N-2$ case, it seems to give the appropriate result. With these discussion, we give the following statement.

$$\begin{aligned} \langle \prod_{j=1}^{N-2+(N-k)d} \mathcal{O}_e(z_i) \rangle_d &= k^{kd+1} (k \leq N-2) \\ &< k^{kd+1} (k \geq N-1) \end{aligned} \tag{3.1.15}$$

3.2 Schubert Calculus

In the $d=1$ case, boundary part comes only from the map η_1, and we can eliminate it by using $Gr(2,N)$ instead of $CP^{2N-1}/Im(\eta_1) = \mathcal{M}_{0,1}^{CP^{N-1}}$. Using this fact we perform exact calculation of $\langle \prod_{j=1}^{2N-k-2} \mathcal{O}_e(z_i) \rangle_1$. (An important difference between $Gr(2,N)$ and $(CP^{2N-1}/Im(\eta_0))$ is that $Gr(2,N)$ is the $SL(2,C)$ quotient space of the latter. Indeed $\dim(Gr(2,N))$ is $2N-4 = (2N-1)-3$.)

There is a map ξ

$$\begin{aligned} \xi : CP^{2N-1}/Im(\eta_0) &\to Gr(2,N) \\ (a_0^1 s + a_1^1 t, \ldots, a_0^N s + a_1^N t) &\mapsto \begin{pmatrix} a_0^1 & a_0^2 & \ldots & a_0^N \\ a_1^1 & a_0^2 & \ldots & a_0^N \end{pmatrix} / GL(2,C) \end{aligned} \tag{3.2.16}$$

Then, we have to decide the condition which corresponds to (3.1.4), i.e the condition for $l \in Gr(2,N)$ to be contained in M_N^k. This condition can be translated into words of cohomology ring $H^*(Gr(2,N))$. Let F_N^k be the defining equation of M_N^k and s_N^k be the section of $Sym^k(U^*)$ defined from the restriction of F_N^k to $l \in Gr(2,N)$, where U is the universal bundle of $Gr(2,N)$ (See Appendix A for the definition of s_N^k. Universal bundle U is the bundle which is given as the vector bundle on $Gr(2,N)$ whose fiber is two dimensional complex vector space in C^N corresponding to the point in $Gr(2,N)$. U^* is dual bundle of U.) $Sym^k(U^*)$ represents k-times tensor product of U^* modulo the action of symmetric group.). Then

$$\begin{aligned} & l \in Gr(2,N) \text{ is contained in } M_N^k \\ \Longleftrightarrow \quad & F_N^k|_l = 0 \\ \Longleftrightarrow \quad & s_N^k = 0 \; at \; l \in Gr(2,N) \\ \Longleftrightarrow \quad & l \in PD(c_T(Sym^k(U^*))) \end{aligned} \tag{3.2.17}$$

In deriving last line from the third one, we used Gauss-Bonnet theorem that says zero locus of a section of vector bundle E is homologically equivalent to $PD(c_T(E))$. Since $rank(Sym^k(U^*))$ equals to $k+1$, $\dim(PD(c_T(Sym^N(U^*))))$ is $(2N-4)-k-1 = 2N-k-5$, which agrees with $\dim(\mathcal{M}_{1,0}^{M_N^k}) = 2N-k-2$ and $SL(2,C)$ equivalence.

Then we have the following formula

$$\langle \prod_{j=1}^{2N-k-2} \mathcal{O}_e(z_i) \rangle_1 = \int_{Gr(2,N)} c_T(Sym^k(U^*)) \wedge PD(\xi_*(PD(\bigwedge_{i=1}^{2N-2-k} \bar{\varphi}_i^*(e))) \tag{3.2.18}$$

We can explicitly calculate $PD(c_T(Sym^k(U^*)))$ using Schubert calculus. Let us introduce Schubert cycles in $Gr(2,N)$. Schubert cycles $\sigma_{a_1,a_2} \subseteq Gr(2,N)$ $(N-2 \geq a_1 \geq a_2 \geq 0)$ form a basis of $H_*(Gr(2,N),Z) (\simeq H^*(Gr(2,N),Z))$ and are given by the following definition.

$$\sigma_{a_1,a_2} = \{l \in Gr(2,N) | \dim_C(l \cap V_{N-2+i-a_i}) \geq i\} \tag{3.2.19}$$

where V_i's are linear subspace of C^N of dimension i satisfying following condition.

$$V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset C^N \tag{3.2.20}$$

Considering these elements of $H^*(Gr(2,N),C)$, multiplication rules of $H^*(Gr(2,N),C)$ are determined thoroughly through the following formula.

$$\text{Pieri's formula} \qquad \sigma_{a,0} \cdot \sigma_{b_1,b_2} = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ c_1+c_2=a+b_1+b_2}} \sigma_{c_1,c_2} \tag{3.2.21}$$

$$\text{Giambelli's formula} \qquad \sigma_{a_1,a_2} = \sigma_{a_1,0}\sigma_{a_2,0} - \sigma_{a_1+1,0}\sigma_{a_2-1,0} \tag{3.2.22}$$

Then we calculate $c_T(Sym^k(U^*))$. We first introduce the following fact.

Fact.

$$c(U) = 1 - \sigma_1 t + \sigma_{1,1} t^2 \Longrightarrow c(U^*) = 1 + \sigma_1 t + \sigma_{1,1} t^2 \tag{3.2.23}$$

where $c(E)$ denotes total Chern class of vector bundle E and U denotes universal bundle of $Gr(2,N)$.

We formally represent U^* as direct sum of line bundles E and F i.e. $U^* = E \oplus F$ and we set

$$\begin{aligned} c(E) &= 1 + xt \\ c(F) &= 1 + yt \\ & \quad (x \text{ and } y \text{ are formal variables}). \end{aligned} \tag{3.2.24}$$

From Fact and (3.2.24), we have $c(U^*) = c(E)c(F) = 1 + (x+y)t + (xy)t^2$, and

$$\begin{aligned} x + y &= \sigma_1 \\ xy &= \sigma_{1,1} \end{aligned} \tag{3.2.25}$$

We can formally decompose $Sym^k(U^*)$ into the form

$$Sym^k(U^*) = E^{\otimes k} \oplus E^{\otimes k-1} \otimes F \oplus \cdots \oplus E^{\otimes k-1} \otimes F \oplus F^{\otimes k} \tag{3.2.26}$$

and we have

$$c(Sym^k(U^*)) = (1+kxt)(1+((k-1)x+y)t)\cdots(1+kyt) \tag{3.2.27}$$

Top Chern class is given as the coefficients of t^{k+1}.

$$c_T(Sym^k(U^*)) = kx((k-1)x+y)((k-2)x+2y)\cdots ky \tag{3.2.28}$$

$c_T(Sym^k(U^*))$ consists of symmetric polynomials of x and y,so from (3.2.25), we can represent $c_T(Sym^N(U^*))$ as polynomials of σ_1 and $\sigma_{1,1}$. The result is

$$\begin{aligned}
c_T(Sym^k(U^*)) &= k(k!)\sum_{i=0}^{\{\frac{k}{2}\}} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sigma_{1,0}^{2i}\sigma_{1,1}^{\{\frac{k}{2}\}-i+1} \\
&\quad (k:\text{odd}) \\
c_T(Sym^k(U^*)) &= k(k!)\sum_{i=0}^{\{\frac{k}{2}\}} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sigma_{1,0}^{2i+1}\sigma_{1,1}^{\{\frac{k}{2}\}-i+1} \\
&\quad (k:\text{even}) \\
\text{where} & \\
\{\frac{k}{2}\} &:= \frac{k}{2}-\frac{1}{2} \quad (k:\text{odd}) \\
&:= \frac{k}{2}-1 \quad (k:\text{even}) \\
\beta_i &= \frac{(k-2i)^2}{(k-i)i} \quad (1\leq i\leq \frac{k}{2}) \\
Sym^j(\beta_i) &:= \sum_{1\leq i_1\leq\cdots\leq i_j\leq\{\frac{k}{2}\}} \beta_{i_1}\cdots\beta_{i_j}
\end{aligned} \tag{3.2.29}$$

From (3.2.21) and (3.2.22), we can derive two formula.

$$\sigma_{1,1}^n = \sigma_{n,n} \quad (n\leq N-2) \tag{3.2.30}$$

$$\begin{aligned}
\sigma_{1,0}^k\cdot\sigma_{n,n} &= \sum_{i=0}^{[\frac{k}{2}]}({}_kC_i - {}_kC_{i-1})\sigma_{k+n-i,n+i} \quad (k+n\leq N-2) \\
\sigma_{1,0}^{N-1}\cdot\sigma_{1,1} &= \sum_{i=2}^{[\frac{N}{2}]}({}_{N-1}C_i - {}_{N-1}C_{i-1})\sigma_{N-i,i+1} \\
\sigma_{1,0}^{N-3}\cdot\sigma_{2,2} &= \sum_{i=1}^{[\frac{N}{2}]-1}({}_{N-3}C_i - {}_{N-3}C_{i-1})\sigma_{N-1-i,i+2}
\end{aligned} \tag{3.2.31}$$

Combination of (3.2.29),(3.2.30) and (3.2.31) leads to the formula,

$$\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i}C_j - {}_{2i}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad (k\leq N-2)
\end{aligned} \tag{3.2.32}$$

$$\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}-1} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i}C_j - {}_{2i}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad +k(k!)\sum_{j=1}^{\{\frac{k}{2}\}}({}_{k-1}C_j - {}_{k-1}C_{j-1})\sigma_{k-j,j+1} \\
&\quad (k=N-1)
\end{aligned} \tag{3.2.33}$$

$$\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}-2} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i}C_j - {}_{2i}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad +k(k!)Sym^1(\beta_m)\sum_{j=1}^{\{\frac{k}{2}\}-1}({}_{k-3}C_j - {}_{k-3}C_{j-1})\sigma_{k-1-j,j+2} \\
&\quad +k(k!)\sum_{j=2}^{\{\frac{k}{2}\}}({}_{k-1}C_j - {}_{k-1}C_{j-1})\sigma_{k-j,j+1} \\
&\quad (k=N) \\
&\quad (\text{for } k:\ \text{odd})
\end{aligned} \tag{3.2.34}$$

$$\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i+1}C_j - {}_{2i+1}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad (k\leq N-2)
\end{aligned} \tag{3.2.35}$$

$$\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}-1} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i+1}C_j - {}_{2i+1}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad +k(k!)\sum_{j=1}^{\{\frac{k}{2}\}}({}_{k-1}C_j - {}_{k-1}C_{j-1})\sigma_{k-j,j+1} \\
&\quad (k=N-1)
\end{aligned} \tag{3.2.36}$$

$$
\begin{aligned}
&PD(c_T(Sym^k(U^*))) = \\
&\quad k(k!)\sum_{i=0}^{\{\frac{k}{2}\}-2} Sym^{\{\frac{k}{2}\}-i}(\beta_m)\sum_{j=0}^{i}({}_{2i+1}C_j - {}_{2i+1}C_{j-1})\sigma_{\{\frac{k}{2}\}+i-j+1,\{\frac{k}{2}\}-i+j+1} \\
&\quad +k(k!)Sym^1(\beta_m)\sum_{j=1}^{\{\frac{k}{2}\}-1}({}_{k-3}C_j - {}_{k-3}C_{j-1})\sigma_{k-1-j,j+2} \\
&\quad +k(k!)\sum_{j=2}^{\{\frac{k}{2}\}}({}_{k-1}C_j - {}_{k-1}C_{j-1})\sigma_{k-j,j+1} \\
&\quad (k=N) \\
&\quad (\text{for } k: \text{ even})
\end{aligned}
\tag{3.2.37}
$$

These formulas represent the exact moduli spaces $\mathcal{M}_{0,1}^{M_N^k}$ divided by $SL(2,C)$.

Next, from (3.2.18), what we have to do is to determine $^\sharp(\sigma_{k-j,j+1}\cap\xi_*(PD(\wedge_{i=1}^{2N-k-2}\tilde{\varphi}_i^*(e))))$. We define α_N^k as $PD(\wedge_{i=1}^{2N-k-2}\tilde{\varphi}_i^*(e))$. α_N^k is constructed from the definition of $\tilde{\varphi}_i$.

$$
\begin{aligned}
\tilde{\varphi}_i : \mathcal{M}_{0,1}^{CP^{N-1}} &\mapsto CP^{N-1} \\
f\in\mathcal{M}_{0,1}^{CP^{N-1}} &\mapsto f(z_i)
\end{aligned}
\tag{3.2.38}
$$

Since $PD(\wedge_{i=1}^{2N-k-2}\tilde{\varphi}_i^*(e)) = \cap_{i=1}^{2N-k-2}PD(\tilde{\varphi}_i^*(e)) = \cap_{i=1}^{2N-k-2}\tilde{\varphi}_{i*}(PD(e))$, α_N^k is constructed as subspace of CP^{2N-1} satisfying the following conditions.

$$
f(z_i)\in PD(e_i)\quad (i=1,\cdots,2N-k-2) \tag{3.2.39}
$$

where

$$
\begin{aligned}
&f\in CP^{2N-1} = \begin{pmatrix} a_0^1 & a_0^2 & \cdots & a_0^N \\ a_1^1 & a_1^2 & \cdots & a_1^N \end{pmatrix}/C^* \\
&z_1=(0:1),\ \ z_2=(1:0),\ \ z_3=(1:-1) \\
&z_i=(c_i:-1)\ \ (4\le i\le 2N-k-2) \\
&PD(e_i)=\{(X_1:\cdots:X_N)\in CP^{N-1}|X_i=0\} \\
&\qquad (1\le i\le N) \\
&PD(e_i)=\{(X_1:\cdots:X_N)\in CP^{N-1}|X_{k+2}=X_{k+2+i-N}\} \\
&\qquad (N+1\le i\le 2N-k-2)
\end{aligned}
$$

By solving (3.2.40) and using map ξ, we can construct cycle α_N^k.

$$
\alpha_N^k := \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_{2N-k-2} & \alpha_{2N-k-1} & \cdots & \alpha_N \\ 0 & \beta_2 & \alpha_3 & c_4\alpha_4 & \cdots & c_{2N-k-2}\alpha_{2N-k-2} & \beta_{2N-k-1} & \cdots & \beta_N \end{pmatrix} \tag{3.2.40}
$$

$$(N-2\le k\le N)$$

$$
\alpha_N^k := \left(\begin{matrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_{k+2} \\ 0 & \beta_2 & \alpha_3 & c_4\alpha_4 & \cdots & c_{k+2}\alpha_{k+2} \end{matrix}\right.
$$

$$
\left.\begin{matrix} \frac{1-c_{N+1}c_{k+2}}{1-c_{N+1}c_{k+3}}\alpha_{k+2} & \cdots & \frac{1-c_{2N-k-2}c_{k+2}}{1-c_{2N-k-2}c_N}\alpha_{k+2} \\ \frac{1-c_{N+1}c_{k+2}}{1-c_{N+1}c_{k+3}}c_{k+3}\alpha_{k+2} & \cdots & \frac{1-c_{2N-k-2}c_{k+2}}{1-c_{2N-k-2}c_N}c_N\alpha_{k+2} \end{matrix}\right) \tag{3.2.41}
$$

$$(1\le k\le N-3)$$

(where α_i, β_i are arbitrary complex numbers)

We can count $^\sharp(\sigma_{k-j,j+1}\cap\alpha_N^k)$ by use of matrix representation of $\sigma_{k-j,j+1}$.

$$
\sigma_{k-j,j+1} = \begin{pmatrix} \gamma_1 & \cdots & \gamma_{N-k+i-2} & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \delta_1 & \cdots & \delta_{N-k+i-2} & 0 & \delta_{N-k+i-1} & \cdots & \delta_{N-i-3} & 1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2.42}
$$

(γ_i, δ_i represents arbitrary complex number and precisely speaking, we have to add boundary points to (3.2.42) to compactify the cycle.)

Then $(\alpha_N^k\cap\sigma_{k-j,j+1})$ are given as the following $k-2i$ points. (In matrix representation $\dim(\alpha_N^k)+\dim(\sigma_{k-j,j+1}) = 2N-4 < 2N = \dim(\text{Matrix})$, so we have to permit multiplying each row of α_N^k by constant and adding one row to another when we calculate intersection number. See Local Appendix B for details.)

$$
\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix},\cdots,\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2.43}
$$

Thus we have

$$
^\sharp(\alpha_N^k\cap\sigma_{k-j,j+1}) = k-2j. \tag{3.2.44}
$$

Combining (3.2.44) with formulas of $PD(c_T(Sym^k(U^*)))$, and using the following identities,

$$
\begin{aligned}
&\sum_{j=0}^{i}({}_{2i}C_j - {}_{2i}C_{j-1})(2j-2i+2) = 2^{2i} \\
&\sum_{j=0}^{i}({}_{2i+1}C_j - {}_{2i+1}C_{j-1})(2j-2i+2) = 2^{2i+1}
\end{aligned}
\tag{3.2.45}
$$

$$
\sum_{j=0}^{\{\frac{k}{2}\}} Sym^{\{\frac{k}{2}\}-j}(\frac{\beta_m}{4}) = \prod_{j=1}^{\{\frac{k}{2}\}}(1+\frac{\beta_j}{4}) \tag{3.2.46}
$$

we get the final result of this section.

$$
\begin{aligned}
\langle \prod_{i=1}^{(N-2)+(N-k)} \mathcal{O}_e(z_i)\rangle_1 &= k^{k+1} \qquad (k\le N-2) \\
&= k^{k+1} - k^2\cdot k! \qquad (k=N-1) \\
&= k^{k+1} - (k-2)\cdot k\cdot k!\cdot(\sum_{j=1}^{k-1}\frac{j}{k-j}) - 2k\cdot k! \\
&\qquad (k=N)
\end{aligned}
\tag{3.2.47}
$$

Note that this result agrees with the statement of (3.1.15). And we can see in $k \geq N-1$ case, boundary part appears as correction term in correlation function.

Appendix A of Section 3.2

Construction of section s_N

Let $\sum_{j=1}^{N} b_i^j X_j = 0 (i = 1, 2, \ldots, N-2)$ be defining equations of $l \in Gr(2, N)$. In matrix form , it can be written as

$$BX = 0 \tag{3.2.48}$$

where $B = (b_i^j)$ and $X = (X_j)$.

We transform B into a simple form by multiplying an $(N-2) \times (N-2)$ invertible matrix D from the left and $N \times N$ invertible matrix C^{-1} from the right, i.e

$$DBC^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.2.49}$$

Then base transformation $CX = X' = (X'_j)$ turns (3.2.48) into a form

$$X'_i = 0 \quad (i = 3, 4, \ldots, N) \tag{3.2.50}$$

From (3.2.50) we can see X'_1 and X'_2 as dual coordinate basis of $l \in Gr(2, N)$.In other words, they are the basis of the fiber of U^* at l.(U denotes universal bundle of $Gr(2, N)$.)

Let $F_N = \sum_{i=1}^{N} X_i^N$ be defining equation of M_N. We substitute X_i in F_N by $(C^{-1})_i^j X'_j$ and set $X'_i = 0 (i = 3, 4, \ldots, N)$. (This operation corresponds to restriction of F_N at l.)

Then we get homogeneous polynomial of X'_1 and X'_2 of degree N, which defines section of $Sym^N(U^*)$ at l.

Appendix B of Section 3.2

*** Counting of $^\sharp(c_T(PD(Sym^N(U^*))) \cap \alpha_N^k)$**

We will limit the discussions to the case of α_N^N. Other cases can be treated in the same manner.

Since we have $c_T(Sym^N(U^*))$ written in terms of Schubert cycles, we only have to determine the intersection number $^\sharp(\sigma_{N-i,i+1} \cap \alpha).(i = 2, 3, \ldots, \{N/2\})$

*We owe this part of discussion to Dr. Hori

As we have said in the body of this paper, by solving (3.2.40) explicitly and using map ξ,we have α represented as $N+1$ dim subspace in the space of $2\times N$ matrix.

$$\alpha_N^N := \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_{N-2} & \alpha_{N-1} & \alpha_N \\ 0 & \beta_2 & \alpha_3 & c_4\alpha_4 & \ldots & c_{N-2}\alpha_{N-2} & \beta_{N-1} & \beta_N \end{pmatrix} \tag{3.2.51}$$

(β_i, γ_i : arbitrary complex number)

On the other hand,$\sigma N-i, i+1$ can also be represented as $N-5$ dim subspace as follows.

$$\sigma_{N-j,j+1} = \begin{pmatrix} \gamma_1 & \cdots & \gamma_{i-2} & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \delta_1 & \cdots & \delta_{i-2} & 0 & \delta_{i-1} & \cdots & \delta_{N-i-3} & 1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2.52}$$

(γ_i, δ_i represents arbitrary complex number, and, precisely speaking, (3.2.52) represents internal points subset of $\sigma_{N-i,i+1}$, so we have to compactify it by adding boundary points.)

Then, what we have to do is to determine the intersection points between (3.2.51) and (3.2.52).

We have two troubles:

1. (3.2.51) and (3.2.52) are in $2\times N$ matrix form and $GL(2,C)$ indeterminate. So in counting intersection points,we can multiply each row vector of (3.2.51) by constant and add one row to the other.

2. In $i \geq 3$ case,(3.2.51) and (3.2.52) do not intersect transversely, i.e intersect in more than one dimension, so we have to substitute α_N^N by the cycles $\alpha_{i,N}^N$ $(i = 2, 3, \ldots, \{\frac{N}{2}\})$ which are homologically equivalent to α_N^N and intersects transversely with $\sigma_{N-i,i+1}$.

$$\begin{aligned} \alpha_{i,N}^N \;=\; &\left(\begin{matrix} \alpha_3 & \alpha_4 & \cdots & \alpha_i & \alpha_1 & 0 & \alpha_{i+1} & \cdots & \alpha_{N-i+1} \\ \alpha_3 & c_4\alpha_4 & \cdots & c_i\alpha_i & 0 & \beta_2 & c_{i+1}\alpha_{i+1} & \cdots & c_{N-i-1}\alpha_{N-i+1} \end{matrix} \right. \\ &\left. \begin{matrix} \alpha_{N-i}+\alpha_i & \cdots & \alpha_3+\alpha_{N-3} & \alpha_{N-2} & \alpha_{N-1} & \alpha_N \\ c_{N-i}\alpha_{N-i}+c_i\alpha_i & \cdots & \alpha_3+c_{N-3}\alpha_{N-3} & c_{N-2}\alpha_{N-2} & \beta_{N-1} & \beta_N \end{matrix} \right) \end{aligned} \tag{3.2.53}$$

Then,$\sigma_{N-i,i+1}$ and α intersects in the following $N-2i$ points.

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2.54}$$

(Notice that intersection points lie in boundary component of $\sigma_{N-i,i+1}$ except for last one.)

Finally we have $^\sharp(\sigma_{N-i,i+1} \cap \alpha) = N - 2i$.

3.3 Gravitational Moduli Space of CP^1

This section is given to prepare for the notion of stable maps which describes moduli space related to topological sigma model (A-model) coupled with gravity. Gravitational,or complex structure moduli space of CP^1 with n-punctures, $\mathcal{M}_{0,n}$ is roughly the positions of distinct n points on CP^1 divided by the automorphism group of CP^1, $SL(2,C)$.

$$\mathcal{M}_{0,n} \simeq \{z_1, z_2, \cdots, z_n\}/SL(2,C) \tag{3.3.55}$$

R.h.s of (3.3.55) is non-compact, since z_i's are distinct. To compactify $\mathcal{M}_{0,n}$, we add stable curves which consist of several component CP^1's glued by double singularities. From a puncture on one component of a stable curve, punctures lying on the other components cannot be distinguished. In this way, we can describe the coincidence of punctures in systematic way. We will review some topological structures of $\mathcal{M}_{0,n}$ following the formalism of [21].

3.3.1 Trees

We introduce trees describing combinatorial structure of $\mathcal{M}_{0,n}$. Their vertices correspond to components, and edges to punctures.

Definition

A (stable) tree τ is a collection of finite sets V_τ (vertices), E_τ (interior edges), T_τ (exterior edges, or tails), and two boundary maps $b : T_\tau \mapsto V_\tau$ (every tail has one end vertex), and $b : E_\tau \mapsto$ {unbranched pairs of distinct vertices}. (every interior edge has exactly two vertices).

The geometric realization of τ must be connected and simply connected. Every vertex must belong to at least three edges, exterior and / or interior (stability).

Definition

A morphism of trees $f : \tau \mapsto \sigma$ is a collection of three maps (notice arrow directions)

$$f_v : V_\tau \mapsto V_\sigma, \quad f^t : T_\sigma \mapsto T_\tau, \quad f^e : E_\sigma \mapsto E_\tau \tag{3.3.56}$$

with the following properties.

a) f_v is surjective, f^t and f^e are injective.

b) If v_1, v_2 are ends of an edge e' of τ, then either $f_v(v_1) = f_v(v_2)$, or $f_v(v_i)$ are ends of an edge e'' of σ : we say that e' covers this edge, and we must then have $e' = f^e(e'')$.

c) If $v' \in V_\tau$ is such a vertex that $f_v(v')$ is the end of $t'' \in T_\tau$, then v' is the end of $f^t(t'')$.

In other words, f contracts interior edges from $E_\tau / f^e(E_\sigma)$ and tails from $T_\tau / f^t(T_\sigma)$, and is one-to-one on the remaining edges. We will denote by $f(e)$ the image of a non-contracted edge.

Flags and dimension

A pair { edge, one end of it } is called a flag. For a tree τ, We denote by F_τ the set of its flags, and by $F_\tau(v)$ the set of flags ending in vertex v. We have $|F_\tau| = 2|E_\tau| + |T_\tau|$.

The dimension of τ is defined by

$$\dim(\tau) := \sum_{v \in V_\tau} (|F_\tau(v) - 3|) = 2|E_\tau| + |T_\tau| - 3|V_\tau| \tag{3.3.57}$$

Gluing

Let $(\tau_i, t_i), i = 1, 2$, be two pairs consisting each of a tree and its tail. Then gluing (t_1 to t_2) produce a pair (τ, E) consisting of a tree and its interior edge:

$$(\tau, e) := (\tau_1, t_1) * (\tau_2, t_2) \tag{3.3.58}$$

Formally:

$$V_\tau = V_{\tau_1} \coprod V_{\tau_2}, \quad E_\tau = E_{\tau_1} \coprod E_{\tau_2} \coprod \{e\} \tag{3.3.59}$$

$$T_\tau = (T_{\tau_1} \coprod T_{\tau_2}) / \{t_1, t_2\}, \quad b(e) = \{b(t_1), b(t_2)\} \tag{3.3.60}$$

This operation is functorial in the following sense: for two morphisms $f_i : \tau_i \mapsto \sigma_i$ not contracting t_i, we have a self explanatory morphism

$$f_1 * f_2 : (\tau_1, t_1) * (\tau_2, t_2) \mapsto (\sigma_1, f_1(t_1)) * (\sigma_2, f_2(t_2)) \tag{3.3.61}$$

Finally, $F_\tau = F_{\tau_1} \coprod F_{\tau_2}$

3.3.2 From trees to moduli spaces

In this subsection, we define a functor

$$\mathcal{M} : \{\text{trees}\} \mapsto \{\text{algebraic manifolds}\} \tag{3.3.62}$$

Objects

Put

$$\mathcal{M}_\tau = \prod_{v \in V_\tau} \mathcal{M}_{0, F_\tau(v)}. \tag{3.3.63}$$

We have $\dim(\mathcal{M}(\tau)) = \dim(\tau)$.

This space parameterizes a family of (generally reducible) stable rational curves $C(\tau)$ with marked points indexed by T_τ. The dual graph of a generic (but not arbitrary) curve of this family is (canonically identified with) τ. To describe it, consider a point $x = (x_v) \in \mathcal{M}(\tau)$, $x_v \in \mathcal{M}_{0, F_\tau(v)}$ and let $C(x_v)$ be the fiber of a universal curve at this point. If v_1, v_2 bound an edge e of τ, $C(x_v)$ contains a point $y(v_i, e)$ marked by the flag (v_i, e). Identify $y(v_1, e)$ with $y(v_2, e)$ in the disjoint union $\coprod_{v \in F_\tau} C(x_v)$ for all e. This will be $C(\tau)(x)$.

Clearly, its remaining special points are marked by T_τ so that we have a canonical morphism (closed embedding) $\mathcal{M}(\tau) \mapsto \mathcal{M}_{0,T_\tau}$. This is a special case of morphisms defined below.

Morphisms

Any morphisms of trees $f : \tau \mapsto \sigma$ contracting no tails induces a closed embedding $\mathcal{M}(\tau) \mapsto \mathcal{M}(\sigma)$. To construct it, identify $T_\tau = T_\sigma = T$ by means of f^t, and denote by ρ the one vertex tree with tails T. Clearly $\mathcal{M}(\rho) = \mathcal{M}_{0,T}$, and by universality, we have embedding of $\mathcal{M}(\sigma)$ and $\mathcal{M}(\tau)$ into $\mathcal{M}(\rho)$. In this embedding, $\mathcal{M}(\sigma) \subset \mathcal{M}(\tau)$ which is seeked for morphism.

Any morphism of one-vertex trees contracting tails induces the forgetful morphism of the respective moduli spaces.

The general construction of moduli space morphism of trees can be can be obtained by combining these two cases: embed $\mathcal{M}(\tau)$ into $\mathcal{M}_{0,T_\tau}$, $\mathcal{M}(\sigma)$ into $\mathcal{M}_{0,T_\sigma}$, and restrict the forgetful map onto $\mathcal{M}(\tau)$.

Gluing

If $(\tau, t) = (\tau_1, t_1) * (\tau_2, t_2)$, we have canonically:

$$\mathcal{M}(\tau) = \mathcal{M}(\tau_1) \times \mathcal{M}(\tau_2) \tag{3.3.64}$$

$$H^*(\mathcal{M}(\tau)) = H^*(\mathcal{M}(\tau_1)) \otimes H^*(\mathcal{M}(\tau_2)) \tag{3.3.65}$$

3.3.3 Homology of moduli spaces

Additive generators

If $T_\tau = \{1, 2, \cdots, n\}$, we will call τ an n-tree. A morphism of n-trees $\tau \mapsto \sigma$ identical on tails will be called n-morphism. If such a morphism exists, it is unique. Let ρ_n be a one-vertex n-tree. Then $\mathcal{M}(\rho_n) = \mathcal{M}_{0,n}$. For any n-tree τ, there exists a unique n-contraction $\tau \mapsto \rho_n$. Let $d_\tau \in H_*(\mathcal{M}_{0,n})$ be the homology class of $\mathcal{M}(\tau)$ corresponding to this contraction. It depends only on the n-isomorphism class of τ. The manifolds $\mathcal{M}(\tau)$ embedded into each other in this way will be called strata. Then the following theorem holds.

Theorem

d_τ span $H_*(\mathcal{M}_{0,n})$

Linear relations

Choose a system $R = (\tau, \{i, j, k, l\}, v)$ where τ is an n-tree,$1 \leq i, j, k, l \leq n$ are its pairwise distinct tails, and $v \in V_\tau$ is such a vertex that paths from v to i, j, k, l start with pairwise distinct edges e_i, e_j, e_k, e_l respectively (some of these edges may be tails themselves).

Consider all n-contractions $\tau' \mapsto \tau$ which contract exactly one edge onto the vertex v and satisfy the following condition: lifts to τ' of e_i, e_j on the one hand, and e_k, e_l on the other, are incident to different vertices of the contracted edge. Below we will denote by $\{ij\tau'kl\}$ the summation over n-isomorphism classes of such contractions, R being fixed.

Lemma

For any R, we have

$$\sum_{\{ij\tau'kl\}} d_{\tau'} = \sum_{\{ik\tau''jl\}} d_{\tau''} \tag{3.3.66}$$

in $H_*(\mathcal{M}_{0,n})$

Proof Consider a morphism of τ contracting all edges and tails except of i, j, k, l. It induces the forgetful morphism $\mathcal{M}(\tau) \mapsto \mathcal{M}_{0,\{i,j,k,l\}} \cong CP^1$. Then both sides of (3.3.66) are mapped to points which are homologically equivalent to each other on CP^1. Two fibers over boundary divisors of the latter moduli space are represented by the cycles $sum_{\{ij\tau'kl\}}\mathcal{M}(\tau')$ and $sum_{\{ik\tau''jl\}}\mathcal{M}(\tau'')$ respectively.

Theorem

Relation (3.3.66) span the space of all linear relations between d_τ.

Lemma

As an algebra, $H^* := H^*(\mathcal{M}_{0,n})$ is generated by the boundary divisorial cohomology classes D_S indexed by unordered partitions S of $\{1, 2, \cdots, n\}$ into two parts S_1, S_2 of cardinality ≥ 2 and satisfying the following generating relations.

$$\sum_{\{ijSkl\}} D_S = \sum_{\{ikTjl\}} D_T \tag{3.3.67}$$

and

$$D_S D_T = 0 \tag{3.3.68}$$

if four sets $S_i \cap T_j$ are pairwise distinct and non-empty. (In this case we will call S and T in compatible).

Classes D_S are dual to the homology classes d_σ where σ run over n-trees with two vertices, and (3.3.67) is a consequence of (3.3.66).

Denote now by H_* the linear space generated by the symbols $[d_\sigma]$ subject to all relations (3.3.66) where σ, τ run over all n-isomorphism classes of n-trees.

There is an obvious surjective map $a : H_* \mapsto H^*$,

$$a([d_\sigma]) := \text{the cohomology class dual to} d_\sigma. \tag{3.3.69}$$

Main Theorem

H_* can be endowed with a structure of cyclic H^*-module generated by $[d_{\rho_n}] := 1$ so that the map

$$b : H^* \mapsto H_*, \quad b(h) = h \cdot 1 \tag{3.3.70}$$

is surjective.

3.4 Stable Map

Let M be a Kähler manifold with $H^{1,1}(M, Z) \simeq Ze$. Moduli space corresponding to topological sigma model coupled to gravity from CP^1 to M is the moduli space of holomorphic maps from punctured CP^1 to M. It is fundamentally constructed as follows.

$$\mathcal{M}^M_{0,d,n} \simeq \{z_1, z_2, \cdots, z_n, f\}/SL(2, C) \tag{3.4.71}$$

where f denotes holomorphic maps from CP^1 to M of degree d. As we have said in the previous section, n-punctures are in distinct positions. Then expression (3.4.71) is non-compact. To compactify $\mathcal{M}^M_{0,d,n}$ we have to introduce the notion of stable map which describe the coincidence of punctures.

Definition. Stable map is a structure $(C; z_1, z_2, \cdots, z_n, f)$ consisting of a connected compact reduced C with $k \geq 0$ pairwise distinct marked non-singular points z_i and at most ordinary double singular points, and a map $f : C \mapsto M$ having no nontrivial first order infinitesimal automorphisms, identical on M and $z_1, \cdots, z_n$ (stability).

