Boundary Integral Equations for the Time-Domain and Time-Independent Analyses of 2D Non-Planar Cracks

（境界積分方程式法で解く
自由形状２次元亀裂の動力学と静力学）

多田 卓

①

学位論文

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Boundary Integral Equations for the Time-Domain and Time-Independent Analyses of 2D Non-Planar Cracks

by

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Abstract

Understanding of the effects of non-planar fault geometry is a crucial key to a better understanding of the dynamics of earthquake rupturing. However, available numerical methods have practically precluded modeling of fault mechanics based on non-planar geometry, except for a few recent pioneering works.

I have derived a set of rigorous boundary integral equations, both time-domain (elastodynamic) and time-independent (elastostatic), for the analysis of arbitrarily shaped 2D anti-plane / in-plane crack(s) located in an infinite homogeneous isotropic medium. The hypersingularities of the integration kernels were removed after the regularization method of Koller, Bonnet and Madariaga (1992) and Cochard and Madariaga (1994). These formulations, rendered in a unified nomenclature, significantly broaden the range of fault mechanics problems to which the boundary integral equation method (BIEM) can be applied. The specific procedure of their derivation is described in full in Chapter 2 of the present paper.

In Chapter 3, the piecewise constant interpolation is introduced as a specific method of numerical implementation of the BIEM formulations, and the numerical results for some simple cases, both time-dependent and time-independent, are verified against the known analytic solutions. Some demonstrative analyses of crack mechanics based on non-planar geometry are described in Chapter 4.

In Chapter 5, it is pointed out that in the case of in-plane shear faulting a smoothly curved crack cannot be represented as a limiting case of a chain of finite line elements as the discretization interval tends to zero, a situation which previous researchers were apparently unaware of. The two geometries may produce different normal traction distributions along the crack, so that care should be taken so as not to misinterpret the numerical results. It is also shown that no similar problem arises in the cases of anti-plane shear and open in-plane faulting.

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1 Introduction

1.1 Background

In the seismological fracture theory, heterogeneous behavior on an earthquake fault have most often been ascribed to heterogeneous distributions of strength, stress drop or slip characteristics on a single fault plane (see review by Dmowska and Rice, 1986). Among others, "barrier" and "asperity" are the two terms that have been most popularly and most extensively used to describe inhomogeneous zones on an earthquake fault plane. Though these two terms have often been used in a vague and ambiguous way, the former might be defined as a site where rupture during an earthquake is impeded or arrested, while the latter might be taken as a region characterized by an exceptionally large moment drop during rupture (Scholz, 1990, Section 4.5). However, it has been implicitly assumed that the concept of fault inhomogeneities is not so much a representation of real changes in the material properties as it is a representation of geometrical irregularities restated in the context of planar geometry (e.g., Das and Scholz, 1981; Andrews, 1989; Cochard and Madariaga, 1994).

Evidently, the source of heterogenous behavior in natural earthquakes must be much more complicated, possibly involving fault bends and bifurcations, activation of subsidiary faults, transfer of slip onto a different fault plane (formation of fault steps), involvement of tensile microcracks, and so on. In fact, in the field survey of the Nojima fault that was ruptured in the 1995 Hyogoken-Nambu earthquake, the surface fault trace has been recognized as a series of generally linear but discontinuous segments, with an evidence of bifurcation near the southwestern end (Nakata and Yomogida, 1995).

Field evidences for tensile cracks as secondary features in shear fault zones are abundant (e.g., Segall and Pollard, 1983; Martel and Pollard, 1989) and kink cracks, possibly involving tensile displacements, that develop at the tip of a shear fault has been theoretically studied (Jeyakumaran, 1995, and references therein). Experimental studies of rock failure in anti-plane (Knauss, 1970; Cox and Scholz, 1988) and in-plane (Petit and Barquins, 1988; Reches and Lockner, 1994) modes even suggest that shear cannot exist as a primary fracture mechanism but can only be a macroscopic fracture phenomenon which must necessarily involve formation of tensile microcracks.

A better understanding of the effects of non-planar fault geometry is thus a crucial key to a better understanding of the dynamics of earthquake rupturing. Elastostatic (time-independent) and quasi-static analyses of interactions among closed in-plane shear fault segments (e.g., Segall and Pollard, 1980; Bilham and King, 1989; Aydin and Du, 1995; Wei and De Bremaecker, 1995b) as well as those among tensile (open in-plane) fault segments (Du and Aydin, 1991; Olson and Pollard, 1991; Reches and Lockner, 1994) are fairly abundant in literature. However, elastodynamic (time-dependent) analysis of crack(s) of non-planar geometry has been carried out only quite recently. Koller *et al.* (1992) proposed a numerical method for the dynamic analysis of 2D anti-plane shear cracks of arbitrary shape, but their numerical application to non-planar crack problems went no further than a mere preliminary one. Other investigations are confined to non-coplanar (mutually parallel) crack problems: the study by Harris and Day (1993) for two non-coplanar 2D cracks in in-plane shear, and those by Yamashita and Umeda (1994), Kame and Yamashita (1996) and Umeda *et al.* (1996) for two or more non-coplanar 2D cracks in anti-plane shear. The present study is devoted to the development of a new comprehensive numerical method for the analyses of 2D cracks of arbitrary geometry.

Three modes of fracture, modes I, II and III, are often referred to in the literature of fracture mechanics. Mode III corresponds to the anti-plane shear motion, mode II to the in-plane shear motion, and mode I to the in-plane tensile motion, as illustrated in Figure 1. Motion of a closed in-plane crack is described by pure mode II motion, whereas that of an open in-plane crack is a mixture of mode I and II motions.

1.2 The boundary integral equation method for crack analysis

So far, three different numerical approaches have been used in the study of dynamic earthquake source mechanics. One of them, the finite difference method (FDM) (Andrews, 1976b; Mikumo and Miyatake, 1978; Day, 1982; Virieux and Madariaga, 1982) discretizes the equations of motion by representing the model domain by a regularly spaced rectangular grid system. A second, the boundary integral equation method (BIEM) (Kostrov, 1966, 1975; Burridge, 1969; Dmowska and Kostrov, 1973; Crouch, 1976; Andrews, 1976a, 1985, 1994; Das and Aki, 1977; Das, 1980; Cheung and Chen, 1987; Fleck, 1991;

Nishimura, 1994; Jeyakumaran and Keer, 1994), solves integral equations that relate the slip on the crack with the stress on the crack, where the rest of the model space is not explicitly concerned in the formulation. The third, the finite element method (FEM) (e.g., Wei and De Bremaecker, 1995a), is akin to the FDM except that the model domain is divided into a mesh of element of simple (e.g. triangular) shape that need not be regularly spaced.

As was discussed by Das and Kostrov (1987) and by Koller *et al.* (1992), the FDM permits the introduction of inhomogeneous properties of the medium, but the principal numerical shortcomings of the FDM concern the resolution of stress near the crack tip, the numerical dispersion by the computational grid, the necessity to solve the equations all over the model domain, and the fact that treatment of non-planar crack geometry is practically prohibited by the configuration of the grid scheme that has two orthogonal axes (although Inoue and Miyatake, 1995, have recently developed an FDM code for kinked fault problem). On the other hand, the BIEM formulation is more efficient and flexible for the problem of cracks in a homogeneous medium, in that the equation has to be solved only inside the crack and also in that non-planar crack geometry is admissible. The FEM can, in principle, deal with non-planar geometry, but its application to dynamic crack propagation analysis has been practically restricted to 2D straight crack problems, because of the difficulties in the remeshing procedures (Geubelle and Rice, 1995). Thus in the sequel *we shall be exclusively concerned with the BIEM*, which is believed to be the most suitable approach to dynamic analyses of non-planar cracks.

The most generalized BIEM formulations for the time-dependent (elastodynamic) analysis of 3D crack of arbitrary geometry was obtained by Zhang and Achenbach (1989) and by Sládek and Sládek (1984), in the *Fourier frequency domain* and the *Laplace domain* respectively. However, since both of these generalized formulations are concerned with the transient response of a stationary crack, they do not lend themselves to the analyses of cracks that propagates with time. Thus a *time-domain* formulation is necessary when the geometry of the studied crack is not stationary.

Kostrov (1966, 1975), Das and Aki (1977), Das (1980) and Andrews (1985, 1994) used integral representations that express the slip (also called displacement discontinuity) Δu as a convolution of the traction T with an operator (kernel) $\hat{K}$:

$$\Delta u(s,t) = \int\int d\xi d\tau\ \hat{K}(s,t;\xi,\tau) * T(\xi,\tau), \tag{1}$$

where s and ξ denote location along the crack and t and τ denote time. In this case, the slip distribution is obtained by a simple forward convolution procedure once the traction distribution is known. However, this class of approach allows only the study of planar crack(s) in an infinite domain, in which case the kernel $\hat{K}$ is explicitly known (Koller *et al.*, 1992).

There is another class of integral representation that expresses the traction as a convolution of the slip with an operator (kernel) $\hat{L}$:

$$T(s,t) = \int\int d\xi d\tau\ \hat{L}(s,t;\xi,\tau) * \Delta u(\xi,\tau). \tag{2}$$

With this formulation, solving for the slip distribution with the traction known is an inverse problem. This class of approach is more versatile in that it is applicable to the general case of non-planar crack problems. Thus *the system (2) shall be used throughout the present study*. Moreover, Das and Kostrov (1987) pointed out that the system (2) requires a smaller integration domain than does the system (1), because the slip vanishes outside the crack and the locked part of the fault may be excluded from the integration domain.

It should be noted that in the formulation (2) hypersingular terms (Martin and Rizzo, 1989) appear in the convolution integral, which requires special attention in the numerical treatment. Use of the Fourier wavelength domain instead of the usual spatial coordinate (Geubelle and Rice, 1995) can suppress the occurrence of such singularities. However, as long as the boundary integral equations (BIEs) are formulated in the usual spatial coordinate, the hypersingular integrals should be numerically evaluated in some way or other. One approach directly evaluates the hypersingular integrals in the sense of Hadamard finite part integrals, while another approach rewrites (or *regularizes*) the hypersingular integrals into an equivalent form which involves only weakly singular integrals, which are at most integrable in terms of Cauchy principal values (Martin and Rizzo, 1989; Koller *et al.*, 1992).

Sládek and Sládek (1984), who took the latter approach, demonstrated how the hypersingular BIEs, for the time-dependent 3D crack analysis in the Laplace domain, may be converted, through integration by parts, into a more weakly singular form. Aided by their method, Koller *et al.* (1992) derived, for the first time, a regularized BIE in the time domain for an arbitrarily shaped 2D anti-plane crack, although their expressions for non-planar cracks were still somewhat cumbersome.

Koller *et al.*'s (1992) work was continued by Cochard and Madariaga (1994), who concentrated on the special case of a straight anti-plane crack. They reduced Koller *et al.*'s (1992) integral equation to a more easily tractable form and devised a sophisticated semi-analytic method of its numerical solution. Their approach was later extended by Yamashita and Fukuyama (1996) and by Kame and Yamashita (1996) to incorporate the case of more than one non-coplanar anti-plane straight cracks. A continuation of these studies in a different direction was realized by Fukuyama and Madariaga (1995), who dealt with both time-dependent and -independent analyses of a 3D plane crack. The previous BIEM studies for crack analysis are summarized and classified in Table 1.

In the present study I enlarge Koller *et al.*'s (1992) and Cochard and Madariaga's (1994) BIEM approach to incorporate a much broader variety of 2D crack mechanics analysis. Sets of BIEM formulations for both *time-domain* and *time-independent* analyses of 2D non-planar both *anti-plane* and *in-plane* cracks are derived. As is evident from Table 1, the formulation for the time-domain analysis of non-planar 2D *in-plane* cracks has first been achieved in the present study. Another importance of the present study consists in that it renders in a unified nomenclature the BIEM formulations for different settings, which have previously been studied separately by different authors based on inconsistent terminologies.

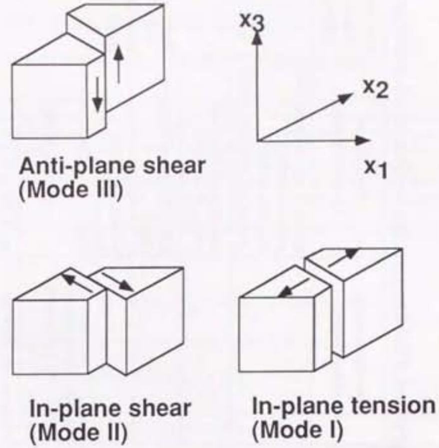

Figure 1: Anti-plane and in-plane motions. The three modes of fracture.

Relevance to previous BIEM approaches to shear crack analyses

	Time-independent (Elastostatic)	Time-domain (Elastodynamic)	Fourier-domain or Laplace-domain (Elastodynamic)
2D planar anti-plane		Kostrov (1966); Burridge (1969) Andrews (1976a) Das and Aki (1977) Cochard and Madariaga (1994)	
2D non-coplanar anti-plane		Yamashita and Fukuyama (1995); Kame (1995)	
2D non-planar anti-plane		Koller, Bonnet and Madariaga (1992)	Sládek and Sládek (1984)
2D planar in-plane		Das and Aki (1977) Andrews (1985)	Zhang and Achenbach (1989)
2D non-planar in-plane	Dmowska and Kostrov (1973) Fleck (1991); Jeyakumaran and Keer (1994)	present study	
3D planar	Fukuyama and Madariaga (1995)	Das (1980) Fukuyama and Madariaga (1995)	
3D non-planar			

Table 1: Classification of previously published BIEM modeling studies of crack mechanics problems. The scope of the present study is denoted by the shaded part of the table.

2 Formulation of the boundary integral equations

2.1 Representation theorem

I start from the *dynamic* (*time-dependent*) representation theorem that expresses the elastic displacement field over the entire medium in terms of the slip distribution along the crack(s). Assuming that the medium is at rest with no slip for time $t \leq 0$ and also that the traction is continuous across the crack(s), we have, for the problem of one or more crack(s) located in an infinite homogeneous isotropic elastic medium (e.g., Aki and Richards, 1980; Section 3.1),

$$\begin{aligned} u_l(\vec{x},t) &= \int_\Gamma d\xi \int_{-\infty}^{\infty} d\tau \Delta u_i(\xi,\tau) c_{ijpq} n_j(\xi) \frac{\partial}{\partial y_q} G_{lp}(\vec{x},t-\tau;\vec{y}(\xi),0) \\ &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau) c_{ijpq} n_j(\xi) \frac{\partial}{\partial x_q} G_{lp}(\vec{x},t-\tau;\vec{y}(\xi),0), \end{aligned} \tag{3}$$

where $u_l(\vec{x},t)$ is the displacement in the l-th direction at position $\vec{x}$ and time t, Γ the whole length of the crack trace(s), ξ the arc length along Γ, $\Delta u_i(\xi,\tau)$ the slip on the crack in the i-th direction at arc length ξ and time τ, c_{ijpq} the elastic constants, $\vec{n}(\xi)$ the unit vector normal to the crack trace at arc length ξ that is directed to the left when seen toward the orientation of the increasing ξ, $\vec{y}(\xi)$ the location of the position on the crack at arc length ξ, $G_{lp}(\vec{x},t-\tau;\vec{y},0)$ the displacement Green function denoting the displacement in the l-th direction observed at position $\vec{x}$ and time $t-\tau$ due to a unit force in the p-th direction applied at position $\vec{y}$ and time 0, and summation over the repeated indices is implied. See Figure 2 for the nomenclature.

This leads to the following integral representation of the stress field in terms of the slip on the crack(s):

$$\begin{aligned} \sigma_{kl}(\vec{x},t) &= \frac{1}{2} c_{klrs} \left(\frac{\partial u_s}{\partial x_r} + \frac{\partial u_r}{\partial x_s} \right) \\ &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau) c_{ijpq} n_j(\xi) \frac{\partial}{\partial x_q} \Sigma_{klp}(\vec{x},t-\tau;\vec{y}(\xi),0), \end{aligned} \tag{4}$$

where $\sigma_{kl}(\vec{x},t)$ is the kl-component of the stress at position $\vec{x}$ and time t and

$$\Sigma_{klp}(\vec{x},t-\tau;\vec{y},0) \equiv c_{klrs} \frac{\partial}{\partial x_s} G_{rp}(\vec{x},t-\tau;\vec{y},0) \tag{5}$$

is the stress Green function denoting the kl-component of the stress observed at position $\vec{x}$ and time $t-\tau$ due to a unit force in the p-th direction applied at position $\vec{y}$ and time 0.

In the *time-independent* problem, a parallel representation theorem holds which does not include time dependence:

$$u_l(\vec{x}) = -\int_\Gamma d\xi \Delta u_i(\xi) c_{ijpq} n_j(\xi) \frac{\partial}{\partial x_q} G_{lp}(\vec{x};\vec{y}(\xi)) \tag{6}$$

$$\sigma_{kl}(\vec{x}) = -\int_\Gamma d\xi \Delta u_i(\xi) c_{ijpq} n_j(\xi) \frac{\partial}{\partial x_q} \Sigma_{klp}(\vec{x};\vec{y}(\xi)). \tag{7}$$

It would be informative to refer here to the equation of motion for the Green functions, which shall be used later in the manipulations of formulae:

$$\rho \frac{\partial^2}{\partial t^2} G_{ik} = \frac{\partial}{\partial x_j} \Sigma_{ijk}, \tag{8}$$

where ρ denotes the density.

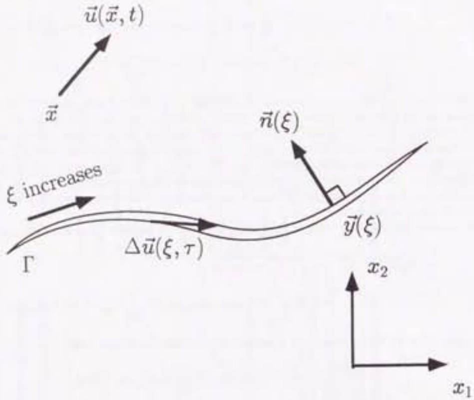

Figure 2: Nomenclature used in the crack analysis. The sign $\vec{u}(\vec{x}, t)$ denotes the displacement at position $\vec{x}$ and time t, Γ the whole length of the crack trace(s), ξ the arc length along Γ, $\Delta\vec{u}(\xi, \tau)$ the slip on the crack at arc length ξ and time τ, $\vec{n}(\xi)$ the unit vector normal to the crack trace at arc length ξ that is directed to the left when seen toward the orientation of the increasing ξ, $\vec{y}(\xi)$ the location of the position on the crack at arc length ξ.

2.2 Time-domain formulation for anti-plane cracking

For the 2D problem, I choose the coordinate axes so that the elastic field variables are independent of the third coordinate. In the anti-plane (mode III) case, the only non-zero displacement and stress components are u_3 and σ_{31}, σ_{32}, so that we have

$$u_3(\vec{x}, t) = -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu \left[n_1(\xi)\frac{\partial}{\partial x_1}G_{33} + n_2(\xi)\frac{\partial}{\partial x_2}G_{33} \right] \tag{9}$$

and

$$\begin{aligned}
\sigma_{31}(\vec{x}, t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu \left[n_1(\xi)\frac{\partial}{\partial x_1}\Sigma_{313} + n_2(\xi)\frac{\partial}{\partial x_2}\Sigma_{313} \right] \\
&= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu^2 \left[\frac{\partial}{\partial x_2}\left(n_2(\xi)\frac{\partial}{\partial x_1} - n_1(\xi)\frac{\partial}{\partial x_2} \right) G_{33} + n_1(\xi)\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}G_{33} \right] \\
&= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu^2 \left[-\frac{\partial}{\partial x_2}\left(n_2(\xi)\frac{\partial}{\partial \xi_1} - n_1(\xi)\frac{\partial}{\partial \xi_2} \right) G_{33} + n_1(\xi)\frac{1}{\beta^2}\frac{\partial^2}{\partial \tau^2}G_{33} \right] \\
&= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu^2 \left[-\frac{\partial}{\partial x_2}\frac{\partial}{\partial \xi}G_{33} + n_1(\xi)\frac{1}{\beta^2}\frac{\partial^2}{\partial \tau^2}G_{33} \right] \\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_3(\xi, \tau)\mu^2 \frac{\partial}{\partial x_2}G_{33} - \\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_3(\xi, \tau)\frac{\mu^2}{\beta^2}n_1(\xi)G_{33},
\end{aligned} \tag{10}$$

where integration by parts has been implemented so as to rewrite (or *regularize*) the hypersingular integral into a more weakly singular form (Sládek and Sládek, 1984; Koller *et al.*, 1992). Likewise,

$$\begin{aligned}
\sigma_{32}(\vec{x}, t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_3(\xi, \tau)\mu \left[n_1(\xi)\frac{\partial}{\partial x_1}\Sigma_{323} + n_2(\xi)\frac{\partial}{\partial x_2}\Sigma_{323} \right] \\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_3(\xi, \tau)\mu^2 \left(-\frac{\partial}{\partial x_1}G_{33} \right) - \\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_3(\xi, \tau)\frac{\mu^2}{\beta^2}n_2(\xi)G_{33},
\end{aligned} \tag{11}$$

where μ is the rigidity, β the S wave velocity and the equation of motion

$$\rho\frac{\partial^2}{\partial t^2}G_{33} = \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) G_{33} \tag{12}$$

was made use of.

With the limiting process $\vec{x} \to \vec{y}(s)$, we get, for the traction in the x_3-direction $T_3(s, t)$ on the crack at arc length s and time t,

$$\begin{aligned}
T_3(s, t) &= n_1(s)\sigma_{31}(\vec{y}(s), t) + n_2(s)\sigma_{32}(\vec{y}(s), t) \\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_3(\xi, \tau)\mu^2 \left(n_1(s)\frac{\partial}{\partial x_2}G_{33} - n_2(s)\frac{\partial}{\partial x_1}G_{33} \right) - \\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_3(\xi, \tau)\frac{\mu^2}{\beta^2}\left(n_1(s)n_1(\xi) + n_2(s)n_2(\xi) \right) G_{33}.
\end{aligned} \tag{13}$$

This is the displacement BIE that expresses the traction on the crack(s) in terms of the slip on the crack(s), equivalent to Equation (6) of Koller *et al.* (1992).

The explicit form of the Green function for the anti-plane case is (Achenbach, 1973, Section 3.10.2; Koller *et al.*, 1992; Cochard and Madariaga, 1994):

$$G_{33}(\vec{x}, t-\tau; \vec{y}, 0) = \frac{1}{2\pi\mu}\frac{1}{\sqrt{(t-\tau)^2 - (r/\beta)^2}}H(t - \tau - \frac{r}{\beta}), \tag{14}$$

where $r \equiv ||\vec{x} - \vec{y}||$ and $H(\cdot)$ is the Heaviside step function.

From this expression it is evident that the integral terms including first-order spatial derivatives of G_{33} are still hypersingular. Cochard and Madariaga (1994) pointed out that these hypersingularities may be removed by noting the following identities:

$$\begin{aligned}\frac{\partial}{\partial x_i}G_{33} &= \frac{1}{2\pi\mu}\frac{r\gamma_i}{\beta^2}\frac{1}{[(t-\tau)^2-(r/\beta)^2]^{3/2}}H(t-\tau-\frac{r}{\beta}) \\ &= \frac{1}{2\pi\mu}\frac{\gamma_i}{r}\frac{\partial}{\partial\tau}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) \qquad (15) \\ &= \frac{1}{2\pi\mu}\frac{-\gamma_i}{r}\frac{\partial^2}{\partial\tau^2}\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta}). \qquad (16)\end{aligned}$$

where

$$\gamma_i \equiv (x_i - y_i)/r. \qquad (17)$$

Substituting Equations (15) and (16) into (9), (10), (11) and (13), we finally get, after performing integration by parts, at the following expressions, where the singular integrals should be interpreted in terms of Cauchy principal values:

$$\begin{aligned}u_3(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right)\times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_3(\xi,\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta}) \qquad (18) \\ \sigma_{31}(\vec{x},t) &= \frac{\mu}{2\pi}\int_\Gamma d\xi\frac{\gamma_2}{r}\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_3(\xi,\tau)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) - \\ &-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi n_1(\xi)\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_3(\xi,\tau)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) \qquad (19) \\ \sigma_{32}(\vec{x},t) &= \frac{\mu}{2\pi}\int_\Gamma d\xi\frac{-\gamma_1}{r}\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_3(\xi,\tau)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) - \\ &-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi n_2(\xi)\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_3(\xi,\tau)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}). \qquad (20)\end{aligned}$$

where

$$\begin{aligned}r &\equiv \sqrt{(x_1-y_1(\xi))^2+(x_2-y_2(\xi))^2} \qquad (21) \\ \gamma_i &\equiv (x_i - y_i(\xi))/r \qquad (22)\end{aligned}$$

and the dot over a variable denotes the time derivative, and

$$\begin{aligned}T_3(s,t) &= \frac{\mu}{2\pi}\int_\Gamma d\xi\left(n_1(s)\frac{\gamma_2}{r}-n_2(s)\frac{\gamma_1}{r}\right)\times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_3(\xi,\tau)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) - \\ &-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi\,(n_1(s)n_1(\xi)+n_2(s)n_2(\xi))\times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_3(\xi,\tau)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}), \qquad (23)\end{aligned}$$

where

$$\begin{aligned}r &\equiv \sqrt{(y_1(s)-y_1(\xi))^2+(y_2(s)-y_2(\xi))^2} \qquad (24) \\ \gamma_i &\equiv (y_i(s)-y_i(\xi))/r. \qquad (25)\end{aligned}$$

Differentiating (18) with respect to t we get the following representation for the displacement velocity:

$$\begin{aligned}\dot{u}_3(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right)\times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_3(\xi,\tau)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}). \qquad (26)\end{aligned}$$

2.3 Time-independent formulation for anti-plane cracking

The time-independent counterpart of the displacement BIEs for the 2D anti-plane crack problem follows by analogy to the time-dependent version:

$$u_3(\vec{x}) = -\int_\Gamma d\xi \Delta u_3(\xi)\mu \left(n_1(\xi)\frac{\partial}{\partial x_1}G_{33} + n_2(\xi)\frac{\partial}{\partial x_2}G_{33}\right) \tag{27}$$

$$\sigma_{31}(\vec{x}) = -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)\mu^2 \frac{\partial}{\partial x_2}G_{33} \tag{28}$$

$$\sigma_{32}(\vec{x}) = -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)\mu^2 \left(-\frac{\partial}{\partial x_1}G_{33}\right) \tag{29}$$

$$T_3(s) = -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)\mu^2 \left(n_1(s)\frac{\partial}{\partial x_2}G_{33} - n_2(s)\frac{\partial}{\partial x_1}G_{33}\right). \tag{30}$$

The set of the 2D Green functions for the time-independent anti-plane case is familiarly known as the Kelvin solution and is given by (Maruyama, 1966; Zhang and Achenbach, 1989)

$$G_{33}(\vec{x};\vec{y}) = \frac{1}{2\pi\mu}(-\log r) \tag{31}$$

$$\frac{\partial}{\partial x_i}G_{33}(\vec{x};\vec{y}) = \frac{1}{2\pi\mu}\frac{-\gamma_i}{r}. \tag{32}$$

Unlike in the time-dependent case, the integral terms including first-order spatial derivatives of G_{33} are not hypersingular, but can be understood as Cauchy principal value integrals.

