Pairing Picture of Quantum Hall Liquids

(ペアリング描像による量子ホール液体の理論的研究)

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Contents

Chapter 1

Introduction to Quantum Hall Effect

1.1 QHE in Single-layer Systems

The two-dimensional interacting electron system is the one of the most attractive problems in condensed matter physics. The quantum Hall effect (QHE) is the representative example of it [1, 2]. The QHE is observed under the strong magnetic field perpendicular to the plane of electrons. When we measure the Hall resistivity ρ_{xy} with varying the external magnetic field, there appear plateaus around points of rational Landau level fillings ν, [1] and the value of ρ_{xy} is equal to $(1/\nu)(h/e^2)$, where h is the Planck constant and e is the elementary electric charge. On the other hand, the longitudinal resistivity ρ_{xx} goes to zero around these plateaus. Therefore, the Hall conductance σ_{xy} is quantized in units of e^2/h and the coefficient of it is the Landau level filling fraction ν, that is,

$$\sigma_{xy} = -\nu \frac{e^2}{h}. \tag{1.1}$$

Astonishingly the Hall conductance is independent of the detail of samples and has a universal value.

At first the quantum Hall effect was discovered at integer Landau level fillings[3]. It is called the integral quantum Hall effect (IQHE). An improvement of samples in GaAs heterojunctions reveals this effect at fractional Landau level fillings [4]. It is called the fractional quantum Hall effect (FQHE). The IQHE occurs with the filled Landau levels and FQHE occurs with the partially filled Landau level.[2] The theory of IQHE usually neglects the effect of interactions on the assumption that electrons in filled Landau levels are inert and we focus on the impurity effect. On the other hand, the interaction is considered to be essential in FQHE. Up to now, we do not have the unified theory of IQHE and FQHE. However, in real experimental situation, the energy scale of the Coulomb interaction and the cyclotron energy are almost the same. Therefore, we can not neglect the Coulomb interaction. In the following, we do not discuss IQHE any more and concentrate on FQHE.

1.1.1 FQHE at $\nu = 1/m$ ($m =$ odd)

The FQHE is observed mostly at Landau level fillings with odd integer denominator. Especially the states at $\nu = 1/m$, where m is an odd integer, are the fundamental states of FQHE. The understanding of the FQHE at these filling fractions was achieved by Laughlin [5]. He obtained the ground state wave function by a variational method and showed that the ground state is incompressible and quasiparticles have fractional charges. The ground state of FQHE is given by

$$\Psi(z_1, z_2, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^m \exp\left(-\frac{1}{4\ell_B^2} \sum_{i=1}^{N} |z_i|^2\right), \tag{1.2}$$

where $z_j = x_j + iy_j$ is the coordinate of the j-th electron confined in the xy-plane and $\ell_B = \sqrt{ch/eB}$ is the magnetic length. Here the applied external magnetic field is assumed to be $-B\hat{e}_z$ with $\hat{e}_z$ being the unit

[1] On counting the Landau level filling factor, it is assumed that spins of electrons are fully polarized.

[2] When we discuss the filling of Landau level, we implicitly assume the existence of Landau levels of interacting electrons.

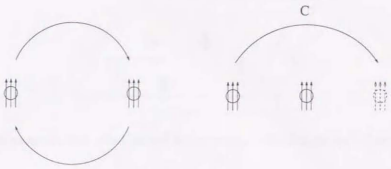

Figure 1.1: The interchange of two composite particles (left figure) corresponds to that one composite particle moves around another composite particle (right figure).

vector of the z axis. The ground state described by this Laughlin wave function (1.2) is an incompressible liquid, which is the most important nature of quantum Hall liquids. Excitations above the ground state described by Eq.(1.2) are not usual ones. Quasiparticles have fractional charges and obey the fractional statistics. These fractional charges are recently observed by the shot noise experiment in a quantum point contact [6, 7].

Next we discuss the effective theory of the FQHE. The key of it was found by Girvin and MacDonald [8]. They showed that Laughlin wave function (1.2) has the off-diagonal long range order (ODLRO), which characterizes phenomena such as superfluidity and superconductivity. Bose fields which have this ODLRO are composite bosons [8, 9, 10]. The basic idea of composite particles is the following. The fact that the spatial dimension of the system is two make it possible to change the statistics of electrons by mapping them into particles attached to fictitious fluxes. We call them composite particles. The direction of fluxes are perpendicular to the plane. When we interchange the position of two neighboring composite particles, they obtain the Aharonov-Bohm phase caused by these fictitious fluxes. Let us consider the composite particle attached to $m\phi_0$ fluxes with charge e, where $\phi_0 = ch/e$ is the flux quantum. The Aharonov-Bohm phase at interchanging the position of two composite particles are given by

$$\frac{e}{\hbar c}\int_C d\mathbf{r}\cdot\mathbf{a} = \pi \times m. \tag{1.3}$$

The path C is shown in fig.1.1, and $\mathbf{a}$ is the Chern-Simons gauge field, which is the gauge field for fictitious flux. The phases caused by interchanging two particles are this Aharonov-Bohm phase and the statistics of composite particles. As the original problem is the electronic problem, that is, fermionic problem, the total phase must be odd times π. Therefore, the problem is mapped into composite bosons (fermions) if choose m as odd (even) integer.

For fractional quantum Hall systems at $\nu = 1/m$, where m is an odd integer, we map electron systems into composite bosons with m fictitious fluxes. In that case, we have m fictitious fluxes and m external fluxes per particle. When we choose the direction of fictitious fluxes as the inverse direction of the external magnetic field, the Chern-Simons gauge field completely cancel the external magnetic field at mean field level. Therefore, we have the two-dimensional boson system in the absence of the external magnetic field. The ground state of this bose system is expected to be the superconducting state. In Fig.1.2, we show that how we can understand the quantum Hall effect by the superconductivity of composite bosons[10]. Because of the fictitious fluxes of composite bosons, the supercurrent of them induces the transverse voltage drop. The FQHE is understood as the result of the combination of the supercurrent and the voltage drop. The boson supercurrent is given by $I = -e \times dN/dt$. The flux current is given by $I_{\rm flux} = (1/\nu) \times \phi_0 \times dN/dt$. Therefore, the voltage drop is given by

$$V = -\frac{1}{c} \times I_{\rm flux} = \frac{1}{\nu} \times \frac{h}{e^2}\, I. \tag{1.4}$$

As a result we obtain,

$$R_H = \frac{1}{\nu} \times \frac{h}{e^2}. \tag{1.5}$$

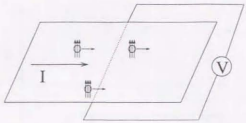

Figure 1.2: The quantum Hall effect caused by the superconducting current of composite bosons.

1.1.2 FQHE at $\nu = m/(mp+1)$ ($p =$ even, $m =$ integer)

The FQHE is also observed at other odd denominator filling fractions:

$$\nu = \frac{m}{mp+1}, \tag{1.6}$$

where m is an integer and p is an even integer. These states are understood by Jain's composite fermion (CF) scheme [11]. In his theory, we map a system of electrons into one of composite particles with even number of fictitious fluxes. Even number of fluxes do not change the statistics of particles because the Aharonov-Bohm phase caused by them is $\pi\times$(even integer). Therefore, the statistics of them corresponds to that of electrons, that is, fermion. When we map the system of electrons into one of CFs by attaching p fluxes, the effective magnetic field for CFs is given by $B_{\rm eff} = B - p\phi_0\overline{\rho}$, where $\overline{\rho}$ is the average of particle numbers. The effective Landau level filling fraction of CFs is given by

$$\nu_{\rm CF} = \frac{\overline{\rho}}{B_{\rm eff}/\phi_0} = m. \tag{1.7}$$

Therefore, the FQHE at $\nu = m/(mp+1)$ corresponds to the IQHE of CFs with m filled Landau levels. The effective theory of these states was constructed by Blok and Wen [12]. To get it, they performed two singular gauge transformations. One is from electrons to CFs. Another is from CFs of m filled Landau levels to composite bosons of m species. Here one flux are attached to each boson. As a result, the system of electron is cast into the one of bosons. The attached fluxes cancel the effective external magnetic field of CFs. These bose fields are expected to be the superconducting states at mean field. Therefore, also at $\nu = m/(mp+1)$, the FQHE is understood by the superconductivity of composite bosons.

Basically, the quantum Hall effect at odd denominator fillings is understood by the superconductivity of composite bosons.

1.2 QHE in Bilayer Systems

The QHE is also observed in bilayer systems [13]. The advantage to investigate such a system is that we have many parameters to control it. In Fig.1.3, we show the setup of bilayer systems. We have parameters to control the system: interlayer separation d, interlayer tunneling t and Landau level indices for each layer. Furthermore, we can apply a different electric field for each layer. In the following, we focus on the case of $\nu_1 = \nu_2$.

The ground state wave function is well understood in bilayer systems. In the absence of an interlayer tunneling and an appropriate interlayer separation, the ground state of the system is well described by the Halperin (m,m,n) wave functions [14]:

$$\begin{aligned}\Psi_{mmn}(z_1^{\uparrow}, z_2^{\uparrow}, \cdots, z_N^{\uparrow}; z_1^{\downarrow}, z_2^{\downarrow}, \cdots, z_N^{\downarrow}) \\ = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^m \prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})^m \prod_{i,j}(z_i^{\uparrow} - z_j^{\downarrow})^n \exp\left[-\frac{1}{4\ell_B^2}\sum_j(|z_j^{\uparrow}|^2 + |z_j^{\downarrow}|^2)\right],\end{aligned} \tag{1.8}$$

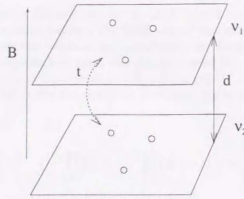

Figure 1.3: The setup for the bilayer system. Contrary to single layer systems, we have additional parameters to control the system: the interlayer separation d, the interlayer tunneling t and the Landau level filling for each layer.

where $\uparrow(\downarrow)$ is the index for upper (lower) layer. The total Landau level filling is related to m and n by $\nu = 2/(m+n)$. Equation (1.8) is an extension of the Laughlin wave function (1.2). Yoshioka *et al.* calculated the overlap between the ground state wave function of a finite system and the Halperin (m, m, n) wave function (1.8) with changing the interlayer separation d in the absence of the interlayer tunneling [15]. For $\nu = 1/2$, the ground state of a finite system is well described by the Halperin $(3, 3, 1)$ wave function around the region $1 < d/\ell_B < 2$. For $\nu = 1$, the Halperin $(1, 1, 1)$ wave function has good overlap with the ground state of a finite system around $0 \lesssim d/\ell_B \lesssim 1$.

Contrary to single layer systems, the FQHE is observed at $\nu = 1/2$ [13]. However, it is understood by extending the Laughlin wave function. In fact, as far as we concern the quantization of the Hall conductance, we can deal with all of the bilayer systems by composite boson theory [16, 17, 18].

