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Electrical Conductivity of Interacting Fermions

（相互作用するフェルミ粒子系の電気伝導度）

平成８年１２月　博士（理学）申請

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Thesis

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Contents

Chapter 1

Introduction

The electrical resistivity due to mutual Coulomb interaction between electrons is one of the most difficult problems in the theory of quantum transport in solids. It is generally believed that the electrical resistivity, $\rho_{dc}(T)$, in metals at low temperature, T, is given by $\rho_{res} + AT^2$, where ρ_{res} and A are the residual resistivity and the coefficient of T^2-term, respectively, and that T^2-term is caused by Umklapp scattering associated with mutual Coulomb interaction resulting from the existence of the crystal lattice.[1)]

Microscopic analysis of this problem has been carried out by Yamada and Yosida,[2)] who emphasized that any theory addressing to the present problem should yield a correct result of the absence of resistivity in the limiting case of continuum in the absence of the impurity scattering. (We will use the term"continuum" to refer to the absence of the crystal lattice.) Their studies are based on the formulation of the Kubo formula[3)] developed by Èliashberg[4)] by use of the perturbative treatment with the Feynman diagram. In the theoretical studies of the Kubo formula, the current-current correlation function is expressed in terms of the one-particle Green's functions with the self-energy corrections, γ, and the vertex corrections to the current operator. There exists an important relationship, the Ward-Takahashi identity,[5-7)] between the self-energy corrections and the vertex corrections. Quite often, the resistivity, the inverse of the conductivity, is thought to be determined basically by γ. However, there are cases where the proportionality of the resistivity to γ is not correct even at a qualitative level. The most typical case is the continuum in the absence of the impurity scattering, as studied by Yamada and Yosida, where there is no resistivity even though γ is finite.

The absence of the resistivity in this limiting case is due to the current conservation through the electron-electron scattering. The memory-function formalism[8)] based on the Mori formula[9,10)] is, as well known, another formulation which yields a correct result of the absence of the resistivity if the total current is conserved, and is powerful for the actual calculation of the resistivity of the various systems in the lowest order of the scattering processes which do not conserve the total current. This formalism, however, generally leads to non-zero resistivity if the total current is not conserved even though the total momentum is conserved, in contradiction to the general belief that the resistivity of the system without any momentum dissipation mechanism should vanish. An example which indicates the existence of the resistivity requires the momentum dissipation mechanism has been studied by Yamada *et al.*[11)] who showed that T^2-term in the resistivity is absent in the isotropic system composed of two kinds of electrons with the interaction between them. In this model, the total current is not conserved since the electrons have different masses but the total momentum is conserved. Hence, the memory-function formalism is not valid in such a case. This failure is due to the high-frequency expansion of the electrical conductivity, $\sigma(\omega)$, inherent to the memory-function formalism.

In the presence of the crystal lattice which leads not only to the Umklapp scattering but also to the band structure, the problem gets more involved and the interrelationship between the conservation of total momentum of two scattering particles and the absence of the resistivity is not clear. (We use the term "band structure" to refer to a general Fermi surface different from sphere.) In this case, because of the effect of the band structure, the current is determined by the group velocity, $v_{\mathbf{p}} = \nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}$, which is not proportional to the momentum, and therefore the total current is not conserved even through the normal scattering processes which conserve the total momentum. An example which automatically includes both effects of the Umklapp scattering and the band structure, is the infinite dimensional ($d = \infty$) Hubbard model since the $d \to \infty$ limit[12,13] can only be formulated on the lattice. In this model, the vertex corrections to the current are seen to vanish[14] and therefore the resistivity[15-17] is determined only by γ, in sharp contrast to the limiting case of continuum. The quasiparticle's damping rate, γ, has contributions from normal processes. The proportionality of the resistivity to γ in $d = \infty$ may imply that normal processes can contribute to the resistivity.

In this thesis, we will investigate the resistivity due to mutual Coulomb interaction of a lattice electron system in the presence of impurity or Umklapp scattering , with a special emphasis on normal processes through which the crystal momentum is conserved but the group velocity is not conserved. The previous theoretical studies on the Kubo formula is formulated in terms of the memory function which is extended to the matrix form on the basis of the Fermi liquid theory. The present theory yields a correct result of the absence of T^2-term in the resistivity in the absence of both impurity and Umklapp scattering , in spite of the fact that the total current is not conserved. It is found that the T^2-term in the resistivity due to mutual Coulomb interaction results even in the absence of the Umklapp scattering processes once the impurity scattering is present. However, the resistivity in this case saturates as the temperature gets high. This is a special case of the breakdown of the Matthiessen's rule. As this fact may imply, we can show that, even if there exist no impurities, the normal scattering processes generally contribute to the resistivity in the presence of Umklapp processes. Once one realizes this fact it is interesting to ask what is the effect of the normal processes on the resistivity in the presence of Umklapp processes in two-dimensional systems without impurities. In this case, it has been pointed out by Fujimoto *et al.*[18] that the resistivity is proportional to T^2 even though the damping rate of the quasi-particle, γ, is proportional to $T^2 \log T$, indicating the different temperature dependences between the two. They claimed that this is because normal processes, which give the $T^2 \log T$ contribution to γ, do not contribute to the resistivity.

In the following we confine ourselves to the case of the d-dimensional lattice with $d \geq 2$ and will not study the case of $d = 1$, since, though interesting, there exist some special features in this case.[19-21] Throughout this thesis, we take units, $\hbar = k_B = 1$.

The structure of the thesis is as follows. In Chap. 2, we review Èliashberg's formulation of the Kubo formula for $\sigma(\omega)$ on the basis of the Fermi liquid theory, and diagrammatical analysis of T^2-term in the resistivity to respect the consistency between the selfenergy and vertex corrections developed by Yamada and Yosida. This leads to the absence of the T^2-term in the limiting case of continuum in the absence of the impurity scattering. We will even point out that, in this limiting case, the generalized Ward-Takahashi identity reflecting the current conservation insures not only the absence of the resistivity at finite temperature but also the absence of the renormalization of, the Drude weight, the coefficient of $1/\omega$-term in $\sigma(\omega)$. In Chap. 3, the electrical conductivity, $\sigma(\omega)$, of the lattice system with short-range Coulomb interaction and the s-wave impurity scattering is expressed by the memory function which is extended to the matrix form. The relationship between our present formulation and the ordinary memory-function formalism is discussed. In Chap. 4, we show that, even in the absence of Umklapp processes, normal processes associated with Coulomb interaction can contribute to the resistivity once the impurity scattering is present, and then make an explicit calculation of the resistivity in the three-dimensional system with low electron number density. In Chap. 5 and Chap. 6, we investigate the coefficient of T^2-term in the resistivity of high-dimensional and two-dimensional lattice systems in the absence of the impurity scattering, respectively. In Chap. 5, we obtain $1/d$-corrections to the T^2-term in the resistivity in the lowest order of the short-range Coulomb interaction. In Chap. 6, we obtain the doping dependence of the coefficient of T^2-term in the resistivity of the Hubbard model with the nearest-neighbor hopping on a two-dimensional square lattice in the lowest order of the short-range Coulomb interaction. Summary and conclusions are given in Chap. 7.

Chapter 2

Electrical Conductivity of Fermi Liquid

2.1 Èliashberg's Formulation of Kubo Formula

In this section, we first review the formulation of the electrical conductivity $\sigma(\omega)$ developed by Èliashberg[4]) on the basis of the Fermi liquid theory.

The Kubo formula for $\sigma_{\mu\nu}(\omega)$ is given by

$$\sigma_{\mu\nu}(\omega) = e^2 \lim_{\mathbf{q}\to 0} \frac{\chi^R_{\mu\nu}(\mathbf{q},\omega) - \chi^R_{\mu\nu}(\mathbf{q},0)}{i\omega}, \tag{2.1}$$

where the retarded current-current correlation function is defined by

$$\chi^R_{\mu\nu}(\mathbf{q},\omega) \equiv i\int_0^\infty \mathrm{d}t e^{i\omega t}\langle [J_{-\mathbf{q}\mu}(t), J_{\mathbf{q}\nu}(0)]\rangle. \tag{2.2}$$

Here, the Heisenberg operator $O(t)$ is defined as

$$O(t) \equiv e^{i(H-\mu N)t} O e^{-i(H-\mu N)t}, \tag{2.3}$$

for an arbitrary operator O and

$$\mathbf{J}_\mathbf{q} = \sum_{\mathbf{p}\sigma} \mathbf{v}_\mathbf{p} c^\dagger_{\mathbf{p}+\mathbf{q}/2\sigma} c_{\mathbf{p}-\mathbf{q}/2\sigma}, \tag{2.4}$$

is a current operator, where $\mathbf{v}_\mathbf{p}$ is group velocity and $c^\dagger_{\mathbf{p}\sigma}$ is the creation operator for an electron with momentum $\mathbf{p}$ and spin σ. $\chi^R_{\mu\nu}(\mathbf{q},\omega)$ can be obtained by the analytic continuation of $\chi_{\mu\nu}(\mathbf{q}, i\omega_m)$ with respect to the frequency as follows,

$$\chi^R_{\mu\nu}(\mathbf{q},\omega) = [\chi_{\mu\nu}(\mathbf{q},w)]_{i\omega_m\to\omega+i0}, \tag{2.5}$$

where

$$\chi_{\mu\nu}(\mathbf{q}, i\omega_m) \equiv \int_0^{1/T} \mathrm{d}\tau \langle T_\tau\{J_{-\mathbf{q}\mu}(\tau) J_{\mathbf{q}\nu}(0)\}\rangle e^{i\omega_m\tau}. \tag{2.6}$$

Here, the τ-Heisenberg operator $O(\tau)$ is defined as

$$O(\tau) \equiv e^{(H-\mu N)\tau} O e^{-(H-\mu N)\tau}, \tag{2.7}$$

for an arbitrary operator O.

By introducing the single particle Green's function $G(\mathbf{p}, i\epsilon_n)$ and the three-point current vertex functions $\Lambda(\mathbf{p}, i\epsilon_n; i\omega_m)$, respectively, defined by

$$G(\mathbf{p}, i\epsilon_n) \equiv -\int_0^{1/T} \mathrm{d}\tau \langle T_\tau\{c_{\mathbf{p}\sigma}(\tau) c^\dagger_{\mathbf{p}\sigma}(0)\}\rangle e^{i\epsilon_n\tau}, \tag{2.8}$$

$$\Lambda_\mu(\mathbf{p}, i\epsilon_n; i\omega_m) G(\mathbf{p}, i\epsilon_n + i\omega_m) G(\mathbf{p}, i\epsilon_n)\delta_{n+m-n'} \equiv T\int_0^{1/T} \mathrm{d}\tau \mathrm{d}\tau_1 \mathrm{d}\tau_1' \langle T_\tau\{J_\mu(\tau) c_{\mathbf{p}\sigma}(\tau_1) c^\dagger_{\mathbf{p}\sigma}(\tau_1')\}\rangle e^{i\omega_m\tau + i\epsilon_n\tau_1 - i\epsilon_{n'}\tau_1'}, \tag{2.9}$$

we can obtain

$$\chi_{\mu\nu}(0, i\omega_m) = -2T\sum_n\sum_\mathbf{p} v_{\mathbf{p}\mu} G(\mathbf{p}, i\epsilon_n + i\omega_m) G(\mathbf{p}, i\epsilon_n) \Lambda_\nu(\mathbf{p}, i\epsilon_n; i\omega_m). \tag{2.10}$$

$\Lambda(\mathbf{p}, i\epsilon_n; i\omega_m)$ is related to the four-point vertex function $\Gamma(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$ as follows,

$$\Lambda_\mu(\mathbf{p}, i\epsilon_n; i\omega_m) = v_{\mathbf{p}\mu} + T\sum_{n'}\sum_{\mathbf{p}'} \Gamma(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m) G(\mathbf{p}', i\epsilon_{n'} + i\omega_m) G(\mathbf{p}', i\epsilon_{n'}) v_{\mathbf{p}'\mu}. \tag{2.11}$$

Here, $\Gamma(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon'_n; i\omega_m)$ satisfies the following Bethe-Salpeter equation,

$$\begin{aligned}\Gamma(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m) &= \Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m) + T\sum_{n''}\sum_{\mathbf{p}''} \Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}'', i\epsilon_{n''}; i\omega_m) \\ &\times G(\mathbf{p}'', i\epsilon_{n''} + i\omega_m) G(\mathbf{p}'', i\epsilon_{n''}) \Gamma(\mathbf{p}'', i\epsilon_{n''}; \mathbf{p}', i\epsilon_{n'}; i\omega_m),\end{aligned} \tag{2.12}$$

where $\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon'_n; i\omega_m)$ is the proper four-point vertex part.

Performing the analytic continuation of eq. (2.10) with respect to the frequency on the real axis, we obtain

$$\begin{aligned}\chi^R_{\mu\nu}(\mathbf{0},\omega) &= \frac{1}{2i}\sum_\mathbf{p}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi} v_{\mathbf{p}\mu}\left[\tanh\left(\frac{\epsilon^-}{2T}\right) g_1(p^+,p^-)\Lambda_{1\nu}(p^+,p^-) + \left\{\tanh\left(\frac{\epsilon^+}{2T}\right)\right.\right. \\ &\left.\left. - \tanh\left(\frac{\epsilon^-}{2T}\right)\right\} g_2(p^+,p^-)\Lambda_{2\nu}(p^+,p^-) - \tanh\left(\frac{\epsilon^+}{2T}\right) g_3(p^+,p^-)\Lambda_{3\nu}(p^+,p^-)\right],\end{aligned} \tag{2.13}$$

where

$$\Lambda_i(p^+,p^-) \equiv \mathbf{v}_\mathbf{p} + \frac{1}{2i}\sum_{\mathbf{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\sum_{i=1}^{3}\mathcal{J}_{ik}(p^+,p'^-;p'^+,p^-) g_k(p'^+,p'^-)\mathbf{v}_{\mathbf{p}'}, \tag{2.14}$$

$$\begin{aligned} g_1(p^+,p^-) &\equiv G^R(p^+)G^R(p^-), \\ g_2(p^+,p^-) &\equiv G^R(p^+)G^A(p^-), \\ g_3(p^+,p^-) &\equiv G^A(p^+)G^A(p^-).\end{aligned} \tag{2.15}$$

Here, we have used the notation: $p = (\mathbf{p},\epsilon)$, $p' = (\mathbf{p}',\epsilon')$, $q = (0,\omega)$ and have put $p^\pm \equiv p \pm q/2$, $p'^\pm \equiv p' \pm q/2$.

The functions $\mathcal{J}_{ik}(\epsilon;\epsilon';\omega)$ (for $i,k=1,2,3$) in eq. (2.14) related to the analytic continuation of $\Gamma(i\epsilon_n;i\epsilon_{n'};i\omega_m)$, are given in ref. 4, where we have dropped the momentum subscripts since we are interested in the analytic properties with respect to frequencies, and we have used the notation; $\mathcal{J}_{ik}(p+q,p'-q;p',p)\equiv\mathcal{J}_{ik}(p,p';q)$. Especially, the functions $\mathcal{J}^{(1)}_{i2}(p^+,p'^-;p'^+,p^-)$ play important roles below, which are expressed for $i=1,3$ as follows,

$$\mathcal{J}^{(1)}_{i2}(p^+,p'^-;p'^+,p^-)=\left\{\tanh\left(\frac{\epsilon^+}{2T}\right)-\tanh\left(\frac{\epsilon^-}{2T}\right)\right\}\Gamma^{(1)}_{i2}(p^+,p'^-;p'^+,p^-). \tag{2.16}$$

Following the conventional Fermi liquid theory,[22] the function $\mathcal{J}^{(1)}_{22}(\epsilon,\epsilon';\omega)$ is expressed by Δ_1 and Δ_2, which are the discontinuities of $\Gamma^{(1)}(z;z';w)$ across the cuts $\mathrm{Im}(z-z')=0$ and $\mathrm{Im}(z+z'+w)=0$, respectively, where z, z' and w are complex variables corresponding to $i\epsilon_n$, $i\epsilon_{n'}$ and $i\omega_m$, respectively, and by the continuous function Γ_c across these cuts,

$$\mathcal{J}^{(1)}_{22}(p^+,p'^-;p'^+,p^-) = \left\{\tanh\left(\frac{\epsilon'^+}{2T}\right)-\tanh\left(\frac{\epsilon'^-}{2T}\right)\right\}\Gamma_c(p^+,p'^-;p'^+,p^-)+i\mathcal{J}'(p^+,p'^-;p'^+,p^-), \tag{2.17}$$

$$\begin{aligned}\mathcal{J}'(p^+,p'^-;p'^+,p^-) &= \frac{1}{2}\left\{2\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)-\tanh\left(\frac{\epsilon'^+}{2T}\right)-\tanh\left(\frac{\epsilon'^-}{2T}\right)\right\}\Delta_1(\epsilon'-\epsilon)\\ &+\frac{1}{2}\left\{2\coth\left(\frac{\epsilon+\epsilon'}{2T}\right)-\tanh\left(\frac{\epsilon'^+}{2T}\right)-\tanh\left(\frac{\epsilon'^-}{2T}\right)\right\}\Delta_2(\epsilon+\epsilon').\end{aligned} \tag{2.18}$$

The functions G^R and G^A in eq. (2.15) are the retarded and advanced Green's functions, respectively, which are given by

$$G^R(p)=[G^A(p)]^*=\frac{1}{\epsilon+\mu-\epsilon_{\mathbf{p}}-\Sigma^R(p)}, \tag{2.19}$$

where μ is the chemical potential and $\Sigma^R(p)$ is the selfenergy. When T is sufficiently low and $\epsilon\lesssim T$, $\epsilon_{\mathbf{p}}-\mu\lesssim T$, they are well described by

$$G^R(p)=[G^A(p)]^*=\frac{a_{\mathbf{p}}}{\epsilon-\epsilon^*_{\mathbf{p}}+i\gamma_{\mathbf{p}}}, \tag{2.20}$$

where

$$\epsilon^*_{\mathbf{p}}=\left[\epsilon_{\mathbf{p}}-\mu+\mathrm{Re}\Sigma^R(p)\right]_{\epsilon=\epsilon^*_{\mathbf{p}}}, \tag{2.21}$$

$$a^{-1}_{\mathbf{p}}=\left[1-\frac{\partial}{\partial\epsilon}\mathrm{Re}\Sigma^R(p)\right]_{\epsilon=\epsilon^*_{\mathbf{p}}}, \tag{2.22}$$

$$\gamma_{\mathbf{p}}=-a_{\mathbf{p}}\left[\mathrm{Im}\Sigma^R(p)\right]_{\epsilon=\epsilon^*_{\mathbf{p}}}. \tag{2.23}$$

When $\omega\ll T$, we can assume that g_1 and g_3 are independent of ω,

$$g_1(p^+,p^-)\simeq[G^R(p)]^2\equiv g_1(p), \tag{2.24}$$

$$g_3(p^+,p^-)\simeq[G^A(p)]^2\equiv g_3(p). \tag{2.25}$$

Only the function $g_2(p)$ depends strongly on ω for small values of ω,

$$g_2(p^+,p^-)\simeq 2\pi i a^2_{\mathbf{p}}\delta(\epsilon-\epsilon^*_{\mathbf{p}})/(\omega+2i\gamma_{\mathbf{p}}). \tag{2.26}$$

Retaining ω-dependences only in g_2 and $\mathcal{J}_{i2}$ in the case $\omega\ll T$, we obtain the conductivity from eqs. (2.1) and (2.13) as follows,

$$\sigma_{\mu\nu}(\omega)=e^2\sum_{\mathbf{p}\sigma}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{2\pi}\left(-\frac{\partial f}{\partial\epsilon}\right)\tilde{v}_{\mathbf{p}\mu}\frac{2\pi i a^2_{\mathbf{p}}\delta(\epsilon-\epsilon^*_{\mathbf{p}})}{\omega+2i\gamma_{\mathbf{p}}}\Lambda_{2\nu}(p^+,p^-), \tag{2.27}$$

where $f\equiv f(\epsilon)\equiv(e^{\epsilon/T}+1)^{-1}$ is the Fermi distribution function and

$$\tilde{v}_{\mathbf{p}\mu}\equiv v_{\mathbf{p}\mu}+\frac{1}{2i}\sum_{\mathbf{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\sum_{k=1,3}\mathcal{J}^{(0)}_{2k}(p,p';0)g_k(p')v_{\mathbf{p}'\mu}, \tag{2.28}$$

and $\mathcal{J}^{(0)}_{ik}$, the whole contributions from diagrams but without g_2 defined by

$$\mathcal{J}^{(0)}_{ik}(p,p';q) = \mathcal{J}^{(1)}_{ik}(p,p';q)+\frac{1}{2i}\sum_{\mathbf{p}''}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon''}{2\pi}\sum_{l=1,3}\mathcal{J}^{(1)}_{il}(p,p'';q)g_l(p'')\mathcal{J}^{(0)}_{lk}(p'',p';q). \tag{2.29}$$

Then, the Bethe-Salpeter equation, eq. (2.12), becomes

$$\Lambda_2(p^+,p^-)=\tilde{v}_{\mathbf{p}\mu}+\frac{1}{2i}\sum_{\mathbf{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\mathcal{J}^{(0)}_{22}(p^+,p'^-;p'^+,p^-)\frac{2\pi i a^2_{\mathbf{p}'}\delta(\epsilon'-\epsilon^*_{\mathbf{p}'})}{\omega+2i\gamma_{\mathbf{p}'}}\Lambda_2(p'^+,p'^-). \tag{2.30}$$

Introducing $\Phi_{\mathbf{p}}$ by

$$\Phi_{\mathbf{p}}\equiv\frac{a_{\mathbf{p}}}{-i\omega+2\gamma_{\mathbf{p}}}[\Lambda_2(p^+,p^-)]_{\epsilon=\epsilon^*_{\mathbf{p}}}, \tag{2.31}$$

where the ω-dependence is not shown explicitly, we obtain

$$\sigma_{\mu\nu}(\omega)=e^2\sum_{\mathbf{p}\sigma}v^*_{\mathbf{p}\mu}\Phi_{\mathbf{p}\nu}\left(-\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=\epsilon^*_{\mathbf{p}}}. \tag{2.32}$$

We note $\tilde{v}_{\mathbf{p}\mu}$ near the Fermi surface is given as follows,

$$\tilde{v}_{\mathbf{p}\mu}\simeq v_{\mathbf{p}\mu}+\left[\frac{\partial}{\partial p_\mu}\mathrm{Re}\Sigma^R(p)\right]_{\epsilon=\epsilon^*_{\mathbf{p}}}, \tag{2.33}$$

and then

$$v^*_{\mathbf{p}\mu}\equiv\frac{\partial\epsilon^*_{\mathbf{p}}}{\partial p_\mu}\simeq a_{\mathbf{p}}\tilde{v}_{\mathbf{p}\mu}. \tag{2.34}$$

From eq. (2.30), $\Phi_{\mathbf{p}}$ is seen to satisfy the following integral equation,

$$-i\omega\Phi_{\mathbf{p}\mu}=v^*_{\mathbf{p}\mu}-2\gamma_{\mathbf{p}}\Phi_{\mathbf{p}\mu}+\frac{1}{2i}\sum_{\mathbf{p}'}a_{\mathbf{p}}[\mathcal{J}^{(0)}_{22}(p^+,p'^-;p'^+,p^-)]_{\epsilon=\epsilon^*_{\mathbf{p}},\epsilon'=\epsilon^*_{\mathbf{p}'}}a_{\mathbf{p}'}\Phi_{\mathbf{p}'\mu}. \tag{2.35}$$

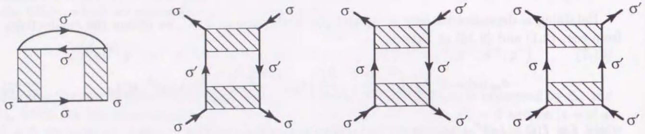

Figure 2.1: Diagrams for selfenergy $\Sigma(\mathbf{p}, i\epsilon_n)$ and for proper four-point vertex part $\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$ giving rise to T^2-term in resistivity. The thick solid line represents the dressed Green's function.