Substituting Equation (32) into (27), (28), (29) and (30), we get at the expressions:

$$u_3(\vec{x}) = \frac{1}{2\pi}\int_\Gamma d\xi \Delta u_3(\xi)(n_1(\xi)\frac{\gamma_1}{r} + n_2(\xi)\frac{\gamma_2}{r}) \tag{33}$$

$$\sigma_{31}(\vec{x}) = \frac{\mu}{2\pi}\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)\frac{\gamma_2}{r} \tag{34}$$

$$\sigma_{32}(\vec{x}) = \frac{\mu}{2\pi}\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)\frac{-\gamma_1}{r} \tag{35}$$

$$T_3(s) = \frac{\mu}{2\pi}\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_3(\xi)(n_1(s)\frac{\gamma_2}{r} - n_2(s)\frac{\gamma_1}{r}). \tag{36}$$

2.4 Time-domain formulation for closed in-plane cracking

In the 2D in-plane (plane strain) shear crack problem to be treated in this section, we assume, for simplicity, that the crack surface is closed everywhere, or that the displacement discontinuity on the crack(s) has only the tangential (purely mode II) component. In fact, no normal displacement discontinuity exists in natural faults as a macroscopic feature, so that consideration of the tangential slip suffices for most of the practical purposes. Denoting the amount of right-lateral shear slip by Δu_t, we obtain the relations

$$\begin{cases} \Delta u_1(\xi,\tau) = n_2(\xi)\Delta u_t(\xi,\tau) \\ \Delta u_2(\xi,\tau) = -\, n_1(\xi)\Delta u_t(\xi,\tau) \end{cases} \tag{37}$$

Using the above notation, we have

$$\begin{aligned} u_1(\vec{x},t) = & -\int_\Gamma d\xi \int_0^t d\tau \left\{ [\Delta u_1(\xi,\tau)(\lambda+2\mu)n_1(\xi) + \Delta u_2(\xi,\tau)\lambda n_2(\xi)]\frac{\partial}{\partial x_1}G_{11} + \right. \\ & + [\Delta u_1(\xi,\tau)\lambda n_1(\xi) + \Delta u_2(\xi,\tau)(\lambda+2\mu)n_2(\xi)]\frac{\partial}{\partial x_2}G_{12} + \\ & \left. + (\Delta u_1(\xi,\tau)\mu n_2(\xi) + \Delta u_2(\xi,\tau)\mu n_1(\xi))\left(\frac{\partial}{\partial x_2}G_{11} + \frac{\partial}{\partial x_1}G_{12}\right)\right\} \\ = & -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \left[2n_1(\xi)n_2(\xi)\left(\frac{\partial}{\partial x_1}G_{11} - \frac{\partial}{\partial x_2}G_{12}\right) + \right. \\ & \left. + (n_2^2(\xi) - n_1^2(\xi))\left(\frac{\partial}{\partial x_2}G_{11} + \frac{\partial}{\partial x_1}G_{12}\right)\right] \end{aligned} \tag{38}$$

$$\begin{aligned} u_2(\vec{x},t) = & -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \left[2n_1(\xi)n_2(\xi)\left(\frac{\partial}{\partial x_1}G_{21} - \frac{\partial}{\partial x_2}G_{22}\right) + \right. \\ & \left. + (n_2^2(\xi) - n_1^2(\xi))\left(\frac{\partial}{\partial x_2}G_{21} + \frac{\partial}{\partial x_1}G_{22}\right)\right] \end{aligned} \tag{39}$$

and, likewise, the stress field is expressed as:

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) - \sigma_{22}(\vec{x},t)) = & -\frac{1}{2}\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \times \\ & \times \left[2n_1(\xi)n_2(\xi)\left(\frac{\partial}{\partial x_1}\Sigma_{111} - \frac{\partial}{\partial x_2}\Sigma_{112} - \frac{\partial}{\partial x_1}\Sigma_{221} + \frac{\partial}{\partial x_2}\Sigma_{222}\right) + \right. \\ & \left. + (n_2^2(\xi) - n_1^2(\xi))\left(\frac{\partial}{\partial x_2}\Sigma_{111} + \frac{\partial}{\partial x_1}\Sigma_{112} - \frac{\partial}{\partial x_2}\Sigma_{221} - \frac{\partial}{\partial x_1}\Sigma_{222}\right)\right] \\ = & -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu^2 \times \\ & \times \left\{2n_1(\xi)n_2(\xi)\left(\frac{\partial^2}{\partial x_1^2}G_{11} + \frac{\partial^2}{\partial x_2^2}G_{22} - 2\frac{\partial^2}{\partial x_1 \partial x_2}G_{12}\right) + \right. \\ & \left. + (n_2^2(\xi) - n_1^2(\xi))\left[\frac{\partial^2}{\partial x_1 \partial x_2}(G_{11} - G_{22}) + \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)G_{12}\right]\right\} \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) + \sigma_{22}(\vec{x},t)) = & -\frac{1}{2}\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \times \\ & \times \left[2n_1(\xi)n_2(\xi)\left(\frac{\partial}{\partial x_1}\Sigma_{111} - \frac{\partial}{\partial x_2}\Sigma_{112} + \frac{\partial}{\partial x_1}\Sigma_{221} - \frac{\partial}{\partial x_2}\Sigma_{222}\right) + \right. \\ & \left. + (n_2^2(\xi) - n_1^2(\xi))\left(\frac{\partial}{\partial x_2}\Sigma_{111} + \frac{\partial}{\partial x_1}\Sigma_{112} + \frac{\partial}{\partial x_2}\Sigma_{221} + \frac{\partial}{\partial x_1}\Sigma_{222}\right)\right] \\ = & -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu(\lambda+\mu) \times \\ & \times \left\{2n_1(\xi)n_2(\xi)\left(\frac{\partial^2}{\partial x_1^2}G_{11} - \frac{\partial^2}{\partial x_2^2}G_{22}\right) + \right. \end{aligned}$$

$$
\begin{aligned}
&+ (n_2^2(\xi) - n_1^2(\xi)) \left[\frac{\partial^2}{\partial x_1 \partial x_2}(G_{11} + G_{22}) + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) G_{12} \right] \Big\} \qquad (41)\\
\sigma_{12}(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau)\mu \left[2n_1(\xi)n_2(\xi) \left(\frac{\partial}{\partial x_1}\Sigma_{121} - \frac{\partial}{\partial x_2}\Sigma_{122} \right) + \right.\\
&\left. + (n_2^2(\xi) - n_1^2(\xi)) \left(\frac{\partial}{\partial x_2}\Sigma_{121} + \frac{\partial}{\partial x_1}\Sigma_{122} \right) \right]\\
&= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau)\mu^2 \times\\
&\times \left\{ 2n_1(\xi)n_2(\xi) \left[\frac{\partial^2}{\partial x_1 \partial x_2}(G_{11} - G_{22}) + \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) G_{12} \right] + \right.\\
&\left. + (n_2^2(\xi) - n_1^2(\xi)) \left(\frac{\partial^2}{\partial x_2^2}G_{11} + \frac{\partial^2}{\partial x_1^2}G_{22} + 2\frac{\partial^2}{\partial x_1 \partial x_2}G_{12} \right) \right\}, \qquad (42)
\end{aligned}
$$

where λ and μ are the Lamé constants and the identity

$$G_{12} = G_{21} \tag{43}$$

was made use of. The other non-zero stress component σ_{33}, for which the formulae shall not be given herein, is related with other components by a simple relation

$$\sigma_{33} = \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11} + \sigma_{22}) = \frac{\alpha^2 - 2\beta^2}{2(\alpha^2 - \beta^2)}(\sigma_{11} + \sigma_{22}). \tag{44}$$

It is convenient, for the purpose of regularization of the above integral equations, to represent the spatial derivative operators and the Green functions in terms of a different coordinate system, so defined at each location ξ on the crack(s) that the first coordinate axis x_t is locally tangent to the crack trace and the second axis x_n is normal to it:

$$
\begin{aligned}
\gamma_t &= n_2(\xi)\gamma_1 - n_1(\xi)\gamma_2 && (45)\\
\gamma_n &= n_1(\xi)\gamma_1 + n_2(\xi)\gamma_2 && (46)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_t} &= n_2(\xi)\frac{\partial}{\partial x_1} - n_1(\xi)\frac{\partial}{\partial x_2} && (47)\\
\frac{\partial}{\partial x_n} &= n_1(\xi)\frac{\partial}{\partial x_1} + n_2(\xi)\frac{\partial}{\partial x_2} && (48)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{tt} &= n_2^2(\xi)G_{11} + n_1^2(\xi)G_{22} - 2n_1(\xi)n_2(\xi)G_{12} && (49)\\
G_{nn} &= n_1^2(\xi)G_{11} + n_2^2(\xi)G_{22} + 2n_1(\xi)n_2(\xi)G_{12} && (50)\\
G_{tn} &= n_1(\xi)n_2(\xi)(G_{11} - G_{22}) + (n_2^2(\xi) - n_1^2(\xi))G_{12}, && (51)
\end{aligned}
$$

which leads to the following expressions:

$$
\begin{aligned}
u_1(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau)\mu \left[n_2(\xi) \left(\frac{\partial}{\partial x_n}G_{tt} + \frac{\partial}{\partial x_t}G_{tn} \right) + \right.\\
&\left. + n_1(\xi) \left(\frac{\partial}{\partial x_n}G_{tn} + \frac{\partial}{\partial x_t}G_{nn} \right) \right] \qquad (52)\\
u_2(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_i(\xi,\tau)\mu \left[-n_1(\xi) \left(\frac{\partial}{\partial x_n}G_{tt} + \frac{\partial}{\partial x_t}G_{tn} \right) + \right.\\
&\left. + n_2(\xi) \left(\frac{\partial}{\partial x_n}G_{tn} + \frac{\partial}{\partial x_t}G_{nn} \right) \right] \qquad (53)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x},t) - \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu^2 \times\\
&\times \left\{ 2n_1(\xi)n_2(\xi) \left(\frac{\partial^2}{\partial x_n^2}G_{tt} + \frac{\partial^2}{\partial x_t^2}G_{nn} + 2\frac{\partial^2}{\partial x_t \partial x_n}G_{tn} \right) + \right.\\
&\left. + (n_2^2(\xi) - n_1^2(\xi)) \left[\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt} - G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} - \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right] \right\} \qquad (54)\\
\frac{1}{2}(\sigma_{11}(\vec{x},t) + \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu(\lambda+\mu) \times\\
&\times \left[\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt} + G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right] \qquad (55)\\
\sigma_{12}(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu^2 \times\\
&\times \left\{ -2n_1(\xi)n_2(\xi) \left[\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt} - G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} - \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right] + \right.\\
&\left. + (n_2^2(\xi) - n_1^2(\xi)) \left(\frac{\partial^2}{\partial x_n^2}G_{tt} + \frac{\partial^2}{\partial x_t^2}G_{nn} + 2\frac{\partial^2}{\partial x_t \partial x_n}G_{tn} \right) \right\}. \qquad (56)
\end{aligned}
$$

Unlike in the anti-plane case, the regularization process for the stress components is rather cumbersome. Although the explicit form of the displacement Green functions for the 2D in-plane case is not available in customary textbooks, an integral representation given by Achenbach (1973, Section 3.10.3) is reducible, after performing the definite integration, to:

$$
\begin{aligned}
G_{11} &= \frac{1}{2\pi\rho}\frac{\partial^2}{\partial x_1^2}(I_\alpha - I_\beta) + G_{33} && (57)\\
G_{22} &= \frac{1}{2\pi\rho}\frac{\partial^2}{\partial x_2^2}(I_\alpha - I_\beta) + G_{33} && (58)\\
G_{12} &= \frac{1}{2\pi\rho}\frac{\partial^2}{\partial x_1 \partial x_2}(I_\alpha - I_\beta), && (59)
\end{aligned}
$$

where G_{33} is the displacement Green function for the anti-plane case (see previous section) and

$$
\begin{aligned}
I_c \equiv\ & \left\{ (t-\tau)\log\left[c(t-\tau)/r + \sqrt{(c(t-\tau)/r)^2 - 1} \right] - \right.\\
& \left. - \sqrt{(t-\tau)^2 - (r/c)^2} \right\} H(t-\tau-r/c) \quad (c = \alpha, \beta). \qquad (60)
\end{aligned}
$$

The explicit form of the Green functions for the in-plane case shall be given elsewhere in the present section.

On the basis of the above derivative representations and the equation of motion

$$\rho\frac{\partial^2}{\partial t^2}G_{33} = \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) G_{33} \tag{61}$$

it can be proven that the following identities hold true:

$$
\begin{aligned}
\frac{\partial}{\partial x_2}G_{11} &= \frac{\partial}{\partial x_1}G_{12} + \frac{\partial}{\partial x_2}G_{33} && (62)\\
\frac{\partial}{\partial x_1}G_{22} &= \frac{\partial}{\partial x_2}G_{12} + \frac{\partial}{\partial x_1}G_{33} && (63)\\
\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) G_{12} &= \frac{\partial^2}{\partial x_1 \partial x_2}(G_{11} - G_{22}) && (64)
\end{aligned}
$$

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right) G_{12} &= \frac{\partial^2}{\partial x_1 \partial x_2}(G_{11}+G_{22}-2G_{33}) && (65)\\
\frac{\partial^2}{\partial x_2^2}G_{11}+\frac{\partial^2}{\partial x_1^2}G_{22} &= 2\frac{\partial^2}{\partial x_1 \partial x_2}G_{12}+\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}G_{33}. && (66)
\end{aligned}$$

It should be kept in mind hereafter that the form of the Green functions for the in-plane case is subject to no modification if we simultaneously replace x_1 with x_t and x_2 with x_n, since the transformation between the two coordinate system can be achieved by simple rotation. For the displacement components we get:

$$\begin{aligned}
u_1(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \left[n_2(\xi)\left(2\frac{\partial}{\partial x_t}G_{tn}+\frac{\partial}{\partial x_n}G_{33}\right)+\right.\\
&\left.\quad + n_1(\xi)\left(2\frac{\partial}{\partial x_n}G_{tn}+\frac{\partial}{\partial x_t}G_{33}\right)\right] && (67)\\
u_2(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu \left[-n_1(\xi)\left(2\frac{\partial}{\partial x_t}G_{tn}+\frac{\partial}{\partial x_n}G_{33}\right)+\right.\\
&\left.\quad + n_2(\xi)\left(2\frac{\partial}{\partial x_n}G_{tn}+\frac{\partial}{\partial x_t}G_{33}\right)\right]. && (68)
\end{aligned}$$

Noting the relation

$$\frac{\partial}{\partial x_t}G_{ij} = -\frac{\partial}{\partial \xi}G_{ij} \qquad (69)$$

and performing integration by parts, we obtain, for the stress components,

$$\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x},t)-\sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left[2n_1(\xi)n_2(\xi)\left(4\frac{\partial^2}{\partial x_t \partial x_n}G_{tn}+\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}G_{33}\right)+\right.\\
&\left.\quad + (n_2^2(\xi)-n_1^2(\xi))2\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt}-G_{nn})\right]\\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left[2n_1(\xi)n_2(\xi)4\frac{\partial}{\partial x_n}G_{tn}+(n_2^2(\xi)-n_1^2(\xi))2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]-\\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_t(\xi,\tau)\frac{\mu^2}{\beta^2}2n_1(\xi)n_2(\xi)G_{33} && (70)\\
\frac{1}{2}(\sigma_{11}(\vec{x},t)+\sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu(\lambda+\mu) \times\\
&\quad \times 2\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt}+G_{nn}-G_{33})\\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) && (71)\\
\sigma_{12}(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left[-2n_1(\xi)n_2(\xi)2\frac{\partial^2}{\partial x_t \partial x_n}(G_{tt}-G_{nn})+\right.\\
&\left.\quad + (n_2^2(\xi)-n_1^2(\xi))\left(4\frac{\partial^2}{\partial x_t \partial x_n}G_{tn}+\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}G_{33}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left[-2n_1(\xi)n_2(\xi)2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})+(n_2^2(\xi)-n_1^2(\xi))4\frac{\partial}{\partial x_n}G_{tn}\right]-\\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_t(\xi,\tau)\frac{\mu^2}{\beta^2}(n_2^2(\xi)-n_1^2(\xi))G_{33}. && (72)
\end{aligned}$$

With the limiting process $\vec{x} \to \vec{y}(s)$, we get, for the tangential traction $T_t(s,t)$ on the crack at arc length s and time t,

$$\begin{aligned}
T_t(s,t) &= 2n_1(s)n_2(s)\frac{1}{2}(\sigma_{11}(\vec{y}(s),t)-\sigma_{22}(\vec{y}(s),t))+(n_2^2(s)-n_1^2(s))\sigma_{12}(\vec{y}(s),t)\\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left\{[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))]4\frac{\partial}{\partial x_n}G_{tn}+\right.\\
&\left.\quad +[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))-(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)]2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right\}-\\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_t(\xi,\tau)\frac{\mu^2}{\beta^2} \times\\
&\quad \times [2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))]G_{33}. && (73)
\end{aligned}$$

This is the displacement BIE that expresses the traction on the crack(s) in terms of the slip on the crack(s). Likewise, the normal traction $T_n(s,t)$ across the crack at arc length s and time t is given by

$$\begin{aligned}
T_n(s,t) &= \frac{1}{2}(\sigma_{11}(\vec{y}(s),t)+\sigma_{22}(\vec{y}(s),t))-\\
&\quad -(n_2^2(s)-n_1^2(s))\frac{1}{2}(\sigma_{11}(\vec{y}(s),t)-\sigma_{22}(\vec{y}(s),t))+2n_1(s)n_2(s)\sigma_{12}(\vec{y}(s),t)\\
&= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_t(\xi,\tau)\mu^2 \times\\
&\quad \times \left\{[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))-(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)]4\frac{\partial}{\partial x_n}G_{tn}-\right.\\
&\quad -[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))]2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})+\\
&\left.\quad +\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33})\right\}-\\
&\quad -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_t(\xi,\tau)\frac{\mu^2}{\beta^2} \times\\
&\quad \times [2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))-(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)]G_{33}. && (74)
\end{aligned}$$

The explicit form of the Green functions for the in-plane case can be derived from the derivative representation which has been given earlier in this section:

$$\begin{aligned}
G_{11}(\vec{x},t-\tau;\vec{y},0) &- G_{22}(\vec{x},t-\tau;\vec{y},0) =\\
&= -\frac{1}{2\pi\mu}(\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}\left[2(t-\tau)^2-\frac{r^2}{\alpha^2}\right]\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})+\\
&\quad +\frac{1}{2\pi\mu}(\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}\left[2(t-\tau)^2-\frac{r^2}{\beta^2}\right]\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) && (75)\\
G_{11}(\vec{x},t-\tau;\vec{y},0) &+ G_{22}(\vec{x},t-\tau;\vec{y},0) =\\
&= \frac{1}{2\pi\mu}\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})+
\end{aligned}$$

$$+\frac{1}{2\pi\mu}\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) \tag{76}$$

$$\begin{aligned} G_{12}(\vec{x},t-\tau;\vec{y},0) &= \frac{1}{2\pi\mu}\gamma_1\gamma_2\frac{\beta^2}{r^2}\left[2(t-\tau)^2-\frac{r^2}{\alpha^2}\right]\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi\mu}\gamma_1\gamma_2\frac{\beta^2}{r^2}\left[2(t-\tau)^2-\frac{r^2}{\beta^2}\right]\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}), \end{aligned} \tag{77}$$

where $r \equiv \|\vec{x}-\vec{y}\|$ and $H(\cdot)$ is the Heaviside step function. The Green function for the anti-plane case is given again for quick reference:

$$G_{33}(\vec{x},t-\tau;\vec{y},0) = \frac{1}{2\pi\mu}\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}). \tag{78}$$

It would be informative, for the sake of interested readers, to note the following simple identity which was utilized in the derivation of the above formulae:

$$\frac{\partial}{\partial r}I_c = -\frac{1}{r}\sqrt{(t-\tau)^2-(r/c)^2}\,H(t-\tau-\frac{r}{c}). \tag{79}$$

Next we shall proceed to the final stage of the regularization process. Making use of the relations

$$\frac{\partial}{\partial x_i}\frac{1}{\sqrt{(t-\tau)^2-(r/c)^2}} = \frac{\gamma_i}{r}\frac{\partial}{\partial\tau}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/c)^2}} \tag{80}$$

$$= \frac{-\gamma_i}{r}\frac{\partial^2}{\partial\tau^2}\sqrt{(t-\tau)^2-(r/c)^2} \tag{81}$$

and

$$\left[2(t-\tau)^2-\frac{r^2}{c^2}\right]\frac{1}{\sqrt{(t-\tau)^2-(r/c)^2}} = -\frac{\partial}{\partial\tau}\left[(t-\tau)\sqrt{(t-\tau)^2-(r/c)^2}\right] \tag{82}$$

$$= \frac{1}{3}\frac{\partial^2}{\partial\tau^2}\left[(t-\tau)^2-(r/c)^2\right]^{3/2} \tag{83}$$

we can prove that the following identities hold true:

$$\begin{aligned} 2\frac{\partial}{\partial x_1}G_{12}+\frac{\partial}{\partial x_2}G_{33} &= \frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial^2}{\partial\tau^2}\left[\frac{2}{3}(\gamma_2^2-3\gamma_1^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\alpha)^2]^{3/2} - \right. \\ &\left. -2\gamma_1^2\frac{\beta^2}{\alpha^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}\right]H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial^2}{\partial\tau^2}\left[\frac{2}{3}(\gamma_2^2-3\gamma_1^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\beta)^2]^{3/2} + \right. \\ &\left. +(\gamma_2^2-\gamma_1^2)\sqrt{(t-\tau)^2-(r/\beta)^2}\right]H(t-\tau-\frac{r}{\beta}) \end{aligned} \tag{84}$$

$$\begin{aligned} 2\frac{\partial}{\partial x_2}G_{12}+\frac{\partial}{\partial x_1}G_{33} &= \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial^2}{\partial\tau^2}\left[\frac{2}{3}(\gamma_1^2-3\gamma_2^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\alpha)^2]^{3/2} - \right. \\ &\left. -2\gamma_2^2\frac{\beta^2}{\alpha^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}\right]H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial^2}{\partial\tau^2}\left[\frac{2}{3}(\gamma_1^2-3\gamma_2^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\beta)^2]^{3/2} - \right. \\ &\left. -(\gamma_2^2-\gamma_1^2)\sqrt{(t-\tau)^2-(r/\beta)^2}\right]H(t-\tau-\frac{r}{\beta}) \end{aligned} \tag{85}$$

and

$$\begin{aligned} 2\frac{\partial}{\partial x_2}G_{12} &= \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial}{\partial\tau}\left[2(3\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} + \right. \\ &\left. +2\gamma_2^2\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right]H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial}{\partial\tau}\left[2(3\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} + \right. \\ &\left. +2\gamma_2^2\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right]H(t-\tau-\frac{r}{\beta}) \end{aligned} \tag{86}$$

$$\begin{aligned} \frac{\partial}{\partial x_2}(G_{11}-G_{22}) &= \frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial}{\partial\tau}\left[2(3\gamma_1^2-\gamma_2^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} - \right. \\ &\left. -(\gamma_2^2-\gamma_1^2)\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right]H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial}{\partial\tau}\left[2(3\gamma_1^2-\gamma_2^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} - \right. \\ &\left. -(\gamma_2^2-\gamma_1^2)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right]H(t-\tau-\frac{r}{\beta}) \end{aligned} \tag{87}$$

$$\begin{aligned} &\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33}) = \\ &= \frac{1}{2\pi\mu}\frac{\gamma_2}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial\tau}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}). \end{aligned} \tag{88}$$

