J. Math. Sci. Univ. Tokyo
23 (2016), 675–726.

Uniformization of Cyclic Quotients of Multiplicative A-singularities

By Kenjiro Sasaki and Shigeru Takamura

Abstract. This work is motivated by the canonical model of degenerations of Riemann surfaces. For a quotient space A_{d-1}/Γ of a 'multiplicative' A-singularity A_{d-1} in $\mathbb{C}^{n+1}$ under a certain cyclic group action Γ on A_{d-1}, we *explicitly* construct a small finite abelian subgroup G of $GL(n, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. A resolution of $\mathbb{C}^n/G$ gives a decomposition of the monodromy (a *higher-dimensional fractional Dehn twist*) of a degeneration $A_{d-1}/\Gamma \to \mathbb{C}$ into subtwists along the exceptional set (it seems that T. Ashikaga's work on resolutions is related to this). Moreover: (1) We give a numerical criterion for a certain subgroup of $GL(n, \mathbb{C})$ to be small. (2) For a certain family of subgroups of $GL(n, \mathbb{C})$, we show that if one subgroup of this family is small, then all subgroups of this family are small (*equi-smallness theorem*).

1. Introduction

Let d be a positive integer and consider the following two complex varieties:

$$V = \{(x_1, x_2, \dots, x_n, t) \in \mathbb{C}^{n+1} \,:\, x_1^2 + x_2^2 + \cdots + x_n^2 = t^d\},$$
$$W = \{(z_1, z_2, \dots, z_n, t) \in \mathbb{C}^{n+1} \,:\, z_1 z_2 \cdots z_n = t^d\}.$$

We say that V is an *additive A-singularity* and W is a *multiplicative A-singularity*. If $n = 2$, they are isomorphic via $(x_1, x_2) = (z_1 + \mathrm{i}z_2,\, z_1 - \mathrm{i}z_2)$. In contrast if $n \geq 3$, they are not isomorphic: The singular locus of V is isolated, while that of W is not isolated — the former is the origin, while the latter is the union of ${}_nC_2$ hyperplanes $H_{ij} = \{z_i = z_j = t = 0\}$, $1 \leq i < j \leq n$.

2010 *Mathematics Subject Classification.* Primary 32Q30; Secondary 14D05.
Key words: Uniformization of singularity, monodromy, degeneration of Riemann surfaces, fractional Dehn twist, small group, pseudo-reflection group.

Now let $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ be projections $f(x_1, x_2, \dots, x_n, t) = t$, $g(z_1, z_2, \dots, z_n, t) = t$. A smooth fiber $f^{-1}(s)$ (resp. $g^{-1}(s)$), as $s \to 0$, degenerates to the singular fiber $f^{-1}(0)$ (resp. $g^{-1}(0)$). When $n = 2$, the topological monodromy of $f : V \to \mathbb{C}$ (and $g : W \to \mathbb{C}$) is a $(-d)$-*Dehn twist* (Figure 1.1). When $n \geq 3$, the topological monodromy of $f : V \to \mathbb{C}$ is a *generalized Dehn twist*, and is described by using the *double covering method* (see [AGV], p.6). The topological monodromy of $g : W \to \mathbb{C}$ is another generalization of a Dehn twist. *In what follows, we exclusively consider W, and write it as A_{d-1}.*

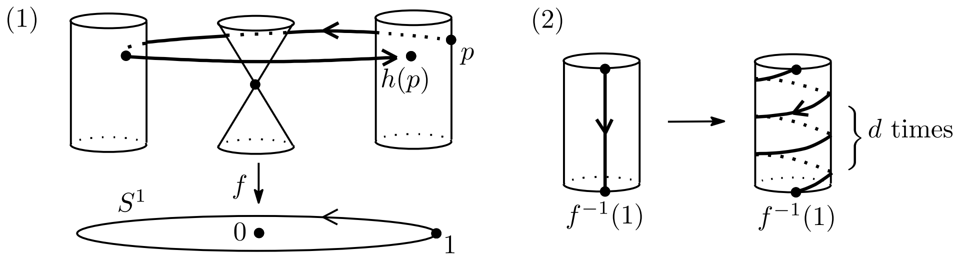

Fig. 1.1. (1) The topological monodromy of $f : V \to \mathbb{C}$. (2) It is a $(-d)$-Dehn twist.

We next introduce a *fractional* Dehn twist. Where a and m $(0 < a < m)$ and b and n $(0 < b < n)$ are two pairs of relatively prime integers, an $\left(\frac{a}{m}, \frac{b}{n}\right)$-*fractional Dehn twist* is a self-homeomorphism of an annulus $[0, 1] \times S^1$ illustrated in Figure 1.2. It is explicitly given by $(t, e^{\mathrm{i}\theta}) \mapsto (t, e^{2\pi\mathrm{i}\{(1-t)a/m - tb/n\}} e^{\mathrm{i}\theta})$.

More generally, where κ is an integer, an $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-*fractional Dehn twist* is defined as the composite map of a $(+\kappa)$-Dehn twist and an $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist (Figure 1.3). If $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, the $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist appears as the topological monodromy of a degeneration: Set $c := \gcd(m, n)$, $m' := m/c$, $n' := n/c$ and $d := n'a + m'b + m'n'c\kappa$, or $d = m'n'c\left(\frac{a}{m} + \frac{b}{n} + \kappa\right)$. Let Γ be the cyclic group acting on A_{d-1} generated by an automorphism $\gamma : (z, w, t) \in A_{d-1} \mapsto (e^{2\pi\mathrm{i}a/m} z, e^{2\pi\mathrm{i}b/n} w, e^{2\pi\mathrm{i}/m'n'c} t) \in A_{d-1}$. The induced map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$

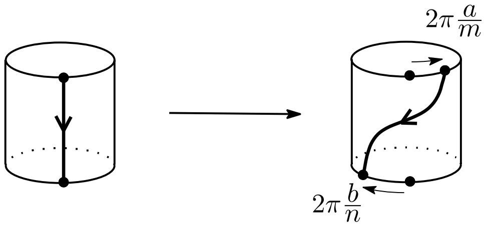

Fig. 1.2. An $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist.

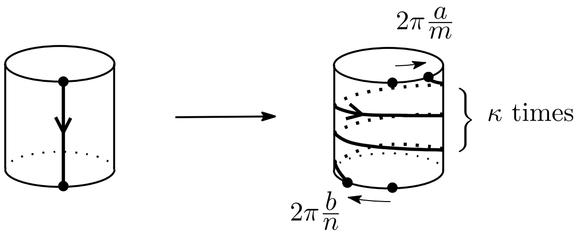

Fig. 1.3. An $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

by a Γ-invariant map $\Phi : (z, w, t) \in A_{d-1} \mapsto t^{m'n'c} \in \mathbb{C}$ is a degeneration whose topological monodromy is the $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

We point out that $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ arises as a *local model* of a degeneration of Riemann surfaces; recall that a proper surjective holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to $\Delta := \{s \in \mathbb{C} : |s| < 1\}$ is a *degeneration of Riemann surfaces* (of genus g) if $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a Riemann surface (of genus g). Figure 1.4 (1) illustrates an example of a singular fiber, which consists of cores, branches and a trunk. Contracting the branches and the trunk of this singular fiber yields the *canonical model* $\pi' : M' \to \Delta$ of $\pi : M \to \Delta$; the branches and the trunk become *cyclic quotient singularities* of M' (because the contraction of a chain of projective lines yields a cyclic quotient singularity). The singular fiber $(\pi')^{-1}(0)$ is thus as illustrated in Figure 1.4 (2). Let $p \in \pi^{-1}(0)$ be

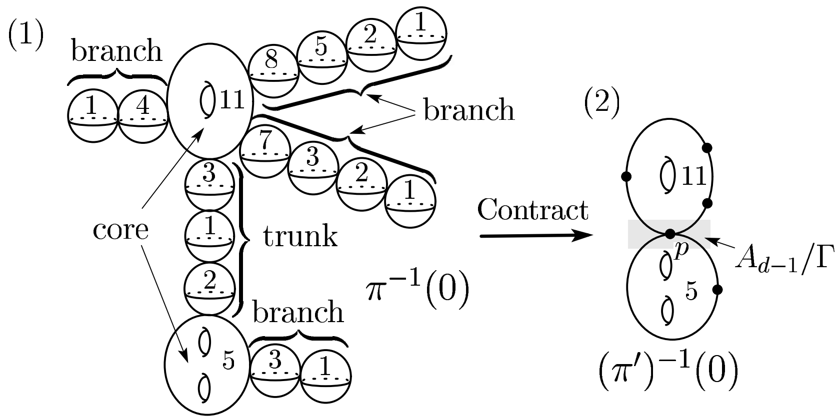

Fig. 1.4. Intersections of irreducible components are *transversal*. The positive integer on an irreducible component denotes the *multiplicity* of that component. The five bold points on $(\pi')^{-1}(0)$ denote the *cyclic quotient singularities* of M'.

the point resulting from the contraction of the trunk. A neighborhood of $p \in M'$ is then isomorphic to A_{d-1}/Γ (for $a/m = 4/11$, $b/n = 3/5$, $\kappa = 0$). Moreover the restriction $\pi'|_{A_{d-1}/\Gamma}$ coincides with $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$, and the topological monodromy of $\pi'|_{A_{d-1}/\Gamma}$ is a $-\left(\frac{4}{11}, \frac{3}{5}, 0\right)$-fractional Dehn twist.

More generally, for any trunk (see Figure 1.5), the same holds: *A neighborhood of its contraction is isomorphic to A_{d-1}/Γ (for some a/m, b/n, κ), and the local topological monodromy is a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist, and A_{d-1}/Γ is a cyclic quotient singularity.*

In the above, the contraction of a trunk yields A_{d-1}/Γ, which is a cyclic quotient singularity. In fact, for *any* Γ (that is, for *any* a/m, b/n, κ), the quotient A_{d-1}/Γ is a cyclic quotient singularity, that is, $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ for some cyclic group $G = \langle g \rangle$, where g is of the form $(u, v) \mapsto (e^{2\pi \mathrm{i}/l}u,\ e^{2\pi \mathrm{i}q/l}v)$ where l and q are some relatively prime positive integers. This is the starting point of our present work — we generalize it to the higher-dimensional case in order to apply it to degenerations of complex manifolds.

Let a_i and m_i $(i = 1, 2, \ldots, n)$ be relatively prime integers such that $0 < a_i < m_i$. Set $c := \gcd(m_1, m_2, \ldots, m_n)$ and $m_i' := m_i/c$. Take an

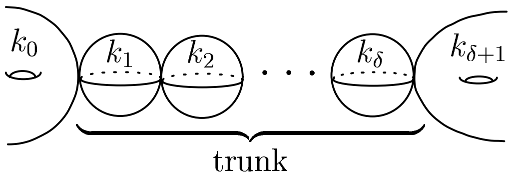

Fig. 1.5. A trunk is a chain of projective lines connecting cores. (k_0, $k_1, \dots, k_{\delta+1}$ are multiplicities.)

integer κ such that $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and set

$$d := \left(\sum_{i=1}^{n} a_i m_1' \cdots \check{m}_i' \cdots m_n' \right) + m_1' m_2' \cdots m_n' c\kappa,$$

where $\check{m}_i'$ means the omission of m_i'. Or

$$d = m_1' m_2' \cdots m_n' c \Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa \Big). \tag{1.1}$$

Now let γ be an automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma : \ (x_1, \dots, x_n, t) \ \longmapsto \ (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ \dots, e^{2\pi \mathrm{i} a_n/m_n} x_n, \ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t).$$

Then (1.1) ensures that γ preserves $A_{d-1} := \big\{ (x_1, x_2, \dots, x_n, t) \in \mathbb{C}^{n+1} : x_1 x_2 \cdots x_n = t^d \big\}$. Let Γ be the cyclic group generated by γ. Let $\Phi : A_{d-1} \to \mathbb{C}$ be a Γ-invariant holomorphic map given by $\Phi(x_1, x_2, \cdots, x_n, t) = t^{m_1' m_2' \cdots m_n' c}$, and $\overline{\Phi}$ denote the holomorphic map on A_{d-1}/Γ induced by Φ. The topological monodromy of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ is called a $-\Big(\frac{a_1}{m_1}, \frac{a_2}{m_2}, \cdots, \frac{a_n}{m_n}, \kappa \Big)$*-fractional Dehn twist*. This will be described in [SaTa].

The present paper shows that the cyclic quotient A_{d-1}/Γ is *uniformized* by a small abelian group. Here a finite subgroup of $GL(n, \mathbb{C})$ is *small* if it contains no pseudo-reflections. The following was originally proved by the second author:

(i) **Uniformization theorem for dimension 2** *There exists a small cyclic group $G \subset GL(2, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$* (Theorem 2.1). (This ensures that the minimal resolution of A_{d-1}/Γ is obtained by the *Hirzebruch-Jung resolution*.)

(ii) *Moreover under this isomorphism,* $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ *corresponds to the map* $\overline{\phi} : \mathbb{C}^2/G \to \mathbb{C}$ *induced by the* G*-invariant map* $\phi : \mathbb{C}^2 \to \mathbb{C}$, $\phi(u,v) = u^n v^m$ (Lemma 2.4).

This is generalized as follows (a diagonal matrix $\begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$ is denoted by $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$):

MAIN THEOREM A. (i) *There exists a small finite abelian group* $G \subset GL(n, \mathbb{C})$ *such that* $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ (Theorem 6.3), *where* G *is cyclic only when* $n = 2$. *Next set* $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ *where* $\check{m}'_i$ *means the omission of* m'_i. *Then* l_i *is a positive integer* (Remark 3.1) *and* G *is generated by the diagonal matrices* Q, R_1, R_2, $\ldots$, R_{n-1} *given by*

$Q = \mathrm{diag}\bigl(e^{2\pi i l_1 a_1/cd},\ e^{2\pi i l_2 a_2/cd}, \ldots, e^{2\pi i l_{n-1} a_{n-1}/cd},\ e^{2\pi i l_n (a_n + m_n \kappa)/cd}\bigr)$ *and*

$R_i = \mathrm{diag}\bigl(1, \ldots, 1,\ e^{2\pi i l_i m'_i/d},\ 1, \ldots, 1,\ e^{-2\pi i l_n m'_n/d}\bigr)$, *where* $e^{2\pi i l_i m'_i/d}$ *lies in the* i*th place* (Corollary 7.13).

(ii) *Under the isomorphism in* (i), $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ *corresponds to the map* $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$ *induced by the* G*-invariant map* $\phi : \mathbb{C}^n \to \mathbb{C}$, $\phi(v_1, v_2, \ldots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$ *where* $k_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$ (Theorem 6.6 (2)).

REMARK. A resolution of $\mathbb{C}^n/G$ gives a decomposition of the monodromy (a *higher-dimensional fractional Dehn twist*) of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ into subtwists along the exceptional set. It seems that T. Ashikaga's work on resolutions [Ash], [AsIs] is related to this.

The construction of G in Main Theorem A uses the following diagram of coverings:

(1.2)
$$\begin{tikzcd} & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}, \\ \mathbb{C}^n & & & \end{tikzcd}$$

where p, q and r are covering maps given by

- $p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$ (note: $p : \widetilde{A}_{d-1} \to \mathbb{C}^n$ is the universal covering of A_{d-1}),

- $q(X_1, X_2, \ldots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$,
- $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$, where l_i is the positive integer appearing in Main Theorem A.

We *lift and descend* Γ with respect to the diagram (1.2): Lift Γ to a group $\widetilde{\Gamma}$ (acting on $\widetilde{A}_{d-1}$), and then descend $\widetilde{\Gamma}$ to a group H (acting on $\mathbb{C}^n$), and next descend H to a group G (acting on $\mathbb{C}^n$). Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$ and $G \subset GL(n, \mathbb{C})$ is a small finite abelian group. We remark that in the case $n = 2$, H is *always* small, so the descent with respect to r is actually unnecessary. Even for $n \geq 3$, it may occur that H is small. Indeed:

Main Theorem B (Theorem 5.14 (2)). *The finite abelian group H is small if and only if* $\gcd(m'_i, m'_j) = 1$ *for any i, j such that $i \neq j$.*

Next let P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections of H. Regard κ as a 'parameter', and write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. Then the following holds:

Main Theorem C (Lemma 6.7 and Theorem 6.8).