Here, $\mathcal{J}_{22}^{(0)}$ is given as follows up to the first order of ω,

$$[\mathcal{J}_{22}^{(0)}(p^+, p'^-; p'^+, p^-)]_{\epsilon=\epsilon^*_{\mathbf{p}}, \epsilon'=\epsilon^*_{\mathbf{p}'}} = i\mathcal{J}'(\mathbf{p}, \mathbf{p}') + \frac{\omega}{2T}\cosh^{-2}\left(\frac{\epsilon^*_{\mathbf{p}'}}{2T}\right)\Gamma^k(\mathbf{p}, \mathbf{p}'), \tag{2.36}$$

where

$$\Gamma^k(p, p') \equiv \Gamma_c(p, p'; 0) + \frac{1}{2i}\sum_{\mathbf{p}''}\int_{-\infty}^{\infty}\frac{d\epsilon''}{2\pi}\sum_{k=1,3}\mathcal{J}_{2k}^{(0)}(p, p''; 0)g_k(p'')\Gamma_{k2}^{(0)}(p'', p'; 0). \tag{2.37}$$

In eq. (2.36), $\mathcal{J}'(\mathbf{p}, \mathbf{p}')$ and $\Gamma^k(\mathbf{p}, \mathbf{p}')$ are those at $\epsilon = \epsilon^*_{\mathbf{p}}$ and $\epsilon' = \epsilon^*_{\mathbf{p}'}$.

The conductivity, eq. (2.32), is obtained by solving the integral equation, eq. (2.35), in principle. For approximate calculations, the guiding principle has been given by Yamada and Yosida.[2)] In their studies on T^2-term in the resistivity due only to mutual Coulomb interaction, they found that the consistency between the selfenergy Σ and the proper four-point vertex part $\Gamma^{(1)}$ is important in order to implement the conservation law where the T^2-term automatically vanishes in the absence of crystal lattice. The selfenergy Σ and the proper four-point vertex part $\Gamma^{(1)}$ giving rise to the T^2-term are shown in Fig. 2.1.

Then, by expanding eq. (2.35) at $\omega = 0$ up to ϵ^2 and T^2, $\sigma_{\mu\nu}(0)$ is given by

$$\sigma_{\mu\nu}(0) = e^2\sum_{\mathbf{p}\sigma} v^*_{\mathbf{p}\mu}\Phi_{\mathbf{p}\nu}\left(-\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=\epsilon^*_{\mathbf{p}}}, \tag{2.38}$$

$$0 = v^*_{\mathbf{p}\mu} + \sum_{\mathbf{p}',\mathbf{k}}\Delta_0(\mathbf{p}, \mathbf{p}'; \mathbf{p}'+\mathbf{k}, \mathbf{p}-\mathbf{k})[\Phi_{\mathbf{p}-\mathbf{k}\mu} + \Phi_{\mathbf{p}'+\mathbf{k}\mu} - \Phi_{\mathbf{p}'\mu} - \Phi_{\mathbf{p}\mu}]. \tag{2.39}$$

Here Δ_0 is defined by

$$\Delta_0(\mathbf{p}, \mathbf{p}'; \mathbf{p}'+\mathbf{k}, \mathbf{p}-\mathbf{k}) \equiv \pi\rho_{\mathbf{p}-\mathbf{k}}(0)\rho_{\mathbf{p}'+\mathbf{k}}(0)\rho_{\mathbf{p}'}(0)$$
$$\times\left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}, \mathbf{p}'; \mathbf{p}'+\mathbf{k}, \mathbf{p}-\mathbf{k}) + \frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}, \mathbf{p}'; \mathbf{p}'+\mathbf{k}, \mathbf{p}-\mathbf{k})\right](\epsilon^2 + (\pi T)^2), \tag{2.40}$$

where $\rho_{\mathbf{p}}(\epsilon)$ is the spectral function,

$$\rho_{\mathbf{p}}(\epsilon) \equiv \rho(p) = a_{\mathbf{p}}\delta(\epsilon - \epsilon^*_{\mathbf{p}}), \tag{2.41}$$

and $\Gamma^A_{\uparrow\uparrow}$ is defined by

$$\Gamma^A_{\uparrow\uparrow}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = \Gamma_{\uparrow\uparrow}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) - \Gamma_{\uparrow\uparrow}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_4, \mathbf{p}_3), \tag{2.42}$$

and $\Gamma_{\sigma\sigma'}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4)$ is the full four-point vertex function evaluated at zero temperature and zero frequencies. If $\Phi_{\mathbf{p}}$ can be obtained by solving the integral equation, eq. (2.39), we can get T^2-term in the resistivity by substituting $\Phi_{\mathbf{p}}$ into eq. (2.38). If there is no crystal lattice, the second term in eq. (2.39) vanishes because $\Phi_{\mathbf{p}}$ can be put to be proportional to the momentum $\mathbf{p}$. Thus, in the limiting case of continuum in the absence of the impurity scattering, the T^2-term exactly vanishes.

2.2 Limiting Case of Continuum in the Absence of Impurity Scattering

In this section, we point out that, in the limiting case of continuum in the absence of the impurity scattering, the generalized Ward-Takahashi identity reflecting the current conservation insures not only the absence of the resistivity at finite temperature but also the absence of the renormalization of, the Drude weight, the coefficient of $1/\omega$-term in $\sigma(\omega)$ in Èliashberg's formulation.

When the total current is conserved, there exists an important relationship between the selfenergy corrections and the vertex corrections given as follows,[6,7)]

$$[\Sigma(\mathbf{p}, i\epsilon_n) - \Sigma(\mathbf{p}, i\epsilon_n + i\omega_m)]v_{\mathbf{p}\mu}$$
$$= T\sum_{n'}\sum_{\mathbf{p}'}\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)[G(\mathbf{p}', i\epsilon_{n'}) - G(\mathbf{p}', i\epsilon_{n'} + i\omega_m)]v_{\mathbf{p}'\mu}, \tag{2.43}$$

which is the generalized Ward-Takahashi identity reflecting the current conservation.(See Appendix A.)

Applying the analytic continuation on both sides of eq. (2.43) and making suitable rearrangement of variables, we get

$$[\Sigma^R(p^-) - \Sigma^R(p^+)]v_{\mathbf{p}\mu}$$
$$= \frac{1}{2i}\sum_{\mathbf{p}}\int_{-\infty}^{\infty}\frac{d\epsilon'}{2\pi}\{\mathcal{J}_{11}^{(1)}(p^+, p'^-; p'^+, p^-)g_1(p'^+, p'^-)[\omega + \Sigma^R(p'^-) - \Sigma^R(p'^+)]$$
$$+\mathcal{J}_{12}^{(1)}(p^+, p'^-; p'^+, p^-)g_2(p'^+, p'^-)[\omega + \Sigma^A(p'^-) - \Sigma^R(p'^+)]$$
$$+\mathcal{J}_{13}^{(1)}(p^+, p'^-; p'^+, p^-)g_3(p'^+, p'^-)[\omega + \Sigma^A(p'^-) - \Sigma^A(p'^+)]\}v_{\mathbf{p}'\mu}, \tag{2.44}$$

$$[\Sigma^A(p^-) - \Sigma^R(p^+)]v_{\mathbf{p}\mu}$$
$$= \frac{1}{2i}\sum_{\mathbf{p}'}\int_{-\infty}^{\infty}\frac{d\epsilon'}{2\pi}\{\mathcal{J}_{21}^{(1)}(p^+, p'^-; p'^+, p^-)g_1(p'^+, p'^-)[\omega + \Sigma^R(p'^-) - \Sigma^R(p'^+)]$$
$$+\mathcal{J}_{22}^{(1)}(p^+, p'^-; p'^+, p^-)g_2(p'^+, p'^-)[\omega + \Sigma^A(p'^-) - \Sigma^R(p'^+)]$$
$$+\mathcal{J}_{23}^{(1)}(p^+, p'^-; p'^+, p^-)g_3(p'^+, p'^-)[\omega + \Sigma^A(p'^-) - \Sigma^A(p'^+)]\}v_{\mathbf{p}'\mu}, \tag{2.45}$$

$$
\begin{aligned}
&[\Sigma^A(p^-) - \Sigma^A(p^+)]v_{\mathrm{p}\mu} \\
&= \frac{1}{2i}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\ \{\mathcal{J}_{31}^{(1)}(p^+,p'^-;p'^+,p^-)g_1(p'^+,p'^-)[\omega+\Sigma^R(p'^-)-\Sigma^R(p'^+)] \\
&\qquad +\mathcal{J}_{32}^{(1)}(p^+,p'^-;p'^+,p^-)g_2(p'^+,p'^-)[\omega+\Sigma^A(p'^-)-\Sigma^R(p'^+)] \\
&\qquad +\mathcal{J}_{33}^{(1)}(p^+,p'^-;p'^+,p^-)g_3(p'^+,p'^-)[\omega+\Sigma^A(p'^-)-\Sigma^A(p'^+)]\}v_{\mathrm{p}'\mu},
\end{aligned} \tag{2.46}
$$

Putting $\omega = 0$ in both sides of eq. (2.45), we get

$$
-2\mathrm{Im}\Sigma(p)v_{\mathrm{p}\mu} - \frac{1}{2}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\mathrm{d}\epsilon'\mathcal{J}'(p,p';0)\rho(p')v_{\mathrm{p}'\mu} = 0. \tag{2.47}
$$

Further, putting $\epsilon = \epsilon^*_{\mathrm{p}}$ in eq. (2.47), we obtain one of useful equations

$$
2\gamma_{\mathrm{p}}v_{\mathrm{p}\mu} - \frac{1}{2}\sum_{\mathrm{p}'}a_{\mathrm{p}}\mathcal{J}'(\mathrm{p},\mathrm{p}')a_{\mathrm{p}'}v_{\mathrm{p}'\mu} = 0. \tag{2.48}
$$

Toyoda[7] obtained, by the analytic continuation of eq. (2.43), the expression corresponding to eq. (2.48) where $\Gamma^{(1)}$ is assumed to be a function of $\mathrm{p}'-\mathrm{p}$, $i\epsilon_{n'} - i\epsilon_n$ and $i\omega_m$. We note, however, that this assumption is not valid in general and then we did not adopt this approximation in obtaining eq. (2.48) as seen in eq. (2.18). In Appendix B, we mention that this difference is important.

On the other hand, from ω-linear terms of eq. (2.44) and eq. (2.46), we obtain

$$
\begin{aligned}
\left[1-\frac{\partial}{\partial\epsilon}\Sigma^R(p)\right]v_{\mathrm{p}\mu} &= v_{\mathrm{p}\mu} + \frac{1}{2i}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\sum_{k=1,3}\mathcal{J}_{1k}^{(0)}(p,p';0)g_k(p')v_{\mathrm{p}'\mu} \\
&+ \frac{1}{2}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\mathrm{d}\epsilon'\Gamma_{12}^{(0)}(p,p';0)\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon'}{2T}\right)\rho(p')v_{\mathrm{p}'\mu},
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
\left[1-\frac{\partial}{\partial\epsilon}\Sigma^A(p)\right]v_{\mathrm{p}\mu} &= v_{\mathrm{p}\mu} + \frac{1}{2i}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\sum_{k=1,3}\mathcal{J}_{3k}^{(0)}(p,p';0)g_k(p')v_{\mathrm{p}'\mu} \\
&+ \frac{1}{2}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\mathrm{d}\epsilon'\Gamma_{32}^{(0)}(p,p';0)\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon'}{2T}\right)\rho(p')v_{\mathrm{p}'\mu}.
\end{aligned} \tag{2.50}
$$

From ω-linear terms of eq. (2.45), we also obtain

$$
\begin{aligned}
\left[1-\frac{\partial}{\partial\epsilon}\mathrm{Re}\Sigma^R(p)\right]v_{\mathrm{p}\mu} &= v_{\mathrm{p}\mu} + \frac{1}{2}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\mathrm{d}\epsilon'\Gamma_c(p,p';0)\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon'}{2T}\right)\rho(p')v_{\mathrm{p}'\mu} \\
&+ \frac{1}{2i}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon'}{2\pi}\left(\mathcal{J}_{21}^{(1)}(p,p';0)g_1(p';0)\left[1-\frac{\partial}{\partial\epsilon'}\Sigma^R(p')\right]\right. \\
&\qquad \left.+\mathcal{J}_{23}^{(1)}(p,p';0)g_3(p';0)\left[1-\frac{\partial}{\partial\epsilon'}\Sigma^A(p')\right]\right)v_{\mathrm{p}'\mu},
\end{aligned} \tag{2.51}
$$

where we have used the fact that the correction terms in ω-expansion in $\mathcal{J}'$ are order of ω^2, which is obvious from eq. (2.18). Substituting eqs. (2.49) and (2.50) into eq. (2.51), we see

$$
\left[1-\frac{\partial}{\partial\epsilon}\mathrm{Re}\Sigma^R(p)\right]v_{\mathrm{p}\mu} = \tilde{v}_{\mathrm{p}\mu} + \frac{1}{2}\sum_{\mathrm{p}'}\int_{-\infty}^{\infty}\mathrm{d}\epsilon'\Gamma^k(p,p')\frac{1}{2T}\cosh^{-2}\left(\frac{\epsilon'}{2T}\right)\rho(p')v_{\mathrm{p}'\mu}. \tag{2.52}
$$

Further, putting $\epsilon = \epsilon^*_{\mathrm{p}}$ in eq. (2.52), we obtain another useful equation,

$$
\begin{aligned}
v_{\mathrm{p}\mu} &= v^*_{\mathrm{p}\mu} + \sum_{\mathrm{p}'}a_{\mathrm{p}}\Gamma^k(\mathrm{p},\mathrm{p}')a_{\mathrm{p}'}\left(-\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=\epsilon^*_{\mathrm{p}'}}v_{\mathrm{p}'\mu}, &(2.53)\\
&= v^*_{\mathrm{p}\mu} + \sum_{\mathrm{p}'}a_{\mathrm{p}}\Gamma^\omega(\mathrm{p},\mathrm{p}')a_{\mathrm{p}'}\left(-\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=\epsilon^*_{\mathrm{p}'}}v^*_{\mathrm{p}'\mu}, &(2.54)
\end{aligned}
$$

where

$$
\Gamma^\omega(\mathrm{p},\mathrm{p}') \equiv \Gamma^k(\mathrm{p},\mathrm{p}') + \sum_{\mathrm{p}''}\Gamma^k(\mathrm{p},\mathrm{p}'')a_{\mathrm{p}''}\left(-\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=\epsilon^*_{\mathrm{p}''}}a_{\mathrm{p}''}\Gamma^\omega(\mathrm{p}'',\mathrm{p}'). \tag{2.55}
$$

The last term in the right hand side of eq. (2.54) expresses the effect of back-flow, or the drag effect. Eq. (2.54) is the finite-temperature version of the well-known relationships between the bare and effective masses at absolute zero,[22] and implies that the adiabatic insertion of interactions does not change the value of total current flow of a noninteracting system in the limiting case of continuum in the absence of the impurity scattering.

At sufficiently low temperature, we can ignore the energy dependence of effective mass m^* and therefore put $v^*_{\mathrm{p}\mu} = p_\mu/m^*$. Hence, corresponding to eqs. (2.48) and (2.53), we obtain

$$
\frac{p_\mu}{m^*} - \sum_{\mathrm{p}'}a\Gamma^k(\mathrm{p},\mathrm{p}')a\left(-\frac{\partial f}{\partial\epsilon'}\right)_{\epsilon'=\epsilon^*_{\mathrm{p}'}}\frac{p'_\mu}{m^*} = \frac{m}{m^*}\frac{p_\mu}{m^*}, \tag{2.56}
$$

$$
2\gamma\frac{p_\mu}{m^*} - \frac{1}{2}\sum_{\mathrm{p}'}a\mathcal{J}'(\mathrm{p},\mathrm{p}')a\frac{p'_\mu}{m^*} = 0, \tag{2.57}
$$

where m is bare electron mass. Here, we put

$$
\Phi_{\mathrm{p}\mu} = \frac{p_\mu}{m^*}F. \tag{2.58}
$$

Then, inserting eqs. (2.56) and (2.57) into eq. (2.35), we obtain

$$
F = -\frac{m^*}{m}\frac{1}{i\omega}. \tag{2.59}
$$

By eq. (2.32), we thus get

$$
\sigma(\omega) = -\frac{m^*}{m}\frac{\chi^*_0}{i\omega} = -\frac{ne^2}{mi\omega}, \tag{2.60}
$$

where

$$
\chi^*_0 \equiv \sum_{\mathrm{p}\sigma}v^{*2}_{\mathrm{p}\mu}\delta(\epsilon^*_{\mathrm{p}}). \tag{2.61}
$$

Although χ^*_0 is renormalized by m/m^*, $1/\omega$-coefficient of $\sigma(\omega)$ recovers the value of noninteracting system due to the drag effect mentioned above. The result of the absence of T^2-term in the resistivity in the limiting case of continuum in the absence of the impurity scattering has been shown by Yamada and Yosida,[2] as was seen in the last section. However, to the best of our knowledge, this is the first to show the absence of the renormalizations of the coefficient of $1/\omega$, *i.e.* Drude weight, in the framework of the Fermi liquid theory.

Chapter 3

From Kubo Formula to Matrix Formulation of Memory Function

In Sec. 2.2, we have seen that Èliashberg's formulation of the Kubo formula for $\sigma(\omega)$ leads to a correct result in the limiting case of continuum in the absence of the impurity scattering. On the other hand, the memory-function formalism[8] based on the Mori formula[9,10] is another formulation which leads to a correct result in this limiting case. This formalism is powerful for actual calculations of the resistivity of the various systems in the lowest order of the scattering processes which do not conserve the total current. In the lattice system, which is of our present interest, the total current is not conserved even when the total momentum is conserved because the current is determined by the group velocity which is not proportional to the momentum. In this case, the memory-function formalism leads to an incorrect result of non-zero resistivity as discussed in Chap. 1. As it turns out, this failure is due to the fact that, in the ordinary memory-function formalism, only the violation of the current conservation is considered but the momentum conservation is not taken into account. This is due to the high-frequency expansion of $\sigma(\omega)$ inherent to the memory-function formalism.

In this chapter, it is shown that with the help of Èliashberg's formulation of the Kubo formula for $\sigma(\omega)$, which is valid at low-frequency, and of the diagrammatic analysis of T^2-term in the resistivity to respect the consistency between the selfenergy and the vertex corrections developed by Yamada and Yosida,[2] the memory function is extended to the matrix form so that it yields a correct result of the vanishing resistivity when the total momentum is conserved.

We consider the following Hamiltonian,

$$H = \sum_{p\sigma} \epsilon_p c^{\dagger}_{p\sigma} c_{p\sigma} + U \sum_{p,p',k\neq 0} c^{\dagger}_{p-k\uparrow} c^{\dagger}_{p'+k\downarrow} c_{p'\downarrow} c_{p\uparrow} + \nu \sum_{p,q\sigma} \rho_q c^{\dagger}_{p+q\sigma} c_{p\sigma}, \tag{3.1}$$

where U is the short-range Coulomb interaction and $\rho_q = \sum_i e^{-iq\cdot r_i}$ where r_i represents the position of the impurity. Here, we also consider s-wave impurity scattering whose strength is ν since the effect of the violation of the current conservation through normal processes gets important once the impurity scattering is present as will be seen in Chap. 4.

3.1 Generalization to Matrix Form of Memory Function

As was seen in Sec. 2.1, the conductivity, eq. (2.32), is obtained by solving the integral equation, eq. (2.35). In this section, we investigate the kernel of the integral equation, eq. (2.35), up to ω, ϵ^2 and T^2 where the effects of the periodic lattice which leads not only to the Umklapp scattering but also to the band structure are fully taken into account. Then, an expression of $\sigma(\omega, T)$ is given in terms of memory function which is extended to the matrix form.

First, we rewrite eq. (2.35) together with eq. (2.36) as

$$i v^*_{p\mu} = \sum_{p'} [\omega \delta_{pp'} + K_{pp'}] \Phi_{p'\mu}, \tag{3.2}$$

where $K_{pp'} = K'_{pp'} + iK''_{pp'}$ and

$$K'_{pp'} = -\omega a_p \Gamma^k(p, p') a_{p'} \left(-\frac{\partial f}{\partial \epsilon'}\right)_{\epsilon'=\epsilon^*_{p'}}, \tag{3.3}$$

$$K''_{pp'} = \delta_{pp'} 2\gamma_p - \frac{1}{2} a_p \mathcal{J}'(p, p') a_{p'}. \tag{3.4}$$

We note that, if the total current is conserved, eq. (2.48), which is obtained from the analytic continuation of the generalized Ward-Takahashi identity reflecting the current conservation, leads to the fact that $K''_{pp'}$ has a zero eigenvalue for the eigenvector, $v_{p'\mu}$, as follows,

$$\sum_{p'} K''_{pp'} v_{p'\mu} = 0. \tag{3.5}$$

This fact implies that the conservation laws are generally related to zero eigenvalues of $K''_{pp'}$. Actually, as has been indicated by Wölfle,[23] the vanishing resistivity of the lattice electron systems in the absence of both Umklapp and impurity scattering is concluded from the existence of a zero eigenvalue of $K''_{pp'}$ reflecting the conservation of the crystal momentum. This will be demonstrated in the following, Sec. 3.3 and Sec. 3.4.

In the case of a general Fermi surface different from sphere, the total current is determined by the group velocity, which is not necessarily proportional to the momentum. In this case, in order to formulate $\sigma(\omega)$ unambiguously, we expand eq. (3.2) by the Fermi surface harmonics, $\{\psi_L(p)\}$, which are the polynomials of the Cartesian components of the group velocity, $v^*_{p\mu}$, first introduced by Allen.[24] $\psi_L(p)$'s are listed in Table 3.1 up to the third order in the case of three dimensions. Although the Fermi surface harmonics are different from the spherical harmonics, they essentially correspond to those in the case of the limiting case of continuum, $v^*_{p\mu} \propto p_\mu$. The Fermi surface harmonics satisfy the orthonormality relation

$$\langle \psi_L | \psi_{L'} \rangle = \delta_{LL'}, \tag{3.6}$$

where the inner product is defined as

$$\langle u|v \rangle \equiv \langle uv \rangle \equiv \sum_p u_p v_p \delta(\epsilon - \epsilon^*_p) / \sum_p \delta(\epsilon - \epsilon^*_p). \tag{3.7}$$

The bracket notation implies

$$\langle u \rangle = \sum_p u_p \delta(\epsilon - \epsilon^*_p) / \sum_p \delta(\epsilon - \epsilon^*_p). \tag{3.8}$$

Table 3.1: Fermi surface harmonics for cubic symmetry for polynomials up to the third order. Normalization factors are omitted for simplicity, which can be formally written down by use of the inner product. We put $v_{\mathbf{p}}^{*2} \equiv v_{\mathbf{p}x}^{*2} + v_{\mathbf{p}y}^{*2} + v_{\mathbf{p}z}^{*2}$.