Then $\mathcal{M}^M_{0,d,n}$ is defined by the

Definition. $\mathcal{M}^M_{0,d,n}$ is the moduli space of stable maps to M of curves of arithmetic genus 0 with $n \geq 0$ marked points such that $f_*[C] = de$.

With this definition, we can naturally define the map $\Pi : \mathcal{M}^M_{0,d,n} \mapsto \mathcal{M}_{0,n}$ as follows.

$$\Pi : \{(z_1, z_2, \cdots, z_n), f\}/SL(2,C) \mapsto (z_1, z_2, \cdots, z_n)/SL(2,C) \qquad (3.4.72)$$

We can easily extend (3.4.72) to boundary part of moduli spaces with the notion of stable curve and stable map.

Let M be a Fano variety ,i.e., $c_1(TM) > 0$ and $W_i (i = 1, \cdots, m)$ be the element of $H^*(M)$. From (3.3.68), we have the following equality.

$$\sum_{\{ijSkl\}} \int_{\mathcal{M}^d_{0,n,M}} \bigwedge_{p=1}^{n} \phi_p^*(W_{m_p}) \wedge \Pi^*(D_S) = \sum_{\{ikTjl\}} \int_{\mathcal{M}^d_{0,n,M}} \bigwedge_{p=1}^{n} \phi_p^*(W_{m_p}) \wedge \Pi^*(D_T) \quad (3.4.73)$$

Then we assume the following formula.

$$\begin{aligned}
&\sum_{\{ijSkl\}} \int_{\mathcal{M}^d_{0,n,M}} \bigwedge_{p=1}^{n} \phi_p^*(W_{m_p}) \wedge \Pi^*(D_S) \\
&= \sum_{A\coprod B=\{1,\cdots,n\}-\{i,j,k,l\}} \sum_{d_1+d_2=d} \int_{\mathcal{M}^{d_1}_{0,\sharp A+3,M}} \phi_i^*(W_{m_i}) \wedge \phi_j^*(W_{m_j}) \bigwedge_{p\in A} \phi_p^*(W_{m_p}) \wedge \phi_a^*(W_\alpha) \\
&\cdot \eta^{\alpha\beta} \int_{\mathcal{M}^{d_2}_{0,\sharp B+3,M}} \phi_b^*(W_\beta) \wedge \phi_k^*(W_{m_k}) \wedge \phi_l^*(W_{m_l}) \bigwedge_{p\in B} \phi_p^*(W_{m_p}) \qquad (3.4.74)
\end{aligned}$$

where

$$\eta_{\alpha\beta} := \int_M W_\alpha \wedge W_\beta \quad \eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_\alpha^\gamma \qquad (3.4.75)$$

ϕ_a and ϕ_b are the evaluation map corresponding to the double singularity punctures z_a and z_b coming from the contracted edge of τ' in (3.3.66). We can understand the meaning of (3.4.74) by physical logic ,i.e., infinite time evolution are described by the insertion of projection operator to ground states. Then combining (3.4.73),(3.4.74)

and the fact that M is Fano variety, we have

$$\begin{aligned}
&\sum_{A\coprod B=\{1,\cdots,n\}-\{i,j,k,l\}} \langle \mathcal{O}_{W_{m_i}} \mathcal{O}_{W_{m_j}} \prod_{p\in A} \mathcal{O}_{W_{m_p}} \mathcal{O}_{W_\alpha} \rangle \eta^{\alpha\beta} \langle \mathcal{O}_{W_\beta} \mathcal{O}_{W_{m_k}} \mathcal{O}_{W_{m_l}} \prod_{p\in B} \mathcal{O}_{W_{m_p}} \rangle \\
= &\sum_{A\coprod B=\{1,\cdots,n\}-\{i,j,k,l\}} \langle \mathcal{O}_{W_{m_i}} \mathcal{O}_{W_{m_k}} \prod_{p\in A} \mathcal{O}_{W_{m_p}} \mathcal{O}_{W_\alpha} \rangle \eta^{\alpha\beta} \langle \mathcal{O}_{W_\beta} \mathcal{O}_{W_{m_j}} \mathcal{O}_{W_{m_l}} \prod_{p\in B} \mathcal{O}_{W_{m_p}} \rangle
\end{aligned} \qquad (3.4.76)$$

We can eliminate the condition $d = d_1 + d_2$ using topological selection rule.

Now we introduce the generating function of correlation functions (free energy).

$$F_M(t_1, t_2, \cdots, t_m) := \sum_{n_p \geq 0} \langle \prod_{p=1}^{m} \mathcal{O}_{W_p}^{n_p} \rangle \prod_{p=1}^{m} \frac{t_p^{n_p}}{n_p!} \qquad (3.4.77)$$

And we have

$$\partial_{t_i}\partial_{t_j}\partial_{t_\alpha} F_M = \sum_{n_p \geq 0} \langle \mathcal{O}_{W_i} \mathcal{O}_{W_j} \mathcal{O}_{W_\alpha} \prod_{p=1}^{m} \mathcal{O}_{W_p}^{n_p} \rangle \prod_{p=1}^{m} \frac{t_p^{n_p}}{n_p!} \qquad (3.4.78)$$

Then using (3.4.76), we derive the following equation.

$$\begin{aligned}
&\partial_{t_i}\partial_{t_j}\partial_{t_\alpha} F_M \eta^{\alpha\beta} \partial_{t_\beta}\partial_{t_k}\partial_{t_l} F_M \\
&= \sum_{0\leq m_p \leq n_p, 0 \leq n_p} \langle \mathcal{O}_{W_i} \mathcal{O}_{W_j} \prod_{p=1}^{m} \mathcal{O}_{W_p}^{m_p} \mathcal{O}_{W_\alpha} \rangle \eta^{\alpha\beta} \langle \mathcal{O}_{W_\beta} \mathcal{O}_{W_k} \mathcal{O}_{W_l} \prod_{p=1}^{m} \mathcal{O}_{W_p}^{n_p - m_p} \rangle \prod_{p=1}^{m} {}_{n_p}C_{m_p} \frac{t_p^{n_p}}{n_p!} \\
&= \partial_{t_i}\partial_{t_k}\partial_{t_\alpha} F_M \eta^{\alpha\beta} \partial_{t_\beta}\partial_{t_j}\partial_{t_l} F_M \qquad (3.4.79)
\end{aligned}$$

This is DWVV equation. We will return to it in Section 4.2.

3.5 Torus Action Method

In this section we introduce torus action method invented by Kontsevich, and perform some explicit calculation of correlation functions of A-model on M_N^N coupled with gravity from the formula (2.4.89). We also give path-integral representation of generating function of correlation functions of A-model on M_N^k coupled with gravity.

3.5.1 Introduction of the Torus Action and the Bott Residue Formula

Torus action method is the strategy to use the Bott residue formula [27] which reduces the integral of Chern classes of vector bundle on X to the one on X_f of the fixed point set of the torus action flow on X to the case where X is $\mathcal{M}^{CP^{N-1}}_{0,d,m}$.

First, let us introduce the torus action flow on CP^{N-1},

$$\begin{aligned} T_t : CP^{N-1} &\mapsto CP^{N-1} \\ (X_1, X_2, \cdots, X_N) &\mapsto (e^{\lambda_1 t} X_1, e^{\lambda_2 t} X_2, \cdots, e^{\lambda_N t} X_N) \\ &\quad (t \in C) \end{aligned} \tag{3.5.80}$$

where $\lambda_i \in C$ is the character of the flow. Then (3.5.80) induce the flow on $\mathcal{M}^{CP^{N-1}}_{0,d,m}$ from the compatibility with the evaluation map.

$$\begin{aligned} &\phi_i(T_t((z_1, z_2, \cdots, z_m, f)/\sim)) \\ :=\ & T_t \circ \phi_i((z_1, z_2, \cdots, z_m, f)/\sim) \\ =\ & T_t \circ f(z_i) \end{aligned} \tag{3.5.81}$$

Next, we introduce the Bott residue formula. For simplicity, we use X for $\mathcal{M}^{CP^{N-1}}_{0,d,m}$. Let $\mathcal{E}_1, \cdots, \mathcal{E}_n$ be a holomorphic vector bundle on X, and X_f be the fixed point set of X under the flow (3.5.81). We can decompose X_f as the sum of the connected components X_γ.

$$X_f = \bigcup_\gamma X_\gamma \tag{3.5.82}$$

Then consider $\mathcal{E}_i|_{X_\gamma}$ and the normal bundle $\mathcal{N}_\gamma \simeq T'X|_{X_\gamma}/T'X_\gamma$ and decompose them into the eigen vector bundle under the torus action T_t, i.e.,

$$\begin{aligned} \mathcal{E}_i|_{X_\gamma} &\cong \bigoplus_{j=1}^{n_{\mathcal{E}_i}} \mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j} \\ \mathcal{N}_\gamma &\cong \bigoplus_{j=1}^{n_{\mathcal{N}}} \mathcal{N}^{\gamma, g_j(\lambda_*)}_j \end{aligned} \tag{3.5.83}$$

where

$$\begin{aligned} T_t(\mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j}) &= e^{f_{i,j}(\lambda_*)t} \mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j} \\ T_t(\mathcal{N}^{\gamma, g_j(\lambda_*)}_j) &= e^{g_j(\lambda_*)t} \mathcal{N}^{\gamma, g_j(\lambda_*)}_j \end{aligned} \tag{3.5.84}$$

and we set

$$\begin{aligned} rank(\mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j}) &= r_{\mathcal{E}}(i,j) \\ rank(\mathcal{N}^{\gamma, g_j(\lambda_*)}_j) &= r_{\mathcal{N}}(j) \end{aligned} \tag{3.5.85}$$

We can represent the total Chern class of $\mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j}$ and $\mathcal{N}^{\gamma, g_j(\lambda_*)}_j$ as the product of first Chern class of formal line bundles as follows.

$$\begin{aligned} c(\mathcal{E}^{\gamma, f_{i,j}(\lambda_*)}_{i,j}|_{X_\gamma}) &= \prod_{j=1}^{n_{\mathcal{E}_i}} \prod_{k=1}^{r_{\mathcal{E}}(i,j)} (1 + t \cdot e^{\gamma, f_{i,j}(\lambda_*)}_{i,j,k}) \\ c(\mathcal{N}_\gamma) &= \prod_{j=1}^{n_{\mathcal{N}}} \prod_{k=1}^{r_{\mathcal{N}}(j)} (1 + t \cdot n^{\gamma, g_j(\lambda_*)}_{j,k}) \end{aligned} \tag{3.5.86}$$

Top Chern classes are given as the coefficient form of t^k of highest degree.

With these preparations, we introduce the Bott residue formula.

$$\begin{aligned} &\int_X \prod_i c_T^{a_i}(\mathcal{E}_i) = \\ &\quad \sum_\gamma \int_{X_\gamma} \frac{\prod_i \prod_{j=1}^{n_{\mathcal{E}_i}} \prod_{k=1}^{r_{\mathcal{E}}(i,j)} (e^{\gamma, f_{i,j}(\lambda_*)}_{i,j,k} + f_{i,j}(\lambda_*))^{a_i}}{\prod_{j=1}^{n_{\mathcal{N}}} \prod_{k=1}^{r_{\mathcal{N}}(i)} (n^{\gamma, g_j(\lambda_*)}_{i,j} + g_j(\lambda_*))} \end{aligned} \tag{3.5.87}$$

3.5.2 Construction of Fixed Point Set

Fixed points of CP^{N-1} under T_t are given by considering the projective equivalence ,

$$p_i := (0, 0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0). \tag{3.5.88}$$

Then, we can find the fundamental maps $l^d_{i,j}$ from $CP^1 \mapsto CP^{N-1}$ which remain fixed under T_t as the degree d maps which connect p_i and p_j.

$$l^d_{i,j} : (s,t) \mapsto (0, \cdots, 0, \overbrace{s^d}^{i}, 0, \cdots, 0, \overbrace{t^d}^{j}, 0, \cdots, 0) \tag{3.5.89}$$

Of course $l^d_{i,j}$ is kept fixed under $SL(2, C)$ equivalence. But now that we have coupled gravity with the theory, we have to consider the boundary components of moduli space of CP^1, i.e., stable curves. Stable curve $\mathcal{C}$ with k-punctures is constructed with the set of CP^1's $\{C_\alpha\}$ with punctures assigned on them and additional punctures of double singularity which connect two components of C_α's. Then we can translate the condition into the condition that the genus of stable curve is zero into imposing its arithmetic genus to be zero. In geometrical language, if we represent C_α as a line and define a figure with lines which intersect at singular punctures, this is equivalent to the non-existence of closed loops in it. This addition makes us to introduce stable maps which map stable curves to CP^{N-1}.

With these considerations, we can label the connected components of the fixed point set $\mathcal{M}^{CP^{N-1}}_{0,d,m,f}$ with a tree graph Γ with the following structure. We denote them by $\mathcal{M}^{CP^{N-1}}_{0,d,m}(\Gamma)$. The rules of correspondences are,

1) The vertices $v \in Vert(\Gamma)$ correspond to the connected component C_v of $f^{-1}(p_1, \cdots, p_N)$. C_v can be a sum of connected irreducible components of $\mathcal{C}$ or be a point.

2) The edges $\alpha \in Edge(\Gamma)$ correspond to the irreducible component $C_\alpha$mapped to $l^d_{i,j}$.

Then we have to add the additional structures to Γ,

1) We label each $v \in Vert(\Gamma)$ by $f_v \in \{1, 2, \cdots, N\}$ which is defined by $p_{f_v} = f(C_v)$.

2) The m-punctures are distributed among the vertices $v \in Vert(\Gamma)$. We represent this distribution by $S_v \in \{1, 2, \cdots, m\}$.

3) We attach degree d_α to each $\alpha \in Edge(\Gamma)$ defined by the degree of $l^d_{i,j}$.

We have to set punctures on the vertices $Vert(\Gamma)$ because if we put punctures on C^α, they move with the flow T_t, which contradicts with the assumption of fixed point sets. Then we can construct $\mathcal{M}^{CP^{N-1}}_{0,d,m}(\Gamma)$ under conditions that emerge from the above three structures,

1) If $\alpha \in Edge(\Gamma)$ connects $v, u \in Vert(\Gamma)$, $f_u \neq f_v$.

2) $\{1, 2, \cdots, m\} = \coprod_{v \in Vert(\Gamma)} S_v$.

3) $\sum_{\alpha \in Edge(\Gamma)} d_\alpha = d$

Then we have

$$\mathcal{M}^{CP^{N-1}}_{0,d,m}(\Gamma) \cong \prod_{v \in Vert(\Gamma)} (\mathcal{M}_{0,S_v})/(Aut(\Gamma)) \tag{3.5.90}$$

where $\mathcal{M}_{0,S_v}$ is the moduli space of complex structure of CP^1 with S_v punctures. It represents the gravitational degree of freedom of C_v. According to Kontsevich, division by $Aut(\Gamma)$ reflects the orbispace structure of $\mathcal{M}^{CP^{N-1}}_{0,d,m}$. It may reflect the multiplicity of the degeneration of stable maps.

3.5.3 Determination of the contribution from Normal and Vector bundles

Contributions from $\mathcal{N}^{abs}_{\mathcal{M}(\Gamma)}$

With these preparations, we determine the contribution from $\mathcal{M}^{CP^{N-1}}_{0,d,m}$ (in the following discussion we abbreviate the notation as $\mathcal{M}(\Gamma)$) to (3.5.87).

First, we calculate the contribution from $\mathcal{N}_{\mathcal{M}(\Gamma)}$. Following Kontsevich,we will use the expression of vector bundles as the K-group [ ], which translates sum and quotient operations into addition and subtraction. Then we have

$$[\mathcal{N}_{\mathcal{M}(\Gamma)}] = [T'\mathcal{M}|_{\mathcal{M}(\Gamma)}] - [T'\mathcal{M}(\Gamma)] \tag{3.5.91}$$

If we set

$$\mathcal{C} = \tilde{\bigcup}_\alpha C_\alpha$$

(where

$$\tilde{\bigcup}_\alpha$$

means a sum with double-singularity gluing operation.), $[T'\mathcal{M}|_{\mathcal{M}(\Gamma)}]$ consists of the following degrees of freedom,

1) Moving $f(\mathcal{C})$ in CP^{N-1}.

2) Resolution of double singularities of $\mathcal{C}$, i.e., from $xy = 0$ to $xy = \epsilon$.

3) Moving puncture degrees of freedom.

And we have

$$\begin{aligned} [T'\mathcal{M}|_{\mathcal{M}(\Gamma)}] &= [H^0(\mathcal{C}, f^*(T'CP^{N-1}))] \\ &+ \sum_{\substack{z \in C_\alpha \cap C_\beta \\ \alpha \neq \beta}} [T'_z C_\alpha \otimes T'_z C_\beta] \\ &+ \sum_{\substack{z \in C_\alpha \cap C_\beta \\ \alpha \neq \beta}} [T'_z C_\alpha] + [T'_z C_\beta] \\ &+ \sum_{z_i, i \in \{1, \cdots, k\}} [T'_{z_i} \mathcal{C}] - \sum_\alpha [H^0(C^\alpha, T'C^\alpha)] \end{aligned} \tag{3.5.92}$$

The last term of (3.5.92) corresponds to devision by $SL(2, C)$ of each component C_α. From (3.5.90) $\mathcal{M}(\Gamma)$ has continuous degrees of freedom which come only from C_α mapped to a point, we have

$$\begin{aligned} [T'\mathcal{M}(\Gamma)] &= \sum_{\substack{z \in C_\alpha \cap C_\beta \\ \alpha \neq \beta \ \alpha, \beta \notin Edge(\Gamma)}} [T'_z C_\alpha \otimes T'_z C_\beta] \\ &+ \sum_{\substack{z \in C_\alpha \cap C_\beta \\ \alpha \neq \beta, \alpha \notin Edge(\Gamma)}} [T'_z C_\alpha] \\ &+ \sum_{z_i, i \in \{1, 2, \cdots, k\}} [T'_{z_i} \mathcal{C}] \\ &- \sum_{\alpha \notin Edge(\Gamma)} [H^0(C^\alpha, T'C^\alpha)] \end{aligned} \tag{3.5.93}$$

where we used the fact that all the punctures lie in the component mapped to a point .

From (3.5.92) and (3.5.93), we have

$$[\mathcal{N}_{\mathcal{M}(\Gamma)}] = [H^0(\mathcal{C}, f^*(T'CP^{N-1}))] + [\mathcal{N}^{abs}_{\mathcal{M}(\Gamma)}] \tag{3.5.94}$$

where

$$[\mathcal{N}^{abs}_{\mathcal{M}(\Gamma)}] := \sum_{\substack{z\in C_\alpha\cap C_\beta \\ \alpha\neq\beta\ \alpha,\beta\in Edge(\Gamma)}} [T'_z C_\alpha \otimes T'_z C_\beta] \tag{3.5.95}$$

$$+ \sum_{\substack{z\in C_\alpha\cap C_\beta \\ \alpha\in Edge(\Gamma),\beta\notin Edge(\Gamma)}} [T'_z C_\alpha \otimes T'_z C_\beta] \tag{3.5.96}$$

$$+ \sum_{\substack{z\in C_\alpha\cap C_\beta \\ \alpha\neq\beta\ \alpha\in Edge(\Gamma)}} [T'_z C_\alpha] - \sum_{\alpha\in Edge(\Gamma)} [H^0(C^\alpha, T'C^\alpha)] \tag{3.5.97}$$

Then we determine the contribution from the first term of (3.5.94), (3.5.96), (3.5.97) and (3.5.97).

First, consider the contribution from (3.5.96). Since $\alpha, \beta \in Edge(\Gamma)$, $T_z C_\alpha$ and $T_z C_\beta$'s are trivial as the line bundle on $\mathcal{M}(\Gamma)$. Let C_α and C_β be mapped to $l^{d_\alpha}_{i,j}$ and $l^{d_\beta}_{i,l}$.

$$C_\alpha : (z_1, z_2) \mapsto (0, \cdots, 0, \overbrace{z_1^{d_\alpha}}^{i}, 0, \cdots, 0, \overbrace{z_2^{d_\alpha}}^{j}, 0, \cdots, 0) \tag{3.5.98}$$

$$C_\beta : (w_1, w_2) \mapsto (0, \cdots, 0, \overbrace{w_1^{d_\beta}}^{i}, 0, \cdots, 0, \overbrace{w_2^{d_\beta}}^{l}, 0, \cdots, 0) \tag{3.5.99}$$

(3.5.100)

Local coordinate around $z \in C_\alpha \cap C_\beta$ on C_α and C_β are $\frac{z_2}{z_1}$ and $\frac{w_2}{w_1}$, and we have

$$T'_z C_\alpha \otimes T'_z C_\beta \cong \frac{d}{d(\frac{z_2}{z_1})} \otimes \frac{d}{d(\frac{w_2}{w_1})} \tag{3.5.101}$$

Definition of torus action (3.5.81) leads us to

$$\begin{aligned} z_1 &\mapsto z_1 e^{\frac{\lambda_i}{d_\alpha}t} \quad z_2 \mapsto z_2 e^{\frac{\lambda_j}{d_\alpha}t} \\ w_1 &\mapsto w_1 e^{\frac{\lambda_i}{d_\beta}t} \quad w_2 \mapsto w_2 e^{\frac{\lambda_l}{d_\beta}t} \end{aligned} \tag{3.5.102}$$

and

$$T'_z C_\alpha \otimes T'_z C_\beta \mapsto e^{(\frac{\lambda_i-\lambda_j}{d_\alpha}+\frac{\lambda_i-\lambda_l}{d_\beta})t} T'_z C_\alpha \otimes T'_z C_\beta \tag{3.5.103}$$

The result is,

$$(\text{Contribution from (3.5.96) to (3.5.87)}) = \prod_{\substack{C_\alpha\cap C_\beta\neq\emptyset \\ \alpha\neq\beta\ \alpha,\beta\in Edge(\Gamma)}} \frac{1}{\frac{\lambda_i-\lambda_j}{d_\alpha}+\frac{\lambda_i-\lambda_l}{d_\beta}} \tag{3.5.104}$$

Again following Kontsevich, we introduce the notation "Flag" $F = (v, \alpha)$ which represents edge α with a direction specified by the source vertex v. We define

$$w_F := \frac{\lambda_{f_v} - \lambda_{f_u}}{d_\alpha} \tag{3.5.105}$$

Then the r.h.s of (3.5.104) can be rewritten as follows.

$$\prod_{\substack{v\in Vert(\Gamma) \\ val(v)=2\quad {}^\sharp S_v=0}} \frac{1}{w_{F_1(v)} + w_{F_2(v)}} \tag{3.5.106}$$

where $val(v)$ represents the valency of v and $F_1(v)$ and $F_2(v)$ are the flags whose sources are v. Note that in this case $f^{-1}(v)$ is a point.

Next we consider the contributions from (3.5.97). Again from (3.5.93), $T_z C_\alpha$ is trivial as the line bundle on $\mathcal{M}(\Gamma)$ but has an eigenvalue w_F as in the derivation of (3.5.103). On the other hand, $T'_z C_\beta$ has trivial torus action (because C_β is mapped to a point) but non trivial line bundle on $\mathcal{M}(\Gamma)$. And if ${}^\sharp$(punctures onC_v) ≥ 3, $\mathcal{M}_{0,S_v}$ is well-defined and we have

$$(\text{Contribution from (3.5.97) to (3.5.87)}) = \prod_{v\in Vert(\Gamma)} (\int_{\mathcal{M}_{0,val(v)+{}^\sharp S_v}} \prod_{\substack{flags \\ F=(v,\alpha)}} \frac{1}{w_F + c_1(T'_{z_F}(C_v))})$$

$$(val(v) + {}^\sharp S_v \geq 3) \tag{3.5.107}$$

where z_F represents the gluing point of C_v and F. We can evaluate the r.h.s of (3.5.107) by expanding in terms of $\frac{1}{w_F}$ and using the fact that $c_1(T'_{z_F} C_v) = -c_1(T'^{*}_{z_F} C_v)$. Expansion coefficients are intersection numbers of Mumford-Morita class on the CP^1-moduli space, which is identified as the correlation function of gravitational descendants by Witten [30]. Continuing the calculation, we have

$$(\text{r.h.s of (3.5.107)}) = \prod_{v\in Vert(\Gamma)} (\sum_{\substack{d_1,\cdots,d_{val(v)}\geq 0 \\ \sum d_i = val(v)+{}^\sharp S_v - 3}} \prod_{\substack{flags \\ F=(v,\alpha)}} w_F^{-d_i-1} \langle \sigma_{d_1} \cdots \sigma_{d_{valv}} \overbrace{P\cdots P}^{{}^\sharp S_v \text{times}} \rangle) \tag{3.5.108}$$

$\langle \sigma_{d_1} \cdots \sigma_{d_{valv}} \overbrace{P\cdots P}^{{}^\sharp S_v \text{times}} \rangle$ is calculated in [29],

$$\langle \sigma_{d_1} \cdots \sigma_{d_{valv}} \overbrace{P\cdots P}^{{}^\sharp S_v \text{times}} \rangle = \frac{(val(v) + {}^\sharp S_v - 3)!}{d_1! \cdots d_{val(v)}!} \tag{3.5.109}$$

Combining (3.5.107),(3.5.108) and (3.5.109), we have

$$(\text{Contribution from (3.5.97) to (3.5.87)}) = \prod_{v\in Vert(\Gamma)} \prod_{\substack{flags \\ F=(v,\alpha)}} w_F^{-1} (\sum_{\substack{flags \\ F=(v,\alpha)}} w_F^{-1})^{val(v)+{}^\sharp S_v - 3}$$

$$(val(v) + {}^\sharp S_v \geq 3) \tag{3.5.110}$$

Then we consider (3.5.97). Contributions of the first terms are, as before

$$\prod_{\substack{C_\alpha\cap C_\beta \\ \alpha\neq\beta\ \alpha\in Edge(\Gamma)}} \frac{1}{w_{F_i(\alpha)}} \tag{3.5.111}$$

where $F_i(\alpha)$'s are two flags having α as their edges.

The second terms that represent the automorphism group degrees of freedom of edge components can be expressed by the tangent bundles on the inverse images of two vertices of the edges and scaling transformation degree of freedom fixing the punctures (We denote it as [0]). In terms of the K-group, we have

$$\begin{aligned} & -\sum_{\alpha\in Edge(\Gamma)} [H^0(C_\alpha, T'C_\alpha)] \\ = & -\sum_{\alpha\in Edge(\Gamma)} ([T'_{z_1(\alpha)}C_\alpha] + [0] + [T'_{z_2(\alpha)}C_\alpha]) \end{aligned} \tag{3.5.112}$$

And contributions to (3.5.87) are

$$\prod_{\alpha\in Edge(\Gamma)} w_{F_1(\alpha)} \cdot w_{F_2(\alpha)} \cdot C([0]) \tag{3.5.113}$$

where $C([0])$ represents the factor from [0]. Multiplying (3.5.111) and (3.5.113), what remains except for $C([0])$ is the products of w_F's whose edges have only one double singularity. In other words, the corresponding $F=(v,\alpha)$ has $val(v)=1$ and $f^{-1}(v)$ is a point. We have

$$(\text{Contributions from (3.5.97)}) = \prod_{\substack{v\in Vert(\Gamma)\\ val(v)=1 \quad \sharp S_v=0}} \prod_{\substack{flags\\ F=(v,\alpha)}} w_F \prod_{\alpha\in Edge(\Gamma)} C([0]) \tag{3.5.114}$$

After all, from (3.5.106),(3.5.110) and (3.5.114), we put all the factors from $[\mathcal{N}^{tabs}_{\mathcal{M}(\Gamma)}]$ into the form,

$$\prod_{v\in Vert(\Gamma)} \prod_{\substack{flags\\ F=(v,\alpha)}} w_F^{-1} \Big(\sum_{\substack{flags\\ F=(v,\alpha)}} w_F^{-1} \Big)^{val(v)+\sharp S_v-3} \prod_{\alpha\in Edge(\Gamma)} C([0]) \tag{3.5.115}$$

Determination of the contributions from $[H^0(\mathcal{C}, f^*(T'CP^{N-1}))]$

Since

$$f(\mathcal{C}) = \bar{\bigcup}_{\alpha\in Edge(\Gamma)} f(C_\alpha)$$

, we can construct $[H^0(\mathcal{C}, f^*(T'CP^{N-1}))]$ by gluing

$$\bigoplus_{\alpha\in Edge(\Gamma)} [H^0(C_\alpha, f^*(T'CP^{N-1}))]$$

at p_{f_v}. This process can be described using exact sequences,

$$\begin{aligned} & 0 \mapsto H^0(\mathcal{C}, f^*(T'CP^{N-1})) \mapsto \\ \bigoplus_{\alpha\in Edge(\Gamma)} H^0(C_\alpha, f^*(T'CP^{N-1})) \mapsto & \bigoplus_{v\in Vert(\Gamma)} C^{val(v)-1} \otimes T'_{p_{f_v}} CP^{N-1} \mapsto 0 \end{aligned} \tag{3.5.116}$$

This contribution is then given as the contribution from the second term divided by the one from from the third term. As the independent basis of $H^0(C_\alpha, f^*(T'CP^{N-1}))$ describing the deformation of $f(C_\alpha)$ in CP^{N-1} where C_α is

$$C_\alpha : \quad (z_1, z_2) \mapsto (0, \cdots, 0, \overbrace{z_1^{d_\alpha}}^{i}, 0, \cdots, 0, \overbrace{z_2^{d_\alpha}}^{j}, 0, \cdots, 0) \tag{3.5.117}$$

, we have

$$(0, \cdots, 0, \overbrace{z_1^{d_\alpha} + \epsilon z_1^m z_2^{d_\alpha - m}}^{i}, 0, \cdots, 0, \overbrace{z_2^{d_\alpha}}^{j}, 0, \cdots, 0) \tag{3.5.118}$$

$$(0, \cdots, 0, \overbrace{z_1^{d_\alpha}}^{i}, 0, \cdots, 0, \overbrace{z_2^{d_\alpha} + \epsilon z_1^m z_2^{d_\alpha - m}}^{j}, 0, \cdots, 0) \tag{3.5.119}$$

$$(0, \cdots, 0, \overbrace{\epsilon z_1^m z_2^{d_\alpha - m}}^{k}, 0, \cdots, 0, \overbrace{z_1^{d_\alpha}}^{i}, 0, \cdots, 0, \overbrace{z_2^{d_\alpha}}^{j}, 0, \cdots, 0). \tag{3.5.120}$$

We can write these basis in more sophisticated form,

$$a) \qquad (\tfrac{z_1}{z_2})^m X_i \tfrac{\partial}{\partial X_i} \qquad (-d_\alpha \leq m \leq d_\alpha) \tag{3.5.121}$$

$$b) \qquad (\tfrac{z_1}{z_2})^m X_j \tfrac{\partial}{\partial X_k} \qquad (0 \leq m \leq d_\alpha) \quad k \neq i, j \tag{3.5.122}$$

This expression directly leads us to

$$(\text{Contribution from (3.5.122)}) = \frac{1}{m \cdot w_F} \qquad (m \neq 0) \tag{3.5.123}$$

$$\frac{1}{C([0])} \qquad (m = 0) \tag{3.5.124}$$

$$(\text{Contribution from (3.5.122)}) = \frac{1}{m \cdot w_F + \lambda_j - \lambda_k} \tag{3.5.125}$$

And from $T'_{p_{f_v}} CP^{N-1} \simeq \oplus_{j\neq p_{f_v}} \frac{\partial}{\partial(\frac{X_j}{X_{f_v}})}$,

$$(\text{Contributions from } C^{val(v)-1} \otimes T'_{p_{f_v}} CP^{N-1}) = \prod_{v\in Vert(\Gamma)} (\lambda_{f_v} - \lambda_j)^{val(v)-1} \tag{3.5.126}$$

Combining (3.5.124), (3.5.125), (3.5.125) and (3.5.126), we have

$$\begin{aligned} & \prod_{flags \quad F=(v,\alpha)} (w_F)^{-d_\alpha} \prod_{\substack{\alpha\in Edge(\Gamma)\\ (u,v):vertices\ of\ \alpha}} \prod_{k\neq f_u, f_v} \prod_{m=0}^{d_\alpha} \Big(\frac{m\lambda_{f_u} + (d_\alpha - m)\lambda_{f_v}}{d_\alpha} - \lambda_k\Big)^{-1} \\ & \prod_{\alpha\in Edge(\Gamma)} (d_\alpha!)^{-2} (C([0]))^{-1} \prod_{v\in Vert(\Gamma)} \Big(\prod_{j: j\neq f_v} (\lambda_{f_v} - \lambda_j) \Big)^{val(v)-1} \end{aligned} \tag{3.5.127}$$

Factors from Vector Bundles $\mathcal{E}_i$

First, we calculate the factors from $\bar{\pi}_m^* \mathcal{E}_{kd+1}$. Since $\bar{\pi}_m$ is merely the operation to forget the operator insertion points, we can consider it as vector bundle $\mathcal{E}_{kd+1}$ on $\mathcal{M}_{0,d,0}^{CP^{N-1}}$. As we have mentioned in section 2, this fiber locally corresponds to $[H^0(\mathcal{C}, f^*(\mathcal{O}(k \cdot H)))]$. We can construct it as in (3.5.115), by the exact sequence,

$$0 \mapsto H^0(\mathcal{C}, f^*(\mathcal{O}(kH))) \mapsto \bigoplus_{\alpha \in Edge(\Gamma)} H^0(C_\alpha, f^*(\mathcal{O}(kH))) \mapsto \bigoplus_{v \in Vert(\Gamma)} C^{val(v)-1} \otimes \mathcal{O}_{p_{f_v}}(kH) \mapsto 0 \quad (3.5.128)$$

Then since the basis of $H^0(C_\alpha, f^*(\mathcal{O}(k \cdot H)))$ are given as

$$\{z_1^{kd_\alpha}, z_1^{kd_\alpha - 1} z_2, \cdots, z_2^{kd_\alpha}\} \quad (3.5.129)$$

and the section of $\mathcal{O}_{p_{f_v}}(kH)$ is $X_{f_v}^k$, we have

$$(\text{Contributions from } c_T(\mathcal{E}_{kd+1})) = \prod_{\substack{\alpha \in Edge(\Gamma) \\ (v_1, v_2): \text{vertices of } \alpha}} \prod_{a=0}^{kd_\alpha} \left(\frac{a\lambda_{f_{v_1}} + (kd_\alpha - a)\lambda_{f_{v_2}}}{d_\alpha}\right) \prod_{v \in Vert(\Gamma)} (k\lambda_{f_v})^{1-val(v)} \quad (3.5.130)$$

Next, we determine the factor from $\phi_i^*(c_1^{j_i}(H))$. From the argument of §3.2, puncture i lies on the vertex $v(i)$ of Γ, and $\phi_i^*(c_1^{j_i}(H))$ reduces to $\mathcal{O}_{p_{f_{v(i)}}}(j_i H)$. This leads us to

$$(\text{Contributions from } \phi_i^*(c_1^{j_i}(H))) = \lambda_{f_{v(i)}}^{j_i} \quad (3.5.131)$$

Local Appendix

We have to divide the above factors by $\sharp Aut(\Gamma)$ coming from (3.5.90) and in practice, we have to multiply a factor $\frac{1}{d_\alpha}$ for each edge α. We cannot justify the reason for this factor at this stage.