(1) *The pseudo-reflection subgroup P_κ of H_κ does not depend on κ: Let κ_0 denote the least integer among κ in the definition of d, then*

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots .$$

(2) (**Equi-smallness**) *If H_{κ_0} is small, then H_κ is small for any κ, and if H_{κ_0} is not small, then H_κ is not small for any κ.*

2. Uniformization Theorem for Dimension 2

Let a and m $(0 < a < m)$ and b and n $(0 < b < n)$ be two pairs of relatively prime integers, and set $c := \gcd(m, n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$. (Note that m' and n' are integers.) Take an integer κ such that $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, and set $d := an' + bm' + m'n'c\kappa$. Let γ be the automorphism of $\mathbb{C}^3$ given by $\gamma : (z, w, t) \mapsto (e^{2\pi \mathrm{i} a/m} z,\ e^{2\pi \mathrm{i} b/n} w,\ e^{2\pi \mathrm{i}/m'n'c} t)$. Then γ preserves $A_{d-1} := \{zw = t^d\}$ in $\mathbb{C}^3$. Let Γ be the cyclic group generated by γ. Then:

THEOREM 2.1 (Uniformization theorem [Tak]). *There exists a small cyclic group* $G \subset GL(2,\mathbb{C})$ *such that* $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$. *Here* G *is explicitly given as follows*: *Let* a^* $(0 < a^* < m)$ *be the integer such that* $aa^* \equiv 1 \bmod m$, *and let* q $(0 < \mathsf{q} < cd)$ *be the integer such that* $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$ (the right hand side is indeed an integer; see Remark 2.2 below). *Then* G *is generated by the automorphism* g *of* $\mathbb{C}^2$ *given by* $g : (u,v) \mapsto (e^{2\pi\mathrm{i}/cd}u,\ e^{2\pi\mathrm{i}\mathsf{q}/cd}v)$.

REMARK 2.2. Substituting $d := an' + bm' + m'n'c\kappa$ into $\dfrac{a^*d - n'}{m'}$ yields $\dfrac{aa^* - 1}{m'}n' + a^*b + a^*n'c\kappa$. Here since $aa^* \equiv 1 \bmod m$, we may write $aa^* - 1 = Km\,(= Km'c)$, where K is an integer. Then $\dfrac{a^*d - n'}{m'} = Kn'c + a^*b + a^*n'c\kappa$.

PROOF. Note first that the universal covering $p : \widetilde{A}_{d-1}\ (= \mathbb{C}^2) \to A_{d-1}$ of A_{d-1} is a d-fold covering given by $p(X,Y) = (X^d, Y^d, XY)$. Next let $q : \widetilde{A}_{d-1} \to \mathbb{C}^2$ be an $m'n'$-fold covering given by $q(X,Y) = (X^{m'}, Y^{n'})$, and consider the following diagram:

$$\begin{tikzcd} & \widetilde{A}_{d-1} = \mathbb{C}^2 \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{C}^2 & & A_{d-1}. \end{tikzcd} \tag{2.1}$$

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to p, and G be the descent of $\widetilde{\Gamma}$ with respect to q. Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^2/G$.

We next show that G is generated by g. For $j = 1,2,\ldots,m'n'c$ and $k = 1,2,\ldots,d$, let $\widetilde{\gamma}_{j,k} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by $\widetilde{\gamma}_{j,k} : (X,Y) \mapsto (e^{2\pi\mathrm{i}(ja+km)/md}X,\ e^{2\pi\mathrm{i}\{j(b+n\kappa)-kn\}/nd}Y)$, and $g_{j,k} : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphism given by $g_{j,k} : (u,v) \mapsto (e^{2\pi\mathrm{i}(ja+km)/cd}u,\ e^{2\pi\mathrm{i}\{j(b+n\kappa)-kn\}/cd}v)$. Then for each $j = 1,2,\ldots,m'n'c$, the set of all lifts of $\gamma^j \in \Gamma$ with respect to p is $\{\widetilde{\gamma}_{j,k} : k = 1,2,\ldots,d\}$, and for any j, k, the descent of $\widetilde{\gamma}_{j,k}$ with respect to q is $g_{j,k}$. Hence $\widetilde{\Gamma}$ and G are explicitly given by

$$\widetilde{\Gamma} = \big\{\widetilde{\gamma}_{j,k} : j = 1,2,\ldots,m'n'c,\ k = 1,2,\ldots,d\big\},$$
$$G = \big\{g_{j,k} : j = 1,2,\ldots,m'n'c,\ k = 1,2,\ldots,d\big\}.$$

Therefore G is generated by the following two automorphisms α, β:

$$\begin{aligned}\alpha:\ (u,v)\ &\longmapsto\ (e^{2\pi\mathrm{i}a/cd}u, e^{2\pi\mathrm{i}(b+n\kappa)/cd}v),\\ \beta:\ (u,v)\ &\longmapsto\ (e^{2\pi\mathrm{i}m'/d}u, e^{-2\pi\mathrm{i}n'/d}v).\end{aligned}$$

Let l $(0 < l < cd)$ be the integer such that $l \equiv \dfrac{1-aa^*}{m}$ mod cd. Then by Corollary 7.17,

$$\alpha^{a^*}\beta^l = g, \qquad g^a = \alpha, \qquad g^m = \beta.$$

Hence $g \in G$ and G is generated by g.

We next show that G is small. Recall that G is generated by $g : (u,v) \mapsto (e^{2\pi\mathrm{i}/cd}u,\ e^{2\pi\mathrm{i}\mathsf{q}/cd}v)$. Here q and cd are relatively prime (Lemma 2.3 (2) below), so G is small. □

Explicit form of $A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$: Since $\widetilde{\Gamma}$ is the lift of Γ with respect to p, the map p induces an isomorphism $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$, and since G is the descent of $\widetilde{\Gamma}$ with respect to q, the map q induces an isomorphism $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^2/G$. The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ in the uniformization theorem (Theorem 2.1) is then given by $\Psi := \overline{q} \circ \overline{p}^{\,-1} :\ A_{d-1}/\Gamma \xrightarrow{\ \cong\ } \mathbb{C}^2/G$. We show that this map is explicitly given by

$$\Psi\big([x,y,t]\big) = \big[x^{m'/d},\ y^{n'/d}\big], \tag{2.2}$$

where $[x,y,t] \in A_{d-1}/\Gamma$ and $\big[x^{m'/d},\ y^{n'/d}\big] \in \mathbb{C}^2/G$ denote the images of $(x,y,t) \in A_{d-1}$ and $\big(x^{m'/d},\ y^{n'/d}\big) \in \mathbb{C}^2$ respectively. To see (2.2), first note that since $p(X,Y) = (X^d, Y^d, XY)$, we have $\overline{p}\big([X,Y]\big) = \big[X^d, Y^d, XY\big]$, so $\overline{p}^{\,-1}\big([x,\ y,\ t]\big) = \big[x^{1/d},\ y^{1/d}\big]$. Next since $q(X,\ Y) = (X^{m'},\ Y^{n'})$, we have $\overline{q}\big([x^{1/d},\ y^{1/d}]\big) = \big[x^{m'/d},\ y^{n'/d}\big]$. Hence $\overline{q} \circ \overline{p}^{\,-1}\big([x,y,t]\big) = \big[x^{m'/d},\ y^{n'/d}\big]$.

Supplement Let a^* $(0 < a^* < m)$ be the integer such that $aa^* \equiv 1$ mod m, and let q $(0 < \mathsf{q} < cd)$ be the integer such that $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'}$ mod cd, where the right hand side is indeed an integer (Remark 2.2). Similarly let b^* $(0 < b^* < n)$ be the integer such that $bb^* \equiv 1$ mod n, and let r $(0 < \mathsf{r} < cd)$ be the integer such that $\mathsf{r} \equiv \dfrac{b^*d - m'}{n'}$ mod cd, where the right hand side is an integer as for q.

LEMMA 2.3.

(1) $\mathsf{qr} \equiv 1 \bmod cd$, *that is,* $\mathsf{r} = \mathsf{q}^*$.

(2) q *and* cd *are relatively prime.*

PROOF. (1): It suffices to show that $\dfrac{a^*d-n'}{m'}\,\dfrac{b^*d-m'}{n'} \equiv 1 \bmod cd$. Here

$$\frac{a^*d-n'}{m'}\,\frac{b^*d-m'}{n'} = d\left(\frac{aa^*-1}{m'}b^* + \frac{bb^*-1}{n'}a^* + a^*b^*c\kappa\right) + 1.$$

Write $aa^*-1 = Km \,(= Km'c)$ and $bb^*-1 = Ln \,(= Ln'c)$. Then

$$\begin{aligned}\frac{a^*d-n'}{m'}\,\frac{b^*d-m'}{n'} &= cd\big(Kb^* + La^* + a^*b^*\kappa\big) + 1\\ &\equiv 1 \bmod cd.\end{aligned}$$

(2): Since $\mathsf{qr} \equiv 1 \bmod cd$, $\mathsf{qr} = 1 + Mcd$ for some integer M. Then $\mathsf{qr} - Mcd = 1$. Here $\gcd(\mathsf{q}, cd)$ divides the left hand side, so divides 1, thus $\gcd(\mathsf{q}, cd) = 1$. $\square$

Correspondence between functions Let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(z,w,t) = t^{m'n'c}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. As explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ in the uniformization theorem, the holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^2/G$. *This map is explicitly given.* First let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic map defined by $\phi(u,v) = u^n v^m$. Then ϕ is G-invariant. To see this, recall that by Theorem 2.1, the cyclic group G is generated by $g : (u,v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a^*d-n'}{m'} \bmod cd$. Then

$$\begin{aligned}\phi\circ g(u,v) &= \phi(e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v) = e^{2\pi \mathrm{i}c(n'+m'\mathsf{q})/cd}u^n v^m\\ &= e^{2\pi \mathrm{i}ca^*d/cd}u^n v^m \qquad \text{by } n' + m'\mathsf{q} \equiv a^*d \bmod cd\\ &= e^{2\pi \mathrm{i}a^*}u^n v^m = u^n v^m\\ &= \phi(u,v).\end{aligned}$$

Thus ϕ is G-invariant, so induces a holomorphic map $\overline{\phi}:\mathbb{C}^2/G\to\mathbb{C}$.

Lemma 2.4 ([Tak]). *Under the isomorphism* $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ *given by* (2.2), $\overline{\Phi}$ *corresponds to* $\overline{\phi}$, *that is,* $\overline{\Phi}=\overline{\phi}\circ\Psi$.

Proof. Note first that

$$\begin{aligned}\overline{\phi}\circ\Psi\big([x,y,t]\big) &= \overline{\phi}\left([x^{m'/d},\, y^{n'/d}]\right)\\ &= x^{m'n/d}y^{n'm/d} = (xy)^{m'n'c/d}.\end{aligned}$$

Here $xy=t^d$ (because $(x,y,t)\in A_{d-1}$), so $\overline{\phi}\circ\Psi\big([x,y,t]\big)=t^{m'n'c}$. Thus $\overline{\phi}\circ\Psi\big([x,y,t]\big)=\overline{\Phi}\big([x,y,t]\big)$. $\Box$

Where $\mathfrak{r}:R\to A_{d-1}/\Gamma$ is the minimal resolution of A_{d-1}/Γ, the composite map $\pi:=\overline{\Phi}\circ\mathfrak{r}:R\to\mathbb{C}$ is a degeneration. As we see immediately, *thanks to* the uniformization theorem, the degeneration $\pi:R\to\mathbb{C}$ is isomorphic to a degeneration which is easy to describe.

Where $\mathfrak{r}':R'\to\mathbb{C}^2/G$ is the minimal resolution of $\mathbb{C}^2/G$, the composite map $\pi':=\overline{\phi}\circ\mathfrak{r}':R'\to\mathbb{C}$ is a degeneration. Since A_{d-1}/Γ and $\mathbb{C}^2/G$ are isomorphic (Theorem 2.1), two minimal resolutions $\mathfrak{r}:R\to A_{d-1}/\Gamma$ and $\mathfrak{r}':R'\to\mathbb{C}^2/G$ are isomorphic, that is, there exists an isomorphism $\widetilde{\Psi}:R\to R'$ that makes the following diagram commute:

(2.3)
$$\begin{array}{ccc} R & \xrightarrow[\cong]{\widetilde{\Psi}} & R' \\ {\scriptstyle\mathfrak{r}}\downarrow & & \downarrow{\scriptstyle\mathfrak{r}'} \\ A_{d-1}/\Gamma & \xrightarrow[\cong]{\Psi} & \mathbb{C}^2/G. \end{array}$$

Theorem 2.5. *The following diagram commutes*:

(2.4)
$$\begin{tikzcd} R \arrow[rr, "\widetilde{\Psi}", "\cong"'] \arrow[dr, "\pi"'] & & R' \arrow[dl, "\pi'"] \\ & \mathbb{C}. & \end{tikzcd}$$

Hence two degenerations $\pi:=\overline{\Phi}\circ\mathfrak{r}:R\to\mathbb{C}$ *and* $\pi':=\overline{\phi}\circ\mathfrak{r}':R'\to\mathbb{C}$ *are isomorphic.*

PROOF. By Lemma 2.4, the following diagram commutes:

$$
(2.5) \qquad \begin{tikzcd} A_{d-1}/\Gamma \arrow[rr, "\Psi", "\cong"'] \arrow[dr, "\overline{\Phi}"'] & & \mathbb{C}^2/G \arrow[dl, "\overline{\phi}"] \\ & \mathbb{C}. & \end{tikzcd}
$$

Combining the commutative diagrams (2.3) and (2.5) yields the commutative diagram (2.4). □

The degeneration $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ may be described as follows: Since G is cyclic, $\mathbb{C}^2/G$ has a (unique) cyclic quotient singularity, which is resolved by a chain of projective lines (*Hirzebruch-Jung resolution*). Accordingly the singular fiber $(\pi')^{-1}(0)$ of $\pi' : R' \to \mathbb{C}$ is as illustrated in Figure 2.1 (see also Remark 2.6).

REMARK 2.6. The multiplicities of the singular fiber $(\pi')^{-1}(0)$ in Figure 2.1 is explicitly determined from m, n, a, b, κ. Let a^* and b^* $(0 < a^* < m,\ 0 < b^* < n)$ be the integers such that $aa^* \equiv 1 \bmod m$ and $bb^* \equiv 1 \bmod n$. Define then two sequences of integers $m_0 > m_1 > \cdots > m_\lambda = 1$ and

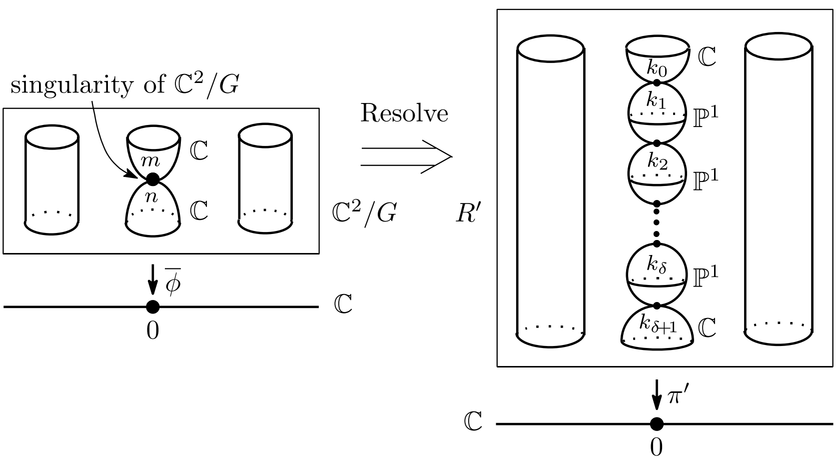

Fig. 2.1. The positive integers $k_0, k_1, \ldots, k_{\delta+1}$ are multiplicities. They are explicitly determined from Γ, more specifically, from m, n, a, b, κ (Remark 2.6).

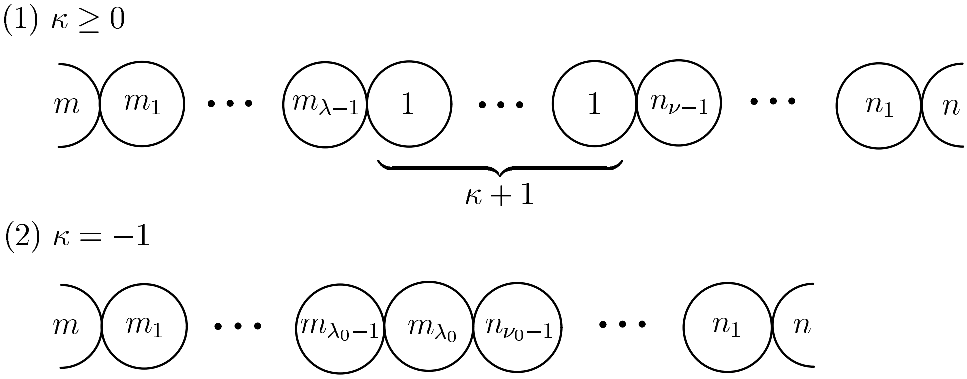

Fig. 2.2. The singular fibers for (1) $\kappa \geq 0$ and (2) $\kappa = -1$. A circle stands for $\mathbb{P}^1$ and a hemicircle for $\mathbb{C}$. (Each intersection is a node.)