Order of polynomial	*Irreducible representation*	*Label of function*	*Explicit form for cubic symmetry except for normalization factor*
0	Γ_0	ψ_0	1
1	Γ_{15}	ψ_x	$v_{\mathbf{p}x}^*$
		ψ_y	$v_{\mathbf{p}y}^*$
		ψ_z	$v_{\mathbf{p}z}^*$
2	$\Gamma_{25'}$	ψ_{xy}	$v_{\mathbf{p}x}^* v_{\mathbf{p}y}^*$
		ψ_{yz}	$v_{\mathbf{p}y}^* v_{\mathbf{p}z}^*$
		ψ_{zx}	$v_{\mathbf{p}z}^* v_{\mathbf{p}x}^*$
	Γ_{12}	$\psi_{3x^2-r^2}$	$3v_{\mathbf{p}x}^{*2} - v_{\mathbf{p}}^{*2}$
		$\psi_{y^2-z^2}$	$v_{\mathbf{p}y}^{*2} - v_{\mathbf{p}z}^{*2}$
	Γ_0	ψ_{r^2}	$v_{\mathbf{p}}^{*2} - \langle v^{*2} \rangle$
3	$\Gamma_{2'}$	ψ_{xyz}	$v_{\mathbf{p}x}^* v_{\mathbf{p}y}^* v_{\mathbf{p}z}^*$
	Γ_{25}	$\psi_{x(y^2-z^2)}$	$v_{\mathbf{p}x}^*(v_{\mathbf{p}y}^{*2} - v_{\mathbf{p}z}^{*2})$
		and	
		2 similar	
	Γ_{15}	ψ_{x^3}	$v_{\mathbf{p}x}^{*3} - v_{\mathbf{p}x}^* \langle v_x^{*4} \rangle / \langle v_x^{*2} \rangle$
		and	
		2 similar	
	Γ_{15}	$\psi_{x(y^2+z^2)}$	$v_{\mathbf{p}x}^* \Big(v_{\mathbf{p}y}^{*2} + v_{\mathbf{p}z}^{*2} - v_{\mathbf{p}x}^{*2} \frac{\langle v_x^{*2}\rangle\langle v_x^{*4}(v_y^{*2}+v_z^{*2})\rangle - \langle v_x^{*4}\rangle\langle v_x^{*2}(v_y^{*2}+v_z^{*2})\rangle}{\langle v_x^{*2}\rangle\langle v_x^{*6}\rangle - \langle v_x^{*4}\rangle^2} - \frac{(\langle v_x^{*2}\rangle\langle v_x^{*6}\rangle - 2\langle v_x^{*4}\rangle^2)\langle v_x^{*2}(v_y^{*2}+v_z^{*2})\rangle + \langle v_x^{*2}\rangle\langle v_x^{*4}\rangle\langle v_x^{*4}(v_y^{*2}+v_z^{*2})\rangle}{\langle v_x^{*2}\rangle^2\langle v_x^{*6}\rangle - \langle v_x^{*2}\rangle\langle v_x^{*4}\rangle^2} \Big)$
		and	
		2 similar	

We will often write it as $\langle u \rangle_\epsilon$ in order to express ϵ-dependence explicitly. We assume that $\Phi_{\mathbf{p}}$ and $M_{\mathbf{p}\mathbf{p}'}$ can be expanded in Fermi surface harmonics

$$\Phi_{\mathbf{p}} = \sum_L \Phi_L(\epsilon)\psi_L(\mathbf{p}), \tag{3.9}$$

$$K_{\mathbf{p}\mathbf{p}'} = \sum_{LL'} K_{LL'}(\epsilon,\epsilon')\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}'). \tag{3.10}$$

The inverse relations are easily obtained from eq. (3.6)

$$\Phi_L(\epsilon) = \sum_{\mathbf{p}} \delta(\epsilon - \epsilon_{\mathbf{p}}^*)\psi_L(\mathbf{p})\Phi_{\mathbf{p}} / \sum_{\mathbf{p}} \delta(\epsilon - \epsilon_{\mathbf{p}}^*), \tag{3.11}$$

$$K_{LL'}(\epsilon,\epsilon') = \sum_{\mathbf{p}\mathbf{p}'} \delta(\epsilon - \epsilon_{\mathbf{p}}^*)\delta(\epsilon' - \epsilon_{\mathbf{p}'}^*)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')K_{\mathbf{p}\mathbf{p}'} \times \left(\sum_{\mathbf{p}\mathbf{p}'} \delta(\epsilon - \epsilon_{\mathbf{p}}^*)\delta(\epsilon' - \epsilon_{\mathbf{p}'}^*) \right)^{-1}. \tag{3.12}$$

Then, from eq. (2.32), $\sigma(\omega)$ (along, say, x-axis) is given by

$$\sigma(\omega) = e^2 \sum_\sigma \int_{-\infty}^{\infty} \mathrm{d}\epsilon N^*(\epsilon) \langle v_x^{*2} \rangle_\epsilon^{1/2} \Phi_1(\epsilon) \left(-\frac{\partial f}{\partial \epsilon} \right), \tag{3.13}$$

and by eq. (3.2)

$$\omega\hat{\Phi}(\epsilon) + \int_{-\infty}^{\infty} \mathrm{d}\epsilon' N^*(\epsilon')\hat{K}(\epsilon,\epsilon')\hat{\Phi}(\epsilon') = i\langle v_x^{*2} \rangle_\epsilon^{1/2} \hat{e}_1, \tag{3.14}$$

where

$$\hat{\Phi}(\epsilon) = {}^t(\Phi_x(\epsilon), \Phi_{x^3}(\epsilon), \Phi_{x(y^2+z^2)}(\epsilon), \cdots), \tag{3.15}$$

$$\hat{e}_1 = {}^t(1, 0, 0, \cdots, 0, \cdots), \tag{3.16}$$

$$\hat{K}(\epsilon,\epsilon') = \begin{pmatrix} K_{x,x}(\epsilon,\epsilon') & K_{x,x^3}(\epsilon,\epsilon') & K_{x,x(y^2+z^2)}(\epsilon,\epsilon') & \cdots \\ K_{x^3,x}(\epsilon,\epsilon') & K_{x^3,x^3}(\epsilon,\epsilon') & K_{x^3,x(y^2+z^2)}(\epsilon,\epsilon') & \cdots \\ K_{x(y^2+z^2),x}(\epsilon,\epsilon') & K_{x(y^2+z^2),x^3}(\epsilon,\epsilon') & K_{x(y^2+z^2),x(y^2+z^2)}(\epsilon,\epsilon') & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.17}$$

Here $N^*(\epsilon) = \sum_{\mathbf{p}} \delta(\epsilon - \epsilon_{\mathbf{p}}^*)$. Note that it is enough to consider $\psi_L(\mathbf{p}) \in \Gamma_{15}$. If $\Phi_1(\epsilon) \equiv \Phi_x(\epsilon)$ can be obtained by solving the integral equation, eq. (3.14), we can get $\sigma(\omega)$ by substituting $\Phi_1(\epsilon)$ into eq. (3.13). In the above, $\sigma(\omega)$ is exactly formulated at small ω in the Fermi liquid theory.

Now, in the spirit of Yamada and Yosida mentioned in Sec. 2.1, we would like to obtain $\hat{\Phi}(\epsilon)$ by expanding $\hat{K}(\epsilon,\epsilon')$ up to ω, ν^2, ϵ^2 and T^2 in eq. (3.14). It is, however, noted that we consider here impurity scattering as well and that we expand $K''_{LL'}(\epsilon,\epsilon')$ up to ν^2 with respect to impurity scattering.

For that purpose, we first consider $K''_{LL'}(\epsilon,\epsilon')$ up to ν^2 and U^2, where the processes shown in Fig. 3.1 are taken into account in the proper four-point vertex part. The corresponding expression

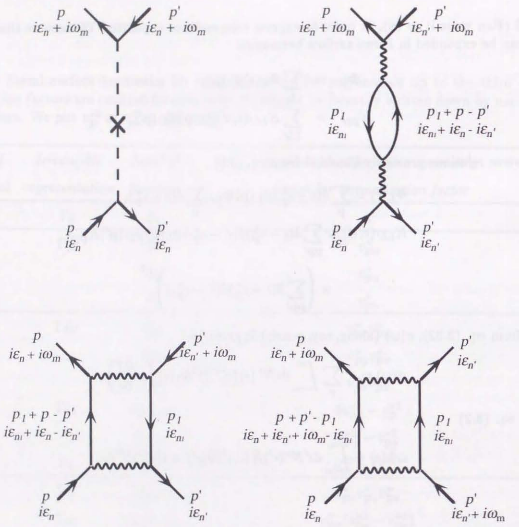

Figure 3.1: Second-order Feynman diagrams with respect to the mutual Coulomb interaction and impurity scattering for proper four-point vertex part $\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$. The solid, wavy and broken lines represent the dressed Green's function, the bare Coulomb interaction and impurity scattering, respectively.

for $\mathcal{J}'$ is given by

$$\begin{aligned}\mathcal{J}'(p,p';0) &= 4\pi n_i \nu^2 \delta(\epsilon-\epsilon') + 4\pi U^2 \sum_{\mathbf{p}_1} \int_{-\infty}^{\infty} d\epsilon_1 \Big[2\left\{ \bar{f}(\epsilon_1) f(\epsilon_1+\epsilon-\epsilon') f(\epsilon') \right. \\ &\left. + f(\epsilon_1)\bar{f}(\epsilon_1+\epsilon-\epsilon')\bar{f}(\epsilon') \right\} \rho(p_1)\rho(p_1+p-p') - \left\{ f(\epsilon_1) f(\epsilon+\epsilon'-\epsilon_1)\bar{f}(\epsilon') \right. \\ &\left. + \bar{f}(\epsilon_1)\bar{f}(\epsilon+\epsilon'-\epsilon_1) f(\epsilon') \right\} \rho(p_1)\rho(p+p'-p_1) \Big], \end{aligned} \tag{3.18}$$

where we put $p_1 = (\mathbf{p}_1, \epsilon_1)$ and $\bar{f}(\epsilon) = 1 - f(\epsilon)$. The imaginary part of Σ^R is given from $\mathcal{J}'$ by the analytic continuation of the ordinary Ward-Takahashi identity reflecting the local conservation of the electron number density[25] as

$$\begin{aligned}-2\mathrm{Im}\Sigma^R(p) &= \frac{1}{2}\sum_{\mathbf{p}'} \int_{-\infty}^{\infty} d\epsilon' \mathcal{J}'(p,p';0)\rho(p') \\ &= 2\pi n_i \nu^2 N(\epsilon) + 2\pi U^2 \sum_{\mathbf{p}',\mathbf{p}_1} \int_{-\infty}^{\infty} d\epsilon' d\epsilon_1 \left\{ \bar{f}(\epsilon_1) f(\epsilon_1+\epsilon-\epsilon') f(\epsilon') \right. \\ &\left. + f(\epsilon_1)\bar{f}(\epsilon_1+\epsilon-\epsilon')\bar{f}(\epsilon') \right\} \rho(p_1)\rho(p_1+p-p')\rho(p'). \end{aligned} \tag{3.19}$$

Here, we put $N(\epsilon) = \sum_{\mathbf{p}} a_{\mathbf{p}} \delta(\epsilon - \epsilon^*_{\mathbf{p}})$. Substituting eqs. (3.18) and (3.19) into eq. (3.12) together with eq. (3.4), we obtain

$$\begin{aligned}K''_{LL'}(\epsilon,\epsilon') &= -\delta(\epsilon-\epsilon')\sum_{\mathbf{p}} 2\mathrm{Im}\Sigma^R(p)\rho(p)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})/N^*(\epsilon)N^*(\epsilon') \\ &-\frac{1}{2}\sum_{\mathbf{p},\mathbf{p}'} \mathcal{J}'(p,p';0)\rho(p)\rho(p')\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')/N^*(\epsilon)N^*(\epsilon') \\ &= 2\pi n_i \nu^2 N(\epsilon)\delta(\epsilon-\epsilon')\sum_{\mathbf{p}}\rho(p)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})/(N^*(\epsilon))^2 \\ &+2\pi U^2\delta(\epsilon-\epsilon')\sum_{\mathbf{p},\mathbf{p}',\mathbf{p}_1}\int_{-\infty}^{\infty} d\epsilon' d\epsilon_1 \left\{ \bar{f}(\epsilon_1) f(\epsilon_1+\epsilon-\epsilon') f(\epsilon') \right. \\ &\left. + f(\epsilon_1)\bar{f}(\epsilon_1+\epsilon-\epsilon')\bar{f}(\epsilon') \right\} \rho(p)\rho(p_1)\rho(p_1+p-p')\rho(p')\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})/(N^*(\epsilon))^2 \\ &-2\pi U^2\sum_{\mathbf{p},\mathbf{p}',\mathbf{p}_1}\int_{-\infty}^{\infty} d\epsilon_1 \left\{ \bar{f}(\epsilon_1) f(\epsilon_1+\epsilon-\epsilon') f(\epsilon') + f(\epsilon_1)\bar{f}(\epsilon_1+\epsilon-\epsilon')\bar{f}(\epsilon') \right\} \\ &\times \rho(p)\rho(p_1)\rho(p_1+p-p')\rho(p')\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')/N^*(\epsilon)N^*(\epsilon') \\ &-2\pi U^2\sum_{\mathbf{p},\mathbf{p}',\mathbf{p}_1}\int_{-\infty}^{\infty} d\epsilon_1 \left\{ \bar{f}(\epsilon_1) f(\epsilon') f(\epsilon_1+\epsilon-\epsilon') + f(\epsilon_1)\bar{f}(\epsilon')\bar{f}(\epsilon_1+\epsilon-\epsilon') \right\} \\ &\times \rho(p)\rho(p_1)\rho(p')\rho(p_1+p-p')\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')/N^*(\epsilon)N^*(\epsilon') \\ &+2\pi U^2\sum_{\mathbf{p},\mathbf{p}',\mathbf{p}_1}\int_{-\infty}^{\infty} d\epsilon_1 \left\{ \bar{f}(\epsilon') f(\epsilon'+\epsilon-\epsilon_1) f(\epsilon_1) + f(\epsilon')\bar{f}(\epsilon'+\epsilon-\epsilon_1)\bar{f}(\epsilon_1) \right\} \\ &\times \rho(p)\rho(p')\rho(p'+p-p_1)\rho(p_1)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')/N^*(\epsilon)N^*(\epsilon'). \end{aligned} \tag{3.20}$$

By expanding the right hand side of eq. (3.20) up to ν^2, ϵ^2 and T^2, eq. (3.14) leads to

$$[\omega\hat{1} + \hat{M}(\epsilon;\omega)]\hat{\Phi}(\epsilon) = i\langle v_x^{*2}\rangle_\epsilon^{1/2}\hat{e}_1, \tag{3.21}$$

where

$$\hat{M}(\epsilon;\omega) = \begin{pmatrix} M_{x,x}(\epsilon;\omega) & M_{x,x^3}(\epsilon;\omega) & M_{x,x(y^2+z^2)}(\epsilon;\omega) & \cdots \\ M_{x^3,x}(\epsilon;\omega) & M_{x^3,x^3}(\epsilon;\omega) & M_{x^3,x(y^2+z^2)}(\epsilon;\omega) & \cdots \\ M_{x(y^2+z^2),x}(\epsilon;\omega) & M_{x(y^2+z^2),x^3}(\epsilon;\omega) & M_{x(y^2+z^2),x(y^2+z^2)}(\epsilon;\omega) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.22}$$

Here, $M_{LL'}(\epsilon;\omega) = M'_{LL'}(\omega) + iM''_{LL'}(\epsilon)$ and $M''_{LL'}(\epsilon)$ is given by

$$\begin{aligned} M''_{LL'}(\epsilon) &= 2\pi n_i \nu^2 N(0) \sum_{\mathbf{p}} \rho_{\mathbf{p}}(0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})/N^*(0) \\ &+\frac{\pi}{4}(\epsilon^2+(\pi T)^2) \sum_{\mathbf{p},\mathbf{p}',\mathbf{k}} U^2(\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-)) \\ &\qquad \times (\psi_{L'}(\mathbf{p}^+)+\psi_{L'}(\mathbf{p}'^-)-\psi_{L'}(\mathbf{p}'^+)-\psi_{L'}(\mathbf{p}^-)) \\ &\qquad \times \rho_{\mathbf{p}^+}(0)\rho_{\mathbf{p}^-}(0)\rho_{\mathbf{p}'^+}(0)\rho_{\mathbf{p}'^-}(0)/N^*(0), \end{aligned} \tag{3.23}$$

where $\mathbf{p}^{\pm} \equiv \mathbf{p} \pm \mathbf{k}/2$ and $\mathbf{p}'^{\pm} \equiv \mathbf{p}' \pm \mathbf{k}/2$.

Next, we consider the higher order terms of $\hat{K}''(\epsilon,\epsilon')$ with respect to U but up to ν^2, ϵ^2 and T^2. The higher order terms can be simply included by taking Σ and $\Gamma^{(1)}$ into account in order not to violate the Ward-Takahashi identity. In a similar way to Yamada and Yosida[2)], we can obtain terms proportional to ϵ^2 and T^2 in $\hat{M}''(\epsilon)$, by replacing U^2 by

$$\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-) + \frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-), \tag{3.24}$$

in the second term in the right hand side of eq. (3.23). (See Fig. 2.1.) Here, $\Gamma_{\sigma\sigma'}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)$ is the full four-point vertex function evaluated at zero temperature and zero frequencies in the absence of the impurity scattering. An example of the higher order diagrams with respect to U but up to ν^2 for Σ and $\Gamma^{(1)}$ are shown in Fig. 3.2. A diagram of Fig. 3.2(a) gives corrections to $\gamma_{\mathbf{p}}$ and diagrams of Fig. 3.2(b)$\sim$(f) give corrections to $\mathcal{J}'(\mathbf{p},\mathbf{p}')$ in eq. (3.4). Here, we note that diagrams of the type Fig. 3.2(c)$\sim$(f) give the contributions to $\hat{M}''$ proportional to $\nu^2\epsilon^2$ and ν^2T^2 and thus these contributions are neglected. On the other hand, The corrections with respect to U in diagrams of the type Fig. 3.2(a) and (b) are included in $\Lambda_0(\mathbf{p},\mathbf{p}')$ where $\Lambda_0(\mathbf{p},\mathbf{p}')$ is the full-three point vertex function evaluated at zero temperature and zero frequencies in the absence of the impurity scattering. (See Fig. 3.3.) We thus obtain

$$M'_{LL'}(\omega) = -\omega \sum_{\mathbf{p},\mathbf{p}'} \Gamma^k(\mathbf{p},\mathbf{p}')\rho_{\mathbf{p}}(0)\rho_{\mathbf{p}'}(0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}')/N^*(0), \tag{3.25}$$

$$\begin{aligned} M''_{LL'}(\epsilon) &= \pi n_i \sum_{\mathbf{p},\mathbf{p}'} (\nu\Lambda_0(\mathbf{p},\mathbf{p}'))^2 (\psi_L(\mathbf{p})-\psi_L(\mathbf{p}'))(\psi_{L'}(\mathbf{p})-\psi_{L'}(\mathbf{p}')) \\ &\qquad \times \rho_{\mathbf{p}}(0)\rho_{\mathbf{p}'}(0)/N^*(0) \\ &+\frac{\pi}{4}(\epsilon^2+(\pi T)^2) \sum_{\mathbf{p},\mathbf{p}',\mathbf{k}} \left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-) + \frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-)\right] \\ &\qquad \times (\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-)) \\ &\qquad \times (\psi_{L'}(\mathbf{p}^+)+\psi_{L'}(\mathbf{p}'^-)-\psi_{L'}(\mathbf{p}'^+)-\psi_{L'}(\mathbf{p}^-)) \\ &\qquad \times \rho_{\mathbf{p}^+}(0)\rho_{\mathbf{p}^-}(0)\rho_{\mathbf{p}'^+}(0)\rho_{\mathbf{p}'^-}(0)/N^*(0), \end{aligned} \tag{3.26}$$

up to ω, ν^2, ϵ^2 and T^2. From eqs. (3.13) and (3.21), the electrical conductivity is given by

$$\sigma(\omega) = ie^2 \sum_{\mathbf{p},\sigma} \left(-\frac{\partial f}{\partial \epsilon^*_{\mathbf{p}}}\right) v^{*2}_{\mathbf{p}x} \left[\frac{1}{\omega\hat{1}+\hat{M}(\epsilon^*_{\mathbf{p}};\omega)}\right]_{11}. \tag{3.27}$$

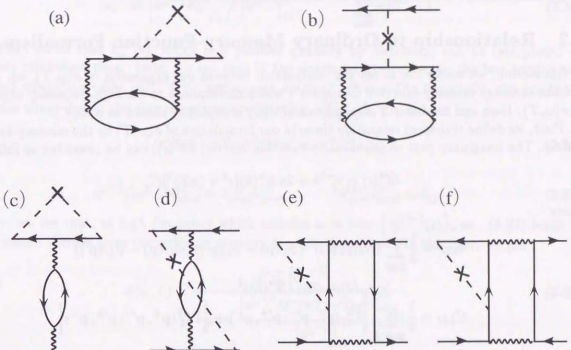

Figure 3.2: An example of higher order Feynman diagrams with respect to U but up to ν^2 for (a) selfenergy $\Sigma(\mathbf{p}, i\epsilon_n)$ and for (b)$\sim$(f) proper four-point vertex part $\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$. The thick solid, wavy and broken lines represent the dressed Green's function, the bare Coulomb interaction and impurity scattering, respectively.

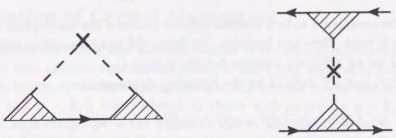

Figure 3.3: Diagrams for selfenergy $\Sigma(\mathbf{p}, i\epsilon_n)$ and for proper four-point vertex part $\Gamma^{(1)}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$ giving rise to term proportional to ν^2 at $T=0$ in $\hat{M}''(\epsilon)$. The thick solid and broken lines represent the dressed Green's function and impurity scattering.

In the above equation, we see that $\hat{M}(\epsilon;\omega)$ corresponds to the memory function which is extended to the matrix form. We call $\hat{M}(\epsilon;\omega)$ the memory-function matrix.

3.2 Relationship to Ordinary Memory-Function Formalism

In this section, we would like to see the relationship between our expression of $\sigma(\omega,T)$, eq. (3.27), and the ordinary memory-function formalism[8] which corresponds to the high-frequency expansion of $\sigma(\omega,T)$. Here and hereafter T-dependence of $\sigma(\omega)$ is explicitly shown as $\sigma(\omega,T)$.

First, we define transport relaxation times in our formulation of $\sigma(\omega,T)$ by the memory-function matrix. The imaginary part of the memory-function matrix, $\hat{M}''(\epsilon)$, can be rewritten as follows,

$$\hat{M}''(\epsilon) = \tau_i^{-1}\hat{a} + 2\pi N^*(0)(\epsilon^2+(\pi T)^2)\hat{C}, \tag{3.28}$$

where

$$a_{LL'} = \frac{1}{2}\sum_{\mathbf{p},\mathbf{p}'}(\Lambda_0(\mathbf{p},\mathbf{p}'))^2(\psi_L(\mathbf{p})-\psi_L(\mathbf{p}'))(\psi_{L'}(\mathbf{p})-\psi_{L'}(\mathbf{p}'))\times\rho_{\mathbf{p}}(0)\rho_{\mathbf{p}'}(0)/(N^*(0))^2 \tag{3.29}$$

$$C_{LL'} = \frac{1}{8}\sum_{\mathbf{p},\mathbf{p}',\mathbf{k}}\left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-)+\frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-)\right] \times(\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-)) \times(\psi_{L'}(\mathbf{p}^+)+\psi_{L'}(\mathbf{p}'^-)-\psi_{L'}(\mathbf{p}'^+)-\psi_{L'}(\mathbf{p}^-)) \times\rho_{\mathbf{p}^+}(0)\rho_{\mathbf{p}^-}(0)\rho_{\mathbf{p}'^+}(0)\rho_{\mathbf{p}'^-}(0)/(N^*(0))^2. \tag{3.30}$$

Here, τ_i is a transport relaxation time due to the impurity scattering defined by

$$\tau_i^{-1} = 2\pi n_i\nu^2 N^*(0). \tag{3.31}$$

We see that $\hat{a}$ is a positive definite matrix and that $\hat{C}$ is a positive semidefinite matrix. This insures the non-negativity of the resistivity. Introducing a matrix, $\hat{a^\omega}$, which is related to the real part of the memory-function matrix, $\hat{M}'(\omega)$, as follows,

$$\hat{a^\omega} \equiv \hat{1} + \left[\frac{\partial}{\partial\omega}\hat{M}'(\omega)\right]_{\omega=0}, \tag{3.32}$$

we see that $\hat{a^\omega}$ can be considered to be a positive definite matrix if the assumption of the convergence of the perturbation is valid. Here and hereafter, we define $\hat{Q}^{\frac{1}{2}}$ to be a positive definite matrix which satisfies $(\hat{Q}^{\frac{1}{2}})^2 = \hat{Q}$ for an arbitrary positive definite matrix $\hat{Q}$.