As has been mentioned elsewhere, it is to be noted that the form of the Green functions for the in-plane case is subject to no modification if we simultaneously replace x_1 with x_t and x_2 with x_n. Substituting Equations (84), (85), (86), (87) and (88) into (67), (68), (70), (71), (72), (73) and (74), we get, after lengthy algebraic manipulations, at the following expressions, in which the singular integrals should be interpreted in the sense of Cauchy principal values:

$$\begin{aligned} u_1(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)\right] \times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_t(\xi,\tau)\frac{2}{3}\frac{\beta^2}{r^2}\left\{\left[(t-\tau)^2-(r/\alpha)^2\right]^{3/2}H(t-\tau-\frac{r}{\alpha}) - \right. \\ &\left. -\left[(t-\tau)^2-(r/\beta)^2\right]^{3/2}H(t-\tau-\frac{r}{\beta})\right\} + \\ &+\frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2-\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}2\gamma_1^2\right] \times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_t(\xi,\tau)\frac{\beta^2}{\alpha^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\ &-\frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(-2\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}(\gamma_1^2-\gamma_2^2)\right] \times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_t(\xi,\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta}) \end{aligned} \tag{89}$$

$$\begin{aligned} u_2(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}(3\gamma_2^2-\gamma_1^2)\right] \times \\ &\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_t(\xi,\tau)\frac{2}{3}\frac{\beta^2}{r^2}\left\{\left[(t-\tau)^2-(r/\alpha)^2\right]^{3/2}H(t-\tau-\frac{r}{\alpha}) - \right. \\ &\left. -\left[(t-\tau)^2-(r/\beta)^2\right]^{3/2}H(t-\tau-\frac{r}{\beta})\right\} + \\ &+\frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(\gamma_1^2-\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}2\gamma_2^2\right] \times \end{aligned}$$

$$
\begin{aligned}
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi,\tau) \frac{\beta^2}{\alpha^2} \sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \\
& -\frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi) n_2(\xi) \frac{\gamma_2}{r} 2\gamma_1^2 + (n_2^2(\xi) - n_1^2(\xi)) \frac{\gamma_1}{r} (\gamma_2^2 - \gamma_1^2) \right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi,\tau) \sqrt{(t-\tau)^2 - (r/\beta)^2} H(t-\tau-\frac{r}{\beta})
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x},t) - \sigma_{22}(\vec{x},t)) \;=\; & \frac{\mu}{\pi} \int_\Gamma d\xi \left[n_1(\xi) \frac{\gamma_1}{r} (\gamma_1^2 - 3\gamma_2^2) + n_2(\xi) \frac{\gamma_2}{r} (3\gamma_1^2 - \gamma_2^2) \right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. -(t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t-\tau-\frac{r}{\beta}) \right] + \\
& + \frac{\mu}{\pi} \int_\Gamma d\xi \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) (\gamma_1^2 - \gamma_2^2) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left[\frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. - \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right] - \\
& - \frac{\mu}{2\pi} \frac{1}{\beta^2} \int_\Gamma d\xi 2n_1(\xi) n_2(\xi) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi,\tau) \frac{1}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta})
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x},t) + \sigma_{22}(\vec{x},t)) \;=\; & \frac{\mu}{\pi} \int_\Gamma d\xi \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left(1 - \frac{\beta^2}{\alpha^2} \right) \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha})
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
\sigma_{12}(\vec{x},t) \;=\; & \frac{\mu}{\pi} \int_\Gamma d\xi \left[n_1(\xi) \frac{\gamma_2}{r} (3\gamma_1^2 - \gamma_2^2) - n_2(\xi) \frac{\gamma_1}{r} (\gamma_1^2 - 3\gamma_2^2) \right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. -(t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t-\tau-\frac{r}{\beta}) \right] + \\
& + \frac{\mu}{\pi} \int_\Gamma d\xi \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) 2\gamma_1\gamma_2 \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left[\frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. - \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right] - \\
& - \frac{\mu}{2\pi} \frac{1}{\beta^2} \int_\Gamma d\xi (n_2^2(\xi) - n_1^2(\xi)) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi,\tau) \frac{1}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta})
\end{aligned} \tag{93}
$$

and

$$
\begin{aligned}
T_t(s,t) \;=\; & 2n_1(s) n_2(s) \frac{1}{2}(\sigma_{11}(\vec{y}(s),t) - \sigma_{22}(\vec{y}(s),t)) + (n_2^2(s) - n_1^2(s)) \sigma_{12}(\vec{y}(s),t)
\end{aligned}
$$

$$
\begin{aligned}
\;=\; & \frac{\mu}{\pi} \int_\Gamma d\xi \left\{ [2n_1(s) n_2(s) n_1(\xi) - (n_2^2(s) - n_1^2(s)) n_2(\xi)] \frac{\gamma_1}{r} (\gamma_1^2 - 3\gamma_2^2) + \right. \\
& \left. + [2n_1(s) n_2(s) n_2(\xi) + (n_2^2(s) - n_1^2(s)) n_1(\xi)] \frac{\gamma_2}{r} (3\gamma_1^2 - \gamma_2^2) \right\} \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. -(t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t-\tau-\frac{r}{\beta}) \right] + \\
& + \frac{\mu}{\pi} \int_\Gamma d\xi [-2n_1(s) n_2(s) (\gamma_2^2 - \gamma_1^2) + (n_2^2(s) - n_1^2(s)) 2\gamma_1\gamma_2] \times \\
& \times \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left[\frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. - \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right] - \\
& - \frac{\mu}{2\pi} \frac{1}{\beta^2} \int_\Gamma d\xi [2n_1(s) n_2(s) 2n_1(\xi) n_2(\xi) + (n_2^2(s) - n_1^2(s))(n_2^2(\xi) - n_1^2(\xi))] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi,\tau) \frac{1}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta})
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
T_n(s,t) \;=\; & \frac{1}{2}(\sigma_{11}(\vec{y}(s),t) + \sigma_{22}(\vec{y}(s),t)) - \\
& -(n_2^2(s) - n_1^2(s)) \frac{1}{2}(\sigma_{11}(\vec{y}(s),t) - \sigma_{22}(\vec{y}(s),t)) + 2n_1(s) n_2(s) \sigma_{12}(\vec{y}(s),t) \\
\;=\; & \frac{\mu}{\pi} \int_\Gamma d\xi \left\{ [2n_1(s) n_2(s) n_1(\xi) - (n_2^2(s) - n_1^2(s)) n_2(\xi)] \frac{\gamma_2}{r} (3\gamma_1^2 - \gamma_2^2) - \right. \\
& \left. - [2n_1(s) n_2(s) n_2(\xi) + (n_2^2(s) - n_1^2(s)) n_1(\xi)] \frac{\gamma_1}{r} (\gamma_1^2 - 3\gamma_2^2) \right\} \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. -(t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t-\tau-\frac{r}{\beta}) \right] + \\
& + \frac{\mu}{\pi} \int_\Gamma d\xi [2n_1(s) n_2(s) 2\gamma_1\gamma_2 + (n_2^2(s) - n_1^2(s))(\gamma_2^2 - \gamma_1^2)] \times \\
& \times \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left[\frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \right. \\
& \left. - \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right] + \\
& + \frac{\mu}{\pi} \int_\Gamma d\xi \left(n_1(\xi) \frac{\gamma_1}{r} + n_2(\xi) \frac{\gamma_2}{r} \right) \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_t(\xi,\tau) \left(1 - \frac{\beta^2}{\alpha^2} \right) \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \\
& - \frac{\mu}{2\pi} \frac{1}{\beta^2} \int_\Gamma d\xi [2n_1(s) n_2(s)(n_2^2(\xi) - n_1^2(\xi)) - (n_2^2(s) - n_1^2(s)) 2n_1(\xi) n_2(\xi)] \times
\end{aligned}
$$

$$\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) \frac{1}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t - \tau - \frac{r}{\beta}). \tag{95}$$

Note that the expressions

$$\frac{\beta^2}{r^2} \left\{ \left[(t-\tau)^2 - (r/\alpha)^2\right]^{3/2} - \left[(t-\tau)^2 - (r/\beta)^2\right]^{3/2} \right\}$$

and

$$\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} - (t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2}\right]$$

converge to finite limit values as $r \to 0$, so that the integral terms that include these expressions are not hypersingular.

Differentiating (89) and (90) with respect to t we get the following representations for the displacement velocity field:

$$\begin{aligned}
\dot{u}_1(\vec{x}, t) \;=\; & \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2 - 3\gamma_2^2) + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_2}{r}(3\gamma_1^2 - \gamma_2^2)\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t - \tau - \frac{r}{\alpha}) - \right. \\
& \left. -(t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t - \tau - \frac{r}{\beta})\right] + \\
& + \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2 - \gamma_2^2) + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_2}{r}2\gamma_1^2\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) \frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t - \tau - \frac{r}{\alpha}) - \\
& - \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(-2\gamma_2^2) + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_2}{r}(\gamma_1^2 - \gamma_2^2)\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t - \tau - \frac{r}{\beta})
\end{aligned} \tag{96}$$

$$\begin{aligned}
\dot{u}_2(\vec{x}, t) \;=\; & \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(3\gamma_1^2 - \gamma_2^2) + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_1}{r}(3\gamma_2^2 - \gamma_1^2)\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) 2\frac{\beta^2}{r^2} \left[(t-\tau)\sqrt{(t-\tau)^2 - (r/\alpha)^2} H(t - \tau - \frac{r}{\alpha}) - \right. \\
& \left. - (t-\tau)\sqrt{(t-\tau)^2 - (r/\beta)^2} H(t - \tau - \frac{r}{\beta})\right] + \\
& + \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(\gamma_1^2 - \gamma_2^2) + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_1}{r}2\gamma_2^2\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) \frac{\beta^2}{\alpha^2} \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\alpha)^2}} H(t - \tau - \frac{r}{\alpha}) - \\
& - \frac{1}{2\pi} \int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}2\gamma_1^2 + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_1}{r}(\gamma_2^2 - \gamma_1^2)\right] \times \\
& \times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_t(\xi, \tau) \frac{t-\tau}{\sqrt{(t-\tau)^2 - (r/\beta)^2}} H(t - \tau - \frac{r}{\beta}).
\end{aligned} \tag{97}$$

2.5 Time-independent formulation for closed in-plane cracking

The time-independent counterpart of the displacement BIEs for the 2D closed in-plane crack problem follows by analogy to the time-dependent version:

$$\begin{aligned}
u_1(\vec{x}) \;=\; & -\int_\Gamma d\xi \Delta u_t(\xi)\mu \left[n_2(\xi)\left(2\frac{\partial}{\partial x_t}G_{tn} + \frac{\partial}{\partial x_n}G_{33}\right) + \right. \\
& \left. + n_1(\xi)\left(2\frac{\partial}{\partial x_n}G_{tn} + \frac{\partial}{\partial x_t}G_{33}\right)\right]
\end{aligned} \tag{98}$$

$$\begin{aligned}
u_2(\vec{x}) \;=\; & -\int_\Gamma d\xi \Delta u_t(\xi)\mu \left[-n_1(\xi)\left(2\frac{\partial}{\partial x_t}G_{tn} + \frac{\partial}{\partial x_n}G_{33}\right) + \right. \\
& \left. + n_2(\xi)\left(2\frac{\partial}{\partial x_n}G_{tn} + \frac{\partial}{\partial x_t}G_{33}\right)\right]
\end{aligned} \tag{99}$$

$$\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x}) - \sigma_{22}(\vec{x})) \;=\; & -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)\mu^2 \times \\
& \times \left[2n_1(\xi)n_2(\xi)4\frac{\partial}{\partial x_n}G_{tn} + (n_2^2(\xi) - n_1^2(\xi))2\frac{\partial}{\partial x_n}(G_{tt} - G_{nn})\right]
\end{aligned} \tag{100}$$

$$\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x}) + \sigma_{22}(\vec{x})) \;=\; & -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)\mu^2 \times \\
& \times \frac{\alpha^2}{\beta^2}\left(1 - \frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt} + G_{nn} - G_{33})
\end{aligned} \tag{101}$$

$$\begin{aligned}
\sigma_{12}(\vec{x}) \;=\; & -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)\mu^2 \times \\
& \times \left[-2n_1(\xi)n_2(\xi)2\frac{\partial}{\partial x_n}(G_{tt} - G_{nn}) + (n_2^2(\xi) - n_1^2(\xi))4\frac{\partial}{\partial x_n}G_{tn}\right]
\end{aligned} \tag{102}$$

$$\begin{aligned}
T_t(s) \;=\; & -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)\mu^2 \times \\
& \times \left\{[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s) - n_1^2(s))(n_2^2(\xi) - n_1^2(\xi))]4\frac{\partial}{\partial x_n}G_{tn} + \right. \\
& \left. + [2n_1(s)n_2(s)(n_2^2(\xi) - n_1^2(\xi)) - (n_2^2(s) - n_1^2(s))2n_1(\xi)n_2(\xi)]2\frac{\partial}{\partial x_n}(G_{tt} - G_{nn})\right\}
\end{aligned} \tag{103}$$

$$\begin{aligned}
T_n(s) \;=\; & -\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)\mu^2 \times \\
& \times \left\{[2n_1(s)n_2(s)(n_2^2(\xi) - n_1^2(\xi)) - (n_2^2(s) - n_1^2(s))2n_1(\xi)n_2(\xi)]4\frac{\partial}{\partial x_n}G_{tn} - \right. \\
& - [2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s) - n_1^2(s))(n_2^2(\xi) - n_1^2(\xi))]2\frac{\partial}{\partial x_n}(G_{tt} - G_{nn}) + \\
& \left. + \frac{\alpha^2}{\beta^2}\left(1 - \frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt} + G_{nn} - G_{33})\right\}.
\end{aligned} \tag{104}$$

The Kelvin solutions, or the Green functions for the time-independent in-plane case, are given by (Maruyama, 1966; Zhang and Achenbach, 1989)

$$G_{11}(\vec{x};\vec{y}) - G_{22}(\vec{x};\vec{y}) = \frac{1}{4\pi\mu}\left[-\left(1 - \frac{\beta^2}{\alpha^2}\right)(\gamma_2^2 - \gamma_1^2)\right] \tag{105}$$

$$G_{11}(\vec{x};\vec{y}) + G_{22}(\vec{x};\vec{y}) = \frac{1}{4\pi\mu}\left[\left(1 - \frac{\beta^2}{\alpha^2}\right) - 2\left(1 + \frac{\beta^2}{\alpha^2}\right)\log r\right] \tag{106}$$

$$G_{12}(\vec{x};\vec{y}) = \frac{1}{4\pi\mu}\left(1 - \frac{\beta^2}{\alpha^2}\right)\gamma_1\gamma_2. \tag{107}$$

The Green function for the anti-plane case is given again for quick reference:

$$G_{33}(\bar{x};\bar{y}) = \frac{1}{4\pi\mu}(-2\log r), \tag{108}$$

and thus we have

$$\begin{aligned}
2\frac{\partial}{\partial x_1}G_{12} + \frac{\partial}{\partial x_2}G_{33} &= \frac{1}{2\pi\mu}\frac{\gamma_2}{r}\left[\left(1-\frac{\beta^2}{\alpha^2}\right)(\gamma_2^2-\gamma_1^2)-1\right] && (109)\\
2\frac{\partial}{\partial x_2}G_{12} + \frac{\partial}{\partial x_1}G_{33} &= \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\left[-\left(1-\frac{\beta^2}{\alpha^2}\right)(\gamma_2^2-\gamma_1^2)-1\right] && (110)\\
2\frac{\partial}{\partial x_2}G_{12} &= \frac{1}{2\pi\mu}\frac{-\gamma_1}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)(\gamma_2^2-\gamma_1^2) && (111)\\
\frac{\partial}{\partial x_2}(G_{11}-G_{22}) &= \frac{1}{2\pi\mu}\frac{-\gamma_2}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)2\gamma_1^2 && (112)
\end{aligned}$$

$$\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33}) = \frac{1}{2\pi\mu}\frac{-\gamma_2}{r}\left(1-\frac{\beta^2}{\alpha^2}\right). \tag{113}$$

As has been mentioned elsewhere, it is to be noted that the form of the Green functions for the in-plane case is subject to no modification if we simultaneously replace x_1 with x_t and x_2 with x_n. Substituting Equations (109), (110), (111), (112) and (113) into (98), (99), (100), (101), (102), (103) and (104), we get, after lengthy algebraic manipulations, at the following expressions, where the singular integrals should be understood as Cauchy principal value problems:

$$\begin{aligned}
u_1(\bar{x}) &= \frac{1}{2\pi}\int_\Gamma d\xi\,\Delta u_t(\xi)\ \times\\
&\times\left\{2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}\left[\gamma_1^2+\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_2^2\right]+\right.\\
&\left.+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}\left[\left(2-\frac{\beta^2}{\alpha^2}\right)\gamma_1^2+\frac{\beta^2}{\alpha^2}\gamma_2^2)\right]\right\} && (114)\\
u_2(\bar{x}) &= \frac{1}{2\pi}\int_\Gamma d\xi\,\Delta u_t(\xi)\ \times\\
&\times\left\{-2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}\left[\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_1^2+\gamma_2^2\right]+\right.\\
&\left.+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}\left[\frac{\beta^2}{\alpha^2}\gamma_1^2+\left(2-\frac{\beta^2}{\alpha^2}\right)\gamma_2^2\right]\right\} && (115)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}(\sigma_{11}(\bar{x})-\sigma_{22}(\bar{x})) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_t(\xi)2\gamma_1\gamma_2(n_2(\xi)\frac{\gamma_1}{r}-n_1(\xi)\frac{\gamma_2}{r}) && (116)\\
\frac{1}{2}(\sigma_{11}(\bar{x})+\sigma_{22}(\bar{x})) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_t(\xi)(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}) && (117)\\
\sigma_{12}(\bar{x}) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_t(\xi)(\gamma_2^2-\gamma_1^2)(n_2(\xi)\frac{\gamma_1}{r}-n_1(\xi)\frac{\gamma_2}{r}) && (118)
\end{aligned}$$

and

$$\begin{aligned}
T_t(s) &= 2n_1(s)n_2(s)\frac{1}{2}(\sigma_{11}(\bar{y}(s))-\sigma_{22}(\bar{y}(s)))+(n_2^2(s)-n_1^2(s))\sigma_{12}(\bar{y}(s))\\
&= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_t(\xi)\ \times\\
&\times[2n_1(s)n_2(s)2\gamma_1\gamma_2+(n_2^2(s)-n_1^2(s))(\gamma_2^2-\gamma_1^2)](n_2(\xi)\frac{\gamma_1}{r}-n_1(\xi)\frac{\gamma_2}{r}) && (119)
\end{aligned}$$

$$\begin{aligned}
T_n(s) &= \frac{1}{2}(\sigma_{11}(\bar{y}(s))+\sigma_{22}(\bar{y}(s)))\ -\\
&-(n_2^2(s)-n_1^2(s))\frac{1}{2}(\sigma_{11}(\bar{y}(s))-\sigma_{22}(\bar{y}(s)))+2n_1(s)n_2(s)\sigma_{12}(\bar{y}(s))\\
&= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_t(\xi)\ \times\\
&\times\left\{(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r})\ +\right.\\
&\left.+[2n_1(s)n_2(s)(\gamma_2^2-\gamma_1^2)-(n_2^2(s)-n_1^2(s))2\gamma_1\gamma_2](n_2(\xi)\frac{\gamma_1}{r}-n_1(\xi)\frac{\gamma_2}{r})\right\}. && (120)
\end{aligned}$$

2.6 Time-domain formulation for open in-plane cracking

For the sake of completeness I now consider the case of open in-plane crack(s), which has both non-zero normal and tangential (mixed modes I and II) displacement discontinuities along its trace. In the present section I derive the expressions for the elastic field induced by the normal displacement discontinuity (opening slip) Δu_n. For simplicity I omit all terms relevant to the tangential displacement discontinuity (shear slip) Δu_t, but *all the formulae that follow should be understood as supplementary terms specific to the case of open cracking that should be added to the formulae for the case of closed in-plane cracking.* Denoting the amount of opening by Δu_n, we obtain the relations

$$\left\{ \begin{array}{lcl} \Delta u_1(\xi,\tau) & = & n_1(\xi)\Delta u_n(\xi,\tau) \\ \Delta u_2(\xi,\tau) & = & n_2(\xi)\Delta u_n(\xi,\tau) \end{array} \right. \tag{121}$$

Following a sequence of algebra similar to that practiced in the case of closed in-plane cracking, we arrive at the expressions

$$\begin{aligned} u_1(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau) \left\{ n_2(\xi) \left[\lambda \frac{\partial}{\partial x_t} G_{tt} + (\lambda+2\mu) \frac{\partial}{\partial x_n} G_{tn} \right] + \right. \\ &\quad \left. + n_1(\xi) \left[\lambda \frac{\partial}{\partial x_t} G_{tn} + (\lambda+2\mu) \frac{\partial}{\partial x_n} G_{nn} \right] \right\} \end{aligned} \tag{122}$$

$$\begin{aligned} u_2(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau) \left\{ -n_1(\xi) \left[\lambda \frac{\partial}{\partial x_t} G_{tt} + (\lambda+2\mu) \frac{\partial}{\partial x_n} G_{tn} \right] + \right. \\ &\quad \left. + n_2(\xi) \left[\lambda \frac{\partial}{\partial x_t} G_{tn} + (\lambda+2\mu) \frac{\partial}{\partial x_n} G_{nn} \right] \right\} \end{aligned} \tag{123}$$

and

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) - \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau)\mu \times \\ &\quad \times \left\{ (n_2^2(\xi) - n_1^2(\xi)) \left[(\lambda+\mu) \left(\frac{\partial^2}{\partial x_t^2} G_{tt} - \frac{\partial^2}{\partial x_n^2} G_{nn} \right) - \right. \right. \\ &\quad \left. - \mu \left(\frac{\partial^2}{\partial x_t^2} G_{tt} + \frac{\partial^2}{\partial x_n^2} G_{nn} - 2 \frac{\partial^2}{\partial x_t \partial x_n} G_{tn} \right) \right] + \\ &\quad + 2n_1(\xi)n_2(\xi) \left[(\lambda+\mu) \left(\frac{\partial^2}{\partial x_t \partial x_n} (G_{tt} + G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right) - \right. \\ &\quad \left. \left. - \mu \left(\frac{\partial^2}{\partial x_t \partial x_n} (G_{tt} - G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} - \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right) \right] \right\} \end{aligned} \tag{124}$$

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) + \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau)(\lambda+\mu) \times \\ &\quad \times \left[(\lambda+\mu) \left(\frac{\partial^2}{\partial x_t^2} G_{tt} + \frac{\partial^2}{\partial x_n^2} G_{nn} + 2 \frac{\partial^2}{\partial x_t \partial x_n} G_{tn} \right) - \right. \\ &\quad \left. - \mu \left(\frac{\partial^2}{\partial x_t^2} G_{tt} - \frac{\partial^2}{\partial x_n^2} G_{nn} \right) \right] \end{aligned} \tag{125}$$

$$\begin{aligned} \sigma_{12}(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau)\mu \times \\ &\quad \times \left\{ -2n_1(\xi)n_2(\xi) \left[(\lambda+\mu) \left(\frac{\partial^2}{\partial x_t^2} G_{tt} - \frac{\partial^2}{\partial x_n^2} G_{nn} \right) - \right. \right. \\ &\quad \left. - \mu \left(\frac{\partial^2}{\partial x_t^2} G_{tt} + \frac{\partial^2}{\partial x_n^2} G_{nn} - 2 \frac{\partial^2}{\partial x_t \partial x_n} G_{tn} \right) \right] + \\ &\quad + (n_2^2(\xi) - n_1^2(\xi)) \left[(\lambda+\mu) \left(\frac{\partial^2}{\partial x_t \partial x_n} (G_{tt} + G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right) - \right. \\ &\quad \left. \left. - \mu \left(\frac{\partial^2}{\partial x_t \partial x_n} (G_{tt} - G_{nn}) + \left(\frac{\partial^2}{\partial x_t^2} - \frac{\partial^2}{\partial x_n^2} \right) G_{tn} \right) \right] \right\}. \end{aligned} \tag{126}$$

Paying attention to the equations of motion

$$\rho \frac{\partial^2}{\partial t^2}(G_{11} - G_{22}) = (\lambda+2\mu) \left(\frac{\partial^2}{\partial x_1^2} G_{11} - \frac{\partial^2}{\partial x_2^2} G_{22} \right) + \mu \left(\frac{\partial^2}{\partial x_2^2} G_{11} - \frac{\partial^2}{\partial x_1^2} G_{22} \right) \tag{127}$$

$$\begin{aligned} \rho \frac{\partial^2}{\partial t^2}(G_{11} + G_{22}) &= (\lambda+2\mu) \left(\frac{\partial^2}{\partial x_1^2} G_{11} + \frac{\partial^2}{\partial x_2^2} G_{22} \right) + \mu \left(\frac{\partial^2}{\partial x_2^2} G_{11} + \frac{\partial^2}{\partial x_1^2} G_{22} \right) + \\ &\quad + 2(\lambda+\mu) \frac{\partial^2}{\partial x_1 \partial x_2} G_{12} \end{aligned} \tag{128}$$

$$\rho \frac{\partial^2}{\partial t^2} G_{33} = \mu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) G_{33} \tag{129}$$

the following new relations can be found:

$$\frac{\partial^2}{\partial x_1^2} G_{11} - \frac{\partial^2}{\partial x_2^2} G_{22} = 2 \frac{\beta^2}{\alpha^2} \frac{\partial^2}{\partial x_1^2} G_{33} + \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} (G_{11} - G_{22} - G_{33}) \tag{130}$$

$$\frac{\partial^2}{\partial x_1^2} G_{11} + \frac{\partial^2}{\partial x_2^2} G_{22} = -2 \frac{\partial^2}{\partial x_1 \partial x_2} G_{12} + \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} (G_{11} + G_{22} - G_{33}). \tag{131}$$