$n_0 > n_1 > \cdots > n_\nu = 1$ inductively by the division algorithm with *negative* residues:

$$\begin{cases} m_0 := m, \quad m_1 := a^*, \\ m_{i-1} = s_i m_i - m_{i+1} \quad (0 < m_{i+1} < m_i), \qquad i = 1, 2, \ldots, \lambda - 1, \end{cases}$$

$$\begin{cases} n_0 := n, \quad n_1 := b^*, \\ n_{i-1} = t_i n_i - n_{i+1} \quad (0 < n_{i+1} < n_i), \qquad i = 1, 2, \ldots, \nu - 1. \end{cases}$$

Then:

(i) If $\kappa \geq 0$, then $(\pi')^{-1}(0)$ is as illustrated in (1) of Figure 2.2.

(ii) If $\kappa = -1$, then there exists a unique pair of integers λ_0 and ν_0 $(0 < \lambda_0 < \lambda,\ 0 < \nu_0 < \nu)$ such that $m_{\lambda_0+1} + n_{\nu_0+1} = m_{\lambda_0} = n_{\lambda_0}$, and $(\pi')^{-1}(0)$ is as illustrated in (2) of Figure 2.2.

3. Lifting and Descent

3.1. Diagram of covering maps

We generalize the uniformization theorem for dimension 2 (Theorem 2.1) to an arbitrary dimension. First let a_i and m_i $(i = 1, 2, \ldots, n)$ be relatively prime integers such that $0 < a_i < m_i$. If κ is an integer satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, then

$$\kappa \geq -n + 1. \tag{3.1}$$

Indeed since $0 < a_i < m_i$, we have $0 < \frac{a_i}{m_i} < 1$ $(i = 1, 2, \ldots, n)$, so $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} < n$, thus $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa < n + \kappa$. Here the left hand side is positive by assumption, so $0 < n + \kappa$, that is, $-n + 1 \leq \kappa$.

Next set $c := \gcd(m_1, m_2, \ldots, m_n)$, $m'_i := m_i/c$ and

$$d := \left(\sum_{i=1}^{n} a_i m'_1 \cdots \check{m}'_i \cdots m'_n \right) + m'_1 m'_2 \cdots m'_n c\kappa, \tag{3.2}$$

where $\check{m}'_i$ means the omission of m'_i. Note that $d > 0$, indeed

$$d = m'_1 m'_2 \cdots m'_n c \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa \right) > 0. \tag{3.3}$$

Rewrite the equation on the left hand side as

$$\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} = \frac{d}{m'_1 m'_2 \cdots m'_n c} - \kappa.$$

Then $e^{2\pi\mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi\mathrm{i} d/m'_1 m'_2 \cdots m'_n c}$. Here $e^{-2\pi\mathrm{i}\kappa} = 1$, so

$$e^{2\pi\mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi\mathrm{i} d/m'_1 m'_2 \cdots m'_n c}. \tag{3.4}$$

Now let γ be an automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma : \ (x_1, \ldots, x_n, t) \ \mapsto \ (e^{2\pi\mathrm{i}a_1/m_1} x_1, \ \ldots, e^{2\pi\mathrm{i}a_n/m_n} x_n, \ e^{2\pi\mathrm{i}/m'_1 m'_2 \cdots m'_n c} t).$$

Then γ preserves $A_{d-1} := \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} \ : \ x_1 x_2 \cdots x_n = t^d\}$, that is, γ maps A_{d-1} to itself. Namely if $x_1 x_2 \cdots x_n = t^d$, then

$$(e^{2\pi\mathrm{i}a_1/m_1} x_1)(e^{2\pi\mathrm{i}a_2/m_2} x_2) \cdots (e^{2\pi\mathrm{i}a_n/m_n} x_n) = (e^{2\pi\mathrm{i}/m'_1 m'_2 \cdots m'_n c} t)^d,$$

that is, $e^{2\pi\mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} x_1 x_2 \cdots x_n = e^{2\pi\mathrm{i} d/m'_1 m'_2 \cdots m'_n c} t^d$. This indeed holds by (3.4). Now let Γ be the cyclic group generated by the automorphism γ of A_{d-1}.

The universal covering $p : \widetilde{A}_{d-1}$ $(= \mathbb{C}^n) \to A_{d-1}$ of A_{d-1} is a d^{n-1}-fold covering given by $p : (X_1, X_2, \ldots, X_n) \mapsto (X_1^d, \ X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$.

Consider the following diagram of coverings:

$$\begin{tikzcd}
 & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\
 & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}, \\
\mathbb{C}^n & & &
\end{tikzcd} \tag{3.5}$$

where

- $q : (X_1, X_2, \ldots, X_n) \mapsto (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$ is an $m'_1 m'_2 \cdots m'_n$-fold covering,
- $r : (u_1, u_2, \ldots, u_n) \mapsto (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$ is an $l_1 l_2 \cdots l_n$-fold covering. Here
$$l_i := \frac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)} \qquad (i = 1, 2, \ldots, n),$$
where $\check{m}'_i$ means the omission of m'_i. Note that l_i is a positive integer (see Remark 3.1 below).

Remark 3.1. $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ is a (positive) integer, because from the definition of lcm, the denominator $\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$ divides the numerator $m'_1 \cdots \check{m}'_i \cdots m'_n$.

Now let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p, H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and G be the descent of H with respect to the covering r. We will show that G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ (the uniformization theorem). We begin with some preparation.

3.2. $\widetilde{\Gamma}$, H and G are finite groups

We first show that $\widetilde{\Gamma}$ is a group.

(i) $1 \in \widetilde{\Gamma}$: This is the trivial lift of $1 \in \Gamma$ (that is the identity map of $\widetilde{A}_{d-1}$).

(ii) $\xi \in \widetilde{\Gamma} \Rightarrow \xi^{-1} \in \widetilde{\Gamma}$: If ξ is a lift of $\gamma^j \in \Gamma$, then ξ^{-1} is a lift of $\gamma^{-j} \in \Gamma$.

(iii) $\xi_1, \xi_2 \in \widetilde{\Gamma} \Rightarrow \xi_1 \xi_2 \in \widetilde{\Gamma}$: If ξ_1, ξ_2 are lifts of γ^j, $\gamma^k \in \Gamma$, then $\xi_1 \xi_2$ is a lift of $\gamma^{j+k} \in \Gamma$.

We next show that H is a group as follows (similarly we can show that G is a group):

(i)' $1 \in H$: This is the descent of $1 \in \widetilde{\Gamma}$.

(ii)' $h \in H \Rightarrow h^{-1} \in H$: If h is the descent of $\xi \in \widetilde{\Gamma}$, then h^{-1} is the descent of $\xi^{-1} \in \widetilde{\Gamma}$.

(iii)' $h_1, h_2 \in H \Rightarrow h_1h_2 \in H$: If h_1, h_2 are the descents of $\xi_1, \xi_2 \in \widetilde{\Gamma}$, then h_1h_2 is the descent of $\xi_1\xi_2 \in \widetilde{\Gamma}$.

The orders of $\widetilde{\Gamma}$, H and G are determined as follows (below, $|\widetilde{\Gamma}|$, $|H|$ and $|G|$ denote the orders):

Order of $\widetilde{\Gamma}$: Since $\widetilde{\Gamma}$ is the lift of Γ with respect to the d^{n-1}-fold covering p, we have $|\widetilde{\Gamma}| = d^{n-1}|\Gamma|$. Here $|\Gamma| = m_1'm_2'\cdots m_n'c$, so $|\widetilde{\Gamma}| = m_1'm_2'\cdots m_n'cd^{n-1}$.

Order of H: Since H is the descent of $\widetilde{\Gamma}$ (or $\widetilde{\Gamma}$ is the lift of H) with respect to the $m_1'm_2'\cdots m_n'$-fold covering q, we have $|\widetilde{\Gamma}| = m_1'm_2'\cdots m_n'|H|$. Here $|\widetilde{\Gamma}| = m_1'm_2'\cdots m_n'cd^{n-1}$ so $|H| = cd^{n-1}$.

Order of G: Since G is the descent of H (or H is the lift of G) with respect to the $l_1l_2\cdots l_n$-fold covering r, we have $|H| = l_1l_2\cdots l_n|G|$. Here $|H| = cd^{n-1}$, so $|G| = \dfrac{cd^{n-1}}{l_1l_2\cdots l_n}$. (This is indeed an integer. See Remark 3.3 below.)

The results obtained in this section are summarized as follows:

PROPOSITION 3.2. *Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and let G be the descent of H with respect to the covering r. Then*:

(1) *The lift $\widetilde{\Gamma}$ of Γ is a finite group of order $m_1'm_2'\cdots m_n'cd^{n-1}$.* (In fact, $\widetilde{\Gamma}$ is abelian. See Lemma 4.7.)

(2) *The descent H of $\widetilde{\Gamma}$ is a finite group of order cd^{n-1}.* (In fact, H is abelian. See Lemma 4.8 (3).)

(3) *The descent G of H is a finite group of order $\dfrac{cd^{n-1}}{l_1l_2\cdots l_n}$.* (In fact, G is abelian. See Lemma 6.1 (C).)

Remark 3.3. The fact that $|G| = \dfrac{cd^{n-1}}{l_1 l_2 \cdots l_n}$ is an integer is reconfirmed as follows (we show this only for $n=3$): Using

$$\begin{cases} d = m_1' m_2' m_3' c \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa \right) \quad \text{(see (3.3))}, \\ l_1 := \dfrac{m_2' m_3'}{\operatorname{lcm}(m_2', m_3')}, \ l_2 := \dfrac{m_1' m_3'}{\operatorname{lcm}(m_1', m_3')}, \ l_3 := \dfrac{m_1' m_2'}{\operatorname{lcm}(m_1', m_2')}, \end{cases}$$

rewrite $|G| = \dfrac{cd^2}{l_1 l_2 l_3}$ as

$$\begin{aligned} |G| &= c\Big\{ \prod_{i \neq j} \operatorname{lcm}(m_i',\, m_j') \Big\} c^2 \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa \right)^2 \\ &= c \prod_{i \neq j} \operatorname{lcm}(m_i', m_j') \left\{ (c\kappa)^2 + \sum_{i=1}^{3} \left(\frac{2a_i c\kappa}{m_i'} + \frac{a_i^2}{(m_i')^2} \right) + \sum_{i \neq j} \frac{2a_i a_j}{m_i' m_j'} \right\}. \end{aligned}$$

Here $\prod_{i \neq j} \operatorname{lcm}(m_i', m_j') = \operatorname{lcm}(m_1',\, m_2')\operatorname{lcm}(m_1',\, m_3')\operatorname{lcm}(m_2',\, m_3')$ is divisible by m_i', $(m_i')^2$, $m_i' m_j'$, so the last expression is indeed an integer.

4. Determination of H

We keep the notation concerning the diagram (3.5). Moreover we adopt the following notation: For $j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c$,

- $\mathrm{Lift}^{(j)}$: The set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.
- $q_*\big(\mathrm{Lift}^{(j)}\big)$: The descent of $\mathrm{Lift}^{(j)}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.
- $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$: The descent of $q_*\big(\mathrm{Lift}^{(j)}\big)$ with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Then

- $\widetilde{\Gamma} = \displaystyle\bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$ is the lift of Γ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.

- $H = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.

- $G = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ is the descent of H with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Actually, $\widetilde{\Gamma} = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$ is a disjoint union. Namely, if $j \neq k$, then $\mathrm{Lift}^{(j)} \cap \mathrm{Lift}^{(k)} = \emptyset$. On the other hand, $H = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ and $G = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ are *not* disjoint unions. In fact, a descent of an element of $\mathrm{Lift}^{(j)}$ may coincide with that of an element of $\mathrm{Lift}^{(k)}$ $(j \neq k)$. In this case, $q_*\bigl(\mathrm{Lift}^{(j)}\bigr) \cap q_*\bigl(\mathrm{Lift}^{(k)}\bigr) \neq \emptyset$, and moreover, $r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr) \cap r_* \circ q_*\bigl(\mathrm{Lift}^{(k)}\bigr) \neq \emptyset$.

In what follows, we write $\widetilde{\Gamma}$ as a disjoint union: $\widetilde{\Gamma} = \coprod_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$.

4.1. The lifts of each element of Γ

We next determine the set $\mathrm{Lift}^{(j)}$ of all lifts of $\gamma^j \in \Gamma$ with respect to the covering p. For $j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c$, we first define a set $\Lambda^{(j)}$ of n-tuples of integers as follows:

$$\Lambda^{(j)} := \Bigl\{ (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \;:\; 0 \leq p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z} \Bigr\}. \tag{4.1}$$

Lemma 4.1. *The number of elements of $\Lambda^{(j)}$ is d^{n-1}.*

Proof. Setting $\Xi := \{(p_1, p_2, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1} \;:\; 0 \leq p_i < d\}$, consider a map $\varphi : \Lambda^{(j)} \to \Xi$ given by $(p_1, p_2, \ldots, p_{n-1}, p_n) \longmapsto (p_1, p_2, \ldots, p_{n-1})$. Here Ξ consists of d^{n-1} elements, thus it suffices to show that φ is bijective.

Surjectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty. Set $N := j\kappa - \sum_{i=1}^{n-1} p_i$ and let p_n $(0 \leq p_n < d)$ be the integer such that $p_n \equiv N \bmod d$. Then $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$. Moreover $\varphi(p_1, p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_{n-1})$, thus $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty.

Injectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. Note that $(p_1, p_2, \ldots, p_n)$ is contained in $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ precisely when p_n satisfies $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$, that is, $p_n \equiv j\kappa - \sum_{i=1}^{n-1} p_i \bmod d$. Such an integer p_n $(0 \leq p_n < d)$ is unique, so $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. $\Box$

Let $\Lambda^{(j)}$ be the set given by (4.1). For each $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ by

$$
\begin{aligned}
(X_1, X_2, \ldots, X_n) \longmapsto& \\
&\big(e^{2\pi\mathrm{i}(ja_1 + m_1 p_1)/m_1 d} X_1,\ e^{2\pi\mathrm{i}(ja_2 + m_2 p_2)/m_2 d} X_2, \ldots, e^{2\pi\mathrm{i}(ja_n + m_n p_n)/m_n d} X_n\big).
\end{aligned}
$$

Lemma 4.2. *For any* $(p_1, p_2, \ldots, p_n)$, $(p'_1, p'_2, \ldots, p'_n) \in \Lambda^{(j)}$, *the following hold*:

(1) *For* $i = 1, 2, \ldots, n$,

$$e^{2\pi\mathrm{i}(ja_i + m_i p_i)/m_i d} = e^{2\pi\mathrm{i}(ja_i + m_i p'_i)/m_i d} e^{2\pi\mathrm{i}(p_i - p'_i)/d}.$$

(2) *If* $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, *say* $p_i \neq p'_i$ *for some* i, *then* $e^{2\pi\mathrm{i}(ja_i + m_i p_i)/m_i d} \neq e^{2\pi\mathrm{i}(ja_i + m_i p'_i)/m_i d}$.

(3) *If* $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, *then* $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \neq \widetilde{\gamma}^{(j)}_{p'_1, p'_2, \ldots, p'_n}$.

Proof. (1): From $\dfrac{ja_i + m_i p_i}{m_i d} - \dfrac{ja_i + m_i p'_i}{m_i d} = \dfrac{p_i - p'_i}{d}$, we have $\dfrac{ja_i + m_i p_i}{m_i d} = \dfrac{ja_i + m_i p'_i}{m_i d} + \dfrac{p_i - p'_i}{d}$, which yields the equation in assertion.

(2): Since $0 \leq p_i < d$ and $0 \leq p'_i < d$, $p_i \neq p'_i$ implies $p_i \not\equiv p'_i \bmod d$, accordingly $\dfrac{p_i - p'_i}{d} \not\equiv 0 \bmod \mathbb{Z}$. Hence $e^{2\pi \mathrm{i}(p_i - p'_i)/d} \neq 1$ in (1), implying that $e^{2\pi \mathrm{i}(ja_i + m_i p_i)/m_i d} \neq e^{2\pi \mathrm{i}(ja_i + m_i p'_i)/m_i d}$.

(3): This follows from (2). □

We next show the following:

COROLLARY 4.3. *The number of elements of* $\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\}$ *is* d^{n-1}.