In terms of $\tau_{tr}^{(j)}(\epsilon)$ and $\hat{u}_{tr}^{(j)}$ defined by the following eigenequation,

$$\hat{a^\omega}^{-\frac{1}{2}}\hat{M}''(\epsilon)\hat{a^\omega}^{-\frac{1}{2}}\hat{u}_{tr}^{(j)} = \tau_{tr}^{(j)-1}(\epsilon)\hat{u}_{tr}^{(j)}, \qquad {}^t\hat{u}_{tr}^{(j)}\hat{u}_{tr}^{(j')} = \delta_{jj'}, \tag{3.33}$$

eq. (3.27) is expressed as follows,

$$\sigma(\omega,T) = e^2\sum_{\mathbf{p},\sigma}\left(-\frac{\partial f}{\partial\epsilon^*_{\mathbf{p}}}\right)v^{*2}_{\mathbf{p}x}\left[\hat{a^\omega}^{-1}\right]_{11}\sum_{j=0}^{\infty}\frac{w_{tr}^{(j)2}}{-i\omega+1/\tau_{tr}^{(j)}(\epsilon^*_{\mathbf{p}})}, \tag{3.34}$$

where

$$w_{tr}^{(j)} \equiv \left[\hat{a^\omega}^{-\frac{1}{2}}\hat{u}_{tr}^{(j)}\right]_1 \Big/ \left[\hat{a^\omega}^{-1}\right]_{11}^{\frac{1}{2}}, \qquad \sum_{j=0}^{\infty}w_{tr}^{(j)2} = 1. \tag{3.35}$$

Eq. (3.34) shows that $\tau_{tr}^{(j)}(\epsilon)$, which is a positive quantity by definition, can be interpreted as transport relaxation time. Here, we see that in the approximation taken in the last section the transport relaxation rates, $\tau_{tr}^{(j)-1}(\epsilon)$'s, are correct up to ν^2, ϵ^2 and T^2. Especially, the transport relaxation times due to the electron-electron scattering, $\tau_{el\text{-}el}^{(j)}(\epsilon)$'s, are defined by

$$1/\tau_{el\text{-}el}^{(j)}(\epsilon) \equiv 2\pi N^*(0)(\epsilon^2+(\pi T)^2)\lambda_{el\text{-}el}^{(j)}, \tag{3.36}$$

where

$$\hat{a^\omega}^{-\frac{1}{2}}\hat{C}\hat{a^\omega}^{-\frac{1}{2}}\hat{u}_{el\text{-}el}^{(j)} = \lambda_{el\text{-}el}^{(j)}\hat{u}_{el\text{-}el}^{(j)}, \qquad {}^t\hat{u}_{el\text{-}el}^{(j)}\hat{u}_{el\text{-}el}^{(j')} = \delta_{jj'}. \tag{3.37}$$

Now, we see that, at high frequency which satisfies $\omega \gg \mathrm{Max}\left\{\tau_{tr}^{(j)-1}(\epsilon)\right\}$, eq. (3.27) leads to an expression derived from the ordinary memory-function formalism. In this case, we get

$$\sigma(\omega,T) \simeq \frac{e^2\chi_0^*\left[\hat{a^\omega}^{-1}\right]_{11}}{-i\omega+\dfrac{\left[\hat{a^\omega}^{-1}\hat{M}''(\pi T/\sqrt{3})\hat{a^\omega}^{-1}\right]_{11}}{\left[\hat{a^\omega}^{-1}\right]_{11}}}. \tag{3.38}$$

Assuming we can take the limit $\omega \to 0$ in eq. (3.38), we obtain for the resistivity $\rho_{dc}(T)$, the inverse of *dc*-conductivity $\sigma_{dc}(T) \equiv \sigma(0,T)$, up to U^2 and ν^2,

$$\rho_{dc}(T) \equiv \sigma_{dc}^{-1}(T) \simeq \left[\hat{M}''(\pi T/\sqrt{3})\right]_{11}\Big/ e^2\sum_{\mathbf{p},\sigma}\left(-\frac{\partial f}{\partial\epsilon_{\mathbf{p}}}\right)v^2_{\mathbf{p}x}. \tag{3.39}$$

It can be easily checked that eq. (3.39) is also derived from the ordinary memory-function formalism. In the absence of both Umklapp processes and the impurity scattering, the ordinary memory-function formalism generally leads to an incorrect result of non-zero resistivity even in the lowest order of U. On the other hand, in our formulation, since the memory function is extended to the matrix form, we can obtain a correct result of the absence of resistivity, as will be explicitly demonstrated in the following.

3.3 Conservation of Crystal Momentum

In this section, we would like to see how the conservation of the crystal momentum through normal processes associated with electron-electron scattering is taken into account in our formulation.

We consider the two-body electron-electron scattering process where two quasiparticles with momenta $\mathbf{p}+\mathbf{k}/2$ and $\mathbf{p}'-\mathbf{k}/2$ are scattered to states with momenta $\mathbf{p}-\mathbf{k}/2$ and $\mathbf{p}'+\mathbf{k}/2$, respectively. Introducing S_0, which is the Fermi surface in the extended zone, and $S_\pm$, which is the surface given by shifting S_0 by $\mp\mathbf{k}/2$, we can express the condition that the initial and final states of the scattering are on the Fermi surface as $\mathbf{p} \in S_+ \cap S_-$, where $\mathbf{p}$ is in the first Brillouin zone. Further, we define $S_0[0]$ and $S_0[\mathbf{G}]$ as the Fermi surface whose centers are at the origin and the reciprocal lattice vector $\mathbf{G}$, respectively, and also define $S_\pm[0]$ and $S_\pm[\mathbf{G}]$ in a similar way to the above. Then, the normal scattering, which conserves momentum in the first Brillouin zone, can be represented by $\mathbf{p},\mathbf{p}' \in S_+[0]\cap S_-[0]$. On the other hand, the Umklapp scattering with the reciprocal lattice vector $\mathbf{G}$ can be represented by $\mathbf{p} \in S_+[0]\cap S_-[0]$ and $\mathbf{p}' \in S_\pm[0]\cap S_\mp[\mathbf{G}]$. Thus, for given Fermi surface, the matrix $\hat{C}$ defined by eq.(3.30), which represents two-body electron-electron scattering processes in $\hat{M}''$, can be always separated into two, $\hat{C}_N$ and $\hat{C}_U$, the former coming from normal processes and the latter coming from Umklapp processes,

$$\hat{C} = \hat{C}_N + \hat{C}_U. \tag{3.40}$$

(In infinite-dimensional systems, this is not true as will be seen in Chap. 5.)

In the system whose Fermi surface is sufficiently small, Umklapp processes do not exist and then $\hat{C}_U = \hat{O}$. (In our formulation, we neglected the N-body scattering processes ($N > 2$) associated with Coulomb interaction, whose contributions to the relaxation rate are roughly estimated to be proportional to $T^{2(N-1)}$.) On the other hand, normal processes always exist independently of the size of the Fermi surface and generally $\hat{C}_N \neq \hat{O}$. (Two dimensions turned out to be special and there exists the case of $\hat{C}_N = \hat{O}$ depending on the shape of the Fermi surface as will be seen in Chap. 6.)

Reflecting the conservation of the crystal momentum through the normal scattering, we find from eq. (3.30) that $\hat{C}_N$ always has a zero eigenvalue,

$$\hat{C}_N\hat{\phi} = \hat{O}, \qquad \phi_L = \sum_{\mathrm{p}} \delta(\epsilon^*_{\mathrm{p}})\psi_L(\mathrm{p})p_x/N^*(0). \tag{3.41}$$

This is in contrast to the ordinary memory-function formalism where the memory function is scalar and thus the momentum conservation cannot be taken into account.

3.4 In the Absence of Momentum Dissipation Mechanism

In this section, we see, as a direct consequence of eq. (3.41), that, in the absence of both Umklapp and impurity scattering ($\nu = 0$ and $\hat{C}_U = \hat{O}$), $\sigma(\omega, T)$ diverges as $1/\omega$ in the limit $\omega \to 0$ even at finite T in spite of the fact that the total current is not conserved. Then we examine the Drude weight, the coefficient of $1/\omega$.

We introduce $\tau^{(j)}_N(\epsilon)$, the transport relaxation time due to normal processes associated with electron-electron scattering, defined by

$$1/\tau^{(j)}_N(\epsilon) \equiv 2\pi N^*(0)(\epsilon^2 + (\pi T)^2)\lambda^{(j)}_N, \tag{3.42}$$

where

$$\hat{a^\omega}^{-\frac{1}{2}}\hat{C}_N\hat{a^\omega}^{-\frac{1}{2}}\hat{u}^{(j)}_N = \lambda^{(j)}_N\hat{u}^{(j)}_N, \qquad {}^{\mathrm{t}}\hat{u}^{(j)}_N\hat{u}^{(j')}_N = \delta_{jj'}. \tag{3.43}$$

Here, we find that $\lambda^{(0)}_N = 0$ for $\hat{u}^{(0)}_N = \hat{a^\omega}^{\frac{1}{2}}\hat{\phi}/({}^{\mathrm{t}}\hat{\phi}\hat{a^\omega}\hat{\phi})^{\frac{1}{2}}$ because the conservation of the crystal momentum through the normal scattering is expressed by eq. (3.41) and that $\lambda^{(j)}_N > 0$ for $j \geq 1$ because there do not exist any conserved quantities with the symmetry of Γ_{15} except the momentum. From eq. (3.27), we then obtain

$$\begin{aligned} \sigma(\omega, T) &= e^2\sum_{\mathrm{p},\sigma}\left(-\frac{\partial f}{\partial \epsilon^*_{\mathrm{p}}}\right)v^{*2}_{\mathrm{p}x}\left[\frac{1}{-i\omega\hat{a^\omega} + 2\pi N^*(0)(\epsilon^{*2}_{\mathrm{p}} + (\pi T)^2)\hat{C}_N}\right]_{11} &(3.44)\\ &= e^2\sum_{\mathrm{p},\sigma}\left(-\frac{\partial f}{\partial \epsilon^*_{\mathrm{p}}}\right)v^{*2}_{\mathrm{p}x}\left[\hat{a^\omega}^{-1}\right]_{11}\left(\frac{w^{(0)2}_N}{-i\omega} + \sum_{j=1}^{\infty}\frac{w^{(j)2}_N}{-i\omega + 1/\tau^{(j)}_N(\epsilon^*_{\mathrm{p}})}\right), &(3.45)\end{aligned}$$

where

$$w^{(j)}_N \equiv \left[\hat{a^\omega}^{-\frac{1}{2}}\hat{u}^{(j)}_N\right]_1 / \left[\hat{a^\omega}^{-1}\right]_{11}^{\frac{1}{2}}, \qquad \sum_{j=0}^{\infty} w^{(j)2}_N = 1. \tag{3.46}$$

In the bracket of eq. (3.45), the first term reflects the conservation of the crystal momentum as mentioned above, while the existence of the second term is due to the fact that the current is determined by the group velocity, which is not proportional to the momentum.

The existence of the first term leads to the absence of T^2-term in the resistivity.

At $T = 0$, both the terms are proportional to $1/\omega$ while, at $T > 0$, only the first term is proportional to $1/\omega$ since $1/\tau^{(j)}_N(\epsilon)$'s are proportional to ϵ^2 and T^2 in the second term. Introducing "Drude weights" D^ω and D^T respectively defined by

$$D^\omega \equiv \pi \lim_{\omega\to 0}\lim_{T\to 0}\omega\mathrm{Im}\sigma(\omega, T) = e^2\chi^*_0\left[\hat{a^\omega}^{-1}\right]_{11}, \tag{3.47}$$

$$D^T \equiv \pi \lim_{T\to 0}\lim_{\omega\to 0}\omega\mathrm{Im}\sigma(\omega, T) = e^2\chi^*_0\left[\hat{a^\omega}^{-1}\right]_{11}w^{(0)2}_N, \tag{3.48}$$

we see the following inequalities,

$$D^\omega > D^T > 0. \tag{3.49}$$

D^T is explicitly given by

$$D^T = \frac{e^2\chi^*_0}{1 - \dfrac{\sum_{\mathrm{p},\mathrm{p}'}\Gamma^k(\mathrm{p},\mathrm{p}')\rho_{\mathrm{p}}(0)\rho_{\mathrm{p}'}(0)p_xp'_x}{\sum_{\mathrm{p}}\delta(\epsilon^{*2}_{\mathrm{p}})p_x^2}}\frac{\left(\sum_{\mathrm{p}}\delta(\epsilon^{*2}_{\mathrm{p}})v^*_{\mathrm{p}x}p_x\right)^2}{\left(\sum_{\mathrm{p}}\delta(\epsilon^{*2}_{\mathrm{p}})p_x^2\right)\left(\sum_{\mathrm{p}}\delta(\epsilon^{*2}_{\mathrm{p}})v^{*2}_{\mathrm{p}x}\right)}. \tag{3.50}$$

The inequality, eq. (3.49) is in contrast to the equality of the above two limits in the limiting case of continuum given by

$$D^\omega = D^T = \frac{\pi n e^2}{m}. \tag{3.51}$$

The inequality, eq. (3.49), has a simple physical interpretation. To see this, we consider a linear response of the current to the unit pulse of the electric field at $t = 0$,

$$j(t;T) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\sigma(\omega, T). \tag{3.52}$$

Here $j(t;T)$ is equal to $j_0 = e^2\chi_0$ at $t = +0$, where

$$\chi_0 \equiv \lim_{\mathrm{q}\to 0}\chi^R_{\mu\mu}(\mathrm{q}, 0), \tag{3.53}$$

which corresponds to the f-sum rule, and is expected to decrease monotonically for $t > 0$ at any temperature because the total current is not conserved. At sufficiently large t, $j(t;T)$ approaches to $j_\infty(T)$ which is given by $j_\infty(0) = D^\omega/\pi$ at absolute zero and $j_\infty(T) \simeq D^T/\pi$ at finite T. Since the current is not conserved even through the normal processes in the electron-electron scattering, the value of $j_\infty(T)$ at $T \neq 0$ is reduced from its value at $T = 0$. However, $j_\infty(T)$ cannot be zero because of the momentum conservation. (Strictly speaking, $j_\infty(T) = 0$ at $t = \infty$ since there exist Umklapp processes in the N-body scattering ($N > 2$) as noted above. However, these processes are ignored here.)

Chapter 4

Resistivity due to Mutual Coulomb Interaction in the Presence of Impurity Scattering

We will show in this chapter that, in the presence of the impurity scattering, there exists generally a finite T^2-term in the resistivity due to Coulomb interaction even in the absence of Umklapp processes in the lattice systems. This is because the current is not conserved even through the normal processes in the electron-electron scattering. This T^2 contribution in the resistivity, however, saturates as temperature gets higher because the momentum conservation processes through the electron-electron scattering become dominant.

4.1 Presence of T^2-term in the Absence of Umklapp Scattering Processes

We consider three-dimensional and two-dimensional systems in the presence of the impurity scattering. In this case, we can introduce $\tilde{\tau}^{(j)}_{el\text{-}el}(\epsilon)$ by

$$1/\tilde{\tau}^{(j)}_{el\text{-}el}(\epsilon) \equiv 2\pi N^*(0)(\epsilon^2 + (\pi T)^2)\tilde{\lambda}^{(j)}_{el\text{-}el}, \tag{4.1}$$

where

$$\hat{a}^{-\frac{1}{2}}\hat{C}\hat{a}^{-\frac{1}{2}}\hat{\tilde{u}}^{(j)}_{el\text{-}el} = \tilde{\lambda}^{(j)}_{el\text{-}el}\hat{\tilde{u}}^{(j)}_{el\text{-}el}, \qquad {}^{t}\hat{\tilde{u}}^{(j)}_{el\text{-}el}\hat{\tilde{u}}^{(j')}_{el\text{-}el} = \delta_{jj'}. \tag{4.2}$$

Here, we note that $\tilde{\tau}^{(j)}_{el\text{-}el}(\epsilon)$'s are similar quantities to the transport relaxation times due to the electron-electron scattering, $\tau^{(j)}_{el\text{-}el}(\epsilon)$'s, defined by eq. (3.36), except a difference of $\hat{a}$ and $\hat{a^{\omega}}$ in their definitions.

At sufficiently low T where $\tau_i^{-1} \gg \mathrm{Max}\left\{\tilde{\tau}^{(j)-1}_{el\text{-}el}(\pi T/\sqrt{3})\right\}$, we obtain in a similar way to the derivation of eq. (3.38)

$$\sigma_{dc}(T) = \frac{e^2\chi_0^*\left[\hat{a}^{-1}\right]_{11}}{\dfrac{1}{\tau_i} + \dfrac{8}{3}\pi N^*(0)(\pi T)^2 \dfrac{\left[\hat{a}^{-1}\hat{C}\hat{a}^{-1}\right]_{11}}{\left[\hat{a}^{-1}\right]_{11}}}. \tag{4.3}$$

Eq. (4.3) is correct whether or not Umklapp processes exist. However, our concern is to show that normal processes contribute to the resistivity once the impurity scattering is present. We thus consider the system with impurities but without Umklapp processes ($\hat{C} = \hat{C}_N$) in the following.

In a similar way to the derivation of eq. (3.45), we obtain

$$\begin{aligned}\sigma_{dc}(T) &= e^2\sum_{\mathbf{p},\sigma}\left(-\frac{\partial f}{\partial \epsilon^*_{\mathbf{p}}}\right)v^{*2}_{\mathbf{p}x}\left[\frac{1}{\tau_i^{-1}\hat{a} + 2\pi N^*(0)(\epsilon^{*2}_{\mathbf{p}} + (\pi T)^2)\hat{C}_N}\right]_{11} \\ &= e^2\sum_{\mathbf{p},\sigma}\left(-\frac{\partial f}{\partial \epsilon^*_{\mathbf{p}}}\right)v^{*2}_{\mathbf{p}x}\left[\hat{a}^{-1}\right]_{11}\tau_i\left(\tilde{w}^{(0)2}_N + \sum_{j=1}^{\infty}\tilde{w}^{(j)2}_N\frac{1}{1+\tau_i/\tilde{\tau}^{(j)}_N(\epsilon^*_{\mathbf{p}})}\right).\end{aligned} \tag{4.4}$$

Here, $\tilde{\tau}^{(j)}_N(\epsilon)$ and $\tilde{w}^{(j)}_N$ are similar quantities to $\tau^{(j)}_N(\epsilon)$ and $w^{(j)}_N$, respectively, and can be given by replacing $\hat{a^{\omega}}$ by $\hat{a}$ in eq. (3.43) as follows,

$$1/\tilde{\tau}^{(j)}_N(\epsilon) \equiv 2\pi N^*(0)(\epsilon^2 + (\pi T)^2)\tilde{\lambda}^{(j)}_N, \tag{4.5}$$

$$\tilde{w}^{(j)}_N \equiv \left[\hat{a}^{-\frac{1}{2}}\hat{\tilde{u}}^{(j)}_N\right]_1 / \left[\hat{a}^{-1}\right]^{\frac{1}{2}}_{11}, \qquad \sum_{j=0}^{\infty}\tilde{w}^{(j)2}_N = 1, \tag{4.6}$$

where

$$\hat{a}^{-\frac{1}{2}}\hat{C}_N\hat{a}^{-\frac{1}{2}}\hat{\tilde{u}}^{(j)}_N = \tilde{\lambda}^{(j)}_N\hat{\tilde{u}}^{(j)}_N, \qquad {}^{t}\hat{\tilde{u}}^{(j)}_N\hat{\tilde{u}}^{(j')}_N = \delta_{jj'}. \tag{4.7}$$

Especially, we get

$$\tilde{w}^{(0)2}_N = \frac{\hat{\phi}_1^2}{{}^{t}\hat{\phi}\hat{a}\hat{\phi}} \Big/ \left[\hat{a}^{-1}\right]_{11}, \tag{4.8}$$

where $\hat{\phi}$ is defined by eq. (3.41).

At sufficiently low T where $\tau_i^{-1} \gg \mathrm{Max}\left\{\tilde{\tau}^{(j)-1}_N(\pi T/\sqrt{3})\right\}$, the resistivity $\rho_{dc}(T)$, the inverse of $\sigma_{dc}(T)$, is given by

$$\rho_{dc}(T) = \rho_{res} + AT^2, \tag{4.9}$$

where

$$\begin{aligned}\rho_{res} &= 1/e^2\chi_0^*\left[\hat{a}^{-1}\right]_{11}\tau_i, \\ A &= \frac{8}{3}\pi^3 N^*(0)\sum_{j=1}^{\infty}\tilde{\lambda}^{(j)}_N\tilde{w}^{(j)2}_N \Big/ e^2\chi_0^*\left[\hat{a}^{-1}\right]_{11}.\end{aligned} \tag{4.10}$$

Here we note that eq. (4.9) is equivalent to eq. (4.3) since $\sum_{j=1}^{\infty}\tilde{\lambda}^{(j)}_N\tilde{w}^{(j)2}_N = \left[\hat{a}^{-1}\hat{C}\hat{a}^{-1}\right]_{11} / \left[\hat{a}^{-1}\right]_{11}$.

At sufficiently "high" temperature where $\tau_i \gg \mathrm{Max}\left(\tilde{\tau}^{(j)}_N(\pi T/\sqrt{3})\right)$, the first term in the bracket of eq. (4.4) gets dominant to $\sigma_{dc}(T)$ if $\tilde{\lambda}^{(j)}_N > 0$ for $j \geq 1$. Then, the resistivity $\rho_{dc}(T)$ is seen by eq. (4.4) to approach ρ_∞ given by

$$\begin{aligned}\rho_\infty &= 1/e^2\chi_0^*\left[\hat{a}^{-1}\right]_{11}\tau_i\tilde{w}^{(0)2}_N \\ &= \rho_{res}/\tilde{w}^{(0)2}_N.\end{aligned} \tag{4.11}$$

We define α by $\rho_\infty = (1+\alpha)\rho_{res}$.

We see that $\tilde{w}^{(0)2}_N < 1$ because the group velocity, $v^*_{\mathbf{p}\mu}$, is not proportional to the momentum, p_μ, in the lattice systems and that $\tilde{\lambda}^{(j)}_N > 0$ for $j \geq 1$ unless $\hat{C}_N = \hat{O}$ because there do not exist any conserved quantities with the symmetry of Γ_{15} except the momentum in normal processes

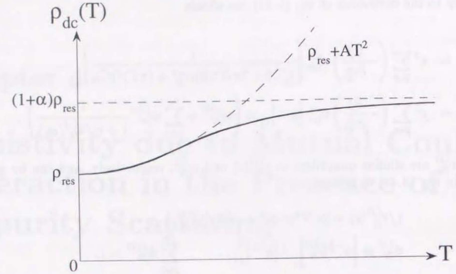

Figure 4.1: Schematic representation of the temperature dependence of $\rho_{dc}(T)$ in three-dimensional systems in the absence of Umklapp processes.

associated with two-body electron-electron scattering. Then, we get $A > 0$ and $\alpha > 0$. In contrast to the general belief, there exists a finite T^2-term in the resistivity even in the absence of the Umklapp scattering processes once the impurity scattering is present. However, this temperature dependence saturates as the temperature gets high. This is very different from the case with the Umklapp scattering.

In three-dimensional systems, $\hat{C}_N$ is not equal to zero as a matrix. Then the temperature dependence of $\rho_{dc}(T)$ is schematically shown in Fig. 4.1. This is a special case of the breakdown of the Matthiessen's rule.