1.3 Mystery of $\nu = 5/2$ State and Pairing Picture of FQHE

As we have seen in Sec.1.1, FQH states with odd denominator Landau level filling fraction are well understood. However, Willett *et al.* discovered the FQHE at $\nu = 5/2$ in 1987 [19]. Contrary to the well-understood FQH states with odd denominator filling fraction, this state has even denominator filling. The theory of the FQHE in single layer systems, such as Laughlin wave function, the composite boson Chern-Simons gauge field theory and Jain's CF theory failed to understand the FQHE at $\nu = 5/2$. From the view point of the Chern-Simons gauge theory, we cannot cancel the external magnetic field by a flax attachment as far as we adopt composite bosons. To cancel the external magnetic field at even denominator fillings, we need composite particles with even number fluxes, that is, CFs.

In addition to the fact that the denominator is an even integer, we must take into account the effect of the filled Landau levels to understand the $\nu = 5/2$ state. The $\nu = 5/2$ state has one filled Landau level of spin $\uparrow$ and that of spin $\downarrow$ because of the small g-factor in GaAs samples. Therefore, it is the second Landau level which is partially filled. On the other hand, the $\nu = 1/2$ state has no filled Landau level and the $\nu = 3/2$ state has one filled Landau level. However, the FQHE at $\nu = 1/2$ and $\nu = 3/2$ have not been observed. Therefore, the theory of $\nu = 5/2$ must be able to explain the reason why we do not observe the FQHE at $\nu = 1/2$ and $\nu = 3/2$ but do at $\nu = 5/2$.

To understand the $\nu = 5/2$ state, we need to seek an alternative picture of the quantum Hall effect. Haldane and Rezayi attacked the $\nu = 5/2$ problem [20]. The effect of the second Landau level was taken into account as the change of the pseudo potential [21]. The pseudo potential of the second Landau level for pairs with zero relative angular momentum is reduced from that of the first Landau level. For that reason, they proposed the hollow core model for the $\nu = 5/2$ state and concluded that the ground state of the system is the spin-singlet d-wave pairing of CFs by a numerical analysis.[3]

[3]Though there are some works on the theory of the $\nu = 5/2$ state, such as non-Abelian Chern-Simons gauge theory [22] and

From the theoretical view point, the existence of the QHE caused by the pairing of CFs is seemed to be the natural consequence. The relation between the QHE caused by composite bosons and the QHE by the pairing of CFs is analogous to the one between the superfluidity of ^{4}He system and that of ^{3}He system [24].

The pairing picture of the quantum Hall effect was also developed [25] in bilayer quantum Hall systems at $\nu = 1/2$. In this system, as numerical works support [15, 26] the ground state of the system is well described by the Halperin $(3,3,1)$ state [14] in the absence of an interlayer tunneling:

$$\Psi_{331}(z_1^\uparrow, z_2^\uparrow, \cdots, z_N^\uparrow; z_1^\downarrow, z_2^\downarrow, \cdots, z_N^\downarrow) = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)^3 \prod_{i<j}(z_i^\downarrow - z_j^\downarrow)^3 \prod_{i,j}(z_i^\uparrow - z_j^\downarrow)^1 \exp\left[-\frac{1}{4\ell_B^2}\sum_j(|z_j^\uparrow|^2 + |z_j^\downarrow|^2)\right]. \quad (1.9)$$

On the other hand, the Pfaffian state [27] is proposed in the strong tunneling limit [28, 29]:

$$\Psi_{\text{Pfaffian}}(z_1, z_2, \cdots, z_N) = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2 \exp\left(-\frac{1}{4\ell_B^2}\sum_j |z_j|^2\right), \quad (1.10)$$

where $z_j = z_j^\uparrow = z_j^\downarrow$ and

$$\text{Pf}(M_{ij}) = \sum_{P\in\sigma_{2N}}(-1)^P \prod_{k=1}^{N} M_{P_{2k-1},P_{2k}}. \quad (1.11)$$

Ho showed that both the $(3,3,1)$ state and the Pfaffian state belong to the family of triplet p-wave pairing states of CFs based on an analysis of wave functions [25]. His second quantized wave function is given by

$$|N,\chi\rangle = \int\prod_{j=1}^{2N} d^2 z_j \prod_{i<j}(z_i - z_j)^2 \mathrm{e}^{-\frac{1}{4\ell_B^2}\sum_{j=1}^{2N}|z_j|^2} \prod_{j=1}^{N}\left[\frac{\chi_{\mu\nu}\psi_\mu^\dagger(z_{2j-1})\psi_\nu^\dagger(z_{2j})}{z_{2j-1} - z_{2j}}\right]|0\rangle, \quad (1.12)$$

where χ is the matrix for pseudo spins (layer indices) and $|0\rangle$ is the vacuum state. For the $(3,3,1)$ state, χ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For the Pfaffian state, χ is given by $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. However, his analysis was limited to the ground state wave function and the origin of the pairing interaction was not discussed.

With regard to the pairing interaction, Greiter, Wen, and Wilczek (GWW) derived such interaction for spinless fermions, which leads to the p-wave pairing [28, 29]. However, GWW used an approximation that the number of CF's fluxes is small and they retained first order term for the Chern-Simons gauge filed and neglected the quadratic term and the Coulomb interaction term. Therefore, how neglected terms affect the pairing state was unclear and the condition of quantum Hall effect was not discussed. Bonesteel *et al.* [30, 31] studied an interaction between CFs mediated by the Chern-Simons gauge field fluctuation. Though they retained the quadratic term of the Chern-Simons gauge field, they used a random phase approximation to obtain the effective interaction and the resulting one had a very complicated form. Therefore, they discussed only that there seemed to be some instability toward pairing of CFs. Up to now, there has been no sufficient theoretical foundation to treat the pairing of CFs and the application of this picture to quantum Hall systems has been limited.

In this thesis, we present the pairing theory of CFs. To begin with, we introduce extended CFs by a non-unitary transformation and derive the Hamiltonian. We analyze this Hamiltonian by the pairing approximation and discuss the pairing state in single-layer systems and bilayer systems. Finally, we summarize the results.

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a scenario of condensation of Skyrmions [23], there are no conclusive work on the $\nu = 5/2$ problem.

Chapter 2

Extended Composite Fermions

In this chapter, we introduce extended CF operators by a non-unitary transformation. To discuss general systems, we consider n species of electrons. We set $n = 1$ for spin-polarized single-layer systems and set $n = 2$ for spin-polarized bilayer systems or spin-unpolarized single-layer systems. The Hamiltonian for extended CFs is derived in this chapter. We also present a classical picture of the attractive interaction between CFs.

2.1 Rajaraman-Sondhi Transformation

The mapping from an electron system into an extended composite particle one in the second quantized form is given by

$$\begin{cases} \phi_\alpha(\mathbf{r}) = \mathrm{e}^{-J_\alpha(\mathbf{r})}\psi_\alpha(\mathbf{r}), \\ \pi_\alpha(\mathbf{r}) = \psi_\alpha^\dagger(\mathbf{r})\,\mathrm{e}^{J_\alpha(\mathbf{r})}, \end{cases} \tag{2.1}$$

where

$$J_\alpha(\mathbf{r}) = \sum_\beta K_{\alpha\beta}\int d^2\mathbf{r}'\rho_\beta(\mathbf{r}')\log(z-z') - \frac{1}{4\ell_B^2}|z|^2, \tag{2.2}$$

with $z = x + \mathrm{i}\,y$, $\ell_B = \sqrt{ch/eB}$ being the magnetic length and K being a $n \times n$ symmetric matrix[32, 33]. Here we assume $\nabla \times \mathbf{A} = -B(B > 0)$ for an external magnetic field.[1] Operators $\phi_\alpha(\mathbf{r})$ and $\pi_\alpha(\mathbf{r})$ satisfy the following relations:

$$\begin{cases} \phi_\alpha(\mathbf{r})\,\pi_\beta(\mathbf{r}') + (-1)^{K_{\alpha\beta}}\pi_\beta(\mathbf{r}')\,\phi_\alpha(\mathbf{r}) = \delta_{\alpha\beta}\delta^{(2)}(\mathbf{r}-\mathbf{r}'), \\ \phi_\alpha(\mathbf{r})\,\phi_\beta(\mathbf{r}') + (-1)^{K_{\alpha\beta}}\phi_\beta(\mathbf{r}')\,\phi_\alpha(\mathbf{r}) = 0, \\ \pi_\alpha(\mathbf{r})\,\pi_\beta(\mathbf{r}') + (-1)^{K_{\alpha\beta}}\pi_\beta(\mathbf{r}')\,\pi_\alpha(\mathbf{r}) = 0. \end{cases} \tag{2.3}$$

From Eqs.(2.3), we see that if we set the components of the matrix K as even integers, we have fermions. On the other hand, if we set them as odd integers, we have bosons.[2] To show these relations, we set

$$F(\lambda) = \mathrm{e}^{-\lambda J_\alpha(\mathbf{r})}\psi_\beta^\dagger(\mathbf{r}')\,\mathrm{e}^{\lambda J_\alpha(\mathbf{r})}. \tag{2.4}$$

Differentiating $F(\lambda)$ with respect to the variable λ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) = -K_{\alpha\beta}\log(z-z')F(\lambda), \tag{2.5}$$

where we have used

$$\left[J_\alpha(\mathbf{r}), \psi_\beta^\dagger(\mathbf{r}')\right] = K_{\alpha\beta}\psi_\beta^\dagger(\mathbf{r}')\log(z-z'). \tag{2.6}$$

[1] If we have $\nabla \times \mathbf{A} = +B$, we must replace z with $z^* = x - \mathrm{i}\,y$.

[2] In bilayer systems, it is possible that particles have fermi statistics in the same layer and boson statistics in the opposite layer, and vice versa. This situation is possible, for instance at $\nu = 2/3$. However, we do not consider such cases.