3.5.4 Some Explicit Calculation of Amplitudes

For some examples, we calculate $\langle \mathcal{O}_{e^{N-4}} \rangle_1$, $\langle \mathcal{O}_{e^{N-4}} \rangle_2$ and $\langle \mathcal{O}_{e^{N-4}} \rangle_3$ for $k = N$ case. First, we write out tree graphs that contribute to the amplitudes up to degree 3. (See **Fig3.1**.) In **Fig3.1**, we omit the external insertion of "punctures". So in calculation, we have to add all the cases of external operator insertions of $\langle \mathcal{O}_{e^{N-4}} \rangle$ to vertices. Note that the two character numbers (for example "i") of neighboring vertices never coincide with each other. Then direct application of the argument of the previous subsections leads us to the following formula.

$$\langle \mathcal{O}_{e^{N-4}} \rangle_1 = \frac{1}{2} \sum_{i \neq j} \left(\prod_{k \neq i,j} (\lambda_i - \lambda_k)^{-1} (\lambda_j - \lambda_k)^{-1} \prod_{a=0}^{N} (a\lambda_i + (N-a)\lambda_j) \left(\frac{\lambda_i^{N-4} - \lambda_j^{N-4}}{w_{F_1}}\right)\right)$$

(from (a))

$$\begin{aligned}
\langle \mathcal{O}_{e^{N-4}} \rangle_2 = {} & \frac{1}{2} \sum_{\substack{i \neq j \\ j \neq k}} \left(\frac{1}{N\lambda_j} \left(\frac{\lambda_i^{N-4} w_{F_4} + \lambda_k^{N-4} w_{F_1}}{w_{F_2} + w_{F_3}} + \lambda_j^{N-4} \right) \right. \\
& \frac{1}{w_{F_1} w_{F_2} w_{F_3} w_{F_4}} \prod_{n \neq j} (\lambda_j - \lambda_n) \\
& \prod_{m_1 \neq i,j} (\lambda_i - \lambda_{m_1})^{-1} (\lambda_j - \lambda_{m_1})^{-1} \prod_{m_2 \neq j,k} (\lambda_j - \lambda_{m_2})^{-1} (\lambda_k - \lambda_{m_2})^{-1} \\
& \prod_{a_1=0}^{N} (a_1\lambda_i + (N-a_1)\lambda_j) \prod_{a_2=0}^{N} (a_2\lambda_j + (N-a_2)\lambda_k)
\end{aligned}$$

(from (c))

$$\begin{aligned}
+ {} & \frac{1}{4} \sum_{i \neq j} ((\lambda_i^{N-4} - \lambda_j^{N-4}) w_{F_2} \frac{1}{w_{F_1}^2 w_{F_2}^2} \\
& \prod_{k \neq i,j} (\lambda_i - \lambda_k)^{-1} (\lambda_j - \lambda_k)^{-1} \left(\frac{\lambda_i + \lambda_j}{2} - \lambda_k\right)^{-1} \\
& \prod_{a=0}^{2N} \left(\frac{a\lambda_j + (2N-a)\lambda_k}{2}\right))
\end{aligned}$$

(from (b))

$$\begin{aligned}
\langle \mathcal{O}_{e^{N-4}} \rangle_3 = {} & \frac{1}{2} \sum_{\substack{i \neq j \\ j \neq k, k \neq l}} ((\lambda_i^{N-4} \frac{1}{w_{F_2} + w_{F_3}} \frac{1}{w_{F_4} + w_{F_5}} w_{F_6} \\
+ {} & w_{F_1} \frac{\lambda_j^{N-4}}{w_{F_2} w_{F_3}} \frac{1}{w_{F_4} + w_{F_5}} w_{F_6} \\
+ {} & w_{F_1} \frac{1}{w_{F_2} + w_{F_3}} \frac{\lambda_k^{N-4}}{w_{F_4} w_{F_5}} w_{F_6} \\
+ {} & w_{F_1} \frac{1}{w_{F_2} + w_{F_3}} \frac{1}{w_{F_4} + w_{F_5}} \lambda_l^{N-4}) \\
& \frac{1}{w_{F_1} w_{F_2} w_{F_3} w_{F_4} w_{F_5} w_{F_6}} \frac{1}{(N\lambda_j N\lambda_k)} \\
& \prod_{n_1 \neq j} (\lambda_j - \lambda_{n_1}) \prod_{n_2 \neq k} (\lambda_k - \lambda_{n_2}) \\
& \prod_{m_1 \neq i,j} (\lambda_i - \lambda_{m_1})^{-1} (\lambda_j - \lambda_{m_1})^{-1} \\
& \prod_{m_2 \neq j,k} (\lambda_j - \lambda_{m_2})^{-1} (\lambda_k - \lambda_{m_2})^{-1}
\end{aligned}$$

$$
\begin{aligned}
& \prod_{m_3\neq k,l}(\lambda_k-\lambda_{m_3})^{-1}(\lambda_l-\lambda_{m_3})^{-1} \\
& \prod_{a_1=0}^{N}(a_1\lambda_i+(N-a_1)\lambda_j)\prod_{a_2=0}^{N}(a_2\lambda_j+(N-a_2)\lambda_k) \\
& \prod_{a_3=0}^{N}(a_3\lambda_k+(N-a_3)\lambda_l)) \\
& \text{(from (f))} \\
+ & \frac{1}{2}\sum_{\substack{i\neq j\\ j\neq k}}(\frac{1}{4}(\lambda_i^{N-4}\frac{1}{w_{F_2}+w_{F_3}}w_{F_4}+w_{F_1}\frac{\lambda_j^{N-4}}{w_{F_2}w_{F_3}}w_{F_4}+w_{F_1}\frac{1}{w_{F_2}+w_{F_3}}\lambda_k^{N-4}) \\
& \frac{1}{N\lambda_j}\frac{1}{w_{F_1}^2w_{F_2}^2}\frac{1}{w_{F_3}w_{F_4}}\prod_{n\neq j}(\lambda_j-\lambda_n) \\
& \prod_{m_1\neq i,j}(\lambda_i-\lambda_{m_1})^{-1}(\lambda_j-\lambda_{m_1})^{-1}(\frac{\lambda_i+\lambda_j}{2}-\lambda_{m_1})^{-1} \\
& \prod_{m_2\neq j,k}(\lambda_j-\lambda_{m_2})^{-1}(\lambda_k-\lambda_{m_2})^{-1} \\
& \prod_{a_1=0}^{2N}(\frac{a_1\lambda_j+(2N-a_1)\lambda_k}{2})\prod_{a_2=0}^{N}(a_2\lambda_j+(N-a_2)\lambda_k)) \\
& \text{(from (e))} \\
+ & \frac{1}{6}\sum_{i\neq j}(\frac{1}{36}(w_{F_2}(\lambda_i^{N-4}-\lambda_j^{N-4})\frac{1}{w_{F_1}^3w_{F_2}^3} \\
& \prod_{m\neq i,j}(\lambda_i-\lambda_m)^{-1}(\frac{2\lambda_i+\lambda_j}{3}-\lambda_m)^{-1}(\frac{\lambda_i+2\lambda_j}{3}-\lambda_m)^{-1}(\lambda_j-\lambda_m)^{-1} \\
& \prod_{a=0}^{3N}(\frac{a\lambda_i+(3N-a)\lambda_j}{3})) \\
& \text{(from (d))} \\
+ & \frac{1}{6}\sum_{\substack{i\neq j\\ i\neq k,i\neq l}}(\lambda_j^{N-4}\frac{1}{w_{F_1}w_{F_3}w_{F_5}}w_{F_4}w_{F_6}+\lambda_l^{N-4}\frac{1}{w_{F_1}w_{F_3}w_{F_5}}w_{F_2}w_{F_6} \\
+ & \lambda_k^{N-4}\frac{1}{w_{F_1}w_{F_3}w_{F_5}}w_{F_2}w_{F_4} \\
+ & w_{F_2}w_{F_4}w_{F_6}\lambda_i^{N-4}\frac{1}{w_{F_1}w_{F_3}w_{F_5}}(\frac{1}{w_{F_1}}+\frac{1}{w_{F_3}}+\frac{1}{w_{F_5}})) \\
& \frac{1}{(N\lambda_i)^2}\frac{1}{w_{F_1}w_{F_2}w_{F_3}w_{F_4}w_{F_5}w_{F_6}} \\
& \prod_{m_1\neq i,j}(\lambda_i-\lambda_{m_1})^{-1}(\lambda_j-\lambda_{m_1})^{-1} \\
& \prod_{m_2\neq i,l}(\lambda_i-\lambda_{m_2})^{-1}(\lambda_l-\lambda_{m_2})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \prod_{m_3\neq i,k}(\lambda_i-\lambda_{m_3})^{-1}(\lambda_k-\lambda_{m_3})^{-1} \\
& \prod_{a_1=0}^{N}(a_1\lambda_i+(N-a_1)\lambda_j) \\
& \prod_{a_2=0}^{N}(a_2\lambda_i+(N-a_2)\lambda_k) \\
& \prod_{a_3=0}^{N}(a_3\lambda_i+(N-a_3)\lambda_l) \\
& \prod_{n\neq i}(\lambda_i-\lambda_n)^2 \\
& \text{(from (g))}
\end{aligned}
\qquad (3.5.132)
$$

These results are generically independent of the values λ_i, so we set λ_i equals to 3^i. Similarly we calculate the amplitudes $\langle\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle_1, \langle\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle_2$, and $\langle\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle_3$ $(\alpha+\beta=N-3)$. The results are collected in **Table 3.1,~ Table 3.4**.

Note that $\langle\mathcal{O}_{e^{N-4}}\rangle_n\cdot n=\langle\mathcal{O}_e\mathcal{O}_{e^{N-4}}\rangle_n$. This implies that the Kähler equation of Gromov- Witten invariants holds for the amplitudes defined by (2.4.89). Assuming this relation for all amplitudes, the results of **Table 3.1~ Table 3.4** coincide with the ones calculated from mirror symmetry [7]. † We calculate amplitudes $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle$ of matter theory on M_N^N for later use. Fusion rules hold in the matter theory, so we can reduce the amplitudes into the products of three-point functions.

Consider the "matter" expansion

$$
\langle\mathcal{O}_e\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle=N+\sum_{k=1}^{\infty}\langle\mathcal{O}_e\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle_k e^{-kt} \qquad (3.5.133)
$$

where t is the deformation parameter coupled to the Kähler form. By using fusion rules, and flat metric $\eta_{\alpha\beta}=N\cdot\delta_{\alpha+\beta,N-2}$,

$$
\begin{aligned}
\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle &= N^{5-N}\prod_{i=1}^{N-4}\langle\mathcal{O}_e\mathcal{O}_{e^i}\mathcal{O}_{e^{N-3-i}}\rangle \\
&= N+\sum_{k=1}^{\infty}\langle\overbrace{\mathcal{O}_e\cdots\mathcal{O}_e}^{N-2\text{times}}\rangle_k e^{-kt}. \qquad (3.5.134)
\end{aligned}
$$

Then for example, $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_1$ can be calculated as

$$
\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle_1 = -\frac{1}{2}\sum_{i\neq j}\prod_{k\neq i,j}(\lambda_i-\lambda_k)^{-1}(\lambda_j-\lambda_k)^{-1}\prod_{a=0}^{N}(a\lambda_i+(N-a)\lambda_j)
$$

†Note that for three-point function, amplitudes of the matter theory and the ones of theory coupled with gravity coincide.

$$((N-4)(\lambda_j^{N-2}-\lambda_i^{N-2})-(N-2)(\lambda_j^{N-3}\lambda_i-\lambda_i^{N-3}\lambda_j))$$
$$\frac{1}{(\lambda_i-\lambda_j)^3} \quad (3.5.135)$$

$$= N^{N+1}-(N-2)\cdot N\cdot N!(\frac{N-1}{1}+\cdots+\frac{1}{N-1})$$
$$-2N\cdot N! \quad (3.5.136)$$

If we set $\lambda_i = i$, we can derive (3.5.135) from (3.5.136) by a rather clumsy but elementary calculation. This agrees with (3.2.47). We write out numerical results of $\langle\prod_{j=1}^{N-2}\mathcal{O}_e(z_j)\rangle$ for $5 \leq N \leq 10$ case.

$$\begin{aligned}
\langle\prod_{j=1}^{3}\mathcal{O}_e(z_j)\rangle &= 5+2875e^{-t}+4876875e^{-2t}+8564575000e^{-3t}+\cdots \\
\langle\prod_{j=1}^{4}\mathcal{O}_e(z_j)\rangle &= 6+120960e^{-t}+4136832000e^{-2t} \\
&\quad +148146924602880e^{-3t}+\cdots \\
\langle\prod_{j=1}^{5}\mathcal{O}_e(z_j)\rangle &= 7+3727381e^{-t}+2637885990187e^{-2t} \\
&\quad +1927092954108108787e^{-3t}+\cdots \\
\langle\prod_{j=1}^{6}\mathcal{O}_e(z_j)\rangle &= 8+106975232e^{-t}+1672023727001660e^{-2t} \\
&\quad +2661169233308169509273 6e^{-3t}+\cdots \\
\langle\prod_{j=1}^{7}\mathcal{O}_e(z_j)\rangle &= 9+3103936929e^{-t}+1165013014173543657e^{-2t} \\
&\quad +441297815019235844688286425e^{-3t}+\cdots \\
\langle\prod_{j=1}^{8}\mathcal{O}_e(z_j)\rangle &= 10+94327552000e^{-t}+930496455109619200000e^{-2t} \\
&\quad +9217712440694086335170560000000e^{-3t}+\cdots \quad (3.5.137)
\end{aligned}$$

3.5.5 Construction of Generating Function

In the previous subsection, we see that we can calculate the amplitudes $\langle * \rangle_{d,grav.alt.}$ for topological sigma model on M_N^N coupled to gravity by torus action method. As we have seen in section 3 and section 4 , this method has a structure of summing over tree graphs, so we can construct a representation of Path-Integral form of the generating function of all amplitudes. In this subsection we treat general M_N^k for later use. Changes occur only in contributions from vector bundles. First, let us write out

explicitly the contribution from $\mathcal{M}(\Gamma)$ to the amplitude $\langle\mathcal{O}_{e^{j_1}}\cdots\mathcal{O}_{e^{j_k}}\rangle_{d,alt.grav.}$

$$\begin{aligned}
&(\text{Contribution from } \mathcal{M}(\Gamma) \text{ to } \langle\mathcal{O}_{e^{j_1}}\cdots\mathcal{O}_{e^{j_k}}\rangle_{n,alt.grav.}) \\
&= \frac{1}{\sharp Aut(\Gamma)}(\prod_{v\in Vert(\Gamma)}\lambda_{f_v(i)}^{j_i} \\
&\prod_{v\in Vert(\Gamma)}\prod_{\substack{flags\\F=(v,\alpha)}} w_F^{-1}(\sum_{\substack{flags\\F=(v,\alpha)}} w_F^{-1})^{val(v)+\sharp S_v-3} \\
&\prod_{v\in Vert(\Gamma)}(\prod_{j\neq f_v}(\lambda_{f_v}-\lambda_j))^{val(v)-1} \\
&\prod_{\alpha\in Edge(\Gamma)}\frac{1}{d_\alpha} \\
&\prod_{\substack{flags\\F=(v,\alpha)}}(w_F)^{-d_\alpha}\prod_{\substack{\alpha\in Edge(\Gamma)\\(u,v):vertices\ of\ \alpha}}\prod_{k\neq f_u,f_v}\prod_{m=0}^{d_\alpha}(\frac{m\lambda_{f_u}+(d_\alpha-m)\lambda_{f_v}}{d_\alpha}-\lambda_k)^{-1} \\
&\prod_{\alpha\in Edge(\Gamma)}(d_\alpha!)^{-2}\prod_{v\in Vert(\Gamma)}(\prod_{j:j\neq f_v}(\lambda_{f_v}-\lambda_j))^{val(v)-1} \\
&\prod_{\substack{\alpha\in Edge(\Gamma)\\(v_1,v_2):vertices\ of\ \alpha}}\prod_{a=0}^{kd_\alpha}(\frac{a\lambda_{f_{v_1}}+(kd_\alpha-a)\lambda_{f_{v_2}}}{d_\alpha})\prod_{v\in Vert(\Gamma)}(k\lambda_{f_v})^{1-val(v)}) \quad (3.5.138)
\end{aligned}$$

Then we classify the factors into two groups. One consists of the factors from edges, and the other from vertices. The factors from the edges are

$$\begin{aligned}
&(i)\ \prod_{\substack{flags\\F=(v,\alpha)}} w_F^{-1} \\
&(ii)\ \prod_{v\in Vert(\Gamma)}(\prod_{j\neq f_v}(\lambda_{f_v}-\lambda_j))^{val(v)} \\
&(iii)\ \prod_{\substack{flags\\F=(v,\alpha)}}(w_F)^{-d_\alpha}\prod_{\substack{\alpha\in Edge(\Gamma)\\(u,v):vertices\ of\ \alpha}}\prod_{k\neq f_u,f_v}\prod_{m=0}^{d_\alpha}(\frac{m\lambda_{f_u}+(d_\alpha-m)\lambda_{f_v}}{d_\alpha}-\lambda_k)^{-1}\prod_{\alpha\in Edge(\Gamma)}(d_\alpha!)^{-2} \\
&(iv)\ \prod_{\substack{\alpha\in Edge(\Gamma)\\(v_1,v_2):vertices\ of\ \alpha}}\prod_{a=0}^{kd_\alpha}(\frac{a\lambda_{f_{v_1}}+(kd_\alpha-a)\lambda_{f_{v_2}}}{d_\alpha})\prod_{v\in Vert(\Gamma)}(k\lambda_{f_v})^{-val(v)} \\
&(v)\ \prod_{Edge(\Gamma)}\frac{1}{d_\alpha} \quad (3.5.139)
\end{aligned}$$

And the factors we can push into the contribution from vertices are,

$$\begin{aligned}
&(i)\ \prod_{v\in Vert(\Gamma)}\lambda_{f_v(i)}^{j_i} \\
&(ii)\ \prod_{v\in Vert(\Gamma)}(\sum_{\substack{flags\\F=(v,\alpha)}} w_F^{-1})^{val(v)+\sharp S_v-3}
\end{aligned}$$

$$
\text{(iii)} \prod_{v\in Vert(\Gamma)} \Big(\prod_{j\neq f_v} (\lambda_{f_v} - \lambda_j)\Big)^{-1}
$$

$$
\text{(iv)} \prod_{v\in Vert(\Gamma)} (k\lambda_{f_v}) \tag{3.5.140}
$$

Then we introduce the field variables $\phi_{ij,d}$,propagator $g_{ij,d;i'j',d'}$, vertex $C_{i_1j_1,d_1;\cdots i_mj_m,d_m}\phi_{i_1j_1,d_1}\cdots\phi_{i_mj_m,d_m}$ and external field source parameters $t_1,\cdots,t_{N-2}$.

In this formulation, field variables correspond to the edges with characters i and j and degree d, $g_{ij,d;i'j',d'}$ remains nonzero only if $i=j', j=i', d=d'$, and the nonzero value of propagator is given as the reciprocal of the product of (3.5.139) (i)~(v). Then we have

$$
g_{ij,d} := g_{ij,d;ji,d} = \frac{-d^{N-2-(N-k)d}(\lambda_i-\lambda_j)^2\prod_{l=1}^{N}\prod_{a=1}^{d-1}(a\lambda_i+(d-a)\lambda_j-d\lambda_l)}{\prod_{a=1}^{kd-1}(a\lambda_i+(kd-a)\lambda_j)} \tag{3.5.141}
$$

Vertex $C_{i_1j_1,d_1;\cdots i_mj_m,d_m}\phi_{i_1j_1,d_1}\cdots\phi_{i_mj_m,d_m}$ are constructed with pairing the factor λ_i^m to t_m as follows.

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{i=1}^{N}\frac{k\lambda_i}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\sum_{\substack{d_1,\cdots d_m,d_*\geq 1\\ j_1,\cdots j_m,j_*\neq i}}\ \sum_{\tilde{j}_1,\cdots \tilde{j}_l\in\{1,2,\cdots,N-2\}}\\
&\quad(v_{ij_1,d_1}+\cdots+v_{ij_m,d_m})^{m+l-3}\phi_{ij_1,d_1}\cdots\phi_{ij_m,d_m}t_{\tilde{j}_1}\cdots t_{\tilde{j}_l}(\lambda_i^{\tilde{j}_1+\cdots+\tilde{j}_l})\\
&\quad=\sum_{i=1}^{N}\frac{k\lambda_i}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{d_1,\cdots d_m,d_*\geq 1\\ j_1,\cdots j_m,j_*\neq i}}(v_{ij_1,d_1}+\cdots+v_{ij_m,d_m})^{m-3}\phi_{ij_1,d_1}\cdots\phi_{ij_m,d_m}\\
&\quad\exp((t_1\lambda_i+\cdots+t_{N-2}\lambda_i^{N-2})(v_{ij_1,d_1}+\cdots+v_{ij_m,d_m}))\\
&\quad v_{ij,d}:=\frac{d}{\lambda_i-\lambda_j}
\end{aligned} \tag{3.5.142}
$$

where $1/m!$ is the factor that produces $1/^{\sharp}Aut(\Gamma)$ and $1/l!$ is the combinatorial factor in the insertions of the external operators. With these preparation, we have the path-integral representation of generating function.

$$
\begin{aligned}
&F_{M_N^k}(t_1,\cdots,t_{N-2})\\
&\quad:=\sum_{n_1,\cdots,n_{N-2}\geq 0}\langle\mathcal{O}_{e^1}^{n_1}\cdots\mathcal{O}_{e^{N-2}}^{n_{N-2}}\rangle\frac{t_1^{n_1}\cdots t_{N-2}^{n_{N-2}}}{n_1!\cdots n_{N-2}!}\\
&\quad=Res_zRes_h(\frac{1}{z}\log(\det(g^{-1})\frac{1}{h}\int d\phi_{ij,d}\\
&\quad\exp(-\frac{1}{2}\sum_{i,j,d}\frac{-d^{N-2-(N-k)d}(z\lambda_i-z\lambda_j)^2\prod_{l=1}^{N}\prod_{a=1}^{d-1}(az\lambda_i+(d-a)z\lambda_j-dz\lambda_l)}{\prod_{a=1}^{kd-1}(az\lambda_i+(kd-a)z\lambda_j)}\phi_{ij,d}\phi_{ji,d}\\
&\quad+\sum_{i=1}^{N}\frac{kz\lambda_i}{\prod_{j\neq i}(z\lambda_i-z\lambda_j)}\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{d_1,\cdots d_m,d_*\geq 1\\ j_1,\cdots j_m,j_*\neq i}}(\frac{v_{ij_1,d_1}}{z}+\cdots+\frac{v_{ij_m,d_m}}{z})^{m-3}\phi_{ij_1,d_1}\cdots\phi_{ij_m,d_m}
\end{aligned}
$$

$$
\exp((t_1\lambda_iz+\cdots+t_{N-2}\lambda_i^{N-2}z^{N-2})(\frac{v_{ij_1,d_1}}{z}+\cdots+\frac{v_{ij_m,d_m}}{z}))))) \tag{3.5.143}
$$

where we introduce h and dummy variable z to pick up the portion that comes from tree graphs and satisfies the topological selection rule (2.4.80). We must make one final remark. As we can easily see from the formulation of this calculation, (3.5.143) represents only quantum part. So we have to add classical part $\frac{1}{6}\int_{M_N^k}(\sum_{i=0}^{N-2}t_ie^i)^3$ by hand to obtain full generating function.

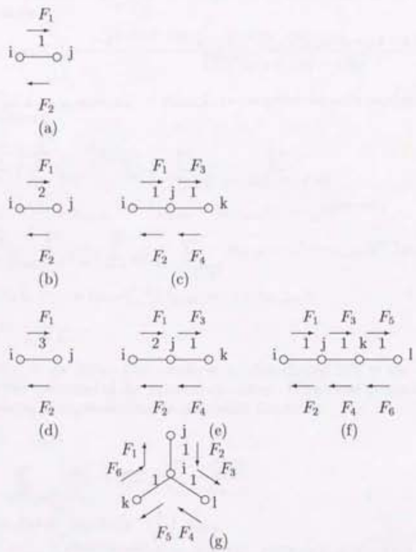

Figure 3.1: Tree Graphs up to Degree 3

Table 3.1: $\langle \mathcal{O}_{e^{N-4}} \rangle_{alt,grav}$

	$\langle \mathcal{O}_{e^{N-4}} \rangle_1$	$\langle \mathcal{O}_{e^{N-4}} \rangle_2$	$\langle \mathcal{O}_{e^{N-4}} \rangle_3$
N=5	2875	$\frac{4876875}{4}$	$\frac{8564575000}{9}$
N=6	60480	440899200	6255156284160
N=7	1009792	122240038536	$\frac{274758045710330728}{9}$
N=8	15984640	33397163702784	$\frac{1386812286427888746496}{9}$
N=9	253490796	9757818404032059	8975606542275623675356800
N=10	4120776000	3151991359959750000	62988860116574026518400000

Table 3.2: $\langle \mathcal{O}_{e^{\alpha}} \mathcal{O}_{e^{\beta}} \rangle_{1,alt,grav}$

N=5	$\langle \mathcal{O}_{e} \mathcal{O}_{e} \rangle_1 = 2875$
N=6	$\langle \mathcal{O}_{e} \mathcal{O}_{e^2} \rangle_1 = 60480$
N=7	$\langle \mathcal{O}_{e} \mathcal{O}_{e^3} \rangle_1 = 1009792$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^2} \rangle_1 = 1707797$
N=8	$\langle \mathcal{O}_{e} \mathcal{O}_{e^4} \rangle_1 = 15984640$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^3} \rangle_1 = 37502976$
N=9	$\langle \mathcal{O}_{e} \mathcal{O}_{e^5} \rangle_1 = 253490796$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^4} \rangle_1 = 763954092$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e\ 3} \rangle_1 = 1069047153$
N=10	$\langle \mathcal{O}_{e} \mathcal{O}_{e^6} \rangle_1 = 4120776000$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^5} \rangle_1 = 15274952000$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e^4} \rangle_1 = 27768048000$

Table 3.3: $\langle \mathcal{O}_{e^\alpha} \mathcal{O}_{e^\beta} \rangle_{2,alt,grav}$

N=5	$\langle \mathcal{O}_e \mathcal{O}_e \rangle_2 = \frac{4876875}{2}$
N=6	$\langle \mathcal{O}_e \mathcal{O}_{e^2} \rangle_2 = 881798400$
N=7	$\langle \mathcal{O}_e \mathcal{O}_{e^3} \rangle_2 = 244480077072$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^2} \rangle_2 = \frac{1021577199083}{2}$
N=8	$\langle \mathcal{O}_e \mathcal{O}_{e^4} \rangle_2 = 66794327405568$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^3} \rangle_2 = 224340722909184$
N=9	$\langle \mathcal{O}_e \mathcal{O}_{e^5} \rangle_2 = 19515636808064118$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^4} \rangle_2 = 93777295510651590$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e^3} \rangle_2 = \frac{312074853388012521}{2}$
N=10	$\langle \mathcal{O}_e \mathcal{O}_{e^6} \rangle_2 = 6303982719919500000$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^5} \rangle_2 = 40342298393756700000$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e^4} \rangle_2 = 100290980414189400000$

Table 3.4: $\langle \mathcal{O}_{e^\alpha} \mathcal{O}_{e^\beta} \rangle_{3,alt,grav}$

N=5	$\langle \mathcal{O}_e \mathcal{O}_e \rangle_3 = \frac{8564575000}{3}$
N=6	$\langle \mathcal{O}_e \mathcal{O}_{e^2} \rangle_3 = 18765468852480$
N=7	$\langle \mathcal{O}_e \mathcal{O}_{e^3} \rangle_3 = \frac{274758045710330728}{3}$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^2} \rangle_3 = \frac{667645611328487267}{3}$
N=8	$\langle \mathcal{O}_e \mathcal{O}_{e^4} \rangle_3 = \frac{1386812286427888746496}{3}$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^3} \rangle_3 = 2000750410187353882624$
N=9	$\langle \mathcal{O}_e \mathcal{O}_{e^5} \rangle_3 = 269268196268268710260704O$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^4} \rangle_3 = 1787389856307036165086834 4$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e^3} \rangle_3 = 3381593580626825378989881 9$
N=10	$\langle \mathcal{O}_e \mathcal{O}_{e^6} \rangle_3 = 18896658034972207955520000000$ $\langle \mathcal{O}_{e^2} \mathcal{O}_{e^5} \rangle_3 = \frac{5244731673388664322694400000000}{3}$ $\langle \mathcal{O}_{e^3} \mathcal{O}_{e^4} \rangle_3 = 54662781193401578550848000000 0$

Chapter 4

Operator Product Approach

In this chapter, we treat approaches using operator product algebra of topological sigma model (A-model). In pure matter theory, we can insert identity operator $|\mathcal{O}_{W_\alpha}\rangle\eta^{\alpha\beta}\langle\mathcal{O}_{W_\beta}|$ into correlation functions because $\mathcal{O}_{W_\alpha}$ span entire Hilbert space (of local vertex operators). This leads us to fusion rules of correlation functions.

$$\langle \mathcal{O}_{W_\alpha}\mathcal{O}_{W_\beta} * \rangle = \langle \mathcal{O}_{W_\alpha}\mathcal{O}_{W_\beta}\mathcal{O}_{W_\gamma}\rangle\eta^{\gamma\delta}\langle\mathcal{O}_{W_\delta} * \rangle \tag{4.0.1}$$

In Section 4.1, under the assumption that fusion rules are closed under the subring spanned by $\mathcal{O}_{e^\alpha}$'s in case of topological sigma models on M_N^k, we construct 1-variable polynomial representation of sub ring of quantum cohomology ring of M_N^k using the results of Section 3.5.

In Section 4.2, we treat operator algebra of A-model coupled to gravity. In this case, because of existence of gravitational descendant states, $|\mathcal{O}_{W_\alpha}\rangle\eta^{\alpha\beta}\langle\mathcal{O}_{W_\beta}|$ is not an identity operator. But operator product algebra defined by

$$\mathcal{O}_{W_\alpha} \cdot \mathcal{O}_{W_\beta} = \langle \mathcal{O}_{W_\alpha}\mathcal{O}_{W_\beta}\mathcal{O}_{W_\gamma}\rangle\eta^{\gamma\delta}\mathcal{O}_{W_\delta} \tag{4.0.2}$$

remains. Assuming this algebra is associative,

$$(\mathcal{O}_{W_\alpha} \cdot \mathcal{O}_{W_\beta}) \cdot \mathcal{O}_{W_\gamma} = \mathcal{O}_{W_\alpha} \cdot (\mathcal{O}_{W_\beta} \cdot \mathcal{O}_{W_\gamma}) \tag{4.0.3}$$

we can rederive DWVV equation derived geometrically in Section 3.4. Using this equation, we solve A-model coupled with gravity on CP^3, CP^4 and $Gr(2,4)$.

4.1 Pure Matter Case

4.1.1 Strategy for Determination of Quantum Cohomology Ring of M_N^k

In this section, we treat pure A-model having target space as degree k hypersurface $(k \leq N)$ in CP^{N-1}, M_N^k.

$$M_N^k := \{(X_1 : X_2 : \cdots : X_N) \in CP^{N-1} | X_1^k + \cdots + X_N^k = 0\} \tag{4.1.4}$$

Since M_N^k is hypersurface in CP^{N-1}, we can choose subring $H_e^*(M_N^k)$ generated by Kahler class $e \in H^{1,1}(M_N^k)$. Correspondingly, we assume that BRST-closed observables $\mathcal{O}_{e^\alpha}$ $(\alpha = 0, 1, \cdots, N-2)$ form closed subalgebra in quantum cohomology ring of M_N^k (Operator algebra in pure matter theory). Then we investigate this subalgebra $H_{e,q}^*(M_N^k)$ in the following way. Operator product algebra is constructed by three point functions and metric.

$$\begin{aligned} \mathcal{O}_{e^\alpha} \cdot \mathcal{O}_{e^\beta} &= \langle \mathcal{O}_{e^\alpha} \mathcal{O}_{e^\beta} \mathcal{O}_{e^\gamma} \rangle \eta^{\gamma\delta} \mathcal{O}_{e^\delta} \\ \eta_{\gamma\delta} &:= \langle \mathcal{O}_{e^0} \mathcal{O}_{e^\gamma} \mathcal{O}_{e^\delta} \rangle = \int_{M_N^k} e^\gamma \wedge e^\delta = k\delta_{\gamma+\delta, N-2} \\ \eta_{\alpha\beta}\eta^{\beta\gamma} &= \delta_\alpha^\gamma \end{aligned} \tag{4.1.5}$$

Correlation functions in pure matter theory satisfy the fusion rule.

$$\langle \mathcal{O}_{e^\alpha} \mathcal{O}_{e^\beta} * \rangle = \langle \mathcal{O}_{e^\alpha} \mathcal{O}_{e^\beta} \mathcal{O}_{e^\gamma} \rangle \eta^{\gamma\delta} \langle \mathcal{O}_{e^\delta} * \rangle \tag{4.1.6}$$

From the above definition we can easily see $\mathcal{O}_{e^0}$ acts trivially on $H_{e,q}^*(M_N^k)$, and we regard $\mathcal{O}_{e^0}$ as identity. Three point functions are determined from the geometrical evaluation of correlation functions of topological sigma model,

$$\begin{aligned} &\langle \mathcal{O}_{e^\alpha}(z_1) \mathcal{O}_{e^\beta}(z_2) \mathcal{O}_{e^\gamma}(z_3) \rangle \\ &= \sum_{d=0}^{\infty} \delta_{\alpha+\beta+\gamma,(N-k)d+N-2} \int_{\mathcal{M}_{0,d}^{M_N^k}} \phi_1^*(e^\alpha) \wedge \phi_2^*(e^\beta) \wedge \phi_3^*(e^\gamma) \cdot q^d \\ &= \sum_{d=0}^{\infty} \delta_{\alpha+\beta+\gamma,(N-k)d+N-2} \int_{\mathcal{M}_{0,d,3}^{M_N^k}} \bar{\phi}_1^*(e^\alpha) \wedge \bar{\phi}_2^*(e^\beta) \wedge \bar{\phi}_3^*(e^\gamma) \cdot q^d \end{aligned} \tag{4.1.7}$$

where

$$\begin{aligned} \phi_i : \mathcal{M}_{0,d}^{M_N^k} &\mapsto M_{N,k} \\ f \in \mathcal{M}_{0,d}^{M_N^k} &\mapsto f(z_i) \\ \bar{\phi}_i : \mathcal{M}_{0,d,3}^{M_N^k} &\mapsto M_{N,k} \\ \{f, z_1, z_2, z_3\}/SL(2,C) \in \mathcal{M}_{0,d,3}^{M_N^k} &\mapsto f(z_i) \\ q &:= e^{-t}. \end{aligned}$$

$\mathcal{M}_{0,d}^{M_N^k}$ and $\mathcal{M}_{0,d,3}^{M_N^k}$ denote moduli spaces of holomorphic maps of degree d. from CP^1 to M_N^k of pure matter theory and of theory coupled to gravity. We insert $\delta_{\alpha+\beta+\gamma,(N-k)d+N-2}$ to represent topological selection rules explicitly. The equality between the first line and the second line of (4.1.7) can be explained as follows. $\mathcal{M}_{0,d}^{M_N^k}$ has internal $SL(2,C)$ which moves $\{f(z_1), f(z_2), f(z_3)\}$ without changing the posiotion of $f(CP^1)$ in M_N^k. In $\mathcal{M}_{0,d,3}^{M_N^k}$, these degrees of are killed by dividing by $SL(2,C)$ but the degrees of freedom that change the position of $\{z_1, z_2, z_3\}$ on CP^1 are added. Since $SL(2,C)$ can be considered as the degrees of freedom which maps $\{0, 1, \infty\}$ to any distinct points $\{z_1, z_2, z_3\}$, this difference cannot be distinguishd under the action of the evaluation maps $\phi_i, \bar{\phi}_i$.