In two-dimensional systems, there exists, however, a special case of $\hat{C}_N = \hat{O}$ in dependence on the shape of the Fermi surface. Then, we have classified cases depending on the shape of the Fermi surface by an integer Z, where Z is the maximum number of the common points of two Fermi surfaces relatively shifted by the transferred momentum **k** and tangent to each other. Examples of $Z = 1$ and $Z = 2$ are shown in Fig. 4.2. As will be shown in Chap. 6, in the case of the Fermi surface with $Z = 1$, $\hat{C}_N = \hat{O}$ and, in the case of the Fermi surface with $Z = 2$, $\hat{C}_N$ is not equal to zero as a matrix and does not have a logarithmic singularity which exists in the coefficient of T^2-term in the damping rate of quasiparticle γ. Therefore, we see by eq. (4.4) that , in the case of $Z = 2$, T^2-term in the resistivity results and this T-dependence saturates as temperature gets high in a similar way to the case of three dimensions. On the other hand, in the case of $Z = 1$, there do not exist such T-dependences even in the presence of the impurity scattering. The situation of the case of the small Fermi surface with $Z \geq 3$ is not certain at present and will be left to the future problem, though the realization of this case in actual system will be rare.

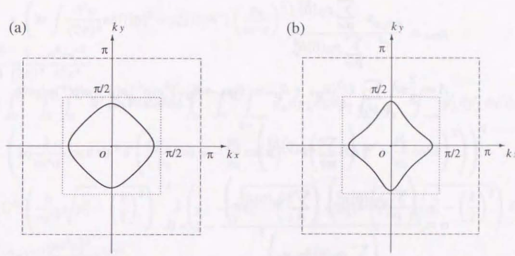

Figure 4.2: Fermi surfaces in two dimensions. (a)$Z = 1$ (b)$Z = 2$

4.2 Three-Dimensional System at Low Electron Number Density

In the last section, we have seen that, in three-dimensional systems with impurities but without Umklapp processes, the resistivity, $\rho_{dc}(T)$, is given by $\rho_{res} + AT^2$ at sufficiently low temperature, and $\rho_{dc}(T)$ approaches $\rho_\infty = (1+\alpha)\rho_{res}$ as temperature gets high. In this section, we would like to make an explicit calculation of A and α in the lowest order of U for a three-dimensional system at the low electron number density in the absence of Umklapp processes. We will also touch upon the case of two dimensions.

We consider the second-order processes with respect to ν and U shown in Fig. 3.1 in the proper four-point vertex part. In this case, the imaginary part of the memory-function matrix, $\hat{M}''(\epsilon)$, is given as follows,

$$\hat{M}''(\epsilon) = \tau_i^{-1}\hat{a} + 2\pi N^*(0)(\epsilon^2 + (\pi T)^2)\hat{C}, \tag{4.12}$$

where

$$a_{LL'} = \frac{1}{2}\sum_{\mathbf{p},\mathbf{p}'} \rho_{\mathbf{p}}(0)\rho_{\mathbf{p}'}(0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})/(N^*(0))^2, \tag{4.13}$$

$$\begin{aligned} C_{LL'} = \frac{1}{8}\sum_{\mathbf{p},\mathbf{p}',\mathbf{k}} & U^2(\psi_L(\mathbf{p}^+) + \psi_L(\mathbf{p}'^-) - \psi_L(\mathbf{p}'^+) - \psi_L(\mathbf{p}^-)) \\ & \times (\psi_{L'}(\mathbf{p}^+) + \psi_{L'}(\mathbf{p}'^-) - \psi_{L'}(\mathbf{p}'^+) - \psi_{L'}(\mathbf{p}^-)) \\ & \times \rho_{\mathbf{p}^+}(0)\rho_{\mathbf{p}^-}(0)\rho_{\mathbf{p}'^+}(0)\rho_{\mathbf{p}'^-}(0)/(N^*(0))^2. \end{aligned} \tag{4.14}$$

Since $\hat{a}^{-1}$ is given by

$$\left[\hat{a}^{-1}\right]_{LL'} = \sum_{\mathbf{p}} \delta(\epsilon^*_{\mathbf{p}})\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})a_{\mathbf{p}}^{-1}/N(0), \tag{4.15}$$

we obtain

$$\rho_{res} = \frac{\pi n_i \nu^2}{e^2} \frac{\sum\limits_{\mathbf{p},\sigma} \rho_{\mathbf{p}}(0)}{\sum\limits_{\mathbf{p},\sigma} \rho_{\mathbf{p}}(0) \tilde{v}_{\mathbf{p}}^2}, \tag{4.16}$$

$$A = \frac{2}{3}\pi^3 \sum_{\mathbf{p},\mathbf{p}',\mathbf{k}} U^2 (\tilde{v}_{\mathbf{p}+} + \tilde{v}_{\mathbf{p}'-} - \tilde{v}_{\mathbf{p}'+} - \tilde{v}_{\mathbf{p}-})^2 \rho(p^+)\rho(p^-)\rho(p'^+)\rho(p'^-) \times \left(e \sum_{\mathbf{p},\sigma} \rho_{\mathbf{p}}(0) \tilde{v}_{\mathbf{p}}^2 \right)^{-2}, \tag{4.17}$$

$$\alpha = \frac{\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) p_x^2\right)\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \tilde{v}_{\mathbf{p}x}^2\right)}{\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \tilde{v}_{\mathbf{p}x} p_x\right)^2} - 1 > 0. \tag{4.18}$$

Here, we noted $\tilde{v}_{\mathbf{p}} = a_{\mathbf{p}}^{-1} v_{\mathbf{p}}^*$.

The important point to note is the existence of a small but finite deviation of the Fermi surface from a spherical surface, $\epsilon_{\mathbf{p}}^* = |\mathbf{p}|^2/2m^*$. Then, we consider a special model where $\tilde{v}_{\mathbf{p}\mu}$ can be approximated by

$$\tilde{v}_{\mathbf{p}\mu} = \frac{p_\mu}{m^* a} + \eta \left(\frac{p_\mu}{p_0}\right)^2 \frac{p_\mu}{m^* a}. \tag{4.19}$$

where η is a small parameter and p_0 is a scale of the Fermi momentum. We thus get

$$\tilde{v}_{\mathbf{p}+\mu} + \tilde{v}_{\mathbf{p}'-\mu} - \tilde{v}_{\mathbf{p}'+\mu} - \tilde{v}_{\mathbf{p}-\mu} = 3\eta \frac{k_\mu}{m^* a} \frac{p_\mu^2 - p_\mu'^2}{p_0^2}. \tag{4.20}$$

Here, we define the following coordinates,

$$\mathbf{k} = k\mathbf{e}_{\parallel}, \tag{4.21}$$

$$\mathbf{p} = p_\perp (\cos\varphi \mathbf{e}_{\perp 1} + \sin\varphi \mathbf{e}_{\perp 2}) + p_\parallel \mathbf{e}_\parallel, \tag{4.22}$$

$$\mathbf{p}' = p'_\perp (\cos\varphi' \mathbf{e}_{\perp 1} + \sin\varphi' \mathbf{e}_{\perp 2}) + p'_\parallel \mathbf{e}_\parallel, \tag{4.23}$$

where

$$\mathbf{e}_\parallel = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{4.24}$$

$$\mathbf{e}_{\perp 1} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{4.25}$$

$$\mathbf{e}_{\perp 2} = (-\sin\phi, \cos\phi, 0). \tag{4.26}$$

Since the right hand side of eq. (4.20) is already proportional to η, by eq. (4.17), we obtain A in the second order of U and in the leading order of η as follows,

$$\begin{aligned}
A &= \frac{2}{3}\pi^3 \int \frac{\mathrm{d}^3 p \mathrm{d}^3 p' \mathrm{d}^3 k}{(2\pi)^9} a^4 (\tilde{v}_{\mathbf{p}+z} + \tilde{v}_{\mathbf{p}'-z} - \tilde{v}_{\mathbf{p}'+z} - \tilde{v}_{\mathbf{p}-z})^2 \\
&\quad \times U^2 \delta((|\mathbf{p}^+|^2 - p_0^2)/2m^*)\delta((|\mathbf{p}'^-|^2 - p_0^2)/2m^*)\delta((|\mathbf{p}'^+|^2 - p_0^2)/2m^*)\delta((|\mathbf{p}^-|^2 - p_0^2)/2m^*) \\
&\quad \times \left(2e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a\delta((|\mathbf{p}|^2 - p_0^2)/2m^*) \left(\frac{p_z}{m^* a}\right)^2 \right)^{-2} \\
&= \frac{\pi^2}{3} \frac{1}{(2\pi)^8} \frac{a^6 m^{*2}}{n^2 e^2} \\
&\quad \times \int_0^{2p_0} \int_0^\pi \int_0^{2\pi} k^2 \sin\theta \mathrm{d}k \mathrm{d}\theta \mathrm{d}\phi \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty p_\perp \mathrm{d}p_\perp \mathrm{d}\varphi \mathrm{d}p_\parallel \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty p'_\perp \mathrm{d}p'_\perp \mathrm{d}\varphi' \mathrm{d}p'_\parallel \\
&\quad \times \left(3\eta \frac{k}{m^* a} \cos\theta \sin^2\theta \left(\left(\frac{p_\perp}{p_0}\cos\varphi - \frac{p_\parallel}{p_0}\cot\theta\right)^2 - \left(\frac{p'_\perp}{p_0}\cos\varphi' - \frac{p'_\parallel}{p_0}\cot\theta\right)^2 \right) \right)^2 \\
&\quad \times U^2 \left(\frac{k}{m^{*2}} \sqrt{p_0^2 - \left(\frac{k}{2}\right)^2} \right)^{-2} \delta\left(p_\perp - \sqrt{p_0^2 - \left(\frac{k}{2}\right)^2}\right) \delta(p_\parallel) \delta\left(p'_\perp - \sqrt{p_0^2 - \left(\frac{k}{2}\right)^2}\right) \delta(p'_\parallel) \\
&= \frac{9}{2}(2\pi)^{-3} \frac{(m^* a)^4}{ne^2} \eta^2 U^2 \\
&\quad \times \int_0^{2p_0} \frac{\mathrm{d}k}{2p_0} \left(\frac{k}{2p_0}\right)^2 \left[1 - \left(\frac{k}{2p_0}\right)^2\right]^2 \int_0^\pi \mathrm{d}\theta \sin^5\theta \cos^2\theta \int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\varphi \mathrm{d}\varphi' (\cos^2\varphi - \cos^2\varphi')^2 \\
&= \frac{8}{\pi} \left(\frac{\eta}{35}\right)^2 U^2 \frac{(m^* a)^4}{ne^2},
\end{aligned} \tag{4.27}$$

where $n = p_0^3/3\pi^2$. This result will be interpreted from Fig. 4.3 where S_0 is the Fermi surface and $S_\pm$ is the surface given by shifting S_0 by $\mp\mathbf{k}/2$. Here, the condition that the initial and final states are on S_0 in the process of two-body scattering can be simply represented by the condition that $\mathbf{p}$ and $\mathbf{p}'$ are on the intersection of S_+ and S_-, as already mentioned in Sec. 3.3. Since the intersection is a closed loop, we introduce angle variables φ and φ', $0 \le \varphi, \varphi' < 2\pi$, to parameterize this loop. Then, $\tilde{v}_{\mathbf{p}}$ is conserved in special processes of normal scattering corresponding to $\varphi' = \varphi$ or $\varphi' = \varphi + \pi$, but not conserved in general processes, and therefore a finite T^2-term in the resistivity results.

We note that, if the same problem is considered in two dimensions, all processes corresponds to either $\varphi' = \varphi$ or $\varphi' = \varphi + \pi$ because of the restriction of the phase space. In this case, therefore, T^2-term in the resistivity does not result as mentioned in Sec. 4.1.

On the other hand, from eq. (4.18), α is given by

$$\alpha = \frac{\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \left(\frac{p_z}{m^* a}\right)^2\right)\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \left(\tilde{v}_{\mathbf{p}z} - \frac{p_z}{m^* a}\right)^2\right) - \left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \left(\tilde{v}_{\mathbf{p}z} - \frac{p_z}{m^* a}\right) \frac{p_z}{m^* a}\right)^2}{\left(\sum\limits_{\mathbf{p}} \rho_{\mathbf{p}}(0) \tilde{v}_{\mathbf{p}z} \frac{p_z}{m^* a}\right)^2}. \tag{4.28}$$

Introducing the quantities, $P^{(n)}$, defined by

$$P^{(n)} \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a\delta\left(\frac{|\mathbf{p}|^2}{2} - \frac{p_0^2}{2}\right)\left(\frac{p_z}{p_0}\right)^{2n} \tag{4.29}$$

$$= \frac{1}{2(2n+1)\pi^2} p_0, \tag{4.30}$$

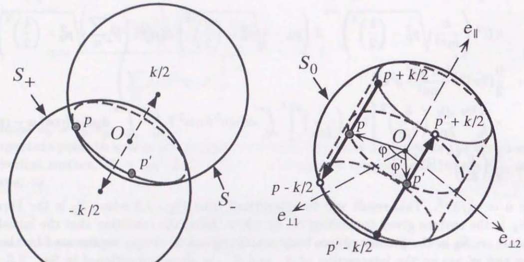

Figure 4.3: Normal scattering process in three dimensions. The left figure represents the condition that the initial and final states in the process are on the Fermi surface. The right figure represents one of the possible processes. The black and white circles correspond to the initial and final states, respectively.

since $\tilde{v}_{pz} - p_z/m^*a$ is already proportional to η, we obtain up to the leading order in η,

$$\alpha = \eta^2 \left(P^{(1)}P^{(3)} - P^{(2)2}\right) / P^{(1)2} \tag{4.31}$$

$$= 84 \left(\frac{\eta}{35}\right)^2 . \tag{4.32}$$

Chapter 5

High-Dimensional Systems

In this chapter and next, we consider the resistivity due only to mutual Coulomb interaction in the absence of the impurity scattering.

It is well known that the resistivity of the infinite dimensional system[15-17] is determined only by the damping of quasiparticle, γ, because the vertex corrections for the current operator vanish and therefore there exists only the self-energy corrections.[14] On the other hand, as seen in Chap. 3, the consistency between the selfenergy and vertex corrections is indispensable in order to implement conservation of crystal momentum. This insures that the resistivity vanishes in the absence of Umklapp processes. In this chapter, we investigate the resistivity of high-dimensional systems by our formulation in terms of the memory-function matrix which respects the consistency between the selfenergy and vertex corrections.

We consider the model with the nearest-neighbor hopping on the d-dimensional hypercubic lattice whose kinetic energy is given by (the band center is taken as the origin of energy)

$$\epsilon_{\mathbf{p}} = -2t\sum_{\mu=1}^{d}\cos p_\mu, \tag{5.1}$$

where we put $2dt^2 = t^{*2}$ and regard t^* as a quantity of the order of 1 in high-dimensions ($d \to \infty$). In the case of $d = \infty$, we can consider that the Fermi surface does not change by U since the selfenergy is independent of momentum.[13] Then, one can expect Umklapp scattering processes when the absolute value of the chemical potential of the noninteracting system, μ_0, is smaller than that of the band energy at the momentum, $\mathbf{p} = (\pi/2, 0, \cdots, 0)$,

$$|\mu_0| \le 2t(d-1) = \sqrt{2}t^*\frac{(d-1)}{\sqrt{d}}. \tag{5.2}$$

Hence, in the limit of $d \to \infty$, the Umklapp processes always exist.[17] The proportionality of the resistivity to γ in $d = \infty$ may imply that normal processes generally contribute to the resistivity once Umklapp scattering processes are present. This is because γ results not only from Umklapp scattering processes but also from normal scattering processes, although the separation into normal processes and Umklapp processes is no longer visible in $d \to \infty$. This is another aspect of the breakdown of the Matthissen's rule.

The vertex corrections are considered to partially recover the momentum conservation and their contributions turns out to be proportional to $1/d$, which will be evaluated in the lowest order of U. Here, the dc-conductivity is rescaled as $\tilde\sigma_{dc}(T) = d\sigma_{dc}(T)$ and is given from eq. (3.27) as follows,

$$\tilde\sigma_{dc}(T) = 2de^2\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\left(-\frac{\partial f}{\partial\epsilon^*_{\mathbf{p}}}\right)\frac{v^{*2}_{\mathbf{p}}}{2\pi N^*(0)(\epsilon^{*2}_{\mathbf{p}}+(\pi T)^2)}\left[\hat C^{-1}\right]_{11} \tag{5.3}$$

$$\simeq e^2\tilde\chi^*_0\left[\hat C^{-1}\right]_{11}\Big/\frac{8}{3}\pi(\pi T)^2N^*(0), \tag{5.4}$$

where $\tilde\chi^*_0 = d\chi^*_0$ and we have evaluated $\epsilon^{*2}_{\mathbf{p}}$ in the denominator as $(\pi T)^2/3$. Then the matrix $\hat C$ can be separated up to U^2 as follows,

$$\begin{aligned} C_{LL'} &= \frac{1}{8}\sum_i\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp_1\mathrm{d}^dp_2\mathrm{d}^dp_3\mathrm{d}^dp_4}{(2\pi)^{4d}}U^2\left(\psi_L(\mathbf{p}_1)+\psi_L(\mathbf{p}_2)-\psi_L(\mathbf{p}_3)-\psi_L(\mathbf{p}_4)\right)\\ &\quad\times\left(\psi_{L'}(\mathbf{p}_1)+\psi_{L'}(\mathbf{p}_2)-\psi_{L'}(\mathbf{p}_3)-\psi_{L'}(\mathbf{p}_4)\right)\rho_{\mathbf{p}_1}(0)\rho_{\mathbf{p}_2}(0)\rho_{\mathbf{p}_3}(0)\rho_{\mathbf{p}_4}(0)\\ &\quad\times e^{i(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\cdot\mathbf{r}_{ij}}/(N^*(0))^2 \end{aligned} \tag{5.5}$$

$$\begin{aligned} &= \frac{1}{2}U^2(N^*(0))^{-2}\sum_i(\rho_{ij}(0))^3\left(\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\rho_{\mathbf{p}}(0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})\cos(\mathbf{p}\cdot\mathbf{r}_{ij})\right)\\ &\quad-\frac{3}{2}U^2(N^*(0))^{-2}\sum_i(\rho_{ij}(0))^2\left(\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\rho_{\mathbf{p}}(0)\psi_L(\mathbf{p})\sin(\mathbf{p}\cdot\mathbf{r}_{ij})\right)\\ &\quad\times\left(\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\rho_{\mathbf{p}}(0)\psi_{L'}(\mathbf{p})\sin(\mathbf{p}\cdot\mathbf{r}_{ij})\right). \end{aligned} \tag{5.6}$$

Here, we used $\psi_L(\mathbf{p}) = -\psi_L(-\mathbf{p})$ and $\rho_{ij}(\epsilon)$ is the spectral function in the site space defined by

$$\rho_{ij}(\epsilon) \equiv \int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\rho(p)e^{i\mathbf{p}\cdot\mathbf{r}_{ij}}, \tag{5.7}$$

whose leading term in the $1/d$-expansion is proportional to $d^{-L_{ij}/2}$ where $L_{ij} = \sum_{\nu=1}^{d}|n_\nu|$ for $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j \equiv (n_1, n_2, \cdots, n_\nu, \cdots, n_d)$, as given in Appendix C. The second term in eq. (5.6) corresponds to the vertex corrections and is the negative definite matrix. This leads to the fact that the vertex corrections decrease T^2-term in the resistivity.

We can neglect all effects of renormalizations since we consider the T^2-term in the lowest order of U. Then, we note

$$\psi_1(\mathbf{p}) = \sin p_\mu\Big/\left(\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\sin^2p_\mu\delta(\epsilon_{\mathbf{p}}-\mu_0)\Big/\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}}-\mu_0)\right)^{\frac{1}{2}}. \tag{5.8}$$

Since $\rho_{ij}(\epsilon) \propto d^{-L_{ij}/2}$, it is convenient to perform the site sum in eq. (5.6) as the sum over L_{ij} as

$$\hat C = \sum_{L_{ij}}\hat C^{(L_{ij})} = \hat C^{(0)} + \hat C^{(1)} + \cdots. \tag{5.9}$$

Then, we obtain

$$\begin{aligned} C^{(0)}_{LL'} &= \frac{1}{2}U^2\left(N^{(d)}(0)\right)\left(\int_{-\pi}^{\pi}\frac{\mathrm{d}^dp}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}}-\mu_0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p})\right)\\ &= \frac{1}{2}U^2\left(N^{(d)}(0)\right)^2\delta_{LL'}. \end{aligned} \tag{5.10}$$

Note that $\hat C^{(0)}$ has no contribution from the second term in eq. (5.6). This corresponds to the absence of the vertex corrections in $d = \infty$. Here, $N^{(d)}(\epsilon)$ is the density of states for given spin in

d-dimensional system given by

$$
\begin{aligned}
N^{(d)}(\epsilon) &= \rho_{ii}(\epsilon) && (5.11)\\
&= \int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d} \delta(\epsilon + \mu_0 - \epsilon_{\mathbf{p}}) && (5.12)\\
&= \frac{1}{\sqrt{2\pi t^{*2}}} \exp\left(-\frac{1}{2}\left(\frac{\epsilon+\mu_0}{t^*}\right)^2 - \frac{1}{16d}\left(3 - 6\left(\frac{\epsilon+\mu_0}{t^*}\right)^2 + \left(\frac{\epsilon+\mu_0}{t^*}\right)^4\right) + O\left(\frac{1}{d^2}\right)\right). && (5.13)
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
C^{(1)}_{LL'} &= \frac{1}{2}U^2 \left(\rho_{<i,j>}(0)\right)^3 \left(N^{(d)}(0)\right)^{-2} \left(\int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}} - \mu_0)\psi_L(\mathbf{p})\psi_{L'}(\mathbf{p}) 2\sum_{\mu=1}^{d} \cos p_\mu\right)\\
&\quad -\frac{3}{2}U^2 \left(\rho_{<i,j>}(0)\right)^2 \left(N^{(d)}(0)\right)^{-2} 2\sum_{\mu=1}^{d}\left(\int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}} - \mu_0)\psi_L(\mathbf{p}) \sin p_\mu\right)\\
&\quad \times \left(\int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}} - \mu_0)\psi_{L'}(\mathbf{p}) \sin p_\mu\right)\\
&= \frac{1}{2}U^2 \left(-\frac{\mu_0}{t}\right) \left(\rho_{<i,j>}(0)\right)^3 \left(N^{(d)}(0)\right)^{-1} \delta_{LL'}\\
&\quad -\frac{3}{4}U^2 \left(\rho_{<i,j>}(0)\right)^2 \left(N^{(d)}(0)\right)^{-1} \tilde{\chi}_0^{(d)} t^{*-2} \delta_{L1}\delta_{1L'}, \qquad (5.14)
\end{aligned}
$$

where we have used eqs. (5.1) and (5.8) for the second equality and the orthogonality relation of the Fermi surface harmonics. Here, $< i, j >$ means the nearest neighbors and we get

$$
\rho_{\langle i,j\rangle}(0) = -\frac{\mu_0}{\sqrt{2d t^{*2}}} N^{(\infty)}(0) + O(d^{-3/2}). \qquad (5.15)
$$

In eq. (5.14), $\tilde{\chi}_0^{(d)}$ is defined by

$$
\begin{aligned}
\tilde{\chi}_0^{(d)} &\equiv \int_{-\pi}^{\pi} \frac{\mathrm{d}^d p}{(2\pi)^d} 4t^{*2} \sin^2 p_\mu \delta(\epsilon_{\mathbf{p}} - \mu_0) && (5.16)\\
&= 2t^{*2} N^{(d)}(0)\left(1 + \frac{1}{4d}\left(1 - \left(\frac{\mu_0}{t^*}\right)^2\right) + O\left(\frac{1}{d^2}\right)\right). && (5.17)
\end{aligned}
$$

Since $\hat{C}^{(2)}$ is proportional to $1/d^2$, we conclude that the off-diagonal elements of $C_{LL'}$ are of the order of $1/d^2$. On the other hand, the diagonal elements of $C_{LL'}$ has terms of the order of $1/d$ in addition to that of the order of 1. Thus, by substituting eqs. (5.10) and (5.14) into eq. (5.9), we obtain

$$
\begin{aligned}
C_{LL'} &= \frac{1}{2}U^2 \left(N^{(\infty)}(0)\right)^2 \left[\left\{1 + \frac{1}{d}\frac{3}{8}\left(\left(\frac{\mu_0}{t^*}\right)^4 + 2\left(\frac{\mu_0}{t^*}\right)^2 - 1\right)\right\}\delta_{LL'}\right.\\
&\quad \left. - \frac{1}{d}\frac{3}{2}\left(\frac{\mu_0}{t^*}\right)^2 \delta_{L1}\delta_{1L'} + O\left(\frac{1}{d^2}\right)\right] \qquad (5.18)
\end{aligned}
$$

Inserting eqs. (5.17) and (5.18) into eq. (5.4), we get T^2-term in the resistivity given as follows

$$
\tilde{\rho}_{dc}(T) \equiv \tilde{\sigma}_{dc}^{-1}(T) = \tilde{A}T^2, \qquad (5.19)
$$

$$
\begin{aligned}
\tilde{A} &= \frac{8}{3}\pi^3 N^{(d)}(0) C_{11} \Big/ e^2 \tilde{\chi}_0^{(d)} + O\left(\frac{1}{d^2}\right)\\
&= \frac{1}{3}\left(\frac{\pi U}{e t^{*2}}\right)^2 \exp\left[-\left(\frac{\mu_0}{t^*}\right)^2 - \frac{1}{d}\left(\frac{5}{8} + \frac{1}{2}\left(\frac{\mu_0}{t^*}\right)^2 - \frac{3}{8}\left(\frac{\mu_0}{t^*}\right)^4\right) + O\left(\frac{1}{d^2}\right)\right]. \qquad (5.20)
\end{aligned}
$$

Chapter 6

Two-Dimensional Systems

In two-dimensional systems without impurities, it has been pointed out by Fujimoto *et al.*[18] that the resistivity is proportional to T^2 even though the damping rate of the quasi-particle, γ, is proportional to $T^2 \log T$, indicating the different temperature dependences between the two. They claimed that this is because normal processes, which give the $T^2 \log T$ contribution to the damping rate of quasiparticle γ, do not contribute to the resistivity. In the previous chapter, however, we observed that, in the case of a general Fermi surface different from sphere, normal processes generally contribute to the resistivity in the presence of Umklapp processes. Once one realizes this fact it is interesting to ask what is the effect of normal processes on the resistivity of two-dimensional systems in the presence of Umklapp processes.