PROOF. By (3) of Lemma 4.2, the number of elements in the set $\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\}$ coincides with that of $\Lambda^{(j)}$, and by Lemma 4.1, it is d^{n-1}. □

Recall that $d = m'_1 m'_2 \cdots m'_n c \left(\sum_{i=1}^{n} \dfrac{a_i}{m_i} + \kappa \right)$ (see (3.3)), so

$$\sum_{i=1}^{n} \frac{a_i}{m_i} = \frac{d}{m'_1 m'_2 \cdots m'_n c} - \kappa. \tag{4.2}$$

LEMMA 4.4. *For any* $(p_1, p_2, \dots, p_n) \in \Lambda^{(j)}$,

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} \equiv \frac{j}{m'_1 m'_2 \cdots m'_n c} \bmod \mathbb{Z}.$$

PROOF. Using (4.2), the left hand side is rewritten as

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} = \frac{j}{m'_1 m'_2 \cdots m'_n c} - \frac{j\kappa}{d} + \sum_{i=1}^{n} \frac{p_i}{d}.$$

Here $\sum_{i=1}^{n} \dfrac{p_i}{d} \equiv \dfrac{j\kappa}{d} \bmod \mathbb{Z}$ (because $(p_1, p_2, \dots, p_n) \in \Lambda^{(j)}$), so $\sum_{i=1}^{n} \dfrac{ja_i + m_i p_i}{m_i d} \equiv \dfrac{j}{m'_1 m'_2 \cdots m'_n c} \bmod \mathbb{Z}$. □

COROLLARY 4.5. *For each j, let* $\mathrm{Lift}^{(j)}$ *be the set of all lifts of* $\gamma^j \in \Gamma$ *with respect to the covering* $p : \widetilde{A}_{d-1} \to A_{d-1}$. *Then the following hold*:

(1) *The number of elements of* $\mathrm{Lift}^{(j)}$ *is* d^{n-1}.

(2) *For any* $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \mathrm{Lift}^{(j)}$.

(3) $\mathrm{Lift}^{(j)} = \bigl\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\bigr\}$.

PROOF. (1): Since the covering p is d^{n-1}-fold, for each j, $\gamma^j \in \Gamma$ has d^{n-1} lifts, so $\mathrm{Lift}^{(j)}$ consists of d^{n-1} elements.

(2): It suffices to show that the following diagram commutes:

$$\begin{tikzcd} \widetilde{A}_{d-1} \arrow[r, "\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}"] \arrow[d, "p"'] & \widetilde{A}_{d-1} \arrow[d, "p"] \\ A_{d-1} \arrow[r, "\gamma^j"] & A_{d-1}. \end{tikzcd}$$

For $(X_1, \ldots, X_n) \in \widetilde{A}_{d-1}$,

$$\begin{aligned} &p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}(X_1, \ldots, X_n) \\ &= p\bigl(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/m_1d} X_1,\, \ldots,\, e^{2\pi \mathrm{i}(ja_n+m_np_n)/m_nd} X_n\bigr) \\ &= \bigl(e^{2\pi \mathrm{i} ja_1/m_1} X_1,\, \ldots,\, e^{2\pi \mathrm{i} ja_n/m_n} X_n,\, e^{2\pi \mathrm{i} \sum_{i=1}^{n}\{(ja_i+m_ip_i)/m_id\}} X_1X_2\cdots X_n\bigr). \end{aligned}$$

Here $e^{2\pi \mathrm{i} \sum_{i=1}^{n}\{(ja_i+m_ip_i)/m_id\}} = e^{2\pi \mathrm{i} j/m'_1m'_2\cdots m'_nc}$ by Lemma 4.4, thus

$$\begin{aligned} &p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}(X_1, \ldots, X_n) \\ &\qquad = \bigl(e^{2\pi \mathrm{i} ja_1/m_1} X_1,\, \ldots,\, e^{2\pi \mathrm{i} ja_n/m_n} X_n,\, e^{2\pi \mathrm{i} j/m'_1m'_2\cdots m'_nc} X_1X_2\cdots X_n\bigr). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\gamma^j \circ p(X_1, \ldots, X_n) \\ &\qquad = \bigl(e^{2\pi \mathrm{i} ja_1/m_1} X_1,\, \ldots,\, e^{2\pi \mathrm{i} ja_n/m_n} X_n,\, e^{2\pi \mathrm{i} j/m'_1m'_2\cdots m'_nc} X_1X_2\cdots X_n\bigr). \end{aligned}$$

Hence $p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} = \gamma^j \circ p$, confirming the assertion.

(3): From (2), $\bigl\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\bigr\} \subset \mathrm{Lift}^{(j)}$. Here "$\subset$" is "$=$", because the numbers of elements of both sets are equal, indeed they consist of d^{n-1} elements ((1) and Corollary 4.3). $\square$

The following will be used in later discussion:

COROLLARY 4.6. *$\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ descends to γ^j. Moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ is of the form $(X_1,\ldots,X_i,\ldots,X_n)\mapsto(X_1,\ldots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\ldots,X_n)$, then it descends to γ^j of the form*

$$(x_1,\ldots,x_n,t)\longmapsto(x_1,\ldots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\ldots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t).$$

PROOF. The first statement follows from Corollary 4.5 (3). The second one is restated as $p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}=\gamma^j\circ p$, which is confirmed as follows:

$$\begin{aligned}p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}&(X_1,\ldots,X_i,\ldots,X_n)\\&=p(X_1,\ldots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\ldots,X_n)\\&=(X_1^d,\ldots,e^{2\pi\mathrm{i}ja_i/m_i}X_i^d,\ldots,X_n^d,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_1X_2\cdots X_n)\\&=\gamma^j\circ p(X_1,\ldots,X_i,\ldots,X_n).\ \square\end{aligned}$$

By Corollary 4.5 (3), $\mathrm{Lift}^{(j)}=\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}\ :\ (p_1,p_2,\ldots,p_n)\in\Lambda^{(j)}\right\}$. Since $\widetilde{\Gamma}=\coprod_{j=1}^{m_1'm_2'\cdots m_n'c}\mathrm{Lift}^{(j)}$ (disjoint union), we have

$$\widetilde{\Gamma}=\coprod_{j=1}^{m_1'm_2'\cdots m_n'c}\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}\ :\ (p_1,p_2,\ldots,p_n)\in\Lambda^{(j)}\right\}.$$

Or

$$\widetilde{\Gamma}=\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}\ :\ (p_1,p_2,\ldots,p_n)\in\Lambda^{(j)},\ j=1,2,\ldots,m_1'm_2'\cdots m_n'c\right\}.$$

We thus obtain:

LEMMA 4.7. *The lift $\widetilde{\Gamma}$ of Γ consists of the automorphisms $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ given by*

$$(X_1,\ldots,X_n)\ \longmapsto\ \left(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/m_1d}X_1,\ldots,e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd}X_n\right),$$

where $(p_1,p_2,\ldots,p_n)\in\Lambda^{(j)}$ and $j=1,2,\ldots m_1'm_2'\cdots m_n'c$. (In particular, any two elements of $\widetilde{\Gamma}$ commute, so $\widetilde{\Gamma}$ is abelian.)

4.2. Determination of H

Recall that $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$, where $\mathrm{Lift}^{(j)}$ denotes the set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Accordingly $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. We determine $q_*\big(\mathrm{Lift}^{(j)}\big)$. To that end, for $j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c$ and $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $h^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(u_1, \ldots, u_n) \longmapsto \big(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd} u_1, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd} u_n\big).$$

Lemma 4.8.

(1) *$h^{(j)}_{p_1,p_2,\ldots,p_n}$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.*

(2) $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(3) $H = \Big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}, j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c\Big\}$. (Thus any two elements of H commute, that is, H is abelian.)

Proof. (1): Indeed since $\big(e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id}\big)^{m'_i} = e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}$, the following diagram commutes:

$$\begin{tikzcd} \widetilde{A}_{d-1} \arrow[r, "\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}"] \arrow[d, "q"'] & \widetilde{A}_{d-1} \arrow[d, "q"] \\ \mathbb{C}^n \arrow[r, "h^{(j)}_{p_1,p_2,\ldots,p_n}"] & \mathbb{C}^n. \end{tikzcd}$$

(2): By Corollary 4.5 (3), $\mathrm{Lift}^{(j)} = \big\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$, accordingly by (1), $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(3): This follows from $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ and (2). $\square$

5. The Pseudo-Reflection Subgroup of H

5.1. Cyclic subgroups Γ_i of Γ and $\widetilde{\Gamma}_i$ of $\widetilde{\Gamma}$

Let $\gamma : A_{d-1} \to A_{d-1}$ be the automorphism given by

$$\begin{aligned} \gamma :\ & (x_1, \ldots, x_n, t) \\ & \longmapsto (e^{2\pi\mathrm{i}a_1/m_1}x_1,\ \ldots,\ e^{2\pi\mathrm{i}a_n/m_n}x_n,\ e^{2\pi\mathrm{i}/m'_1m'_2\cdots m'_nc}t). \end{aligned} \tag{5.1}$$

(The order of γ is $m'_1m'_2\cdots m'_nc$.) Consider the cyclic group Γ generated by γ:

$$\Gamma = \{\gamma^j\ :\ j = 1, 2, \ldots, m'_1m'_2\cdots m'_nc\}.$$

Let Γ_i $(i = 1, 2, \ldots, n)$ be the subgroup of Γ consisting of automorphisms of the form

$$(x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto (x_1,\ \ldots, e^{2\pi\mathrm{i}ja_i/m_i}x_i, \ldots, x_n,\ e^{2\pi\mathrm{i}j/m'_1m'_2\cdots m'_nc}t),$$

that is,

$$(\sharp) \qquad e^{2\pi\mathrm{i}ja_k/m_k} = 1 \quad (k = 1, 2, \ldots, \check{i}, \ldots, n).$$

LEMMA 5.1. *For $j \in \mathbb{Z}$,*

$$\gamma^j \in \Gamma_i \iff j \textit{ is a multiple of } \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c.$$

PROOF. $\Longrightarrow$: If $\gamma^j \in \Gamma_i$, then from $(\sharp)$, ja_k is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$. Here a_k and m_k are relatively prime, so j is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$. In particular, j is a multiple of $\mathrm{lcm}(m_1, \ldots, \check{m}_i, \ldots m_n) = \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$.

$\Longleftarrow$: If j is a multiple of $\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$, then j is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$, so $\dfrac{ja_k}{m_k}$ is an integer. Thus $e^{2\pi\mathrm{i}ka_k/m_k} = 1$ $(k = 1, 2, \ldots, \check{i}, \ldots, n)$, so $\gamma^j \in \Gamma_i$. □

From Lemma 5.1, the following holds:

COROLLARY 5.2. *Γ_i is generated by $\gamma_i := \gamma^{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c}$.*

This element is explicitly given by

$$\gamma_i:\ (x_1,\ldots,x_i,\ldots,x_n,t)\longmapsto \Big(x_1,\ \ldots, e^{2\pi \mathrm{i} a_i \mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)/m'_i}x_i,\ldots,x_n,\ e^{2\pi \mathrm{i}\,\mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)/m'_1m'_2\cdots m'_n}t\Big).$$

Here $e^{2\pi \mathrm{i}\,\mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)/m'_1m'_2\cdots m'_n}=e^{2\pi\mathrm{i}/m'_il_i}$, because

$$\frac{\mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)}{m'_1m'_2\cdots m'_n}=\frac{1}{m'_i}\frac{\mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)}{m'_1\cdots\check{m}'_i\cdots m'_n}=\frac{1}{m'_il_i}.$$

Thus

$$\gamma_i:\ (x_1,\ldots,x_i,\ldots,x_n,t)\longmapsto \Big(x_1,\ \ldots, e^{2\pi \mathrm{i} a_i \mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)/m'_i}x_i,\ldots,x_n,\ e^{2\pi\mathrm{i}/m'_il_i}t\Big).$$

Set $L_i:=\mathrm{lcm}(m'_1,\ldots,\check{m}'_i,\ldots,m'_n)$, then

$$\gamma_i:\ (x_1,\ldots,x_n,t)\longmapsto \big(x_1,\ \ldots, e^{2\pi\mathrm{i}a_iL_i/m'_i}x_i,\ldots,x_n,\ e^{2\pi\mathrm{i}/m'_il_i}t\big).$$

For $k\in\mathbb{Z}$,

$$\gamma_i^k:\ (x_1,\ldots,x_i,\ldots,x_n,t)\longmapsto \big(x_1,\ \ldots, e^{2\pi\mathrm{i}a_iL_ik/m'_i}x_i,\ldots,x_n,\ e^{2\pi\mathrm{i}k/m'_il_i}t\big).$$

In particular,

$$\gamma_i^k=\mathrm{id}\ \text{if and only if}\ e^{2\pi\mathrm{i}a_iL_ik/m'_i}=1\ \text{and}\ e^{2\pi\mathrm{i}k/m'_il_i}=1.$$

Here:

(A) $e^{2\pi\mathrm{i}a_iL_ik/m'_i}=1$ if and only if $\dfrac{L_ik}{m'_i}$ is an integer (because a_i and m'_i are relatively prime).

(B) $e^{2\pi\mathrm{i}k/m'_il_i}=1$ if and only if $\dfrac{k}{m'_il_i}$ is an integer.

We restate (A). First write $\dfrac{L_i}{m'_i}$ as $\dfrac{L'_i}{m''_i}$ where L'_i and m''_i are relatively prime positive integers (or, $L'_i:=\dfrac{L_i}{\gcd(L_i,m'_i)}$ and $m''_i:=\dfrac{m'_i}{\gcd(L_i,m'_i)}$). Then $\dfrac{L_ik}{m'_i}\left(=\dfrac{L'_ik}{m''_i}\right)$ is an integer if and only if m''_i divides k. Thus (A) is restated as:

(A)' $e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1$ if and only if m''_i divides k.

From (A)' and (B),

$$\gamma_i^k = \mathrm{id} \text{ if and only if } k \text{ is a common multiple of } m''_i \text{ and } m'_i l_i.$$

Here m'_i is a multiple of m''_i (because $m''_i := \dfrac{m'_i}{\gcd(L_i, m'_i)}$). Thus any common multiple of m''_i and $m'_i l_i$ is necessarily a multiple of $m'_i l_i$. Therefore:

LEMMA 5.3. $\gamma_i^k = \mathrm{id}$ *if and only if k is a multiple of $m'_i l_i$. In particular, the order of γ_i is $m'_i l_i$.*

We summarize the above results (Corollary 5.2 and Lemma 5.3) as follows:

COROLLARY 5.4. *For each $i = 1, 2, \ldots, n$, let Γ_i be the subgroup of Γ consisting of automorphisms of the form*

$$(x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto (x_1, \ldots, e^{2\pi \mathrm{i} j a_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} j/m'_1 m'_2 \cdots m'_n c} t).$$

Then Γ_i is a cyclic group of order $m'_i l_i$ generated by the automorphism

$$\gamma_i : (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto \big(x_1, \ldots, e^{2\pi \mathrm{i} a_i L_i/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}/m'_i l_i} t\big),$$

where $L_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$ and $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{L_i}$.

Let $p : \widetilde{A}_{d-1} \, (= \mathbb{C}^n) \to A_{d-1}$ be the covering of A_{d-1} given by

$$p(X_1, X_2, \ldots, X_n) = (X_1^d,\, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n),$$

and $\widetilde{\Gamma}$ be the lift of Γ with respect to p. Next let $\xi_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by

$$\xi_i : (X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \ldots, X_n). \tag{5.2}$$

Then:

LEMMA 5.5.

(1) *The order of ξ_i is $m'_i l_i$.* (The order of γ_i is also $m'_i l_i$ by Lemma 5.3.)