Taking into account the initial condition $F(0) = \psi_\beta^\dagger(\mathbf{r}')$, we solve the differential equation (2.5). The solution is given by

$$F(\lambda) = \psi_\beta^\dagger(\mathbf{r}')(z - z')^{-\lambda K_{\alpha\beta}}. \tag{2.7}$$

Therefore, we obtain,

$$e^{-\lambda J_\alpha(\mathbf{r})}\psi_\beta^\dagger(\mathbf{r}')e^{\lambda J_\alpha(\mathbf{r})} = \psi_\beta^\dagger(\mathbf{r}')(z - z')^{-\lambda K_{\alpha\beta}}. \tag{2.8}$$

Taking similar steps, we obtain

$$e^{-\lambda J_\alpha(\mathbf{r})}\psi_\beta(\mathbf{r}')e^{\lambda J_\alpha(\mathbf{r})} = \psi_\beta(\mathbf{r}')(z - z')^{\lambda K_{\alpha\beta}}. \tag{2.9}$$

Using Eqs.(2.8), (2.9) and

$$[J_\alpha(\mathbf{r}), J_\beta(\mathbf{r}')] = 0, \tag{2.10}$$

which is shown by using the relation $[\rho_\alpha(\mathbf{r}), \rho_\beta(\mathbf{r}')] = 0$, we obtain

$$\begin{aligned}\phi_\alpha(\mathbf{r})\pi_\beta(\mathbf{r}') &= e^{-J_\alpha(\mathbf{r})}\psi_\alpha(\mathbf{r})\psi_\beta^\dagger(\mathbf{r}')e^{J_\beta(\mathbf{r}')} \\ &= \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}') - e^{-J_\alpha(\mathbf{r})}\psi_\beta^\dagger(\mathbf{r}')\psi_\alpha(\mathbf{r})e^{J_\beta(\mathbf{r}')} \\ &= \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}') - (-1)^{K_{\alpha\beta}}\pi_\beta(\mathbf{r}')\phi_\alpha(\mathbf{r}).\end{aligned} \tag{2.11}$$

This proves the first equation of Eqs.(2.3). Other two equations are proved by taking the similar steps.

The case of $n = 1$ and K_{11} being odd integer is analyzed by Rajaraman and Sondhi [32]. They obtained the Laughlin wave function within a mean field level. When we use the usual Chern-Simons singular gauge transformations, we must take into account a Gaussian fluctuation to obtain the Laughlin wave function[10]. The case of $n = 2$ and all components of $K_{\alpha\beta}$ being even integers was considered by Rajaraman [33]. He analyzed bilayer systems following the theory of Lopez and Fradkin [34], where the QHE is understood as the IQHE of CFs.

In these works, the interaction term between CFs was not discussed. We derive it in the next section.

2.2 Hamiltonian

Now we derive the Hamiltonian for the extended CFs. In this section, we focus on the kinetic energy term and neglect the Coulomb interaction. The second quantized kinetic energy term for n species of electrons is given by

$$H^0 = \sum_{\alpha=1}^{n} \frac{1}{2m_b} \int d^2\mathbf{r}\psi_\alpha^\dagger(\mathbf{r}) \left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right)^2 \psi_\alpha(\mathbf{r}), \tag{2.12}$$

where m_b is a band mass for electrons. Performing the extended CF transformations: Eqs.(2.1), we obtain

$$H^0 = \sum_\alpha \frac{1}{2m_b} \int d^2\mathbf{r}\pi_\alpha(\mathbf{r}) \left(-i\hbar\nabla + \frac{c}{e}\mathbf{A} - i\hbar J_\alpha(\mathbf{r})\right)^2 \phi_\alpha(\mathbf{r}). \tag{2.13}$$

Now we introduce the Chern-Simons gauge field by

$$\mathbf{a}_\alpha(\mathbf{r}) = \frac{c\hbar}{e}\sum_\beta K_{\alpha\beta} \int d^2\mathbf{r}'\rho_\beta(\mathbf{r}')\nabla \mathrm{Im}\log(z - z'). \tag{2.14}$$

Taking the rotation of $\mathbf{a}_\alpha(\mathbf{r})$, we obtain

$$\nabla \times \mathbf{a}_\alpha(\mathbf{r}) = \phi_0 \sum_\beta K_{\alpha\beta}\rho_\beta(\mathbf{r}). \tag{2.15}$$

This is the relation between the fictitious fluxes and the particle density. Using the field $\mathbf{a}_\alpha$, we obtain

$$\frac{e}{c}\mathbf{A} - i\hbar\nabla J_\alpha(\mathbf{r}) = \frac{e}{c}\left\{(\mathbf{A} + \mathbf{a}_\alpha) + i\tilde{e}_z \times (\mathbf{A} + \mathbf{a}_\alpha)\right\}. \tag{2.16}$$

In deriving this equation, we have used Cauchy-Riemann relations. Furthermore, $\mathbf{A}+\mathbf{a}_\alpha$ is transformed into the following form:

$$\begin{aligned}\mathbf{A}+\mathbf{a}_\alpha &= \frac{\phi_0}{2\pi}\sum_\beta K_{\alpha\beta}\int d^2\mathbf{r}'\delta\rho_\beta(\mathbf{r}')\nabla\mathrm{Im}\log(z-z') \\ &\equiv \delta\mathbf{a}_\alpha \end{aligned} \tag{2.17}$$

Using the field $\delta\mathbf{a}_\alpha$, the Hamiltonian is given by

$$H^0 = H^0_{\mathrm{CF}} + V^H + V^{NH}, \tag{2.18}$$

where

$$H^0_{\mathrm{CF}} = \sum_\alpha \int d^2\mathbf{r}\ \pi_\alpha(\mathbf{r})\left(-\frac{\hbar^2}{2m_b}\nabla^2\right)\phi_\alpha(\mathbf{r}), \tag{2.19}$$

$$V^H = \sum_\alpha \frac{e}{2m_bc}\int d^2\mathbf{r}\pi_\alpha(\mathbf{r})\left\{-i\hbar\nabla,\delta\mathbf{a}_\alpha\right\}\phi_\alpha(\mathbf{r}), \tag{2.20}$$

$$V^{NH} = \sum_\alpha \frac{e}{2m_bc}\int d^2\mathbf{r}\pi_\alpha(\mathbf{r})\left\{-i\hbar\nabla,i\hat{e}_z\times\delta\mathbf{a}_\alpha\right\}\phi_\alpha(\mathbf{r}). \tag{2.21}$$

Here, $\{\mathbf{A},\mathbf{B}\}=\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}$. Note that when we introduce the current operator for CFs:

$$\mathbf{j}^{\mathrm{CF}}_\alpha(\mathbf{r}) = \frac{\hbar}{2m_bi}\left[\pi_\alpha(\mathbf{r})\nabla\phi_\alpha(\mathbf{r})-(\nabla\pi_\alpha(\mathbf{r}))\phi_\alpha(\mathbf{r})\right], \tag{2.22}$$

Eqs.(2.20) and (2.21) become

$$V^H = \sum_\alpha \frac{e}{c}\int d^2\mathbf{r}\ \delta\mathbf{a}\cdot\mathbf{j}^{\mathrm{CF}}_\alpha, \tag{2.23}$$

$$V^{NH} = \sum_\alpha \frac{e}{c}\int d^2\mathbf{r}\ i\hat{e}_z\times\delta\mathbf{a}\cdot\mathbf{j}^{\mathrm{CF}}_\alpha. \tag{2.24}$$

Performing Fourier transformations:

$$\phi_\alpha(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}} e^{i\mathbf{k}\,\mathbf{r}}\phi_{\mathbf{k}\alpha}, \tag{2.25}$$

$$\pi_\alpha(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\,\mathbf{r}}\pi_{\mathbf{k}\alpha}, \tag{2.26}$$

$$\delta\mathbf{a}_\alpha(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{q}} e^{i\mathbf{q}\,\mathbf{r}}\delta\mathbf{a}_{\mathbf{q}\alpha}, \tag{2.27}$$

where Ω is the area of the system, Eqs.(2.19), (2.20) and (2.21) become

$$H^0_{\mathrm{CF}} = \sum_{\mathbf{k}\alpha}\frac{\hbar^2k^2}{2m_b}\pi_{\mathbf{k}\alpha}\phi_{\mathbf{k}\alpha}, \tag{2.28}$$

$$V^H = \frac{1}{2\Omega}\sum_{\alpha\beta}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\neq 0} K_{\alpha\beta}V^H_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\pi_{\mathbf{k}_2+\mathbf{q},\alpha}\pi_{\mathbf{k}_1,\beta}\phi_{\mathbf{k}_1+\mathbf{q},\beta}\phi_{\mathbf{k}_2,\alpha}, \tag{2.29}$$

$$V^{NH} = \frac{1}{2\Omega}\sum_{\alpha\beta}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\neq 0} K_{\alpha\beta}V^{NH}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\pi_{\mathbf{k}_2+\mathbf{q},\alpha}\pi_{\mathbf{k}_1,\beta}\phi_{\mathbf{k}_1+\mathbf{q},\beta}\phi_{\mathbf{k}_2,\alpha}, \tag{2.30}$$

where

$$V^H{}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} = -\frac{2\pi\mathrm{i}}{m_b}\cdot\frac{\mathbf{q}\times(\mathbf{k}_1-\mathbf{k}_2)}{q^2}, \tag{2.31}$$

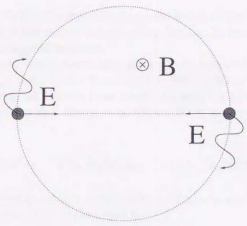

Figure 2.1: The most fundamental motion in quantum Hall systems.

$$V^{NH}{}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} = -\frac{2\pi}{m_b}\cdot\frac{\mathbf{q}\cdot(\mathbf{k}_1-\mathbf{k}_2)}{q^2}. \tag{2.32}$$

Finally, the Hamiltonian for extended CFs is given by

$$\begin{aligned} H_{\rm CF} &= \sum_{\mathbf{k}\alpha}\frac{\hbar^2k^2}{2m_b}\pi_{\mathbf{k}\alpha}\phi_{\mathbf{k}\alpha} \\ &\quad +\frac{1}{2\Omega}\sum_{\alpha\beta}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\neq 0} K_{\alpha\beta}V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\pi^{\dagger}_{\mathbf{k}_2+\mathbf{q},\alpha}\pi^{\dagger}_{\mathbf{k}_1,\beta}\phi_{\mathbf{k}_1+\mathbf{q},\beta}\phi_{\mathbf{k}_2,\alpha}, \end{aligned} \tag{2.33}$$

where

$$V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} = V^{H}{}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} + V^{NH}{}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}. \tag{2.34}$$

Note that we do not apply any approximations to obtain this Hamiltonian. However, there is the non-Hermitian term V^{NH} because we perform the non-unitrary transformation.

In the next section, we explain the reason why we introduce the extended CFs.

2.3 Meaning of Extended Composite Fermions

To discuss the meaning of extended composite fermions, one important question must be considered: What is the most fundamental motion in quantum Hall systems?

It is a well-known fact that if we have a charged particle in the magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ where $\mathbf{B}$ is perpendicular to $\mathbf{E}$, it goes along the direction of $\mathbf{E}\times\mathbf{B}$. Then, how about two charged electrons interacting by the Coulomb interaction in the magnetic field? Let us consider a system of two electrons confined in the xy-plane and subjected to the magnetice field $-B\hat{e}_z$. If we take into account the Coulomb interaction between electrons, each electron feels electric field $\mathbf{E}$ caused by another electron. Therefore, each electron moves around another electron (see Fig.2.1). This motion is the most fundamental one in quantum Hall systems. In fact, the Laughlin wave function (1.2) is constructed of this correlation only. The Laughlin wave function is of Jastrow type wave function and is only constructed of the two-body correlation of relative angular momentum m, where m is related to the Landau level filling fraction by $\nu = 1/m$. Furthermore, in this scheme the importance of the Coulomb interaction in quantum Hall effect is naturally understood.