6.1 Damping Rate of Quasiparticle and Transport Relaxation Rates

First, we investigate two-body scattering processes which give $T^2 \log T$ contribution to γ. Here, we expand γ, which is defined by the average of the damping rate of quasiparticle, $\gamma_{\mathbf{p}}$, over the Fermi surface, up to T^2 as follows,

$$\begin{aligned}
\gamma &\equiv \sum_{\mathbf{p}} \left(-\frac{\partial f}{\partial \epsilon}\right)_{\epsilon=\epsilon^*_{\mathbf{p}}} \gamma_{\mathbf{p}} \Big/ \sum_{\mathbf{p}} \left(-\frac{\partial f}{\partial \epsilon}\right)_{\epsilon=\epsilon^*_{\mathbf{p}}} \\
&\simeq (\pi T)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}^2 p \mathrm{d}^2 p' \mathrm{d}^2 k}{(2\pi)^6} \pi \left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+, \mathbf{p}'^-; \mathbf{p}'^+, \mathbf{p}^-) + \frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+, \mathbf{p}'^-; \mathbf{p}'^+, \mathbf{p}^-)\right] \\
&\quad \times a_{\mathbf{p}^+} a_{\mathbf{p}'^-} a_{\mathbf{p}'^+} a_{\mathbf{p}^-} \delta(\epsilon^*_{\mathbf{p}^+})\delta(\epsilon^*_{\mathbf{p}'^-})\delta(\epsilon^*_{\mathbf{p}'^+})\delta(\epsilon^*_{\mathbf{p}^-}) \Big/ 6N^*(0) \\
&= (\pi T)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}^2 k}{(2\pi)^6} \sum_{\mathbf{p},\mathbf{p}' \in S_+ \cap S_-} \pi \left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+, \mathbf{p}'^-; \mathbf{p}'^+, \mathbf{p}^-) + \frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+, \mathbf{p}'^-; \mathbf{p}'^+, \mathbf{p}^-)\right] \\
&\quad \times \frac{a_{\mathbf{p}^+} a_{\mathbf{p}'^-} a_{\mathbf{p}'^+} a_{\mathbf{p}^-}}{|v^*_{\mathbf{p}^+ x} v^*_{\mathbf{p}^- y} - v^*_{\mathbf{p}^- x} v^*_{\mathbf{p}^+ y}||v^*_{\mathbf{p}'^+ x} v^*_{\mathbf{p}'^- y} - v^*_{\mathbf{p}'^- x} v^*_{\mathbf{p}'^+ y}|} \Big/ 6N^*(0),
\end{aligned} \tag{6.1}$$

where we note that $\mathbf{p}$ and $\mathbf{p}'$, which are common points of $S_\pm$ in the first Brillouin zone, are the vector functions of the transferred momentum $\mathbf{k}$. Here, we see that the normal scattering can be represented by $\mathbf{p}, \mathbf{p}' \in S_+[0] \cap S_-[0]$ and that the Umklapp scattering with the reciprocal lattice vector $\mathbf{G}$ can be represented by $\mathbf{p} \in S_+[0] \cap S_-[0]$ and $\mathbf{p}' \in S_\pm[0] \cap S_\mp[\mathbf{G}]$. The definitions $S_\pm$, $S_\pm[0]$ and $S_\pm[\mathbf{G}]$ have been already given in Sec. 3.3.

In eq. (6.1), $|v^*_{\mathbf{p}^+ x} v^*_{\mathbf{p}^- y} - v^*_{\mathbf{p}^- x} v^*_{\mathbf{p}^+ y}|$ and $|v^*_{\mathbf{p}'^+ x} v^*_{\mathbf{p}'^- y} - v^*_{\mathbf{p}'^- x} v^*_{\mathbf{p}'^+ y}|$ are Jacobians of δ-functions which represent the fact that the initial and final states in two-body scattering are on the Fermi surface. These Jacobians are equal to the areas of the parallelograms set up by the vectors, $\mathbf{v}^*_{\mathbf{p}^+}$ and $\mathbf{v}^*_{\mathbf{p}^-}$, and the vectors, $\mathbf{v}^*_{\mathbf{p}'^+}$ and $\mathbf{v}^*_{\mathbf{p}'^-}$, respectively, and thus are equal to 0 in the following cases: (1)S_+ and S_- coincide with each other, that is, $\mathbf{k} = 0$; (2)S_+ and S_- are tangent to each other at $\mathbf{p} = \mathbf{p}_0$ and $\mathbf{p}' = \mathbf{p}'_0$ for $\mathbf{k} = \mathbf{k}_0$. We can consider that the integrand in eq. (6.1) does not have other singularities except the contributions from these Jacobians, by the assumption of the Fermi liquid. Let us examine the behaviors of $|v^*_{\mathbf{p}^+ x} v^*_{\mathbf{p}^- y} - v^*_{\mathbf{p}^- x} v^*_{\mathbf{p}^+ y}|$ and $|v^*_{\mathbf{p}'^+ x} v^*_{\mathbf{p}'^- y} - v^*_{\mathbf{p}'^- x} v^*_{\mathbf{p}'^+ y}|$ around their zero points in the above cases(1) and (2), respectively. The former corresponds to the normal scattering processes with small momentum transfer and the latter belongs to the scattering with large momentum transfer including both Umklapp processes and normal processes.

For small momentum transfer, we consider the vicinity of singularity of the integrand in eq. (6.1). Here, we assume S_0 can be approximated by the circle of some finite curvature around $\mathbf{p}$, which satisfies the condition that $\mathbf{v}^*_{\mathbf{p}}$ is perpendicular to $\mathbf{k}$. Then, we can evaluate for small transferred momentum $\mathbf{k}$ as

$$|v^*_{\mathbf{p}^+ x} v^*_{\mathbf{p}^- y} - v^*_{\mathbf{p}^- x} v^*_{\mathbf{p}^+ y}| \propto \sqrt{k_x^2 + k_y^2}. \tag{6.2}$$

In a similar way, we can also evaluate

$$|v^*_{\mathbf{p}'^+ x} v^*_{\mathbf{p}'^- y} - v^*_{\mathbf{p}'^- x} v^*_{\mathbf{p}'^+ y}| \propto \sqrt{k_x^2 + k_y^2}. \tag{6.3}$$

Thus, we find that this region of the integral gives a logarithmic singularity to γ as follows,

$$\int\int_{k_x^2 + k_y^2 < \Lambda_c^2} \frac{\mathrm{d}k_x \mathrm{d}k_y}{k_x^2 + k_y^2}. \tag{6.4}$$

Here Λ_c is momentum cut-off. This leads to $T^2 \log T$-term in γ.

For large momentum transfer, we consider the vicinity of singularity of the integrand in eq. (6.1). Here, we assume S_+ and S_- can be approximated around $\mathbf{p}_0$ by the circles with different curvatures in general. Then, we can evaluate for small $\tilde{\mathbf{k}} = \mathbf{k} - \mathbf{k}_0$ as

$$|v^*_{\mathbf{p}^+ x} v^*_{\mathbf{p}^- y} - v^*_{\mathbf{p}^- x} v^*_{\mathbf{p}^+ y}| \propto \sqrt{\cos\theta \tilde{k}_x + \sin\theta \tilde{k}_y}, \tag{6.5}$$

where θ is the angle of the vector which turns from center of curvature in the side of S_- to that in the side of S_+, and which is parallel to $\mathbf{v}^*_{\mathbf{p}_0 \pm \mathbf{k}_0/2}$.(See Fig. 6.1) In a similar way, we can also evaluate

$$|v^*_{\mathbf{p}'^+ x} v^*_{\mathbf{p}'^- y} - v^*_{\mathbf{p}'^- x} v^*_{\mathbf{p}'^+ y}| \propto \sqrt{\cos\theta' \tilde{k}_x + \sin\theta' \tilde{k}_y}. \tag{6.6}$$

Thus, we can evaluate the contributions to γ from the vicinity of this singularity of the integrand,

$$\int\int_{\Omega} \frac{\mathrm{d}\tilde{k}_x \mathrm{d}\tilde{k}_y}{\sqrt{(\cos\theta \tilde{k}_x + \sin\theta \tilde{k}_y)(\cos\theta' \tilde{k}_x + \sin\theta' \tilde{k}_y)}}, \tag{6.7}$$

where Ω is the region which satisfies $-\pi - k_{0x} < \tilde{k}_x \le \pi - k_{0x}$, $-\pi - k_{0y} < \tilde{k}_y \le \pi - k_{0y}$, $0 < \cos\theta \tilde{k}_x + \sin\theta \tilde{k}_y < \Lambda_c$ and $0 < \cos\theta' \tilde{k}_x + \sin\theta' \tilde{k}_y < \Lambda_c$. Here, we can classify normal processes or Umklapp processes by the condition that $\tan\theta \tan\theta' > 0$ or $\tan\theta \tan\theta' < 0$, respectively. Thus, Umklapp processes do not give a logarithmic singularity to γ. But normal processes with large momentum transfer with $\theta = \theta'$ give a logarithmic singularity to γ as follows,

$$\int\int_{\Omega} \frac{\mathrm{d}\tilde{k}_x \mathrm{d}\tilde{k}_y}{\cos\theta \tilde{k}_x + \sin\theta \tilde{k}_y}. \tag{6.8}$$

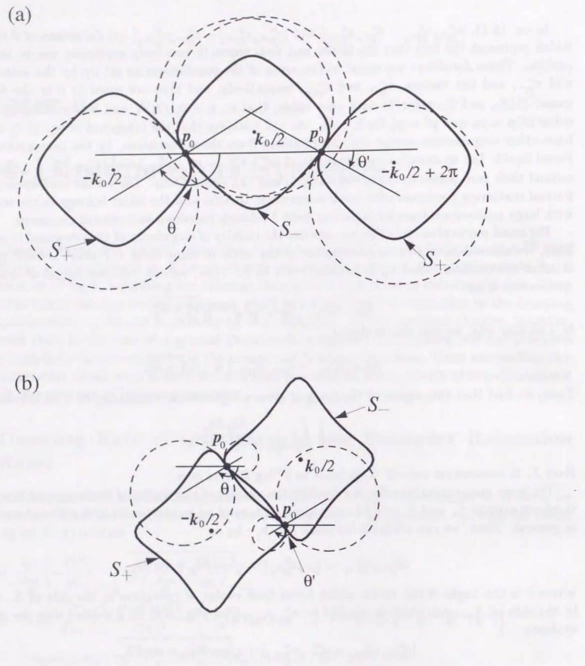

Figure 6.1: Examples of scattering processes with large momentum transfer which give divergent contributions to integrand in eq. (6.1). (a) Umklapp processes. (b) normal processes.

This leads to $T^2 \log T$-term in γ.

Next, we investigate $1/\tau^{(j)}_{el\text{-}el}(\pi T/\sqrt{3})$, the transport relaxation rate due to electron-electron scattering defined by eq. (3.36). If the matrix elements of $\hat{C}$ have no logarithmic singularity, $1/\tau^{(j)}_{el\text{-}el}(\pi T/\sqrt{3})$'s are proportional to T^2 even in two dimensions where the quasiparticle's damping, γ, has $T^2 \log T$-dependence. In this case, the resistivity is proportional to T^2. Thus, we investigate the matrix $\hat{C}$. From eq. (3.30), we obtain

$$\begin{aligned} C_{LL'} &= \frac{1}{8}\int_{-\pi}^{\pi}\frac{\mathrm{d}^2k}{(2\pi)^6}\sum_{\mathbf{p},\mathbf{p}'\in S_+\cap S_-}\left[\Gamma^2_{\uparrow\downarrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-)+\frac{1}{2}\Gamma^{A2}_{\uparrow\uparrow}(\mathbf{p}^+,\mathbf{p}'^-;\mathbf{p}'^+,\mathbf{p}^-)\right] \\ &\quad \times a_{\mathbf{p}^+}a_{\mathbf{p}'^-}a_{\mathbf{p}'^+}a_{\mathbf{p}^-}\,(N^*(0))^{-2} \\ &\quad \times\frac{(\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-))(\psi_{L'}(\mathbf{p}^+)+\psi_{L'}(\mathbf{p}'^-)-\psi_{L'}(\mathbf{p}'^+)-\psi_{L'}(\mathbf{p}^-))}{|v^*_{\mathbf{p}^+x}v^*_{\mathbf{p}^-y}-v^*_{\mathbf{p}^-x}v^*_{\mathbf{p}^+y}||v^*_{\mathbf{p}'^+x}v^*_{\mathbf{p}'^-y}-v^*_{\mathbf{p}'^-x}v^*_{\mathbf{p}'^+y}|}. \end{aligned} \tag{6.9}$$

The important fact to note is the existence of the factors, $\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-)$'s, in this expression which does not exist in eq. (6.1). For $\hat{C}_U$, we take the summation with respect to $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{G}$ which satisfy $\mathbf{p}$ (or $\mathbf{p}'$) $\in S_+[\mathbf{0}]\cap S_-[\mathbf{0}]$ and $\mathbf{p}'$ (or $\mathbf{p}$) $\in S_\pm[\mathbf{0}]\cap S_\mp[\mathbf{G}]$. For $\hat{C}_N$, we take the summation with respect to $\mathbf{p}$ and $\mathbf{p}'$ which satisfy $\mathbf{p},\mathbf{p}' \in S_+[\mathbf{0}]\cap S_-[\mathbf{0}]$. From the above examination of γ, it is obvious that $\hat{C}_U$ does not have the logarithmic singularity. We shall thus confine our attention to $\hat{C}_N$. Here, we classify the Fermi surface by an integer Z which is the maximum number of the common points of $S_+[\mathbf{0}]$ and $S_-[\mathbf{0}]$ when each is tangent to the other for $-\pi < k_x, k_y \le \pi$, as mentioned in Sec. 4.1. We show examples of the Fermi surfaces with $Z = 1$ and $Z = 2$ in Fig. 6.2(a) and (b), respectively.

In the case of the Fermi surface with $Z = 1$ which corresponds to the closed Fermi surface without inflexion points, the number of the intersections is necessarily equal to two when $S_+[\mathbf{0}]$ and $S_-[\mathbf{0}]$ intersect each other, as shown in Fig. 6.2(a). Then, there exist only the processes of normal scattering with $\mathbf{p}' = \pm\mathbf{p}$, since the inversion symmetry of the Fermi surface $\epsilon^*_{\mathbf{p}} = \epsilon^*_{-\mathbf{p}} = 0$. We thus obtain $\psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-) = 0$ for all present processes in the normal scattering in eq. (6.9). This leads to an interesting result that $\hat{C}_N$ is equal to zero as a matrix. The model with dispersion $\epsilon_{\mathbf{p}} = -2t(\cos p_x + \cos p_y)$ belongs to this type. We will actually calculate T^2-term in the resistivity of this model in the whole region of the electron number density in Sec. 7.2.

In the case of the Fermi surface with $Z \ge 2$, $\hat{C}_N$ is not equal to zero as a matrix since there exist the normal scattering processes with $\mathbf{p}' \ne \pm\mathbf{p}$, an example of which is shown in Fig. 6.2(b). It is enough to consider the case of small momentum transfer and the case of large momentum transfer with $\theta = \theta'$ which lead to logarithmic singularity of γ.

For small momentum transfer $\mathbf{k}$, we get

$$\begin{aligned} \psi_L(\mathbf{p}^+)+\psi_L(\mathbf{p}'^-)-\psi_L(\mathbf{p}'^+)-\psi_L(\mathbf{p}^-) &\simeq (\nabla\psi_L(\mathbf{p})-\nabla\psi_L(\mathbf{p}'))\cdot\mathbf{k} \\ &\propto \sqrt{k_x^2+k_y^2} \end{aligned} \tag{6.10}$$

Thus, $\hat{C}_N$ does always have singular contributions from the small transferred momentum, independently of the shape of the Fermi surface.

For large momentum transfer with $\theta = \theta'$, if $\psi_L(\mathbf{p}_0 + \mathbf{k}_0/2) + \psi_L(\mathbf{p}'_0 - \mathbf{k}_0/2) - \psi_L(\mathbf{p}'_0 + \mathbf{k}_0/2) - \psi_L(\mathbf{p}_0 - \mathbf{k}_0/2) \ne 0$, $\hat{C}_N$ has logarithmically singular contributions as follows,

$$\hat{C}_N \sim \int\!\!\int_{\Omega}\frac{\mathrm{d}\tilde{k}_x\mathrm{d}\tilde{k}_y}{\cos\theta\tilde{k}_x+\sin\theta\tilde{k}_y}. \tag{6.11}$$

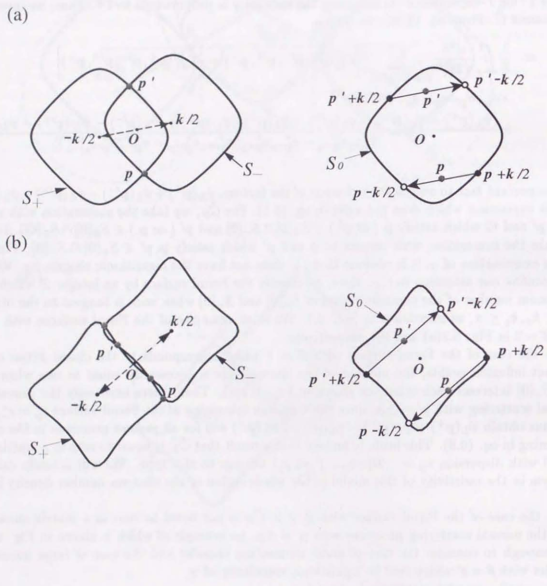

Figure 6.2: Normal scattering processes in two dimensions. (a) There exist only the processes with $\mathbf{p}' = \pm\mathbf{p}$. (b) An example of the process with $\mathbf{p}' \neq \pm\mathbf{p}$.

But as far as the case of the Fermi surface with $Z = 2$ is concerned, the singularity can be removed as will be seen below.

In the case of $Z = 2$, as seen Fig. 6.1(b), $\mathbf{p}_0' = \pm\mathbf{p}_0$ necessarily holds although $\theta' = \theta$. It is enough to discuss the case of $\mathbf{p}_0' = \mathbf{p}_0$ because of the inversion symmetry of the Fermi surface. We then obtain for small $\mathbf{p} - \mathbf{p}'$

$$\begin{aligned} &\psi_L(\mathbf{p}^+) + \psi_L(\mathbf{p}'^-) - \psi_L(\mathbf{p}'^+) - \psi_L(\mathbf{p}^-) \\ &\simeq \left(\nabla\psi_L\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) - \nabla\psi_L\left(\mathbf{p}' - \frac{\mathbf{k}}{2}\right)\right) \cdot (\mathbf{p} - \mathbf{p}'). \end{aligned} \tag{6.12}$$

On the other hand, by approximating S_+ and S_- around $\mathbf{p}_0$ again by the circles of curvature, we obtain

$$|\mathbf{p} - \mathbf{p}'| \propto \sqrt{\cos\theta \tilde{k}_x + \sin\theta \tilde{k}_y}. \tag{6.13}$$

Therefore, the logarithmic singularity is removed and thus we find that $\hat{C}_N$ does not have the logarithmic singularity.

Finally, we give summary of this section. $\hat{C}_U$ does not have a logarithmic singularity independently of the shape of the Fermi surface. The case of the Fermi surface with $Z = 1$ is special in the sense that $\hat{C}_N = \hat{O}$. In the case of the Fermi surface with $Z = 2$, $\hat{C}_N$ does not have a logarithmic singularity and $\hat{C}_N \neq \hat{O}$. Thus, we conclude that, in the case of the Fermi surface with $Z = 1$ or $Z = 2$, the transport relaxation rates, $1/\tau^{(j)}_{el\text{-}el}(\pi T/\sqrt{3})$'s are all proportional to T^2. On the other hand, in the case of the Fermi surface with $Z \geq 3$, $1/\tau^{(j)}_{el\text{-}el}(\pi T/\sqrt{3})$ may have a logarithmic singularity. However, even in this case, the resistivity is proportional to T^2 at sufficiently low T, as will be seen in the next section.

6.2 Resistivity

In this section, we consider two-dimensional and three-dimensional systems in the presence of Umklapp processes and neglect the impurity potentials. We then obtain in a similar way to eq. (4.4) since $\hat{C}_U$ is a positive definite matrix,

$$\begin{aligned} \sigma_{dc}(T) &= e^2 \sum_{\mathbf{p},\sigma} \left(-\frac{\partial f}{\partial \epsilon^*_{\mathbf{p}}}\right) v^{*2}_{\mathbf{p}x} \left[\frac{1}{2\pi N^*(0)(\epsilon^{*2}_{\mathbf{p}} + (\pi T)^2)(\hat{C}_N + \hat{C}_U)}\right]_{11} \\ &= e^2 \sum_{\mathbf{p},\sigma} \left(-\frac{\partial f}{\partial \epsilon^*_{\mathbf{p}}}\right) \frac{v^{*2}_{\mathbf{p}x}}{2\pi N^*(0)(\epsilon^{*2}_{\mathbf{p}} + (\pi T)^2)} \left[\hat{C}_U^{-1}\right]_{11} \left(\tilde{w}^{(0)2}_{N,U} + \sum_{j=1}^{\infty} \tilde{w}^{(j)2}_{N,U} \frac{1}{1 + \tilde{\lambda}^{(j)}_{N,U}}\right) \end{aligned} \tag{6.14}$$

where

$$\hat{C}_U^{-\frac{1}{2}} \hat{C}_N \hat{C}_U^{-\frac{1}{2}} \hat{\tilde{u}}^{(j)}_{N,U} = \tilde{\lambda}^{(j)}_{N,U} \hat{\tilde{u}}^{(j)}_{N,U}, \qquad {}^t\hat{\tilde{u}}^{(j)}_{N,U} \hat{\tilde{u}}^{(j')}_{N,U} = \delta_{jj'}, \tag{6.15}$$

$$\tilde{w}^{(j)}_{N,U} \equiv \left[\hat{C}_U^{-\frac{1}{2}} \hat{\tilde{u}}^{(j)}_{N,U}\right]_1 \Big/ \left[\hat{C}_U^{-1}\right]_{11}^{\frac{1}{2}}, \qquad \sum_{j=0}^{\infty} \tilde{w}^{(j)2}_{N,U} = 1. \tag{6.16}$$

Here, we note that $\tilde{\lambda}^{(0)}_{N,U} = 0$, $\hat{\tilde{u}}^{(0)}_{N,U} = \hat{C}_U^{\frac{1}{2}} \hat{\phi} / ({}^t\hat{\phi} \hat{C}_U \hat{\phi})^{\frac{1}{2}}$, and $\tilde{\lambda}^{(j)}_{N,U} > 0$ for $j \geq 1$. Thus, $\tilde{w}^{(0)2}_{N,U}$ is given by

$$\tilde{w}^{(0)2}_{N,U} = \frac{\hat{\phi}_1^2}{{}^t\hat{\phi} \hat{C}_U \hat{\phi}} \Big/ \left[\hat{C}_U^{-1}\right]_{11}. \tag{6.17}$$

We easily see from eq.(6.14) that, in the presence of Umklapp processes, normal processes contribute to the resistivity unless $\hat{C}_N = \hat{O}$, since $\tilde{w}_{N,U}^{(0)2} < 1$ and $\lambda_{N,U}^{(j)} > 0$ for $j \geq 1$. Here, Umklapp processes play a similar role of impurities which are necessary to the the finite T^2-term in the resistivity resulting from normal processes as was seen in Chap. 4. This is another aspect of the breakdown of the Matthiessen's rule.