(2) $\xi_i \in \widetilde{\Gamma}$. *In fact*, ξ_i *is a lift of* $\gamma_i \in \Gamma_i\,(\subset \Gamma)$, *that is, the following diagram commutes*:

$$\begin{tikzcd} \widetilde{A}_{d-1} \arrow[r, "\xi_i"] \arrow[d, "p"'] & \widetilde{A}_{d-1} \arrow[d, "p"] \\ A_{d-1} \arrow[r, "\gamma_i"] & A_{d-1}. \end{tikzcd}$$

PROOF. (1) is clear. We show (2). It suffices to show that $p \circ \xi_i = \gamma_i \circ p$. Note first that

$$\begin{aligned} & p \circ \xi_i(X_1, \dots, X_i, \dots, X_n) \\ & \qquad = (X_1^d, \dots, e^{2\pi \mathrm{i} d/m_i' l_i} X_i^d, \dots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \gamma_i \circ p(X_1, \dots, X_i, \dots, X_n) \\ & \qquad = (X_1^d, \dots, e^{2\pi \mathrm{i} a_i L_i/m_i'} X_i^d, \dots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

Thus to show that $p \circ \xi_i = \gamma_i \circ p$, it suffices to show that $e^{2\pi \mathrm{i} d/m_i' l_i} = e^{2\pi \mathrm{i} a_i L_i/m_i'}$, that is,

$$\frac{d}{m_i' l_i} \equiv \frac{a_i L_i}{m_i'} \mod \mathbb{Z}. \tag{5.3}$$

Since $d = m_1' m_2' \cdots m_n' c\Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa\Big)$ and $l_i = \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{L_i}$, the left hand side of (5.3) is

$$\begin{aligned} \frac{d}{m_i' l_i} &= \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} + c\kappa L_i \\ &\equiv \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} \mod \mathbb{Z}. \end{aligned}$$

Here $L_i := \mathrm{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')$ is divisible by m_k' ($k = 1, 2, \dots, \check{i}, \dots, n$), so $\frac{a_k L_i}{m_k'} \in \mathbb{Z}$, that is, $\frac{a_k L_i}{m_k'} \equiv 0 \mod \mathbb{Z}$ ($k = 1, 2, \dots, \check{i}, \dots, n$), hence $\frac{d}{m_i' l_i} \equiv \frac{a_i L_i}{m_i'} \mod \mathbb{Z}$, confirming (5.3). $\square$

As we saw in the paragraph above Lemma 4.7,

$$\widetilde{\Gamma} = \left\{ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)},\ j=1,2,\ldots,m'_1m'_2\cdots m'_nc \right\},$$

where $\Lambda^{(j)} = \left\{ (p_1,p_2,\ldots,p_n) \in \mathbb{Z}^n \,:\, 0 \le p_i < d,\ \sum_{i=1}^n \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z} \right\}$ and $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is the automorphism given by

$$(X_1,\ldots,X_n) \longmapsto \left(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/m_1d} X_1, \ldots, e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd} X_n \right).$$

Here Corollary 4.6 states that (i) $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ descends to γ^j and (ii) moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ is of the form $(X_1,\ldots,X_n) \mapsto (X_1,\ldots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i, \ldots,X_n)$, then it descends to γ^j of the form

$$(x_1,\ldots,x_n,t) \longmapsto (x_1,\ldots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\ldots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t).$$

Note the following:

Lemma 5.6. *In the case of* (ii)*, there exists an integer s_i such that $e^{2\pi\mathrm{i}ja_i/m_i} = e^{2\pi\mathrm{i}a_iL_is_i/m'_i}$ and $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id} = e^{2\pi\mathrm{i}s_i/m'_il_i}$.*

Proof. Since the γ^j in (ii) is an element of Γ_i, and Γ_i is generated by γ_i (Corollary 5.4), there exists an integer s_i such that $\gamma^j = \gamma_i^{s_i}$. Here

$$\begin{cases} \gamma^j :\ (x_1,\ldots,x_n,t) \mapsto (x_1,\ldots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\ldots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t), \\ \gamma_i^{s_i} :\ (x_1,\ldots,x_n,t) \mapsto \left(x_1,\ldots,e^{2\pi\mathrm{i}a_iL_is_i/m'_i}x_i,\ldots,x_n,e^{2\pi\mathrm{i}s_i/m'_il_i}t\right), \end{cases}$$

so $e^{2\pi\mathrm{i}ja_i/m_i} = e^{2\pi\mathrm{i}a_iL_is_i/m'_i}$ and $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id} = e^{2\pi\mathrm{i}s_i/m'_il_i}$. $\square$

Let $\xi_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by

$$\xi_i :\ (X_1,\ldots,X_i,\ldots,X_n) \longmapsto (X_1,\ldots,e^{2\pi\mathrm{i}/m'_il_i}X_i,\ldots,X_n).$$

Then $\xi_i \in \widetilde{\Gamma}$ (Lemma 5.5 (2)). In fact, $\xi_i \in \widetilde{\Gamma} \cap \Xi_i$, where Ξ_i $(i=1,2,\ldots,n)$ is the multiplicative group of automorphisms consisting of scalar multiplication of the ith coordinate of $\widetilde{A}_{d-1}$ $(=\mathbb{C}^n)$:

$$\Xi_i := \left\{ (X_1,\ldots,X_i,\ldots,X_n) \longmapsto (X_1,\ldots,\lambda X_i,\ldots,X_n) \,:\, \lambda \in \mathbb{C}^\times \right\}.$$

Setting $\widetilde{\Gamma}_i := \widetilde{\Gamma} \cap \Xi_i$, we claim that ξ_i in fact generates $\widetilde{\Gamma}_i$, that is, any element of $\widetilde{\Gamma}_i$ is a power of ξ_i. To see this, note that $\widetilde{\Gamma}_i$ consists of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$\begin{aligned}\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} :\ &(X_1,\dots,X_i,\dots,X_n)\\ &\longmapsto (X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n).\end{aligned}$$

Here for each $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}_i$, there exists an integer s_i such that $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id} = e^{2\pi\mathrm{i}s_i/m_i'l_i}$ (Lemma 5.6). Then

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} :\ (X_1,\dots,X_i,\dots,X_n) \longmapsto (X_1,\dots,e^{2\pi\mathrm{i}s_i/m_i'l_i}X_i,\dots,X_n),$$

so $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \xi_i^{s_i}$, confirming that ξ_i generates $\widetilde{\Gamma}_i$. Here the order of ξ_i is $m_i'l_i$ (Lemma 5.5 (1)), so the order of the cyclic group $\widetilde{\Gamma}_i$ is $m_i'l_i$.

We formalize the above result as follows:

Proposition 5.7. *For each $i = 1,2,\dots,n$, let $\widetilde{\Gamma}_i$ be the subgroup of $\widetilde{\Gamma}$ consisting of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ of the form*

$$(X_1,\dots,X_i,\dots,X_n) \longmapsto (X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n).$$

Then $\widetilde{\Gamma}_i$ is a cyclic group of order $m_i'l_i$ generated by the automorphism

$$\xi_i : (X_1,\dots,X_i,\dots,X_n) \longmapsto (X_1,\dots,e^{2\pi\mathrm{i}/m_i'l_i}X_i,\dots,X_n),$$

where $l_i := \dfrac{m_1'\cdots\check{m}_i'\cdots m_n'}{\mathrm{lcm}(m_1',\dots,\check{m}_i',\dots,m_n')}$.

5.2. Cyclic subgroups H_i of H

We have described cyclic subgroups $\widetilde{\Gamma}_i$ $(i = 1,2,\dots,n)$ of $\widetilde{\Gamma}$. We next describe subgroups of H corresponding to them. Here H is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1}\,(=\mathbb{C}^n) \to \mathbb{C}^n$ given by

$$q(X_1,X_2,\dots,X_n) = (X_1^{m_1'},\, X_2^{m_2'},\dots,X_n^{m_n'}).$$

Explicitly H is given by (Lemma 4.8 (3)):

$$H = \left\{ h^{(j)}_{p_1,p_2,\dots,p_n} \,:\, (p_1,p_2,\dots,p_n) \in \Lambda^{(j)},\ j = 1,2,\dots,m_1'm_2'\cdots m_n'c \right\},$$

where $h^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is the automorphism given by

$$(u_1,\dots,u_n) \longmapsto \left(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/cd}u_1,\dots,e^{2\pi\mathrm{i}(ja_n+m_np_n)/cd}u_n\right).$$

Now let H_i $(i=1,2,\dots,n)$ be the subgroup of H consisting of $h^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$(5.4)\qquad (u_1,\dots,u_i,\dots,u_n) \longmapsto (u_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/cd}u_i,\dots,u_n).$$

Let $h_i : \mathbb{C}^n \to \mathbb{C}^n$ be the automorphism given by

$$(5.5)\qquad h_i : (u_1,\dots,u_i,\dots,u_n) \longmapsto (u_1,\dots,e^{2\pi\mathrm{i}/l_i}u_i,\dots,u_n).$$

Then $h_i \in H$. In fact, h_i is the descent of $\xi_i \in \widetilde{\Gamma}_i\,(\subset \widetilde{\Gamma})$, that is, the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{A}_{d-1} & \xrightarrow{\ \xi_i\ } & \widetilde{A}_{d-1} \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
\mathbb{C}^n & \xrightarrow{\ h_i\ } & \mathbb{C}^n.
\end{array}$$

Since $\widetilde{\Gamma}_i$ is a cyclic group generated by ξ_i (Proposition 5.7) and h_i is the descent of ξ_i with respect to q, the descent of $\widetilde{\Gamma}_i$ is a cyclic group generated by h_i. As we show subsequently, this cyclic group coincides with H_i.

To show this, it suffices to show that for any $h^{(j)}_{p_1,p_2,\dots,p_n} \in H_i$, there exists an element of $\widetilde{\Gamma}_i$ that descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. Here

$$\begin{cases} h^{(j)}_{p_1,p_2,\dots,p_n} : \ (u_1,\dots,u_n) \longmapsto (u_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/cd}u_i,\dots,u_n), \\ q : \ (X_1,X_2,\dots,X_n) \longmapsto (X_1^{m_1'},X_2^{m_2'},\dots,X_n^{m_n'}). \end{cases}$$

Thus an automorphism $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by

$$(5.6)\qquad \zeta : (X_1,\dots,X_n) \longmapsto (X_1,\dots,e^{2\pi\mathrm{i}\{(ja_i+m_ip_i)/m_id\}}X_i,\dots,X_n)$$

descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. We show that in fact $\zeta \in \widetilde{\Gamma}$ (then from the form of ζ, $\zeta \in \widetilde{\Gamma}_i$, so ζ is a lift of $h^{(j)}_{p_1,p_2,\dots,p_n}$).

Step 1. Since $q(X_1,X_2,\dots,X_n) = (X_1^{m_1'},X_2^{m_2'},\dots,X_n^{m_n'})$, the set of all lifts of $h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1,\dots,u_n) \mapsto (u_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/cd}u_i,\dots,u_n)$ with respect to the covering q consists of automorphisms

$$(X_1, \dots, X_n) \\ \mapsto (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i+m_ip_i)/m_id+k_i/m'_i\}} X_i, \dots, e^{2\pi \mathrm{i} k_n/m'_n} X_n),$$

where $k_1, k_2, \dots, k_n$ are integers.

Step 2. Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}$ is a lift of $h^{(j)}_{p_1,p_2,\dots,p_n}$ with respect to q (Lemma 4.8 (1)), $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ coincides with one of the automorphisms in Step 1. Namely for some integers $k_1, k_2, \dots, k_n$,

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (X_1, \dots, X_i, \dots, X_n) \longmapsto \\ (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i+m_ip_i)/m_id+k_i/m'_i\}} X_i, \dots, e^{2\pi \mathrm{i} k_n/m'_n} X_n).$$

Next for each $k = 1, 2, \dots, n$, take the automorphism

$$\xi_k : (X_1, \dots, X_n) \mapsto (X_1, \dots, e^{2\pi \mathrm{i}/m'_k l_k} X_k, \dots, X_n).$$

The composite automorphism $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \xi_1^{-l_1k_1} \xi_2^{-l_2k_2} \cdots \xi_n^{-l_nk_n}$ is then given by

$$(X_1, \dots, X_i, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i+m_ip_i)/m_id\}} X_i, \dots, X_n).$$

This coincides with the automorphism ζ given by (5.6), thus

$$\zeta = \widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \xi_1^{-l_1k_1} \xi_2^{-l_2k_2} \cdots \xi_n^{-l_nk_n}.$$

Step 3. Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}$ and $\xi_k \in \widetilde{\Gamma}$ $(k = 1, 2, \dots, n)$, we have $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \xi_1^{-l_1k_1} \xi_2^{-l_2k_2} \cdots \xi_n^{-l_nk_n} \in \widetilde{\Gamma}$. Hence $\zeta \in \widetilde{\Gamma}$, confirming the assertion.

We thus obtained the following:

LEMMA 5.8. *For each $h^{(j)}_{p_1,p_2,\dots,p_n} \in H_i$, there exists an element of $\widetilde{\Gamma}_i$ that descends to it* (*with respect to the covering q*). *In fact, the automorphism $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by $\zeta : (X_1, \dots, X_n) \mapsto (X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i+m_ip_i)/m_id\}} X_i, \dots, X_n)$ is an element of $\widetilde{\Gamma}_i$ that descends to*

$$h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd} u_i, \dots, u_n).$$

COROLLARY 5.9. *H_i is the descent of $\widetilde{\Gamma}_i$ with respect to the covering q.*

The descent of $\widetilde{\Gamma}_i$ with respect to the covering q is a cyclic group generated by h_i in (5.5). On the other hand, this descent coincides with H_i (Corollary 5.9). Thus:

LEMMA 5.10. *H_i is a cyclic group generated by the automorphism $h_i : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i, \dots, u_n)$. Thus the order of H_i is l_i.*

5.3. The pseudo-reflection subgroup of H

We retain the notation above. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. Let H_i $(i = 1, 2, \dots, n)$ be the subgroup of H consisting of $h^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$(5.7) \qquad h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1, \dots, u_i, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \dots, u_n).$$

In fact, H_i is a cyclic group of order l_i generated by h_i (Lemma 5.10). Note that if $i \neq j$, then $H_i \cap H_j = \{1\}$. In particular,

$$(5.8) \qquad H_1 H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n.$$

Note also that the set of all pseudo-reflections in H is given by $\left(\bigcup_{i=1}^{n} H_i \right) \setminus \{1\}$.

Here a *pseudo-reflection* is a diagonalizable matrix such that one of its eigenvalues is a root of unity (distinct from 1) and all other eigenvalues are 1. Note that the identity matrix is *not* a pseudo-reflection.

Now let P be the *pseudo-reflection subgroup* of H that is the subgroup generated by all pseudo-reflections in H, that is, by $\left(\bigcup_{i=1}^{n} H_i \right) \setminus \{1\}$. Here H_i $(i = 1, 2, \dots, n)$ is a cyclic group generated by h_i, so P is generated by $h_1, h_2, \dots, h_n$, thus $P = H_1 H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n$ (see (5.8)). Since the order of H_i is l_i, the order of P is $l_1 l_2 \cdots l_n$. This confirms the following:

PROPOSITION 5.11. *Where H_i $(i = 1, 2, \dots, n)$ is a cyclic subgroup of H generated by the automorphism $h_i : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i,$*

$\dots, u_n)$, the pseudo-reflection subgroup P of H is the direct product $P = H_1 \times H_2 \times \cdots \times H_n$ and the order of P is $l_1 l_2 \cdots l_n$.

In particular, $P = \{1\}$ if and only if $l_1 = l_2 = \cdots = l_n = 1$. Thus:

COROLLARY 5.12. *H is small if and only if $l_1 = l_2 = \cdots = l_n = 1$.*

Now let G be the descent of H with respect to the $l_1 l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$ given by $r(u_1, u_2, \dots, u_n) = (u_1^{l_1}, u_2^{l_2}, \dots, u_n^{l_n})$. Then $l_1 = l_2 = \cdots = l_n = 1$ *if and only if r is the identity map, or equivalently $H = G$.* This, combined with Corollary 5.12, gives the following:

LEMMA 5.13.

$$\begin{aligned} H \text{ is small} &\iff l_1 = l_2 = \cdots = l_n = 1 \\ &\iff r \text{ is the identity map} \\ &\iff H = G. \end{aligned}$$

The following arithmetic results are proved later (Corollary 5.19):

(1) If $n = 2$, then $l_1 = l_2 = 1$.

(2) If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(m'_j, m'_k) = 1$ for any $j \neq k$.

This, combined with Lemma 5.13, yields the following:

THEOREM 5.14 (Numerical criterion of smallness).

(1) *If $n = 2$, then H is always small.*

(2) *If $n \geq 3$, then H is small if and only if $\gcd(m'_i, m'_j) = 1$ for any i, j such that $i \neq j$.*

Example 5.15. If $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$, $m_3 = 6$ and $\kappa = 0$, then $c = \gcd(m_1, m_2, m_3) = 2$, $m'_1 = 1$, $m'_2 = 2$, $m'_3 = 3$ and $d = 2{+}3{+}6 = 11$. In this case, Γ is generated by the automorphism γ of A_{d-1} ($= A_{10}$) given by $\gamma(x_1, x_2, x_3, t) \mapsto (e^{2\pi i/2}x_1,\ e^{2\pi i/4}x_2,\ e^{2\pi i/6}x_3,\ e^{2\pi i/12}t)$. Let $\widetilde{\Gamma}$

be the lift of Γ with respect to the covering $p : \widetilde{A}_{10} \to A_{10}$, $p(X_1, X_2, X_3) = (X_1^{11}, X_2^{11}, X_3^{11}, X_1X_2X_3)$, and let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{10} \to \mathbb{C}^3$, $q(X_1, X_2, X_3) = (X_1,\ X_2^2,\ X_3^3)$. Then, since $\gcd(m_1', m_2') = 1$, $\gcd(m_1', m_3') = 1$ and $\gcd(m_2', m_3') = 1$, Theorem 5.14 ensures that H is small.

5.4. Supplement: Arithmetic result

This section is devoted to proving an arithmetic result used in §5.3.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive integers such that $\gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1$, where $n \geq 2$. Set $l_i := \dfrac{\lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n}{\operatorname{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n)}$, where $\check{\lambda}_i$ means the omission of λ_i. Note that l_i is a positive integer (cf. Remark 3.1). We show that if $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$.