Then we determine $H_{e,q}^*(M_N^k)$ with the following strategy.

1. Using equality of 4.1.7, we evaluate all the three point functions using torus action method with the following equation (3.5.143).

$$\begin{aligned} &\int_{\mathcal{M}_{0,d,3}^{M_N^k}} \bar{\phi}_1^*(e^\alpha) \wedge \bar{\phi}_2^*(e^\beta) \wedge \bar{\phi}_3^*(e^\gamma) \\ &= \int_{\mathcal{M}_{0,d,3}^{CP^{N-1}}} c_T(\bar{\pi}_3^*(\mathcal{E}_{kd+1})) \wedge \bar{\varphi}_1^*(c_1^\alpha(H)) \wedge \bar{\varphi}_2^*(c_1^\beta(H)) \wedge \bar{\varphi}_3^*(c_1^\gamma(H)) \\ &= \partial_{t_\alpha}\partial_{t_\beta}\partial_{t_\gamma} Res_z Res_h (\frac{1}{z} \log(\det((g_{ij,j'i',d})^{-1}) \frac{1}{h} \int d\phi_{ij,d} \\ &\exp(-\frac{1}{2}\sum_{i,j,d} \frac{-d^{(N-2-(N-k)d)}(5^i z - 5^j z)^2 \prod_{l=1}^{N} \prod_{a=1}^{d-1}(5^i az + 5^j(d-a)z - 5^l dz)}{\prod_{a=1}^{kd-1}(5^i az + 5^j(kd-a)z)} \phi_{ij,d}\phi_{ji,d} \\ &+ \sum_{i=1}^{N} \frac{5^i kz}{\prod_{j\neq i}(5^i z - 5^j z)} \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{d_1,\cdots,d_l,d_* \geq 1 \\ j_1,\cdots,j_l,j_* \neq i}} (\frac{v_{ij_1,d_1}}{z} + \cdots + \frac{v_{ij_l,d_l}}{z})^{l-3} \phi_{ij_1,d_1} \cdots \phi_{ij_l,d_l} \\ &\exp((5^i t_1 z + \cdots + 5^{i(N-2)} t_{N-2} z^{N-2})(\frac{v_{ij_1,d_1}}{z} + \cdots + \frac{v_{ij_l,d_l}}{z}))))|_{t_*=0} \end{aligned} \tag{4.1.8}$$

where

$$g_{ij,j'i',d} := \frac{-d^{(N-2-(N-k)d)}(5^i z - 5^j z)^2 \prod_{l=1}^{N} \prod_{a=1}^{d-1}(5^i az + 5^j(d-a)z - d5^l z)}{\prod_{a=1}^{kd-1}(5^i az + 5^j(kd-a)z)} \tag{4.1.9}$$

$$v_{ij,d} := \frac{d}{5^i - 5^j} \tag{4.1.10}$$

$$\begin{aligned} \bar{\varphi}_i : \mathcal{M}_{0,d,3}^{CP^{N-1}} &\mapsto CP^{N-1} \\ \{(z_1, z_2, z_3), f\}/SL(2,C) \in \mathcal{M}_{0,d,3}^{CP^{N-1}} &\mapsto f(z_i) \end{aligned} \tag{4.1.11}$$

where H is hyperplane bundle on CP^{N-1}, $\bar{\pi}_3$ is 3-fold forgetful map from $\mathcal{M}_{0,d,3}^{CP^{N-1}}$ to $\mathcal{M}_{0,d,0}^{CP^{N-1}}$ and $\mathcal{E}_{kd+1}$ denotes direct image sheaf $R_{\pi_1}^0(\bar{\varphi}_1^*(kH))$ coming from forgetful map π_1 from $\mathcal{M}_{0,d,1}^{CP^{N-1}}$ to $\mathcal{M}_{0,d,0}^{CP^{N-1}}$.

2. We can consider $\mathcal{O}_e$ as the generator of $H^*_{e,q}(M_{N,k})$, and we only have to determine multiplication rules for $\mathcal{O}_e$. In other words, if we set

$$F^{N,k}_\alpha : \mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \sum_{d=0}^{[\frac{1+\alpha}{N-k}]} \langle \mathcal{O}_e \mathcal{O}_{e^\alpha} \mathcal{O}_{e^{N-3-\alpha+(N-k)d}} \rangle \frac{1}{k} \mathcal{O}_{e^{1+\alpha-(N-k)d}} \quad (k < N) \tag{4.1.12}$$

$$F^{N,N}_\alpha : \mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \langle \mathcal{O}_e \mathcal{O}_{e^\alpha} \mathcal{O}_{e^{N-3-\alpha}} \rangle \frac{1}{N} \mathcal{O}_{e^{1+\alpha}} \quad (k = N) \tag{4.1.13}$$

,$H^*_q(M^k_N)$ is constructed as follows.

$$C[\mathcal{O}_e, \mathcal{O}_{e^2}, \cdots, \mathcal{O}_{e^{N-2}}]/I[F^{N,k}_1, F^{N,k}_2, \cdots, F^{N,k}_{N-2}] \tag{4.1.14}$$

where $C[\mathcal{O}_e, \mathcal{O}_{e^2}, \cdots, \mathcal{O}_{e^{N-2}}]$ denotes the polynomial ring generated by $\mathcal{O}_{e^\alpha}$ and $I[F^{N,k}_1, F^{N,k}_2, \cdots, F^{N,k}_{N-2}]$ is the ideal generated by $F^{N,k}_\alpha$'s.

We calculate $F^{N,k}_\alpha$ for $k \le N-2$ and $N \le 9$ case and find the ideal includes the following relation.

$$(\mathcal{O}_e)^{N-1} - k^k \cdot q \cdot (\mathcal{O}_e)^{k-1} = 0 \tag{4.1.15}$$

Numerical results are shown in Table A. In this case, using (4.1.15) and $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_0 = k$, we can easily see

$$\langle \prod_{i=1}^{N-2+(N-k)d} \mathcal{O}_e(z_i) \rangle = k^{kd+1} \cdot q^d \tag{4.1.16}$$

This agrees with the statement (3.1.15).

4.1.2 Reformulation as One Variable Polynomial Algebra

With some algebra, we can rewrite the relations (4.1.14) into the form

$$G^{N,k}_\alpha : \mathcal{O}_{e^\alpha} = (\mathcal{O}_e)^\alpha - \sum_{d=1}^{[\frac{\alpha}{N-k}]} \gamma^{N,k}_{\alpha,d} (\mathcal{O}_e)^{\alpha-(N-k)d} \cdot q^d \tag{4.1.17}$$

$$(2 \le \alpha \le N-2) \tag{4.1.18}$$

$$G^{N,k}_0 : 0 = (\mathcal{O}_e)^{N-1} - \sum_{d=1}^{[\frac{N-1}{N-k}]} \delta^{N,k}_d (\mathcal{O}_e)^{N-1-(N-k)d} \cdot q^d \quad (N > k) \tag{4.1.19}$$

$$G^{N,N}_\alpha : \mathcal{O}_{e^\alpha} = (\prod_{j=1}^{\alpha} \gamma^{N,N}_j(q))(\mathcal{O}_e)^\alpha \quad (2 \le \alpha \le N-2) \tag{4.1.20}$$

$$G^{N,N}_0 : 0 = (\prod_{j=1}^{N-2} \gamma^{N,N}_j(q))(\mathcal{O}_e)^{N-1} \quad (N = k) \tag{4.1.21}$$

where

$$\gamma^{N,N}_j(q) := N/\langle \mathcal{O}_e \mathcal{O}_{e^{j-1}} \mathcal{O}_{e^{N-2-j}} \rangle. \tag{4.1.22}$$

Then we can realize $H^*_{q,e}(M^k_N)$ as one variable polynomial algebra by regarding $\mathcal{O}_e$ as X, $\mathcal{O}_{e^\alpha}$ as r.h.s of $G^{N,k}_\alpha$, and $G^{N,k}_0$ as a relation. And if we define r.h.s of $G^{N,k}_\alpha$ as $f^{N,k}_\alpha(X)$ for $\alpha = 0, 2, 3, \cdots, N-2$ and X as $f_1(X)$, correlation functions are written in the residue form which follows from (4.1.6) as is well known in [5] .

$$\langle \mathcal{O}_{e^{\alpha_1}} \mathcal{O}_{e^{\alpha_2}} \cdots \mathcal{O}_{e^{\alpha_l}} \rangle_{M^k_N} = \mathrm{res}_{X=\infty} \left(\frac{f^{N,k}_{\alpha_1}(X) \cdot f^{N,k}_{\alpha_2}(X) \cdots f^{N,k}_{\alpha_l}(X)}{f_0(X)} \right) \cdot k \tag{4.1.23}$$

These results are collected in Table B. At first sight, this reformulation seems to be superficial, but we find some curious relation between $\gamma^{N,k}_{\alpha,d}$ for $k \le N-2$ case.

relation 1

$$\gamma^{N,k}_{\alpha,1} = \gamma^{N-1,k}_{\alpha-1,1} \quad (k \le N-2) \tag{4.1.24}$$

relation 2

$$\gamma^{N,k}_{N-2,2} = \frac{(\gamma^{N,k}_{N-k,1})^2}{2} \quad ((N-k)2 = N-2) \tag{4.1.25}$$

$$\gamma^{N,k}_{N-3,2} = \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+1,1}}{2} - \frac{\gamma^{N,k}_{N-k,1}}{4}\right) \tag{4.1.26}$$

$$\gamma^{N,k}_{N-2,2} = \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+1,1}}{2} + \frac{\gamma^{N,k}_{N-k,1}}{4}\right) \quad ((N-k)2 = N-3) \tag{4.1.27}$$

$$\gamma^{N,k}_{N-4,2} = \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2} - \frac{\gamma^{N,k}_{N-k+1,1}}{4} - \frac{\gamma^{N,k}_{N-k,1}}{8}\right) \tag{4.1.28}$$

$$\gamma^{N,k}_{N-3,2} = \frac{(\gamma^{N,k}_{N-k+1,1})^2}{2} \tag{4.1.29}$$

$$\gamma^{N,k}_{N-2,2} = \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2} + \frac{\gamma^{N,k}_{N-k+1,1}}{4} + \frac{\gamma^{N,k}_{N-k,1}}{8}\right) \quad ((N-k)2 = N-4) \tag{4.1.30}$$

$$\gamma^{N,k}_{N-5,2} = \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+3,1}}{2} - \frac{\gamma^{N,k}_{N-k+2,1}}{4} - \frac{\gamma^{N,k}_{N-k+1,1}}{8} - \frac{\gamma^{N,k}_{N-k,1}}{16}\right) \tag{4.1.31}$$

$$\begin{aligned}\gamma^{N,k}_{N-4,2} &= \gamma^{N,k}_{N-k+1,1}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2}-\frac{\gamma^{N,k}_{N-k+1,1}}{4}\right)\\ &+\frac{\gamma^{N,k}_{N-k,1}}{2}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2}-\frac{\gamma^{N,k}_{N-k+1,1}}{4}-\frac{\gamma^{N,k}_{N-k,1}}{8}\right) \quad (4.1.32)\\ \gamma^{N,k}_{N-3,2} &= \gamma^{N,k}_{N-k+1,1}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2}+\frac{\gamma^{N,k}_{N-k+1,1}}{4}\right)\\ &-\frac{\gamma^{N,k}_{N-k,1}}{2}\left(\frac{\gamma^{N,k}_{N-k+2,1}}{2}-\frac{\gamma^{N,k}_{N-k+1,1}}{4}-\frac{\gamma^{N,k}_{N-k,1}}{8}\right) \quad (4.1.33)\\ \gamma^{N,k}_{N-2,2} &= \gamma^{N,k}_{N-k,1}\left(\frac{\gamma^{N,k}_{N-k+3,1}}{2}+\frac{\gamma^{N,k}_{N-k+2,1}}{4}+\frac{\gamma^{N,k}_{N-k+1,1}}{8}+\frac{\gamma^{N,k}_{N-k,1}}{16}\right) \quad (4.1.34)\\ &((N-k)2=N-5)\end{aligned}$$

relation 3

$$\begin{aligned}\gamma^{N,k}_{N-2,3} &= \gamma^{N,k}_{N-k,1}\gamma^{N,k}_{2(N-k),2} \quad (4.1.35)\\ &((N-k)3=N-2)\end{aligned}$$

We can reconstruct some of the above relations from the compatibility of the expansion form of (2.12) and relation (4.1.15), but we are not sure that all of them follow from it at this stage. With these relations, we can figure out some characteristic feature of $H^*_{q,e}(M^k_N)$.
First, quantum correction of degree 1 to $H^*_{q,e}(M^k_N)$ does not depend on N, which can be easily seen from relation 1. So we think these correction coefficients $\gamma^k_{\alpha,1}(:=\gamma^{N,k}_{N-k+\alpha-1,1})$ play central role in the ring when $k \leq N-2$. In other words, we expect all the higher degree quantum correction coefficients are determined by $\gamma^k_{\alpha,1}$. Relation 2 are found from these speculations. Second, from the expansion form of (4.1.18), degree d coefficients of $\mathcal{O}_{e^\alpha}$ occur when $\alpha \geq (N-k)d$ holds. Then if $k \leq [\frac{N}{2}]+1$, no corrections occur from sectors with degree greater than 1. But degree 1 corrections remain stable since they exist as long as α is no less than $N-k$. This seems to support our first speculation. We will show some examples of these features using the results of $H^*_{q,e}(M^6_N)$.

$H^*_{q,e}(M^6_8)$

$$\begin{aligned}\mathcal{O}_{e} &= X\\ \mathcal{O}_{e^2} &= X^2-\gamma^6_{1,1}q\\ \mathcal{O}_{e^3} &= X^3-\gamma^6_{2,1}Xq\\ \mathcal{O}_{e^4} &= X^4-\gamma^6_{3,1}X^2q-\gamma^6_{1,1}\left(\frac{\gamma^6_{3,1}}{2}-\frac{\gamma^6_{2,1}}{4}-\frac{\gamma^6_{1,1}}{8}\right)q^2\\ \mathcal{O}_{e^5} &= X^5-\gamma^6_{4,1}X^3q-\frac{(\gamma^6_{2,1})^2}{2}Xq^2\\ \mathcal{O}_{e^6} &= X^6-\gamma^6_{5,1}X^4q-\gamma^6_{1,1}\left(\frac{\gamma^6_{3,1}}{2}+\frac{\gamma^6_{2,1}}{4}+\frac{\gamma^6_{1,1}}{8}\right)X^2q^2\end{aligned}$$

$$-(\gamma^6_{1,1})^2\left(\frac{\gamma^6_{3,1}}{2}-\frac{\gamma^6_{2,1}}{4}-\frac{\gamma^6_{1,1}}{8}\right)q^3 \quad (4.1.36)$$

$H^*_{q,e}(M^6_9)$

$$\begin{aligned}\mathcal{O}_{e} &= X\\ \mathcal{O}_{e^2} &= X^2\\ \mathcal{O}_{e^3} &= X^3-\gamma^6_{1,1}q\\ \mathcal{O}_{e^4} &= X^4-\gamma^6_{2,1}X^1q\\ \mathcal{O}_{e^5} &= X^5-\gamma^6_{3,1}X^2q\\ \mathcal{O}_{e^6} &= X^6-\gamma^6_{4,1}X^3q-\gamma^6_{1,1}\left(\frac{\gamma^6_{2,1}}{2}-\frac{\gamma^6_{1,1}}{4}\right)q^2\\ \mathcal{O}_{e^7} &= X^7-\gamma^6_{5,1}X^4q-\gamma^6_{1,1}\left(\frac{\gamma^6_{2,1}}{2}+\frac{\gamma^6_{1,1}}{4}\right)Xq^2 \quad (4.1.37)\end{aligned}$$

$H^*_{q,e}(M^6_{10})$

$$\begin{aligned}\mathcal{O}_{e} &= X\\ \mathcal{O}_{e^2} &= X^2\\ \mathcal{O}_{e^3} &= X^3\\ \mathcal{O}_{e^4} &= X^4-\gamma^6_{1,1}q\\ \mathcal{O}_{e^5} &= X^5-\gamma^6_{2,1}Xq \quad (4.1.38)\\ \mathcal{O}_{e^6} &= X^6-\gamma^6_{3,1}X^2q\\ \mathcal{O}_{e^7} &= X^7-\gamma^6_{4,1}X^3q\\ \mathcal{O}_{e^8} &= X^8-\gamma^6_{5,1}X^4q-\frac{(\gamma^6_{1,1})^2}{2}q^2 \quad (4.1.39)\end{aligned}$$

$H^*_{q,e}(M^6_N)$ $(N \geq 11)$

$$\begin{aligned}\mathcal{O}_{e^k} &= X^k \quad (1 \leq k \leq N-7)\\ \mathcal{O}_{e^{N-7+\alpha}} &= X^{N-7+\alpha}-\gamma^6_{\alpha,1}X^{\alpha-1}q(1 \leq \alpha \leq 5) \quad (4.1.40)\end{aligned}$$

where

$$\gamma^6_{1,1}=720, \gamma^6_{2,1}=6984, \gamma^6_{3,1}=23328, \gamma^6_{4,1}=39672, \gamma^6_{5,1}=45936 \quad (4.1.41)$$

Table A. Multiplication Rules of $H^*_{q,e}(M^k_N)$

$H^*_{q,e}(M^1_N)$

$$\mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \mathcal{O}_{e^{\alpha+1}} \quad (0 \leq \alpha \leq N-3) \quad \mathcal{O}_e \cdot \mathcal{O}_{e^{N-2}} = q \tag{4.1.42}$$

$H^*_{q,e}(M^2_N)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \mathcal{O}_{e^{\alpha+1}} \quad (0 \leq \alpha \leq N-4) \quad \mathcal{O}_e \cdot \mathcal{O}_{e^{N-3}} = \mathcal{O}_{e^{N-2}} + 2q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-2}} = 2\mathcal{O}_e q \end{aligned} \tag{4.1.43}$$

$H^*_{q,e}(M^3_5)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} + 6q \quad \mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} + 15\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = 6\mathcal{O}_{e^2} q + 36q^2 \end{aligned} \tag{4.1.44}$$

$H^*_{q,e}(M^3_5) \quad (N \geq 6)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \mathcal{O}_{e^{\alpha+1}} \quad (0 \leq \alpha \leq N-5) \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-4}} = \mathcal{O}_{e^{N-3}} + 6q \quad \mathcal{O}_e \cdot \mathcal{O}_{e^{N-3}} = \mathcal{O}_{e^{N-2}} + 15\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-2}} = 6\mathcal{O}_{e^2} q \end{aligned} \tag{4.1.45}$$

$H^*_{q,e}(M^4_6)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} + 24q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} + 104\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = \mathcal{O}_{e^4} + 104\mathcal{O}_{e^2} q + 2784q^2 \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^4} = 24\mathcal{O}_{e^3} q + 2784\mathcal{O}_e q^2 \end{aligned} \tag{4.1.46}$$

$H^*_{q,e}(M^4_7)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} + 24q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = \mathcal{O}_{e^4} + 104\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^4} = \mathcal{O}_{e^5} + 104\mathcal{O}_{e^2} q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^5} = 24\mathcal{O}_{e^3} q + 576q^2 \end{aligned} \tag{4.1.47}$$

$H^*_{q,e}(M^4_N)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} = \mathcal{O}_{e^{\alpha+1}} \quad (0 \leq \alpha \leq N-6) \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-5}} = \mathcal{O}_{e^{N-4}} + 24q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-4}} = \mathcal{O}_{e^{N-3}} + 104\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-3}} = \mathcal{O}_{e^{N-2}} + 104\mathcal{O}_{e^2} q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^{N-2}} = 24\mathcal{O}_{e^3} q \end{aligned} \tag{4.1.48}$$

$H^*_{q,e}(M^5_7)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} + 120q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} + 770\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = \mathcal{O}_{e^4} + 1345\mathcal{O}_{e^2} q + 211200q^2 \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^4} = \mathcal{O}_{e^5} + 770\mathcal{O}_{e^3} q + 692500\mathcal{O}_e q^2 \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^5} = 120\mathcal{O}_{e^4} q + 211200\mathcal{O}_{e^2} q^2 \end{aligned} \tag{4.1.49}$$

$H^*_{q,e}(M^5_8)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} + 120q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = \mathcal{O}_{e^4} + 770\mathcal{O}_e q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^4} = \mathcal{O}_{e^5} + 1345\mathcal{O}_{e^2} q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^5} = \mathcal{O}_{e^6} + 770\mathcal{O}_{e^3} q + 99600q^2 \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^6} = 120\mathcal{O}_{e^4} q + 99600\mathcal{O}_e q^2 \end{aligned} \tag{4.1.50}$$

$H^*_{q,e}(M^5_9)$

$$\begin{aligned} &\mathcal{O}_e \cdot \mathcal{O}_e = \mathcal{O}_{e^2} \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^2} = \mathcal{O}_{e^3} \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^3} = \mathcal{O}_{e^4} + 120q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^4} = \mathcal{O}_{e^5} + 770\mathcal{O}_e q \\ &\mathcal{O}_e 0 \cdot \mathcal{O}_{e^5} = \mathcal{O}_{e^6} + 1345\mathcal{O}_{e^2} q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^6} = \mathcal{O}_{e^7} + 770\mathcal{O}_{e^3} q \\ &\mathcal{O}_e \cdot \mathcal{O}_{e^7} = +120\mathcal{O}_{e^4} q + 14400q^2 \end{aligned} \tag{4.1.51}$$

$H^*_{q,e}(M^5_N) \quad (N \geq 10)$

$$\begin{aligned} \mathcal{O}_e \cdot \mathcal{O}_{e^\alpha} &= \mathcal{O}_{e^{\alpha+1}} \quad (0 \leq \alpha \leq N-7) \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-6}} &= \mathcal{O}_{e^{N-5}} + 120q \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-5}} &= \mathcal{O}_{e^{N-4}} + 770\mathcal{O}_e q \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-4}} &= \mathcal{O}_{e^{N-3}} + 1345\mathcal{O}_{e^2} q \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-3}} &= \mathcal{O}_{e^{N-2}} + 770\mathcal{O}_{e^3} q \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-2}} &= 120\mathcal{O}_{e^4} q \end{aligned} \tag{4.1.52}$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_8)\\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2} + 720q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + 6264\mathcal{O}_e q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 16344\mathcal{O}_{e^2} q + 18843840q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 16344\mathcal{O}_{e^3} q + 1314584640\mathcal{O}_e q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 6264\mathcal{O}_{e^4} q + 1314584640\mathcal{O}_{e^2} q^2\\
&\quad +144069995520q^3\\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= +720\mathcal{O}_{e^5} q + 18843840\mathcal{O}_{e^3} q^2\\
&\quad +144069995520\mathcal{O}_e q^3
\end{aligned} \tag{4.1.53}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_9)\\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2}\\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + 720q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 6264\mathcal{O}_e q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 16344\mathcal{O}_{e^2} q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 16344\mathcal{O}_{e^3} q + 14152320q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= \mathcal{O}_{e^7} + 6264\mathcal{O}_{e^4} q + 44006976\mathcal{O}_e q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^7} &= 720\mathcal{O}_{e^5} q + 14152320\mathcal{O}_{e^2} q^2
\end{aligned} \tag{4.1.54}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_{10})\\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2}\\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3}\\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 720q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 6264\mathcal{O}_e q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 16344\mathcal{O}_{e^2} q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= \mathcal{O}_{e^7} + 16344\mathcal{O}_{e^3} q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^7} &= \mathcal{O}_{e^8} + 6264\mathcal{O}_{e^4} q + 4769280\mathcal{O}_e q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^8} &= 720\mathcal{O}_{e^5} q + 4769280\mathcal{O}_{e^2} q^2
\end{aligned} \tag{4.1.55}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^7_9)\\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2} + 5040q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + 56196\mathcal{O}_e q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 200452\mathcal{O}_{e^2} q + 205625920q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 300167\mathcal{O}_{e^3} q + 24699506832\mathcal{O}_e q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 200452\mathcal{O}_{e^4} q + 53751685624\mathcal{O}_{e^2} q^2\\
&\quad +534155202302400q^3\\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= \mathcal{O}_{e^7} + 56196\mathcal{O}_{e^5} q + 24699506832\mathcal{O}_{e^3} q^2\\
&\quad +1920365635990032\mathcal{O}_e q^3\\
\mathcal{O}_e \cdot \mathcal{O}_{e^7} &= 5040\mathcal{O}_{e^6} q + 2056259520\mathcal{O}_{e^4} q^2\\
&\quad +534155202302400\mathcal{O}_{e^2} q^3\\
&\quad +5112982794486067200q^4
\end{aligned} \tag{4.1.56}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^7_{10})\\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2}\\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + 5040q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 56196\mathcal{O}_e q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 200452\mathcal{O}_{e^2} q\\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 300167\mathcal{O}_{e^3} q + 2091962880q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= \mathcal{O}_{e^7} + 200452\mathcal{O}_{e^4} q + 13570681320\mathcal{O}_e q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^7} &= \mathcal{O}_{e^8} + 56196\mathcal{O}_{e^5} q + 13570681320\mathcal{O}_{e^2} q^2\\
\mathcal{O}_e \cdot \mathcal{O}_{e^8} &= 5040\mathcal{O}_{e^6} q + 2091962880\mathcal{O}_{e^3} q^2\\
&\quad +13462263763200q^3
\end{aligned} \tag{4.1.57}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^8_{11}) \\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2} \\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + 40320q \\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4} + 554112\mathcal{O}_e q \\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5} + 2552192\mathcal{O}_{e^2} q \\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= \mathcal{O}_{e^6} + 5241984\mathcal{O}_{e^3} q + 345655618560q^2 \\
\mathcal{O}_e \cdot \mathcal{O}_{e^6} &= \mathcal{O}_{e^7} + 5241984\mathcal{O}_{e^4} q + 3857214283776\mathcal{O}_e q^2 \\
\mathcal{O}_e \cdot \mathcal{O}_{e^7} &= \mathcal{O}_{e^8} + 2552192\mathcal{O}_{e^5} q + 8150222448640\mathcal{O}_{e^2} q^2 \\
\mathcal{O}_e \cdot \mathcal{O}_{e^8} &= \mathcal{O}_{e^9} + 554112\mathcal{O}_{e^6} q + 3857214283776\mathcal{O}_{e^3} q^2 \\
&\quad +235354398279598080q^3 \\
\mathcal{O}_e \cdot \mathcal{O}_{e^9} &= 40320\mathcal{O}_{e^7} q + 345655618560\mathcal{O}_{e^4} q^2 \\
&\quad +235354398279598080q^3\mathcal{O}_e
\end{aligned}
\tag{4.1.58}
$$

Table B. One Variable Polynomial Representation of $H^*_{q,e}(M^k_N)$

$$
\begin{aligned}
&H^*_{q,e}(M^1_N) \\
f_0(X) &= X^{N-1} - q \\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (0 \le \alpha \le N-2)
\end{aligned}
\tag{4.1.59}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^2_N) \\
f_0(X) &= X^{N-1} - 2^2 Xq \\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (0 \le \alpha \le N-3) \\
\mathcal{O}_{e^{N-2}} &= X^{N-2} - 2qX
\end{aligned}
\tag{4.1.60}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^3_N) \\
f_0(X) &= X^{N-1} - 3^3 X^2 q \\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (0 \le \alpha \le N-4) \\
\mathcal{O}_{e^{N-3}} &= X^{N-3} - 6q \\
\mathcal{O}_{e^{N-2}} &= X^{N-2} - 21Xq
\end{aligned}
\tag{4.1.61}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^4_6) \\
f_0(X) &= X^5 - 4^4 X^3 q \\
\mathcal{O}_{e^0} &= 1 \\
\mathcal{O}_e &= X \\
\mathcal{O}_{e^2} &= X^2 - 24q \\
\mathcal{O}_{e^3} &= X^3 - 128Xq \\
\mathcal{O}_{e^4} &= X^4 - 232X^2 q - 288q^2
\end{aligned}
\tag{4.1.62}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^4_N) \quad (N \ge 7) \\
f_0(X) &= X^{N-1} - 4^4 X^3 q \\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (0 \le \alpha \le N-5) \\
\mathcal{O}_{e^{N-4}} &= X^{N-4} - 24q \\
\mathcal{O}_{e^{N-3}} &= X^{N-3} - 128Xq \\
\mathcal{O}_{e^{N-2}} &= X^{N-2} - 232X^2 q
\end{aligned}
\tag{4.1.63}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^5_7)\\
f_0(X) &= X^6 - 5^5X^4q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2 - 120q\\
\mathcal{O}_{e^3} &= X^3 - 890Xq\\
\mathcal{O}_{e^4} &= X^4 - 2235X^2q - 49800q^2\\
\mathcal{O}_{e^5} &= X^5 - 3005X^3q - 57000Xq^2
\end{aligned}
\tag{4.1.64}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^5_8)\\
f_0(X) &= X^7 - 5^5X^4q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2\\
\mathcal{O}_{e^3} &= X^3 - 120q\\
\mathcal{O}_{e^4} &= X^4 - 890Xq\\
\mathcal{O}_{e^5} &= X^5 - 2235X^2q\\
\mathcal{O}_{e^6} &= X^6 - 3005X^3q - 7200q^2
\end{aligned}
\tag{4.1.65}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^5_N) \quad (N \leq 9)\\
f_0(X) &= X^{N-1} - 5^5X^4q\\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (0 \leq \alpha \leq N-6)\\
\mathcal{O}_{e^{N-5}} &= X^{N-5} - 120q\\
\mathcal{O}_{e^{N-4}} &= X^{N-4} - 890Xq\\
\mathcal{O}_{e^{N-3}} &= X^{N-3} - 2235X^2q\\
\mathcal{O}_{e^{N-2}} &= X^{N-2} - 3005X^3q
\end{aligned}
\tag{4.1.66}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_8)\\
f_0(X) &= X^7 - 6^6X^5q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2 - 720q\\
\mathcal{O}_{e^3} &= X^3 - 6984Xq\\
\mathcal{O}_{e^4} &= X^4 - 23328X^2q - 7076160q^2\\
\mathcal{O}_{e^5} &= X^5 - 39672X^3q - 24388128Xq^2\\
\mathcal{O}_{e^6} &= X^6 - 45936X^4q - 9720000X^2q^2\\
&\quad -5094835200q^3
\end{aligned}
\tag{4.1.67}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_9)\\
f_0(X) &= X^8 - 6^6X^5q\\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (1 \leq \alpha \leq 2)\\
\mathcal{O}_{e^3} &= X^3 - 720q\\
\mathcal{O}_{e^4} &= X^4 - 6984Xq\\
\mathcal{O}_{e^5} &= X^5 - 23328X^2q\\
\mathcal{O}_{e^6} &= X^6 - 39672X^3q - 2384640q^2\\
\mathcal{O}_{e^7} &= X^7 - 45936X^4q - 2643840Xq^2
\end{aligned}
\tag{4.1.68}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^6_{10})\\
f_0(X) &= X^9 - 6^6X^5q\\
\mathcal{O}_{e^\alpha} &= X^\alpha \quad (1 \leq \alpha \leq 3)\\
\mathcal{O}_{e^4} &= X^4 - 720q\\
\mathcal{O}_{e^5} &= X^5 - 6984Xq\\
\mathcal{O}_{e^6} &= X^6 - 23328X^2q\\
\mathcal{O}_{e^7} &= X^7 - 39672X^3q\\
\mathcal{O}_{e^8} &= X^8 - 45936X^4q - 259200Xq^2
\end{aligned}
\tag{4.1.69}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^7_9)\\
f_0(X) &= X^8 - 7^7 X^6 q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2 - 5040q\\
\mathcal{O}_{e^3} &= X^3 - 61236Xq\\
\mathcal{O}_{e^4} &= X^4 - 261688X^2q - 1045981440q^2\\
\mathcal{O}_{e^5} &= X^5 - 561855X^3q - 7364461860Xq^2\\
\mathcal{O}_{e^6} &= X^6 - 762307X^4q - 8660264508X^2q^2\\
&\quad -53577635146560q^3\\
\mathcal{O}_{e^7} &= X^7 - 818503X^5q - 1785767760X^3q^2\\
&\quad -47590972087680q^3
\end{aligned}
\tag{4.1.70}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^7_{10})\\
f_0(X) &= X^9 - 7^7 X^6 q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2\\
\mathcal{O}_{e^3} &= X^3 - 5040q\\
\mathcal{O}_{e^4} &= X^4 - 61236Xq\\
\mathcal{O}_{e^5} &= X^5 - 261688X^2q\\
\mathcal{O}_{e^6} &= X^6 - 561855X^3q - 579121200q^2\\
\mathcal{O}_{e^7} &= X^7 - 762307X^4q - 1874923848Xq^2\\
\mathcal{O}_{e^8} &= X^8 - 818503X^5q - 739786320X^2q^2
\end{aligned}
\tag{4.1.71}
$$

$$
\begin{aligned}
&H^*_{q,e}(M^8_{11})\\
f_0(X) &= X^{10} - 8^8 X^7 q\\
\mathcal{O}_{e^0} &= 1\\
\mathcal{O}_{e} &= X\\
\mathcal{O}_{e^2} &= X^2\\
\mathcal{O}_{e^3} &= X^3 - 40320q\\
\mathcal{O}_{e^4} &= X^4 - 594432Xq\\
\mathcal{O}_{e^5} &= X^5 - 3146624X^2q\\
\mathcal{O}_{e^6} &= X^6 - 8388608X^3q - 134298823680q^2\\
\mathcal{O}_{e^7} &= X^7 - 13630592X^4q - 875510074368Xq^2\\
\mathcal{O}_{e^8} &= X^8 - 16182784X^5q - 994943923200X^2q^2\\
\mathcal{O}_{e^9} &= X^9 - 16736896X^6q - 203929850880X^3q^2\\
&\quad -5414928570777600q^3
\end{aligned}
\tag{4.1.72}
$$

4.2 Gravitational Case

4.2.1 Meaning of the Correlation Function

In the topological sigma model (A-model) which describes maps from CP^1 to the target space M, BRST-closed observables are constructed from elements of $H^*(M)$. We denote the BRST-closed observable constructed from $W \in H^*(M)$ as $\mathcal{O}_W$. Witten showed in the pure matter case [8] (without coupling to gravity) topological correlation functions are given in terms of intersection numbers of holomorphic maps from CP^1 to M as follows.

$$\begin{aligned} \langle \mathcal{O}_{W_{i_1}}(z_1)\mathcal{O}_{W_{i_2}}(z_2)\cdots\mathcal{O}_{W_{i_k}}(z_k)\rangle &= \int_{\mathcal{M}^M_{0,d}} \chi(\nu)\prod_{j=1}^{k}\phi_j^*(W_{i_j}) \\ \phi_j : \mathcal{M}^M_{0,d} &\mapsto M \\ f \in \mathcal{M}^M_{0,d} &\mapsto f(z_j) \in M \quad j=1,\cdots,k \end{aligned} \tag{4.2.1}$$

($\mathcal{M}^M_{0,d}$ is the moduli space of holomorphic maps from CP^1 to M of degree d,and $(z_1,\dots,z_k)$ are "fixed" distinct points on CP^1. Degree d is related to the sum of $dim_C(W_{i_j})$ by the topological selection rule which we will introduce later).