In two-dimensional systems, it depends on the shape of the Fermi surface whether or not normal processes contribute to the resistivity in the presence of Umklapp processes. In the case of $Z = 1$, only Umklapp processes contribute to a finite T^2-term in the resistivity since $\hat{C}_N = \hat{O}$. However, in the case of $Z = 2$, all $\lambda_{N,U}^{(j)}$'s$(j \geq 1)$ in eq. (6.14) are finite since $\hat{C}_N$ has no logarithmic singularity. Then, normal processes, which are known to give the $T^2 \log T$ contribution to γ, do not change the fact that the resistivity is proportional to T^2 but they give a finite contribution to the T^2-term in the resistivity.

In the case of the Fermi surface with $Z \geq 3$, although $\lambda_{N,U}^{(j)}$'s$(j \geq 1)$ in the denominator of the second term in eq. (6.14) may have logarithmic singularities, the resistivity is proportional to T^2 at sufficiently low T since the first term in eq. (6.14) is dominant to $\sigma_{dc}(T)$.

6.3 Hubbard Model with Nearest-Neighbor Hopping

The resistivity of the two-dimensional Hubbard model with the nearest-neighbor hoppings ($\epsilon_{\mathbf{p}} = -2t(\cos p_x + \cos p_y)$) has the contributions only from the Umklapp scattering processes in the absence of the impurity scattering as mentioned in the previous sections. In this case, the ordinary memory-function formalism, which corresponds to an approximation truncating $\hat{C}$ up to the first-order polynomials in the lowest order of U, leads to a correct result of the vanishing T^2-term in the resistivity in the absence of Umklapp processes. Thus, we calculate T^2-term in the resistivity by using the ordinary memory-function formalism. We consider here the case of less than half-filling. There are two kinds of Umklapp scattering processes depending on the direction of reciprocal lattice vector involved in the scattering processes; $\mathbf{G} = (\pm 2\pi, 0)$ and $\mathbf{G} = (0, \pm 2\pi)$. We call the former (the latter) as the Umklapp scattering processes along the x-axis (y-axis). We note that in the present approximation, the resistivity is given by the simple sum of the contributions from these two kinds of the Umklapp scattering processes.

Then, we get the coefficient of T^2-term in the resistivity, A, (along, say, the x-axis) as follows,

$$\rho_{dc}(T) \equiv \sigma_{dc}^{-1}(T) = AT^2,$$
$$A = \frac{U^2}{24\pi e^2 \chi_0^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \sum_{\substack{i,j=\pm \\ i',j'=\pm}} \left[\frac{(v_{\mathbf{p}+x} + v_{\mathbf{p}'-x} - v_{\mathbf{p}'+x} - v_{\mathbf{p}-x})^2}{|v_{\mathbf{p}+x}v_{\mathbf{p}-y} - v_{\mathbf{p}-x}v_{\mathbf{p}+y}||v_{\mathbf{p}'+x}v_{\mathbf{p}'-y} - v_{\mathbf{p}'-x}v_{\mathbf{p}'+y}|} \right]_{\substack{\mathbf{p}=\mathbf{p}^{(i,j)} \\ \mathbf{p}'=\mathbf{p}^{(i',j')}}}. \tag{6.18}$$

$\mathbf{p}^{(i,j)} = (p_x^{(i,j)}, p_y^{(i,j)})$ is given by the set of solution of the following simultaneous equations,

$$\begin{cases} \cos(p_x + k_x/2) + \cos(p_y + k_y/2) = u \\ \cos(p_x - k_x/2) + \cos(p_y - k_y/2) = u. \end{cases} \tag{6.19}$$

We have used $u = |\mu|/2t$ for $-4t \leq \mu \leq 0$. Further, $p_x^{(i)} \geq 0$ and $p_y^{(i)} \geq 0$ are introduced by $p_x^{(i,\pm)} \equiv \pm p_x^{(i)}$ and $p_y^{(i,\pm)} \equiv \mp \mathrm{sgn}(k_x k_y) p_y^{(i)}$, respectively. Then, we get

$$(\cos p_x^{(\pm)}, \cos p_y^{(\pm)}) = \left(\frac{-u\cos\frac{k_x}{2}\sin^2\frac{k_y}{2} \pm w\cos\frac{k_y}{2}}{\sin^2\frac{k_x}{2} - \sin^2\frac{k_y}{2}}, \frac{-u\cos\frac{k_y}{2}\sin^2\frac{k_x}{2} \pm w\cos\frac{k_x}{2}}{\sin^2\frac{k_y}{2} - \sin^2\frac{k_x}{2}} \right), \tag{6.20}$$

$$\left| v_{\mathbf{p}+x}v_{\mathbf{p}-y} - v_{\mathbf{p}-x}v_{\mathbf{p}+y} \right|_{\mathbf{p}=\mathbf{p}^{(\pm,j)}} = 2(2t)^2 w \sqrt{\frac{\left(w \pm u\cos\frac{k_x}{2}\cos\frac{k_y}{2}\right)^2 - u^2}{\left(\sin^2\frac{k_x}{2} - \sin^2\frac{k_y}{2}\right)^2}}, \tag{6.21}$$

where

$$\begin{aligned} w &= w(\mathbf{k}; u) \\ &= \sqrt{\left(\sin^2\frac{k_x}{2} - \sin^2\frac{k_y}{2}\right)^2 + u^2 \sin^2\frac{k_x}{2}\sin^2\frac{k_y}{2}}. \end{aligned} \tag{6.22}$$

$\mathbf{p}^{(+,\pm)}$ exist only when $\mathbf{k} \in \Omega_0$ where the region Ω_0 is given by

$$\Omega_0 : \cos\frac{k_x}{2} + \cos\frac{k_y}{2} > u. \tag{6.23}$$

On the other hand, $\mathbf{p}^{(-,\pm)}$ exist only when $\mathbf{k} \in \Omega_\parallel \bigcup \Omega_\perp$ where the regions $\Omega_\parallel$ and $\Omega_\perp$ are respectively given by

$$\Omega_\parallel \;:\; \cos\frac{k_x}{2} + u < \cos\frac{k_y}{2}, \tag{6.24}$$

$$\Omega_\perp \;:\; \cos\frac{k_y}{2} + u < \cos\frac{k_x}{2}. \tag{6.25}$$

The regions Ω_0, $\Omega_\parallel$ and $\Omega_\perp$ are shown in Fig. 6.3. Note that if $0 \leq u \leq 1$, the regions Ω_0, $\Omega_\parallel$ and $\Omega_\perp$ all exist and the condition $\Omega_\parallel \bigcup \Omega_\perp \subset \Omega_0$ is always satisfied. On the other hand, if $1 \leq u \leq 2$, only the region Ω_0 exists. We can further check the following three facts: (1) If $\mathbf{k} \in \Omega_0$, a particle with momentum $\mathbf{p}^{(+,j)} \pm \mathbf{k}/2$ is scattered to a state with $\mathbf{p}^{(+,j)} \mp \mathbf{k}/2$ always within the zone-boundaries; (2) If $\mathbf{k} \in \Omega_\parallel$, a particle with momentum $\mathbf{p}^{(-,j)} \pm \mathbf{k}/2$ is scattered to a state with $\mathbf{p}^{(-,j)} \mp \mathbf{k}/2$ always across the zone-boundaries $p_x = \pm\pi$; (3) If $\mathbf{k} \in \Omega_\perp$, a particle with momentum $\mathbf{p}^{(-,j)} \pm \mathbf{k}/2$ is scattered to a state with $\mathbf{p}^{(-,j)} \mp \mathbf{k}/2$ always across the zone-boundaries $p_y = \pm\pi$. Since only the Umklapp scattering processes, which correspond to the processes with $\mathbf{p} = \mathbf{p}^{(\pm,j)}$ and $\mathbf{p}' = \mathbf{p}^{(\mp,j')}$, contribute to A, the above facts show that A is non-zero only when $0 < u < 1$, that is, $-2t < \mu < 0$.

We separate A into $A_\parallel$ and $A_\perp$ which is the contributions from the Umklapp scattering processes along the x-axis and the y-axis, respectively, as follows,

$$A = A_\parallel + A_\perp, \tag{6.26}$$

$$A_\parallel = \frac{U^2}{3\pi e^2 \chi_0^2} \int\int_{\mathbf{k}\in\Omega_\parallel} \frac{d^2k}{(2\pi)^2} \left[\frac{(v_{\mathbf{p}+x} + v_{\mathbf{p}'-x} - v_{\mathbf{p}'+x} - v_{\mathbf{p}-x})^2}{|v_{\mathbf{p}+x}v_{\mathbf{p}-y} - v_{\mathbf{p}-x}v_{\mathbf{p}+y}||v_{\mathbf{p}'+x}v_{\mathbf{p}'-y} - v_{\mathbf{p}'-x}v_{\mathbf{p}'+y}|} \right]_{\substack{\mathbf{p}=\mathbf{p}^{(+)} \\ \mathbf{p}'=\mathbf{p}^{(-)}}}, \tag{6.27}$$

$$A_\perp = \frac{U^2}{3\pi e^2 \chi_0^2} \int\int_{\mathbf{k}\in\Omega_\perp} \frac{d^2k}{(2\pi)^2} \left[\frac{(v_{\mathbf{p}+x} + v_{\mathbf{p}'-x} - v_{\mathbf{p}'+x} - v_{\mathbf{p}-x})^2}{|v_{\mathbf{p}+x}v_{\mathbf{p}-y} - v_{\mathbf{p}-x}v_{\mathbf{p}+y}||v_{\mathbf{p}'+x}v_{\mathbf{p}'-y} - v_{\mathbf{p}'-x}v_{\mathbf{p}'+y}|} \right]_{\substack{\mathbf{p}=\mathbf{p}^{(+)} \\ \mathbf{p}'=\mathbf{p}^{(-)}}}. \tag{6.28}$$

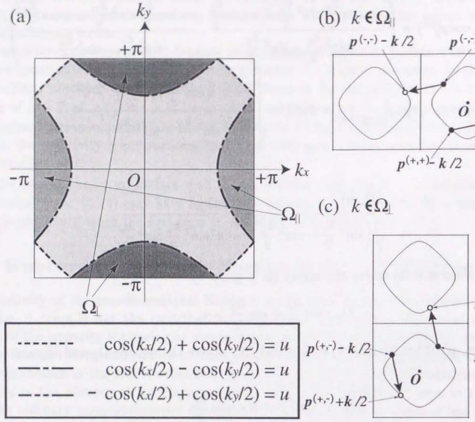

Figure 6.3: (a)The region where $p^{(i,j)}$ exists and the corresponding Umklapp scattering processes. The colored region represents Ω_0 which includes $\Omega_\parallel$ and $\Omega_\perp$. Two possible Umklapp scattering processes are shown for (b) k in $\Omega_\parallel$ and (c) k in $\Omega_\perp$, where the solid closed lines represent the Fermi surface.

From eqs (6.20)-(6.21), we get

$$A_\parallel = \frac{U^2}{3\pi^3 e^2 t^2 \chi_0^2} \int_0^{2\cos^{-1} u} \mathrm{d}k_x \int_{2\cos^{-1}(\cos\frac{k_x}{2} - u)}^{\pi} \mathrm{d}k_y \times \frac{\cos^2\frac{k_x}{2}\sin^2\frac{k_y}{2}}{\left(\cos^2\frac{k_x}{2} - \cos^2\frac{k_y}{2}\right)\sqrt{\left(\cos^2\frac{k_x}{2} - \cos^2\frac{k_y}{2}\right)^2 - 2u^2\left(\cos^2\frac{k_x}{2} + \cos^2\frac{k_y}{2}\right) + u^4}}, \tag{6.29}$$

$$A_\perp = \frac{U^2}{3\pi^3 e^2 t^2 \chi_0^2} \int_0^{2\cos^{-1} u} \mathrm{d}k_x \int_{2\cos^{-1}(\cos\frac{k_x}{2} - u)}^{\pi} \mathrm{d}k_y \times \frac{\sin^2\frac{k_x}{2}\cos^2\frac{k_y}{2}}{\left(\cos^2\frac{k_x}{2} - \cos^2\frac{k_y}{2}\right)\sqrt{\left(\cos^2\frac{k_x}{2} - \cos^2\frac{k_y}{2}\right)^2 - 2u^2\left(\cos^2\frac{k_x}{2} + \cos^2\frac{k_y}{2}\right) + u^4}}. \tag{6.30}$$

Note that we always have $A_\parallel > A_\perp$. We change the integration variables from k_x and k_y to k'_x and k'_y as follows,

$$\sin\frac{k_x}{2} = \sqrt{1-u^2}\sin\frac{k'_x}{2}, \tag{6.31}$$

$$\cos\frac{k_y}{2} = \left(\cos\frac{k_x}{2} - u\right)\cos\frac{k'_y}{2}. \tag{6.32}$$

Then, we obtain

$$A_\parallel = \frac{U^2}{3\pi^3 e^2 t^2 \chi_0^2}\sqrt{1-u^2}\int_0^{\pi}\int_0^{\pi} \mathrm{d}k'_x \mathrm{d}k'_y \times \frac{s\cos\frac{k'_x}{2}}{s^2 - (s-u)^2\cos^2\frac{k'_y}{2}}\sqrt{\frac{1-(s-u)^2\cos^2\frac{k'_y}{2}}{(s+u)^2 - (s-u)^2\cos^2\frac{k'_y}{2}}}, \tag{6.33}$$

$$A_\perp = \frac{U^2}{3\pi^3 e^2 t^2 \chi_0^2}\sqrt{1-u^2}\int_0^{\pi}\int_0^{\pi} \mathrm{d}k'_x \mathrm{d}k'_y \times \frac{s^{-1}(1-s^2)(s-u)^2\cos\frac{k'_x}{2}\cos^2\frac{k'_y}{2}}{\left(s^2-(s-u)^2\cos^2\frac{k'_y}{2}\right)\sqrt{\left(1-(s-u)^2\cos^2\frac{k'_y}{2}\right)\left((s+u)^2-(s-u)^2\cos^2\frac{k'_y}{2}\right)}}, \tag{6.34}$$

where

$$s = s(k'_x; u) = \sqrt{\cos^2\frac{k'_x}{2} + u^2\sin^2\frac{k'_x}{2}}. \tag{6.35}$$

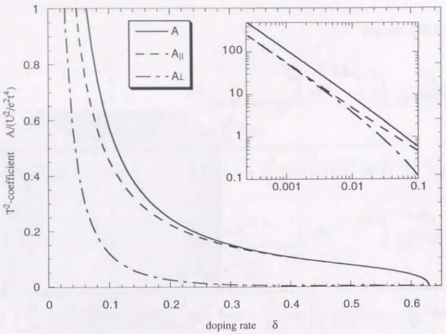

Figure 6.4: δ-dependences of $A_{\|}$, $A_{\perp}$ and A. The inset shows a log-log plot near half-filling.

Finally, let us examine the limiting behavior of $A_{\|}$ and $A_{\perp}$ for $\mu \to -2t$ and $\mu \to 0$. From eqs. (6.33)-(6.34), we get for $\mu \to -2t$

$$A_{\|} \propto \frac{U^2}{e^2t^4}\left(1-\frac{|\mu|}{2t}\right)^{\frac{1}{2}}, \tag{6.36}$$

$$A_{\perp} \propto \frac{U^2}{e^2t^4}\left(1-\frac{|\mu|}{2t}\right)^{\frac{7}{2}}. \tag{6.37}$$

This implies that far away from half-filling, A has main contributions from $A_{\|}$. We also get for $\mu \to 0$

$$A_{\|} \sim A_{\perp} \sim \frac{U^2}{e^2t^4}\frac{\pi^2}{192}\frac{t}{|\mu|}, \tag{6.38}$$

$$A \sim \frac{U^2}{e^2t^4}\frac{\pi^2}{96}\frac{t}{|\mu|}. \tag{6.39}$$

As our system approaches the half-filling($\mu \to 0$), A diverges because of the flatness of the Fermi surface. The doping rate δ is related to μ by $\delta \propto (|\mu|/t)\log(|\mu|/t)$ near the half-filling where the logarithmic factor reflects the Van Hove singularity. Therefore, $A_{\|}$, $A_{\perp}$ and A diverge as $[\delta\log(1/\delta)]^{-1}$ when $\delta \to 0$. This limiting behavior for $\delta \to 0$ is not due to the approximation of the memory-function formalism since the singularity arises from the Jacobians in eq. (6.18) which is always present in the contributions of the order of U^2. The overall δ-dependence of $A_{\|}$, $A_{\perp}$ and A are shown in Fig. 6.4.

Chapter 7

Summary and Conclusions

Electrical conductivity, $\sigma(\omega)$ due to mutual Coulomb interaction of a lattice electron system has been studied on the basis of the Fermi liquid theory. Based on the Kubo formula to respect the consistency between the selfenergy and the vertex corrections, a formulation is given in terms of the memory function $\hat{M}(\epsilon;\omega)$ which is extended to the matrix form, and then a general expression of $\hat{M}(\epsilon;\omega)$ for the d-dimensional systems ($d \geq 2$) with the short-range Coulomb interaction, U, and s-wave impurity scattering, ν, at finite temperature, T, is given up to ω, ν^2, ϵ^2 and T^2. In our formulation, the effects of the periodic lattice which leads not only to the Umklapp scattering but also to the band structure are fully taken into account.

Because of the latter effects, the total current is not conserved even in the absence of momentum dissipation mechanism. In spite of this fact, it is shown that, in the absence of momentum dissipation mechanism, the present theory yields a correct result of the absence of T^2-term in the resistivity since $\hat{M}''(\epsilon)$, the imaginary part of $\hat{M}$, has a zero eigenvalue reflecting the conservation of crystal momentum. This is in contrast to the ordinary memory-function formalism which corresponds to the high-frequency expansion of $\sigma(\omega)$ and leads to an incorrect result of non-zero resistivity.

On the other hand, the effects of the band structure lead to the fact that the Drude weights D^{ω} and D^{T}, which are limits of $\pi\omega\mathrm{Im}\sigma(\omega)$ in the absence of momentum dissipation mechanism in $\omega \to 0$, $T/\omega \to 0$ and in $T \to 0$, $\omega/T \to 0$, respectively, are not equivalent. This is in contrast to the equivalence of the above two limits in the limiting case of continuum where the total current is conserved.

In the presence of momentum dissipation mechanism through impurity or Umklapp scattering, normal processes generally contribute to the resistivity as another aspect of the effects of the band structure. This can be considered as a special case of breakdown of the Matthiessen's rule.

In three-dimensional systems in the absence of the Umklapp scattering but in the presence of the impurity scattering, the resistivity $\rho_{dc}(T)$ is shown to be given by $\rho_{res} + AT^2$ at sufficiently low temperature. A is finite due to the violation of the current conservation. We have therefore concluded that, in contrast to the general belief, a finite T^2-term in the resistivity results even in the absence of the Umklapp scattering processes once the impurity scattering is present. However, this temperature dependence due to Coulomb interaction saturates as temperature gets higher and approaches $\rho_{\infty} = (1+\alpha)\rho_{res}$ due to the conservation of crystal momentum. The quantity α is due to the difference of the shape of the Fermi surface from sphere. In the case of low electron number density in three dimensions, we have explicitly calculated A and α in the lowest order of U.

Two-dimensional systems in the absence of the Umklapp scattering but in the presence of the impurity scattering turned out to be special in the sense that it depends on the shape of the Fermi surface whether or not the T^2-term in the resistivity results. Then, we have classified cases depending on the shape of the Fermi surface by an integer Z, where Z is the maximum number of the common points of two Fermi surfaces relatively shifted by the transferred momentum k and tangent to each other. In the case of the Fermi surface with $Z = 2$, the resistivity has similar T-dependences to the case of three dimensions. But, in the case of the Fermi surface with $Z = 1$, which corresponds to a closed Fermi surface without inflexion points, the resistivity does not have such temperature dependences because normal processes do not contribute to the resistivity.

Schematic representation of the temperature dependence of the resistivity in the absence of the Umklapp scattering but in the presence of the impurity scattering is summarized in Fig. 7.1.

Next, we investigated systems in the presence of the Umklapp scattering but in the absence of the impurity scattering.

We indicated that the resistivity of the $d = \infty$ Hubbard model, which is determined by the damping rate of quasiparticle, γ, can be naturally understood by the fact that there exist Umklapp processes for any electron number density and thus the normal processes contribute to the resistivity as well. The $1/d$-correction to the resistivity has been also calculated in the second order of U.

In two-dimensions, in the case of $Z = 1$, only the Umklapp scattering leads to T^2-term in the resistivity while, in the case of $Z = 2$, the normal scattering has a finite contribution. However, in both cases, the transport relaxation rates, the inverse of the transport relaxation times, and therefore the resistivity are proportional to T^2 in contrast to γ, which is proportional to $T^2 \log T$. We have also found that, in the case of $Z \geq 3$, the resistivity is proportional to T^2 at sufficiently low temperature even if there exists the transport relaxation rate which has a logarithmic singularity. Especially, the model with nearest-neighbor hopping which belongs to the case of $Z = 1$ is investigated in detail in the second order of U with a special emphasis on the flatness of the Fermi surface at the half-filling. It is found that T^2-coefficient of the resistivity diverges as $[\delta \log(1/\delta)]^{-1}$ when the doping rate, δ, approaches zero.

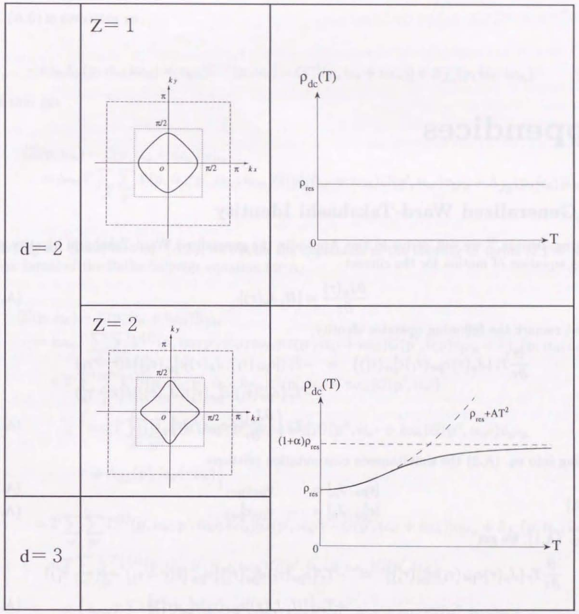

Figure 7.1: Schematic representation of the temperature dependence of the resistivity in the absence of the Umklapp scattering but in the presence of the impurity scattering for two-dimensional Fermi surfaces with $Z = 1$ and $Z = 2$ and for arbitrary three-dimensional Fermi surface different from sphere.