Making use of these formulae and of those derived in a previous section, we arrive at the following expressions:

$$\begin{aligned} u_1(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau)\mu \times \\ &\quad \times \left\{ n_2(\xi) \left[\frac{\alpha^2}{\beta^2} \left(1 - 2\frac{\beta^2}{\alpha^2} \right) \frac{\partial}{\partial x_t} (G_{tt} + G_{nn} - G_{33}) + 2 \frac{\partial}{\partial x_n} G_{tn} \right] + \right. \\ &\quad \left. + n_1(\xi) \left[\frac{\alpha^2}{\beta^2} \frac{\partial}{\partial x_n} (G_{tt} + G_{nn} - G_{33}) - 2 \frac{\partial}{\partial x_t} G_{tn} \right] \right\} \end{aligned} \tag{132}$$

$$\begin{aligned} u_2(\vec{x},t) &= -\int_\Gamma d\xi \int_0^t d\tau \Delta u_n(\xi,\tau)\mu \times \\ &\quad \times \left\{ -n_1(\xi) \left[\frac{\alpha^2}{\beta^2} \left(1 - 2\frac{\beta^2}{\alpha^2} \right) \frac{\partial}{\partial x_t} (G_{tt} + G_{nn} - G_{33}) + 2 \frac{\partial}{\partial x_n} G_{tn} \right] + \right. \\ &\quad \left. + n_2(\xi) \left[\frac{\alpha^2}{\beta^2} \frac{\partial}{\partial x_n} (G_{tt} + G_{nn} - G_{33}) - 2 \frac{\partial}{\partial x_t} G_{tn} \right] \right\} \end{aligned} \tag{133}$$

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) - \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta u_n(\xi,\tau)\mu^2 \times \\ &\quad \times \left\{ (n_2^2(\xi) - n_1^2(\xi)) \left[4 \frac{\partial}{\partial x_n} G_{tn} + \left(1 - \frac{\beta^2}{\alpha^2} \right) 2 \frac{\partial}{\partial x_t} G_{33} \right] + 2n_1(\xi)n_2(\xi) \times \right. \\ &\quad \left. \times \left[\frac{\alpha^2}{\beta^2} \left(1 - \frac{\beta^2}{\alpha^2} \right) 2 \frac{\partial}{\partial x_n} (G_{tt} + G_{nn} - G_{33}) - 2 \frac{\partial}{\partial x_n} (G_{tt} - G_{nn}) \right] \right\} - \\ &\quad - \int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2} \Delta u_n(\xi,\tau) \frac{\mu^2}{\beta^2} (n_2^2(\xi) - n_1^2(\xi)) \times \\ &\quad \times \left[\left(1 - \frac{\beta^2}{\alpha^2} \right) (G_{tt} - G_{nn} - G_{33}) - \frac{\beta^2}{\alpha^2} (G_{tt} + G_{nn} - G_{33}) \right] \end{aligned} \tag{134}$$

$$\begin{aligned} \frac{1}{2}(\sigma_{11}(\vec{x},t) + \sigma_{22}(\vec{x},t)) &= -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta u_n(\xi,\tau)\mu^2 \left(1 - \frac{\beta^2}{\alpha^2} \right) \left(-2 \frac{\partial}{\partial x_t} G_{33} \right) - \\ &\quad - \int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2} \Delta u_n(\xi,\tau) \frac{\mu^2}{\beta^2} \times \end{aligned}$$

$$
\times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)^2 (G_{tt}+G_{nn}-G_{33}) - \left(1-\frac{\beta^2}{\alpha^2}\right)(G_{tt}-G_{nn}-G_{33})\right] \quad (135)
$$

$$
\begin{aligned}
\sigma_{12}(\vec{x},t) = & -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_n(\xi,\tau)\mu^2 \times \\
& \times \left\{-2n_1(\xi)n_2(\xi)\left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + (n_2^2(\xi)-n_1^2(\xi)) \times \right. \\
& \left. \times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]\right\} - \\
& -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_n(\xi,\tau)\frac{\mu^2}{\beta^2}(-2n_1(\xi)n_2(\xi)) \times \\
& \times \left[\left(1-\frac{\beta^2}{\alpha^2}\right)(G_{tt}-G_{nn}-G_{33}) - \frac{\beta^2}{\alpha^2}(G_{tt}+G_{nn}-G_{33})\right]. \qquad (136)
\end{aligned}
$$

With the limiting process $\vec{x} \to \vec{y}(s)$, we get, for the tangential traction $T_t(s,t)$ on the crack at arc length s and time t,

$$
\begin{aligned}
T_t(s,t) = & \; 2n_1(s)n_2(s)\frac{1}{2}(\sigma_{11}(\vec{y}(s),t) - \sigma_{22}(\vec{y}(s),t)) + (n_2^2(s)-n_1^2(s))\sigma_{12}(\vec{y}(s),t) \\
= & -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_n(\xi,\tau)\mu^2 \times \\
& \times \{[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi)) - (n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
& \times \left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + \\
& +[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
& \times \left.\left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]\right\} - \\
& -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_n(\xi,\tau)\frac{\mu^2}{\beta^2} \times \\
& \times [2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi)) - (n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
& \times \left[\left(1-\frac{\beta^2}{\alpha^2}\right)(G_{tt}-G_{nn}-G_{33}) - \frac{\beta^2}{\alpha^2}(G_{tt}+G_{nn}-G_{33})\right]. \qquad (137)
\end{aligned}
$$

This is the displacement BIE that expresses the traction on the crack(s) in terms of the slip on the crack(s). Likewise, the normal traction $T_n(s,t)$ across the crack at arc length s and time t is given by

$$
\begin{aligned}
T_n(s,t) = & \; \frac{1}{2}(\sigma_{11}(\vec{y}(s),t) + \sigma_{22}(\vec{y}(s),t)) - \\
& -(n_2^2(s)-n_1^2(s))\frac{1}{2}(\sigma_{11}(\vec{y}(s),t) - \sigma_{22}(\vec{y}(s),t)) + 2n_1(s)n_2(s)\sigma_{12}(\vec{y}(s),t) \\
= & -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial \xi}\Delta u_n(\xi,\tau)\mu^2 \times \\
& \times \{-[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
& \times \left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + \\
& +[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi)) - (n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
& \times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right] - \\
& \left.-\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right\} -
\end{aligned}
$$

$$
\begin{aligned}
& -\int_\Gamma d\xi \int_0^t d\tau \frac{\partial^2}{\partial \tau^2}\Delta u_n(\xi,\tau)\frac{\mu^2}{\beta^2} \times \\
& \times \{-[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
& \times \left[\left(1-\frac{\beta^2}{\alpha^2}\right)(G_{tt}-G_{nn}-G_{33}) - \frac{\beta^2}{\alpha^2}(G_{tt}+G_{nn}-G_{33})\right] + \\
& \left.+\left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)^2(G_{tt}+G_{nn}-G_{33}) - \left(1-\frac{\beta^2}{\alpha^2}\right)(G_{tt}-G_{nn}-G_{33})\right]\right\}. \qquad (138)
\end{aligned}
$$

Since we can prove that the following identities hold true:

$$
\begin{aligned}
& \frac{\alpha^2}{\beta^2}\left(1-2\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}(G_{11}+G_{22}-G_{33}) + 2\frac{\partial}{\partial x_2}G_{12} = \\
& = \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial^2}{\partial \tau^2}\left[\frac{2}{3}(\gamma_1^2-3\gamma_2^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\alpha)^2]^{3/2} + \right. \\
& \left. + \left(2\gamma_1^2\frac{\beta^2}{\alpha^2}-1\right)\sqrt{(t-\tau)^2-(r/\alpha)^2}\right] H(t-\tau-\frac{r}{\alpha}) - \\
& -\frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial^2}{\partial \tau^2}\left[\frac{2}{3}(\gamma_1^2-3\gamma_2^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\beta)^2]^{3/2} - \right. \\
& \left. -2\gamma_2^2\sqrt{(t-\tau)^2-(r/\beta)^2}\right] H(t-\tau-\frac{r}{\beta}) \qquad (139)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\alpha^2}{\beta^2}\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33}) - 2\frac{\partial}{\partial x_1}G_{12} = \\
& = -\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial^2}{\partial \tau^2}\left[\frac{2}{3}(\gamma_2^2-3\gamma_1^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\alpha)^2]^{3/2} - \right. \\
& \left. - \left(2\gamma_1^2\frac{\beta^2}{\alpha^2}-1\right)\sqrt{(t-\tau)^2-(r/\alpha)^2}\right] H(t-\tau-\frac{r}{\alpha}) + \\
& +\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial^2}{\partial \tau^2}\left[\frac{2}{3}(\gamma_2^2-3\gamma_1^2)\frac{\beta^2}{r^2}[(t-\tau)^2-(r/\beta)^2]^{3/2} - \right. \\
& \left. -2\gamma_1^2\sqrt{(t-\tau)^2-(r/\beta)^2}\right] H(t-\tau-\frac{r}{\beta}) \qquad (140)
\end{aligned}
$$

$$
\begin{aligned}
& 2\frac{\partial}{\partial x_2}G_{12} + \left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}G_{33} = \\
& = \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial}{\partial \tau}\left[2(3\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} + \right. \\
& \left. +2\gamma_2^2\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right] H(t-\tau-\frac{r}{\alpha}) - \\
& -\frac{1}{2\pi\mu}\frac{\gamma_1}{r}\frac{\partial}{\partial \tau}\left[2(3\gamma_2^2-\gamma_1^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} + \right. \\
& \left. + \left(\gamma_2^2-\gamma_1^2+\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right] H(t-\tau-\frac{r}{\beta}) \qquad (141)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33}) - \frac{\partial}{\partial x_2}(G_{11}-G_{22}) = \\
& = -\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial}{\partial \tau}\left[2(3\gamma_1^2-\gamma_2^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} + \right. \\
& \left. + \left(2\gamma_1^2\frac{\beta^2}{\alpha^2}-1\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right] H(t-\tau-\frac{r}{\alpha}) +
\end{aligned}
$$

$$+\frac{1}{2\pi\mu}\frac{\gamma_2}{r}\frac{\partial}{\partial\tau}\left[2(3\gamma_1^2-\gamma_2^2)\frac{\beta^2}{r^2}(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}\;-\right.$$
$$\left.-(\gamma_2^2-\gamma_1^2)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right]H(t-\tau-\frac{r}{\beta}) \qquad (142)$$

$$\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}G_{33} \;=\; \frac{1}{2\pi\mu}\frac{\gamma_1}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial\tau}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) \qquad (143)$$

and

$$\left(1-\frac{\beta^2}{\alpha^2}\right)(G_{11}-G_{22}-G_{33})-\frac{\beta^2}{\alpha^2}(G_{11}+G_{22}-G_{33})=$$
$$=\;-\frac{1}{2\pi\mu}\left\{2(\gamma_2^2-\gamma_1^2)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\beta^2}{r^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}\;+\right.$$
$$\left.+\left[\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_1^2+\gamma_2^2\right]\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right\}H(t-\tau-\frac{r}{\alpha})\;+$$
$$+\frac{1}{2\pi\mu}\left[2(\gamma_2^2-\gamma_1^2)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\beta^2}{r^2}\sqrt{(t-\tau)^2-(r/\beta)^2}\;-\right.$$
$$\left.-2\gamma_1^2\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right]H(t-\tau-\frac{r}{\beta}) \qquad (144)$$

$$\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)^2(G_{11}+G_{22}-G_{33})-\left(1-\frac{\beta^2}{\alpha^2}\right)(G_{11}-G_{22}-G_{33})=$$
$$=\;\frac{1}{2\pi\mu}\left\{2(\gamma_2^2-\gamma_1^2)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\beta^2}{r^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}\;+\right.$$
$$\left.+\left[-\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_1^2+\gamma_2^2\right]\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}\right\}H(t-\tau-\frac{r}{\alpha})\;-$$
$$-\frac{1}{2\pi\mu}\left[2(\gamma_2^2-\gamma_1^2)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\beta^2}{r^2}\sqrt{(t-\tau)^2-(r/\beta)^2}\;-\right.$$
$$\left.-2\gamma_1^2\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}\right]H(t-\tau-\frac{r}{\beta}), \qquad (145)$$

we get, after lengthy algebraic manipulations, at the following expressions, in which the singular integrals are to be interpreted in terms of Cauchy principal values:

$$u_1(\vec{x},t) \;=\; \frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{2}{3}\frac{\beta^2}{r^2}\left\{\left[(t-\tau)^2-(r/\alpha)^2\right]^{3/2}H(t-\tau-\frac{r}{\alpha})\;-\right.$$
$$\left.-\left[(t-\tau)^2-(r/\beta)^2\right]^{3/2}H(t-\tau-\frac{r}{\beta})\right\}\;+$$
$$+\frac{1}{2\pi}\int_\Gamma d\xi\left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\gamma_1}{r}\;+\right.$$
$$\left.+2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}2\gamma_1^2+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}(\gamma_2^2-\gamma_1^2)\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{\beta^2}{\alpha^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha})\;-$$
$$-\frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(\gamma_1^2-\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}2\gamma_2^2\right]\times$$

$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta}) \qquad (146)$$

$$u_2(\vec{x},t) \;=\; \frac{1}{2\pi}\int_\Gamma d\xi\left[-2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{2}{3}\frac{\beta^2}{r^2}\left\{\left[(t-\tau)^2-(r/\alpha)^2\right]^{3/2}H(t-\tau-\frac{r}{\alpha})\;-\right.$$
$$\left.-\left[(t-\tau)^2-(r/\beta)^2\right]^{3/2}H(t-\tau-\frac{r}{\beta})\right\}\;+$$
$$+\frac{1}{2\pi}\int_\Gamma d\xi\left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\gamma_2}{r}\;+\right.$$
$$\left.+2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}2\gamma_2^2+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}(\gamma_2^2-\gamma_1^2)\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{\beta^2}{\alpha^2}\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha})\;-$$
$$-\frac{1}{2\pi}\int_\Gamma d\xi\left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_2^2-\gamma_1^2)-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}2\gamma_1^2\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta}) \qquad (147)$$

$$\frac{1}{2}(\sigma_{11}(\vec{x},t)-\sigma_{22}(\vec{x},t))$$
$$=\;\frac{\mu}{\pi}\int_\Gamma d\xi\left[n_1(\xi)\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)-n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)\right]\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)2\frac{\beta^2}{r^2}\left[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha})\;-\right.$$
$$\left.-(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\right]\;+$$
$$+\frac{\mu}{\pi}\int_\Gamma d\xi\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right)2\gamma_1\gamma_2\;\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left[\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})\;-\right.$$
$$\left.-\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\right]\;+$$
$$+\frac{\mu}{\pi}\int_\Gamma d\xi 2n_1(\xi)n_2(\xi)\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right)\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})\;+$$
$$+\frac{\mu}{\pi}\int_\Gamma d\xi[-(n_2^2(\xi)-n_1^2(\xi))]\left(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}\right)\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\;-$$
$$-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[-(n_2^2(\xi)-n_1^2(\xi))]\;\times$$
$$\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\left[\frac{\beta^4}{\alpha^4}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})\;+\right.$$

$$
\begin{aligned}
&+\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[-(n_2^2(\xi)-n_1^2(\xi))][2n_1(\xi)n_2(\xi)2\gamma_1\gamma_2+(n_2^2(\xi)-n_1^2(\xi))(\gamma_2^2-\gamma_1^2)] \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\Big\{2\frac{\beta^2}{r^2}\Big[\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\
&-\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})-\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big\} \qquad (148)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}(\sigma_{11}(\vec{x},t)+\sigma_{22}(\vec{x},t)) \\
&= \frac{\mu}{\pi}\int_\Gamma d\xi\left(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}\right) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\Big[\left(1-\frac{\beta^2}{\alpha^2}\right)^2\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) + \\
&+\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[2n_1(\xi)n_2(\xi)2\gamma_1\gamma_2+(n_2^2(\xi)-n_1^2(\xi))(\gamma_2^2-\gamma_1^2)] \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\Big\{2\frac{\beta^2}{r^2}\Big[\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\
&-\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})-\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big\} \qquad (149)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{12}(\vec{x},t) &= \frac{\mu}{\pi}\int_\Gamma d\xi\left[-n_1(\xi)\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)-n_2(\xi)\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)\right] \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)2\frac{\beta^2}{r^2}\Big[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\
&-(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right)(\gamma_2^2-\gamma_1^2) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\Big[\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) - \\
&-\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi(n_2^2(\xi)-n_1^2(\xi))\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi 2n_1(\xi)n_2(\xi)\left(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}\right) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi 2n_1(\xi)n_2(\xi) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\Big[\frac{\beta^4}{\alpha^4}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) + \\
&+\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi 2n_1(\xi)n_2(\xi)[2n_1(\xi)n_2(\xi)2\gamma_1\gamma_2+(n_2^2(\xi)-n_1^2(\xi))(\gamma_2^2-\gamma_1^2)] \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\Big\{2\frac{\beta^2}{r^2}\Big[\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\
&-\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha})-\frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big\} \qquad (150)
\end{aligned}
$$

and

$$
\begin{aligned}
&T_1(s,t) = 2n_1(s)n_2(s)\frac{1}{2}(\sigma_{11}(\vec{y}(s),t)-\sigma_{22}(\vec{y}(s),t))+(n_2^2(s)-n_1^2(s))\sigma_{12}(\vec{y}(s),t) \\
&= \frac{\mu}{\pi}\int_\Gamma d\xi\Big\{[2n_1(s)n_2(s)n_1(\xi)-(n_2^2(s)-n_1^2(s))n_2(\xi)]\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2) - \\
&-[2n_1(s)n_2(s)n_2(\xi)+(n_2^2(s)-n_1^2(s))n_1(\xi)]\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)\Big\} \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)2\frac{\beta^2}{r^2}\Big[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) - \\
&-(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[2n_1(s)n_2(s)2\gamma_1\gamma_2+(n_2^2(s)-n_1^2(s))(\gamma_2^2-\gamma_1^2)]\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\Big[\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) - \\
&-\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta})\Big] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))]\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right) \times \\
&\times\int_0^t d\tau\frac{\partial}{\partial\xi}\Delta\dot{u}_n(\xi,\tau)\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[-2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))+(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)]\left(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}\right) \times
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_n(\xi,\tau) \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[-2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))+(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \left[\frac{\beta^4}{\alpha^4}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) + \right. \\
&\left. + \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta})\right] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[-2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))+(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
&\times[2n_1(\xi)n_2(\xi)2\gamma_1\gamma_2+(n_2^2(\xi)-n_1^2(\xi))(\gamma_2^2-\gamma_1^2)] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \left(1-\frac{\beta^2}{\alpha^2}\right) \left\{ 2\frac{\beta^2}{r^2}\left[\sqrt{(t-\tau)^2-(r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right.\right. \\
&\left. -\sqrt{(t-\tau)^2-(r/\beta)^2} H(t-\tau-\frac{r}{\beta})\right] + \\
&\left. +\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right\} \qquad (151)
\end{aligned}
$$

$$
\begin{aligned}
T_n(s,t) &= \frac{1}{2}(\sigma_{11}(\vec{y}(s),t)+\sigma_{22}(\vec{y}(s),t)) - \\
&-(n_2^2(s)-n_1^2(s))\frac{1}{2}(\sigma_{11}(\vec{y}(s),t)-\sigma_{22}(\vec{y}(s),t)) + 2n_1(s)n_2(s)\sigma_{12}(\vec{y}(s),t) \\
&= \frac{\mu}{\pi}\int_\Gamma d\xi \left\{ [-2n_1(s)n_2(s)n_1(\xi)+(n_2^2(s)-n_1^2(s))n_2(\xi)]\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2) - \right. \\
&\left. -[2n_1(s)n_2(s)n_2(\xi)+(n_2^2(s)-n_1^2(s))n_1(\xi)]\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2) \right\} \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_n(\xi,\tau) 2\frac{\beta^2}{r^2}\left[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
&\left. -(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} H(t-\tau-\frac{r}{\beta})\right] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[2n_1(s)n_2(s)(\gamma_2^2-\gamma_1^2)-(n_2^2(s)-n_1^2(s))2\gamma_1\gamma_2)]\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right) \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_n(\xi,\tau) \left[\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \right. \\
&\left. -\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta})\right] + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))+(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)]\left(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}\right) \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_n(\xi,\tau) \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) + \\
&+\frac{\mu}{\pi}\int_\Gamma d\xi[1+2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))]\left(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}\right) \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \xi} \Delta \dot{u}_n(\xi,\tau) \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \left[\frac{\beta^4}{\alpha^4}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) + \right. \\
&\left. + \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta})\right] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \left[\left(1-\frac{\beta^2}{\alpha^2}\right)^2 \frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) + \right. \\
&\left. + \left(1-\frac{\beta^2}{\alpha^2}\right) \frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta})\right] - \\
&-\frac{\mu}{2\pi}\frac{1}{\beta^2}\int_\Gamma d\xi[1+2n_1(s)n_2(s)2n_1(\xi)n_2(\xi)+(n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
&\times[2n_1(\xi)n_2(\xi)2\gamma_1\gamma_2+(n_2^2(\xi)-n_1^2(\xi))(\gamma_2^2-\gamma_1^2)] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \left(1-\frac{\beta^2}{\alpha^2}\right) \left\{ 2\frac{\beta^2}{r^2}\left[\sqrt{(t-\tau)^2-(r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right.\right. \\
&\left. -\sqrt{(t-\tau)^2-(r/\beta)^2} H(t-\tau-\frac{r}{\beta})\right] + \\
&\left. +\frac{\beta^2}{\alpha^2}\frac{1}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \frac{1}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \right\}. \qquad (152)
\end{aligned}
$$

Differentiating (146) and (147) with respect to t we get the following representations for the displacement velocity field:

$$
\begin{aligned}
\dot{u}_1(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)\right] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) 2\frac{\beta^2}{r^2}\left[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
&\left. -(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} H(t-\tau-\frac{r}{\beta})\right] + \\
&+\frac{1}{2\pi}\int_\Gamma d\xi \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\gamma_1}{r} + \right. \\
&\left. +2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}2\gamma_1^2+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}(\gamma_2^2-\gamma_1^2)\right] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}} H(t-\tau-\frac{r}{\alpha}) - \\
&-\frac{1}{2\pi}\int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}(\gamma_1^2-\gamma_2^2)+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}2\gamma_2^2\right] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta \dot{u}_n(\xi,\tau) \frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}} H(t-\tau-\frac{r}{\beta}) \qquad (153) \\
\dot{u}_2(\vec{x},t) &= \frac{1}{2\pi}\int_\Gamma d\xi \left[-2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_1^2-3\gamma_2^2)-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}(3\gamma_1^2-\gamma_2^2)\right] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial \tau} \Delta u_n(\xi,\tau) 2\frac{\beta^2}{r^2}\left[(t-\tau)\sqrt{(t-\tau)^2-(r/\alpha)^2} H(t-\tau-\frac{r}{\alpha}) - \right. \\
&\left. -(t-\tau)\sqrt{(t-\tau)^2-(r/\beta)^2} H(t-\tau-\frac{r}{\beta})\right] + \\
&+\frac{1}{2\pi}\int_\Gamma d\xi \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\gamma_2}{r} + \right.
\end{aligned}
$$

$$
\begin{aligned}
&+2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}2\gamma_2^2 + (n_2^2(\xi) - n_1^2(\xi))\frac{\gamma_2}{r}(\gamma_2^2 - \gamma_1^2)\Big] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{\beta^2}{\alpha^2}\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\alpha)^2}}H(t-\tau-\frac{r}{\alpha}) - \\
&-\frac{1}{2\pi}\int_\Gamma d\xi \left[2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}(\gamma_2^2-\gamma_1^2) - (n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}2\gamma_1^2\right] \times \\
&\times \int_0^t d\tau \frac{\partial}{\partial\tau}\Delta\dot{u}_n(\xi,\tau)\frac{t-\tau}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}).
\end{aligned} \tag{154}
$$