The difficulty of the quantum Hall effect is caused by this correlation effect. Every pair of electrons has strong correlation. The Chern-Simons gauge theory is a method to take into account such a correlation effect. The phases caused by above correlation effect are taken into account by replacing it with Aharonov-Bohm phase caused by fictitious fluxes. After the Chern-Simons singular gauge transformation, the resulting composite particles do not experience such a two-body correlation effect any more. However, the usual

Chern-Simons gauge theory only takes into account the phase factor of the two-body correlation. Therefore, the Hamiltonian has unexpected terms caused by remaining factors. In fact, to discuss interactions between CFs we are forced to adopt some approximations.

If we perform a transformation which takes into account the two-body correlation effect perfectly, we expect that we will have a simple form of the Hamiltonian. The transformation given by Eq.(2.1) is such a transformation. In the following, we explain that point. To simplify the discussion, we consider spinless CFs. Suppose a state for N particles and denote it as $|\Psi\rangle_N$. If we use the field operators of electrons, the $|\Psi\rangle_N$ is described by

$$|\Psi\rangle_N = \frac{1}{N!}\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \cdots d^2\mathbf{r}_N \Psi_{\rm el}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\cdots\psi^\dagger(\mathbf{r}_N)|0\rangle, \tag{2.35}$$

where $\Psi_{\rm el}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ is the first quantized wave function for electrons. On the other hand, if we use the field operators of extended CFs, the $|\Psi\rangle_N$ is described by

$$|\Psi\rangle_N = \frac{1}{N!}\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \cdots d^2\mathbf{r}_N \Psi_{\rm CF}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)\pi(\mathbf{r}_1)\pi(\mathbf{r}_2)\cdots\pi(\mathbf{r}_N)|0\rangle, \tag{2.36}$$

where $\Psi_{\rm CF}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ is the first quantized wave function for extended CFs. Let us find the relation between $\Psi_{\rm el}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ and $\Psi_{\rm CF}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ [32, 33]. Using Eq.(2.8), we obtain

$$\begin{aligned}
\pi(\mathbf{r}_1)\pi(\mathbf{r}_2)\cdots\pi(\mathbf{r}_N)|0\rangle &= \psi^\dagger(\mathbf{r}_1)e^{J(\mathbf{r}_1)}\psi^\dagger(\mathbf{r}_2)e^{J(\mathbf{r}_2)}\cdots\psi^\dagger(\mathbf{r}_N)e^{J(\mathbf{r}_N)}|0\rangle \\
&= \prod_{i<j}(z_i-z_j)^\phi\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\cdots\psi^\dagger(\mathbf{r}_N)e^{J(\mathbf{r}_1)}e^{J(\mathbf{r}_2)}\cdots e^{J(\mathbf{r}_N)}|0\rangle \\
&= \prod_{i<j}(z_i-z_j)^\phi e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^N|z_j|^2}\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\cdots\psi^\dagger(\mathbf{r}_N)|0\rangle,
\end{aligned} \tag{2.37}$$

where we have set $K_{11}=\phi$ and used $e^{J(\mathbf{r}_1)}e^{J(\mathbf{r}_2)}\cdots e^{J(\mathbf{r}_N)}|0\rangle = e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^N|z_j|^2}|0\rangle$. Substituting Eq.(2.37) into Eq.(2.36), we obtain

$$\begin{aligned}
|\Psi\rangle_N &= \frac{1}{N!}\int d^2\mathbf{r}_1 d^2\mathbf{r}_2\cdots d^2\mathbf{r}_N \prod_{i<j}(z_i-z_j)^\phi e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^N|z_j|^2}\Psi_{\rm CF}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N) \\
&\quad\times\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\cdots\psi^\dagger(\mathbf{r}_N)|0\rangle.
\end{aligned} \tag{2.38}$$

From Eq.(2.35) and Eq.(2.38), we obtain

$$\Psi_{\rm el}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N) = \prod_{i<j}(z_i-z_j)^\phi e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^N|z_j|^2}\Psi_{\rm CF}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N). \tag{2.39}$$

Therefore, the non-unitary transformation (2.1) perfectly takes into account the two-body correlation effect.

We can derive the Eq(2.39) in general cases [33]. In that case, we obtain

$$\Psi_{\rm el}(\{z_1^\alpha,z_2^\alpha,\cdots,z_N^\alpha\}) = \prod_{\alpha\beta}\prod_{i<j}(z_i^\alpha-z_j^\beta)^{K_{\alpha\beta}}\exp\left(-\frac{1}{4\ell_B^2}\sum_{j=1}^N|z_j|^2\right)\Psi_{\rm CF}(\{z_1^\alpha,z_2^\alpha,\cdots,z_N^\alpha\}). \tag{2.40}$$

2.4 Pairing Interaction

After taking into account the most fundamental two-body correlation effect, what is the remaining effect between composite particles? The answer is that the fluctuation causes significant effects. In fact, as we see in the following composite particles feel an attractive interaction between them. Here we present a heuristic explanation of the pairing interaction. To simplify the discussion, we focus on the case of one component CFs. As we have seen in Sec.2.2, Eq. (2.23) has the form of the minimal coupling between charged particles

and the gauge field. Therefore, V^H causes an interaction like the Lorentz force. The equation of motion derived from Eq.(2.23) is given by

$$\begin{aligned} m_b \frac{d}{dt}\mathbf{v} &= -e\mathbf{v} \times (\nabla \times \delta\mathbf{a}) \\ &= -e\phi_0\phi\mathbf{v} \times \hat{e}_z \delta\rho. \end{aligned} \tag{2.41}$$

Equation (2.41) shows that a CF passing by another CF in the counterclockwise direction from the view point at positive z-axis feels the attractive force toward it because $\delta\rho > 0$ around the composite particles. On the other hand, if a CF passes by another CF in the clockwise direction, the repulsive interaction is caused between them. For that reason, a pairing state with positive angular momentum is expected to exist. Note that the angular momentum of this pairing state and that of the cyclotron motion is in the opposite direction. This is naturally understood as in the following. Around a CF, the cancellation between the external magnetic field and the fictitious fluxes is not complete and the latter is larger than the former. Therefore, a CF which comes around another CF feels the magnetic field in the direction of the Chern-Simons gauge field. However, $\nabla \times \mathbf{a}$ and $\nabla \times \mathbf{A}$ are in the opposite direction. Hence the cyclotron motion caused by the applied external magnetic field and the cyclotron motion caused by the Chern-Simons gauge field are in the opposite direction.

Note also that the attractive interaction is relevant only in the case of CFs. Because of the Fermi statistics, we always have particles with nonzero velocity. However, in composite boson cases, this attractive interaction is no more relevant when the bose condensation occurs.

Let us examine the effect of V^{NH} by calculating the force caused by V^{NH}. Of course it is not a real force but an imaginary force. It is obtained by replacing $\delta\mathbf{a}$ with $i\hat{e}_z \times \delta\mathbf{a}$ in Eq.(2.41). The resulting imaginary force is given by

$$\mathbf{F}_{\text{imaginary}} = -ie(\mathbf{v} \times \hat{e}_z)(\nabla \cdot \delta\mathbf{a}). \tag{2.42}$$

We find that $\mathbf{F}_{\text{imaginary}}$ is equal to zero because we obtain $\nabla \cdot \delta\mathbf{a} = 0$ from Eq.(2.17). Within a classical analysis given here, V^{NH} seems to have no effect to the system.

Chapter 3

Pairing States in Single-layer Quantum Hall Systems

In this chapter, we discuss the possibility of the quantum Hall effect at $\nu = 1/m$, where m is an even integer, in single-layer system. Though the quantum Hall effect is not observed in this system[35], the possibility of the quantum Hall effect in it is still a controversial problem[36]. Furthermore, this problem is closely related to the $\nu = 5/2$ state since the difference between them is the existence of filled Landau levels.

We analyze the Hamiltonian obtained in the last chapter by the pairing approximation and derive gap equations for the pairing state. First, we consider the effect of the pairing potential V^H. In that case, we find that the ground state is a p-wave pairing state of CFs. Next we take into account V^{NH} and discuss that it is irrelevant for pairing states. The effect of the Coulomb interaction is considered and we derive the condition of the pairing state. With regard to the real spin degrees of freedom, we show that the spin polarized state has lower energy than the spin unpolarized pairing state.

3.1 Hamiltonian

For spinless fermions, we set $n = 1$ in Eq.(2.33) and choose an even integer for $K_{11} = \phi$. The form of the Coulomb interaction between electrons are not changed by the mapping from an electron system to a CF system. The interaction between CFs derived from the Coulomb interaction is given by

$$V^C = \int d^2\mathbf{r} d^2\mathbf{r}' \, \frac{e^2}{\epsilon|\mathbf{r}-\mathbf{r}'|} \delta\rho(\mathbf{r}) \, \delta\rho(\mathbf{r}') \,, \tag{3.1}$$

where ϵ is a dielectric constant and $\delta\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\,\psi(\mathbf{r}) - \bar{\rho} = \pi^\dagger(\mathbf{r})\,\phi(\mathbf{r}) - \bar{\rho}$.

To discuss the pairing state, we concentrate on the interaction for the pair with zero total momentum. Then we set $\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2 = 0$ in Eq.(2.33). The Hamiltonian is given by

$$H = \sum_{\mathbf{k}} \xi_k \pi^\dagger_{\mathbf{k}} \phi_{\mathbf{k}} + \frac{1}{2\Omega} \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} V_{\mathbf{k}_1\mathbf{k}_2} \pi^\dagger_{\mathbf{k}_1} \pi^\dagger_{-\mathbf{k}_1} \phi_{-\mathbf{k}_2} \phi_{\mathbf{k}_2} \,, \tag{3.2}$$

where $\xi_k = k^2/2m_b$ and

$$V_{\mathbf{k}_1\mathbf{k}_2} = \phi \, V^H_{\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_1+\mathbf{k}_2} + \phi \, V^{NH}_{\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_1+\mathbf{k}_2} + V^C_{\mathbf{k}_1-\mathbf{k}_2} \,, \tag{3.3}$$

is the two-body interaction for zero total momentum pairs. Here,

$$V^C_{\mathbf{q}} = \frac{\pi\alpha}{4m_b} \sqrt{\frac{2}{\phi}} \frac{k_F}{q} \,, \tag{3.4}$$

with $\alpha = \left(e^2/\epsilon\ell_B\right)/\epsilon_F$ and $k_F\ell_B = \sqrt{2/\phi}$.