ν is the additional degree of freedom which arises when f can be decomposed as $f = \tilde{f}\circ\alpha$ where α is a map from CP^1 to CP^1 of degree $\frac{d}{j}$ and $\tilde{f}$ a map from CP^1 to M of degree $j(d|j)$. But as we will discuss later, we have to consider ν only when M is C.Y manifold, i.e. $c_1(T'M)=0$.

Since $\phi_j^*(W_{i_j})$ defines $dim_C(W_{i_j})$ form on $\mathcal{M}^M_{0,d}$, in generic case when ν is trivial, $\langle\mathcal{O}_{W_{i_1}}\cdots\mathcal{O}_{W_{i_k}}\rangle$ doesn't vanish only when the following conditions are satisfied.

$$\begin{aligned} \sum_{j=1}^{k} dim_C(W_{i_j}) &= dim_C(\mathcal{M}^M_{0,d}) \\ &= dim H^0(f^*(T'M)) \\ &= dc_1(T'M) + dim_C(M) \end{aligned} \tag{4.2.2}$$

In deriving the third line from the second line, we used Rieman-Roch theorem and assumed $H^1(f^*(T'M))=0$.

If we take W_{i_j} as the form which has a delta-function support on the Poincare dual of W_{i_j}, $PD(W_{i_j})$, we can interpret $\phi_i^*(W_{i_j})$ as the following constraint on $\mathcal{M}_d(M)$.

$$f(z_j) \in PD(W_{i_j}) \tag{4.2.3}$$

We can easily see the above condition imposes $dim_C(W_i)$ independent constraints on $\mathcal{M}^M_{0,d}$ (use count degrees of freedom in the complex sense). Since we have to use $(dim_C(M) - dim_C(f(CP^1)) - dim_C(PD(W_{i_j}))$ degrees of freedom to let $f(CP^1)\cap$

$PD(W_{i_j}) \neq \emptyset$ and in case $dim_C(f(CP^1)) = 1$, we have to use one further degree of freedom to let z_j to lie on $f(CP^1)\cap PD(W_{i_j})$. Condition (4.2.3) tells us that by imposing all the constraints $i=1,\cdots,k$, we have zero degrees of freedom and topological correlation functions reduce to

$$\begin{aligned} &\langle \mathcal{O}_{W_{i_1}}(z_1)\cdots\mathcal{O}_{W_{i_k}}(z_k)\rangle_{generic} \\ &= \sharp\{f : CP^1 \xrightarrow{hol} M | f(z_j) \in PD(W_{i_j}), j=1,\cdots,k\} \end{aligned} \tag{4.2.4}$$

At this point, we consider the case of multiple cover map. From the above argument, multiple cover map $f=\tilde{f}\circ\alpha$ also has to satisfy the condition (4.2.3) which restricts the motion of $f(CP^1)=\tilde{f}(CP^1)$ in the target space M. But since $\tilde{f}$ is a map of degree j, it has as many as $jc_1(T'M)+dim_C(M)(< dc_1(T'M)+dim_C(M))$ freedom in M and this is imcompatible with (4.2.3). Only when $c_1(T'M)=0$ i.e. M is C.Y manifold, compatibility of (4.2.2) and (4.2.3) holds in the case of multiple cover map and we have to integrate the additional ν. Then we conclude that when $c_1(T'M)>0$,we can neglect $\chi(\nu)$ and only consider the generic case.

Next, let us consider what happens if we couple topological gravity to the above topological sigma model. Roughly speaking, we have to integrate over moduli space of CP^1 with punctures. Since the moduli space of CP^1 with k-punctures are given by the position of k-distinct points on CP^1 divided by $SL(2,C)$, which is the internal symmetry group of CP^1, the difinition (4.2.2) is modified as follows.

$$\begin{aligned} \langle\mathcal{O}_{W_{i_1}}\cdots\mathcal{O}_{W_{i_k}}\rangle &= \int_{\mathcal{M}_{d,0,k}(M)}\prod_{j=1}^{k}\bar{\phi}_j^{\,*}(W_{i_j}) \\ \bar{\phi}_j : \mathcal{M}^M_{0,d,k} &\mapsto M \\ \{(z_1,z_2,\cdots,z_k),f\}/SL(2,C) &\mapsto f(z_j) \end{aligned} \tag{4.2.5}$$

where the action of $u\in SL(2,C)$ is defined as follows.

$$u\circ\{(z_1,z_2,\cdots,z_k),f\} = \{(u(z_1),u(z_2),\cdots,u(z_k)),(u^{-1})^*f\} \tag{4.2.6}$$

This action is natural in the sense that the image of $\bar{\phi}_j$ remains invariant under $SL(2,C)$. Main difference between (4.2.4) and (4.2.6) is that in the former case, we keep z_i "fixed" on CP^1 but in the latter they move. Then we have $dim_C(\mathcal{M}^M_{0,d,k}) = k-3+dc_1(TM)+dim_C(M)$ and modify (4.2.2) as follows.

$$\begin{aligned} &\sum_{j=1}^{k} dim_C(W_{i_j}) = dc_1(T'M)+dim_C(M)+k-3 \\ \Longleftrightarrow &\sum_{j=1}^{k} dim_C(W_{i_j}-1) = dc_1(T'M)+dim_C(M)-3 \end{aligned} \tag{4.2.7}$$

Integrating over the positions of k-punctures the codition (4.2.3) changes into

$$f(CP^1)\cap PD(W_{i_j}) \neq \emptyset. \tag{4.2.8}$$

Under the condition (4.2.7), $f(CP^1) \cap PD(W_{i_j})$ must be a finite point set for each j and z_i integration contributes $(f(CP^1) \cap PD(W_{i_j}))^\sharp$ to the correlation function. Then we have

$$\langle \mathcal{O}_{W_{i_1}} \cdots \mathcal{O}_{W_{i_k}} \rangle = \sum_f \prod_{j=1}^{k} (f(CP^1) \cap PD(W_{i_j}))^\sharp$$
$$\{f : CP^1 \xrightarrow{hol} M | f(CP^1) \cap PD(W_{i_j}) \neq \emptyset \quad j = 1, 2, \cdots, k\} \tag{4.2.9}$$

4.2.2 Set up of the calculation

Topological Sigma Model (A-Model) can be constructed as the twisted version of $N = 2$ super conformal field theory [28]. We can perturb topological field theory by adding the terms $\sum_i t_i \mathcal{O}_{W_i}$ to the lagrangian and correlation functions depend on variables $\{t_i\}$ [24].

$$\begin{aligned} &\langle \mathcal{O}_{W_{i_1}} \mathcal{O}_{W_{i_2}} \cdots \mathcal{O}_{W_{i_k}} (t_1, t_2, \ldots, t_{dim(H^*(M))}) \rangle \\ &= \int \mathcal{D}X e^{-L(X) + \sum_i t_i \mathcal{O}_{W_i}} \mathcal{O}_{W_{i_1}} \mathcal{O}_{W_{i_2}} \cdots \mathcal{O}_{W_{i_k}} \end{aligned} \tag{4.2.1}$$

where X denotes the field variables of the A-Model. We set $D = dim(H^*(M))$. As the twisted version of N=2 SCFT (coupled with gravity and on small phase space),$\{\mathcal{O}_{W_i}\}$ has ring structure which can be determined from three point correlation functions.

$$\mathcal{O}_{W_i} \mathcal{O}_{W_j} = C_{ij}^k (t_1, t_2, \ldots, t_D) \mathcal{O}_{W_k} \tag{4.2.2}$$
$$\text{where} \quad C_{ij}^k = C_{ijl} \eta^{lk} \tag{4.2.3}$$
$$C_{ijl} = \langle \mathcal{O}_{W_i} \mathcal{O}_{W_j} \mathcal{O}_{W_l} (t_1, t_2, \ldots, t_D) \rangle \tag{4.2.4}$$
$$\eta^{kl} \eta_{lm} = \delta_m^k \tag{4.2.5}$$
$$\eta_{lm} = C_{1lm} (t_1, t_2, \ldots, t_D) \tag{4.2.6}$$

In our notation W_1 corresponds to $1 \in H^*(M)$ and we set W_2 to the Kähler form of M (in our case where M is CP^3, CP^4 or $Gr(2,4)$, $dim(H^2(M)) = dim(H^{1,1}(M)) = 1$,this notation is O.K). We assume that t_i's are flat coordinates and η_{lm} do not depend on them and determined by classical intersection number $\int_M W_l \wedge W_m$. Next, we impose associativity condition on this algebra.(This relation is DWVV eq.)

$$\begin{aligned} & (\mathcal{O}_{W_i} \mathcal{O}_{W_j}) \mathcal{O}_{W_k} = \mathcal{O}_{W_i} (\mathcal{O}_{W_j} \mathcal{O}_{W_k}) \\ \Longleftrightarrow\ & C_{ij}^l \mathcal{O}_{W_l} \mathcal{O}_{W_k} = \mathcal{O}_{W_i} C_{jk}^m \mathcal{O}_{W_m} \\ \Longleftrightarrow\ & C_{ij}^l C_{lk}^n \mathcal{O}_{W_n} = C_{jk}^m C_{im}^n \mathcal{O}_{W_n} \\ \Longleftrightarrow\ & C_{ij}^l C_{lk}^n = C_{jk}^m C_{im}^n \\ \Longleftrightarrow\ & C_{ijm} \eta^{lm} C_{lkn} = C_{jkm} \eta^{lm} C_{imn} \end{aligned} \tag{4.2.7}$$

And there exists a free energy (prepotential) $F_M(t_1, \ldots, t_D)$ which satisfies following conditions.

$$C_{ijk}(t_1, t_2, \ldots, t_D) = \partial_i \partial_j \partial_k F_M \tag{4.2.8}$$
$$(\partial_i := \frac{\partial}{\partial t_i}) \tag{4.2.9}$$

Combining (4.2.7) and (4.2.9), we obtain a series of partial differential equations for F_M.

$$\eta^{lm} \partial_i \partial_j \partial_m F_M \partial_l \partial_k \partial_n F_M = \eta^{lm} \partial_j \partial_k \partial_m F_M \partial_i \partial_l \partial_n F_M \tag{4.2.10}$$

We can also consider prepotential as the generating function of all topological correlation functions.

$$F_M(t_1, \ldots, t_D) = \sum_{n_1, \cdots, n_D \geq 0} \langle \mathcal{O}_{W_1}^{n_1} \cdots \mathcal{O}_{W_D}^{n_D} \rangle \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_D^{n_D}}{n_D!} \tag{4.2.11}$$

($\mathcal{O}_{W_i}^{n_i}$ represents $\overbrace{\mathcal{O}_{W_i} \cdots \mathcal{O}_{W_i}}^{n_i \text{ times}}$ and should not be confused with $\mathcal{O}_{(W_i)^{n_i}}$). At the topological point (i.e., all the t_i's are set to zero) correlation functions become intersection numbers on moduli spaces of holomorphic maps from CP^1 (with k- marked points) to target space M.

Holomorphic maps f are characterized by their degree which equal the intersection number of $f^*(CP^1)$ with the Kähler form of target space M. Then $\langle \mathcal{O}_{W_1}^{n_1} \cdots \mathcal{O}_{W_D}^{n_D} \rangle$ remains non-zero only when the following topological selection rule is satisfied.

$$\begin{aligned} & \sum_{i=1}^{D} n_i dim_C(W_i) = \sum_{i=1}^{D} n_i - 3 + dim H^0(CP^1, \phi^*(T'M)) \\ \Longleftrightarrow & \sum_{i=1}^{D} n_i (dim_C(W_i) - 1) = -3 + dc_1(T'M) + dim_C(M) \end{aligned} \tag{4.2.12}$$

Here d is the degree of holomorphic map and we used Riemann–Roch theorem in deriving the second line from the first one. When d equals zero, f is a constant map to the target space and moduli space becomes just the direct product of target space and moduli space of CP^1 with $\sum_{i=1}^{D} n_i$ punctures. Then selection rules (4.2.12) decomposes to

$$\sum_{i=1}^{D} n_i dim_C(W_i) = dim_C(M) \tag{4.2.13}$$

and

$$\sum_{i=1}^{D} n_i = 3 \tag{4.2.14}$$

From (4.2.14), we conclude that in $d=0$ case only 3-point functions survive and correlation functions are just classical intersection numbers $\int_M W_i \wedge W_j \wedge W_k$.

From the flat metric condition, insertion of W_1 remains non-zero only for three point functions from constant maps because one and two point functions including W_1 cannot remein nonzero when $c_1(M) \geq 1$ and $d \geq 1$ and if we suppose $n(\geq 3)$ point functions remain nonzero in $d \geq 1$ sector, flat metric condition is broken(in three point functions in $d \geq 1$ sector, we take into account of the insertion of operator $\mathcal{O}_{W_2}(dim_C(W_2)=1)$ which we will discuss later).

With these considerations, expansion of the free energy becomes

$$F_M(t_1,\ldots,t_D) = \frac{1}{6}\int_M (\sum_{i=1}^{D} t_i W_i)^3 + \sum_{d=1}^{\infty}\sum_{n_2,\cdots,n_D\geq 0} \frac{{t_2}^{n_2}}{n_2!}\cdots\frac{{t_D}^{n_D}}{n_D!}\langle \mathcal{O}_{W_2}^{n_2}\cdots\mathcal{O}_{W_D}^{n_D}\rangle$$
$$(\sum_{i=2}^{D} n_i(dim_C(W_i)-1) = -3 + dc_1(T'M) + dim_C(M)) \quad (4.2.15)$$

(where the product of the first term of r.h.s. means taking the wedge product of $H^*(M)$).

Next, we consider the insertion of the operator $\mathcal{O}_{W_2}$ which corresponds to the Kähler form of the target space . Since $codim_C(PD(W_2))=1$, holomorphic map f of degree d always intersects with it in d-points,and the condition $f(CP^1)\cap PD(W_2) \neq \emptyset$ imposes no constraint. Then from (4.2.9) the insertion of $\mathcal{O}_{W_2}$ results in the multiplication by a factor d,

$$\langle \mathcal{O}_{W_2}^{n_2}\mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle = d^{n_2}\langle \mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle \quad (4.2.16)$$

Combining (4.2.15) and (4.2.16) we obtain the following expansion

$$F_M(t_1,\ldots,t_D) = \frac{1}{6}\int_M (\sum_{i=1}^{D} t_i W_i)^3 + \sum_{d=1}^{\infty}\sum_{n_3,\cdots,n_D\geq 0} \frac{{t_3}^{n_3}}{n_3!}\cdots\frac{{t_D}^{n_D}}{n_D!}\langle \mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle e^{dt_2}$$
$$(\sum_{i=3}^{D} n_i(dim_C(W_i)-1) = -3 + dc_1(T'M) + dim_C(M)) \quad (4.2.17)$$

Then by combining (4.2.10) and (4.2.17), we determine the correlation functions in the case where target space is CP^3, CP^4 and $Gr(2,4)$.

4.2.3 The Calculations

The non zero Betti numbers of CP^3, CP^4 and $Gr(2,4)$ are

$$b_{00} = b_{11} = b_{22} = b_{33} = 1 \quad (4.2.1)$$

$$b_{00} = b_{11} = b_{22} = b_{33} = b_{44} = 1 \quad (4.2.2)$$

$$b_{00} = b_{11} = b_{33} = b_{44} = 1, \qquad b_{22} = 2 \quad (4.2.3)$$

respectively. By means of wedge product we obtain an associative, commutative ring $H^*(M,Q)$ for each manifold M.For CP^3, CP^4, and $Gr(2,4)$ we have the multiplication table as follows.

Table 4.1: **The ring of** CP^3

	W_1	W_2	W_3	W_4
W_1	W_1	W_2	W_3	W_4
W_2	W_2	W_3	W_4	0
W_3	W_3	W_4	0	0
W_4	W_4	0	0	0

$dim_C(W_1)$=0 $dim_C(W_2)$=1

$dim_C(W_3)$=2 $dim_C(W_4)$=3

Table 4.2: **The ring of** CP^4

	W_1	W_2	W_3	W_4	W_5
W_1	W_1	W_2	W_3	W_4	W_5
W_2	W_2	W_3	W_4	W_5	0
W_3	W_3	W_4	W_5	0	0
W_4	W_4	W_5	0	0	0
W_5	W_5	0	0	0	0

$dim_C(W_1)$=0 $dim_C(W_2)$=1

$dim_C(W_3)$=2 $dim_C(W_4)$=3 $dim_C(W_5)$=4

Table 4.3: **The ring of** $Gr(2,4)$

	W_1	W_2	W_3	W_4	W_5	W_6
W_1	W_1	W_2	W_3	W_4	W_5	W_6
W_2	W_2	W_3+W_4	W_5	W_5	W_6	0
W_3	W_3	W_5	W_6	0	0	0
W_4	W_4	W_5	0	0	0	0
W_5	W_5	W_6	0	0	0	0
W_6	W_6	0	0	0	0	0

$dim_C(W_1)$=0 $dim_C(W_2)$=1 $dim_C(W_3)$=2

$dim_C(W_4)$=2 $dim_C(W_5)$=3 $dim_C W_6$=4

Dual to each cohomology class is a class of cycles (e.g. for the case of CP^3 W_3 is dual to a point, W_2 is dual to a line, W_1 is dual to the CP^3).

As a point intersects the CP^3 in a point and a line intersects the plane by a point. Thus we have for CP^3,

$$< W_1, W_4 >= 1, < W_2, W_3 >= 1 \tag{4.2.4}$$

For CP^4

$$< W_1, W_5 >= 1, \quad < W_2, W_4 >= 1, \quad < W_3, W_3 >= 1 \tag{4.2.5}$$

and for Gr(2,4) it becomes

$$< W_1, W_6 >= 1, \quad < W_2, W_5 >= 1 \tag{4.2.6}$$

$$< W_3, W_3 >= 1, \quad < W_4, W_4 >= 1 \tag{4.2.7}$$

All other intersections on generators being zero. The CP^3, CP^4 ring can be identified the ring of polynomials in one indeterminate C[x] modulo the gradient of

$$W(x) = x^4/4, \qquad W(x) = x^5/5 \tag{4.2.8}$$

Ine the case of Grassmanians their cohomology $H^*(Gr, Q)$ can't be generated by $H^2(Gr, Q)$. The cohomology ring of Grassmanian $Gr(2,4)$ for instance, can be written as the singularity ring generated by a single potential[26]

$$W(x_i) = \frac{1}{5}x_1^5 - x_1^3 x_2 + x_2^2 x_1 \tag{4.2.9}$$

Where x_1 correspond to W_2 and x_2 correspond to $\frac{1}{2}(W_3 + W_4)$.

From (3.6) one can split F_M into a classical part and instanton correction part as

$$F_M = f_{cl} + f_M \tag{4.2.10}$$

So for CP^3, CP^4 and $Gr(2,4)$ we have

$$F_{CP^3} = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + \frac{1}{6}t_2^3 + f_{CP^3}(t_2, t_3, t_4) \tag{4.2.11}$$

$$F_{CP^4} = \frac{1}{2}t_1^2 t_5 + \frac{1}{2}t_1 t_4^2 + \frac{1}{2}t_3^2 t_4 + t_1 t_2 t_5 + f_{CP^4}(t_2, t_3, t_4, t_5) \tag{4.2.12}$$

$$F_{Gr(2,4)} = \frac{1}{2}t_1^2 t_6 + \frac{1}{2}t_1 t_3^2 + \frac{1}{2}t_1 t_4^2 + t_1 t_1 t_5 + \frac{1}{2}t_2^2 t_3 + \frac{1}{2}t_2^2 t_4 + f_{Gr(2,4)}(t_2, t_3, t_4, t_5, t_6) \tag{4.2.13}$$

From (2.7) the Riemann-Roch theorem tell us

$$dimH^0(CP^1, f^*(T'M)) - 3 = (dimM - 3) + dc_1(T'M) \tag{4.2.14}$$

Once we specify the target space , we know its frist Chern class, then the above formula give the dimension of its moduli space. In case of CP^3 $c_1(T'CP^3) = 4$ so,

$$dimH^0(CP^1, f^*(T'CP^3)) - 3 = 4d \tag{4.2.15}$$

For CP^4 and $Gr(2,4)$, the frist Chern class are

$$c_1(T'CP^4) = 5, \qquad c_1(T'Gr(2,4)) = 4 \tag{4.2.16}$$

Thus

$$dimH^0(CP^1, f^*(T'CP^4)) - 3 = 5d + 1 \tag{4.2.17}$$

$$dimH^0(CP^1, f^*(T'Gr(2,4))) - 3 = 4d + 1 \tag{4.2.18}$$

From (4.2.17) we can expond f_M further as follows

$$f_{CP^3} = \sum_{d=1}^{\infty} \sum_{n_3+2n_4=4d} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} >}{n_3! n_4!} t_3^{n_3} t_4^{n_4} e^{dt_2}$$

$$= \sum_{d=1}^{\infty} \sum_{n_4} \frac{< \mathcal{O}_{W_3}^{4d-2n_4} \mathcal{O}_{W_4}^{n_4} >}{(4d-2n_4)! n_4!} t_3^{4d-2n_4} t_4^{n_4} e^{dt_2} \tag{4.2.19}$$

$$f_{CP^4} = \sum_{d=1}^{\infty} \sum_{n_3+2n_4+3n_5=5d+1} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >}{n_3! n_4! n_5!} t_3^{n_3} t_4^{n_4} t_5^{n_5} e^{dt_2}$$

$$= \sum_{d=1}^{\infty} \sum_{n_4, n_5} \frac{< \mathcal{O}_{W_3}^{5d-2n_4-3n_5+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >}{(5d-2n_4-3n_5+1)! n_4! n_5!} t_3^{5d-2n_4-3n_5+1} t_4^{n_4} t_5^{n_5} e^{dt_2} \tag{4.2.20}$$

$$f_{Gr(2,4)} = \sum_{d=1}^{\infty} \sum_{n_3+n_4+2n_5+3n_6=4d+1} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >}{n_3! n_4! n_5! n_6!} t_3^{n_3} t_4^{n_4} t_5^{n_5} t_6^{n_6} e^{dt_2}$$

$$= \sum_{d=1}^{\infty} \sum_{n_4, n_5, n_6} \frac{< \mathcal{O}_{W_3}^{4d-n_4-2n_5-3n_6+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >}{(4d-n_4-2n_5-3n_6+1)! n_4! n_5!} t_3^{4d-n_4-2n_5-3n_6+1} t_4^{n_4} t_5^{n_5} t_6^{n_6} e^{dt_2} \tag{4.2.21}$$

Then, we abbreviate the notion in the following calculation as

$$< \mathcal{O}_{W_3}^{4d-2n_4} \mathcal{O}_{W_4}^{n_4} >_{CP^3} = N_{n_4}^d \tag{4.2.22}$$

$$< \mathcal{O}_{W_3}^{5d-2n_4-3n_5+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >_{CP^4} = N_{n_4, n_5}^d \tag{4.2.23}$$

$$< \mathcal{O}_{W_3}^{4d-n_4-2n_5-3n_6+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >_{Gr(2,4)} = N_{n_4, n_5, n_6}^d \tag{4.2.24}$$

We let $t_2 = x, t_3 = y, t_4 = z$ for CP^3,$t_2 = w, t_3 = x, t_4 = y, t_5 = z$ for CP^4 and $t_2 = v, t_3 = w, t_4 = x, t_5 = y, t_6 = z$ for $Gr(2,4)$. A deformation of the multiplication table (table 1, table 2, table 3) become the fusion rules for the quantum cohomology ring with $\mathcal{O}_{W_i}$'s substituted for t's as

$$\mathcal{O}_{W_i} \circ \mathcal{O}_{W_j} = \partial_i \partial_j \partial_l F_M \eta^{lk} \mathcal{O}_{W_k} \tag{4.2.25}$$

The structure constants of the quantum cohomology obey the so called WDVV equation which satisfying the requirements[3]

(i)commutativity

(ii)associativity

(iii)existence of a unit $\mathcal{O}_{W_1}$

Commutativity follows from the definition, while condition(3.6) (equivalently(3.17) expresses that $\mathcal{O}_{W_1}$ plays the role of unit. The crucial assumption is the associativity which imposes strong conditions on f_M. Now let us introduce some more notations, by $f_{M,xyz}$ we mean $\partial_x \partial_y \partial_z f_M$. In the following we will simply omit the index "M", and just denote it as f_{xyz}.

The quantum ring of CP^3 is

$$\begin{aligned}
\mathcal{O}_{W_2}\mathcal{O}_{W_2} &= f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + (f_{xxx}+1)\mathcal{O}_{W_3}, && (4.2.26a)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_3} &= f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + (f_{xxy})\mathcal{O}_{W_3} + \mathcal{O}_{W_4}, && (4.2.26b)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_4} &= f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + (f_{xxz})\mathcal{O}_{W_3}, && (4.2.26c)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_3} &= f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + (f_{xyy})\mathcal{O}_{W_3}, && (4.2.26d)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_4} &= f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + (f_{xyz})\mathcal{O}_{W_3}, && (4.2.26e)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_4} &= f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + (f_{xzz})\mathcal{O}_{W_3}. && (4.2.26f)
\end{aligned}$$

The quantum ring of CP^4 is

$$\begin{aligned}
\mathcal{O}_{W_2}\mathcal{O}_{W_2} &= f_{wwz}\mathcal{O}_{W_1} + f_{wwy}\mathcal{O}_{W_2} + f_{wwx}\mathcal{O}_{W_3} + (f_{www}+1)\mathcal{O}_{W_4}, && (4.2.27a)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_3} &= f_{wxz}\mathcal{O}_{W_1} + f_{wxy}\mathcal{O}_{W_2} + f_{wxx}\mathcal{O}_{W_3} + f_{wwx}\mathcal{O}_{W_4}, && (4.2.27b)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_4} &= f_{wyz}\mathcal{O}_{W_1} + f_{wyy}\mathcal{O}_{W_2} + f_{wxy}\mathcal{O}_{W_3} + f_{wwy}\mathcal{O}_{W_4} + \mathcal{O}_{W_5}, && (4.2.27c)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_5} &= f_{wzz}\mathcal{O}_{W_1} + f_{wyz}\mathcal{O}_{W_2} + f_{wxz}\mathcal{O}_{W_3} + f_{wwz}\mathcal{O}_{W_4}, && (4.2.27d)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_3} &= f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + f_{xxx}\mathcal{O}_{W_3} + f_{wxx}\mathcal{O}_{W_4} + \mathcal{O}_{W_5}, && (4.2.27e)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_4} &= f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + f_{xxy}\mathcal{O}_{W_3} + f_{wxy}\mathcal{O}_{W_4}, && (4.2.27f)
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_{W_3}\mathcal{O}_{W_5} &= f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + f_{xxz}\mathcal{O}_{W_3} + f_{wxz}\mathcal{O}_{W_4}, && (4.2.27g)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_4} &= f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + f_{xyy}\mathcal{O}_{W_3} + f_{wyy}\mathcal{O}_{W_4}, && (4.2.27h)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_5} &= f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + f_{xyz}\mathcal{O}_{W_3} + f_{wyz}\mathcal{O}_{W_4}, && (4.2.27i)\\
\mathcal{O}_{W_5}\mathcal{O}_{W_5} &= f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + f_{xzz}\mathcal{O}_{W_3} + f_{wzz}\mathcal{O}_{W_4}. && (4.2.27j)
\end{aligned}$$

The quantum ring of $Gr(2,4)$ is

$$\begin{aligned}
\mathcal{O}_{W_2}\mathcal{O}_{W_2} &= f_{vvz}\mathcal{O}_{W_1} + f_{vvy}\mathcal{O}_{W_2} + (f_{vvw}+1)\mathcal{O}_{W_3}\\
&+ (f_{vvx}+1)\mathcal{O}_{W_4} + f_{vvv}\mathcal{O}_{W_5}, && (4.2.28a)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_3} &= f_{vwz}\mathcal{O}_{W_1} + f_{vwy}\mathcal{O}_{W_2} + f_{vww}\mathcal{O}_{W_3}\\
&+ f_{vwx}\mathcal{O}_{W_4} + (f_{vvw}+1)\mathcal{O}_{W_5}, && (4.2.28b)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_4} &= f_{vxz}\mathcal{O}_{W_1} + f_{vxy}\mathcal{O}_{W_2} + f_{vwx}\mathcal{O}_{W_3} + f_{vxx}\mathcal{O}_{W_4} + (f_{vvx}+1)\mathcal{O}_{W_5}, && (4.2.28c)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_5} &= f_{vyz}\mathcal{O}_{W_1} + f_{vyy}\mathcal{O}_{W_2} + f_{vwy}\mathcal{O}_{W_3} + f_{vxy}\mathcal{O}_{W_4} + f_{vvy}\mathcal{O}_{W_5}, && (4.2.28d)\\
\mathcal{O}_{W_2}\mathcal{O}_{W_6} &= f_{vzz}\mathcal{O}_{W_1} + f_{vyz}\mathcal{O}_{W_2} + f_{vwz}\mathcal{O}_{W_3} + f_{vxz}\mathcal{O}_{W_4} + f_{vvz}\mathcal{O}_{W_5}, && (4.2.28e)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_3} &= f_{wwz}\mathcal{O}_{W_1} + f_{wwy}\mathcal{O}_{W_2} + f_{www}\mathcal{O}_{W_3}\\
&+ f_{wwx}\mathcal{O}_{W_4} + f_{vww}\mathcal{O}_{W_5} + \mathcal{O}_{W_6}, && (4.2.28f)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_4} &= f_{wxz}\mathcal{O}_{W_1} + f_{wxy}\mathcal{O}_{W_2} + f_{wwx}\mathcal{O}_{W_3} + f_{wxx}\mathcal{O}_{W_4} + f_{vwx}\mathcal{O}_{W_5}, && (4.2.28g)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_5} &= f_{wyz}\mathcal{O}_{W_1} + f_{wyy}\mathcal{O}_{W_2} + f_{wwy}\mathcal{O}_{W_3} + f_{wxy}\mathcal{O}_{W_4} + f_{vwy}\mathcal{O}_{W_5}, && (4.2.28h)\\
\mathcal{O}_{W_3}\mathcal{O}_{W_6} &= f_{wzz}\mathcal{O}_{W_1} + f_{wyz}\mathcal{O}_{W_2} + f_{wwz}\mathcal{O}_{W_3} + f_{wxz}\mathcal{O}_{W_4} + f_{vwz}\mathcal{O}_{W_5}, && (4.2.28i)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_4} &= f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + f_{wxx}\mathcal{O}_{W_3}\\
&+ f_{xxx}\mathcal{O}_{W_4} + f_{vxx}\mathcal{O}_{W_5} + \mathcal{O}_{W_6}, && (4.2.28j)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_5} &= f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + f_{wxy}\mathcal{O}_{W_3} + f_{xxy}\mathcal{O}_{W_4} + f_{vxy}\mathcal{O}_{W_5}, && (4.2.28k)\\
\mathcal{O}_{W_4}\mathcal{O}_{W_6} &= f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + f_{wxz}\mathcal{O}_{W_3} + f_{xxz}\mathcal{O}_{W_4} + f_{vxz}\mathcal{O}_{W_5}, && (4.2.28l)\\
\mathcal{O}_{W_5}\mathcal{O}_{W_5} &= f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + f_{wyy}\mathcal{O}_{W_3} + f_{xyy}\mathcal{O}_{W_4} + f_{vyy}\mathcal{O}_{W_5}, && (4.2.28m)\\
\mathcal{O}_{W_5}\mathcal{O}_{W_6} &= f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + f_{wyz}\mathcal{O}_{W_3} + f_{xyz}\mathcal{O}_{W_4} + fvyz\mathcal{O}_{W_5}, && (4.2.28n)
\end{aligned}$$

$$\mathcal{O}_{W_6}\mathcal{O}_{W_6} = f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + f_{wzz}\mathcal{O}_{W_3} + f_{xzz}\mathcal{O}_{W_4} + f_{vzz}\mathcal{O}_{W_5}. \quad (4.2.28o)$$

Associativity condition (3.7) implies the free energy of CP^3 must satifying the following constraint equation

$$-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxz} = 0, \quad (4.2.29a)$$

$$-f_{xxz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0, \quad (4.2.29b)$$

$$f_{yzz} - f_{xxz}f_{yyy} + f_{xxy}f_{yyz} = 0, \quad (4.2.29c)$$

$$-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0, \quad (4.2.29d)$$

$$f_{zzz} - f_{xyz}^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0, \quad (4.2.29e)$$

$$f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0. \quad (4.2.29f)$$

For CP^4 there are 17 independent constraint equations. We just write down five of them which are enough to determine the correlation functions of CP^4

$$-f_{wwz} - f_{wwy}f_{wxx} + f_{wxx}^2 + 2f_{wwx}f_{xxx} - f_{www}f_{xxy} = 0, \quad (4.2.30a)$$

$$f_{wxy}^2 + f_{wwy}f_{wyy} + 2f_{wyz} - f_{wwx}f_{xyy} - f_{www}f_{yyy} = 0, \quad (4.2.30b)$$

$$f_{wxy}f_{wxz} + f_{wwz}f_{wyy} + f_{wzz} - f_{wwx}f_{xyz} - f_{www}f_{yyz} = 0, \quad (4.2.30c)$$

$$-f_{wxz}f_{xyy} + f_{wxy}f_{xyz} - f_{wwz}f_{yyy} + f_{wwy}f_{yyz} + f_{yzz} = 0, \quad (4.2.30d)$$

$$f_{wxy}f_{xxy} + f_{wwy}f_{xyy} - f_{wxx}f_{xyy} + f_{xyz} - f_{wwx}f_{yyy} = 0. \quad (4.2.30e)$$