2.7 Time-independent formulation for open in-plane cracking

The time-independent counterpart of the displacement BIEs for the 2D open in-plane crack problem follows by analogy to the time-dependent version. As in Section 2.6, *all the formulae that follow should be understood as supplementary terms specific to the case of open cracking that should be added to the formulae for the case of closed in-plane cracking.*

$$
\begin{aligned}
u_1(\bar{x}) &= -\int_\Gamma d\xi \Delta u_n(\xi)\mu \times \\
&\times \left\{n_2(\xi)\left[\frac{\alpha^2}{\beta^2}\left(1-2\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_t}(G_{tt}+G_{nn}-G_{33}) + 2\frac{\partial}{\partial x_n}G_{tn}\right] + \right. \\
&\left. + n_1(\xi)\left[\frac{\alpha^2}{\beta^2}\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_t}G_{tn}\right]\right\}
\end{aligned} \tag{155}
$$

$$
\begin{aligned}
u_2(\bar{x}) &= -\int_\Gamma d\xi \Delta u_n(\xi)\mu \times \\
&\times \left\{-n_1(\xi)\left[\frac{\alpha^2}{\beta^2}\left(1-2\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_t}(G_{tt}+G_{nn}-G_{33}) + 2\frac{\partial}{\partial x_n}G_{tn}\right] + \right. \\
&\left. + n_2(\xi)\left[\frac{\alpha^2}{\beta^2}\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_t}G_{tn}\right]\right\}
\end{aligned} \tag{156}
$$

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11}(\bar{x}) - \sigma_{22}(\bar{x})) &= -\int_\Gamma d\xi \frac{\partial}{\partial\xi}\Delta u_n(\xi)\mu^2 \times \\
&\times \left\{(n_2^2(\xi)-n_1^2(\xi))\left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + 2n_1(\xi)n_2(\xi) \times \right. \\
&\left. \times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]\right\}
\end{aligned} \tag{157}
$$

$$
\frac{1}{2}(\sigma_{11}(\bar{x}) + \sigma_{22}(\bar{x})) = -\int_\Gamma d\xi \frac{\partial}{\partial\xi}\Delta u_n(\xi)\mu^2\left(1-\frac{\beta^2}{\alpha^2}\right)\left(-2\frac{\partial}{\partial x_t}G_{33}\right) \tag{158}
$$

$$
\begin{aligned}
\sigma_{12}(\bar{x}) &= -\int_\Gamma d\xi \frac{\partial}{\partial\xi}\Delta u_n(\xi)\mu^2 \times \\
&\times \left\{-2n_1(\xi)n_2(\xi)\left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + (n_2^2(\xi)-n_1^2(\xi)) \times \right. \\
&\left. \times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]\right\}
\end{aligned} \tag{159}
$$

$$
\begin{aligned}
T_t(s) &= -\int_\Gamma d\xi \frac{\partial}{\partial\xi}\Delta u_n(\xi)\mu^2 \times \\
&\times \{[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi)) - (n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
&\times \left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] + \\
&+[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
&\times \left.\left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33}) - 2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right]\right\}
\end{aligned} \tag{160}
$$

$$
\begin{aligned}
T_n(s) &= -\int_\Gamma d\xi \frac{\partial}{\partial\xi}\Delta u_n(\xi)\mu^2 \times \\
&\times \{-[2n_1(s)n_2(s)2n_1(\xi)n_2(\xi) + (n_2^2(s)-n_1^2(s))(n_2^2(\xi)-n_1^2(\xi))] \times \\
&\times \left[4\frac{\partial}{\partial x_n}G_{tn} + \left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\right] +
\end{aligned}
$$

$$
\begin{aligned}
&+[2n_1(s)n_2(s)(n_2^2(\xi)-n_1^2(\xi))-(n_2^2(s)-n_1^2(s))2n_1(\xi)n_2(\xi)] \times \\
&\times \left[\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_n}(G_{tt}+G_{nn}-G_{33})-2\frac{\partial}{\partial x_n}(G_{tt}-G_{nn})\right] - \\
&-\left(1-\frac{\beta^2}{\alpha^2}\right)2\frac{\partial}{\partial x_t}G_{33}\bigg\}. \qquad (161)
\end{aligned}
$$

Substituting the explicit forms of the Kelvin solutions, we have

$$
\begin{aligned}
\frac{\alpha^2}{\beta^2}\left(1-2\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}(G_{11}+G_{22}-G_{33})+2\frac{\partial}{\partial x_2}G_{12} &= \frac{1}{2\pi\mu}\frac{-\gamma_1}{r}\left[\left(1-\frac{\beta^2}{\alpha^2}\right)2\gamma_2^2-\frac{\beta^2}{\alpha^2}\right] && (162)\\
\frac{\alpha^2}{\beta^2}\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33})-2\frac{\partial}{\partial x_1}G_{12} &= \frac{1}{2\pi\mu}\frac{-\gamma_2}{r}\left[\left(1-\frac{\beta^2}{\alpha^2}\right)2\gamma_2^2+\frac{\beta^2}{\alpha^2}\right] && (163)\\
2\frac{\partial}{\partial x_2}G_{12}+\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}G_{33} &= \frac{1}{2\pi\mu}\frac{-\gamma_1}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)2\gamma_2^2 && (164)\\
\frac{\alpha^2}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_2}(G_{11}+G_{22}-G_{33})-\frac{\partial}{\partial x_2}(G_{11}-G_{22}) &= \frac{1}{2\pi\mu}\frac{-\gamma_2}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)(\gamma_2^2-\gamma_1^2) && (165)\\
\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial x_1}G_{33} &= \frac{1}{2\pi\mu}\frac{-\gamma_1}{r}\left(1-\frac{\beta^2}{\alpha^2}\right). && (166)
\end{aligned}
$$

In view of the above expressions we get, after lengthy algebraic manipulations, at the expressions, where the singular integrals should be in terms of Cauchy principal values:

$$
\begin{aligned}
u_1(\vec{x}) &= \frac{1}{2\pi}\int_\Gamma d\xi \Delta u_n(\xi)\left\{\frac{\gamma_1}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)+\right.\\
&+2n_1(\xi)n_2(\xi)\frac{\gamma_2}{r}\left[\left(2-\frac{\beta^2}{\alpha^2}\right)\gamma_1^2+\frac{\beta^2}{\alpha^2}\gamma_2^2)\right]-\\
&\left.-(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_1}{r}\left[\gamma_1^2+\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_2^2\right]\right\} && (167)\\
u_2(\vec{x}) &= \frac{1}{2\pi}\int_\Gamma d\xi \Delta u_n(\xi)\left\{\frac{\gamma_2}{r}\left(1-\frac{\beta^2}{\alpha^2}\right)+\right.\\
&+2n_1(\xi)n_2(\xi)\frac{\gamma_1}{r}\left[\frac{\beta^2}{\alpha^2}\gamma_1^2+\left(2-\frac{\beta^2}{\alpha^2}\right)\gamma_2^2\right]+\\
&\left.+(n_2^2(\xi)-n_1^2(\xi))\frac{\gamma_2}{r}\left[\left(2\frac{\beta^2}{\alpha^2}-1\right)\gamma_1^2+\gamma_2^2\right]\right\} && (168)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11}(\vec{x})-\sigma_{22}(\vec{x})) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_n(\xi)2\gamma_1\gamma_2(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}) && (169)\\
\frac{1}{2}(\sigma_{11}(\vec{x})+\sigma_{22}(\vec{x})) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_n(\xi)(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r}) && (170)\\
\sigma_{12}(\vec{x}) &= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_n(\xi)(\gamma_2^2-\gamma_1^2)(n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}) && (171)
\end{aligned}
$$

and

$$
\begin{aligned}
T_t(s) &= 2n_1(s)n_2(s)\frac{1}{2}(\sigma_{11}(\vec{y}(s))-\sigma_{22}(\vec{y}(s)))+(n_2^2(s)-n_1^2(s))\sigma_{12}(\vec{y}(s))\\
&= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_n(\xi)\times\\
&\times[2n_1(s)n_2(s)2\gamma_1\gamma_2+(n_2^2(s)-n_1^2(s))(\gamma_2^2-\gamma_1^2)](n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r}) && (172)
\end{aligned}
$$

$$
\begin{aligned}
T_n(s) &= \frac{1}{2}(\sigma_{11}(\vec{y}(s))+\sigma_{22}(\vec{y}(s)))-\\
&-(n_2^2(s)-n_1^2(s))\frac{1}{2}(\sigma_{11}(\vec{y}(s))-\sigma_{22}(\vec{y}(s)))+2n_1(s)n_2(s)\sigma_{12}(\vec{y}(s))\\
&= \frac{\mu}{\pi}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_n(\xi)\times\\
&\times\left\{(n_1(\xi)\frac{\gamma_2}{r}-n_2(\xi)\frac{\gamma_1}{r})+\right.\\
&\left.+[2n_1(s)n_2(s)(\gamma_2^2-\gamma_1^2)-(n_2^2(s)-n_1^2(s))2\gamma_1\gamma_2](n_1(\xi)\frac{\gamma_1}{r}+n_2(\xi)\frac{\gamma_2}{r})\right\}. && (173)
\end{aligned}
$$

2.8 Relevance to other methods of time-independent in-plane crack analysis

Muskhelishvili (1953, 1963) expressed the *time-independent* plane-strain elastic field in terms of two analytic functions $\phi(z)$ and $\psi(z)$, often called the Goursat functions, where $z \equiv x_1 + ix_2$:

$$\sigma_{11} + \sigma_{22} = 2[\phi'(z) + \overline{\phi'(z)}] \tag{174}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\phi''(z) + \psi'(z)] \tag{175}$$

$$2\mu(u_1 + iu_2) = \frac{\lambda + 3\mu}{\lambda + \mu}\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}. \tag{176}$$

Muskhelishvili's (1953, 1963) complex potential method has been applied to a considerable variety of analytic and semi-analytic analyses of *open in-plane* (mixed modes I and II) cracks (see review by Isida, 1976). In a typical open crack problem in which no body force is present, the Goursat functions are so determined that the normal and tangential tractions T_n, T_t may be equal to the prescribed values along the the crack and along the outer boundary of the medium (or at infinity, if the medium is infinite). The two degrees of freedom of the stress field are matched by the set of two independent boundary conditions.

In the *closed in-plane* (pure mode II) crack analysis, on the other hand, a different class of constraints are to be imposed on the stress functions. The boundary conditions along the crack trace are the zero tangential traction T_t and the continuous normal displacement u_n. This category of problem is beyond the reach of Muskhelishvili's (1953, 1963) complex potential method, except for some cases of a straight crack and a system of co-planar cracks in which certain symmetry holds.

A different class of BIEM formulations for the *time-independent* in-plane closed / open crack analysis is found in the studies of Cheung and Chen (1987), Fleck (1991), Jeyakumaran and Keer (1994) and by Jeyakumaran (1995). The underlying concept of their method is practically equivalent to mine, except that they did not give their formulation so explicitly as I have done. Their formulation is applicable to 2D in-plane cracks of arbitrary shape, although their method cannot deal with the time-dependent crack problems.

3 Numerical implementation and corroborative examples

3.1 Discretization with the piecewise constant interpolation

As is evident from the appearance of the system (2) that I adopted as my approach of the present study, *solving for the slip with the traction known is an inverse problem. Once the slip is known, solving for the stress and displacement fields is a forward problem.*

$$\text{solve for } \Delta u_i(s[,t]) \quad \underset{\longleftarrow}{\text{inverse}} \quad T_i(s[,t]) \text{ known}$$

$$\text{once } \Delta u_i(s[,t]) \text{ known} \quad \underset{\longrightarrow}{\text{forward}} \quad \text{solve for } u_i(\vec{x}[,t]) \text{ and } \sigma_{ij}(\vec{x}[,t])$$

Hereafter I describe how the BIEM formulations are translated into the scheme of numerical implementations.

The crack trace(s) are discretized by a set of equally spaced nodal points with an interval of Δs and, if the problem is time-dependent, time is also discretized by a set of equally spaced time steps with an interval of Δt. My choice for $\beta\Delta t/\Delta s$ is 0.5 (Koller *et al.*, 1992; Cochard and Madariaga, 1994) in all modes of fracture. In addition, I postulate that $\alpha/\beta = \sqrt{3}$ in all the numerical calculations that follow.

The slip rate $\Delta\dot{u}$ or the slip Δu is approximated by a linear combination of a set of properly chosen basis functions, as is a common practice in FEM. This takes the following form for the time-dependent case (Koller *et al.*, 1992):

$$\Delta\dot{u}(s,t) = v^j(s)\theta_n(t)\phi_j^n \tag{177}$$

and for the time-independent case:

$$\Delta u(s) = v^j(s)\phi_j, \tag{178}$$

where $v^j(s)$ and $\theta_n(t)$ are the spatial and temporal basis functions respectively, and ϕ_j^n or ϕ_j is the discretized slip at the i-th nodal point and the m-th time step.

Two of the most representative choices of the basis function are the *piecewise constant* interpolation, in which the approximate function is assumed to be constant across an element and is discontinuous between elements, and the *piecewise linear* interpolation, in which the approximate function is assumed to vary linearly across an element and is continuous between elements. The former has been in use by Crouch (1976), Das and Aki (1977), Das (1980), Andrews (1985), Cochard and Madariaga (1994), Yamashita and Fukuyama (1996) and Kame and Yamashita (1996), whereas the latter is illustrated by the studies of Gerasoulis and Srivastav (1981), Cheung and Chen (1987), Koller *et al.* (1992) and Jeyakumaran and Keer (1994).

The basis functions of the piecewise constant interpolation are (Figure 3):

$$v^j(s) \equiv \begin{cases} 1 & \text{if } |s - s_j| < \Delta s/2 \\ 0 & \text{if } |s - s_j| > \Delta s/2 \end{cases}, \tag{179}$$

with s_j being the position of the j-th nodal point and

$$\theta_n(t) \equiv \begin{cases} 1 & \text{if } |t - t_n| < \Delta t/2 \\ 0 & \text{if } |t - t_n| > \Delta t/2 \end{cases}, \tag{180}$$

with t_n being the time of the n-th time step.

The basis functions of the piecewise linear interpolation are (Figure 3):

$$v^j(s) \equiv \begin{cases} 1 - |s - s_j|/\Delta s & \text{if } |s - s_j| < \Delta s \\ 0 & \text{otherwise} \end{cases} \tag{181}$$

$$\theta_n(t) \equiv \begin{cases} 1 - |t - t_n|/\Delta t & \text{if } |t - t_n| < \Delta t \\ 0 & \text{otherwise} \end{cases} \tag{182}$$

One should now pay special attention to the fact that the regularization procedure, utilized in the derivation of our BIEM formulations, relies on the assumption that the first-order spacial derivative of the slip function $(\partial/\partial s)\Delta u(s,t)$ (or the dislocation density) is continuous. Thus the numerical evaluation of the regularized Cauchy principal value integrals is valid only for collocation points at which $(\partial/\partial s)\Delta u(s,t)$ is continuous. Hence Koller *et al.* (1992), who used the piecewise linear interpolation, had to take the collocation points at the midpoints of the nodal points $(s_{j-1}+s_j)/2$ $(j=1,2,\ldots,N)$. This led to N constraining equations at the N collocation points for $N-1$ degrees of freedom at the $N-1$ nodal points. Koller *et al.* (1992) thus chose to solve an overdetermined system of equations in the least squares sense.

In order to match the number of degrees of freedom with that of the constraining equations, *I use the piecewise constant interpolation in all the numerical calculations that follow*, with the collocation points coinciding with the nodal points s_j $(j=1,2,\ldots,N-1)$. This choice is justified because $(\partial/\partial s)\Delta u(s,t)$ is continuous at the nodal points with the piecewise constant interpolation.

The discretization reduces the integral equation (23), for instance, to a set of simultaneous linear algebraic equations:

$$\frac{2\pi}{\mu}T_i^m = \sum_j \sum_{n=1}^{m} A_{ij}^{m-n}\phi_j^n, \tag{183}$$

where T_i^m and ϕ_i^m are the discretized traction and slip, respectively, on the crack at the i-th nodal point and the m-th time step and

$$\begin{aligned} A_{ij}^n \equiv\ & \int_\Gamma d\xi \left(n_1(s_i)\frac{\gamma_{2i}}{r_i} - n_2(s_i)\frac{\gamma_{1i}}{r_i}\right)\frac{\partial v^j(\xi)}{\partial \xi} \times \\ & \times \int_{-\Delta t}^{\Delta t} d\tau\,\theta_0(\tau)\frac{n\Delta t-\tau}{\sqrt{(n\Delta t-\tau)^2-(r_i/\beta)^2}}H(n\Delta t-\tau-\frac{r_i}{\beta}) - \\ & -\frac{1}{\beta^2}\int_\Gamma d\xi (n_1(s_i)n_1(\xi)+n_2(s_i)n_2(\xi))v^j(\xi) \times \\ & \times \int_{-\Delta t}^{\Delta t} d\tau \frac{\partial\theta_0(\tau)}{\partial\tau}\frac{1}{\sqrt{(n\Delta t-\tau)^2-(r_i/\beta)^2}}H(n\Delta t-\tau-\frac{r_i}{\beta}) \end{aligned} \tag{184}$$

$$r_i \equiv \sqrt{(y_1(s_i)-y_1(\xi))^2+(y_2(s_i)-y_2(\xi))^2} \tag{185}$$

$$\gamma_{ki} \equiv (y_k(s_i)-y_k(\xi))/r_i. \tag{186}$$

Note that the integrand, with the piecewise constant interpolation, has to be evaluated only at the ends of the element for all the time-domain calculations, since the derivatives of the basis functions exhibit a delta-function type behavior at the element ends and are equal to zero elsewhere:

$$\frac{\partial v^j(s)}{\partial s} = \begin{cases} \delta(s) & \text{at } s=s_j-\Delta s/2 \\ -\delta(s) & \text{at } s=s_j+\Delta s/2 \\ 0 & \text{otherwise} \end{cases} \tag{187}$$

and likewise for $\partial\theta_n(t)/\partial t$. The numerical integration was carried out with a modified trapezoidal law in a general form

$$\int_a^b f(x)dx \simeq \frac{b-a}{N}\sum_{n=1}^{N} f\left(a+\frac{n-1/2}{N}(b-a)\right). \tag{188}$$

The endpoints of the integration interval are avoided in the summation because the integrands usually behave like a step function there. Note that in the time-dependent case some of the elements may be cut across by one (or both) of the wavefront $t-\tau=r/\beta$, in which case the integrand is non-zero only over that part of the element which lies within the wavefronts.

In a time-marching numerical scheme, ripple noises tend to arise due to the abrupt progresses of the fracture front along the discretized fault trace, which may later lead to numerical instabilities. In order to suppress such numerical instabilities that evolve with time, artificial damping is introduced (Appendix A). This is a common practice in time-marching FDM schemes (e.g. Virieux and Madariaga, 1982) and was also used in the time-marching BIEM scheme of Koller *et al.* (1992). Stronger artificial damping suppresses the ripples, but tends to oversuppress the quantity of slip, so that there is a problem of trade-off in the decision of the damping factor.

3.2 Kostrov's self-similar crack evolution problem in three modes of fracture

I verify my numerical solutions for some simple problems against known analytic solutions. Kostrov's (1964) self-similar crack evolution problem is one of the few cases in which the elastic field around a propagating crack is known analytically.

Suppose that in an infinite homogeneous isotropic elastic medium a straight crack, either in anti-plane shear, in-plane shear or in tension, begins to form at the instant $t=0$ along the x_3-axis and then propagates in the x_1x_3-plane bilaterally from the origin with a constant speed v, the stress drop being equal to σ_0 everywhere on the crack plane (Figure 4). The crack tip velocity v should be less than the S wave velocity β for anti-plane shear, or less than the Rayleigh wave velocity for in-plane shear and tension. The anti-plane shear and in-plane shear problems, as pointed out by Kostrov (1964), can be analytically solved on the basis of Smirnov and Sobolev's ingenious method of functional-invariant solutions, and the explicit solutions given by Kikuchi (1976) are duplicated in Appendix B.1 and B.2. The tension problem can be solved in an analogous way (Cherepanov and Afanas'ev, 1974; Freund, 1990) and the solution I obtained is described in Appendix B.3.

Figures 5, 6 and 7 compare the numerical and analytic solutions of the self-similar crack evolution problem, in anti-plane shear, in-plane shear and in tension respectively, with various values of v/β. The slip at the center of the crack is compared quantitatively at an instant when the crack has reached a length of 19.0, or 19 discrete elements. The parameters assumed are $\sigma_0=1.0$, $\mu=1.0$ and $\alpha/\beta=\sqrt{3}$. The verifications show that the numerical results with an adequate choice of the artificial damping coefficient C (Appendix A) are in good agreement with the analytic solutions.

Cochard and Madariaga (1994) and Madariaga (1995) gave time-domain BIEs in a form somewhat different from those given here, for the straight anti-plane crack and straight in-plane shear crack problems respectively (Appendix C). Numerical results based on their formulations are also shown in the figures, which are as good as those according to mine, except that a slightly lighter artificial damping is appropriate for their formulations. Note that no artificial damping was necessary with the unique semi-analytic method of numerical implementation used by Cochard and Madariaga (1994), but that method was not used in the present study because it is not applicable to curved crack cases.

3.3 A straight crack in three modes of time-independent stress

Next I verify the numerical solution for the time-independent crack problem, either in anti-plane shear, in-plane shear or in tension, against the known analytic solution for the case of an isolated straight crack in an infinite homogeneous isotropic medium. As before, the stress drop is equal to σ_0 everywhere on the crack.

Figures 8, 9 and 10 verify the numerical solutions against the classical analytic solutions taken from Pollard and Segall (1987). The parameters assumed are $\sigma_0=1.0$, $\mu=1.0$ and $\alpha/\beta=\sqrt{3}$, and the length of the crack is taken equal to 19.0 or 19 discrete elements. The good agreement between the numerical and analytic results is recognized in the figures.

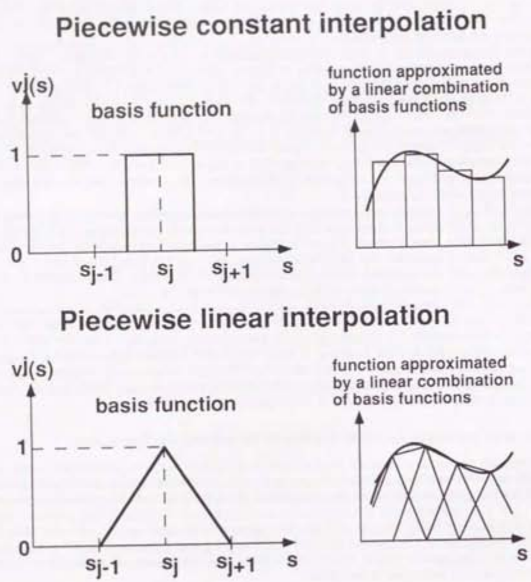

Figure 3: The basis functions of the piecewise constant and piecewise linear interpolations. A function is approximated by a linear combination of the basis functions.

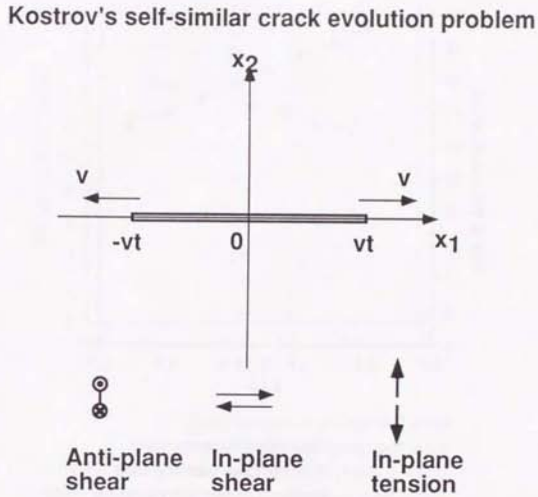

Figure 4: Kostrov's self-similar crack evolution problem (Kostrov, 1964; Cherepanov and Afanas'ev, 1974). In an infinite homogeneous isotropic elastic medium a straight shear crack begins to form at the instant $t = 0$ along the x_3-axis and then propagates in the $x_1 x_3$-plane bilaterally from the origin with a constant speed v, the stress drop either in anti-plane shear, in-plane shear or tension being equal to σ_0 everywhere on the crack plane. The speed v is less than the S wave velocity for the anti-plane case and less than the Rayleigh wave velocity for the in-plane cases.

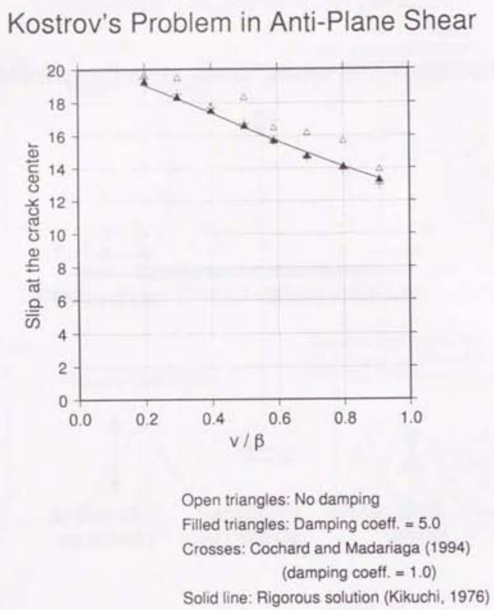

Figure 5: Comparison of the numerical and analytic solutions to Kostrov's self-similar crack evolution problem in anti-plane shear, with $\sigma_0 = 1.0$, $\mu = 1.0$ and for various values of v/β. The slip at the crack center is compared at an instant when the crack has reached a length of 19.0, or 19 discrete elements. See Appendix A for the definition of the damping coefficient, and Appendix C for the formulation of Cochard and Madariaga (1994).