Appendices

A Generalized Ward-Takahashi Identity

Following Toyoda,[6] we will derive in this Appendix the generalized Ward-Takahashi identity reflecting equation of motion for the current

$$\frac{\partial J_\mu(\tau)}{\partial \tau} = [H, J_\mu(\tau)]. \tag{A.1}$$

We first remark the following operator identity,

$$\begin{aligned}
\frac{\partial}{\partial \tau} T_\tau\{J_\mu(\tau) c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\} &= -T_\tau\{[c_{\mathrm{p}\sigma}(\tau_1), J_\mu(\tau)] c^\dagger_{\mathrm{p}\sigma}(\tau_1')\}\delta(\tau-\tau_1) \\
&\quad -T_\tau\{c_{\mathrm{p}\sigma}(\tau_1)[c^\dagger_{\mathrm{p}\sigma}(\tau_1'), J_\mu(\tau)]\}\delta(\tau-\tau_1') \\
&\quad +T_\tau\left\{\frac{\partial J_\mu(\tau)}{\partial \tau} c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\right\}.
\end{aligned} \tag{A.2}$$

Inserting into eq. (A.2) the simultaneous commutation relations

$$[c_{\mathrm{p}\sigma}, J_\mu] = v_{\mathrm{p}\mu} c_{\mathrm{p}\sigma}, \tag{A.3}$$

$$[c^\dagger_{\mathrm{p}\sigma}, J_\mu] = -v_{\mathrm{p}\mu} c^\dagger_{\mathrm{p}\sigma}, \tag{A.4}$$

and eq. (A.1), we get

$$\begin{aligned}
\frac{\partial}{\partial \tau} T_\tau\{J_\mu(\tau) c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\} &= -T_\tau\{c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\} v_{\mathrm{p}\mu}\left(\delta(\tau-\tau_1) - \delta(\tau-\tau_1')\right) \\
&\quad +T_\tau\left\{[H, J_\mu(\tau)] c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\right\}.
\end{aligned} \tag{A.5}$$

Taking the thermal average and performing the Fourier transformation, we obtain

$$\begin{aligned}
&-i\omega_m \int_0^{1/T} d\tau d\tau_1 d\tau_1' \langle T_\tau\{J_\mu(\tau) c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\}\rangle e^{i\omega_m \tau + i\epsilon_n \tau_1 - i\epsilon_{n'}\tau_1'} \\
&= -1/T \delta_{n+m-n'} \int_0^{1/T} d(\tau_1 - \tau_1') \langle T_\tau\{c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\}\rangle v_{\mathrm{p}\mu}\left(e^{i(\epsilon_n + \omega_m)(\tau_1 - \tau_1')} - e^{i\epsilon_n(\tau_1 - \tau_1')}\right) \\
&\quad + \int_0^{1/T} d\tau d\tau_1 d\tau_1' \langle T_\tau\{[H, J_\mu(\tau)] c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\}\rangle e^{i\omega_m \tau + i\epsilon_n \tau_1 - i\epsilon_{n'}\tau_1'},
\end{aligned} \tag{A.6}$$

where we have performed the partial integration in the left hand side of eq. (A.6). Introducing $\Lambda_j(\mathrm{p}, i\epsilon_n; i\omega_m)$ defined by

$$\begin{aligned}
&\Lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m) G(\mathrm{p}, i\epsilon_n + i\omega_m) G(\mathrm{p}, i\epsilon_n)\delta_{n+m-n'} \\
&\equiv T \int_0^{1/T} d\tau d\tau_1 d\tau_1' \langle T_\tau\{[H, J_\mu(\tau)] c_{\mathrm{p}\sigma}(\tau_1) c^\dagger_{\mathrm{p}\sigma}(\tau_1')\}\rangle e^{i\omega_m \tau + i\epsilon_n \tau_1 - i\epsilon_{n'}\tau_1'},
\end{aligned} \tag{A.7}$$

eq. (A.6) is rewritten as

$$-i\omega_m \Lambda_\mu(\mathrm{p}, i\epsilon_n; i\omega_m) = v_{\mathrm{p}\mu}[G^{-1}(\mathrm{p}, i\epsilon_n) - G^{-1}(\mathrm{p}, i\epsilon_n + i\omega_m)] + \Lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m). \tag{A.8}$$

We thus get

$$\begin{aligned}
&[\Sigma(\mathrm{p}, i\epsilon_n) - \Sigma(\mathrm{p}, i\epsilon_n + i\omega_m)] v_{\mathrm{p}\mu} \\
&= i\omega_m T \sum_{n'}\sum_{\mathrm{p}'} \Gamma(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m) G(\mathrm{p}', i\epsilon_{n'} + i\omega_m) G(\mathrm{p}', i\epsilon_{n'}) v_{\mathrm{p}'\mu} + \Lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m).
\end{aligned} \tag{A.9}$$

Inserting eq. (2.12) into eq. (A.9), we obtain the expression of the identity in terms of $\Gamma^{(1)}$ which is the kernel of the Bethe-Salpeter equation for Λ,

$$\begin{aligned}
&[\Sigma(\mathrm{p}, i\epsilon_n) - \Sigma(\mathrm{p}, i\epsilon_n + i\omega_m)] v_{\mathrm{p}\mu} \\
&= i\omega_m T \sum_{n'}\sum_{\mathrm{p}'} \Gamma^{(1)}(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m) G(\mathrm{p}', i\epsilon_{n'} + i\omega_m) G(\mathrm{p}', i\epsilon_{n'}) v_{\mathrm{p}'\mu} + \lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m) \\
&\quad + T \sum_{n'}\sum_{\mathrm{p}'} \Gamma^{(1)}(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m) G(\mathrm{p}', i\epsilon_{n'} + i\omega_m) G(\mathrm{p}', i\epsilon_{n'}) \\
&\quad \times \Big\{ i\omega_m T \sum_{n''}\sum_{\mathrm{p}''} \Gamma(\mathrm{p}', i\epsilon_{n'}; \mathrm{p}'', i\epsilon_{n''}; i\omega_m) G(\mathrm{p}'', i\epsilon_{n''} + i\omega_m) G(\mathrm{p}'', i\epsilon_{n''}) v_{\mathrm{p}''\mu} \\
&\quad + \Lambda_{j_\mu}(\mathrm{p}', i\epsilon_{n'}; i\omega_m) \Big\}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
&= T \sum_{n'}\sum_{\mathrm{p}'} \Gamma^{(1)}(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m)[G(\mathrm{p}', i\epsilon_{n'}) - G(\mathrm{p}', i\epsilon_{n'} + i\omega_m)] v_{\mathrm{p}'\mu} + \lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m) \\
&\quad + T \sum_{n'}\sum_{\mathrm{p}'} \Gamma^{(1)}(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m) G(\mathrm{p}', i\epsilon_{n'} + i\omega_m) G(\mathrm{p}', i\epsilon_{n'}) \\
&\quad \times \Big\{ i\omega_m T \sum_{n''}\sum_{\mathrm{p}''} \Gamma(\mathrm{p}', i\epsilon_{n'}; \mathrm{p}'', i\epsilon_{n''}; i\omega_m) G(\mathrm{p}'', i\epsilon_{n''} + i\omega_m) G(\mathrm{p}'', i\epsilon_{n''}) v_{\mathrm{p}''\mu} \\
&\quad + \Lambda_{j_\mu}(\mathrm{p}', i\epsilon_{n'}; i\omega_m) - [\Sigma(\mathrm{p}', i\epsilon_{n'}) - \Sigma(\mathrm{p}', i\epsilon_{n'} + i\omega_m)] v_{\mathrm{p}'\mu} \Big\},
\end{aligned} \tag{A.11}$$

where the function $\lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m)$ is the proper part of $\Lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m)$. As the last term in the right hand side of eq. (A.11) vanishes, we get

$$\begin{aligned}
&[\Sigma(\mathrm{p}, i\epsilon_n) - \Sigma(\mathrm{p}, i\epsilon_n + i\omega_m)] v_{\mathrm{p}\mu} - T \sum_{n'}\sum_{\mathrm{p}'} \Gamma^{(1)}(\mathrm{p}, i\epsilon_n; \mathrm{p}', i\epsilon_{n'}; i\omega_m)[G(\mathrm{p}', i\epsilon_{n'}) \\
&- G(\mathrm{p}', i\epsilon_{n'} + i\omega_m)] v_{\mathrm{p}'\mu} = \lambda_{j_\mu}(\mathrm{p}, i\epsilon_n; i\omega_m).
\end{aligned} \tag{A.12}$$

If the total current operator commutes with the Hamiltonian, we have $\lambda_{j_\mu} = 0$ and eq. (A.12) leads to eq. (2.43).

B Conserving Approximation

We will show in this Appendix that the generalized Ward-Takahashi identity reflecting equation of motion for the current, which is given by eq. (A.12), always holds for an arbitrary skeleton diagram contributing to Σ from which the skeleton diagrams contributing to $\Gamma^{(1)}$ are constructed by cutting the internal G line in all possible ways.

We consider Σ_γ which represents an arbitrary skeleton diagram labeled by γ contributing to Σ. $\Sigma_\gamma(\mathbf{p}, i\epsilon_n)$, containing l bare interaction lines and $2l-1$ internal G lines, is schematically given by

$$\begin{aligned}\Sigma_\gamma(p) &= (-1)^l \frac{T^l}{(2\pi)^{dl}}(2s+1)^F(-1)^F \sum_{\epsilon_1}\int \mathrm{d}^d p_1 \sum_{\epsilon_2}\int \mathrm{d}^d p_2 \cdots \sum_{\epsilon_{2l-1}}\int \mathrm{d}^d p_{2l-1} \\ &\quad \times \mathcal{V}(p, p_{i_1}; p_{i_2}, p_{i_3})\mathcal{V}(p_{i_4}, p_{i_5}; p_{i_6}, p_{i_7}) \cdots \mathcal{V}(p_{i_{4(l-1)}}, p_{i_{4(l-1)+1}}; p_{i_{4(l-1)+2}}, p) \\ &\quad \times G(p_1)G(p_2)\cdots G(p_{2l-1}),\end{aligned} \tag{B.13}$$

where $\mathcal{V}$ is related to the bare electron-electron interaction V as follows,

$$\mathcal{V}(p_1, p_2; p_3, p_4) \equiv V(\mathbf{p}_1 - \mathbf{p}_4)\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\delta_{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4}, \tag{B.14}$$

and F is the number of closed Fermion loops, s represents spin and $p = (\mathbf{p}, i\epsilon_n)$. In eq. (B.13), the sequence $(i_1, i_2, \cdots, i_{2(2l-1)})$ denotes a certain permutation of the sequence $(1, 1, 2, 2, \cdots, 2l-1, 2l-1)$ which specifies the diagram.

Then, $\Gamma^{(1)}_{\gamma,r}(\mathbf{p}, i\epsilon_n; \mathbf{p}', i\epsilon_{n'}; i\omega_m)$, which contributes to the proper four-point vertex part and is constructed by cutting the r-th internal G line($r = 1, 2, \cdots, 2l-1$) from $\Sigma_\gamma(\mathbf{p}, i\epsilon_n)$, is schematically given by

$$\begin{aligned}\Gamma^{(1)}_{\gamma,r}(p, p'; q) &= (-1)^l \frac{T^{(l-1)}}{(2\pi)^{d(l-1)}}(2s+1)^F(-1)^F \sum_{\epsilon_1}\int \mathrm{d}^d p_1 \sum_{\epsilon_2}\int \mathrm{d}^d p_2 \cdots \sum_{\epsilon_{r-1}}\int \mathrm{d}^d p_{r-1} \\ &\quad \times \sum_{\epsilon_{r+1}}\int \mathrm{d}^d p_{r+1} \sum_{\epsilon_{r+2}}\int \mathrm{d}^d p_{r+2} \cdots \sum_{\epsilon_{2l-1}}\int \mathrm{d}^d p_{2l-1} \\ &\quad \times \mathcal{V}(p_{i_0}, p_{i_1}; p_{i_2}, p_{i_3})\mathcal{V}(p_{i_4}, p_{i_5}; p_{i_6}, p_{i_7}) \cdots \mathcal{V}(p_{i_{4k}}, p_r; p_{i_{4k+2}}, p_{i_{4k+3}}) \cdots \\ &\quad \times \mathcal{V}(p_{i_{4k'}}, p_{i_{4k'+1}}; p_r + q, p_{i_{4k'+3}}) \cdots \mathcal{V}(p_{i_{4(l-1)}}, p_{i_{4(l-1)+1}}; p_{i_{4(l-1)+2}}, p_{i_{4(l-1)+3}}) \\ &\quad \times G(p_1)G(p_2)\cdots G(p_{r-1})G(p_{r+1})\cdots G(p_{2l-1}),\end{aligned} \tag{B.15}$$

where $p_{i_0} \equiv p + q$, $p_{i_{4(l-1)+3}} \equiv p$, $p_r \equiv p'$ and we put $q = (\mathbf{0}, i\omega_m)$.

Here, we note

$$\mathcal{V}(p_1 - q, p_2; p_3, p_4) = \mathcal{V}(p_1, p_2; p_3, p_4 + q), \tag{B.16}$$

$$\mathcal{V}(p_1, p_2 - q; p_3, p_4) = \mathcal{V}(p_1, p_2; p_3 + q, p_4). \tag{B.17}$$

We also note

$$\mathcal{V}(p_1, p_2; p_3 + q, p_4) = \mathcal{V}(p_1, p_2; p_3, p_4 + q), \tag{B.18}$$

since q has zero momentum. We therefore obtain

$$\begin{aligned}&[\Sigma_\gamma(p) - \Sigma_\gamma(p+q)]v_{\mathbf{p}\mu} - T\sum_{\epsilon'}\int \frac{\mathrm{d}^d p'}{(2\pi)^d}\sum_{r=1}^{2l-1}\Gamma^{(1)}_{\gamma,r}(p, p'; q)[G(p') - G(p'+q)]v_{\mathbf{p}'\mu} \\ &= (-1)^l \frac{T^l}{(2\pi)^{dl}}(2s+1)^F(-1)^F \sum_{\epsilon_1}\int \mathrm{d}^d p_1 \sum_{\epsilon_2}\int \mathrm{d}^d p_2 \cdots \sum_{\epsilon_{2l-1}}\int \mathrm{d}^d p_{2l-1} \\ &\quad \times \Big[(v_{\mathbf{p}\mu} + v_{\mathbf{p}_{i_1}\mu} - v_{\mathbf{p}_{i_2}\mu} - v_{\mathbf{p}_{i_3}\mu})\mathcal{V}(p_{i_0}, p_{i_1}; p_{i_2}, p_{i_3} + q) \\ &\qquad \times \mathcal{V}(p_{i_4}, p_{i_5}; p_{i_6}, p_{i_7}) \cdots \mathcal{V}(p_{i_{4(l-1)}}, p_{i_{4(l-1)+1}}; p_{i_{4(l-1)+2}}, p_{i_{4(l-1)+3}}) \\ &\quad + \cdots\cdots \\ &\quad + \mathcal{V}(p_{i_0}, p_{i_1}; p_{i_2}, p_{i_3}) \cdots \mathcal{V}(p_{i_{4(k-1)}}, p_{i_{4(k-1)+1}}; p_{i_{4(k-1)+2}}, p_{i_{4(k-1)+3}}) \\ &\qquad \times (v_{\mathbf{p}_{i_{4k}}\mu} + v_{\mathbf{p}_{i_{4k+1}}\mu} - v_{\mathbf{p}_{i_{4k+2}}\mu} - v_{\mathbf{p}_{i_{4k+3}}\mu})\mathcal{V}(p_{i_{4k}}, p_{i_{4k+1}}; p_{i_{4k+2}}, p_{i_{4k+3}} + q) \\ &\qquad \times \mathcal{V}(p_{i_{4(k+1)}}, p_{i_{4(k+1)+1}}; p_{i_{4(k+1)+2}}, p_{i_{4(k+1)+3}}) \cdots \mathcal{V}(p_{i_{4(l-1)}}, p_{i_{4(l-1)+1}}; p_{i_{4(l-1)+2}}, p_{i_{4(l-1)+3}}) \\ &\quad + \cdots\cdots \\ &\quad + \mathcal{V}(p_{i_0}, p_{i_1}; p_{i_2}, p_{i_3}) \cdots \mathcal{V}(p_{i_{4(l-2)}}, p_{i_{4(l-2)+1}}; p_{i_{4(l-2)+2}}, p_{i_{4(l-2)+3}}) \\ &\qquad \times (v_{\mathbf{p}_{i_{4(l-1)}}\mu} + v_{\mathbf{p}_{i_{4(l-1)+1}}\mu} - v_{\mathbf{p}_{i_{4(l-1)+2}}\mu} - v_{\mathbf{p}\mu})\mathcal{V}(p_{i_{4(l-1)}}, p_{i_{4(l-1)+1}}; p_{i_{4(l-1)+2}}, p_{i_{4(l-1)+3}} + q)\Big] \\ &\quad \times G(p_1)G(p_2)\cdots G(p_{2l-1}).\end{aligned} \tag{B.19}$$

The right hand side of eq. (B.19) is equal to the contributions of order l to λ_{j_μ} corresponding to Σ_γ and it is easily seen to vanish in the limiting case of continuum where $v_{\mathbf{p}\mu}$ is proportional to the momentum of each electron.

If we choose the three second order skeleton diagrams shown in Fig. 3.1 as $\Gamma^{(1)}_{\gamma,r}$ $(r = 1, 2, 3)$, we see that not only the particle-hole diagrams, which are functions of $\mathbf{p}$, $\mathbf{p}'$, $i\epsilon_{n'} - i\epsilon_n$ and $i\omega_m$, but also the particle-particle diagram, which is a function of $\mathbf{p}$, $\mathbf{p}'$, $i\epsilon_{n'} + i\epsilon_n$ and $i\omega_m$, are indispensable for eq. (B.19) to hold. This is the reason why we did not adopt in Chap. 2 the assumption that $\Gamma^{(1)}$ is a function only of $\mathbf{p}' - \mathbf{p}$, $i\epsilon_{n'} - i\epsilon_n$ and $i\omega_m$.

C $1/d$-Expansion

First, let us assume the $\mathbf{p}$-dependence of $\Sigma^R(p)$ as follows,

$$\Sigma^R(p) = \Sigma'(\epsilon) + \frac{1}{d}\frac{\epsilon_{\mathbf{p}}}{t^*}\Sigma''(\epsilon), \tag{C.20}$$

up to the leading correction with respect to $1/d$. We see later this expression is self-consistent. Then, we get

$$\epsilon_{\mathbf{p}} - \mu + \mathrm{Re}\Sigma(\mathbf{p}, 0) = -\frac{2\tilde{t}}{\sqrt{2d}}\sum_{\nu=1}^{d}\cos p_\nu - \tilde{\mu}, \tag{C.21}$$

where

$$\tilde{t} \equiv t^* + d^{-1}\mathrm{Re}\Sigma''(0), \tag{C.22}$$

$$\tilde{\mu} \equiv \mu - \mathrm{Re}\Sigma'(0). \tag{C.23}$$

Then,

$$\begin{aligned}\rho_{ij}(0) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}^d p}{(2\pi)^d}\delta(\epsilon_{\mathbf{p}} - \mu + \mathrm{Re}\Sigma(\mathbf{p}, 0))e^{i\mathbf{p}\cdot\mathbf{r}_{ij}} \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}s}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}^d p}{(2\pi)^d}\exp\left(is\left(-\frac{2\tilde{t}}{\sqrt{2d}}\sum_{\nu=1}^{d}\cos p_\nu - \tilde{\mu}\right) + i\sum_{\nu=1}^{d}p_\nu n_\nu\right) \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}s}{2\pi}e^{-is\tilde{\mu}}\prod_{\nu=1}^{d}(-i)^{n_\nu}J_{n_\nu}(2s\tilde{t}/\sqrt{2d}),\end{aligned} \tag{C.24}$$

where $J_n(z)$ is the n-th order Bessel function given by

$$J_n(z) = \int_{-\pi}^{\pi} \frac{dp}{2\pi i^n} e^{iz\cos p} \cos np = \sum_{l=0}^{\infty} \frac{(-1)^l z^{n+2l}}{2^{n+2l} l!(n+l)!}, \tag{C.25}$$

and

$$\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j \equiv (n_1, n_2, \cdots, n_\nu, \cdots, n_d). \tag{C.26}$$

Here, we introduce L_{ij} given by

$$L_{ij} = \sum_{\nu=1}^{d} |n_\nu| = \sum_{m=0}^{\infty} m N_m, \tag{C.27}$$

where N_m is the number of directions whose coordinate, $|n_\nu|$, is equal to m. We consider the case of $\sum_{m=1}^{\infty} N_m \sim O(1)$. Then, we obtain the leading term of $\rho_{ij}(0)$ with respect to the $1/d$-expansion as follows,

$$\begin{aligned}
\rho_{ij}(0) &= \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{-is\tilde{\mu}} \prod_{m=0}^{\infty} \left((-i)^m J_m(2s\tilde{t}/\sqrt{2d}) \right)^{N_m} \\
&\simeq \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{-is\tilde{\mu}} \left(1 - \frac{(s\tilde{t})^2}{2d}\right)^{d - \sum_{m=1}^{\infty} N_m} \prod_{m=1}^{\infty} \left(\frac{(-is\tilde{t}/\sqrt{2d})^m}{m!} \right)^{N_m} \\
&\simeq \exp\left(-\sum_{m=1}^{\infty} N_m \log(m!)\right) \left(\frac{\tilde{t}}{\sqrt{2d}} \frac{\partial}{\partial \tilde{\mu}} \right)^{L_{ij}} \int_{-\infty}^{\infty} \frac{ds}{2\pi} \exp\left(-\frac{(s\tilde{t})^2}{2} - is\tilde{\mu} \right) \\
&= \left(-\frac{1}{\sqrt{2d}} \right)^{L_{ij}} \exp\left(-\sum_{m=1}^{\infty} N_m \log(m!)\right) H_{L_{ij}}(\tilde{\mu}/\tilde{t}) \frac{1}{\sqrt{2\pi\tilde{t}^2}} \exp\left(-\frac{\tilde{\mu}^2}{2\tilde{t}^2}\right),
\end{aligned} \tag{C.28}$$

where $H_L(x)$'s are Hermite polynomials. In a similar way, we can also estimate the Green's function G_{ij},

$$G_{ij}(z) = \int_{-\infty}^{\infty} d\epsilon \frac{\rho_{ij}(\epsilon)}{z + \mu - \epsilon} \sim O\left(d^{-L_{ij}/2}\right). \tag{C.29}$$

Eq. (C.29) leads to the following two facts: (1)$\Sigma^R(p)$ is independent of momentum for $d = \infty$ because the self-energy in the real space, Σ^R_{ij}, vanishes if $i \neq j$. (2)The momentum dependence of the leading correction to $\Sigma^R(p)$ in the $1/d$-expansion is $d^{-3/2} \sum_{\nu=1}^{d} \cos p_\nu$ because it originates from Σ^R_{ij} where i-th and j-th sites are nearest neighbors and there are at least three particle lines which connect i-th and j-th sites in the Feynman diagrams of Σ^R_{ij}. By (1) and (2), we can check the self-consistency of the assumption, eq. (C.20).

Especially, since we can evaluate $\tilde{t}$ and $\tilde{\mu}$ by their values at $d = \infty$, respectively, $\tilde{t} = t^*$ and $\tilde{\mu} = \mu_0$ as far as the leading term in $\rho_{ij}(0)$ is concerned, we obtain

$$\rho_{ii}(0) = \frac{1}{\sqrt{2\pi t^{*2}}} \exp\left(-\frac{\mu_0^2}{2t^{*2}}\right) + O(d^{-1}), \tag{C.30}$$

$$\rho_{\langle i,j \rangle}(0) = -\frac{\mu_0}{\sqrt{2d t^{*2}}} \rho_{ii}(0) + O(d^{-3/2}), \tag{C.31}$$

where $\langle i,j \rangle$ expresses that i-th and j-th sites are nearest neighbors.

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（相互作用するフェルミ粒子系の電気伝導度）

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