For the case of $Gr(2,4)$ there are fifty-six independent equations. We also write down nine of them that determine the corelation functions of $Gr(2,4)$

$$\begin{aligned} &-f_{vvz} - f_{vvy}f_{vww} + f_{vww}^2 + f_{vwx}^2 + 2f_{vvw}f_{vwy} \\ &-f_{vvw}f_{www} - f_{vvx}f_{wwx} - f_{vvv}f_{wwy} = 0, \end{aligned} \quad (4.2.31a)$$

$$\begin{aligned} &-f_{vvz} - f_{vvy}f_{vxx} + f_{vwx}^2 + f_{vxx}^2 + 2f_{vvx}f_{vxy} \\ &-f_{vvw}f_{wxx} - f_{vvx}f_{xxx} - f_{vvv}f_{xxy} = 0, \end{aligned} \quad (4.2.31b)$$

$$\begin{aligned} &-f_{vxz} - f_{vww}f_{vxy} + f_{vwx}f_{vwy} - f_{vwx}f_{www} + f_{vww}f_{wwx} \\ &-f_{vxx}f_{wwx} - f_{vvx}f_{wwy} + f_{vwx}f_{wxx} + f_{vvw}f_{wxy} = 0, \end{aligned} \quad (4.2.31c)$$

$$\begin{aligned} &-f_{wwz} - f_{xxz} - f_{vxx}f_{wwy} + f_{wwx}^2 + f_{wxx}^2 + 2f_{vwx}f_{wxy} \\ &-f_{www}f_{wxx} - f_{wwx}f_{xxx} - f_{vww}f_{xxy} = 0, \end{aligned} \quad (4.2.31d)$$

$$\begin{aligned} &-f_{xyz} - f_{vxy}f_{wwy} + f_{wwx}f_{wwy} + f_{vwy}f_{wxy} + f_{wxx}f_{wxy} \\ &-f_{www}f_{wxy} + f_{vwx}f_{wyy} - f_{wwx}f_{xxy} - f_{vww}f_{xyy} = 0, \end{aligned} \quad (4.2.31e)$$

$$\begin{aligned} &-f_{xxz} - f_{vxz}f_{wwy} + f_{wwx}f_{wwz} + f_{vwz}f_{wxy} + f_{wxx}f_{wxz} \\ &-f_{www}f_{wxz} + f_{vwx}f_{wyz} - f_{wwx}f_{xxz} - f_{vww}f_{xyz} = 0, \end{aligned} \quad (4.2.31f)$$

$$\begin{aligned} &f_{wxy} + f_{vwy}f_{wwy} + f_{vxy}f_{wxy} + f_{vvy}f_{wyy} \\ &-f_{vww}f_{wyy} - f_{vwx}f_{xyy} - f_{vvw}f_{yyy} = 0, \end{aligned} \quad (4.2.31g)$$

$$\begin{aligned} &f_{wxz} - f_{vwy}f_{vxy} + f_{vwx}f_{vyy} + f_{vwy}f_{wwx} + f_{vxy}f_{wxx} \\ &-f_{vwx}f_{wwy} + f_{vvy}f_{wxy} - f_{vxx}f_{wxy} - f_{vvx}f_{wyy} = 0, \end{aligned} \quad (4.2.31h)$$

$$\begin{aligned} &-f_{yyz} + f_{vwz}f_{wyy} - f_{vwy}f_{wyz} + f_{vxz}f_{xyy} \\ &+f_{vvz}f_{yyy} - f_{vxy}f_{xyz} - f_{vvy}f_{yyz} = 0. \end{aligned} \quad (4.2.31i)$$

Substituting the free energy (4.2.19–4.2.21) into (4.29),(4.30) and (4.31) one obtains the recursion relations of correlation functions. For CP^3 one has

$$2dN^d_{m+1} - N^d_m = \sum_{\substack{f+g=d \\ n+n'=m}} \binom{m}{n} \left[-\binom{4d-2m-3}{4f-2n-2} f N^f_n N^g_{n'} + \binom{4d-2m-3}{4f-2n-3} g^3 N^f_n N^g_{n'}\right], \quad (4.2.32a)$$

$$dN^d_{m+2} - N^d_{m+1} = \sum_{\substack{f+g=d \\ n+n'=m}} \binom{m}{n} \left[\binom{4d-2m-4}{4f-2n-4} g^3 N^f_{n+1} N^g_{n'} - \binom{4d-2m-4}{4f-2n-2} f g^2 N^f_n N^g_{n'+1}\right] \quad (4.2.32b)$$

$$N^d_{m+2} = \sum_{\substack{f+g=d \\ n+n'=m}} \binom{m}{n} \left[\binom{4d-2m-5}{4f-2n-2} f^2 N^f_{n+1} N^g_{n'} - \binom{4d-2m-5}{4f-2n-1} f^2 N^f_n N^g_{n'+1}\right]. \quad (4.2.32c)$$

For the case of CP^4 the recursion relations read as follows

$$d^2 N^d_{m_1,m_2+1} - 2dN^d_{m_1,m_2} + N^d_{m_1,m_2}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Bigg[-\binom{5d-2m_1-3m_2-2}{5f-2n_1-3n_2-1} f^2 g N^f_{n_1+1,n_2} N^g_{n_1',n_2'}$$

$$+\binom{5d-2m_2-3m_3-2}{5f-2n_1-3n_2-1} fg N^f_{n_1,n_2} N^g_{n_1',n_2'}$$

$$+2\binom{5d-2m_1-3m_2-2}{5f-2n_1-3n_2} f^2 g N^f_{n_1,n_2} N^g_{n_1'+1,n_2'}$$

$$-\binom{5d-2m_1-3m_2-2}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2} N^g_{n_1',n_2'}$$

$$-\binom{5d-2m_1-3m_2-2}{5f-2n_1-3n_2+1} f^3 N^f_{n_1,n_2} N^g_{n_1',n_2'+1}\Bigg], \tag{4.2.33a}$$

$$N^d_{m_1+1,m_2+1} - dN^d_{m_1,m_2+2}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Bigg[\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2-2} fg N^f_{n_1+1,n_2} N^g_{n_1',n_2'+1}$$

$$+\binom{5d-2m_2-3m_3-5}{5f-2n_1-3n_2-2} f^2 g N^f_{n_1,n_2+1} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2} N^g_{n_1'+1,n_2'+1}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2+1} f^3 N^f_{n_1,n_2} N^g_{n_1'+2,n_2'+1}\Bigg], \tag{4.2.33b}$$

$$N^d_{m_1+2,m_2} - 2dN^d_{m_1+1,m_2+1}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Bigg[\binom{5d-2m_1-3m_2-4}{5f-2n_1-3n_2-2} fg N^f_{n_1+1,n_2} N^g_{n_1'+1,n_2'}$$

$$+\binom{5d-2m_2-3m_3-4}{5f-2n_1-3n_2-1} f^2 g N^f_{n_1+1,n_2} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-4}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-4}{5f-2n_1-3n_2+1} f^3 N^f_{n_1,n_2} N^g_{n_1'+3,n_2'}\Bigg], \tag{4.2.33c}$$

$$N^d_{m_1+1,m_2+2}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Bigg[\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2-3} f N^f_{n_1,n_2+1} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_2-3m_3-7}{5f-2n_1-3n_2-2} f N^f_{n_1+1,n_2} N^g_{n_1'+1,n_2'+1}$$

$$+\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2+1} N^g_{n_1'+3,n_2'}$$

$$-\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2-1} f^2 N^f_{n_1+1,n_2} N^g_{n_1'+2,n_2'+1}\Bigg]. \tag{4.2.33d}$$

For the case of $Gr(2,4)$ the recursion relation of becomes

$$N^d_{m_1+3,m_2} - N^d_{m_1+1,m_2+1}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Bigg[\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2-2} f N^f_{n_1+1,n_2} N^g_{n_1'+1,n_2'}$$

$$+\binom{5d-2m_2-3m_3-5}{5f-2n_1-3n_2-1} f^2 N^f_{n_1+1,n_2} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2-1} f N^f_{n_1,n_2} N^g_{n'_1+2,n'_2}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2} N^g_{n'_1+3,n'_2}\Big], \tag{4.2.34a}$$

$$d^2 N^d_{m_1,m_2,m_3+1} - 2d N^d_{m_1,m_2+1,m_3} + N^d_{m_1,m_2,m_3} + N^d_{m_1+1,m_2,m_3}$$

$$= \sum_{\substack{f+g=d, n_1+n'_1=m_1 \\ n_2+n'_2=m_2, n_3+n'_3=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} f^2 g N^f_{n_1,n_2+1,n_3} N^g_{n'_1,n'_2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1,n_2,n_3} N^g_{n'_1,n'_2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$+2\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 g N^f_{n_1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1,n_2,n_3} N^g_{n'_1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3+1} f^3 N^f_{n_1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}\Big], \tag{4.2.34b}$$

$$d^2 N^d_{m_1,m_2,m_3+1} - 2d N^d_{m_1+1,m_2+1,m_3} + N^d_{m_1+2,m_2,m_3} + N^d_{m_1+3,m_2,m_3}$$

$$= \sum_{\substack{f+g=d, n_1+n'_1=m_1 \\ n_2+n'_2=m_2, n_3+n'_3=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} f^2 g N^f_{n_1,n_2+1,n_3} N^g_{n'_1+2,n'_2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1+2,n_2,n_3} N^g_{n'_1+2,n'_2,n'_3}$$

$$+2\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 g N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1,n_2,n_3} N^g_{n'_1+2,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1+1,n_2,n_3} N^g_{n'_1+3,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3+1} f^3 N^f_{n_1,n_2,n_3} N^g_{n'_1+2,n'_2+1,n'_3}\Big], \tag{4.2.34c}$$

$$d N^d_{m_1+1,m_2,m_3+1} + N^d_{m_1,m_2+1,m_3} - N^d_{m_1+1,m_2+1,m_3}$$

$$= \sum_{\substack{f+g=d, n_1+n'_1=m_1 \\ n_2+n'_2=m_2, n_3+n'_3=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} fg N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+2,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+1,n_2,n_3} N^g_{n_1'+2,n_2',n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-3}{4f-n_1-2n_2-3n_3} f^2 N^f_{n_1,n_2,n_3} N^g_{n_1'+1,n_2'+1,n_3'}\Bigg], \qquad (4.2.34d)$$

$$N^d_{m_1,m_2,m_3+1} + N^d_{m_1+2,m_2,m_3+1}$$

$$= \sum_{\substack{f+g=d, n_1+n_1'=m_1\\ n_2+n_2'=m_2, n_3+n_3'=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Bigg[\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+1,n_2,n_3} N^g_{n_1'+1,n_2',n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+2,n_2,n_3} N^g_{n_1',n_2'+1,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2} N^f_{n_1,n_2,n_3} N^g_{n_1'+2,n_2',n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+2,n_2,n_3} N^g_{n_1'+2,n_2',n_3'}$$

$$+2\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+1,n_2,n_3} N^g_{n_1'+1,n_2'+1,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+1,n_2,n_3} N^g_{n_1'+3,n_2',n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1,n_2,n_3} N^g_{n_1'+2,n_2'+1,n_3'}\Bigg], \qquad (4.2.34e)$$

$$N^d_{m_1+1,m_2+1,m_3+1}$$

$$= \sum_{\substack{f+g=d, n_1+n_1'=m_1\\ n_2+n_2'=m_2, n_3+n_3'=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Bigg[-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} f N^f_{n_1+1,n_2+1,n_3} N^g_{n_1',n_2'+1,n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+1,n_2,n_3} N^g_{n_1',n_2'+1,n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} f N^f_{n_1,n_2+1,n_3} N^g_{n_1'+1,n_2'+1,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} N^f_{n_1,n_2,n_3} N^g_{n_1'+1,n_2'+1,n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+2,n_2,n_3} N^g_{n_1'+1,n_2'+1,n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1+1,n_2,n_3} N^g_{n_1',n_2'+2,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+1,n_2,n_3} N^g_{n_1'+2,n_2'+1,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-1} f N^f_{n_1,n_2,n_3} N^g_{n_1'+1,n_2'+2,n_3'}\Bigg], \qquad (4.2.34f)$$

$$N^d_{m_1+1,m_2,m_3+2}$$

$$= \sum_{\substack{f+g=d, n_1+n_1'=m_1\\ n_2+n_2'=m_2, n_3+n_3'=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Bigg[-\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-3} f N^f_{n_1+1,n_2,n_3+1} N^g_{n_1',n_2'+1,n_3'}$$

$$+\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-2} N^f_{n_1+1,n_2,n_3} N^g_{n_1',n_2',n_3'+1}$$

$$+\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-3} f N^f_{n_1,n_2,n_3+1} N^g_{n_1'+1,n_2'+1,n_3'}$$

$$-\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-2} N^f_{n_1,n_2,n_3} N^g_{n_1'+1,n_2',n_3'+1}$$

$$+\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-2}N^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2,n'_3+1}$$

$$+\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+1,n'_3+1}$$

$$-\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-2}N^f_{n_1+1,n_2,n_3}N^g_{n'_1+2,n'_2,n'_3+1}$$

$$-\binom{4d-m_1-2m_2-3m_3-6}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1,n_2,n_3}N^g_{n'_1+1,n'_2+1,n'_3+1}\Big], \qquad (4.2.34g)$$

$$N^d_{m_1,m_2+3,m_3}-N^d_{m_1,m_2+1,m_3+1}$$

$$=\sum_{\substack{f+g=d,n_1+n'_1=m_1\\ n_2+n'_2=m_2,n_3+n'_3=m_3}}\binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2}fN^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+1,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-2}fN^f_{n_1+1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-1}f^2N^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2+2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-5}{4f-n_1-2n_2-3n_3}f^2N^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}\Big], \qquad (4.2.34h)$$

$$N^d_{m_1,m_2+2,m_3}-N^d_{m_1+1,m_2,m_3+1}$$

$$=\sum_{\substack{f+g=d,n_1+n'_1=m_1\\ n_2+n'_2=m_2,n_3+n'_3=m_3}}\binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2}fgN^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1}fgN^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2}fN^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+1,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-2}fN^f_{n_1+1,n_2+1,n_3}N^g_{n'_1+2,n'_2,n'_3}$$

$$+\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1}f^2N^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3-1}fN^f_{n_1+2,n_2,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-4}{4f-n_1-2n_2-3n_3}f^2N^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}\Big], \qquad (4.2.34i)$$

$$N^d_{m_1,m_2+1,m_3+2}$$

$$=\sum_{\substack{f+g=d,n_1+n'_1=m_1\\ n_2+n'_2=m_2,n_3+n'_3=m_3}}\binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[\binom{4d-m_1-2m_2-3m_3-7}{4f-n_1-2n_2-3n_3-3}fN^f_{n_1,n_2,n_3+1}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\binom{4d-m_1-2m_2-3m_3-7}{4f-n_1-2n_2-3n_3-2}fN^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+1,n'_3+1}$$

$$
+\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-3 \end{pmatrix} fN^{f}_{n_1+1,n_2,n_3+1}N^{g}_{n'_1+1,n'_2+2,n'_3}
$$

$$
-\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^{f}_{n_1+1,n_2+1,n_3}N^{g}_{n'_1+1,n'_2+1,n'_3+1}
$$

$$
+\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} f^2N^{f}_{n_1,n_2,n_3+1}N^{g}_{n'_1,n'_2+3,n'_3}
$$

$$
-\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f^2N^{f}_{n_1,n_2+1,n_3}N^{g}_{n'_1,n'_2+2,n'_3+1}\Big]. \quad (4.2.34j)
$$

In these recursion relations, d, f, and g are all greater or equal to one. So when d equals one, r,h,s of these equations vanish since g+f ≥ 2. Then we have a set of linear relations for N^1_*'s. We can use these linear relations to determine all the the $< \mathcal{O}_{W_4}\mathcal{O}_{W_4} >=1$ for CP^3, $< \mathcal{O}_{W_5}\mathcal{O}_{W_5} >=1$ for CP^4 and $< \mathcal{O}_{W_6}\mathcal{O}_{W_6} >=1$ for $Gr(2,4)$. Then we put these degree 1 correlation functions to the r,h,s of (4.35),(4.36),(4.37) and obtain linear relations for N^2_*'s. This time, these linear relations thoroughly determine them. For higher degree, the process is the same as d=2 case. We observe that recursion relations we have written down are suffcient for determination. We checked the compatiable condition in the case of d ≤ 4. It seems that the over determined system of WDVV equation work well in all degrees of maps in the case of CP^3, CP^4 and $Gr(2,4)$. The intersection numbers of moduli space of d ≤ 4 are given in the tables.

Table 4.4: D=1 CP^3

$N_0 = 2$	$N_1 = 1$	$N_2 = 1$

Table 4.5: D=2 CP^3

$N_0 = 92$	$N_1 = 18$	$N_2 = 4$	$N_3 = 1$	$N_4 = 0$

Table 4.6: D=3 CP^3

$N_0 = 80160$	$N_1 = 9864$	$N_2 = 1312$	$N_3 = 190$	$N_4 = 30$	$N_5 = 5$	$N_6 = 1$

Table 4.7: D=4 CP^3

$N_0 = 383306880$	$N_1 = 34382544$	$N_2 = 3259680$	$N_3 = 327888$	$N_4 = 35104$
$N_5 = 4000$	$N_6 = 480$	$N_7 = 58$	$N_8 = 4$	

Table 4.8: D=5 CP^3

$N_0 = 6089786376960$	$N_1 = 429750191232$	$N_2 = 31658432256$	$N_3 = 2440235712$
$N_4 = 197240400$	$N_5 = 16744080$	$N_6 = 1492616$	$N_7 = 139098$
$N_8 = 13354$	$N_9 = 1265$	$N_{10} = 105$	

Table 4.9: D=6 CP^3

$N_0 = 2442744889809623 04$	$N_1 = 14207926965714432$	$N_2 = 855909223176192$
$N_3 = 53486265350784$	$N_4 = 3472451647488$	$N_5 = 234526910784$
$N_6 = 16492503552$	$N_7 = 1207260512$	$N_8 = 91797312$
$N_9 = 7200416$	$N_{10} = 573312$	$N_{11} = 44416$
$N_{12} = 2576$		

Table 4.10: D=1 CP^4

$N_{00}=5$	$N_{10}=3$	$N_{20}=2$	$N_{30}=1$	$N_{01}=1$	$N_{11}=1$	$N_{02}=1$

Table 4.11: D=2 CP^4

$N_{00}=6620$	$N_{10}=1734$	$N_{20}=473$	$N_{30}=132$	$N_{40}=36$	$N_{50}=10$
$N_{01}=219$	$N_{11}=67$	$N_{21}=21$	$N_{31}=6$	$N_{41}=2$	
$N_{02}=11$	$N_{12}=4$	$N_{22}=1$			
$N_{03}=1$	$N_{13}=0$				

Table 4.12: D=3 CP^4

$N_{00}=213709980$	$N_{01}=2770596$	$N_{02}=45954$	$N_{03}=1011$	$N_{04}=30$
$N_{05}=0$	$N_{10}=35806494$	$N_{11}=511012$	$N_{12}=9386$	$N_{13}=225$
$N_{14}=5$	$N_{20}=6165822$	$N_{21}=96548$	$N_{22}=1931$	$N_{23}=45$
$N_{24}=1$	$N_{30}=1085892$	$N_{31}=18469$	$N_{32}=385$	$N_{33}=9$
$N_{40}=194024$	$N_{41}=3512$	$N_{42}=76$		
$N_{50}=34780$	$N_{51}=664$	$N_{52}=16$		
$N_{60}=6216$	$N_{61}=128$			
$N_{70}=1108$				
$N_{80}=188$				

Table 4.13: D=4 CP^4

$N_{00}=47723447905060$	$N_{01}=327439797532$	$N_{02}=2679044142$	$N_{03}=26578256$
$N_{04}=324764$	$N_{05}=4830$	$N_{06}=61$	$N_{07}=1$
$N_{10}=5876564125104$	$N_{11}=43242657488$	$N_{12}=380720598$	$N_{13}=4063860$
$N_{14}=52507$	$N_{15}=732$	$N_{16}=9$	
$N_{20}=738764469204$	$N_{21}=5823161346$	$N_{22}=54948346$	$N_{23}=622980$
$N_{24}=8133$	$N_{25}=107$		
$N_{30}=94605276228$	$N_{31}=796460052$	$N_{32}=7990720$	$N_{33}=94104$
$N_{34}=1218$	$N_{35}=14$		
$N_{40}=12302188692$	$N_{41}=110031632$	$N_{42}=1159218$	$N_{43}=13962$
$N_{44}=178$			
$N_{50}=1617593360$	$N_{51}=15251816$	$N_{52}=166936$	$N_{53}=2056$
$N_{60}=213984472$	$N_{61}=2110864$	$N_{62}=23968$	$N_{63}=320$
$N_{70}=28346212$	$N_{71}=291632$	$N_{72}=3516$	
$N_{80}=3748804$	$N_{81}=40492$		
$N_{90}=343260$	$N_{91}=5552$		
$N_{100}=63740$			

Table 4.14: D=1 Gr(2,4)

$N_{000}=0$	$N_{100}=0$	$N_{200}=1$	$N_{300}=1$	$N_{400}=0$	$N_{500}=0$
$N_{001}=0$	$N_{101}=1$	$N_{201}=0$			
$N_{010}=0$	$N_{110}=1$	$N_{210}=1$	$N_{310}=0$		
$N_{011}=1$					
$N_{020}=1$	$N_{120}=1$				

Table 4.15: D=2 Gr(2,4)

$N_{000}=2$	$N_{100}=6$	$N_{200}=18$	$N_{300}=34$	$N_{400}=42$	$N_{500}=42$	$N_{600}=34$	$N_{700}=18$
$N_{800}=6$	$N_{900}=2$						
$N_{001}=1$	$N_{101}=3$	$N_{201}=5$	$N_{301}=5$	$N_{401}=5$	$N_{501}=3$	$N_{601}=1$	
$N_{002}=1$	$N_{102}=1$	$N_{202}=1$	$N_{302}=1$				
$N_{003}=1$							
$N_{010}=3$	$N_{110}=9$	$N_{210}=17$	$N_{310}=21$	$N_{410}=21$	$N_{510}=17$	$N_{610}=9$	$N_{710}=3$
$N_{011}=2$	$N_{111}=3$	$N_{211}=3$	$N_{311}=3$	$N_{411}=2$			
$N_{012}=1$	$N_{112}=1$						
$N_{020}=5$	$N_{120}=9$	$N_{220}=11$	$N_{320}=11$	$N_{420}=9$	$N_{520}=5$		
$N_{021}=2$	$N_{121}=2$	$N_{221}=2$					
$N_{030}=5$	$N_{130}=6$	$N_{230}=6$	$N_{330}=5$				
$N_{031}=1$							
$N_{040}=3$	$N_{140}=3$						

Table 4.16: D=3 Gr(2,4)

$N_{000} = 504$	$N_{001} = 100$	$N_{002} = 25$	$N_{003} = 6$	$N_{004} = 2$
$N_{100} = 1824$	$N_{101} = 307$	$N_{102} = 55$	$N_{103} = 9$	$N_{104} = 2$
$N_{200} = 5159$	$N_{201} = 676$	$N_{202} = 83$	$N_{203} = 10$	
$N_{300} = 11319$	$N_{301} = 1109$	$N_{302} = 101$	$N_{303} = 9$	
$N_{400} = 19512$	$N_{401} = 1460$	$N_{402} = 101$	$N_{403} = 6$	
$N_{500} = 27472$	$N_{501} = 1605$	$N_{502} = 83$		
$N_{600} = 32517$	$N_{601} = 1460$	$N_{602} = 55$		
$N_{700} = 32517$	$N_{701} = 1109$	$N_{702} = 25$		
$N_{800} = 27472$	$N_{801} = 676$			
$N_{900} = 19512$	$N_{901} = 307$			
$N_{1000} = 11319$	$N_{1001} = 100$			
$N_{1100} = 5159$				
$N_{1200} = 1824$				
$N_{1300} = 504$				
$N_{010} = 538$	$N_{011} = 109$	$N_{012} = 23$	$N_{013} = 5$	
$N_{110} = 1603$	$N_{111} = 246$	$N_{112} = 35$	$N_{113} = 5$	
$N_{210} = 3607$	$N_{211} = 403$	$N_{212} = 42$	$N_{213} = 5$	
$N_{310} = 6278$	$N_{311} = 528$	$N_{312} = 42$		
$N_{410} = 8864$	$N_{411} = 579$	$N_{412} = 35$		
$N_{510} = 10499$	$N_{511} = 528$	$N_{512} = 23$		
$N_{610} = 10499$	$N_{611} = 403$			
$N_{710} = 8864$	$N_{711} = 246$			
$N_{810} = 6278$	$N_{811} = 109$			
$N_{910} = 3609$				
$N_{1010} = 1603$				
$N_{1110} = 538$				
$N_{020} = 523$	$N_{021} = 94$	$N_{022} = 16$	$N_{023} = 2$	
$N_{120} = 1203$	$N_{121} = 153$	$N_{122} = 18$		
$N_{220} = 2100$	$N_{221} = 198$	$N_{222} = 18$		
$N_{320} = 2960$	$N_{321} = 216$	$N_{322} = 16$		
$N_{420} = 3501$	$N_{421} = 198$			
$N_{520} = 3501$	$N_{521} = 153$			
$N_{620} = 2960$	$N_{621} = 94$			
$N_{720} = 2100$				
$N_{820} = 1203$				
$N_{920} = 523$				
$N_{030} = 420$	$N_{031} = 61$	$N_{032} = 7$		
$N_{130} = 729$	$N_{131} = 76$	$N_{132} = 7$		
$N_{230} = 1019$	$N_{231} = 82$			
$N_{330} = 1200$	$N_{331} = 76$			
$N_{430} = 1200$	$N_{431} = 61$			
$N_{530} = 1019$				
$N_{630} = 729$				
$N_{730} = 420$				
$N_{040} = 262$	$N_{041} = 28$			
$N_{140} = 358$	$N_{141} = 30$			
$N_{240} = 418$	$N_{241} = 28$			
$N_{340} = 418$				
$N_{440} = 358$				
$N_{540} = 262$				
$N_{050} = 124$	$N_{051} = 10$			
$N_{150} = 144$				
$N_{250} = 144$				
$N_{350} = 124$				
$N_{060} = 48$				
$N_{160} = 48$				

Table 4.17: D=4 Gr(2,4)

$N_{000} = 1044120$	$N_{001} = 93726$	$N_{002} = 9970$	$N_{003} = 1170$	$N_{004} = 138$	$N_{005} = 20$
$N_{100} = 3094440$	$N_{101} = 251402$	$N_{102} = 22570$	$N_{103} = 2058$	$N_{104} = 190$	$N_{105} = 20$
$N_{200} = 8093840$	$N_{201} = 570998$	$N_{202} = 4179$	$N_{203} = 2998$	$N_{204} = 214$	$N_{205} = 20$
$N_{300} = 18245976$	$N_{301} = 1086890$	$N_{302} = 64434$	$N_{303} = 3690$	$N_{304} = 214$	
$N_{400} = 35219976$	$N_{401} = 1752446$	$N_{402} = 84818$	$N_{403} = 3942$	$N_{404} = 190$	
$N_{500} = 58571280$	$N_{501} = 2434530$	$N_{502} = 96894$	$N_{503} = 3690$	$N_{504} = 138$	
$N_{600} = 84843440$	$N_{601} = 2951174$	$N_{602} = 96894$	$N_{603} = 2998$		
$N_{700} = 108066120$	$N_{701} = 3143726$	$N_{702} = 84818$	$N_{703} = 2058$		
$N_{800} = 121770480$	$N_{801} = 2951174$	$N_{802} = 64434$	$N_{803} = 1170$		
$N_{900} = 121770480$	$N_{901} = 2434530$	$N_{902} = 41794$			
$N_{1000} = 108066120$	$N_{1001} = 1752446$	$N_{1002} = 22570$			
$N_{1100} = 84843440$	$N_{1101} = 1086890$	$N_{1102} = 9970$			
$N_{1200} = 58571280$	$N_{1201} = 570998$				
$N_{1300} = 35219976$	$N_{1301} = 251402$				
$N_{1400} = 18245976$	$N_{1401} = 93726$				
$N_{1500} = 8093840$					
$N_{1600} = 3094440$					
$N_{1700} = 1044120$					
$N_{010} = 692760$	$N_{011} = 63904$	$N_{012} = 6528$	$N_{013} = 675$	$N_{014} = 74$	$N_{015} = 6$
$N_{110} = 1852184$	$N_{111} = 147070$	$N_{112} = 12060$	$N_{113} = 976$	$N_{114} = 80$	
$N_{210} = 4249660$	$N_{211} = 281764$	$N_{212} = 18506$	$N_{213} = 1181$	$N_{214} = 80$	
$N_{310} = 8297556$	$N_{311} = 454858$	$N_{312} = 24196$	$N_{313} = 1254$	$N_{314} = 74$	
$N_{410} = 13886500$	$N_{411} = 631136$	$N_{412} = 27526$	$N_{413} = 1181$		
$N_{510} = 20177804$	$N_{511} = 764000$	$N_{512} = 27526$	$N_{513} = 976$		
$N_{610} = 25736664$	$N_{611} = 813396$	$N_{612} = 24196$	$N_{613} = 675$		
$N_{710} = 29015656$	$N_{711} = 764000$	$N_{712} = 18506$			
$N_{810} = 29015656$	$N_{811} = 631136$	$N_{812} = 12060$			
$N_{910} = 25736664$	$N_{911} = 454858$	$N_{912} = 6528$			
$N_{1010} = 20177804$	$N_{1011} = 281764$				
$N_{1110} = 13886500$	$N_{1111} = 147070$				
$N_{1210} = 8297556$	$N_{1211} = 63904$				
$N_{1310} = 4249660$					
$N_{1410} = 1852184$					
$N_{1510} = 692760$					
$N_{020} = 440638$	$N_{021} = 39460$	$N_{022} = 3624$	$N_{023} = 332$	$N_{024} = 26$	
$N_{120} = 1025894$	$N_{121} = 75712$	$N_{122} = 5512$	$N_{123} = 338$	$N_{124} = 26$	
$N_{220} = 2019894$	$N_{221} = 121884$	$N_{222} = 7112$	$N_{223} = 408$		
$N_{320} = 3391958$	$N_{321} = 168332$	$N_{322} = 8032$	$N_{323} = 338$		
$N_{420} = 4932358$	$N_{421} = 203048$	$N_{422} = 8032$	$N_{423} = 332$		
$N_{520} = 6290046$	$N_{521} = 215904$	$N_{522} = 7112$			
$N_{620} = 7089646$	$N_{621} = 203048$	$N_{622} = 5512$			
$N_{720} = 7089646$	$N_{721} = 168332$	$N_{722} = 3624$			
$N_{820} = 6290046$	$N_{821} = 121884$				
$N_{920} = 4932358$	$N_{921} = 75712$				
$N_{1020} = 3391958$	$N_{1021} = 39460$				
$N_{1120} = 2019894$					
$N_{1220} = 1025894$					
$N_{1320} = 440638$					

Table 4.18: D=4 Gr(2,4)

$N_{030} = 256946$	$N_{031} = 21072$	$N_{032} = 1695$	$N_{033} = 121$		
$N_{130} = 508026$	$N_{131} = 33665$	$N_{132} = 2131$	$N_{133} = 126$		
$N_{230} = 852818$	$N_{231} = 46042$	$N_{232} = 2379$	$N_{233} = 121$		
$N_{330} = 1237234$	$N_{331} = 55181$	$N_{332} = 2379$			
$N_{430} = 1574370$	$N_{431} = 58548$	$N_{432} = 2131$			
$N_{530} = 1772374$	$N_{531} = 55181$	$N_{532} = 1695$			
$N_{630} = 1772374$	$N_{631} = 46042$				
$N_{730} = 1574370$	$N_{731} = 33665$				
$N_{830} = 1237234$	$N_{831} = 21072$				
$N_{930} = 852818$					
$N_{1030} = 508026$					
$N_{1130} = 256946$					
$N_{040} = 131874$	$N_{041} = 9540$	$N_{042} = 626$	$N_{043} = 36$		
$N_{140} = 220250$	$N_{141} = 12808$	$N_{142} = 690$			
$N_{240} = 317466$	$N_{241} = 15196$	$N_{242} = 690$			
$N_{340} = 402090$	$N_{341} = 16072$	$N_{342} = 626$			
$N_{440} = 451610$	$N_{441} = 15196$				
$N_{540} = 451610$	$N_{541} = 12808$				
$N_{640} = 402090$	$N_{641} = 9540$				
$N_{740} = 317466$					
$N_{840} = 220250$					
$N_{940} = 131874$					
$N_{050} = 58170$	$N_{051} = 3544$	$N_{052} = 190$			
$N_{150} = 82790$	$N_{151} = 4156$	$N_{152} = 190$			
$N_{250} = 104070$	$N_{251} = 4380$				
$N_{350} = 116486$	$N_{351} = 4156$				
$N_{450} = 116486$	$N_{451} = 3544$				
$N_{550} = 104070$					
$N_{650} = 82790$					
$N_{750} = 58170$					
$N_{060} = 21638$	$N_{061} = 1104$				
$N_{160} = 26958$	$N_{161} = 1160$				
$N_{260} = 30062$	$N_{261} = 1104$				
$N_{360} = 30062$					
$N_{460} = 26958$					
$N_{560} = 21638$					
$N_{070} = 6888$	$N_{071} = 290$				
$N_{170} = 7664$					
$N_{270} = 7664$					
$N_{370} = 6888$					
$N_{080} = 1916$					
$N_{180} = 1916$					

4.2.4 Appendix of Section 4.2:Derivation of Initial Conditions and Some Direct Counting of Amplitudes

We first show $\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle = 1$ for CP^3 (resp. $\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle = 1$ for CP^4). From (2.1.16) this is just number of lines passing through two points of CP^3(resp. CP^4), so it equals to 1 trivially. But we derive this result using schubert calculus of $Gr(2,4)$ (resp.$Gr(2,5)$) which corresponds to the space of lines in CP^3 (resp.CP^4). Schubert cycles $\sigma_{a_1,a_2} \subseteq Gr(2,N)$ $(N-2 \geq a_1 \geq a_2 \geq 0)$ form a basis of $H^*(Gr(2,N),Z)$ and are given by following definition.

$$\sigma_{a_1,a_2} = \{l \in Gr(2,N) | dim_C(l \cap V_{N-2+i-a_i}) \geq i\} \tag{4.2.35}$$

where V_i's are linear subspace of C^N of dimension i satisfying following condition.

$$V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset C^N \tag{4.2.36}$$

From this definition, subset of $Gr(2,N)$ passing through a point of CP^{N-1} is given as $\sigma_{N-2,0}$ because this condition is equivalent to $dim_C(l \cap V_1) = 1$. Then we can calculate $\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle$ for CP^3 (resp.$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle$ for CP^4) as follows.

$$\begin{aligned} \langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle &= \sharp(\sigma_{2,0}\cdot\sigma_{2,0})_{Gr(2,4)} = \sharp(\sigma_{2,2})_{Gr(2,4)} = 1 \\ \langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle &= \sharp(\sigma_{3,0}\cdot\sigma_{3,0})_{Gr(2,5)} = \sharp(\sigma_{3,3})_{Gr(2,5)} = 1 \end{aligned} \tag{4.2.37}$$

In this derivation, we used Pieri's formula

$$\sigma_{a,0}\cdot\sigma_{b_1,b_2} = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ c_1+c_2=a+b_1+b_2}} \sigma_{c_1,c_2} \tag{4.2.38}$$

and $\sigma_{N-2,N-2}$ corresponds to a point of $Gr(2,N)$.