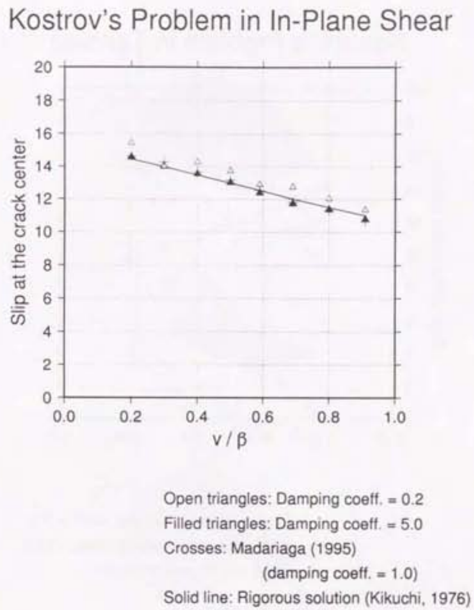

Figure 6: Comparison of the numerical and analytic solutions to Kostrov's self-similar crack evolution problem in in-plane shear, with $\sigma_0 = 1.0$, $\mu = 1.0$, $\alpha/\beta = \sqrt{3}$ and for various values of v/β. The slip at the crack center is compared at an instant when the crack has reached a length of 19.0, or 19 discrete elements. See Appendix A for the definition of the damping coefficient, and Appendix C for the formulation of Madariaga (1995).

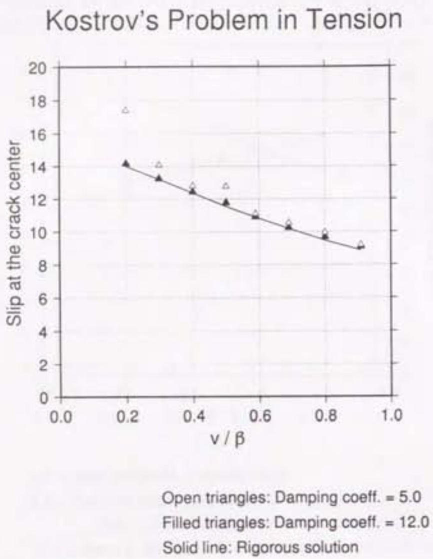

Figure 7: Comparison of the numerical and analytic solutions to Kostrov's self-similar crack evolution problem in tension, with $\sigma_0 = 1.0$, $\mu = 1.0$, $\alpha/\beta = \sqrt{3}$ and for various values of v/β. The slip at the crack center is compared at an instant when the crack has reached a length of 19.0, or 19 discrete elements. See Appendix A for the definition of the damping coefficient.

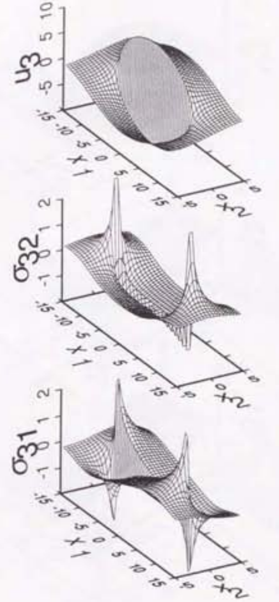

Figure 8: (a) Numerical solution to the time-independent straight crack problem in anti-plane shear, with $\sigma_0 = 1.0$ and $\mu = 1.0$. The length of the crack is equal to 19.0, or 19 discrete elements.

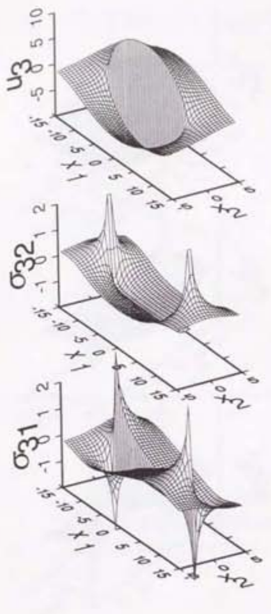

Figure 8: (b) Analytic solution to the time-independent straight crack problem in anti-plane shear (Pollard and Segall, 1987), with $\sigma_0 = 1.0$ and $\mu = 1.0$.

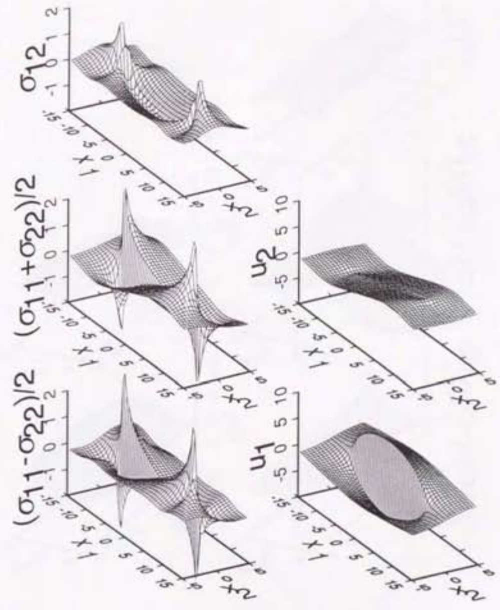

Figure 9: (a) Numerical solution to the time-independent straight crack problem in in-plane shear, with $\sigma_0 = 1.0$, $\mu = 1.0$ and $\alpha/\beta = \sqrt{3}$. The length of the crack is equal to 19.0, or 19 discrete elements.

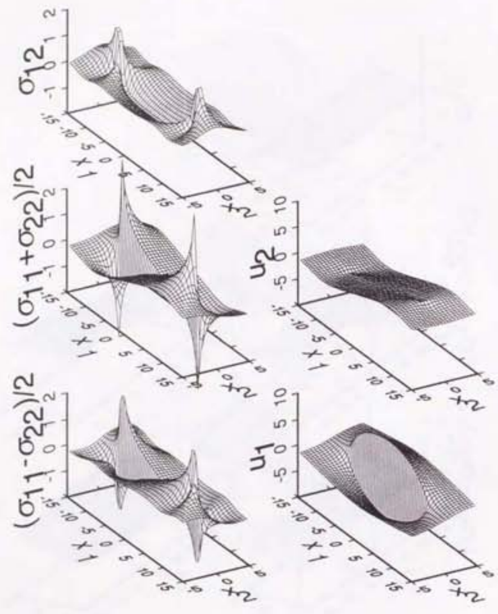

Figure 9: (b) Analytic solution to the time-independent straight crack problem in in-plane shear (Pollard and Segall, 1987), with $\sigma_0 = 1.0$, $\mu = 1.0$ and $\alpha/\beta = \sqrt{3}$.

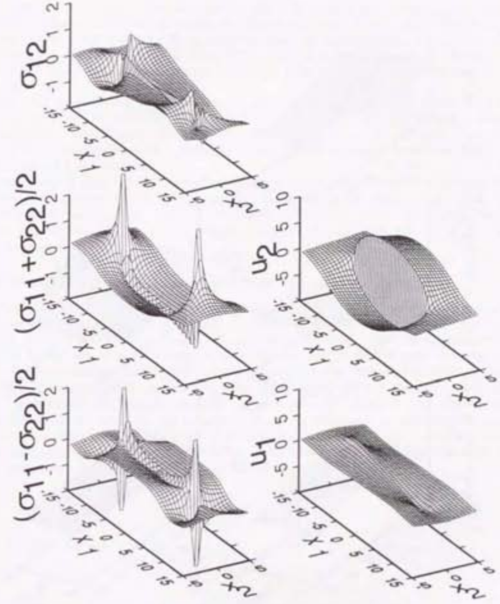

Figure 10: (a) Numerical solution to the time-independent straight crack problem in tension, with $\sigma_0 = 1.0$, $\mu = 1.0$ and $\alpha/\beta = \sqrt{3}$. The length of the crack is equal to 19.0, or 19 discrete elements.

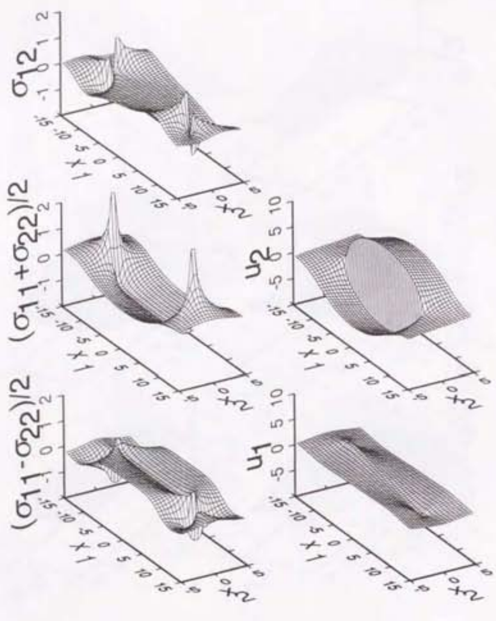

Figure 10: (b) Analytic solution to the time-independent straight crack problem in tension (Pollard and Segall, 1987), with $\sigma_0 = 1.0$, $\mu = 1.0$ and $\alpha/\beta = \sqrt{3}$.

3.4 Three radial cracks and a circular arc crack in time-independent anti-plane shear

Next my formulations are corroborated in two types of non-planar crack problem in time-independent anti-plane shear, for which analytic solutions are given by Sih (1965) by use of a complex potential method (also outlined in Isida, 1976). One of the models is a set of three equally spaced radial cracks with equal length a (Figure 11), while the other is a crack of length $2a$ in the shape of a quarter part of a circular circumference (Figure 12), each subject to an anti-plane shear stress of unit magnitude with the principal axis oriented in arbitrary direction. My numerical results were verified against the analytic solutions in terms of the normalized stress intensity factor (SIF) at one of the crack tips, where the reference SIF is that at the tip of a straight crack with length $2a$ subject to an anti-plane shear stress of unit magnitude normal to the crack.

It is known from analytical considerations that the slip distribution decays toward a crack tip in a square root curve, with a coefficient of steepness proportional to the SIF (Appendix D). Based on this fact, I modified the basis functions for the discrete elements located at the end of the crack to represent the square-root spatial variation as in Koller *et al.* (1992), and evaluated the relative SIFs in terms of the slip on the crack-end element.

Figures 11 and 12 show how the normalized SIF at one of the crack tips varies as the principal axis of the anti-plane shear stress is rotated. In the three radial cracks model, the length a of each branch was divided into 39 discrete elements, while in the circular arc crack model the entire crack length $2a$ was discretized into 79 elements. As is evident from the figures, the numerical SIFs are in good agreement with the analytic solutions.

3.5 Geometrical compatibility at a junction of crack branches

Some remarks on the slip behavior on different crack branches that meet at a junction shall be made in this subsection. It follows from geometrical considerations that at a junction point the slips on each branch are not independent of one another but are required to satisfy a special logic (Andrews, 1989).

Suppose N branches of a 2D crack meet at a junction, and let us define the anti-plane slip Δu_3^i, or the in-plane slip components Δu_1^i and Δu_2^i, as the displacement discontinuity across the i-th branch of the crack in the vicinity of the junction point, measured on the left-hand side with reference to the right-hand side as seen from the junction. Let us make a circuit trip counterclockwise around the junction, measuring and accumulating the amount of the displacement discontinuity each time we come across one of the crack branches. Geometrical compatibility requires that the accumulated displacement discontinuity should have reduced to zero when we come back to the original location which we started from. In mathematical terms, this requirement is expressed by

$$\sum_{i=1}^{N} \Delta u_3^i = 0 \tag{189}$$

for the anti-plane case and

$$\sum_{i=1}^{N} \Delta u_1^i = \sum_{i=1}^{N} \Delta u_2^i = 0 \tag{190}$$

for the in-plane case. An equivalent of (190) for the special case of a *triple* junction in in-plane shear is found in Andrews (1989). Note that in the *in-plane shear* crack problem any attempted slipping at a junction is accompanied by a volume change (Andrews, 1989), which may be resisted by the surrounding medium.

In the numerical implementation based on the piecewise constant interpolation, however, no constraint need be imposed at crack junctions explicitly, because each discrete element is an independent calculational unit and the compatibility relations (189) and (190) are satisfied automatically (Crouch, 1976; Cheung and Chen, 1987; Andrews, 1989). As an example, the numerically obtained slips Δu_3^i on each of the three radial anti-plane cracks that meet at a junction, in the same setting as treated in Section 3.4, is plotted in Figure 13. One may confirm that the numerically calculated slips on the three branches always sum up to make zero, as is predicted by the compatibility relation (189).

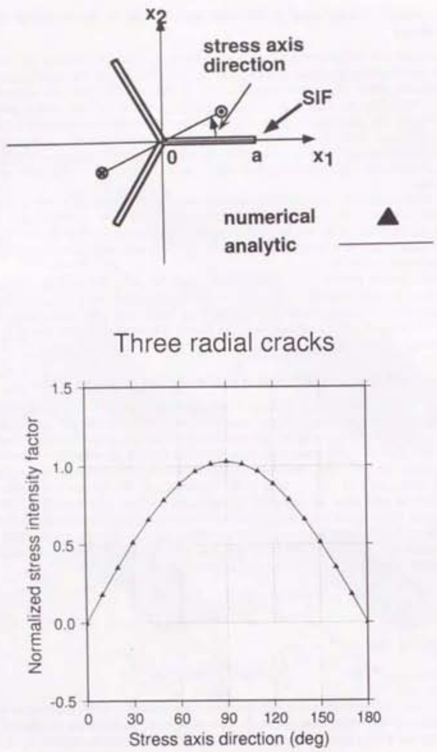

Figure 11: Numerical and analytic results of the time-independent analysis of three radial cracks of length a under an anti-plane shear stress of unit magnitude (Sih, 1965), in terms of the normalized stress intensity factor (SIF) at one of the crack tips. The reference SIF is that at the tip of a straight crack with length $2a$ subject to an anti-plane shear stress of unit magnitude normal to the crack. The length a was discretized into 39 elements.

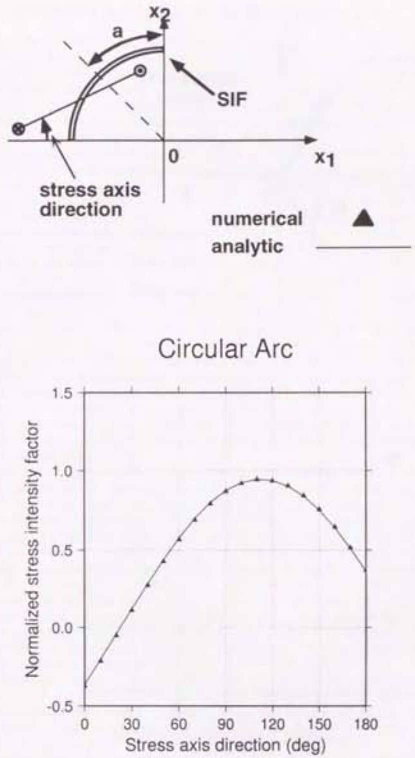

Figure 12: Numerical and analytic results of the time-independent analysis of a curved crack of length $2a$ in the shape of a quarter part of a circumference of a circle, under an anti-plane shear stress of unit magnitude (Sih, 1965), in terms of the normalized stress intensity factor (SIF) at one of the crack tips. The reference SIF is that at the tip of a straight crack with length $2a$ subject to an anti-plane shear stress of unit magnitude normal to the crack. The length $2a$ was discretized into 79 elements.

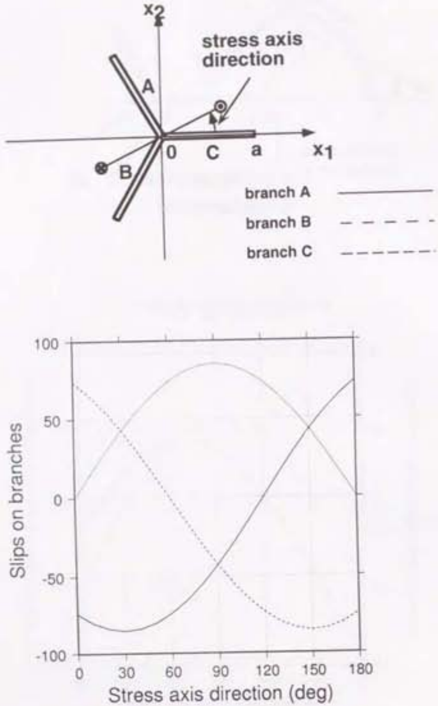

Figure 13: The numerically calculated slips on three radial cracks in time-independent anti-plane shear that meet at a junction, in the same setting as in Figure 11. The slip values on the nodal points closest to the branch junction are plotted against the varying stress axis direction. The anti-plane shear stress is of unit magnitude, $\mu = 1.0$ and the branch length a is 39.0 or 39 discrete elements.

4 Demonstrative analyses of hackly cracks

As mentioned in the Introduction section, numerical *elastodynamic* analysis of 2D cracks has been, until recently, practically confined to the straight crack problems, due to the limitations of the available numerical methods. Dynamic analysis of non-planar cracks has been impossible, except for the cases of non-coplanar (mutually parallel) cracks that have been enabled recently (Harris and Day, 1993; Yamashita and Umeda, 1994; Kame and Yamashita, 1996; Umeda *et al.*, 1996). Koller *et al.* (1992) proposed a numerical method for the dynamic analysis of 2D anti-plane shear cracks of arbitrary shape, but their numerical application goes no further than a mere preliminary one.

However, the complexity of crack geometry along natural faults and fault zones can hardly be fully represented by a model consisting of non-coplanar cracks. Curves, kinks and bifurcations along natural faults and fault zones are documented abundantly in field surveys (e.g., Segall and Pollard, 1980; Nakata and Yomogida, 1995). Furthermore, theory predicts that a shear crack, even when isolated, is likely to deviate from its initial plane of propagation during fast dynamic propagation, because for a sufficiently large propagation speed the maximum shear stress at the crack tip is known to occur in an off-plane direction that is inclined at a non-zero angle to the plane of crack propagation (Freund, 1990). In order to investigate and more properly understand the role of geometrical complexity in the crack dynamics, fault models that incorporate off-plane crack segments should be developed.

The BIEM formulations derived in the present study, which enable the numerical dynamic analysis of arbitrary non-planar 2D cracks, mark an important step toward this objective. In the present section, a 2D crack model with small off-plane side-branches on its sides (called a hackly crack in the sequel) is introduced as an idealized model of a complex crack geometry, and its dynamics is numerically analyzed on the basis of the BIEM developed in the present study. The specific geometry and time evolution history studied are shown in Figure 14.

All the side-branches are inclined at 30 deg from the orientation of the central crack plane. The rupture is assumed to initiate at a quarter part from the left of the final crack length, propagating from there bilaterally at a speed of 0.8 times the S wave velocity. In the calculations, $\alpha/\beta = \sqrt{3}$ was assumed and the artificial damping coefficient C (Appendix A) was taken equal to 5.0. The final crack length was divided into 40 discrete elements.

In examining the numerical results, I specifically concentrate on the stress concentration level at the rupture front, with a view to addressing the problem of how rupture propagation stops. Rupture is expected to decelerate and finally come to an arrest as the crack-tip stress concentration is decreased, but, as long as the crack is assumed to grow in the initial plane of propagation, the stress concentration level grows infinitely large with further growth of the crack. Thus practically all existing models of earthquake arrest, which were based on planar crack geometry, failed to explain the spontaneous arrest of rupture unless some kind of inhomogeneities in the fracture properties was postulated *a priori* near the site of arrest (e.g., Andrews, 1975; Husseini *et al.*, 1975; Das and Scholz, 1981). Umeda *et al.*'s (1996) modeling studies suggested that the interactions among non-coplanar cracks can reduce the crack-tip stress concentration only by an insignificant amount, which does not obviate the need for the *a priori* postulation of inhomogeneities. In the present study I specifically concentrate on how hackly crack geometry can influence the crack-tip stress concentration level and, consequently, the tendency for the acceleration or deceleration of rupture.

Figure 15 shows the temporal variations, for both the anti-plane and in-plane cases, of the shear stress concentration level at the nodal point that is located on the immediate right of the rupture front. The shear stress is assumed to be equal to 1.0 with the principal axis oriented parallel to the central branch of the crack, and the crack surface is free of shear traction, so that the shear stress drop on the central branch equals 1.0. The results of the analysis of the hackly crack, the solid line, is compared to the case of a planar crack with no side-branches, represented by the broken line. The irregular fluctuations of the lines are due to the abrupt progresses of the rupture front along the discretized fault trace.

One may notice that the stress concentration level is lowered by the presence of the side-branches, especially in the in-plane shear case, in which the stress concentration significantly decreases in spite of the crack that ever grows longer. This effect is due to the partitioning of the slip into multiple crack branches that takes place at each subsequent branch point. The numerical results strongly suggest that branching of the crack plays an important role in the deceleration and arrest of earthquake rupturing.

For the sake of completeness, I also calculated the *elastostatic* stress concentration levels at the tip of the hackly crack in time-independent anti-plane shear and in-plane shear. Figure 16 shows the results for different geometries in a normalized appearance, the reference value being the stress concentration at the tip of an isolated straight crack with no side-branches. One can observe the decrease in the stress concentration, which compares well with the results of the elastodynamic analyses.

The mechanics of hackly cracks, as modeled in the present chapter, belongs to a class of problem to which no previous numerical methods were applicable. As illustrated in the present example, the new BIEM is hoped to find its broad applications in the analyses of 2D cracks of various complex geometries, including more realistic models of fault zones that exist in nature.

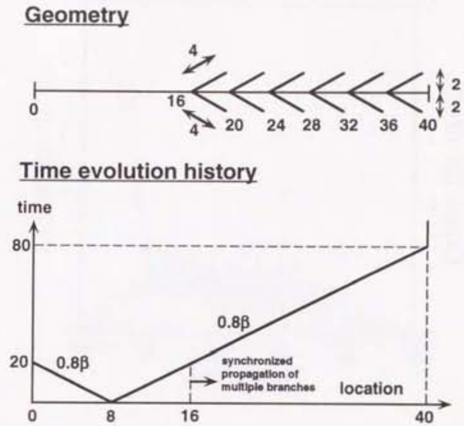

Figure 14: Geometry and time evolution history of the hackly crack analyzed, with the numerals denoting the abscissal locations. Each discrete element has a unit length, so that the total crack length 40.0 was divided into 40 discrete elements. $\alpha/\beta = \sqrt{3}$.

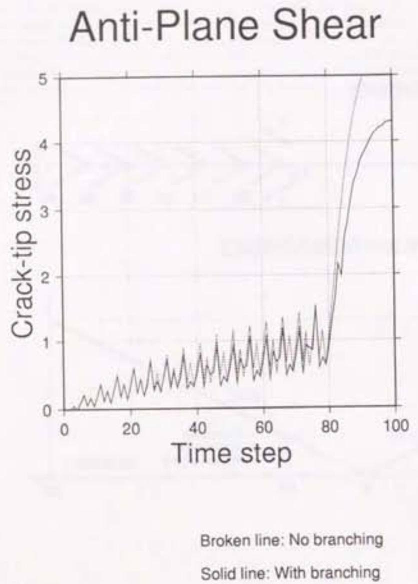

Figure 15: (a) The temporal variation of the stress concentration level at the tip of the propagating hackly crack (Figure 14) in anti-plane shear. The ordinate shows the shear stress on the nodal point that is located on the immediate right of the rupture front. The shear stress is equal to 1.0 with the principal axis oriented parallel to the central branch of the crack, and the crack surface is free of shear traction.

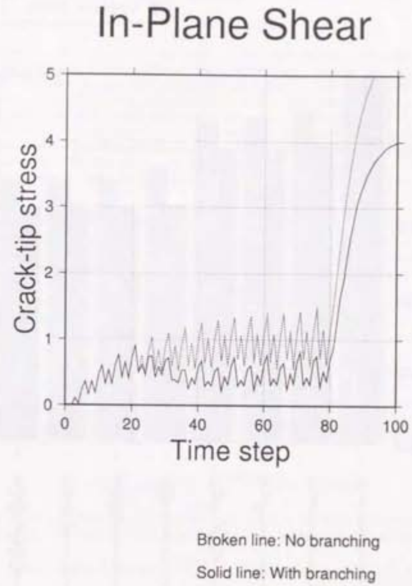

Figure 15: (b) The temporal variation of the stress concentration level at the tip of the propagating hackly crack (Figure 14) in in-plane shear. The ordinate shows the shear stress on the nodal point that is located on the immediate right of the rupture front. The shear stress is equal to 1.0 with the principal axis oriented parallel to the central branch of the crack, and the crack surface is free of shear traction.

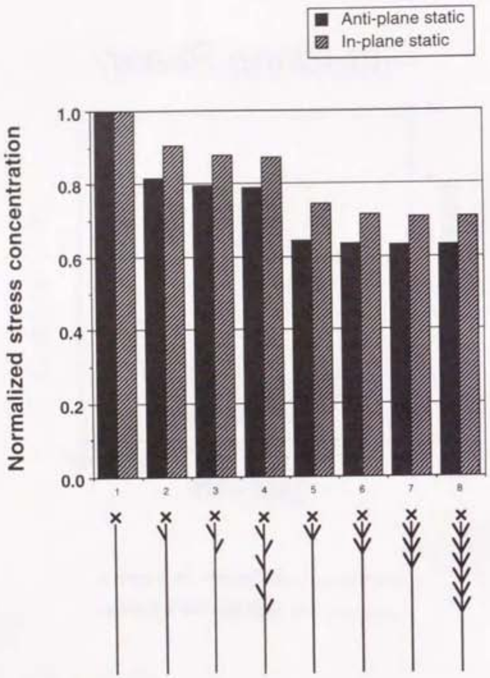

Figure 16: The normalized stress concentration levels at the tip of hackly cracks (Figure 14) in time-independent anti-plane shear and in-plane shear. The reference level is that for an isolated straight crack with no side-branches.