Remark 5.16. If $n = 2$, this equivalence is vacuous, because $l_1 = l_2 = 1$ *always* holds (and $\gcd(\lambda_1, \lambda_2) = 1$ by assumption). In fact $l_1 = \dfrac{\lambda_1}{\gcd(\lambda_1)} = 1$ and $l_2 = \dfrac{\lambda_2}{\gcd(\lambda_2)} = 1$.

We begin with some preparation:

Lemma 5.17. *For any i, j, k such that i, j and k are distinct, $l_i \geq \gcd(\lambda_j, \lambda_k)$.*

Proof. We only show the assertion for $i = 1$, $j = 2$ and $k = 3$ (the assertion for other cases are similarly shown). Note first that $\lambda_2\lambda_3 = \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3)$. Multiplying $\lambda_4 \cdots \lambda_n$ to this yields:

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n = \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n.$$

Here, since $\operatorname{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n \geq \operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n)$,

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n \geq \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n).$$

Dividing this by $\operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n)$,

$$\frac{\lambda_2\lambda_3\lambda_4 \cdots \lambda_n}{\operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4 \cdots \lambda_n)} \geq \gcd(\lambda_2, \lambda_3).$$

Since the left hand side is l_1, we have $l_1 \geq \gcd(\lambda_2, \lambda_3)$. (Note: If $n = 3$, then the equality holds. In fact, $l_1 = \dfrac{\lambda_2\lambda_3}{\operatorname{lcm}(\lambda_2, \lambda_3)} = \gcd(\lambda_2, \lambda_3)$.) $\square$

We next show that:

LEMMA 5.18. *For each $i = 1, 2, \ldots, n$,*

$$l_i = 1 \iff \gcd(\lambda_j, \lambda_k) = 1 \textit{ for any } j \neq k \textit{ (distinct from } i\textit{)}.$$

PROOF. $\Longrightarrow$: By Lemma 5.17, for any i, j, k such that i, j and k are distinct, $l_i \geq \gcd(\lambda_j, \lambda_k)$. In particular if $l_i = 1$, then $\gcd(\lambda_j, \lambda_k) = 1$.

$\Longleftarrow$: If $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$ such that j and k distinct from i, then $\operatorname{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n) = \lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n$, and thus $l_i = 1$. $\square$

From Lemma 5.18, $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$. (Actually if $n = 2$, then $l_1 = l_2 = 1$ always holds (Remark 5.16).)

Now let $m_1, m_2, \ldots, m_n$ be positive integers. Set $c := \gcd(m_1, m_2, \ldots, m_n)$ and $m'_i := \frac{m_i}{c}$ $(i = 1, 2, \ldots, n)$. Then $m'_1, m'_2, \ldots, m'_n$ are positive integers such that $\gcd(m'_1, m'_2, \ldots, m'_n) = 1$. So we may apply the above to obtain the following:

COROLLARY 5.19. *Let $m_1, m_2, \ldots, m_n$ be positive integers. Set $c := \gcd(m_1, m_2, \ldots, m_n)$, $m'_i := \frac{m_i}{c}$ and $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$, where $\check{m}'_i$ means the omission of m'_i.* (Note that l_i is a positive integer (cf. Remark 3.1).) *Then the following hold*:

(1) *If $n = 2$, then $l_1 = l_2 = 1$.*

(2) *If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if* $\gcd(m'_j, m'_k) = 1$ *for any $j \neq k$.*

6. Uniformization Theorem for Arbitrary Dimension

6.1. Determination of G

Recall the diagram (3.5) for the covering maps p, q, r:

$$\begin{tikzcd}[column sep=small] & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}. \\ \mathbb{C}^n & & & \end{tikzcd} \tag{6.1}$$

Then

- $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$ (disjoint union) is the lift of Γ with respect to p, where $\mathrm{Lift}^{(j)}$ is the set of all lifts of $\gamma^j \in \Gamma$.
- $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\widetilde{\Gamma}$ with respect to q, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$.
- $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of H with respect to r, where $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $q_*\big(\mathrm{Lift}^{(j)}\big)$.

Here $\mathrm{Lift}^{(j)} = \big\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\big\}$ (Corollary 4.5 (3)) and $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\big\}$ (Lemma 4.8 (2)). We next determine $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$. For $j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c$ and $(p_1,\, p_2,\, \dots, p_n) \in \Lambda^{(j)}$, define an automorphism $g^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(v_1, \dots, v_n) \longmapsto \big(e^{2\pi \mathrm{i} l_1 (j a_1 + m_1 p_1)/cd} v_1, \dots, e^{2\pi \mathrm{i} l_n (j a_n + m_n p_n)/cd} v_n\big).$$

Then as for Lemma 4.8, we can show the following:

LEMMA 6.1.

(A) *$g^{(j)}_{p_1,p_2,\dots,p_n}$ is the descent of $h^{(j)}_{p_1,p_2,\dots,p_n}$ with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$.*

(B) $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{g^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\big\}$.

(C) $G = \left\{ g^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c \right\}$. (In particular, any two elements of G commute, so G is abelian.)

6.2. Uniformization theorem

Let H be the descent of $\widetilde{\Gamma}$ with respect to the $m'_1 m'_2 \cdots m'_n$-fold covering $q : \mathbb{C}^n \to \mathbb{C}^n$ and P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections in H. The descent G of H with respect to the $l_1 l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$ is regarded as the quotient group H/P. Indeed the kernel of the surjective homomorphism $r_* : H \to G$ (given by $r_*(h) :=$ descent of h) is P, so $G \cong H/P$. Thus G is obtained from H by collapsing the pseudo-reflections in H, consequently:

Proposition 6.2. *G contains no pseudo-reflections, that is, is a small group.*

Now $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$. Here G is a finite abelian group (Proposition 3.2 (3)) and small (Proposition 6.2). The following is thus established:

Theorem 6.3 (Uniformization theorem). *Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by*

$$\gamma : (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi i a_1/m_1} x_1, \ldots, e^{2\pi i a_n/m_n} x_n,\ e^{2\pi i/m'_1 m'_2 \cdots m'_n c} t).$$

Then there exists a small finite abelian group $G \subset GL(n, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$.

We explicitly give the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem. The covering maps p, q and r appearing in the diagram (6.1) induce isomorphisms $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$ and $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^n/H$ and $\overline{r} : \mathbb{C}^n/H \to \mathbb{C}^n/G$. The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem (Theorem 6.3) is then given by

$$\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{\,-1} :\ A_{d-1}/\Gamma \xrightarrow{\ \cong\ } \mathbb{C}^n/G. \tag{6.2}$$

Explicitly:

Lemma 6.4. $\Psi\big([x_1, \ldots, x_n, t]\big) = \big[x_1^{m'_1 l_1/d}, \ldots, x_n^{m'_n l_n/d}\big]$, *where* $[x_1,\ldots,x_n,t]\in A_{d-1}/\Gamma$ *and* $\bigl[x_1^{m'_1l_1/d},\ldots,x_n^{m'_nl_n/d}\bigr]\in\mathbb{C}^n/G$ *denote the images of* $(x_1,\ldots,x_n,t)\in A_{d-1}$ *and* $\bigl(x_1^{m'_1l_1/d},\ldots,x_n^{m'_nl_n/d}\bigr)\in\mathbb{C}^n$ *respectively.*

PROOF. Since $p(X_1,X_2,\ldots,X_n)=(X_1^d,X_2^d,\ldots,X_n^d,X_1X_2\cdots X_n)$, we have $\overline{p}\bigl([X_1,X_2,\ldots,X_n]\bigr)=[X_1^d,X_2^d,\ldots,X_n^d,X_1X_2\cdots X_n]$, so

$$\overline{p}^{\,-1}\bigl([x_1,x_2,\ldots,x_n,t]\bigr)=\bigl[x_1^{1/d},\,x_2^{1/d},\,\ldots,x_n^{1/d}\bigr].$$

Next since $q(X_1,X_2,\ldots,X_n)=(X_1^{m'_1},X_2^{m'_2},\ldots,X_n^{m'_n})$ and $r(u_1,u_2,\ldots,u_n)=(u_1^{l_1},u_2^{l_2},\ldots,u_n^{l_n})$, we have $\overline{q}\bigl([X_1,X_2,\ldots,X_n]\bigr)=[X_1^{m'_1},X_2^{m'_2},\ldots,X_n^{m'_n}]$ and $\overline{r}\bigl([u_1,u_2,\ldots,u_n]\bigr)=[u_1^{l_1},u_2^{l_2},\ldots,u_n^{l_n}]$, so

$$\begin{aligned}\overline{r}\circ\overline{q}\bigl([x_1^{1/d},x_2^{1/d},\ldots,x_n^{1/d}]\bigr)&=\overline{r}\bigl([x_1^{m'_1/d},x_2^{m'_2/d},\ldots,x_n^{m'_n/d}]\bigr)\\&=\bigl[x_1^{m'_1l_1/d},x_2^{m'_2l_2/d},\ldots,x_n^{m'_nl_n/d}\bigr].\end{aligned}$$

Hence $\Psi:=\overline{r}\circ\overline{q}\circ\overline{p}^{\,-1}$ is explicitly given by

$$\Psi\bigl([x_1,x_2,\ldots,x_n,t]\bigr)=\bigl[x_1^{m'_1l_1/d},\,x_2^{m'_2l_2/d},\ldots,x_n^{m'_nl_n/d}\bigr].\ \square$$

6.3. Correspondence between functions

We use the notation in §6.2. Besides, let $\Phi:A_{d-1}\to\mathbb{C}$ be a holomorphic map given by $\Phi(x_1,x_2,\ldots,x_n,t)=t^{m'_1m'_2\cdots m'_nc}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi}:A_{d-1}/\Gamma\to\mathbb{C}$. As we explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\left(\frac{a_1}{m_1},\,\frac{a_2}{m_2},\ldots,\frac{a_n}{m_n},\,\kappa\right)$-*fractional Dehn twist*: If $n=2$, then the topological monodromy of $\overline{\Phi}$ is the $-\left(\frac{a_1}{m_1},\,\frac{a_2}{m_2},\,\kappa\right)$-fractional Dehn twist.

Under the isomorphism $\Psi:A_{d-1}/\Gamma\xrightarrow{\cong}\mathbb{C}^n/G$ in (6.2), $\overline{\Phi}:A_{d-1}/\Gamma\to\mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^n/G$. We describe this map. To that end, we need the following:

LEMMA 6.5. *For an element* $g\in G$ *given by*

$$(v_1,\ldots,v_n)\ \longmapsto\ \bigl(e^{2\pi\mathrm{i}l_1(ja_1+m_1p_1)/cd}v_1,\,\ldots,e^{2\pi\mathrm{i}l_n(ja_n+m_np_n)/cd}v_n\bigr),$$

write $\eta_i = e^{2\pi \mathrm{i} l_i (ja_i + m_i p_i)/cd}$ $(i = 1, 2, \ldots, n)$. *Next for* $i = 1, 2, \ldots, n$, *set* $k_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$, *where* $\check{m}'_i$ *means the omission of* m'_i. *Then* $\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_n^{k_n} = 1$.

Proof. Since $l_i = \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$, we have $k_i l_i = m'_1 \cdots \check{m}'_i \cdots m'_n c$, so

$$\begin{aligned}\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_n^{k_n} &= e^{2\pi \mathrm{i} k_1 l_1 (ja_1 + m_1 p_1)/cd} e^{2\pi \mathrm{i} k_2 l_2 (ja_2 + m_2 p_2)/cd} \cdots \\ &\qquad \cdots e^{2\pi \mathrm{i} k_n l_n (ja_n + m_n p_n)/cd} \\ &= e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^{n} (ja_i/m_i + p_i)/d}.\end{aligned}$$

Here $\sum_{i=1}^{n} p_i/d = j\kappa/d$ (because $(p_1, p_2, \ldots p_n) \in \Lambda^{(j)}$), so

$$\begin{aligned}e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^{n} (ja_i/m_i + p_i)/d} &= e^{2\pi \mathrm{i} j m'_1 m'_2 \cdots m'_n c \left(\sum_{i=1}^{n} a_i/m_i + \kappa\right)/d} \\ &= e^{2\pi \mathrm{i} j} \qquad \text{by (3.3).}\end{aligned}$$

Hence $\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_n^{k_n} = e^{2\pi \mathrm{i} j} = 1$. □

We next show the following (this generalizes Lemma 2.4):

Theorem 6.6. *Let* $\phi : \mathbb{C}^n \to \mathbb{C}$ *be a holomorphic map given by* $\phi(v_1, v_2, \ldots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$, *where* $k_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$. *Then*:

(1) ϕ *is* G*-invariant. In particular, this induces a holomorphic map* $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.

(2) *Under the isomorphism* $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ *in* (6.2), $\overline{\Phi}$ *corresponds to* $\overline{\phi}$, *that is,* $\overline{\Phi} = \overline{\phi} \circ \Psi$.

Proof. (1): For $(v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$ and an element $g \in G$ given by $g : (v_1, v_2, \ldots, v_n) \mapsto (\eta_1 v_1, \eta_2 v_2, \ldots, \eta_n v_n)$,

$$\begin{aligned}\phi \circ g(v_1, v_2, \ldots, v_n) &= \phi(\eta_1 v_1,\, \eta_2 v_2, \ldots, \eta_n v_n) \\ &= (\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_n^{k_n}) v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \\ &= \eta_1^{k_1}\eta_2^{k_2}\cdots\eta_n^{k_n} \phi(v_1, v_2, \ldots, v_n) \\ &= \phi(v_1, v_2, \ldots, v_n) \qquad \text{by Lemma 6.5.}\end{aligned}$$

Thus $\phi \circ g = \phi$, confirming the assertion.

(2): Note first that

$$\begin{aligned}
&\overline{\phi} \circ \Psi\big([x_1, x_2, \ldots, x_n, t]\big) \\
&\quad = \overline{\phi}\big([x_1^{m_1' l_1/d}, x_2^{m_2' l_2/d}, \ldots, x_n^{m_n' l_n/d}]\big) \quad \text{(Lemma 6.4)} \\
&\quad = x_1^{m_1' l_1 k_1/d} x_2^{m_2' l_2 k_2/d} \cdots x_n^{m_n' l_n k_n/d}.
\end{aligned}$$

Here since $k_i l_i = m_1' \cdots \check{m}_i' \cdots m_n' c$, we have $m_i' l_i k_i = m_1' m_2' \cdots m_n' c$. Thus the last expression is rewritten as

$$\begin{aligned}
x_1^{m_1' l_1 k_1/d} x_2^{m_2' l_2 k_2/d} \cdots x_n^{m_n' l_n k_n/d} &= (x_1 x_2 \cdots x_n)^{m_1' m_2' \cdots m_n' c/d} \\
&= t^{m_1' m_2' \cdots m_n' c} \qquad \text{because } x_1 x_2 \cdots x_n = t^d.
\end{aligned}$$

Hence $\overline{\phi} \circ \Psi\big([x_1, x_2, \ldots, x_n, t]\big) = \overline{\Phi}\big([x_1, x_2, \ldots, x_n, t]\big)$. $\square$

6.4. Equi-smallness theorem

Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$\gamma : \ (x_1, \ldots, x_n, t) \ \longmapsto \ (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n, \ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t),$$

where $d := \sum_{k=1}^{n} a_k m_1' \cdots \check{m}_k' \cdots m_n' + m_1' m_2' \cdots m_n' c\kappa$. Here κ is an integer satisfying $(*)$ $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$. Then $\kappa \geq -n+1$ (see (3.1)).

Let $\widetilde{\Gamma}$ be the lift of Γ and H is the descent of $\widetilde{\Gamma}$. The pseudo-reflection subgroup P of H is generated by the automorphisms $h_i : \mathbb{C}^n \to \mathbb{C}^n$ ($i = 1, 2, \ldots, n$) given by $h_i : (u_1, \ldots, u_i, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i} u_i, \ldots, u_n)$ (Proposition 5.11). Here $l_i = \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')}$ does not depend on κ. Thus:

LEMMA 6.7. *The pseudo-reflection subgroup P of H does not depend on κ.*

In what follows, regarding κ as a 'parameter', write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. These are subgroups of $GL(n, \mathbb{C})$. From Lemma 6.7,

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots , \tag{6.3}$$

where κ_0 denotes the least integer in the set S of integers κ satisfying $(*)$. If H_{κ_0} is small, then $P_{\kappa_0} = \{1\}$ and by (6.3), $P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots = \{1\}$. Thus H_κ is small for any $\kappa \in S$. This confirms the following:

THEOREM 6.8 (Equi-smallness). *Let S be the set of integers κ satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and let κ_0 denote the least integer in S. Then H_{κ_0} is small $\Longleftrightarrow$ H_κ is small for any $\kappa \in S$. (In other words, H_{κ_0} is not small $\Longleftrightarrow$ H_κ is not small for any $\kappa \in S$.)*

Example 6.9. (i): When $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$ and $m_3 = 6$, $c = \gcd(m_1, m_2, m_3) = 2$, $m_1' = 1$, $m_2' = 2$ and $m_3' = 3$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$, and thus $\frac{11}{12} + \kappa > 0$. Hence $\kappa_0 = 0$. Here by Example 5.15, H_{κ_0} is small. Thus by Theorem 6.8, H_κ is small for any integer κ such that $\kappa \geq 0$.