3.2 Pairing Approximation

To discuss the pairing states, we apply the pairing approximation to the Hamiltonian (3.2):

$$H \simeq \sum_{\mathbf{k}} \xi_k \pi_{\mathbf{k}} \phi_{\mathbf{k}} + \frac{1}{2\Omega} \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} V_{\mathbf{k}_1 \mathbf{k}_2} \Big[\langle \pi_{\mathbf{k}_1} \pi_{-\mathbf{k}_1} \rangle \phi_{-\mathbf{k}_2} \phi_{\mathbf{k}_2} + \pi_{\mathbf{k}_1} \pi_{-\mathbf{k}_1} \langle \phi_{-\mathbf{k}_2} \phi_{\mathbf{k}_2} \rangle - \langle \pi_{\mathbf{k}_1} \pi_{-\mathbf{k}_1} \rangle \langle \phi_{-\mathbf{k}_2} \phi_{\mathbf{k}_2} \rangle \Big]. \tag{3.5}$$

Now we introduce the gap functions:

$$\Delta_{\mathbf{k}} \equiv -\frac{1}{\Omega} \sum_{\mathbf{k}'(\neq \mathbf{k})} V_{\mathbf{k}\mathbf{k}'} \langle \phi_{-\mathbf{k}'} \phi_{\mathbf{k}'} \rangle, \tag{3.6}$$

$$\overline{\Delta}_{\mathbf{k}} \equiv -\frac{1}{\Omega} \sum_{\mathbf{k}'(\neq \mathbf{k})} V_{\mathbf{k}'\mathbf{k}} \langle \pi_{-\mathbf{k}'} \pi_{\mathbf{k}'} \rangle. \tag{3.7}$$

Using these fields, Eq.(3.5) becomes

$$\begin{aligned} H &\simeq \sum_{\mathbf{k}} \xi_k \pi_{\mathbf{k}} \phi_{\mathbf{k}} - \frac{1}{2} \sum_{\mathbf{k}} \left(\pi_{\mathbf{k}} \pi_{-\mathbf{k}} \Delta_{\mathbf{k}} + \overline{\Delta}_{\mathbf{k}} \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \right) + \text{const.} \\ &= \sum_{\mathbf{k}}{}' \pi^{\mathbf{k}} \mathcal{E}^{\mathbf{k}} \phi^{\mathbf{k}} + \text{const.}, \end{aligned} \tag{3.8}$$

where $\sum_{\mathbf{k}}{}'$ denotes $\sum_{k_x>0, k_y}$ and

$$\pi^{\mathbf{k}} = \begin{bmatrix} \pi_{\mathbf{k}} & \phi_{-\mathbf{k}} \end{bmatrix}, \tag{3.9}$$

$$\phi^{\mathbf{k}} = \begin{bmatrix} \phi_{\mathbf{k}} \\ \pi_{-\mathbf{k}} \end{bmatrix}, \tag{3.10}$$

$$\mathcal{E}^{\mathbf{k}} = \begin{bmatrix} \xi_k & -\Delta_{\mathbf{k}} \\ -\overline{\Delta}_{\mathbf{k}} & -\xi_k \end{bmatrix}. \tag{3.11}$$

The next step is the diagonalization of the matrix $\mathcal{E}^{\mathbf{k}}$. The eigenvalues of the matrix $\mathcal{E}^{\mathbf{k}}$ are given by $\pm E_{\mathbf{k}}$, where

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + \overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}}}. \tag{3.12}$$

This is the excitation energy for quasiparticles. The eigenvector for the eigenvalue $E_{\mathbf{k}}$ is given by

$$\mathbf{u}_1^{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}(E_{\mathbf{k}} + \xi_k)}} \begin{bmatrix} -\xi_k - E_{\mathbf{k}} \\ \overline{\Delta}_{\mathbf{k}} \end{bmatrix}, \tag{3.13}$$

and the eigenvector for the eigenvalue $-E_{\mathbf{k}}$ is given by

$$\mathbf{u}_2^{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}(E_{\mathbf{k}} + \xi_k)}} \begin{bmatrix} \Delta_{\mathbf{k}} \\ \xi_k + E_{\mathbf{k}} \end{bmatrix}. \tag{3.14}$$

The matrix $\mathcal{E}^{\mathbf{k}}$ is diagonalized by matrices $U^{\mathbf{k}}$ and $\overline{U}^{\mathbf{k}}$:

$$U^{\mathbf{k}} = \begin{bmatrix} \mathbf{u}_1^{\mathbf{k}} & \mathbf{u}_2^{\mathbf{k}} \end{bmatrix}, \tag{3.15}$$

$$\overline{U}^{\mathbf{k}} = {}^T \begin{bmatrix} \overline{\mathbf{u}}_1^{\mathbf{k}} & \overline{\mathbf{u}}_2^{\mathbf{k}} \end{bmatrix}, \tag{3.16}$$

where,

$$\overline{\mathbf{u}}_1^{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}(E_{\mathbf{k}}+\xi_k)}} \begin{bmatrix} -\xi_k - E_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \end{bmatrix}, \tag{3.17}$$

$$\overline{\mathbf{u}}_2^{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}(E_{\mathbf{k}}+\xi_k)}} \begin{bmatrix} \overline{\Delta}_{\mathbf{k}} \\ \xi_k + E_{\mathbf{k}} \end{bmatrix}, \tag{3.18}$$

or

$$U^{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}(E_{\mathbf{k}}+\xi_k)}} \begin{bmatrix} -\xi_k - E_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \overline{\Delta}_{\mathbf{k}} & \xi_k + E_{\mathbf{k}} \end{bmatrix} = \overline{U}^{\mathbf{k}}. \tag{3.19}$$

Introducing the quasiparticle field operators:

$$q^{\mathbf{k}} = \begin{bmatrix} q_{\mathbf{k}} \\ p_{-\mathbf{k}} \end{bmatrix} = \overline{U}^{\mathbf{k}} \phi^{\mathbf{k}}, \tag{3.20}$$

$$p^{\mathbf{k}} = \begin{bmatrix} p_{\mathbf{k}} & q_{-\mathbf{k}} \end{bmatrix} = \pi^{\mathbf{k}} U^{\mathbf{k}}, \tag{3.21}$$

we obtain

$$H \simeq \sum_{\mathbf{k}}{}' p^{\mathbf{k}} \begin{bmatrix} E_{\mathbf{k}} & 0 \\ 0 & -E_{\mathbf{k}} \end{bmatrix} q^{\mathbf{k}} + \text{const.}. \tag{3.22}$$

The fields $p_{\mathbf{k}}$ and $q_{\mathbf{k}}$ satisfy the following anti-commutation relations:

$$\begin{cases} \{q_{\mathbf{k}}, p_{\mathbf{k}'}\} = \delta_{\mathbf{k}\mathbf{k}'}, \\ \{q_{\mathbf{k}}, q_{\mathbf{k}'}\} = 0, \\ \{p_{\mathbf{k}}, p_{\mathbf{k}'}\} = 0. \end{cases} \tag{3.23}$$

As an example we show the first equation of Eqs.(3.23):

$$\begin{aligned} \{q_{\mathbf{k}}, p_{\mathbf{k}'}\} &= \left\{ \left(\overline{U}^{\mathbf{k}} \phi^{\mathbf{k}} \right)_1, \left(\pi^{\mathbf{k}'} U^{\mathbf{k}'} \right)_1 \right\} \\ &= \sum_{i,j} \overline{U}^{\mathbf{k}}_{1i} U^{\mathbf{k}'}_{j1} \left\{ \phi^{\mathbf{k}}_i, \pi^{\mathbf{k}'}_j \right\} \\ &= \delta_{\mathbf{k},\mathbf{k}'} \sum_j \overline{U}^{\mathbf{k}}_{1j} U^{\mathbf{k}}_{j1} \\ &= \delta_{\mathbf{k},\mathbf{k}'}. \end{aligned} \tag{3.24}$$

To find the gap equation, we calculate the value $\langle \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \rangle$:

$$\begin{aligned} \langle \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \rangle &= \langle \left(p^{\mathbf{k}} \overline{U}^{\mathbf{k}} \right)_2 \left(U^{\mathbf{k}} q^{\mathbf{k}} \right)_1 \rangle \\ &= \sum_j \overline{U}^{\mathbf{k}}_{i2} U^{\mathbf{k}}_{ij} \langle p^{\mathbf{k}}_i q^{\mathbf{k}}_j \rangle \\ &= \sum_j \left(\overline{\mathbf{u}}^{\mathbf{k}}_j \right)_2 \left(\mathbf{u}^{\mathbf{k}}_j \right)_1 \langle p^{\mathbf{k}}_j q^{\mathbf{k}}_j \rangle \\ &= \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_B T} \right). \end{aligned} \tag{3.25}$$

In the same way, we obtain

$$\langle \pi_{\mathbf{k}} \pi_{-\mathbf{k}} \rangle = \frac{\overline{\Delta}_{\mathbf{k}}}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_B T} \right). \tag{3.26}$$

Substituting these equations into Eqs.(3.6) and (3.7), the gap equations are given by

$$\Delta_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}\tanh\left(\frac{E_{\mathbf{k}'}}{2k_BT}\right), \tag{3.27}$$

$$\overline{\Delta}_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V_{\mathbf{k}'\mathbf{k}}\frac{\overline{\Delta}_{\mathbf{k}'}}{E_{\mathbf{k}'}}\tanh\left(\frac{E_{\mathbf{k}'}}{2k_BT}\right). \tag{3.28}$$

At zero temperature, these equations become

$$\Delta_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}, \tag{3.29}$$

$$\overline{\Delta}_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V_{\mathbf{k}'\mathbf{k}}\frac{\overline{\Delta}_{\mathbf{k}'}}{E_{\mathbf{k}'}}. \tag{3.30}$$

Though we have derived the gap equations neglecting the strong coupling effect, we take into account it by replacing the electron band mass m_b with the CF's effective mass M. The determination of it is very tough problem. Apparently we cannot apply the strong coupling theory of BCS[37] superconductivity because we do not have any cutoff energy such as the Debye energy. We have no guarantee of applying the Migdal's theorem[38] and must take into account all order of diagrams to calculate M. Therefore, we take it as the parameter of our theory. However, it is easy to estimate the effective mass M in an extremal situation, such as the strong magnetic field limit or the weak magnetic field limit. As we will show later, we estimate it in such cases.