5 Can a curved 2D crack be represented as a limiting case of a chain of finite line elements? — a BIEM viewpoint

5.1 Concept

In time-independent analyses of curved 2D cracks, it has often been a common practice to represent a curved bend by a chain of a large number of finite straight line elements (e.g., Crouch, 1976; Cheung and Chen, 1987; Bilham and King, 1989; Aydin and Du, 1995). In the present chapter, however, I point out, in a context of the BIEM, that it is only in the cases of anti-plane shear faulting and of open in-plane faulting that a smoothly curved bend can be represented as a limiting case of a chain of finite line elements (see Figure 17). In the case of closed in-plane faulting the two geometries may produce different normal traction distributions along the crack, so that care should be taken so as not to misinterpret the numerical results. The difference can be important, e.g., when one is concerned with a friction law that depends on the relation between the normal and tangential tractions along the crack.

5.2 Closed in-plane crack

Suppose a curved closed crack is under time-independent in-plane shear, which consists of two straight segments and a smooth circular arc segment connecting them. Jeyakumaran and Keer (1994) dealt with this problem with a BIEM formulation based on Muskhelishvili's (1953, 1963) complex potential method and a piecewise linear interpolation approach of Gerasoulis and Srivastav (1981). According to their results, when a concave-upward crack is subject to a right-lateral in-plane shear, extensional normal traction across the crack appears on the right of the curved bend and compressional one appears on the left. This result was confirmed by a model calculation based on the BIEs derived in the present paper (Figure 18, left).

However, when the curved part of the crack is represented as a chain of a large number of finite line elements, numerical simulations revealed a normal traction along the crack with the opposite sign, compressional on the right and extensional on the left (Figure 18, right). Such a pattern of normal traction distribution was confirmed by repeated calculations with different discretization intervals.

The objective of the present section is to point out that the discrepancy in the solutions of the inverse problem

$$\text{solve for } \Delta u_i(s[,t]) \quad \overset{\text{inverse}}{\longleftarrow} \quad T_i(s[,t]) \text{ known}$$

corresponding to the two situations is an intrinsic one, resulting from the different natures of the BIEs corresponding to the two cases. To prove this, I demonstrate analytically that the forward problem

$$\text{once } \Delta u_i(s[,t]) \text{ known} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{solve for } u_i(\vec{x}[,t]) \text{ and } \sigma_{ij}(\vec{x}[,t])$$

has completely different solutions for the two situations even for an identical set of input data. I choose the forward formulation because it allows rigorous treatment of the limiting case of a chain of finite line elements as the discretization interval tends to zero. Rigorous consideration of such a limiting case is prohibited in the numerical treatment of the inverse problem.

Suppose a smoothly curved concave-upward closed in-plane crack on the x_1x_2-plane extends indefinitely in both left and right directions, and that a uniform slip $\Delta u_i(\xi) \equiv \Delta u_{i0}$ is prescribed on it (Figure 19, top). In this case, no traction is induced by the presence of the slip, since $n_1(\xi)$ and $n_2(\xi)$ are continuous everywhere and the kernels of our BIEs for the stress components are equal to zero everywhere along the crack.

Consider next a concave-upward in-plane crack with a single abrupt kink and let each of the left and right straight segments, extending to infinity, be inclined at an angle η to the x_1-axis. Define the arc length s along the crack so that s increases from left to right and that $s = 0$ corresponds to the kink point (Figure 19, middle). Prescribe a uniform slip $\Delta u_i(\xi) \equiv \Delta u_{i0}$ along the crack and assume that any point on the crack surface that originally belongs to the left segment is allowed to slip only in the orientation of the left segment and that likewise for the right segment (a requisite assumption of the linear elasticity theory). Then our BIE kernels for the stress components exhibit a delta-function type behavior at the location of the kink. Denoting the BIE kernels for the tangential traction $T_t(s)$ and that for the normal traction $T_n(s)$ across the crack due to tangential slip component $\Delta u_t(\xi)$ by

$$
\begin{aligned}
F_{tt}(\xi, s) &\equiv [2n_1(s)n_2(s)2\gamma_1\gamma_2 + (n_2^2(s) - n_1^2(s))(\gamma_2^2 - \gamma_1^2)](n_2(\xi)\frac{\gamma_1}{r} - n_1(\xi)\frac{\gamma_2}{r}) && (191)\\
F_{tn}(\xi, s) &\equiv (n_1(\xi)\frac{\gamma_1}{r} + n_2(\xi)\frac{\gamma_2}{r}) + \\
&\quad +[2n_1(s)n_2(s)(\gamma_2^2 - \gamma_1^2) - (n_2^2(s) - n_1^2(s))2\gamma_1\gamma_2](n_2(\xi)\frac{\gamma_1}{r} - n_1(\xi)\frac{\gamma_2}{r}), && (192)
\end{aligned}
$$

the distribution of the traction along the crack is given by

$$
\begin{aligned}
T_t(s) &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)F_{tt}(\xi, s)\\
&= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}[F_{tt}(\xi, s)|_{\xi\to+0} - F_{tt}(\xi, s)|_{\xi\to-0}] && (193)\\
T_n(s) &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\int_\Gamma d\xi \frac{\partial}{\partial \xi}\Delta u_t(\xi)F_{tn}(\xi, s)\\
&= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}[F_{tn}(\xi, s)|_{\xi\to+0} - F_{tn}(\xi, s)|_{\xi\to-0}], && (194)
\end{aligned}
$$

which after some algebra becomes

$$
\begin{aligned}
T_t(s) &= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{|s|}2\sin^2\eta && (195)\\
T_n(s) &= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{s}\sin 2\eta. && (196)
\end{aligned}
$$

Let us proceed to consider a case in which the crack bend consists of a finite number N of straight line elements mutually connected at abrupt kinks. Let the radius of curvature of the crack bend be R and let each of the left and right straight segments outside the crack bend be inclined at an angle η to the x_1-axis. Define the arc length s along the crack so that s increases from left to right and that $s = 0$ corresponds to the right extremity of the curved part of the crack (Figure 19, bottom). A uniform slip $\Delta u_t(\xi) \equiv \Delta u_{t0}$ is again prescribed along the crack under the assumption that any point on the crack surface is allowed to slip only in the orientation of the straight segment (or element) to which it originally belongs (a requisite assumption of the linear elasticity theory). The relative location of the i-th kink from the left relative to the center of curvature is given by a vector

$$
\left(+R\sec\frac{\eta}{N}\sin\frac{2i - N - 1}{N}\eta,\ -R\sec\frac{\eta}{N}\cos\frac{2i - N - 1}{N}\eta\right), \qquad (197)
$$

the crack orientation changing discontinuously at each subsequent kink by an angle η/N.

Consider the traction across the crack at a sufficient distance $s \gg R$ to the right of the crack bend and suppose the number of the abrupt kinks along the bend $N \gg 1$ is sufficiently large. Neglecting second and higher order terms with respect to R/s and $1/N$, we obtain, after cumbersome algebra, the following expression for the contribution $T_t^i(s)$ and $T_n^i(s)$ of the i-th kink unto the traction at arc length s:

$$
\begin{aligned}
T_t^i(s) &= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{2\eta}{N}\frac{1}{s}\left(\sin\frac{2i - 2N - 1}{N}\eta + \right.\\
&\quad \left. + \frac{R}{s}2\sin\frac{2i - 2N - 1}{2N}\eta\sin\frac{3(2i - 2N - 1)}{2N}\eta\right) + O\left((R/s)^2, (1/N)^2\right) && (198)\\
T_n^i(s) &= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{2\eta}{N}\frac{1}{s}\left(\cos\frac{2i - 2N - 1}{N}\eta + \right.\\
&\quad \left. + \frac{R}{s}\sin\frac{2i - 2N - 1}{N}\eta\right) + O\left((R/s)^2, (1/N)^2\right). && (199)
\end{aligned}
$$

The total traction at arc length s as the sum of all contributions from the N kinks is given, in the limit $N \to \infty$, by

$$
\begin{aligned}
T_t(s) &= \lim_{N\to\infty}\sum_{i=1}^{N} T_t^i(s)\\
&= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{s}\int_{-2\eta}^{0} d\theta\left(\sin\theta + \frac{R}{s}2\sin\frac{1}{2}\theta\sin\frac{3}{2}\theta\right) + O\left((R/s)^2\right)\\
&= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{s}2\sin^2\eta\left(1 - \frac{R}{s}\sin 2\eta\right) + O\left((R/s)^2\right) \text{ for } s > 0 && (200)\\
T_n(s) &= \lim_{N\to\infty}\sum_{i=1}^{N} T_n^i(s)\\
&= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{s}\int_{-2\eta}^{0} d\theta\left(\cos\theta + \frac{R}{s}\sin\theta\right) + O\left((R/s)^2\right)\\
&= -\frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right)\Delta u_{t0}\frac{1}{s}\left(\sin 2\eta - \frac{R}{s}2\sin^2\eta\right) + O\left((R/s)^2\right) \text{ for } s > 0. && (201)
\end{aligned}
$$

It follows from symmetry that $T_t(-s) = T_t(s)$ and that $T_n(-s) = -T_n(s)$. The formulae (200) and (201) in fact coincide with the earlier derived solutions (195) and (196) for the single kink case in the limit of infinite remoteness $s/R \to \infty$, and become equal to zero in the limit of a straight crack case $\eta \to 0$, which is a matter of natural consequence.

Thus it is evident that the BIE, solved as a forward problem

$$
\text{once } \Delta u_i(s[, t]) \text{ known} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{solve for } u_i(\vec{x}[, t]) \text{ and } \sigma_{ij}(\vec{x}[, t]),
$$

has completely different solutions for the two geometries, a smooth curve and a chain of finite line elements, even for an identical set of input data. The BIE has distinct natures for the two cases. This means that *a smoothly curved closed in-plane crack is not equivalent to the limiting case of a chain of finite line elements as the discretization interval tends to zero.* This fact may be more readily understood by a plainer statement that abrupt kinks at the junctions of finite line elements behave as a suppressor to tangential slipping along an in-plane fault (Andrews, 1989).

Incidentally, Jeyakumaran and Keer (1994) and Jeyakumaran (1995) were apparently unaware of this situation, so that care must be taken in the interpretation of their numerical results.

5.3 Anti-plane shear crack

I shall next discuss the case of anti-plane faulting. It can be shown by analogy to the case of the previous section that a uniform slip $\Delta u_3(\xi) \equiv \Delta u_{30}$ along an infinitely long smoothly curved crack induces no traction. Now consider an anti-plane crack with exactly the same geometry as in the previous case, and prescribe a uniform slip $\Delta u_3(\xi) \equiv \Delta u_{30}$. Because the BIE kernel for the shear traction $T_3(s)$ as denoted by

$$
F_{33}(\xi, s) \equiv n_1(s)\frac{\gamma_2}{r} - n_2(s)\frac{\gamma_1}{r} \qquad (202)
$$

does not explicitly include $n_1(\xi)$ nor $n_2(\xi)$, no additional traction along the crack is induced by an abrupt kink at $s = 0$, since

$$
\begin{aligned}
T_3(s) &= \frac{\mu}{2\pi}\int_\Gamma d\xi\frac{\partial}{\partial\xi}\Delta u_3(\xi)F_{33}(\xi, s)\\
&= \frac{\mu}{2\pi}\Delta u_{30}[F_{33}(\xi, s)|_{\xi\to+0} - F_{33}(\xi, s)|_{\xi\to-0}]\\
&= 0. && (203)
\end{aligned}
$$

Thus in the case of anti-plane faulting there is no difference in the nature of the BIE whether a curve along a crack occurs in a smooth or an abrupt way. *A smoothly curved anti-plane crack may safely be approximated by a chain of finite straight line elements.*

5.4 Open in-plane crack

I finally discuss the case of open in-plane faulting. Again, a uniform slip with components $\Delta u_1(\xi) \equiv \Delta u_{10}$ and $\Delta u_2(\xi) \equiv \Delta u_{20}$ along an infinitely long smoothly curved crack induces no traction, as is evident from the fact that the integral representations for the traction components in terms of $(\partial/\partial\xi)\Delta u_t(\xi)$ and $(\partial/\partial\xi)\Delta u_n(\xi)$ can be rewritten in the form with $(\partial/\partial\xi)\Delta u_1(\xi)$ and $(\partial/\partial\xi)\Delta u_2(\xi)$ under the integral sign.

Consider an open in-plane crack with exactly the same geometry as in the previous cases, and prescribe a uniform slip with components $\Delta u_1(\xi) \equiv \Delta u_{10}$ and $\Delta u_2(\xi) \equiv \Delta u_{20}$. This means that

$$\left\{ \begin{array}{lcl} \Delta u_t(\xi) & = & \Delta u_{10}(\xi)\cos\eta + \Delta u_{20}(\xi)\sin\eta \\ \Delta u_n(\xi) & = & -\,\Delta u_{10}(\xi)\sin\eta + \Delta u_{20}(\xi)\cos\eta \end{array} \right. \quad (s > 0) \tag{204}$$

and

$$\left\{ \begin{array}{lcl} \Delta u_t(\xi) & = & \Delta u_{10}(\xi)\cos\eta - \Delta u_{20}(\xi)\sin\eta \\ \Delta u_n(\xi) & = & \Delta u_{10}(\xi)\sin\eta + \Delta u_{20}(\xi)\cos\eta \end{array} \right. \quad (s < 0)\,. \tag{205}$$

Denoting the BIE kernels for the tangential traction $T_t(s)$ and that for the normal traction $T_n(s)$ across the crack due to normal slip component $\Delta u_n(\xi)$ by

$$F_{nt}(\xi, s) \equiv [2n_1(s)n_2(s)2\gamma_1\gamma_2 + (n_2^2(s) - n_1^2(s))(\gamma_2^2 - \gamma_1^2)](n_1(\xi)\frac{\gamma_1}{r} + n_2(\xi)\frac{\gamma_2}{r}) \tag{206}$$

$$\begin{aligned} F_{nn}(\xi, s) \equiv\ & (n_1(\xi)\frac{\gamma_2}{r} - n_2(\xi)\frac{\gamma_1}{r}) + \\ & + [2n_1(s)n_2(s)(\gamma_2^2 - \gamma_1^2) - (n_2^2(s) - n_1^2(s))2\gamma_1\gamma_2](n_1(\xi)\frac{\gamma_1}{r} + n_2(\xi)\frac{\gamma_2}{r}), \end{aligned} \tag{207}$$

the distribution of the traction along the crack is given by

$$\begin{aligned} T_t(s) &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right) \int_\Gamma d\xi \left[\frac{\partial}{\partial\xi}\Delta u_t(\xi)F_{tt}(\xi, s) + \frac{\partial}{\partial\xi}\Delta u_n(\xi)F_{nt}(\xi, s)\right] \\ &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right) \{[\Delta u_t F_{tt}(\xi, s) + \Delta u_n F_{nt}(\xi, s)]_{\xi \to +0} - \\ &\quad - [\Delta u_t F_{tt}(\xi, s) + \Delta u_n F_{nt}(\xi, s)]_{\xi \to -0}\} \end{aligned} \tag{208}$$

$$\begin{aligned} T_n(s) &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right) \int_\Gamma d\xi \left[\frac{\partial}{\partial\xi}\Delta u_t(\xi)F_{tn}(\xi, s) + \frac{\partial}{\partial\xi}\Delta u_n(\xi)F_{nn}(\xi, s)\right] \\ &= \frac{\mu}{\pi}\left(1 - \frac{\beta^2}{\alpha^2}\right) \{[\Delta u_t F_{tn}(\xi, s) + \Delta u_n F_{nn}(\xi, s)]_{\xi \to +0} - \\ &\quad - [\Delta u_t F_{tn}(\xi, s) + \Delta u_n F_{nn}(\xi, s)]_{\xi \to -0}\}\,, \end{aligned} \tag{209}$$

which after some algebra becomes

$$T_t(s) = 0 \tag{210}$$

$$T_n(s) = 0. \tag{211}$$

This means that no additional traction along the crack is induced by an abrupt kink at $s = 0$. In the case of open in-plane faulting there is no difference in the nature of the BIE whether a curve along a crack occurs in a smooth or an abrupt way. *A smoothly curved open in-plane crack may safely be approximated by a chain of finite straight line elements.* This justifies the approach of Crouch (1976) and Cheung and Chen (1987), who approximated a curved open crack in in-plane tension by a chain of finite line elements.

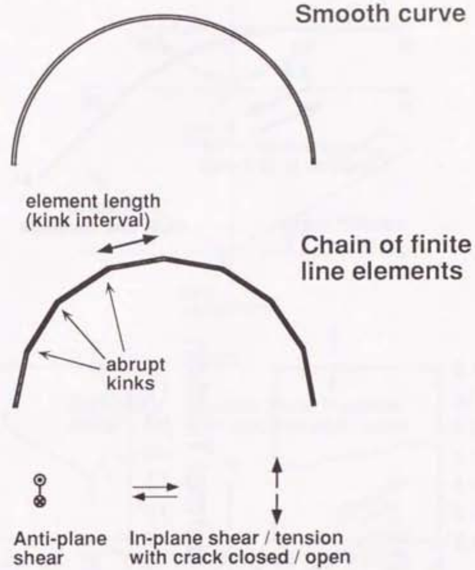

Figure 17: A smoothly curved crack and a chain of finite line elements, in three modes of fracture.

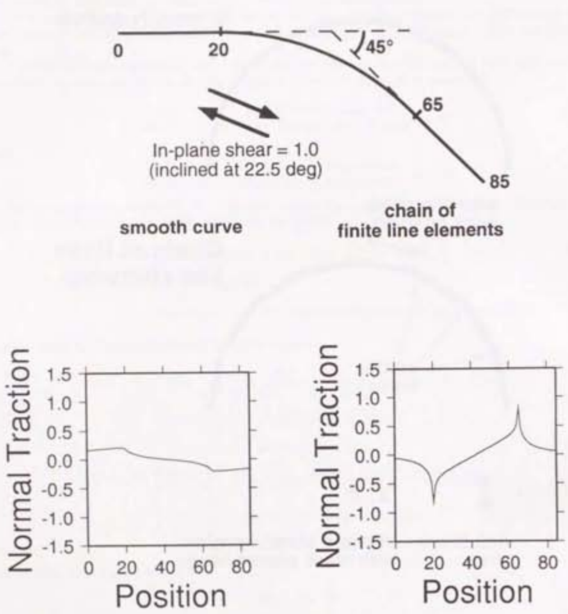

Figure 18: Normal stress distribution along a curved in-plane shear crack, for the cases of a smooth curve and a chain of finite line elements. A positive ordinate denotes dilatation, while a negative one denotes compression. Each discrete element has a unit length, so that the curved part with length 45 was divided into 45 elements.

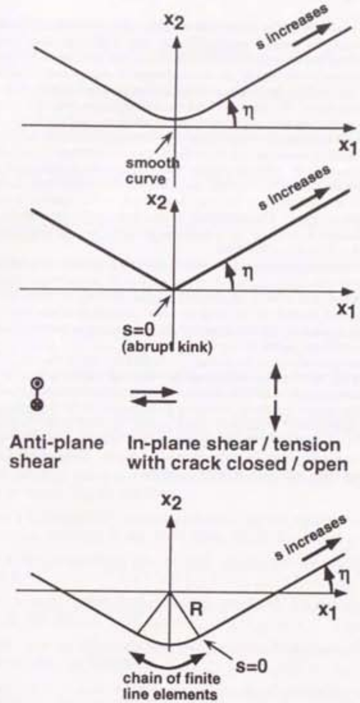

Figure 19: Simplified conceptual models of a chain of finite line elements studied in this chapter.

6 Conclusion

In the present study I have derived a set of rigorous BIEs, both time-domain (elastodynamic) and time-independent (elastostatic), for the analysis of arbitrarily shaped 2D anti-plane / in-plane crack(s) located in an infinite homogeneous isotropic medium. Among others, the formulation for the time-domain analysis of non-planar 2D in-plane cracks, which has been obtained in enlargement of Koller *et al.*'s (1992) and Cochard and Madariaga's (1994) approach, is a first achievement in this field of study and belongs to an ingenuity of the present paper. The approach is characterized by the use of the method of regularization, or removal, of the hypersingularities in the integrals. In addition, although the formulation for the time-independent cases is not a new achievement of the present study, an advantage consists in that I have for the first time used a unified nomenclature for both the time-domain and time-independent cases, which used to be studied separately by different authors based on inconsistent terminologies.

For the sake of numerical implementation, the BIEs are discretized with the piecewise constant interpolation, in which the approximate function is assumed to be constant across an element and is discontinuous between elements. The numerical results were verified against the known analytic solutions for a time-dependent self-similar crack evolution problem and for two types of time-independent non-planar crack problem in anti-plane shear.

The new BIEM numerical approach has subsequently been applied to both time-domain and time-independent analyses of a hackly crack, or a 2D crack in either in-plane shear or anti-plane shear consisting of a straight main branch and a set of small side-branches splaying out from it. The numerical results demonstrated a significant decrease in the crack-tip stress concentration level with reference to the case of a straight crack, thus suggesting that branching of the crack plays an important role in the deceleration and arrest mechanisms of earthquake rupturing.

Finally, it has been pointed out, in the context of the BIEM, that in the case of in-plane shear faulting a smoothly curved crack may not be represented as a limiting case of a chain of finite line elements as the discretization interval tends to zero, a situation which previous researchers were apparently unaware of. The equations that govern the crack mechanics have distinct forms depending on whether the crack orientation changes continuously or abruptly along a bend, however small the kink interval may be set. The two geometries may produce different normal traction distributions along the crack, so that care should be taken so as not to misinterpret the numerical results. However, no similar problem arises in the cases of anti-plane shear and open in-plane faulting.

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Appendices

A Artificial damping applied in the time-marching scheme

Here I outline the procedure of artificial damping which is introduced so as to suppress the numerical noises and instabilities in the time-marching modeling scheme, which arise due to the abrupt progresses of the fracture front along the discretized fault trace. This damping procedure is practiced at every time step. Denoting the slip rate at the i-th nodal point and the m-th time step before the damping by ϕ_i^m and that after the damping by $(\phi_i^m)'$, we solve the following set of simultaneous linear equations:

$$(\phi_i^m)' = \phi_i^m + C[(\phi_{i+1}^m)' + (\phi_{i-1}^m)' - 2(\phi_i^m)'] \text{ for each } i, \tag{A1}$$

where $C > 0$ is a positive constant. As long as the piecewise constant interpolation is used, no special treatment of the damping matrix is required at fault junctions.

Stronger artificial damping or a bigger value of C suppresses the ripples more effectively, but tends to distort the slip distribution, so that there is a problem of trade-off in the decision of the damping factor.

B Analytic solution to Kostrov's self-similar crack evolution problem in three modes of fracture

B.1 Anti-plane shear

Described below is the analytic solution to Kostrov's (1964) self-similar crack evolution problem in anti-plane shear, which was obtained by Kikuchi (1976). I duplicate his results, partly because I have replaced part of his integral representations of non-analytic functions with a combination of complete elliptic integrals, partly because his paper is not written in English.

Suppose that in an infinite homogeneous isotropic elastic medium a straight shear crack begins to form at the instant $t = 0$ along the x_3-axis and then propagates in the x_1x_3-plane bilaterally from the origin with a constant speed v (which is less than the S wave velocity β), the shear stress drop being equal to σ_0 everywhere on the crack plane (Figure 4). We introduce the complex variables θ_α and θ_β (α denotes the P wave velocity and β that of the S wave) defined by

$$\begin{aligned} t - \theta_c x_1 - \sqrt{c^{-2} - \theta_c^2}\, x_2 = 0 \ \ (c = \alpha, \beta) \\ \mathrm{Im}\, \theta_c > 0 \ \text{ for } y > 0 \end{aligned} \tag{B1}$$

or, denoting $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$, by

$$\theta_c = \begin{cases} (t/r)\cos\varphi + i\sqrt{(t/r)^2 - c^{-2}}\sin\varphi & (r < ct) \\ (t/r)\cos\varphi - \sqrt{c^{-2} - (t/r)^2}\sin\varphi & (r \geq ct) \end{cases} \tag{B2}$$

so that the θ_c plane may have branch cuts on the real axis at $|\theta_c| > c^{-1}$ and that the upper half of the x_1x_2-plane $x_2 > 0$ may be mapped onto the upper half of the complex θ_c-plane $\mathrm{Im}\,\theta_c > 0$. Note that the variable θ_α does not appear in the solution of the anti-plane problem since the P wave field is not involved.