Next we derive $\langle \mathcal{O}_{W_5}\mathcal{O}_{W_6}\rangle = 1$ for $Gr(2,4)$. Using Plücker map, Gr(2,4) can be embedded in CP^5 as a quadratic hypersurface. This embedding is constructed as follows. We map a line $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}_C$ in $CP^3(C^4)$ to a multivector $\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \in \bigwedge^2 C^4$ This map (we call it ι) is injective and conversely the image of a line in $\bigwedge^2 C^4$ is characterized by decomposability, i.e. $\omega \in \bigwedge^2 C^4$ is in $Im(\iota)$ iff ω can be written as $\omega = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2$. It can be shown that this condition is equiivalent to $\omega \wedge \omega = 0$. So using a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of C^4 and expanding ω as follows,

$$\omega = \lambda_{12}\boldsymbol{e}_1\wedge\boldsymbol{e}_2 + \lambda_{13}\boldsymbol{e}_1\wedge\boldsymbol{e}_3 + \lambda_{14}\boldsymbol{e}_1\wedge\boldsymbol{e}_4 + \lambda_{23}\boldsymbol{e}_2\wedge\boldsymbol{e}_3 + \lambda_{24}\boldsymbol{e}_2\wedge\boldsymbol{e}_4 + \lambda_{34}\boldsymbol{e}_3\wedge\boldsymbol{e}_4 \tag{4.2.39}$$

we can realize $Gr(2,4)(\simeq Im(\iota))$ in CP^5 as follows.

$$\begin{aligned} &\omega\wedge\omega = 0 \\ \Longleftrightarrow\ &\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0 \end{aligned} \tag{4.2.40}$$

In summary, we can see $Gr(2,4)$ as a quadratic hypersurface G in CP^5. Then we want to find the realization of $\sigma_{2,1}(=W_5)$ and $\sigma_{2,2}(=W_6)$ in G. From the study of the structure of G (see Chap.6 of Griffith Harris [19]) $\sigma_{2,1}$ corresponds to a line in G and trivially $\sigma_{2,2}$ to a point. If we consider plane h (resp. line l) in CP^5, quadratic feature of G makes the intersection $(h \cap G)$ (resp.$(l \cap G)$) into conic of G (resp. two points of G).Then we have to devide them by factor 2,i.e.

$$\sigma_{2,1} \leftrightarrow \frac{1}{2}(h \cap G) \tag{4.2.41}$$

$$\sigma_{2,2} \leftrightarrow \frac{1}{2}(l \cap G) \tag{4.2.42}$$

The space of lines in G (we denote it as L_G) is constructed as the subspace of $Gr(2,6)$(space of lines in CP^5) using bundle calculation (see [3]).

$$L_G = c_T(Sym^2(U^*)) = 4\tilde{\sigma}_{2,1} \tag{4.2.43}$$

where U is the universal bundle of $Gr(2,6)$.

(We denote schubert cycles of $Gr(2,6)$ as $\tilde{\sigma}_{a_1,a_2}$ in order to distinguish them from the ones of $Gr(2,4)$).

From (4.2.42), in L_G, to count the number of lines which passes through $\sigma_{2,1}$ and $\sigma_{2,2}$ are equivalent to picking up the lines which passes through h and l (multiplied by factor $\frac{1}{2}$). Then we have

$$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_6}\rangle = \sharp(\frac{1}{2}\tilde{\sigma}_3 \cdot \frac{1}{2}\tilde{\sigma}_2 \cdot 4\tilde{\sigma}_{2,1}) = 1 \tag{4.2.44}$$

Lastly, using this technique, we calculate the topological amplitude of $d = 1$ sector for CP^3 and CP^4.

<u>CP^3</u>

$$\mathcal{O}_{W_3} \leftrightarrow \sigma_1 \quad \mathcal{O}_{W_4} \leftrightarrow \sigma_2 \quad (\text{in} Gr(2,4))$$
$$\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle = \sharp(\sigma_2 \cdot \sigma_2) = 1$$
$$\langle \mathcal{O}^2_{W_3}\mathcal{O}_{W_4}\rangle = \sharp(\sigma_1^2 \cdot \sigma_2) = 1$$
$$\langle \mathcal{O}^4_{W_3} = \sharp(\sigma_1^4) = 2$$

<u>CP^4</u>

$$\mathcal{O}_{W_3} \leftrightarrow \sigma_1 \quad \mathcal{O}_{W_4} \leftrightarrow \sigma_2 \quad \mathcal{O}_{W_5} \leftrightarrow \sigma_3 \quad (\text{in} Gr(2,5))$$
$$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle = \sharp(\sigma_3 \cdot \sigma_3) = 1$$
$$\langle \mathcal{O}^3_{W_3}\mathcal{O}_{W_5}\rangle = \sharp(\sigma_1^3 \cdot \sigma_3) = 1$$
$$\langle \mathcal{O}^3_{W_4}\rangle = \sharp(\sigma_2^3) = 1$$
$$\langle \mathcal{O}_{W_3}\mathcal{O}_{W_4}\mathcal{O}_{W_5}\rangle = \sharp(\sigma_1 \cdot \sigma_2 \cdot \sigma_3) = 1$$
$$\langle \mathcal{O}^2_{W_3}\mathcal{O}^2_{W_4}\rangle = \sharp(\sigma_1^2 \cdot \sigma_2^2) = 2$$
$$\langle \mathcal{O}^4_{W_3}\mathcal{O}_{W_4}\rangle = \sharp(\sigma_1^4 \cdot \sigma_2) = 3$$
$$\langle \mathcal{O}^5_{W_3}\rangle = \sharp(\sigma_1^5) = 5 \tag{4.2.45}$$

Chapter 5

Mirror Symmetry

So far, we have discussed A-model, but in case that target space is a Calabi-Yau manifold, there is another way of twisting N=2 super symmetric sigma model. We call this topological sigma model as B-model.

Mirror symmetry is the conjectural symmetry which asserts that for Calabi-Yau manifold M, there is another Calabi-Yau manifold M^* which satisfy $H^{p,q}(M) = H^{q,dim(M)-p}(M^*)$, and in addition, correlation function of A-model (resp. B-model) on M and the ones of B-model (resp. A-model) on M^* are coincide. To be more precise, we have a way of translating the correlation functions of B-model on M^* into the ones of A-model on M by identifying deformation parameters of both models. Reverse operation is not well-defined in the present circumstances. In this chapter, we treat B-model on M_N^* which is mirror counterpart of degree N hypersurface in CP^{N-1}, the only one Calabi-Yau manifold in it and calculate $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)\rangle$. We will see the complete coincidence with the previous result and the evidence for the conjecture in the case of general dimensional Calabi-Yau manifold.

5.1 Construction of M_N^*

Construction of a mirror manifold of Calabi-Yau manifold is systematically done with the toric geometry. Let us consider an n-dimensional convex integral polyhedron $\Delta \in R^n$ containing the origin $\nu_0 = (0,0,\cdots,0)$. An integral polyhedron is a polyhedron whose vertices are integral, and is called reflexive if its dual defined by

$$\Delta^* = \{(x_1, x_2, \cdots, x_n) | \sum_{i=1}^{n} x_i y_i \geq -1 \text{ for all } (y_1, y_2, \cdots, y_n) \in \Delta\} \tag{5.1.1}$$

is again an integral polyhedron. Note if Δ is reflexive, then Δ^* is also reflexive since $(\Delta^*)^* = \Delta$. From Δ we can construct toric ambient space P_Δ (for detailed construction see [18]). We introduce complex variable z_i $(i = 1, 2, \cdots, n)$ corresponding to integral base vectors $e^i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ of R^n. Denote ν_j $(j = 0, 1, \cdots, s)$ the integral points in Δ and consider the zero locus Z_f of the Laurent polynomial

$$f(a, Z) = \sum_{j=0}^{s} a_j z_1^{\nu_j^1} z_2^{\nu_j^2} \cdots z_n^{\nu_j^n} \tag{5.1.2}$$

in the algebraic torus $(C^\times)^n \subset P_\Delta$. Fundamentally, this gives the defining equation of Calabi-Yau manifold M_Δ in P_Δ. Operation which translate (5.1.2) into homogeneous polynomial form are discussed later. Correspondingly, mirror counterpart of M_Δ is constructed as Calabi-Yau manifold M_{Δ^*} in P_{Δ^*}.

Then we turn to our CP^{N-1} case. CP^{N-1} is constructed from the polyhedron Δ^N in R^{N-1} with vertices

$$\begin{aligned} \nu_1^N = (N-1, -1, \cdots, -1) \quad \nu_2^N = (-1, N-1, -1, \cdots, -1) \\ \cdots, \nu_{N-1}^N = (-1, -1, \cdots, -1, N-1), \nu_2^N = (-1, -1, -1, \cdots, -1) \end{aligned} \tag{5.1.3}$$

Then defining equation for M_{Δ^N} becomes

$$f_{\Delta_N}(a, Z_i) = \sum_{0 \le d_i, \sum_{i=1}^{N-1} d_i \le N} a_{d_1, d_2, \cdots, d_{N-1}} z_1^{d_1 - 1} z_2^{d_2 - 1} \cdots z_{N-1}^{d_N - 1} \tag{5.1.4}$$

Coordinates z_i are identified with the homogeneous coordinates X_i of CP^{N-1} via

$$(z_1 : z_2 : \cdots : z_{N-1} : 1) = (\frac{X_1}{X_N} : \frac{X_2}{X_N} : \cdots : \frac{X_{N-1}}{X_N} : 1) \tag{5.1.5}$$

Then we have the well known form of defining equation of M_{Δ^N} as follows

$$\begin{aligned} (X_1 X_2 \cdots X_N) f_{\Delta^N}(a, \frac{X_i}{X_N}) &= \sum_{0 \le d_i,\ \sum_{i=1}^{N-1} d_i \le N} a_{d_1 d_2 \cdots d_{N-1}} X_1 X_2 \cdots X_{N-1}^{d_{N-1}} X_N^{N - \sum_{i=1}^{N-1} d_i} \\ &= \sum_{0 \le d_i,\ \sum_{i=1}^{N} d_i = N} \tilde{a}_{d_1 d_2 \cdots d_{N-1}} X_1^{d_1} X_2^{d_2} \cdots X_{N-1}^{d_{N-1}} X_N^{d_N} \quad (5.1.6) \\ &\quad (\tilde{a}_{d_1 d_2 \cdots d_N} := a_{d_1 d_2 \cdots d_{N-1}}) \end{aligned} \tag{5.1.7}$$

(5.1.7) defines a family of Calabi-Yau Manifolds in CP^{N-1} with different complex structure, but A-model doesn't distinguish this difference. Then we can choose simple representation of defining equation of $M_{\Delta^N} := M_N$.

$$M_N := \{(X_1 : X_2 : \cdots : X_N) \in CP^{N-1} | X_1^N + X_2^N + \cdots + X_N^N = 0\} \tag{5.1.8}$$

Next, we consider $M_{\Delta^{N*}}$.Δ^{N*} is the polyhedron whose vertices are

$$\begin{aligned} \nu_1^{N*} = (1, 0, \cdots, 0), \nu_2^{N*} = (0, 1, 0, \cdots, 0), \nu_{N-1}^{N*} = (0, \cdots, 0, 1) \\ \nu_N^{N*} = (-1, -1, \cdots, -1). \end{aligned} \tag{5.1.9}$$

And defining equation of $M_{\Delta^{N*}}$ is

$$\sum_{j=1}^{N-1} a_j z_j + a_0 + a_N z_1^{-1} z_2^{-1} \cdots z_{N-1}^{-1} = 0 \tag{5.1.10}$$

In this case, the toric variety $P_{\Delta^{N*}}$ can be identified with

$$\begin{aligned} P_{\Delta^{N*}} &= H^N \\ &= \{(U_0 : U_1 : \cdots : U_N) \in CP^N | \prod_{i=1}^{N} U_i = U_0^N\} \end{aligned} \tag{5.1.11}$$

where the variables z_i are related to U_i by

$$(1 : z_1 : z_2 : \cdots : z_{N-1} : \frac{1}{\prod_{i=1}^{N-1} z_i}) = (1 : \frac{U_1}{U_0} : \cdots : \frac{U_N}{U_0}) \tag{5.1.12}$$

Then we can rewrite (5.1.10) in terms of U_i as follows.

$$\begin{aligned} U_0(\sum_{j=1}^{N-1} a_j \frac{U_j}{U_0} + a_0 + a_N \frac{U_N}{U_0}) = 0 \\ \Longleftrightarrow \sum_{j=0}^{N} a_j U_j = 0 \end{aligned} \tag{5.1.13}$$

And we have

$$M_{\Delta^{N*}} = \{(U_0 : U_1 : \cdots : U_N) \in CP^N | \sum_{j=0}^{N} a_j U_j = 0, \prod_{i=1}^{N} U_i = U_0^N\} \tag{5.1.14}$$

But the second condition in (5.1.14) is rewritten in more convenient form using etale map.

$$\begin{aligned} \phi : CP^{N-1} &\mapsto H^N \\ (X_1 : X_2 : \cdots : X_N) &\mapsto (X_1 X_2 \cdots X_N : X_1^N : X_2^N : \cdots : X_N^N) \end{aligned} \tag{5.1.15}$$

This map is equivalent to dividing CP^{N-1} by discrete group $Z_N^{\otimes N-2}$ generated by

$$g_1 = (e^{\frac{2\pi i}{N}}, 1, \cdots, 1, e^{-\frac{2\pi i}{N}}), g_2 = (1, e^{\frac{2\pi i}{N}}, 1, \cdots, 1, e^{-\frac{2\pi i}{N}}), \cdots, g_{N-1} = (1, \cdots, 1, e^{\frac{2\pi i}{N}}, e^{-\frac{2\pi i}{N}}) \tag{5.1.16}$$

Combining (5.1.14),(5.1.15) and (5.1.16), we have

$$M_{\Delta^* N} = \{(X_0 : X_1 : \cdots : X_N) \in CP^{N-1}/Z_N^{\otimes N-2} | \sum_{j=1}^{N} a_j X_j^N + a_0 \prod_{j=1}^{N} X_j\} \tag{5.1.17}$$

(5.1.17) also defines the family of Calabi-Yau manifold with different complex structure. Since B-model describes the deformation of complex structure of target space, we cannot ignore a_i's in contrast to M_N. But we can set $a_i = 1, (i = 1, \cdots, N)$ using the linear transformation compatible with the action of $Z_N^{\otimes N-2}$. For later convenience we define a_0 as $-N\psi$ and we have the following family of $M_N^* := M_{\Delta^{N*}}$.

$$M_N^* := \{(X_1 : X_2 : \cdots : X_N) \in CP^{N-1}/Z_N^{\otimes N-2} | X_1^N + X_2^N + \cdots + X_N^N - N\psi X_1 \cdots X_N\} \tag{5.1.18}$$

5.2 B-model

The B-model is obtained by *twisting* a $N = 2$ non-linear sigma model defined on a Calabi-Yau space [8] in a way different from A-twist. $N = 2$ non-linear sigma model is defined as follows. let M be a n-dimensional Calabi-Yau manifold and ϕ^i be a holomorphic coordinate on M $(i = 1, \cdots n)$(and $\phi^{\bar{i}}$ be a anti-holomorphic coordinate), Σ be a Riemann surface, which is restricted to genus zero, and z be a holomorphic coordinate on Σ. The Lagrangian is

$$L = 2t \int_\Sigma d^2 z(\frac{1}{2}g_{i\bar{j}}(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}}) + \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + i\psi_-^{\bar{i}} D_z \psi_-^i g_{i\bar{i}} + i\psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i g\bar{i}i + R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}}) \tag{5.2.19}$$

where $\phi^i(z)$ is a map from Σ to M. Spin quantum numbers are already explained in Section 2.1. This Lagrangian possesses $N = 2$ super symmetry. In terms of fermionic parameter $\alpha_-, \bar{\alpha}_-$, and $\alpha_+, \bar{\alpha}_+$, the super transformation laws are given as follows.

$$\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \\
\delta\phi^{\bar{i}} &= i\bar{\alpha}_-\psi_+^{\bar{i}} + i\bar{\alpha}_+\psi_-^{\bar{i}} \\
\delta\psi_+^i &= -\bar{\alpha}_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma_{jm}^i\psi_+^m \\
\delta\psi_+^{\bar{i}} &= -\alpha_-\partial_{\bar{z}}\phi^{\bar{i}} - i\bar{\alpha}_+\psi_-^{\bar{j}}\Gamma_{\bar{j}\bar{m}}^{\bar{i}}\psi_+^{\bar{m}} \\
\delta\psi_-^i &= -\bar{\alpha}_+\partial_z\phi^i - i\alpha_-\psi_+^j\Gamma_{jm}^i\psi_-^m \\
\delta\psi_-^{\bar{i}} &= -\alpha_+\partial_{\bar{z}}\phi^{\bar{i}} - i\bar{\alpha}_-\psi_+^{\bar{j}}\Gamma_{\bar{j}\bar{m}}^{\bar{i}}\psi_-^{\bar{m}}
\end{aligned} \tag{5.2.20}$$

B-model is obtained by twisting the above Lagrangian as follows;

$$\begin{aligned}
\psi_+^i &\mapsto \rho_{\bar{z}}^i \\
\psi_-^i &\mapsto \rho_z^i \\
(\psi_+^{\bar{i}} + \psi_-^{\bar{i}}) &\mapsto \eta^{\bar{i}} \\
(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}) &\mapsto \theta^{\bar{i}}
\end{aligned} \tag{5.2.21}$$

Here for convenience we redefine the variables;

$$\begin{aligned}
\eta^{\bar{i}} &= \psi_+^{\bar{i}} + \psi_-^{\bar{i}} \\
\theta_i &= g_{i\bar{i}}(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}) \\
\rho_z^i &= \psi_+^i \\
\rho_{\bar{z}}^i &= \psi_-^i
\end{aligned}$$

The B-model Lagrangian is

$$\begin{aligned}
L = {} & t\int_\Sigma d^2z(\frac{1}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^{\bar{j}} + i\frac{1}{2}\eta^{\bar{i}}(D_z\rho_{\bar{z}}^i + D_{\bar{z}}\rho_z^i)g_{i\bar{i}} + i\frac{1}{2}\theta_i(D_{\bar{z}}\rho_z^i - D_z\rho_{\bar{z}}^i) \\
& -\frac{1}{2}R_{i\bar{i}j\bar{j}}\rho_z^i\rho_{\bar{z}}^j\eta^{\bar{i}}\theta_k g^{k\bar{j}})
\end{aligned} \tag{5.2.22}$$

Since the canonical bundle K (or $\bar{K}$) is trivial, the twisting does nothing at all at least locally. Therefore the transformation law (5.2.20) should be still valid. But to keep up with the change of the spin of ψ_+^i, ψ_-^i etc., we also have to change the spin of α_+, α_- etc. By the B-twisting the infinitesimal parameter α_-,$\bar{\alpha}_-$, α_+, and $\bar{\alpha}_+$ turn followingly;

$$\alpha_-, \tilde{\alpha_-} : \text{spin } -1/2 \rightarrow \begin{cases} \alpha_- : \text{spin } -1 \\ \tilde{\alpha_-} : \text{spin } 0 \text{ on } \Sigma \end{cases}$$

$$\alpha_+, \tilde{\alpha_+} : \text{spin } +1/2 \rightarrow \begin{cases} \alpha_+ : \text{spin } +1 \\ \tilde{\alpha_+} : \text{spin } 0 \text{ on } \Sigma \end{cases}$$

According to Eguchi-Yang [28], reinterpretating the above scalar transformations by $\bar{\alpha}_-$, $\bar{\alpha}_+$ as BRST transformation, we can obtain a topological field theory.

That is , the BRST transformation is obtained from (5.2.20) by setting $\alpha_- = \alpha_+ = 0$ and setting $\bar{\alpha}_- = \bar{\alpha}_+ = \alpha = \text{constant}$. The topological transformations are

$$\begin{aligned}
\delta\phi^i &= 0 \\
\delta\phi^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\
\delta\eta^{\bar{i}} &= \delta\theta_i = 0 \\
\delta\rho^i &= -\alpha d\phi^i
\end{aligned} \tag{5.2.23}$$

Also we can introduce the BRST operator Q which generate topological transformation such that $\delta W = -i\{Q, V\}$ for any field V. Q satisfies the condition $Q^2 = 0$ modulo equation of motion. In terms of this BRST operator Q, we can rewrite the Lagrangian (5.2.22) as;

$$L = it\int\{Q, R\} + tW \ ,$$

where

$$R = g_{i\bar{j}}(\rho_z^i\partial_{\bar{z}}\phi^{\bar{j}} + \rho_{\bar{z}}^i\partial_z\phi^{\bar{j}}) \ ,$$

and

$$W = \int_\Sigma(-\theta_i D\rho^i - i/2R_{i\bar{i}j\bar{j}}\rho^i \wedge \rho^j\eta^{\bar{i}}\theta_k g^{k\bar{i}}) \ ,$$

here D is the exterior derivative on Σ and extended to act on forms with values in $\Phi^*(T^{1,0}M)$ by using the pull-back of the Levi-Civita connection on M. Then we can take weak coupling limit $t \to \infty$. Since W is homogeneous with respect to the variable θ, we rescale θ into $\frac{\theta}{t}$. Then the theory does not depend on W and we can conclude that correlation function does not depend on coupling constant t. Instead, B-model is deformed by the variation of complex structure of target space M, which is the key to the later discussion.

5.3 The Observables

While the BRST-invariant observables of A-model form De-Rahm(Dolbeaut) cohomology on M, those of B-model are given by $\bar{\partial}$-cohomology of $A^p(M, \wedge^q T^{1,0}M)$: $A^p(M, \wedge^q T^{1,0}M)$ is the set of $(0,p)$ forms on M which take values in $\wedge^q T^{1,0}M$. (Here $\wedge^q T^{1,0}M$ means the q-th exterior power of holomorphic tangent bundle on M, $T^{1,0}M$.) Such an object can be written;

$$A^p(M, \wedge^q T^{1,0}M) \ni V = V^{j_1 j_2 \cdots j_q}_{\bar{i}_1 \bar{i}_2 \cdots \bar{i}_p} dz^{\bar{i}_1} \wedge dz^{\bar{i}_2} \wedge \cdots \wedge dz^{\bar{i}_p} \partial_{j_1} \wedge \partial_{j_2} \wedge \cdots \wedge \partial_{j_q} \quad (5.3.24)$$

Of course, the sheaf cohomology group $H^p(M, \wedge^q T^{1,0}M)$ is defined by the quotient space (module) of $Z^p(M, \wedge^q T^{1,0}M)$ which is the space of solution of $\bar{\partial}V = 0$ modulo $B^p(M, \wedge^q T^{1,0}M)$ which is the set of S's such that $\bar{\partial}S = 0$ for all $S \in A^{p-1}(M, \wedge^q T^{1,0}M)$. Given any point $z \in \Sigma$, we can give the correspondence of every elements V of $A^{p-1}(M, \wedge^q T^{1,0}M)$ to the quantum field theory operator $\mathcal{O}_V(z)$;

$$\mathcal{O}_V(z) = V = V^{j_1 j_2 \cdots j_q}_{\bar{i}_1 \bar{i}_2 \cdots \bar{i}_p}(z) \eta^{\bar{i}_1} \eta^{\bar{i}_2} \cdots \eta^{\bar{i}_p} \theta_{j_1} \theta_{j_2} \cdots \theta_{j_q}, \quad (5.3.25)$$

and we can find that

$$\{Q, \mathcal{O}_V\} = -\mathcal{O}_{\bar{\partial}V}.$$

Therefore $\mathcal{O}_V$ is BRST-invariant if and only if $\bar{\partial}V = 0$, and $\mathcal{O}_V$ is BRST-exact if and if only $V = \bar{\partial}S$ for some S.

This correspondence gives a natural map from $\oplus_{p,q=0}^{p,q=n} H^p(M, \wedge^q T^{1,0}M)$ to the BRST cohomology or the observables of B-model. This map, in fact, is isomorphic.

5.4 n-point Correlation Function

We describe the observable which corresponds to the elements $V_a \in H^{P_a}(M, \wedge^{Q_a} T^{1,0}M)$ as $\mathcal{O}_{V_a}$. Our concern is the formulation of the correlation function

$$\langle \prod_i \mathcal{O}_{V_i}(z_i) \rangle, \quad (5.4.26)$$

where z_i's are arbitrary points on Σ. The operator $\mathcal{O}_{V_i}$ has a left-moving ghost number P_i and a right-moving one Q_i. In order for (5.4.26) not to vanish, the conservation law between the left(or right)-moving ghost number and back ground ghost charge demands that

$$\sum_i P_i = \sum_i Q_i = \dim(M) = n, \quad (5.4.27)$$

In this section, we especially concern the correlation function of the type

$$\langle \prod_{i=1}^{n} \mathcal{O}_B(z_i) \rangle,$$

where $B \in H^1(M, T^{1,0}M)$. According to Vafa [26] and Witten [8], it is revealed that the path-integral of B-model can be reduced to the integral on the space of the constant maps from the world sheet Σ to its target M, i.e., the integral on the target space M itself. The *Physical Proof* is the following. Let Υ be some function space on which we wish to path-integrate. Consider the theory we are dealing with (of course, in this case B-model) has a group symmetry G. Suppose that G acts freely on Υ. Then there is a fibration $\Upsilon \to \Upsilon/G$, and we can reduce the path-integral over Υ to Υ/G. If we consider only G invariant observables $\mathcal{O}$, we have the formula,

$$\int_\Upsilon \mathcal{D}\phi e^{-S} \mathcal{O} = (volG) \int_{\Upsilon/G} \mathcal{D}\phi e^{-S} \mathcal{O}, \quad (5.4.28)$$

where $volG$ is the volume of the group G. We can apply this formula to the case in which G is a super group generated by the BRST charge Q. but this case is rather strange, since for any fermionic variables θ

$$\int d\theta \cdot 1 = 0,$$

the volume of any super group G becomes zero. Do we have to conclude any correlation function of BRST invariant operators are all zero ? In general, the group G does not act *freely*. In almost all cases there are some fixed point sets Υ_0. The nonzero contribution to the correlation function comes from only Υ_0 . Since on $\Upsilon - \Upsilon_0$ G acts *freely* then we can apply (5.4.28) there. In the B-model Q-invariant points must satisfy from (5.2.23) that

$$\delta\rho^i = -\alpha d\phi^i = 0,$$

therefore

$$d\phi = 0.$$

This means that the maps $\Phi : \Sigma \to M$ should be constant maps on M. Thus we have succeeded in reducing the path-integral over Υ to the integral over the space of constant map on M, that is, the integral on target space M itself. Next we will consider how to calculate n-point correlation function $\langle \prod_{i=1}^{d} \mathcal{O}_B(z_i) \rangle$ concretely.

$$\begin{aligned} \mathcal{O}_B(z_k) &= b^j_{\bar{i}} \eta^{\bar{i}} \theta_j \\ \prod_{k=1}^{n} \mathcal{O}_B(z_k) &= \prod_{k=1}^{n} b^{j_k}_{\bar{i}_k} \cdot (\eta^{\bar{i}_k} \theta_{j_k}) \\ &\equiv \theta_Y. \end{aligned}$$

Corresponding to the map $\otimes^n \theta_B \mapsto \theta_Y$, we can construct the map:

$$B^{\otimes^n} \mapsto Y$$

$$\otimes^n H^1(M, T^{1,0}M) \to H^n(M, \wedge^n T^{1,0}M),$$

It is apparent that this map is merely a classical wedge product.
Now in order to carry out the integral over Calabi-Yau manifold M, we need to transform the element of $H^n(M, \wedge^n T^{1,0}M)$ to the element of $H^n(M, \Omega^n M)$ (here $\Omega^n M$ means the sheaf of $(n,0)$ form.) We can realize the requested transformation by operating the square of the holomorphic $(n,0)$-form on M. According to the general theory of Calabi-Yau manifold, the holomorphic $(n,0)$-form on n-dimensional M exists uniquely up to constant. So we don't have to worry about how to select the holomorphic forms. Therefore we can formulate n-point correlation function up to constant by the integral on M itself,

$$\langle \prod_{i=1}^{n} \mathcal{O}_B(z_i) \rangle = \int_M \Omega \wedge b^{i_1} \wedge b^{i_2} \wedge \cdots \wedge b^{i_n} \Omega_{i_1 i_2 \cdots i_n}, \tag{5.4.29}$$

where

$$\begin{aligned} b^{i_k} &= b^{i_k}_{\bar{j}} dz^{\bar{j}} \\ \Omega &= \Omega_{j_1 j_2 \cdots j_n} dz^{j_1} \wedge dz^{j_2} \wedge \cdots \wedge dz^{j_n} \in H^{n,0}(M). \end{aligned}$$

Note that this formula is defined only up to a constant.

5.5 Kodaira-Spencer equation

It is well-known that $b^i \in H^1(M, T^{1,0}M), (i = 1 \cdots \dim(H^1_{\bar\partial}(M, T^{1,0}M)))$ form a basis of the tangent space to the moduli space of the complex structure of M (we will denote it $\mathcal{M}_{comp.}$);

$$T\mathcal{M}_{comp.}|_M = H^1(M, T^{1,0}M).$$

Kodaira and Spencer [11], [12] showed that the complex-structure moduli space $\mathcal{M}_{comp}$ itself is also a complex manifold.
Let $z^\alpha(\alpha = 1, \cdots, \dim(\mathcal{M}_{comp.}) = \dim(H^1(M, T^{1,0}M)))$ be a holomorphic coordinate on $\mathcal{M}_{comp.}$ (In this section we will deal only with the case $\dim(\mathcal{M}_{comp.}) = 1$).
The deformation equation of Kodaira and Spencer is the following;

$$\frac{\partial \Omega}{\partial z^\alpha} = k_\alpha \Omega + \chi_\alpha, \tag{5.5.30}$$

where k_α depends only on z_α, but not on the coordinate of M, and $\chi_\alpha \in H^{n-1,1}(M)$. That is, this means a decomposition:

$$\frac{\partial \Omega}{\partial z^\alpha} \in H^{n,0} \oplus H^{n-1,1}.$$

Let us guess what happens here. We can describe the holomorphic $(n,0)$-form Ω in terms of a holomorphic local coordinate x^μ of M as

$$\Omega = \frac{1}{n!} h(x) \epsilon_{\mu\nu\cdots\rho} dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\rho. \tag{5.5.31}$$

Noting that the holomorphic local coordinate x^μ depends on the complex structure z^α, derivate the both sides of (5.5.31) with z^α. Then

$$\frac{\partial \Omega}{\partial z^\alpha} = \frac{1}{n!} \frac{\partial h(x)}{\partial z^\alpha} \epsilon_{\mu\nu\cdots\rho} dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\rho + \frac{1}{(n-1)!} h(x) \epsilon_{\mu\nu\cdots\rho} \frac{\partial dx^\mu}{\partial z^\alpha} \wedge dx^\nu \wedge \cdots \wedge dx^\rho. \tag{5.5.32}$$

The first term is apparently a pure $(n,0)$-form. But the second term is a direct linear combination of $(n,0)$-form and $(n-1,1)$-form, and especially, we should note the term $\frac{\partial dx^\mu}{\partial z^\alpha}$. This term consists of $(1,0)$-form part and $(0,1)$-form part. Thus,

$$\begin{aligned} \chi_\alpha &= \left. \frac{\partial \Omega}{\partial z^\alpha} \right|_{(n-1,1)\text{-form part}} \\ &= \left. \frac{1}{(n-1)!} h(x) \epsilon_{\mu\nu\cdots\rho} \frac{\partial dx^\mu}{\partial z^\alpha} \wedge dx^\nu \wedge \cdots \wedge dx^\rho \right|_{(n-1,1)\text{-form part}} \\ &= \frac{1}{(n-1)!} h(x) \epsilon_{\mu\nu\cdots\rho} \left. \frac{\partial dx^\mu}{\partial z^\alpha} \right|_{(0,1)\text{-form part}} \wedge dx^\nu \wedge \cdots \wedge dx^\rho \end{aligned} \tag{5.5.33}$$

It is also well known that $H^{n-1,1}(M)$ and $H^1(M, T^{1,0}M)$ are isomorphic each other with the help of the holomorphic $(n,0)$-form Ω. There is the map from $H^1_{\bar\partial}(M, TM)$ to $H^{n-1,1}(M)$;

$$H^1_{\bar\partial}(M, TM) \mapsto H^{n-1,1}(M) \tag{5.5.34}$$

$$\chi^\mu_\alpha = \chi^\mu_{\alpha,\bar\rho} dx^{\bar\rho} \mapsto \chi_\alpha = \chi^\mu_{\alpha,\bar\rho} \Omega_{\mu\rho\cdots\sigma} dx^{\bar\rho} \wedge dx^\rho \wedge \cdots \wedge dx^\sigma. \tag{5.5.35}$$

We can inverse this map;

$$H^{n-1,1}(M) \mapsto H^1_{\bar\partial}(M, TM) \tag{5.5.36}$$

$$\chi_\alpha = \chi_{\alpha,\bar\rho\mu\rho\cdots\sigma} dx^{\bar\rho} \wedge dx^\rho \wedge \cdots \wedge dx^\sigma \mapsto \chi^\mu_{\alpha,\bar\rho} = \frac{1}{2||\Omega||^2} \bar\Omega^{\mu\rho\cdots\sigma} \chi_{\alpha,\bar\rho\rho\cdots\sigma} dx^{\bar\rho}$$

The original Kodaira-Spencer equation (5.5.30) can be rewritten in terms of the element of $H^1_{\bar\partial}(M, TM)$,

$$\frac{\partial \Omega}{\partial z^\alpha} = k_\alpha \Omega + \chi^\mu_{\alpha\bar\rho} \Omega_\mu \ , \tag{5.5.37}$$

where $\chi^\mu_{\alpha\bar\rho} \in H^1_{\bar\partial}(M, TM)$, and Ω_μ means

$$\Omega_\mu \equiv \Omega_{\mu\nu\cdots\rho} dx^\nu \wedge \cdots \wedge dx^\rho. \tag{5.5.38}$$

From (5.5.33) and (5.5.38) we can infer that

$$\chi^\mu_{\alpha\bar\rho} = \left. \frac{\partial dx^\mu}{\partial z^\alpha} \right|_{(0,1)\text{-form part}}. \tag{5.5.39}$$

Therefore from all the fact above we can immediately derive

$$\begin{aligned} &\langle \prod_{i=1}^{n} \mathcal{O}_{B_i}(z_i) \rangle \\ &\equiv \int_M \Omega \wedge b^{i_1}_{\alpha} \wedge b^{i_2}_{\beta} \wedge \cdots \wedge b^{i_n}_{\gamma} \Omega_{i_1 i_2 \cdots i_n} &(5.5.40) \\ &= \int_M \Omega \wedge \underbrace{\frac{\partial^n \Omega}{\partial z^{\alpha} \partial z^{\beta} \cdots \partial z^{\gamma}}}_{n \text{ times}}. &(5.5.41) \end{aligned}$$

It is because obviously from (5.5.33)

$$\left. \underbrace{\frac{\partial^m \Omega}{\partial z^{\alpha} \partial z^{\beta} \cdots \partial z^{\rho}}}_{m \text{ times}} \right|_{(n-m,m)\text{-form part}} = \chi^{i_1}_{\alpha} \wedge \chi^{i_2}_{\beta} \wedge \cdots \wedge \chi^{i_m}_{\rho} \Omega_{i_1 i_2 \cdots i_m \cdot}, \quad (5.5.42)$$

where $\Omega_{i_1 i_2 \cdots i_m \cdot} := \Omega_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_{n-m}} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{n-m}}$. We should also note that after integrating over M only $(0,n)$-form part of $\underbrace{\frac{\partial^n \Omega}{\partial z^{\alpha} \partial z^{\beta} \cdots \partial z^{\rho}}}_{n \text{ times}}$ remains non-zero. Thus all we have to do for the calculation of the n-point Yukawa coupling is to calculate the holomorphic $(n,0)$-form Ω on M as a function on the complex-structure moduli space (roughly speaking, in our model as a function of ψ).