Now we discuss the pairing state of the ground state. To solve the gap equations (3.29) and (3.30), we set

$$\Delta_{\mathbf{k}} = \Delta_k e^{-i\ell\theta_{\mathbf{k}}}, \tag{3.31}$$

$$\overline{\Delta}_{\mathbf{k}} = \overline{\Delta}_k e^{i\ell\theta_{\mathbf{k}}}, \tag{3.32}$$

for the ℓ-wave pairing state. Obviously this choice is not general one. However, as shown in Sec.2.4, the chirality of the pairing state is fixed by the direction of the Chern-Simons gauge flux. In fact, we show in the following, the attractive interaction is caused only in the case of $\ell > 0$. Therefore, it is enough to consider the pairing state by the form of Eqs.(3.31) and (3.32). Substituting them into Eq.(3.29) and (3.30) respectively, we obtain

$$\Delta_k = \frac{\phi}{2M}\int_0^\infty dk'\frac{k'\Delta_{k'}}{E_{k'}}\left[I_\ell^H(\lambda) + I_\ell^{NH}(\lambda) - I_\ell^C(k,k')\right], \tag{3.33}$$

$$\overline{\Delta}_k = \frac{\phi}{2M}\int_0^\infty dk'\frac{k'\overline{\Delta}_{k'}}{E_{k'}}\left[I_{-\ell}^H(\lambda) + I_{-\ell}^{NH}(\lambda) - I_{-\ell}^C(k,k')\right], \tag{3.34}$$

where $\lambda = (k^2+k'^2)/2kk'$ and

$$I_\ell^H(\lambda) = -\int_0^{2\pi}\frac{d\theta}{2\pi i}\frac{\sin\theta}{\lambda-\cos\theta}e^{-i\ell\theta}, \tag{3.35}$$

$$I_\ell^{NH}(\lambda) = \frac{k^2-k'^2}{2kk'}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{\lambda-\cos\theta}e^{-i\ell\theta}, \tag{3.36}$$

$$I_\ell^C(k,k') = \frac{k_F}{8}\sqrt{\frac{2}{\phi}}\alpha\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{-i\ell\theta}}{k^2+k'^2-2kk'\cos\theta}. \tag{3.37}$$

In the following, we estimate Eqs.(3.35) and (3.36) in the cases of (i)$\ell > 0$, (ii)$\ell = 0$ and (iii)$\ell < 0$.

(i)For $\ell > 0$, we set $e^{-i\ell\theta} = z$ and we obtain

$$I_\ell^H(\lambda) = \oint_C\frac{dz}{2\pi i}\frac{z^2-1}{z^2-2\lambda z+1}z^{\ell-1}, \tag{3.38}$$

where C denotes the path along the unit circle in the counterclockwise direction. The pole of the integrand in Eq.(3.38) is $z_0 = \lambda - \sqrt{\lambda^2 - 1} = k_</k_>$ where $k_<$ ($k_>$) is larger (smaller) value of k and k'. Therefore, we obtain,

$$I_\ell^H(\lambda) = \left(\frac{k_<}{k_>}\right)^\ell. \tag{3.39}$$

On the other hand, we obtain

$$\begin{aligned} I_\ell^{NH}(\lambda) &= -\frac{k^2 - k'^2}{2kk'} \oint_C \frac{dz}{2\pi i} \frac{2}{z^2 - 2\lambda z + 1} z^\ell \\ &= -\frac{k^2 - k'^2}{k_<^2 - k_>^2} \left(\frac{k_<}{k_>}\right)^\ell \\ &= \begin{cases} (k'/k)^\ell & \text{for} \quad k > k', \\ -(k/k')^\ell & \text{for} \quad k < k'. \end{cases} \end{aligned} \tag{3.40}$$

(ii)For $\ell = 0$, Eq.(3.40) is also applicable in this case. With regard to $I_0^H(\lambda)$, we obtain

$$I_0^H(\lambda) = \oint_C \frac{dz}{2\pi i} \frac{z^2 - 1}{z^2 - 2\lambda z + 1} \frac{1}{z} = 0. \tag{3.41}$$

(iii)For $\ell < 0$, we obtain

$$\begin{aligned} I_\ell^H(\lambda) &= -\left(\int_0^{2\pi} \frac{d\theta}{2\pi i} \frac{\sin\theta}{\lambda - \cos\theta} e^{-i|\ell|\theta}\right)^* \\ &= -\left(\frac{k_<}{k_>}\right)^\ell, \end{aligned} \tag{3.42}$$

$$I_\ell^{NH}(\lambda) = \left(I_{|\ell|}^{NH}(\lambda)\right)^* = I_{|\ell|}^{NH}(\lambda). \tag{3.43}$$

Table 3.1 summarizes above results. The attractive interaction occurs for the case of $\ell > 0$. This is consistent with the discussion in Sec.2.4. Before analyzing the gap equations and the pairing states, we calculate the ground state energy for pairing states.

3.3 Ground State Energy

Within the pairing approximation, the ground state energy of the system is given by

$$\langle H \rangle \simeq \sum_{\mathbf{k}} \xi_k \langle \pi_{\mathbf{k}} \phi_{\mathbf{k}} \rangle + \frac{1}{2\Omega} \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} V_{\mathbf{k}_1 \mathbf{k}_2} \langle \pi_{\mathbf{k}_1} \pi_{-\mathbf{k}_1} \rangle \langle \phi_{-\mathbf{k}_2} \phi_{\mathbf{k}_2} \rangle. \tag{3.44}$$

Table 3.1: The summary of function $I_\ell^H(\lambda)$ and $I_\ell^{NH}(\lambda)$.

	$\ell > 0$	$\ell = 0$	$\ell < 0$
$I_\ell^H(\lambda)$	$(k_</k_>)^\ell$	0	$-(k_</k_>)^\ell$
$I_\ell^{NH}(\lambda)$	$\begin{cases} (k'/k)^\ell & \text{for} \quad k > k' \\ -(k/k')^\ell & \text{for} \quad k < k' \end{cases}$	$\begin{cases} 1 & \text{for} \quad k > k' \\ -1 & \text{for} \quad k < k' \end{cases}$	$\begin{cases} (k'/k)^{\lvert\ell\rvert} & \text{for} \quad k > k' \\ -(k/k')^{\lvert\ell\rvert} & \text{for} \quad k < k' \end{cases}$
$I_\ell^H(\lambda) + I_\ell^{NH}(\lambda)$	$\begin{cases} 2(k'/k)^\ell & \text{for} \quad k > k' \\ 0 & \text{for} \quad k < k' \end{cases}$	$\begin{cases} 1 & \text{for} \quad k > k' \\ -1 & \text{for} \quad k < k' \end{cases}$	$\begin{cases} 0 & \text{for} \quad k > k' \\ -2(k/k')^\ell & \text{for} \quad k < k' \end{cases}$

For the ground state, we obtain

$$\langle \pi_{\mathbf{k}} \phi_{\mathbf{k}} \rangle = \frac{1}{2}\left(1 - \frac{\xi_k}{E_{\mathbf{k}}}\right), \quad \langle \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \rangle = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}, \quad \langle \pi_{\mathbf{k}} \pi_{-\mathbf{k}} \rangle = \frac{\overline{\Delta}_{\mathbf{k}}}{2E_{\mathbf{k}}}. \tag{3.45}$$

Substituting these equations and Eq.(3.29) into Eq.(3.44), we obtain

$$\begin{aligned} \langle H \rangle &\simeq \frac{1}{2}\sum_{\mathbf{k}} \xi_k \left(1 - \frac{\xi_k}{E_{\mathbf{k}}}\right) - \frac{1}{4}\sum_{\mathbf{k}} \frac{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}}{E_{\mathbf{k}}} \\ &= -\frac{1}{4}\sum_{\mathbf{k}} \frac{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}} - \xi_k}{E_{\mathbf{k}} + \xi_k}. \end{aligned} \tag{3.46}$$

For no-pairing states, Eq.(3.46) is calculated exactly and we obtain $\langle H \rangle_{\overline{\Delta}\Delta=0} = -M\Omega\epsilon_F^2/4\pi$.

3.4 Pure Pairing State

To clarify the nature of the pairing state, we neglect the interaction V^{NH} and the Coulomb interaction for a while. As we will see later, V^{NH} term and the Coulomb interaction term causes the pair-breaking effect. In next sections we take into account them and discuss the effect of the pair-breaking in detail.

3.4.1 Gap Equation

In the absence of V^{NH} term and the Coulomb interaction term, the gap equation is given by

$$\Delta_k = \frac{\phi}{2M}\int_0^k dk' \frac{k'\Delta_{k'}}{E_{k'}}\left(\frac{k'}{k}\right)^\ell + \frac{\phi}{2M}\int_k^\infty dk' \frac{k'\Delta_{k'}}{E_{k'}}\left(\frac{k}{k'}\right)^\ell. \tag{3.47}$$

In the absence of V^{NH} term, $\overline{\Delta}_{\mathbf{k}}$ is the complex conjugate of $\Delta_{\mathbf{k}}$. Therefore, it is enough to concentrate on the analysis of Eq.(3.47). The gap equation (3.47) was discussed by Greiter *et al.*[29] in detail. The asymptotic form of the function Δ_k is found from Eq.(3.47):

$$\Delta_k \to k^\ell \quad \text{for} \quad k \to +0, \tag{3.48}$$

$$\Delta_k \to k^{-\ell} \quad \text{for} \quad k \to +\infty. \tag{3.49}$$

Taking into account these asymptotic form, we approximate the gap function by

$$\Delta_k = \begin{cases} \epsilon_F \Delta \, (k/k_F)^\ell & \text{for} \quad k < k_F, \\ \epsilon_F \Delta \, (k_F/k)^\ell & \text{for} \quad k > k_F. \end{cases} \tag{3.50}$$

To determine the remaining parameter Δ, we consider the gap equation at $k = k_F$. It is given by

$$\frac{1}{\phi} = \int_0^1 dx \frac{x^{2\ell+1}}{\sqrt{(x^2-1)^2 + \Delta^2 x^{2\ell}}} + \int_1^\infty dx \frac{x^{1-2\ell}}{\sqrt{(x^2-1)^2 + \Delta^2 x^{-2\ell}}} \equiv F_\ell(\Delta). \tag{3.51}$$

In Fig.3.1, we show the gap Δ dependence of $F_\ell(\Delta)$ in Eq.(3.51) for various ℓ. The largest value of Δ is obtained in the case of $\ell = 1$. Although it is not shown in the figure, the gap for $\ell \geq 7$ is lower than that of $\ell = 5$. Table 3.2 summarizes the values of Δ for various ϕ and ℓ. With any ϕ, the largest gap Δ is obtained when $\ell = 1$. Furthermore, the gap Δ grows up with increasing ϕ.

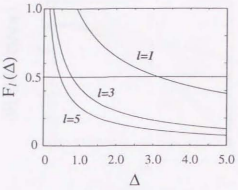

Figure 3.1: The gap Δ dependence of $F_\ell(\Delta)$ in Eq.(3.51) for various ℓ. The gap Δ is determined by the point where curves cross the line given by $1/\phi$. The figure shows the case of $\phi = 2$. The largest gap is given when $\ell = 1$, that is, the p-wave pairing case.