(ii): When $n = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$, $c = \gcd(m_1, m_2, m_3) = 1$, $m_1' = 2$, $m_2' = 3$ and $m_3' = 4$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$, and thus $\frac{23}{12} + \kappa > 0$. Hence $\kappa_0 = -1$. Here since $\gcd(m_1', m_3') = 2$, Theorem 5.14 ensures that H_{κ_0} is not small. Thus by Theorem 6.8, H_κ is not small for any integer κ such that $\kappa \geq -1$.

7. Generators of $\widetilde{\Gamma}$, H and G

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and G be the descent of H with respect to the covering r. Then G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. We explicitly give generators of $\widetilde{\Gamma}$, H, G.

7.1. Generators of $\widetilde{\Gamma}$

Recall that (see the paragraph above Lemma 4.7)

$$\widetilde{\Gamma} = \left\{ \widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c \right\},$$

where $\Lambda^{(j)} = \left\{ (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \,:\, 0 \leq p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z} \right\}$ and $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is an automorphism given by

$$(X_1, \ldots, X_n) \longmapsto \left(e^{2\pi \mathrm{i}(ja_1 + m_1 p_1)/m_1 d} X_1, \ldots, e^{2\pi \mathrm{i}(ja_n + m_n p_n)/m_n d} X_n \right).$$

Recall that Γ is generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$\gamma : (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n, e^{2\pi \mathrm{i}/(m_1' m_2' \cdots m_n' c)} t).$$

The automorphism $\delta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by

$$\begin{aligned}&(X_1, X_2, \ldots, X_n)\\ &\qquad \longmapsto (e^{2\pi \mathrm{i} a_1/m_1 d} X_1, e^{2\pi \mathrm{i} a_2/m_2 d} X_2, \ldots, e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d} X_n)\end{aligned}$$

is a lift of $\gamma \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Hence $\delta \in \widetilde{\Gamma}$. The automorphism $\eta_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ $(i = 1, 2, \ldots, n-1)$ given by

$$(X_1, X_2, \ldots, X_n) \longmapsto (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d} X_n)$$

is a lift of the identity $1 \in \Gamma$ with respect to the covering p. Hence $\eta_i \in \widetilde{\Gamma}$ $(i = 1, 2, \ldots, n-1)$.

LEMMA 7.1. *Any element of $\widetilde{\Gamma}$ is expressed by δ, η_1, $\eta_2, \ldots, \eta_{n-1} \in \widetilde{\Gamma}$. In fact, $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \in \widetilde{\Gamma}$ is expressed as $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} = \delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}}$.*

PROOF. It suffices to show that $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \delta^{-j} \eta_1^{-p_1} \eta_2^{-p_2} \cdots \eta_{n-1}^{-p_{n-1}}$ is the identity. For brevity, express $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n}(\vec{x}) = A\vec{x}$, $\delta(\vec{x}) = B\vec{x}$ and $\eta_i(\vec{x}) = C_i \vec{x}$, where
$A = \mathrm{diag}\big(e^{2\pi \mathrm{i}(j a_1 + m_1 p_1)/m_1 d},\ e^{2\pi \mathrm{i}(j a_2 + m_2 p_2)/m_2 d}, \ldots, e^{2\pi \mathrm{i}(j a_n + m_n p_n)/m_n d}\big)$,
$B = \mathrm{diag}\big(e^{2\pi \mathrm{i} a_1/m_1 d},\ e^{2\pi \mathrm{i} a_2/m_2 d}, \ldots, e^{2\pi \mathrm{i} a_{n-1}/m_{n-1} d},\ e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d}\big)$,
$C_i = \mathrm{diag}\big(1, \ldots, 1,\ e^{2\pi \mathrm{i}/d},\ 1, \ldots, 1,\ e^{-2\pi \mathrm{i}/d}\big)$, where $e^{2\pi \mathrm{i}/d}$ lies in the ith place. Then $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \delta^{-j} \eta_1^{-p_1} \eta_2^{-p_2} \cdots \eta_{n-1}^{-p_{n-1}}(\vec{x}) = AB^{-j} C_1^{-p_1} C_2^{-p_2} \cdots C_{n-1}^{-p_{n-1}} \vec{x}$. It thus suffices to show that the matrix $D := AB^{-j} C_1^{-p_1} C_2^{-p_2} \cdots C_{n-1}^{-p_{n-1}}$ is the identity matrix. Since A, B, C_i are diagonal, D is also diagonal, so it suffices to show that any of its diagonal entries is 1. This is confirmed as follows:

- For $i = 1, 2, \ldots, n-1$, the (i, i) entry of D is
$$e^{2\pi \mathrm{i}(j a_i + m_i p_i)/m_i d} \big(e^{2\pi \mathrm{i} a_i/m_i d}\big)^{-j} \big(e^{2\pi \mathrm{i}/d}\big)^{-p_i} = 1.$$

- The (n, n) entry of D is
$$\begin{aligned}&e^{2\pi \mathrm{i}(j a_n + m_n p_n)/m_n d} \big(e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d}\big)^{-j} \big(e^{2\pi \mathrm{i}/d}\big)^{p_1 + \cdots + p_{n-1}}\\ &\qquad = e^{2\pi \mathrm{i}(p_1 + \cdots + p_n - j\kappa)/d}.\end{aligned}$$

Here since $(p_1, p_2, \dots, p_n) \in \Lambda^{(j)}$, we have $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$, and thus $e^{2\pi \mathrm{i}(p_1+\cdots+p_n-j\kappa)/d} = 1$. $\square$

Lemma 7.1 implies that:

Corollary 7.2. *$\widetilde{\Gamma}$ is generated by δ, $\eta_1, \eta_2, \dots, \eta_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices B, C_1, $C_2, \dots, C_{n-1}$ appearing in the proof of Lemma 7.1.*

7.2. Relations among generators of $\widetilde{\Gamma}$

Recall that $\widetilde{\Gamma}$ is a finite abelian group of order $m'_1 m'_2 \cdots m'_n c d^{n-1}$ (Proposition 3.2 (1)) and is generated by δ, η_1, η_2, $\dots$, η_{n-1} (Corollary 7.2). These generators are generally *not* independent. In fact, the following holds (the proof is the same as that of Lemma 7.1):

Lemma 7.3. $\delta^{m'_1 m'_2 \cdots m'_n c} = \eta_1^{a_1 m'_2 m'_3 \cdots m'_n} \eta_2^{a_2 m'_1 m'_3 \cdots m'_n} \cdots \eta_{n-1}^{a_{n-1} m'_1 \cdots m'_{n-2} m'_n}.$

If the order of δ is $m'_1 m'_2 \cdots m'_n c$, then this relation is actually vacuous. To see this, we need the following:

Lemma 7.4.

(1) *Express $\delta(\vec{x}) = B\vec{x}$, where B is the matrix appearing in the proof of Lemma 7.1. Then $\det B = e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c}$.*

(2) *If $\delta^k = 1$, then k is a multiple of $m'_1 m'_2 \cdots m'_n c$. In particular, the order of δ is a multiple of $m'_1 m'_2 \cdots m'_n c$.*

(3) $\mathrm{lcm}(m'_1, m'_2, \dots, m'_n)cd$ *is a multiple of* $m'_1 m'_2 \cdots m'_n c$, *and* $\delta^{\mathrm{lcm}(m'_1, m'_2, \dots, m'_n)cd} = 1$.

(4) *Write* $\mathrm{lcm}(m'_1, m'_2, \dots, m'_n)cd = N m'_1 m'_2 \cdots m'_n c$ *where N is a positive integer. Then the order of δ is $l m'_1 m'_2 \cdots m'_n c$ for some positive integer l $(1 \le l \le N)$.*

PROOF. We show the assertions only for $n=3$ (the proof is the same for any n).

(1): Since $B = \begin{pmatrix} e^{2\pi\mathrm{i}a_1/m_1 d} & 0 & 0 \\ 0 & e^{2\pi\mathrm{i}a_2/m_2 d} & 0 \\ 0 & 0 & e^{2\pi\mathrm{i}(a_3+m_3\kappa)/m_3 d} \end{pmatrix}$, we have $\det B = e^{2\pi\mathrm{i}a_1/m_1 d} e^{2\pi\mathrm{i}a_2/m_2 d} e^{2\pi\mathrm{i}(a_3+m_3\kappa)/m_3 d} = e^{2\pi\mathrm{i}(a_1/m_1+a_2/m_2+a_3/m_3+\kappa)/d}$. Here $(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)/d = 1/m_1' m_2' m_3' c$ (because $d := m_1' m_2' m_3' c(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)$), confirming the assertion.

(2): If $\delta^k = 1$, then $B^k = I$ (the identity matrix), so $\det(B^k) = 1$. Then $e^{2\pi\mathrm{i}k/m_1' m_2' m_3' c} = 1$ by (1). Thus k is a multiple of $m_1' m_2' m_3' c$.

(3): We first show that $\mathrm{lcm}(m_1', m_2', m_3')cd$ is a multiple of $m_1' m_2' m_3' c$, for which it is sufficient to demonstrate that $\dfrac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1' m_2' m_3' c}$ is an integer. Using $d := m_1' m_2' m_3' c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right)$, we rewrite:

$$\begin{aligned}
\frac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1' m_2' m_3' c} &= \mathrm{lcm}(m_1', m_2', m_3')c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right) \\
&= \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_1} a_1 + \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_2} a_2 + \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_3} a_3 \\
&\qquad + \mathrm{lcm}(m_1', m_2', m_3')c\kappa.
\end{aligned}$$

Since $m_i = m_i' c$, the last expression is equal to

$$\begin{aligned}
&\frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_1'} a_1 + \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_2'} a_2 + \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_3'} a_3 \\
&\qquad + \mathrm{lcm}(m_1', m_2', m_3')c\kappa.
\end{aligned}$$

This is an integer, because

$$\frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_1'},\ \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_2'},\ \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_3'} \text{ are integers.}$$

Thus $\dfrac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1' m_2' m_3' c}$ is an integer, confirming the assertion.

We next show that $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} = 1$. For an integer k, the automorphism $\delta^k : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by

$$(X_1, X_2, X_3) \longmapsto (e^{2\pi\mathrm{i}a_1 k/m_1 d} X_1, e^{2\pi\mathrm{i}a_2 k/m_2 d} X_2, e^{2\pi\mathrm{i}(a_3+m_3\kappa)k/m_3 d} X_3).$$

Here if $k = \mathrm{lcm}(m_1', m_2', m_3')cd$, then

$$k/m_1d = \mathrm{lcm}(m_1', m_2', m_3')/m_1', \qquad k/m_2d = \mathrm{lcm}(m_1', m_2', m_3')/m_2',$$
$$k/m_3d = \mathrm{lcm}(m_1', m_2', m_3')/m_3',$$

hence k/m_1d, k/m_2d, k/m_3d are integers, consequently $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} : (X_1, X_2, X_3) \longmapsto (X_1, X_2, X_3)$, so $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} = 1$.

(4): This follows from (2) and (3). $\square$

Since $\eta_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ $(i = 1, 2, \ldots, n-1)$ is given by

$$(X_1, X_2, \ldots, X_n) \ \longmapsto\ (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d} X_n),$$

the order of η_i is d.

Lemma 7.5.

(1) *There is no nontrivial relation among $\eta_1, \eta_2, \ldots, \eta_{n-1}$: If $\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_{n-1}^{k_{n-1}} = 1$, then $\eta_1^{k_1} = \eta_2^{k_2} = \cdots = \eta_{n-1}^{k_{n-1}} = 1$.*

(2) *Let k be an integer such that $\delta^k \neq 1$. If δ^k is expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$, that is, $\delta^k = \eta_1^{l_1}\eta_2^{l_2}\cdots\eta_{n-1}^{l_{n-1}}$ for some integers $l_1, l_2, \ldots, l_{n-1}$, then k is a multiple of $m_1'm_2'\cdots m_n'c$.*

(3) *If an integer k is not a multiple of $m_1'm_2'\cdots m_n'c$, then $\delta^k \neq 1$. Moreover δ^k cannot be expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$.*

(4) *Let $\langle\delta\rangle$ and $\langle\eta_1, \eta_2, \ldots, \eta_{n-1}\rangle$ denote the subgroups of $GL(n, \mathbb{C})$ generated by δ and $\eta_1, \eta_2, \ldots, \eta_{n-1}$ respectively. If the order of δ is $m_1'm_2'\cdots m_n'c$, then $\langle\delta\rangle \cap \langle\eta_1, \eta_2, \ldots, \eta_{n-1}\rangle = \{1\}$.*

Proof. We show this for $n = 3$ (the proof is the same for any n).

(1): The automorphism $\eta_1^{k_1}\eta_2^{k_2} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by $(X_1, X_2, X_3) \mapsto (e^{2\pi \mathrm{i} k_1/d}X_1,\ e^{2\pi \mathrm{i} k_2/d}X_2,\ e^{-2\pi \mathrm{i}(k_1+k_2)/d}X_n)$. If $\eta_1^{k_1}\eta_2^{k_2} = 1$, then $e^{2\pi \mathrm{i} k_1/d} = 1$, $e^{2\pi \mathrm{i} k_2/d} = 1$, $e^{-2\pi \mathrm{i}(k_1+k_2)/d} = 1$. Accordingly $\eta_1^{k_1} = 1$ and $\eta_2^{k_2} = 1$ hold.

(2): Suppose that $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$. Here since $\delta \in \widetilde{\Gamma}$ is a lift of $\gamma \in \Gamma$, $\delta^k \in \widetilde{\Gamma}$ is a lift of $\gamma^k \in \Gamma$ and since $\eta_1, \eta_2 \in \widetilde{\Gamma}$ are lifts of $1 \in \Gamma$, $\eta_1^{l_1}\eta_2^{l_2} \in \widetilde{\Gamma}$ is a lift of $1 \in \Gamma$. The relation $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$ thus implies that δ^k is a lift of both γ^k

and 1, so $\gamma^k = 1$. Since the order of γ is $m'_1 m'_2 m'_3 c$, this implies that k is a multiple of $m'_1 m'_2 m'_3 c$.

(3): Since the order of δ is a multiple of $m'_1 m'_2 m'_3 c$ (Lemma 7.4 (2)), if an integer k is not a multiple of $m'_1 m'_2 m'_3 c$, then $\delta^k \neq 1$. The rest is a restatement of (2).

(4): This can be shown by contradiction. If $\langle \delta \rangle \cap \langle \eta_1, \eta_2 \rangle \neq \{1\}$, then there exist elements $\delta^k \neq 1$ of $\langle \delta \rangle$ and $\eta_1^{l_1} \eta_2^{l_2} \neq 1$ of $\langle \eta_1, \eta_2 \rangle$ such that $\delta^k = \eta_1^{l_1} \eta_2^{l_2}$. Then (2) implies that k is a multiple of $m'_1 m'_2 m'_3 c$. But $\delta^{m'_1 m'_2 m'_3 c} = 1$ by assumption, accordingly $\delta^k = 1$. This contradicts that $\delta^k \neq 1$. $\square$

By (4) of Lemma 7.4, the order of δ is $l m'_1 m'_2 \cdots m'_n c$ for some positive integer l $(1 \leq l \leq N)$, where $N = \dfrac{\operatorname{lcm}(m'_1, m'_2, \ldots, m'_n) cd}{m'_1 m'_2 \cdots m'_n c}$. The following holds:

COROLLARY 7.6.

(1) *If the order of δ is $m'_1 m'_2 \cdots m'_n c$, then the relation in Lemma 7.3 is vacuous, that is, $\delta^{m'_1 m'_2 \cdots m'_n c} = \eta_1^{a_1 m'_2 m'_3 \cdots m'_n} = \cdots = \eta_{n-1}^{a_{n-1} m'_1 \cdots m'_{n-2} m'_n} = 1$.*

(2) *If the order of δ is $m'_1 m'_2 \cdots m'_n c$, then $\widetilde{\Gamma}$ is isomorphic to the product of cyclic groups $\langle \delta \rangle \times \langle \eta_1 \rangle \times \langle \eta_2 \rangle \times \cdots \times \langle \eta_{n-1} \rangle$, where $\langle \delta \rangle$ and $\langle \eta_i \rangle$ denote the cyclic groups generated by δ and η_i respectively.*

PROOF. We show this for $n = 3$ (the proof is the same for other cases).