5.6 B-model on M_N^*

We will apply the formalism of the previous section to B-model on M_N^*. First, we have to determine the observables. Complex structure of M_N^* is parameterized by the coefficients $(a_0, \cdots, a_N)$ of the defining equation modulo linear transformation of variables compatible with the action of $Z_N^{\otimes N-2}$. As we have said in section 5.1, it kills N degrees of freedom of $N+1$ parameters. Thus we can take $X_1 X_2 \cdots X_N$ (or a_o) as the basis of $H^1(M_N^*, T^{1,0} M_N^*)$, and we have the following relation.

$$\begin{aligned} H^1(M_N^*, T^{1,0} M_N^*) &\simeq C[X_1 X_2 \cdots X_N] &(5.6.43) \\ \dim(H^1(M_N^*, T^{1,0} M_N^*)) &= \dim(H^{1,1}(M^N, C)) = 1 &(5.6.44) \end{aligned}$$

By extending (5.5.35) to the case of $H^m(M, \wedge^m T^{1,0} M)$, we can identify B-model observables as elements of $H^{N-2-m,m}(M_N^*, C)$. We define this subring of $H^{N-2}(M_N^*)$ as $H^{N-2}_{comp.}(M_N^*)$. Then we can generalize (5.6.44) into isomorphism between $H^{N-2}_{comp.}(M_N^*)$ and $H_e^*(M_N)$. It is based on the following fact.

Fact. Let W be defining equation of M_N^*, $X_1^N + X_2^N + \cdots + X_N^N - N\psi X_1 X_2 \cdots X_N$. $H^{N-2}_{comp.}(M_N^*) = \oplus_{m=0}^{N-2} H^{N-2-m,m}_{comp.}(M_N^*)$ is constructed from $Z_N^{\otimes N-2}$ invariant homogeneous polynomial of degree $Nm \quad (0 \le m \le N-2)$ modulo $\partial_i W$ by use of the

map,

$$P_{Nm}(X_1, X_2, \cdots, X_N) \mapsto m! \int_{\gamma} \frac{P_{Nm}(X)}{W^{m+1}} \omega \in H^{N-2-m,m}_{comp.}(M_N^*). \quad (5.6.45)$$

where

$$\omega := \sum_{i=1}^{N} (-1)^i X^i dX^1 \wedge \cdots \wedge d\hat{X}^i \wedge \cdots \wedge dX^N \quad (5.6.46)$$

(γ is a small one dimensional cycle winding around the hypersurface in CP^{N-1} defined by $W=0$. In (5.6.46), dividing by $Z_N^{\otimes N-2}$ can be considered as multiplication factor of $(\frac{1}{N})^{N-2}$ because singular locus caused by $Z_N^{\otimes N-2}$ is measure zero in the integrand. We will omit this factor from now on.)

Now, using this fact, we will determine the structure of $H^{N-2}_{comp.}(M_N^*)$. $Z_N^{\otimes N-2}$ invariant homogeneous polynomials $P_{Nm}(X) \quad (0 \le m \le N-2)$ are generated by $Z_N^{\otimes N-2}$ invariant homogeneous monomials of degree N.

$$X_1^N,\ X_2^N,\ \cdots,\ X_N^N,\ X_1 \cdots X_N \quad (5.6.47)$$

But we have to identify these monomials via relations $\partial_i W$.

$$\partial_i W = N \cdot X_i^{N-1} - N\psi X_1 \cdots \hat{X}_i \cdots X_N \quad (5.6.48)$$

In particular, (5.6.48) tells us that

$$X_i^N = \psi X_1 \cdots X_N \quad (\text{modulo } \partial_i W). \quad (5.6.49)$$

Thus we can choose $X_1 \cdots X_N$ as the generator of $P_{Nm}(X)/\partial_i W$. And we have the following result.

$$\begin{aligned} H^{n-2-m,m}_{comp.}(M_N^*) &\simeq C[m! \int_{\gamma} \frac{(X_1 \cdots X_N)^m}{W^{m+1}} \omega] &(5.6.50) \\ \dim(H^{N-2-m,m}_{comp.}(M_N^*)) &= 1 \end{aligned}$$

(5.6.51) tells us that $H^{N-2}_{comp.}(M_N^*) \simeq H_e^*(M_N)$.

5.7 Construction of Holomorphic $(N-2,0)$ form Ω

From **Fact.**, holomrphic $(N-2,0)$ form Ω is given as follows.

$$\Omega = \int_{\gamma} \frac{1}{W} \omega \quad (5.7.51)$$

Then we expand Ω by integral basis α_i $(i = 1, 2, \cdots, N-2)$ of $H^{N-2}_{comp.}(M^*_N, Z)$.

$$\begin{aligned} \Omega = (\int_{\Gamma_i} \Omega)\alpha_i &= (\int_{\Gamma_i}\int_{\gamma} \frac{\omega}{W})\alpha_i \\ &:= w_i(\psi)\alpha_i \end{aligned} \tag{5.7.52}$$

where Γ_i denotes $PD_{M^*_N}(\alpha_i)$. Since we integrate out all the form variables, $w_i(\psi)$'s are merely functions of ψ.

$w_i(\psi)$'s are known to satisfy certain differential equation (Picard-Fuchs equation), derived from the definition (5.7.53). We will determine the form of this differential equation. First, we return to the representation (5.1.17) and consider $w_i(a_0, a_1, \cdots, a_N)$ instead of $w_i(\psi)$.

$$w_i(a) := \int_{\Gamma_i}\int_{\gamma} \frac{\omega}{\sum_{j=1}^N a_j X_j^N + a_0 \prod_{j=1}^N X_j} \tag{5.7.53}$$

From the form of (5.7.53), we can easily see $w_i(a)$ satisfies the following equation.

$$(\sum_{i=1}^N a_i \frac{\partial}{\partial a_i} + 1) w_i(a) = 0 \tag{5.7.54}$$

$$(\frac{\partial}{\partial a_1}\frac{\partial}{\partial a_2}\cdots\frac{\partial}{\partial a_N} - (\frac{\partial}{\partial a_0})^N) w_i(a) = 0 \tag{5.7.55}$$

The invariance under rescaling of integration variables leads us to,

$$(a_i \frac{\partial}{\partial a_i} - a_N \frac{\partial}{\partial a_N}) w_i(a) = 0 \quad (i = 1, 2, \cdots, N-1) \tag{5.7.56}$$

Equations (5.7.54) and (5.7.56) are satisfied by making the ansatz,

$$w_i(a) = \frac{1}{a_0} W_i(\frac{a_1 a_2 \cdots a_N}{(a_0)^N}) \tag{5.7.57}$$

Then equation (5.7.55) becomes,

$$\Theta_y(\Theta_y^{N-1} + (-1)^{N-1} N y (N\Theta_y + 1)(N\Theta_y + 2)\cdots(N\Theta_y + N - 1)) W_i(y) = 0 \tag{5.7.58}$$

where $y = \frac{a_1 a_2 \cdots a_N}{(a_0)^N}$ and $\Theta_y = y\frac{d}{dy}$. We can ignore the factored operator Θ_y since it adds constant factor to the solution which have to be set to zero in calculation of $N-2$-point correlation function. If we set $a_1 = \cdots = a_N = 1$ and $a_0 = -N\psi$, we obtain the following equation for $w_i(\psi)$.

$$\begin{aligned} ((z\frac{d}{dz})^{N-1} - z(z\frac{d}{dz} + \frac{1}{N})(z\frac{d}{dz} + \frac{2}{N})\cdots(z\frac{d}{dz} + \frac{N-1}{N})) W_i(z) &= 0 \\ w_i(\psi) = z^{\frac{1}{N}} W_i(z) \qquad z &:= \frac{1}{\psi^N} \end{aligned} \tag{5.7.59}$$

The solutions of The Picard-Fuchs equation

We will adopt the D.Morrison's recipe [14] for the construction of the mirror map on Calabi-Yau 3-fold, also in our $(N-2)$-dimensional case. At first we will calculate the series solution around $z = 0$.
We substitute a series solution $W^0 = \sum_{n=0}^\infty a_n z^n$ into (5.7.60),and obtain a recursion relation such that

$$\begin{aligned} a_n &= \frac{(N(n-1)+1)(N(n-1)+2)\cdots(N(n-1)+N-1)}{n^{N-1}N^{N-1}} a_{n-1} \\ (\ n &= 1, 2, \cdots) \end{aligned} \tag{5.7.60}$$

Fixing the first term as $a_0 = 1$, we obtain that

$$a_n = \frac{(Nn)!}{(n!)^N N^{Nn}}. \tag{5.7.61}$$

Thus the series solution around $z = 0$ is

$$W_0 = \sum_{n=0}^\infty \frac{(Nn)!}{(n!)^N N^{Nn}} z^n. \tag{5.7.62}$$

The Picard-Fuchs equation (5.7.60) has $(N-1)$ solutions with singularities around $z = 0$ such as $(\log z)^0, (\log z)^1, \cdots, (\log z)^{N-1}$, since it is a ordinary differential equation of degree $(N-1)$. Now we want to obtain all of them. We introduce the following ansatz;

$$W_x \equiv \sum_{n=0}^\infty \frac{\{N(n+x)\}!}{(n+x)!^N N^{N(n+x)}} z^{n+x}. \tag{5.7.63}$$

In other words we have shifted all the n in (5.7.62) to $n+x$. Since (5.7.63) satisfies (5.7.61) for $n \geq 1$, W_x satisfies that

$$\left[(z\frac{\partial}{\partial z})^{N-1} - z(z\frac{\partial}{\partial z} + \frac{1}{N})(z\frac{\partial}{\partial z} + \frac{2}{N})\cdots(z\frac{\partial}{\partial z} + \frac{N-1}{N})\right] W_x = \frac{(Nx)!}{(x!)^N N^{Nx}} X^{N-1} z^x.$$

Differentiate both sides of the above equation i-th times with x, then set $x = 0$. Noting that the right-hand side becomes zero for $0 \leq i \leq N-2$, we find that $\partial_x^i W_x|_{x=0}$ for $0 \leq i \leq N-2$ is a solution of the Picard-Fuchs equation (5.7.60). Further calculation shows that

$$W^i \equiv \partial_x^i W_x\Big|_{x=0} = \sum_{j=0}^i {}_iC_j \sum_{n=0}^\infty \frac{\partial^j}{\partial x^j} a_n(0) \frac{z}{N}^n (\log\frac{z}{N})^{i-j}, \tag{5.7.64}$$

where

$$a_n(x) \equiv \frac{(N(n+x))!}{(n+x)!^N} = \frac{\Gamma(N(n+x)+1)}{\Gamma(n+x+1)^N}.$$

That means $\partial_x^i W_x|_{x=0}$ has the singularity of $(\log z)^i$ at $z=0$. From (5.7.64), we can easily see that W_i has the form,

$$W_i(z) = \sum_{j=1}^{i} {}_iC_j (\log z)^j y_{i-j}(z) + y_i(z), \tag{5.7.65}$$

where $y_i(z)$ is the non-negative power series of z. In particular,

$$W_0(z) = \sum_{n=0}^{\infty} \frac{(Nn)!}{(n!)^N N^{Nn}} z^n \tag{5.7.66}$$

$$W_1(z) = W_0(z)\log z + \sum_{n=1}^{\infty} \frac{(Nn)!}{(n!)^N N^{Nn}} \left(\sum_{i=1}^{n}\sum_{k=1}^{N-1} \frac{k}{i(Ni-k)}\right) z^n \tag{5.7.67}$$

5.8 Calculation of $\langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i) \rangle$

From the equation (5.5.37) and (5.7.53), we have

$$\begin{aligned} \langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(x) \rangle &= \int_{M_N^*} \Omega \wedge \frac{\partial^{N-2}}{\partial x^{N-2}} \Omega \\ &= w_i(x) \frac{d^{N-2}}{dx^{N-2}} w_j(x) \int_{M_N^*} \alpha_i \wedge \alpha_j \\ x &:= \log z \end{aligned} \tag{5.8.68}$$

We introduce new variable x for later convenience using the fact that (5.5.37) doesn't specify the choice of deformation parameter of complex structure. To proceed further, we have to determine $\int_{M_N^*} \alpha_i \wedge \alpha_j$. This can be done by using the following equation which follows from the fact that $\Omega \wedge \Omega$ doesn't include anti-holomorphic variables, or Kodaira-Spencer equation.

$$\int_{M_N^*} \Omega \wedge \Omega = z^{\frac{2}{N}} W_i(z) W_j(z) \int_{M_N^*} \alpha_i \wedge \alpha_j = 0 \tag{5.8.69}$$

Then we can determine $\int_{M_N^*} \alpha_i \wedge \alpha_j$ uniquely if we demand all the $(\log z)^i$ terms vanish when we expand $\int_{M_N^*} \Omega \wedge \Omega$ in terms of y_i's. And we have

$$\int_{M_N^*} \alpha_i \wedge \alpha_j = \delta_{i+j,N-2} ({}_{N-2}C_j)(-1)^j. \tag{5.8.70}$$

Combination of (5.8.69) and (5.8.70) leads us to,

$$\langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(x) \rangle = \sum_{j=0}^{N-2} {}_{N-2}C_j (-1)^j w_j(x) \frac{d^{N-2}}{dx^{N-2}} w_{N-2-j}(x) \tag{5.8.71}$$

Next, we will evaluate $\langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(x) \rangle$ as the function of x. Let us introduce the following integral,

$$\begin{aligned} R_k &= \int_{M_N^*} \Omega \wedge \frac{\partial^k}{\partial x^k} \Omega \\ &= \sum_{j=0}^{N-2} {}_{N-2}C_j (-1)^j w_j(x) \frac{d^k}{dx^k} w_{N-2-j}(x) \end{aligned} \tag{5.8.72}$$

R_k equals to zero for $k = 0, 1, \cdots, N-3$ because of Kodaira-Spencer equation. Then by using Leibnitz rule successively , we obtain the formula,

$$\begin{aligned} (-1)^N R_{N-1} &= \sum_{i=1}^{N-1} (-1)^{N-1-i} {}_{N-1}C_i \frac{d^i}{dx^i} R_{N-1-i} \\ &= (-1)^{N-1} R_{N-1} + (-1)^{N-2}(N-1) \frac{d}{dx} R_{N-2} \\ \Longleftrightarrow \quad & R_{N-1} = \frac{N-1}{2} \frac{d}{dx} R_{N-2}. \end{aligned} \tag{5.8.73}$$

We can verify another relation between R_{N-1} and R_{N-2}.
Since w_i's are solutions of the following differential equation obtained from (5.7.60),

$$\begin{aligned} ((\partial_x - \frac{1}{N})^{N-1} - e^x \prod_{j=0}^{N-2} (\partial_x + \frac{j}{N})) w_j(x) &= 0 \\ \Longleftrightarrow ((1-e^x)(\partial_x)^{N-1} - \frac{N-1}{N}((1-e^x) + \frac{N}{2}e^x)(\partial_x)^{N-2} & \\ + \sum_{j=0}^{N-3} b_j(x)(\partial_x)^j) w_j(x) &= 0 \end{aligned} \tag{5.8.74}$$

we have

$$(1-e^x) R_{N-1} - \frac{N-1}{N}((1-e^x) + \frac{N}{2}e^x) R_{N-2} = 0. \tag{5.8.75}$$

Combining (5.8.73) and (5.8.75), we get the ordinary differential equation of $R_{N-2} = \langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(x) \rangle$.

$$\frac{d}{dx} R_{N-2} = (\frac{2}{N} + \frac{e^x}{1-e^x}) R_{N-2} \tag{5.8.76}$$

We can solve (5.8.76) explicitly and we reach the final result of B-model in this Chapter.

$$\langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(x) \rangle = \text{const.} \frac{e^{\frac{2}{N}x}}{1-e^x} \tag{5.8.77}$$

5.9 The mirror map and the translation into A-model

Let us now construct a mirror map between the moduli space of A and B models. According to Morrison [14] the mirror map can be obtained by the following process. Let the solution of the Picard-Fuchs equation which is regular at maximally unipotent point, say, $z=0$ be $w_{reg.}(x)$ (in our case $w_0 = e^{\frac{1}{N}x}W_0(x)$). And let $w_{log.}(x)$ be the solution which has a singularity of $\log z$ at $z=0$ (also in our case, $w_1 = e^{\frac{1}{N}x}W_1(x)$). The mirror map is

$$t = \frac{w_{log.}(x)}{w_{reg.}(x)} = \frac{w_1(x)}{w_0(x)} = \frac{W_1(x)}{W_0(x)} \tag{5.9.78}$$

where t is a coordinate of the moduli space of A-model on M_N, or as we have mentioned in Chapter 1, coupling constant of A-model.
We will adapt his idea in the arbitrary dimensional case. Now we want to translate the $(N-2)$-point correlation function of B-model on M_N^* to the one of A-model on M_N. The $(N-2)$-point correlation function of B-model is given by

$$\langle \prod_{i=1}^{N-2} \mathcal{O}_B(z_i)(t) \rangle_{M_N^*} = \int_M \Omega \wedge \frac{\partial^{N-2}}{\partial x^{N-2}} \Omega = \frac{e^{\frac{2}{N}x}}{1-e^x}. \tag{5.9.79}$$

We should note that the correlation function on the B-model is not a scalar on the moduli space but take the value on the square of the line bundle on which the holomorphic $(N-2,0)$-form lives. Therefore we should consider not only the effect of the transformation of the coordinate but also the gauge choice of Ω. Following Candelas et.al. and Morrison, we will adapt the gauge ;

$$\Omega \to \frac{\Omega}{w_0(x)}.$$

The B-model operator $\mathcal{O}_B$ (in our it is represented by $X_1X_2\cdots X_N$) corresponds to the A-model operator $\mathcal{O}_e$ induced from Kähler form e on M_N. Hence we have

$$\begin{aligned}
\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle_{M_N} &= \int_{M_N^*} \frac{\Omega}{w_0(x)} \wedge \frac{\partial^{N-2}}{\partial t^{N-2}} \frac{\Omega}{w_0(x)} \\
&= \frac{1}{w_0(x)^2} \int_{M_N^*} \Omega \wedge \frac{\partial^{N-2}}{\partial t^{N-2}} \Omega \\
&= \frac{1}{w_0(x)^2} \int_{M_N^*} \Omega \wedge (\frac{\partial x}{\partial t}\frac{\partial}{\partial x})^{N-2} \Omega \\
&= \frac{1}{w_0(x)^2} (\frac{dx}{dt})^{N-2} \int_{M_N^*} \Omega \wedge \frac{\partial^{N-2}}{\partial x^{N-2}} \Omega
\end{aligned} \tag{5.9.80}$$

All the nontrivial equivalences are guaranteed by the argument of section 5.5. Hence combining (5.9.79) and (5.9.80), we obtain the following formula.

$$\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle = \frac{1}{W_0(x(t))^2} \frac{1}{1-e^{x(t)}} (\frac{dx}{dt})^{N-2} (\text{const.}). \tag{5.9.81}$$

Combining (5.9.78) and (2.4.77) we obtain

$$\begin{aligned}
t &= N \log N - (\sum_{n=1}^{\infty} b_n e^{nx})/(\sum_{n=0}^{\infty} a_n e^{nx}) - x \\
a_n &= \frac{(Nn)!}{(n!)^N N^{Nn}}, \quad b_n = a_n (\sum_{i=1}^{n} \sum_{k=1}^{N-1} \frac{k}{i(Ni-k)})
\end{aligned} \tag{5.9.82}$$

If we can represent e^x as power series of e^{-t}, $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle$ takes the form of (2.2.43). This can be done as follows. We rewrite (5.9.82) in a more convenient form.

$$-t = -N \log N + x + \sum_{n=1}^{\infty} c_n e^{nx} \tag{5.9.83}$$

where

$$\sum_{n=1}^{\infty} c_n z^n = (\sum_{n=1}^{\infty} b_n z^n)/(\sum_{n=0}^{\infty} a_n z^n) \tag{5.9.84}$$

Then we assume the following expansion,

$$x = -t + N \log N + \sum_{n=1}^{\infty} \gamma_n e^{-nt} \tag{5.9.85}$$

γ_n can be determined from compatibility of (5.9.83) and (5.9.85). We put (5.9.85) into (5.9.81) and determine the constant with the assumption that constant term of $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle$ coincides the classical value $\int_{M_N} e^{N-2} = N$. This assumption is equivalent to large radius limit which asserts the theory turns into classical one when $t \to \infty$.
Then we have $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle$ represented in the form of (2.2.43),

$$\begin{aligned}
&\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i)(t) \rangle \\
&\quad = N + \langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_1 e^{-t} + \langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_2 e^{-2t} + \langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_3 e^{-3t} + \cdots
\end{aligned}$$

where

$$\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_1 = N^{N+1}(1 - 2a_1 - {}_{N-2}C_1(b_1))$$

$$
\begin{aligned}
&= N^{N+1} - (N-2)N(N!)\sum_{i=1}^{N-1}\frac{N-i}{i} - 2N(N!) \\
&\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_2 = N^{2N+1}\{1 - 2a_1 - b_1 + 3a_1^2 - 2a_2 + 2a_1 b_1 + \\
&{}_{N-2}C_1(-b_1 + 4a_1 b_1 + 2b_1^2 - 2b_2) + {}_{N-2}C_2 b_1^2\} \\
&\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_3 = N^{3N+1}\{1 - 2a_1 - 2b_1 + 3a_1^2 + 5a_1 b_1 + \frac{3}{2}b_1^2 - 2a_2 - b_2 \\
&-4a_1^3 - 8a_1^2 b_1 - 3a_1 b_1^2 + 6a_1 a_2 + 4a_2 b_1 + 2a_1 b_2 - 2a_3 \\
&+{}_{N-2}C_1(-b_1 + 3b_1^2 + 4a_1 b_1 - 2b_2 - 10a_1^2 b_1 - 15a_1 b_1^2 \\
&-\frac{9}{2}b_1^3 + 7a_1 b_2 + 5a_2 b_1 + 9b_1 b_2 - 3b_3) \\
&+{}_{N-2}C_2(b_1^2 - 6a_1 b_1^2 - 4b_1^3 + 4b_1 b_2) \\
&+{}_{N-2}C_3(-b_1^3)\} \qquad (5.9.86)
\end{aligned}
$$

Then we write out numerical results from $N = 5$ to $N = 10$.

$$
\begin{aligned}
\langle \prod_{j=1}^{3} \mathcal{O}_e(z_j) \rangle &= 5 + 2875e^{-t} + 4876875e^{-2t} + 8564575000e^{-3t} + \cdots \\
\langle \prod_{j=1}^{4} \mathcal{O}_e(z_j) \rangle &= 6 + 120960e^{-t} + 4136832000e^{-2t} \\
&\quad +148146924602880e^{-3t} + \cdots \\
\langle \prod_{j=1}^{5} \mathcal{O}_e(z_j) \rangle &= 7 + 3727381e^{-t} + 2637885990187e^{-2t} \\
&\quad +19270929541081087 87e^{-3t} + \cdots \\
\langle \prod_{j=1}^{6} \mathcal{O}_e(z_j) \rangle &= 8 + 106975232e^{-t} + 1672023727001660e^{-2t} \\
&\quad +2661169233308169509 2736e^{-3t} + \cdots \\
\langle \prod_{j=1}^{7} \mathcal{O}_e(z_j) \rangle &= 9 + 3103936929e^{-t} + 1165013014173543657e^{-2t} \\
&\quad +441297815019235844688286425e^{-3t} + \cdots \\
\langle \prod_{j=1}^{8} \mathcal{O}_e(z_j) \rangle &= 10 + 94327552000e^{-t} + 930496455109619200000e^{-2t} \\
&\quad +9217712440694086335170560000000e^{-3t} + \cdots \qquad (5.9.87)
\end{aligned}
$$

We can see complete coincidence with the result of geometrical (A-model) calculation (3.5.137) and (3.2.47)! We see that in general $\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_d$ has the structure

$$
\langle \prod_{i=1}^{N-2} \mathcal{O}_e(z_i) \rangle_d = N^{dN+1} - (\text{correction terms}) \qquad (5.9.88)
$$

Note that this result naturally represents the structure of correction terms argued at Section 3.1. Morrison and Plesser proposed that the top term is explained from the 1-loop level effective action of Gauged Linear Sigma Model.
We hope these structures are explained in the framework of Section 3.1 in the future. Of course, by complete coincidence with the result of Geometrical Calculations, we give practical proof of Mirror Symmetry Conjecture at correlation function level in case of A-model on M_N and B-model on M_N^*.

Chapter 6

Conclusion

In this thesis, we solved topological sigma model(A-model) from CP^1 to M_N^k both in pure A-model case and in case coupled to gravity. We gave integral representation of generating function of correlation functions for gravity coupled theory and one variable polynomial representations of quantum cohomology algebra for pure A-model.

The fundamental strategy is the use of the fact that algebraic hypersurface in CP^{N-1} is realized as zero locus of homogeneous algebraic equation of CP^{N-1}. In other words, the assertion of this thesis is that topological sigma model on these hypersurface should be treated as the natural extension of these algebraic constraint to the moduli space of embedding space, i.e., CP^{N-1}. The main feature of the discussion in the algebraic category is that in that category, topological invariants are counted as the number of equation of algebraic equations. We can never reach this point of view in the category of differential geometry, or just seeing the local connection or curvature. Because of this feature, which is expressed as the "solidity" of algebraic manifolds or corresponding moduli space, we can compute the correlation functions by " geometrical approach". Of course, in this thesis, these algebraic equations on moduli space are written in terms of Chern classes of "holomorphic" vector bundles, but the spirit is the same. With this fundamental understanding, what is needed is the technical developments and these are the "fruits" of recent developments in two-dimensional topological field theory. And we can pursue the analysis of topological sigma model as the algebraic geometry of moduli space. We think that the pursuit of this point of view is never seen in other works because of technical difficulty in classical algebraic geometry.

This spirit is reflected in another flow of this thesis , which is the search for the structure of $\langle \prod_{j=1}^{(N-k)d+N-2} \mathcal{O}_e(z_j)\rangle_d$ for pure matter theory. Our result tells us that our speculation in Section 3.1 which argue that moduli space is realized as the zero locus of algebraic equations derived from the defining equation of classical target space is right. We think our result of $\mathcal{O}_e^{N-1} = k^k \mathcal{O}_e^{k-1} e^{-t}$ is the reflection

of the solidity. What remains to show is to calculate $\langle \prod_{j=1}^{N-2} \mathcal{O}_e(z_j)\rangle_{d,M_N^N}$ in view of analysis of pure matter moduli space from CP^1 to CP^{N-1}. We think this problem is equivalent to the exact construction of $\mathcal{M}_{0,d}^{CP^{N-1}}$.

Generalization of our strategy to various weighted projective space is interesting. Because this case reduces to changing of embedding space into weighted projective space. It is treated in [32] at the level of Section 3.1. So more accurate treatment is expected. Another interesting question is that the search for field theory counter part of this solidity of algebraic manifolds. Witten's gauged linear sigma model is one of the approaches in these flow. But accurate treatment is yet to be done. We think the relation of this model and our remaining problem should be pursued further.

Lastly, we have to mention the mirror symmetry. This tells us that our notion of solidity and the structure of $N = 2$ super conformal field theory (special geometry) have deep relation. Because the result from mirror symmetry naturally reflects the structure speculated in section 3.1. And obviously, the calculation of correlation functions from mirror symmetry is very rigid, with no wasting part. So we also have to search for the geometrical (in A-model) meaning of period integral. In other words, reverse transformation from A-model to B-model is very important for deeper understanding of this symmetry.

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Bibliography

[1] M.Kontsevich. *Enumeration of Rational Curves via Torus Actions* hep-th 9405035

[2] M.Nagura and K.Sugiyama, "*Mirror Symmetry of K3 Surface*", UT-663, to appear in Int.J. of Mod. Phys.

[3] M.Jinzenji and M.Nagura, "*Mirror Symmetry and an Exact Calculation of $N-2$ point Correlation Function on Calabi-Yau Manifold Embedded in CP^{N-1}* UT-680, to appear in Int.J. of Mod. Physics.

[4] A.Bertram. *Modular Schubert Calculus* University of Utah Preprint (1994)

[5] K.Intriligator. *Fusion residues* Modern Physics letters A6 (1991), Number 38, pp. 3543-3556.

[6] M.Jinzenji. Construction of Free Energy of Calabi-Yau Manifold embedded in CP^{N-1} via Torus Actions UT-HEP-703-95

[7] B.R. Greene, D.R. Morrison and M.R. Plesser, Mirror Manifolds in Higher Dimension, CLNS-93/1253, IASSNS-HEP-94/2, YCTP-P31-92

[8] E. Witten *Mirror Manifolds and Topological Field Theory*, in *Essay on Mirror Manifolds*, ed. S. -T. Yau, (Int. Press. Co., Hong Kong, 1992), pp.120-180.

[9] E. Witten Toplogical Quantum Field Theory IAS-PUB-IASSNS-HEP-87/72

[10] E. Witten Topological Sigma Models Commun. Math. Phys. 118, 411-449

[11] K.Kodaira and D.C.Spencer, Ann. Math. 67 (1958) 382; 67 (1958) 403; 71 (1960) 43.

[12] K.Kodaira, Complex manifold and Deformation of complex structure Iwanami

[13] W. Lerche, D. Smit and N. Warner, Nucl. Phys. **B372** (1992) 87.

[14] D.Morrison,"*Picards-Fuchs Equations and Mirror Maps For Hypersursaces*", in "*Essays on Mirror Manifolds*, ed. S. -T. Yau, (Int. Press. Co., Hong Kong, 1992).

[15] P. Candelas and X. de la Ossa, Nucl. Phys. **B355** (1991) 415.

[16] P. Candelas, X. de la Ossa, P. Green and L. Parkes, Phys. Lett. **258B** (1991) 118; Nucl. Phys. **B359** (1991) 21.

[17] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau. Mirror Symmetry, mirror map and applications to Calabi-Yau hypersurfaces HUTMP-93/0801, LMU-TPW-93-22 (hep-th/9308122)

[18] T. Oda, H.S. Park. Tôhoku Math J.43 (1991) 375.

[19] P. Griffiths and J. Harrith, *Principles of Algebraic Geometry*, (Wiley, 1978)

[20] S. Katz "*Rational curves on Calabi-Yau manifolds: verifying predictions of Mirror Symmetry*" Oklahoma State University preprint OSU-M-92-3,1992

[21] M.Kontsevich , Y.Manin. *Gromov-Witten Classes,Quantum Cohomology,and Enumerative Geometry* Commun.Math.Phys.164 (1994) 525;

[22] R.Dijkgraaf, E.Witten. Nuclear Physics B342 (1990) 486;

[23] C.Itzykson. *Counting rational curves on rational surfaces* Saclay preprint T94/001

[24] S.Cecotti,C.Vafa. Nuclear Physics B367 (1991) 359;

[25] B.Dubrovin. *Geometry of 2D Topological Field Theory* Preprint SISSA-89/94/FM hep-th/9407018

[26] C.Vafa. *Topological Mirrors and Quantum Rings* HUTP-91/A059

[27] R.Bott *A residue formula for holomorphic vector fields* Jour. Diff. Geom. 1 (1967) 311-330

[28] T.Eguchi and S.K.Yang Mod. Phys. Lett. A, Vol.5, No.21 (1990) 1693-1701

[29] R.Dijkgraaf *Intersection Theory, Integrable Hierarchies and Topological Field Theory* IASSNS-HEP-91/91

[30] E.Witten *On the topological phase of two dimensional gravity* Nucl.Phys.**B340** (1990) 281

[31] M.Kontsevich *Intersection theory on the moduli space of curves and the matrix Airy function* Commun.Math.Phys.**147** (1992),1-23

[32] D.R.Morrison and M.R.Plesser *Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties* DUKE-TH-94-78 IASSNS-HEP-94/82 hep-th/9412236

[33] K.Hori *Constraints for Topological Strings in* $D \geq 1$ UT-694

[34] P.S.Aspinwall and D.R.Morrison *Topological field theory and rational curves* Commun.Math.Phys. **151** (1993),245-262

[35] C.Vafa and E.Witten *A strong coupling test of S-duality* hep-th/940874,HUTS-94/A017,IASSNS-HEP-94-54

[36] S.Keel. *Intersection theory of moduli spaces of n-stable pointed curves of genus zero* Trans. Ams, 330 (1992), 545-574.

[37] V.V.Batyrev. *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties* Preprint, Universität-GHS-Essen,1992