Let us consider the ground state energy (3.46). Substituting Eq.(3.50) into Eq.(3.46), we obtain

$$\begin{aligned}
\langle H \rangle &= -\frac{M\Omega}{4\pi}\epsilon_F^2 \overline{\Delta}\Delta \left[\int_0^1 dx \, \frac{x^{2\ell+1}}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell}}} \cdot \frac{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell}} - x^2 + 1}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell}} + x^2 - 1} \right. \\
&\quad \left. + \int_1^\infty dx \, \frac{x^{-2\ell+1}}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{-2\ell}}} \cdot \frac{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{-2\ell}} - x^2 + 1}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{-2\ell}} + x^2 - 1} \right] \\
&= -\frac{M\Omega}{4\pi}\epsilon_F^2 \int_0^1 dx \left[\frac{x}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell}}} \left(\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell}} - x^2 + 1 \right)^2 \right. \\
&\quad \left. + \frac{x^{4\ell+1}}{\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell+4}}} \frac{(\overline{\Delta}\Delta)^2}{\left(\sqrt{(x^2-1)^2 + \overline{\Delta}\Delta x^{2\ell+4}} + 1 - x^2 \right)^2} \right].
\end{aligned} \tag{3.52}$$

To discuss that whether the ground state is a pairing state or not, we consider the energy difference between pairing state and the no-pairing state, that is, $\delta E \equiv \langle H \rangle_{\overline{\Delta}\Delta} - \langle H \rangle_{\overline{\Delta}\Delta=0}$. For the no-pairing state, the ground

Table 3.2: The gap Δ for various ϕ and ℓ.

ϕ	ℓ	Δ
2	1	3.128
2	3	.791
2	5	.465
2	7	.330
4	1	9.903
4	3	2.029
4	5	1.171
4	7	.828
6	1	19.147
6	3	3.384
6	5	1.928
6	7	1.358

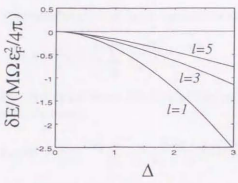

Figure 3.2: The energy difference δE between a pairing state and the no-pairing state. The value of δE always has negative values. Therefore the ground state is the pairing state as long as $\overline{\Delta}\Delta \neq 0$.

state energy is equal to $-M\Omega\epsilon_F^2/4\pi$ as mentioned above. In Fig.3.2, we show the gap Δ dependence of δE for various ℓ. It is seen that δE always has negative values and is the monotonically decreasing function with respect to Δ for any ℓ. Therefore, any pairing state always has lower energy than the no-pairing state and the larger value of Δ gives the lower value of δE. Furthermore, the value of ℓ which gives the lowest δE is $\ell = 1$. From the discussion of gap equations, we have found that the pairing state with the largest value of gap Δ is $\ell = 1$. Putting it all together, the ground state is the p-wave pairing state of CFs.

3.4.2 Wave Function

Next we discuss the ground state wave function of the pairing state. Setting the ground state as the ket $|GS\rangle$, it satisfies the following equations:

$$q_{\mathbf{k}}|GS\rangle = 0, \qquad q_{-\mathbf{k}}|GS\rangle = 0. \tag{3.53}$$

Substituting the expression of $q_{\mathbf{k}}$ and $q_{-\mathbf{k}}$ by fields $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$, which are given by Eqs.(3.20) and (3.21), we obtain

$$\left(\xi_k + E_{\mathbf{k}}\right)\phi_{\mathbf{k}}|GS\rangle = \Delta_{\mathbf{k}}\pi_{-\mathbf{k}}|GS\rangle, \tag{3.54}$$

$$\Delta_{\mathbf{k}}\pi_{\mathbf{k}}|GS\rangle = -\left(\xi_k + E_{\mathbf{k}}\right)\phi_{-\mathbf{k}}|GS\rangle. \tag{3.55}$$

We can replace $\phi_{\mathbf{k}}$ with the Grassmannian differentiation $\partial/\partial\pi_{\mathbf{k}}$ because $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ satisfy anti-commutation relations [39]. Therefore, Eq.(3.54) is transformed into

$$\frac{\partial}{\partial\pi_{\mathbf{k}}}|GS\rangle = \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}} + \xi_k}\pi_{-\mathbf{k}}|GS\rangle. \tag{3.56}$$

The solution of Eq.(3.56) is given by

$$|GS\rangle = \text{const.} \times \exp\left(\frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}} + \xi_k}\pi_{\mathbf{k}}\pi_{-\mathbf{k}}\right)|0\rangle, \tag{3.57}$$

where $|0\rangle$ is the vacuum state. Equation (3.57) also satisfies Eq.(3.55). Equation (3.54) and (3.55) hold for any $\mathbf{k}$. Therefore, the ground state of the system is given by

$$|GS\rangle = \exp\left[\sum_{\mathbf{k}}{}' \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}} + \xi_k}\pi_{\mathbf{k}}\pi_{-\mathbf{k}}\right]|0\rangle. \tag{3.58}$$

The first quantized wave function of $|GS\rangle$ for $2N$ CFs is given by

$$\begin{aligned}\Psi_{\text{CF}}(z_1, z_2, \cdots, z_{2N}) &= \langle 0|\phi_{orb}(z_{2N})\phi_{orb}(z_{2N-1})\cdots\phi_{orb}(z_1)|GS\rangle \\ &= \text{Pf}\ \phi_{orb}(z_{2j-1} - z_{2j}) \\ &= \mathcal{A}\ \phi_{orb}(z_1 - z_2)\phi_{orb}(z_3 - z_4)\cdots\phi_{orb}(z_{2N-1} - z_{2N}),\end{aligned} \tag{3.59}$$

where $\mathcal{A}$ is the operator for the antisymmetrization of functions and the orbital function $\phi_{orb}(\mathbf{r})$ is given by

$$\phi_{orb}(\mathbf{r}) = \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}+\xi_k} e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.60}$$

Equation (3.59) is nothing but the real space wave function of the pairing state for $2N$ particles [40]. Substituting Eq.(3.31) into Eq.(3.60), we obtain

$$\phi_{orb}(\mathbf{r}) = \frac{i}{2\pi} e^{-i\theta_{\mathbf{r}}} \int_0^\infty dk \frac{k\Delta_k}{E_k+\xi_k} J_1(kr), \tag{3.61}$$

where we have used the formula: $\int_0^\pi d\theta e^{ix\cos\theta} = i\pi J_1(x)$ with $J_1(x)$ being the Bessel function of the first order. For the pure pairing state, we obtain

$$\begin{aligned}\phi_{orb}(\mathbf{r}) &= \frac{i}{2\pi} e^{i\theta_{\mathbf{r}}} \left[\int_0^{k_F} dk \frac{k\Delta\epsilon_F(k/k_F)}{\sqrt{\xi_k^2+\Delta^2\epsilon_F^2(k/k_F)^2}+\xi_k} J_1(kr) \right. \\ &\left. + \int_{k_F}^\infty dk \frac{k\Delta\epsilon_F(k_F/k)}{\sqrt{\xi_k^2+\Delta^2\epsilon_F^2(k_F/k)^2}+\xi_k} J_1(kr) \right].\end{aligned} \tag{3.62}$$

Equation (3.62) has a complicated form but when the condition $\Delta = 2$ is satisfied and we take the limit $rk_F \propto r/\ell_B \to \infty$, we obtain the simple form of it:

$$\phi_{orb}(\mathbf{r}) \propto \frac{e^{-i\theta_{\mathbf{r}}}}{r} = \frac{1}{z}. \tag{3.63}$$

In that case, the ground state wave function for CFs is given by

$$\Psi_{\rm CF}(z_1, z_2, \cdots, z_{2N}) = {\rm Pf}\left(\frac{1}{z_{2j-1}-z_{2j}}\right). \tag{3.64}$$

From Eq.(2.39), the ground state wave function for electrons is given by

$$\Psi_{el}(z_1, z_2, \cdots, z_{2N}) = {\rm Pf}\left(\frac{1}{z_{2j-1}-z_{2j}}\right) \prod_{i<j}(z_i-z_j)^2 e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^{2N}|z_j|^2}. \tag{3.65}$$

3.5 Effect of Imaginary Vector Potential

Let us take into account the anti-Hermitian term V^{NH} in the absence of the Coulomb interaction. Before discussing the effect of it, we remark on the relevance between V^{NH} and the three-body interaction term in the usual Chern-Simons gauge theory. When we perform the usual Chern-Simons gauge transformation, the Hamiltonian is given by

$$H = \int d^2\mathbf{r}\, \phi^\dagger(\mathbf{r}) \left(-\frac{1}{2m_b}\nabla^2\right) \phi(\mathbf{r}) + V^H + V^3, \tag{3.66}$$

where V^H is given by Eqs.(2.30) and (2.31) and V^3 is the three-body potential term, which is given by

$$V^3 = \frac{m^2}{2M} \cdot \left(\frac{\phi_0}{2\pi}\right)^2 \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 d^2\mathbf{r}_3 \nabla_1 {\rm Im}\log(z_1-z_2)\nabla_1 {\rm Im}\log(z_1-z_3)\ \psi^\dagger(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\delta\rho(\mathbf{r}_3)\psi(\mathbf{r}_1). \tag{3.67}$$

On the other hand, if we perform the Rajaraman-Sondhi's non-unitary transformation discussed in chap.2, the Hamiltonian is given by

$$H = \int d^2\mathbf{r}\, \pi(\mathbf{r}) \left(-\frac{1}{2m_b}\nabla^2\right) \phi(\mathbf{r}) + V^H + V^{NH}, \tag{3.68}$$

where V^{NH} is given in Eqs.(2.30) and (2.32). Comparing Eq.(3.66) with Eq.(3.68), we see that the anti-Hermitian term V^{NH} corresponds to the three-body interaction term V^3 in the Chern-Simons gauge theory.

The three-body interaction term V^3 was neglected in the analysis of Greiter *et al.* [29] because it is proportional to the square of the number of the Chern-Simons gauge fluxes, which was assumed to be small in Ref.[29]. Of course the number of fluxes is larger than one. Therefore, we do not have any guarantee to neglect it. For that reason, we must take into account the three-body interaction term V^3, which is poorly understood in condensed matter physics.

However, as we have shown, the three-body interaction term V^3 corresponds to the anti-Hermitian term V^{NH}. We can discuss the effect of the former through the latter. The advantages of using V^{NH} instead of V^3 is the following. First, the anti-Hermitian term V^{NH} corresponds to an imaginary vector potential

$$-i\hat{e}_z \times \delta\mathbf{a}, \tag{3.69}$$

as we have seen in Sec.2.2. The effect of the imaginary vector potential was discussed in the depinning phenomena of the vortex lines in the superconductivity with columnar defects[41]. Therefore, we can discuss the effect of it qualitatively by comparing with the depinning phenomena of the vortex lines. Second, the anti-Hermitian term V^{NH} is of the two-body potential form. We can take account of it into the gap equations without any approximations. On the other hand, in the Chern-Simons theory we do some approximations to include the effect of V^3 [42].