The field variables at any point on the x_1x_2-plane are represented in a normalized appearance:

$$v_i = \frac{v\sigma_0}{\mu P(v)} \mathrm{Im}\, v_i' \tag{B3}$$

$$\sigma_{ij} = \frac{\sigma_0}{P(v)} \mathrm{Im}\, \sigma_{ij}', \tag{B4}$$

where the normalized displacement velocity and stress components are given as follows:

$$v_3' = \frac{\theta_\beta}{\sqrt{v^{-2} - \theta_\beta^2}} \tag{B5}$$

$$\sigma'_{31} = \frac{-1}{v\sqrt{v^{-2}-\theta_\beta^2}} \tag{B6}$$

$$\sigma'_{32} = \frac{-1}{v}\int_0^{\theta_\beta}(\beta^{-2}-\theta^2)^{1/2}(v^{-2}-\theta^2)^{-3/2}d\theta \tag{B7}$$

and the normalization factor $P(v)$ is

$$P(v) \equiv -\mathrm{Im}\,\sigma'_{32}|_{\theta=+i\infty} = E_\beta, \tag{B8}$$

where

$$E_c \equiv E(\frac{\pi}{2}, k_c),\ k_c \equiv \sqrt{1-\left(\frac{v}{c}\right)^2} \tag{B9}$$

is a complete elliptic integral of the second kind. The slip across the crack is given by

$$\Delta u_3 = \frac{2\sigma_0}{\mu P(v)}\sqrt{v^2t^2-x_1^2}. \tag{B10}$$

Though the shear stress component σ_{32} is not expressible in terms of an analytic function, it is reducible to a relatively simple expression on the axes of symmetry. On the x_2 or the y-axis, for $0 \le |y| < \beta t$, denoting

$$\xi_c^y \equiv 1/\sqrt{1-(v/c)^2+(vt/y)^2},\ z_c^y \equiv \sqrt{1-(y/ct)^2} \tag{B11}$$

and

$$E_c^y \equiv E(\sin^{-1} z_c^y, k_c), \tag{B12}$$

we have

$$\mathrm{Im}\,\sigma'_{32} = -E_\beta^y + k_\beta^2\xi_\beta^y z_\beta^y. \tag{B13}$$

On the x_1 or the x-axis, for $vt < |x| < \beta t$, denoting

$$\xi^x \equiv 1/\sqrt{1-(vt/x)^2},\ z_c^x \equiv \sqrt{1-(x/ct)^2} \tag{B14}$$

and

$$E_c^x \equiv E(\sin^{-1}(z_c^x/k_c), k_c), \tag{B15}$$

we have

$$\mathrm{Im}\,\sigma'_{32} = -E_\beta^x + \xi^x z_\beta^x. \tag{B16}$$

This leads to the following expression for the stress intensity factor (Appendix D):

$$K_{III} = \frac{\sigma_0\sqrt{\pi a}}{P(v)}\sqrt{1-(v/\beta)^2}, \tag{B17}$$

where $a \equiv vt$ is the half length of the crack. By using the equations (B10) and (B17), one can easily confirm that Freund's (1990) formula (D9), relating the slip rate near the tip of a running crack with the SIF, is satisfied. On the other hand, noting that $P(v) \to 1$ as $v/\beta \to 0$, K_{III} tends in the static limit to $\sigma_0\sqrt{\pi a}$, which coincides with the familiar expression for the SIF at the tip of a stationary straight crack of length a.

B.2 In-plane shear

Here I describe the analytic solution to Kostrov's (1964) self-similar crack evolution problem in in-plane shear, which was obtained by Kikuchi (1976). The setting of the problem and the nomenclature are parallel to that used for the anti-plane shear case. The discussion of this problem is also found in Freund (1990).

The normalized displacement velocity and stress components are

$$v_i = \frac{v\sigma_0}{\mu P(v)}\mathrm{Im}\,v'_i \tag{B18}$$

$$\sigma_{ij} = \frac{\sigma_0}{P(v)}\mathrm{Im}\,\sigma'_{ij}, \tag{B19}$$

where σ_0 is now the in-plane shear stress drop, are given as follows:

$$\begin{aligned} v'_1 = {} & \frac{2\beta^2}{v^2}\frac{\theta_\alpha}{\sqrt{v^{-2}-\theta_\alpha^2}} + \left(1-\frac{2\beta^2}{v^2}\right)\frac{\theta_\beta}{\sqrt{v^{-2}-\theta_\beta^2}} + \\ & +\frac{2\beta^2}{v^2}i\log\frac{i\theta_\beta-\sqrt{v^{-2}-\theta_\beta^2}}{i\theta_\alpha-\sqrt{v^{-2}-\theta_\alpha^2}} \end{aligned} \tag{B20}$$

$$\begin{aligned} v'_2 = {} & \frac{2\beta^2}{v^2}\frac{\sqrt{\alpha^{-2}-\theta_\alpha^2}}{\sqrt{v^{-2}-\theta_\alpha^2}} + \frac{1-2(\beta/v)^2}{1-(v/\beta)^2}\frac{\sqrt{\beta^{-2}-\theta_\beta^2}}{\sqrt{v^{-2}-\theta_\beta^2}} + \\ & +\frac{2\beta^2}{v^2}\log\frac{\sqrt{v^{-2}-\theta_\beta^2}+\sqrt{\beta^{-2}-\theta_\beta^2}}{\sqrt{v^{-2}-\theta_\alpha^2}+\sqrt{\alpha^{-2}-\theta_\alpha^2}} \end{aligned} \tag{B21}$$

$$\begin{aligned} \frac{1}{2}(\sigma'_{11}-\sigma'_{22}) = {} & 2\left(\frac{\beta^2}{\alpha^2}-\frac{2\beta^2}{v^2}\right)\frac{1}{v\sqrt{v^{-2}-\theta_\alpha^2}} - 2\left(1-\frac{2\beta^2}{v^2}\right)\frac{1}{v\sqrt{v^{-2}-\theta_\beta^2}} + \\ & +\frac{4\beta^2}{v}\left(\sqrt{v^{-2}-\theta_\beta^2}-\sqrt{v^{-2}-\theta_\alpha^2}\right) \end{aligned} \tag{B22}$$

$$\frac{1}{2}(\sigma'_{11}+\sigma'_{22}) = -2\left(1-\frac{\beta^2}{\alpha^2}\right)\frac{1}{v\sqrt{v^{-2}-\theta_\alpha^2}} \tag{B23}$$

$$\begin{aligned} \sigma'_{12} = {} & \frac{-4\beta^2}{v}\int_0^{\theta_\alpha}\theta^2(\alpha^{-2}-\theta^2)^{1/2}(v^{-2}-\theta^2)^{-3/2}d\theta + \\ & +\frac{-4\beta^2}{v}\int_0^{\theta_\beta}(\frac{1}{2}\beta^{-2}-\theta^2)^2(\beta^{-2}-\theta^2)^{-1/2}(v^{-2}-\theta^2)^{-3/2}d\theta, \end{aligned} \tag{B24}$$

and the normalization factor $P(v)$ is

$$P(v) \equiv -\mathrm{Im}\,\sigma'_{12}|_{\theta=+i\infty} = \frac{v^2}{\beta^2-v^2}(K_\beta-E_\beta)+\frac{8\beta^2}{v^2}(E_\alpha-E_\beta)+4\left(K_\beta-\frac{\beta^2}{\alpha^2}K_\alpha\right), \tag{B25}$$

where

$$K_c \equiv K(\frac{\pi}{2}, k_c), E_c \equiv E(\frac{\pi}{2}, k_c),\ k_c \equiv \sqrt{1-\left(\frac{v}{c}\right)^2} \tag{B26}$$

are complete elliptic integrals of the first and second kinds respectively. The slip across the crack is given by

$$\Delta u_1 = \Delta u_t = \frac{2\sigma_0}{\mu P(v)}\sqrt{v^2t^2-x_1^2}. \tag{B27}$$

Though the shear stress component σ_{12} is not expressible in terms of an analytic function, it is reducible to a relatively simple expression on the axes of symmetry. On the x_2 or the y-axis, for $0 \le |y| < \beta t$, denoting

$$\xi_c^y \equiv 1/\sqrt{1-(v/c)^2+(vt/y)^2},\ z_c^y \equiv \sqrt{1-(y/ct)^2} \tag{B28}$$

and

$$K_c^y \equiv K(\sin^{-1} z_c^y, k_c), E_c^y \equiv E(\sin^{-1} z_c^y, k_c), \tag{B29}$$

we have

$$\begin{aligned} \mathrm{Im}\,\sigma'_{12} = {} & \frac{v^2}{\beta^2-v^2}(E_\beta^y-K_\beta^y-k_\beta^2\xi_\beta^y z_\beta^y)+\frac{4\beta^2}{v^2}(2E_\beta^y-2E_\alpha^y - \\ & -k_\beta^2\xi_\beta^y z_\beta^y + k_\alpha^2\xi_\alpha^y z_\alpha^y - \frac{z_\beta^y}{\xi_\beta^y} + \frac{z_\alpha^y}{\xi_\alpha^y}\right)+4\left(\frac{\beta^2}{\alpha^2}K_\alpha^y - K_\beta^y\right). \end{aligned} \tag{B30}$$

The expressions for $\beta t \leq |y| < \alpha t$ can be obtained by dropping all the terms that are accompanied with the index β.

On the x_1 or the x-axis, for $vt < |x| < \beta t$, denoting

$$\xi^x \equiv 1/\sqrt{1-(vt/x)^2},\ z_c^x \equiv \sqrt{1-(x/ct)^2} \tag{B31}$$

and

$$K_c^x \equiv K(\sin^{-1}(z_c^x/k_c), k_c), E_c^x \equiv E(\sin^{-1}(z_c^x/k_c), k_c), \tag{B32}$$

we have

$$\begin{aligned} \mathrm{Im}\,\sigma'_{12} &= \frac{v^2}{\beta^2 - v^2}(E_\beta^x - K_\beta^x - \xi^x z_\beta^x) + \frac{4\beta^2}{v^2}(2E_\beta^x - 2E_\alpha^x - \\ &\quad -\xi^x z_\beta^x + \xi^x z_\alpha^x - \frac{z_\beta^x}{\xi^x} + \frac{z_\alpha^x}{\xi^x}\Big) + 4\left(\frac{\beta^2}{\alpha^2}K_\alpha^x - K_\beta^x\right). \end{aligned} \tag{B33}$$

The expressions for $\beta t \leq |y| < \alpha t$ can be obtained by dropping all the terms that are accompanied with the index β. The above leads to the following expression for the stress intensity factor (Appendix D):

$$K_{II} = \frac{\sigma_0\sqrt{\pi a}}{P(v)}\left(\frac{\beta}{v}\right)^2\left\{4\sqrt{1-(v/\alpha)^2} - \frac{[2-(v/\beta)^2]^2}{\sqrt{1-(v/\beta)^2}}\right\}, \tag{B34}$$

where $a \equiv vt$ is the half length of the crack. By using the equations (B27) and (B34), one can easily confirm that Freund's (1990) formula (D10), relating the slip rate near the tip of a running crack with the SIF, is satisfied. On the other hand, noting that $P(v) \to 2(1-\beta^2/\alpha^2)$ as $v/\beta \to 0$, K_{II} tends in the static limit to $\sigma_0\sqrt{\pi a}$, which coincides with the familiar expression for the SIF at the tip of a stationary straight crack of length a.

B.3 Tension

Now I describe the analytic solution to Kostrov's self-similar crack evolution problem in tension, discussed by Cherepanov and Afanas'ev (1974) and Freund (1990), which I derived following the method of Kikuchi (1976). The setting of the problem and the nomenclature are parallel to those used in the anti-plane shear and in-plane shear cases.

The normalized displacement velocity and stress components are

$$v_i = \frac{v\sigma_0}{\mu P(v)}\,\mathrm{Im}\,v'_i \tag{B35}$$

$$\sigma_{ij} = \frac{\sigma_0}{P(v)}\,\mathrm{Im}\,\sigma'_{ij}, \tag{B36}$$

where σ_0 is now the tensile stress drop, are given as follows:

$$\begin{aligned} v'_1 &= \frac{-2\beta^2}{v^2}\frac{\sqrt{\beta^{-2}-\theta_\beta^2}}{\sqrt{v^{-2}-\theta_\beta^2}} - \frac{1-2(\beta/v)^2}{1-(v/\alpha)^2}\frac{\sqrt{\alpha^{-2}-\theta_\alpha^2}}{\sqrt{v^{-2}-\theta_\alpha^2}} + \\ &\quad + \frac{2\beta^2}{v^2}\log\frac{\sqrt{v^{-2}-\theta_\beta^2}+\sqrt{\beta^{-2}-\theta_\beta^2}}{\sqrt{v^{-2}-\theta_\alpha^2}+\sqrt{\alpha^{-2}-\theta_\alpha^2}} \end{aligned} \tag{B37}$$

$$\begin{aligned} v'_2 &= \frac{2\beta^2}{v^2}\frac{\theta_\beta}{\sqrt{v^{-2}-\theta_\beta^2}} + \left(1-\frac{2\beta^2}{v^2}\right)\frac{\theta_\alpha}{\sqrt{v^{-2}-\theta_\alpha^2}} - \\ &\quad - \frac{2\beta^2}{v^2}\, i\log\frac{i\theta_\beta - \sqrt{v^{-2}-\theta_\beta^2}}{i\theta_\alpha - \sqrt{v^{-2}-\theta_\alpha^2}} \end{aligned} \tag{B38}$$

$$\begin{aligned} \frac{1}{2}(\sigma'_{11}-\sigma'_{22}) &= \frac{4\beta^2}{v}\int_0^{\theta_\beta}\theta^2(\beta^{-2}-\theta^2)^{1/2}(v^{-2}-\theta^2)^{-3/2}d\theta + \\ &\quad + \frac{4\beta^2}{v}\int_0^{\theta_\alpha}(\frac{1}{2}\alpha^{-2}-\theta^2)(\frac{1}{2}\beta^{-2}-\theta^2)(\alpha^{-2}-\theta^2)^{-1/2}(v^{-2}-\theta^2)^{-3/2}d\theta \end{aligned} \tag{B39}$$

$$\frac{1}{2}(\sigma'_{11}+\sigma'_{22}) = \frac{-2}{v}\left(1-\frac{\beta^2}{\alpha^2}\right)\int_0^{\theta_\alpha}(\frac{1}{2}\beta^{-2}-\theta^2)(\alpha^{-2}-\theta^2)^{-1/2}(v^{-2}-\theta^2)^{-3/2}d\theta \tag{B40}$$

$$\begin{aligned} \sigma'_{12} &= 2\left(1-\frac{2\beta^2}{v^2}\right)\left(\frac{1}{v\sqrt{v^{-2}-\theta_\beta^2}} - \frac{1}{v\sqrt{v^{-2}-\theta_\alpha^2}}\right) + \\ &\quad + \frac{4\beta^2}{v}\left(\sqrt{v^{-2}-\theta_\alpha^2} - \sqrt{v^{-2}-\theta_\beta^2}\right), \end{aligned} \tag{B41}$$

and the normalization factor $P(v)$ is

$$P(v) \equiv -\mathrm{Im}\,\sigma'_{22}|_{\theta=+i\infty} = \frac{\alpha^2v^2 - 4\alpha^2\beta^2 + 4\beta^4}{\beta^2(\alpha^2-v^2)}(K_\alpha - E_\alpha) + \frac{8\beta^2}{v^2}(E_\beta - E_\alpha) + 4(K_\alpha - K_\beta), \tag{B42}$$

where

$$K_c \equiv K(\frac{\pi}{2}, k_c), E_c \equiv E(\frac{\pi}{2}, k_c), k_c \equiv \sqrt{1-\left(\frac{v}{c}\right)^2} \tag{B43}$$

are complete elliptic integrals of the first and second kinds respectively. The slip across the crack is given by

$$\Delta u_2 = \Delta u_n = \frac{2\sigma_0}{\mu P(v)}\sqrt{v^2t^2 - x_1^2}. \tag{B44}$$

Though the diagonal stress components $\sigma_{11} \pm \sigma_{22}$ are not expressible in terms of analytic functions, they are reducible to relatively simple expressions on the axes of symmetry.

On the crack surface, i.e. on the x_1 or the x-axis for $|x| < vt$,

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11}-\sigma'_{22}) = \frac{2\alpha^2-2\beta^2-v^2}{\alpha^2-v^2}(E_\alpha - K_\alpha) - \frac{8\beta^2}{v^2}(E_\alpha - E_\beta) + 2\left(\frac{\beta^2}{\alpha^2}+1\right)K_\alpha - 4K_\beta \tag{B45}$$

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11}+\sigma'_{22}) = \frac{(\alpha^2-\beta^2)(v^2-2\beta^2)}{\beta^2(\alpha^2-v^2)}(E_\alpha - K_\alpha) + 2\left(\frac{\beta^2}{\alpha^2}-1\right)K_\alpha. \tag{B46}$$

On the x_2 or the y-axis, for $0 \leq |y| < \beta t$, denoting

$$\xi_c^y \equiv 1/\sqrt{1-(v/c)^2+(vt/y)^2},\ z_c^y \equiv \sqrt{1-(y/ct)^2} \tag{B47}$$

and

$$K_c^y \equiv K(\sin^{-1} z_c^y, k_c), E_c^y \equiv E(\sin^{-1} z_c^y, k_c), \tag{B48}$$

we have

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11}-\sigma'_{22}) = \frac{2\alpha^2-2\beta^2-v^2}{\alpha^2-v^2}(E_\alpha^y - K_\alpha^y - k_\alpha^2\xi_\alpha^y z_\alpha^y) - \frac{4\beta^2}{v^2}\Big(2E_\alpha^y - 2E_\beta^y - \tag{B49}$$

$$-k_\alpha^2\xi_\alpha^y z_\alpha^y + k_\beta^2\xi_\beta^y z_\beta^y - \frac{z_\alpha^y}{\xi_\alpha^y} + \frac{z_\beta^y}{\xi_\beta^y}\Big) + 2\left(\frac{\beta^2}{\alpha^2}+1\right)K_\alpha^y - 4K_\beta^y \tag{B50}$$

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11}+\sigma'_{22}) = \frac{(\alpha^2-\beta^2)(v^2-2\beta^2)}{\beta^2(\alpha^2-v^2)}(E_\alpha^y - K_\alpha^y - k_\alpha^2\xi_\alpha^y z_\alpha^y) + 2\left(\frac{\beta^2}{\alpha^2}-1\right)K_\alpha^y. \tag{B51}$$

The expressions for $\beta t \leq |y| < \alpha t$ can be obtained by dropping all the terms that are accompanied with the index β.

On the x_1 or the x-axis, for $vt < |x| < \beta t$, denoting

$$\xi^x \equiv 1/\sqrt{1-(vt/x)^2},\ z_c^x \equiv \sqrt{1-(x/ct)^2} \tag{B52}$$

and

$$K_c^x \equiv K(\sin^{-1}(z_c^x/k_c), k_c), E_c^x \equiv E(\sin^{-1}(z_c^x/k_c), k_c), \tag{B53}$$

we have

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11} - \sigma'_{22}) = \frac{2\alpha^2 - 2\beta^2 - v^2}{\alpha^2 - v^2}(E_\alpha^x - K_\alpha^x - \xi^x z_\alpha^x) - \frac{4\beta^2}{v^2}(2E_\alpha^x - 2E_\beta^x - \tag{B54}$$

$$-\xi^x z_\alpha^x + \xi^x z_\beta^x - \frac{z_\alpha^x}{\xi^x} + \frac{z_\beta^x}{\xi^x}\Big) + 2\left(\frac{\beta^2}{\alpha^2} + 1\right) K_\alpha^x - 4K_\beta^x \tag{B55}$$

$$\frac{1}{2}\mathrm{Im}\,(\sigma'_{11} + \sigma'_{22}) = \frac{(\alpha^2 - \beta^2)(v^2 - 2\beta^2)}{\beta^2(\alpha^2 - v^2)}(E_\alpha^x - K_\alpha^x - \xi^x z_\alpha^x) + 2\left(\frac{\beta^2}{\alpha^2} - 1\right) K_\alpha^x. \tag{B56}$$

The expressions for $\beta t \le |y| < \alpha t$ can be obtained by dropping all the terms that are accompanied with the index β. The above leads to the following expression for the stress intensity factor (Appendix D):

$$K_I = \frac{\sigma_0\sqrt{\pi a}}{P(v)}\left(\frac{\beta}{v}\right)^2 \left\{4\sqrt{1-(v/\beta)^2} - \frac{[2-(v/\beta)^2]^2}{\sqrt{1-(v/\alpha)^2}}\right\}, \tag{B57}$$

where $a \equiv vt$ is the half length of the crack. By using the equations (B44) and (B57), one can easily confirm that Freund's (1990) formula (D11), relating the slip rate near the tip of a running crack with the SIF, is satisfied. On the other hand, noting that $P(v) \to 2(1 - \beta^2/\alpha^2)$ as $v/\beta \to 0$, K_I tends in the static limit to $\sigma_0\sqrt{\pi a}$, which coincides with the familiar expression for the SIF at the tip of a stationary straight crack of length a.

C Formulation of Cochard and Madariaga for the time-domain analysis of a straight 2D shear crack

Here I cite the boundary integral equations for the time-domain analysis of a *straight* 2D shear cracks, derived by Cochard and Madariaga (1994) and by Madariaga (1995) respectively, on the basis of the double Laplace transform and the Cagniard-de Hoop method. The results for the cases of anti-plane shear and in-plane shear, respectively, are:

$$T_3(s,t) = -\frac{\mu}{2\beta}\Delta\dot{u}_3(s,t) - $$
$$-\frac{\mu}{2\pi}\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial\xi}\Delta\dot{u}_3(\xi,\tau)\frac{\sqrt{(t-\tau)^2-(r/\beta)^2}}{(t-\tau)(s-\xi)}H(t-\tau-\frac{r}{\beta}) \tag{C1}$$

$$T_t(s,t) = -\frac{\mu}{2\beta}\Delta\dot{u}_t(s,t) - $$
$$-\frac{\mu}{2\pi}\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial\xi}\Delta\dot{u}_t(\xi,\tau)\frac{4\beta^2(t-\tau)}{(s-\xi)^3}\sqrt{(t-\tau)^2-(r/\alpha)^2}H(t-\tau-\frac{r}{\alpha}) +$$
$$+\frac{\mu}{2\pi}\int_\Gamma d\xi \int_0^t d\tau \frac{\partial}{\partial\xi}\Delta\dot{u}_t(\xi,\tau)\frac{\beta^2}{(s-\xi)^3(t-\tau)}\frac{[2(t-\tau)^2-(r/\beta)^2]^2}{\sqrt{(t-\tau)^2-(r/\beta)^2}}H(t-\tau-\frac{r}{\beta}) \tag{C2}$$

with

$$r \equiv |s - \xi| \tag{C3}$$

and the singular integrals should be interpreted in terms of Cauchy principal values (it can be easily shown that the hypersingular parts of the second and third terms of the right hand side of Equation (C2) cancel out each other). Note that Cochard and Madariaga (1994) proposed a unique semi-analytic method of numerical implementation, with the use of which no artificial damping (Appendix A) was necessary in the time-marching numerical scheme. However, that method was not used in the present study because it is not applicable to curved crack cases.

D Stress intensity factor

The stress intensity factor (SIF) is an index of the stress concentration at a crack tip and is defined as a linear multiplier appearing in the asymptotic expression of the stress field in the close vicinity of the crack tip. The SIFs can be defined independently for the three modes of fracture, which are usually denoted K_I, K_{II} and K_{III}. Suppose a local 2D rectangular coordinate system x_1, x_2 is defined in such a way that the crack tip corresponds to the origin with the crack trace occupying the negative part of the x_1-axis near the crack tip. A local polar coordinate system r, θ is also defined by $r \equiv \sqrt{x_1^2 + x_2^2}$ and $\tan\theta \equiv x_2/x_1$. According to the conventional definition of the SIF, the asymptotic expression of the stress field is

$$\sigma_{31} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2} \tag{D1}$$

$$\sigma_{32} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2} \tag{D2}$$

$$\frac{1}{2}(\sigma_{11} - \sigma_{22}) = -\frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2} - \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(1 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \tag{D3}$$

$$\frac{1}{2}(\sigma_{11} + \sigma_{22}) = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2} - \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2} \tag{D4}$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{D5}$$

(e.g., Isida, 1976; Freund, 1990). Note that Sih's (1965) definition of the SIF differs from the conventional one by a factor of $\sqrt{\pi}$. In the *time-independent* case, the asymptotic form of the slip near the crack tip is expressed by

$$\Delta u_3 = \frac{2}{\mu}\sqrt{\frac{2r}{\pi}}K_{III} \tag{D6}$$

$$\Delta u_1 = \Delta u_t = \frac{2(\lambda + 2\mu)}{\mu(\lambda+\mu)}\sqrt{\frac{2r}{\pi}}K_{II} \tag{D7}$$

$$\Delta u_2 = \Delta u_n = \frac{2(\lambda + 2\mu)}{\mu(\lambda+\mu)}\sqrt{\frac{2r}{\pi}}K_I. \tag{D8}$$

Freund (1990) gives the asymptotic elastic field in the proximity of the crack tip that is *running* with speed v. According to his expressions, the asymptotic form of the slip rate near the crack tip is

$$\Delta\dot{u}_3 = \frac{2v}{\mu}\frac{K_{III}}{\sqrt{2\pi r}}\frac{1}{\sqrt{1-(v/\beta)^2}} \tag{D9}$$

$$\Delta\dot{u}_1 = \Delta\dot{u}_t = \frac{2v}{\mu}\frac{K_{II}}{\sqrt{2\pi r}}\frac{(v/\beta)^2\sqrt{1-(v/\beta)^2}}{4\sqrt{1-(v/\alpha)^2}\sqrt{1-(v/\beta)^2}-[2-(v/\beta)^2]^2} \tag{D10}$$

$$\Delta\dot{u}_2 = \Delta\dot{u}_n = \frac{2v}{\mu}\frac{K_I}{\sqrt{2\pi r}}\frac{(v/\beta)^2\sqrt{1-(v/\alpha)^2}}{4\sqrt{1-(v/\alpha)^2}\sqrt{1-(v/\beta)^2}-[2-(v/\beta)^2]^2}. \tag{D11}$$