(1): If the order of δ is $m'_1 m'_2 m'_3 c$, then $\delta^{m'_1 m'_2 m'_3 c} = 1$, so $\eta_1^{a_1 m'_2 m'_3} \eta_2^{a_2 m'_1 m'_3} = 1$ by Lemma 7.3. Consequently $\eta_1^{a_1 m'_2 m'_3} = \eta_2^{a_2 m'_1 m'_3} = 1$ by Lemma 7.5 (1), confirming the assertion.

(2): By Lemma 7.5 (4), if the order of δ is $m'_1 m'_2 m'_3 c$, then $\langle \delta \rangle \cap \langle \eta_1, \eta_2 \rangle = \{1\}$. Since $\widetilde{\Gamma}$ is generated by δ, η_1, η_2 (Corollary 7.2), we obtain $\widetilde{\Gamma} \cong \langle \delta \rangle \times \langle \eta_1, \eta_2 \rangle$. Here $\langle \eta_1, \eta_2 \rangle = \langle \eta_1 \rangle \times \langle \eta_2 \rangle$ because there is no nontrivial relation between η_1 and η_2 (Lemma 7.5 (1)). Hence $\widetilde{\Gamma} \cong \langle \delta \rangle \times \langle \eta_1 \rangle \times \langle \eta_2 \rangle$, confirming the assertion. $\square$

REMARK 7.7. If the order of δ is greater than $m'_1 m'_2 \cdots m'_n c$, then $\widetilde{\Gamma}$ is not isomorphic to $\langle \delta \rangle \times \langle \eta_1 \rangle \times \langle \eta_2 \rangle \times \cdots \times \langle \eta_{n-1} \rangle$, because there is a nontrivial relation among δ, η_1, $\eta_2, \ldots, \eta_{n-1}$ (Lemma 7.3).

7.3. Generators of H and relations among them

Recall that (see Lemma 4.8 (3))

$$H = \left\{ h^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $h^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$(u_1, \dots, u_n) \longmapsto \left(e^{2\pi \mathrm{i}(ja_1 + m_1 p_1)/cd} u_1, \dots, e^{2\pi \mathrm{i}(ja_n + m_n p_n)/cd} u_n \right).$$

Recall that $\widetilde{\Gamma}$ is generated by δ, η_1, $\eta_2, \dots, \eta_{n-1}$ (Corollary 7.2), where

$$\begin{aligned}
\delta : & (X_1, X_2, \dots, X_n) \\
& \longmapsto (e^{2\pi \mathrm{i} a_1/m_1 d} X_1, e^{2\pi \mathrm{i} a_2/m_2 d} X_2, \dots, e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d} X_n), \\
\eta_i : & (X_1, X_2, \dots, X_n) \longmapsto (X_1, \dots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \dots, e^{-2\pi \mathrm{i}/d} X_n).
\end{aligned}$$

Let α, β_i $(i = 1, 2, \dots, n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$\begin{aligned}
\alpha : (u_1, u_2, \dots, u_n) &\longmapsto (e^{2\pi \mathrm{i} a_1/cd} u_1, e^{2\pi \mathrm{i} a_2/cd} u_2, \dots, e^{2\pi \mathrm{i}(a_n + m_n \kappa)/cd} u_n), \\
\beta_i : (u_1, u_2, \dots, u_n) &\longmapsto (u_1, \dots, u_{i-1}, e^{2\pi \mathrm{i} m'_i/d} u_i, u_{i+1}, \dots, e^{-2\pi \mathrm{i} m'_n/d} u_n).
\end{aligned}$$

They are respectively the descents of δ, $\eta_i \in \widetilde{\Gamma}$ (with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$), hence α, $\beta_i \in H$.

Lemma 7.8. *Any element of H is expressed by α, β_1, $\beta_2, \dots, \beta_{n-1}$. In fact, $h^{(j)}_{p_1,p_2,\dots,p_n} \in H$ is expressed as $h^{(j)}_{p_1,p_2,\dots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$.*

Proof. Since α, $\beta_i \in H$ are the descents of δ, $\eta_i \in \widetilde{\Gamma}$ respectively, $\alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}} \in H$ is the descent of $\delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}} \in \widetilde{\Gamma}$. On the other hand, $h^{(j)}_{p_1,p_2,\dots,p_n} \in H$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}$ (Lemma 4.8 (1)). The relation $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}}$ (in Lemma 7.1) then implies $h^{(j)}_{p_1,p_2,\dots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$. $\square$

Lemma 7.8 implies that H is generated by α, β_1, $\beta_2, \dots, \beta_{n-1}$. Here α and β_i are expressed by the following diagonal matrices:
$S = \mathrm{diag}\big(e^{2\pi \mathrm{i} a_1/cd},\ e^{2\pi \mathrm{i} a_2/cd}, \dots, e^{2\pi \mathrm{i} a_{n-1}/m_{n-1} d},\ e^{2\pi \mathrm{i}(a_n + m_n \kappa)/cd}\big)$ and
$T_i = \mathrm{diag}\big(1, \dots, 1,\ e^{2\pi \mathrm{i} m'_i/d},\ 1, \dots, 1,\ e^{-2\pi \mathrm{i} m'_n/d}\big)$, where $e^{2\pi \mathrm{i} m'_i/d}$ lies in the ith place. Thus:

COROLLARY 7.9. *H is generated by α, β_1, $\beta_2, \dots, \beta_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices S, T_1, $T_2, \dots, T_{n-1}$.*

Here $\alpha, \beta_1, \beta_2, \dots, \beta_{n-1}$ are actually *not* independent. In fact, there are relations among them:

LEMMA 7.10. *The generators α, β_1, $\beta_2, \dots, \beta_{n-1}$ of H satisfy the following relations*:

(a) $\alpha^{m'_1 m'_2 \cdots m'_{n-1} c} = \beta_1^{a_1 m'_2 m'_3 \cdots m'_{n-1}} \beta_2^{a_2 m'_1 m'_3 \cdots m'_{n-1}} \cdots \beta_{n-1}^{a_{n-1} m'_1 m'_2 \cdots m'_{n-2}}$.

(b) *For $i = 1, 2, \dots, n-1$,*

$$\begin{aligned}\alpha^{m'_1 \cdots \check{m}'_i \cdots m'_n c} = \beta_1^{a_1 m'_2 \cdots \check{m}'_i \cdots m'_n} \cdots \beta_i^{(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i} \cdots \\ \cdots \beta_{n-1}^{a_{n-1} m'_1 \cdots \check{m}'_i \cdots m'_{n-2} m'_n},\end{aligned}$$

where note that $(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i$ is an integer.

REMARK 7.11. The existence of nontrivial relations among α, β_1, β_2, $\dots, \beta_{n-1}$ implies that $H = \langle \alpha, \beta_1, \beta_2, \dots, \beta_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle \alpha \rangle \times \langle \beta_1 \rangle \times \langle \beta_2 \rangle \times \cdots \times \langle \beta_{n-1} \rangle$.

7.4. Generators of G and relations among them

Recall that (see Lemma 6.1 (C))

$$G = \left\{ g^{(j)}_{p_1, p_2, \dots, p_n} \, : \, (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}, \, j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $g^{(j)}_{p_1, p_2, \dots, p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$(v_1, \dots, v_n) \longmapsto \left(e^{2\pi \mathrm{i} l_1 (j a_1 + m_1 p_1)/cd} v_1, \dots, e^{2\pi \mathrm{i} l_n (j a_n + m_n p_n)/cd} v_n \right).$$

Recall that H is generated by α, β_1, $\beta_2, \dots, \beta_{n-1}$ (Corollary 7.9), where

$$\begin{aligned}\alpha &: (u_1, u_2, \dots, u_n) \longmapsto (e^{2\pi \mathrm{i} a_1/cd} u_1, e^{2\pi \mathrm{i} a_2/cd} u_2, \dots, e^{2\pi \mathrm{i} (a_n + m_n \kappa)/cd} u_n), \\ \beta_i &: (u_1, u_2, \dots, u_n) \longmapsto (u_1, \dots, u_{i-1}, e^{2\pi \mathrm{i} m'_i/d} u_i, u_{i+1}, \dots, e^{-2\pi \mathrm{i} m'_n/d} u_n).\end{aligned}$$

Let $f, g_i \, (i = 1, 2, \dots, n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$\begin{aligned}f &: (v_1, v_2, \dots, v_n) \longmapsto (e^{2\pi \mathrm{i} l_1 a_1/cd} v_1, e^{2\pi \mathrm{i} l_2 a_2/cd} v_2, \dots, e^{2\pi \mathrm{i} l_n (a_n + m_n \kappa)/cd} v_n), \\ g_i &: (v_1, v_2, \dots, v_n) \longmapsto (v_1, \dots, v_{i-1}, e^{2\pi \mathrm{i} l_i m'_i/d} v_i, v_{i+1}, \dots, e^{-2\pi \mathrm{i} l_n m'_n/d} v_n).\end{aligned}$$

They are respectively the descents of α, $\beta_i \in H$ (with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$), hence $f, g_i \in G$. As for Lemma 7.8, we can show the following:

LEMMA 7.12. *Any element of G is expressed by f, g_1, $g_2, \ldots, g_{n-1}$. In fact, $g^{(j)}_{p_1,p_2,\ldots,p_n} \in G$ is expressed as $g^{(j)}_{p_1,p_2,\ldots,p_n} = f^j g_1^{p_1} g_2^{p_2} \cdots g_{n-1}^{p_{n-1}}$.*

Lemma 7.12 implies that:

COROLLARY 7.13. *G is generated by $f, g_1, g_2, \ldots, g_{n-1}$, where f and g_i are expressed by the diagonal matrices*
$Q = \mathrm{diag}\big(e^{2\pi \mathrm{i} l_1 a_1/cd},\ e^{2\pi \mathrm{i} l_2 a_2/cd}, \ldots, e^{2\pi \mathrm{i} l_{n-1} a_{n-1}/cd},\ e^{2\pi \mathrm{i} l_n (a_n + m_n \kappa)/cd}\big)$ *and*
$R_i = \mathrm{diag}\big(1, \ldots, 1,\ e^{2\pi \mathrm{i} l_i m'_i/d},\ 1, \ldots, 1,\ e^{-2\pi \mathrm{i} l_n m'_n/d}\big)$, *where $e^{2\pi \mathrm{i} l_i m'_i/d}$ lies in the ith place.*

Here $f, g_1, g_2, \ldots, g_{n-1}$ are actually *not* independent. In fact, there are relations among them:

LEMMA 7.14. *The generators f, g_1, $g_2, \ldots, g_{n-1}$ of G satisfy the following ralations*:

(a) $$f^{\mathrm{lcm}(m'_1, m'_2, \ldots, m'_{n-1})c} = g_1^{a_1 \mathrm{lcm}(m'_1, m'_2, \ldots, m'_{n-1})/m'_1} \cdots g_{n-1}^{a_{n-1} \mathrm{lcm}(m'_1, m'_2, \ldots, m'_{n-1})/m'_{n-1}},$$
where note that $a_k \mathrm{lcm}(m'_1, m'_2, \ldots, m'_{n-1})/m'_k$ $(k = 1, 2, \ldots, n-1)$ is an integer (because m'_k divides $\mathrm{lcm}(m'_1, m'_2, \ldots, m'_{n-1})$*).*

(b) *For $i = 1, 2, \ldots, n-1$,*
$$\begin{aligned} &f^{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c} \\ &\quad = g_1^{a_1 \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_1} \cdots g_i^{(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/l_i m'_i} \cdots \\ &\qquad \cdots g_{n-1}^{a_{n-1} \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_{n-1}}, \end{aligned}$$
where note that $(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/l_i m'_i$ is an integer and for $k = 1, 2, \ldots, \check{i}, \ldots, n$, $a_k \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_k$ is an integer (because m'_k divides $\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$*).*

REMARK 7.15. The existence of nontrivial relations among f, g_1, g_2, $\ldots, g_{n-1}$ implies that $G = \langle f, g_1, g_2, \ldots, g_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle f \rangle \times \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_{n-1} \rangle$.

CASE $n = 2$. Let a_1^* $(0 < a_1^* < m_1)$ be the integer such that $a_1 a_1^* \equiv 1 \bmod m_1$. If $n = 2$, then G is a cyclic group generated by $g : (u_1, u_2) \mapsto (e^{2\pi \mathrm{i}/cd} u_1,\, e^{2\pi \mathrm{i}\mathsf{q}/cd} u_2)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a_1^* d - m_2'}{m_1'} \bmod cd$ (Theorem 2.1). Note that $\dfrac{a_1^* d - m_2'}{m_1'}$ is an integer (cf. Lemma 2.3 (1)). Here the automorphism g is expressed by the matrix $P := \begin{pmatrix} e^{2\pi \mathrm{i}/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}\mathsf{q}/cd} \end{pmatrix}$, and as a subgroup of $GL(2, \mathbb{C})$, G is generated by P. On the other hand by Corollary 7.13, G is generated by two matrices $Q = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix}$ and $R_1 = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix}$. Note that $l_1 = l_2 = 1$, thus $G = H$, $f = \alpha$, $g_1 = \beta_1$. We describe the relations among P and Q, R_1.

For simplicity, write m_1, m_2, a_1, a_2, a_1^*, β_1, R_1 as m, n, a, b, a^*, β, R, and set $c := \gcd(m, n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$ and $d := an' + bm' + m'n'c\kappa$.

PROPOSITION 7.16. *The matrices P, Q, $R \in GL(2, \mathbb{C})$ expressing the automorphisms g, α, β are related as follows*:

(1) $P^a = Q$, $P^m = R$.

(2) *Noting that $\dfrac{1 - aa^*}{m}$ is an integer* (*because $aa^* \equiv 1 \bmod m$*), *let l* ($0 < l < cd$) *be the integer such that $l \equiv \dfrac{1 - aa^*}{m} \bmod cd$. Then $Q^{a^*} R^l = P$.*

PROOF. (1): We first show $P^a = Q$. Since $a\mathsf{q} \equiv \dfrac{a(a^* d - n')}{m'} \equiv \dfrac{d - an'}{m'} \equiv b + n\kappa \bmod cd$,

$$P^a = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}a\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix} = Q.$$

We next show $P^m = R$. Since $m\mathsf{q} \equiv \dfrac{m(a^* d - n')}{m'} \equiv a^* cd - cn' \equiv -cn' \bmod cd$,

$$P^m = \begin{pmatrix} e^{2\pi \mathrm{i}m/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}m\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & 0 \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix} = R.$$

(2): We first show $P^{aa^*+ml} = P$. Since $l \equiv \dfrac{1-aa^*}{m} \bmod cd$ and $aa^* + m\dfrac{1-aa^*}{m} = 1$, we have $aa^* + ml \equiv 1 \bmod cd$. Hence

$$e^{2\pi \mathrm{i}(aa^*+ml)/cd} = e^{2\pi \mathrm{i}/cd}, \qquad e^{2\pi \mathrm{i}(aa^*+ml)\mathsf{q}/cd} = e^{2\pi \mathrm{i}\mathsf{q}/cd}.$$

Accordingly, $P^{aa^*+ml} = P$. Then $(P^a)^{a^*}(P^m)^l = P$. Here since $P^a = Q$ and $P^m = R$ hold by (1), $Q^{a^*}R^l = P$. The assertion is thus confirmed. □

Corollary 7.17. *The automorphisms g, α, $\beta : \mathbb{C}^2 \to \mathbb{C}^2$ are related as follows*:

(1) $g^a = \alpha$, $g^m = \beta$.

(2) *Noting that* $\dfrac{1-aa^*}{m}$ *is an integer* (*because* $aa^* \equiv 1 \bmod m$), *let* l ($0 < l < cd$) *be the integer such that* $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. *Then* $\alpha^{a^*}\beta^l = g$.

References

[Ash] Ashikaga, T., Toric modification of cyclic orbifolds and extened Zagier reciprocity for Dedekind sums, Preprint (2012).

[AsIs] Ashikaga, T. and M. Ishizaka, A geometric proof of the reciprocity law of Dedekind sum, Preprint (2009).

[AGV] Arnold, V. I., Gusein-Zade, S. M. and A. N. Varchenko, *Singularities of differentiable maps, II*, Birkhäuser (1988).

[MaMo] Matsumoto, Y. and J. M. Montesinos-Amilibia, Pseudo-periodic maps and degeneration of Riemann surfaces, Springer Lecture Notes in Math. 2030 (2011).

[SaTa] Sasaki, K. and S. Takamura, Singularities and higher-dimensional fractional Dehn twists, in preparation.

[Tak] Takamura, S., Towards the classification of atoms of degenerations, II, (Linearization of degenerations of complex curves), RIMS Preprint 1344 (2001).

(Received May 28, 2015)
(Revised August 7, 2015)

Department of Mathematics
Graduate School of Science
Kyoto University
Oiwakecho, Kitashirakawa, Sakyo-ku
Kyoto 606-8502, JAPAN
E-mail: kjr-ssk@math.kyoto-u.ac.jp
takamura@math.kyoto-u.ac.jp