Applying the discussion about the nature of the imaginary vector potential in the localization-delocalization phenomena [41] to the CF pairing theory, we understand that it causes a pair-breaking effect. Hatano and Nelson discussed the Hamiltonian:

$$H = \frac{1}{2m}\left(\mathbf{p} + i\mathbf{h}\right)^2 + V\left(\mathbf{r}\right), \tag{3.70}$$

where $i\mathbf{h}$ is an imaginary vector potential and $V\left(\mathbf{r}\right)$ is a random potential. When we discuss the depinning of the vortex lines in the superconductivity with columnar defects, the imaginary vector potential $i\mathbf{h}$ is proportional to the transverse magnetic field $\mathbf{H}_\perp$. In the absence of the imaginary vector potential, it is believed that all eigenstates are localized in two-dimensional noninteracting systems[43]. On the other hand, in a large $\mathbf{H}_\perp$ region we expect that flux lines are depinned from defects. They concluded that particles are localized in the region $|\mathbf{h}| < h\kappa$, where κ is the inverse localization length. On the other hand, particles are delocalized in the region $|\mathbf{h}| > h\kappa$.

Now we apply these results to the problem of CF pairings. In the absence of V^{NH}, or the imaginary vector potential, we have the pairing state as we have seen in Sec.3.4. This pairing state corresponds to a localized state if we take a pairing state as a bound state of particles. When we take into account the effect of V^{NH}, we expect that it causes some pair-breaking effect because the no-pairing state, or no bound state, corresponds to a delocalized state. If we adopt the usual Chern-Simons singular gauge transformation, it is difficult to capture this effect from V^3. To deal with the effect of V^3, we need the propagator of the Chern-Simons gauge field, which is derived from the diamagnetic part. However, to get it we approximate the action by expanding it with respect to the Chern-Simons gauge field to the second order after the integration of the CF fields and we use the approximated form of fermion propagators which is valid in the long-wave length and the low-energy limit. After that, we understand that it causes the pair-breaking effect [42]. Bonesteel discussed the effect of V^3 taking above approximations and concluded that V^3 has the pair-breaking effect[42]. His conclusion is similar to our results. However, the ℓ dependence was lost in his analysis for the ℓ-wave pairing state. Therefore, the effect of the short-range interaction was overestimated. Apparently, the effect of the δ-function like repulsive interaction is weak for the p-wave pairing state. For that reason, using V^{NH} of the Rajaraman-Sondhi's non-unitary transformation is more straightforward than using V^3 in the usual Chern-Simons theory.

Now we take account of V^{NH} in the gap equations. Contrary to the usual Chern-Simons gauge theory, we need not to adopt any approximations [30] because V^{NH} is of the two-body interaction form. We can take it into account in the gap equations directly. Furthermore, as we will show below, we can solve the gap equations exactly. This is the most remarkable point of using the Rajaraman-Sondhi's non-unitary transformation. When we include V^{NH} in the gap equations, they are given by

$$\Delta_k = \frac{\phi}{M}\int_0^k dk' \frac{k'\Delta_{k'}}{E_{k'}}\left(\frac{k'}{k}\right)^\ell, \tag{3.71}$$

$$\overline{\Delta}_k = \frac{\phi}{M}\int_k^\infty dk' \frac{k'\Delta_{k'}}{E_{k'}}\left(\frac{k}{k'}\right)^\ell . \tag{3.72}$$

These gap equations are solved exactly as following. To simplify the form of those equations, we introduce dimensionless fields $\Delta_k = \epsilon_F f(k/k_F)$ and $\Delta_k = \epsilon_F g(k/k_F)$, and we obtain

$$f(x) = 2\phi\int_0^x dy \frac{yf(y)}{\sqrt{(y^2-1)^2 + f(y)g(y)}}\left(\frac{y}{x}\right)^\ell , \tag{3.73}$$

$$g(x) = 2\phi\int_x^\infty dy \frac{yg(y)}{\sqrt{(y^2-1)^2 + f(y)g(y)}}\left(\frac{x}{y}\right)^\ell . \tag{3.74}$$

Setting $F(x) = x^\ell f(x)$ and $G(x) = g(x)/x^\ell$, we obtain

$$F(x) = 2\phi\int_0^x dy \frac{yF(y)}{\sqrt{(y^2-1)^2 + F(y)G(y)}}, \tag{3.75}$$

$$G(x) = 2\phi\int_x^\infty dy \frac{yG(y)}{\sqrt{(y^2-1)^2 + F(y)G(y)}}. \tag{3.76}$$

Differentiating $F(x)$ and $G(x)$ with respect to x, we obtain

$$\frac{dF(x)}{dx} = 2\phi\frac{xF(x)}{\sqrt{(x^2-1)^2 + F(x)G(x)}}, \tag{3.77}$$

$$\frac{dG(x)}{dx} = -2\phi\frac{xG(x)}{\sqrt{(x^2-1)^2 + F(x)G(x)}}. \tag{3.78}$$

From Eqs.(3.77) and (3.78), we obtain

$$\frac{d}{dx}\left[F(x)G(x)\right] = 0. \tag{3.79}$$

That is, $F(x)G(x) = C$ (constant). Substituting it into Eq.(3.77), we solve Eq.(3.77) and obtain

$$F(x) = \text{const.} \times \left[x^2 - 1 + \sqrt{(x^2-1)^2 + C}\right]^\phi . \tag{3.80}$$

On the other hand, setting $x = 0$ in Eq.(3.75) we find $F(0) = 0$. Substituting it into Eq.(3.80), we find $C = 0$. As a result, $F(x)$ is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 1, \\ \text{const.} \times (x^2-1)^\phi & \text{for} \quad x > 1. \end{cases} \tag{3.81}$$

On the other hand, $G(x) = 0$ for $x > 1$ because $F(x)G(x) = 0$. For $x < 1$, from Eq.(3.78) we obtain

$$\frac{dG(x)}{dx} = 2\phi\frac{x}{x^2-1}G(x). \tag{3.82}$$

The solution of this equation is given by

$$G(x) = \text{const.} \times (1-x^2)^\phi . \tag{3.83}$$

Finally, the functions $f(x)$ and $g(x)$ are given by

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 1, \\ C_f(x^2-1)^\phi/x^\ell & \text{for} \quad x > 1, \end{cases} \tag{3.84}$$

$$g(x) = \begin{cases} C_g x^t (1-x^2)^\phi & \text{for} \quad x < 1, \\ 0 & \text{for} \quad x > 1, \end{cases} \tag{3.85}$$

where C_f and C_g are constants. As mentioned above, $f(x)g(x) = 0$, or $\overline{\Delta}_k \Delta_k = 0$ for any x. That means this state is a gapless state. However, each of Δ_k and $\overline{\Delta}_k$ is not zero. Therefore, this state is still a pairing state. As we have discussed in the beginning of this section, we expect that the effect of V^{NH} is a pair-breaking. However, it has no ability to break pairings. The effect of it is to make the gap of pairing state go to zero at most.

This conclusion that all of the pairing states are gapless if we neglect the Coulomb interaction is the natural one. In the absence of the Coulomb interaction, we do not have to take into account the two-body correlation discussed in Sec.2.3. Therefore, we do not have the pairing interaction derived from it. Of course, the Rajaraman-Sondhi transformation is the exact one, therefore, there is no reason to prevent to perform such transformation. However, we cannot solve the problem exactly. It depends on the choice of the starting point, that is, the Hamiltonian and the approximation to solve the problem whether we can capture the point of the problem or not. In the absence of the Coulomb interaction, the Hamiltonian obtained by the Rajaraman-Sondhi transformation is not a good starting point.

In above discussion of gap equations, we do not consider the ground state energy. We must examine it to discuss the stability of the gapless pairing state. From Eq.(3.46), we find that the ground state energy of gapless states is the same as that of no-pairing states because $\overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}} \equiv 0$. However, states with $\overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}} \equiv 0$ are not stable. Considering the variation of $\langle H \rangle$ with respect to $\overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}}$, we obtain

$$\langle H \rangle_{\overline{\Delta}_k \Delta_k + \delta\left(\overline{\Delta}_k \Delta_k\right)} - \langle H \rangle_{\overline{\Delta}_k \Delta_k} = -\frac{1}{8} \sum_{\mathbf{k}} \frac{\overline{\Delta}_k \Delta_k}{\left(\overline{\Delta}_k \Delta_k + \xi_k^2\right)^{3/2}} \delta\left(\overline{\Delta}_k \Delta_k\right). \tag{3.86}$$

The coefficient $\overline{\Delta}_k \Delta_k / \left(\overline{\Delta}_k \Delta_k + \xi_k^2\right)^{3/2}$ is not lower than zero. Therefore, the function $\overline{\Delta}_k \Delta_k \equiv 0$ is the relative maximum of $\langle H \rangle$. Hence states with $\overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}} \equiv 0$ are not stable. Any perturbation will change the ground state of the system from the gapless pairing state to the gapful pairing state. If we take into account the Coulomb interaction, this gapless pairing state is not to be the ground state any more. Therefore, we expect that V^{NH} is irrelevant for pairing states.

The irrelevance of V^{NH} was shown in the bosonic Chern-Simons theory of FQHE at $\nu = 1/m$, where m is an odd integer [44]. It was shown in Ref.[44] that three body interaction term is irrelevant at zero temperature if we assume the condensation of composite bosons by applying the renormalization group. We do not have the exact proof for the irrelevance of the three body interaction term with regard to the pairing state of CFs. However, we expect that it is irrelevant because the pairing state of fermions is described by the Ginzburg-Landau theory, and the order parameter of it has the role of bosons in the composite boson Chern-Simons theory. Therefore, we can apply the discussion in Ref.[44] in that case. In the following discussion, we assume that the anti-Hermitian term V^{NH} is irrelevant and neglect it.

3.6 Stability of Pairing State

Now we discuss the effect of the Coulomb interaction and the stability of the p-wave pairing state [45]. When we take into account the Coulomb interaction and neglect the anti-Hermitian term V^{NH}, the gap equation is given by

$$\begin{aligned} \Delta_k &= \frac{\phi}{2M} \int_0^k dk' \frac{k' \Delta_{k'}}{E_{k'}} \left[\frac{k'}{k} - \frac{\alpha'}{4\pi} \frac{k_F}{k} G\left(\frac{k'}{k}\right) \right] \\ &\quad + \frac{\phi}{2M} \int_k^\infty dk' \frac{k' \Delta_{k'}}{E_{k'}} \left[\frac{k}{k'} - \frac{\alpha'}{4\pi} \frac{k_F}{k'} G\left(\frac{k}{k'}\right) \right], \end{aligned} \tag{3.87}$$

where $\alpha' = \alpha \times \sqrt{2/\phi}$ and

$$G(x) = x \left[K(x) - \int_0^1 d\xi \xi K(x\xi) \right], \tag{3.88}$$

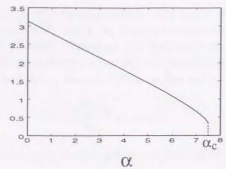

Figure 3.3: The parameter α dependence of the gap Δ at $\phi = 2$, or $\nu = 1/2$. The gap has a finite value in the region $\alpha < \alpha_c$, where $\alpha_c \simeq 7.5$.

with $K(x) = \int_0^{\pi/2} d\phi/\sqrt{1 - x^2 \sin\phi}$ being the complete elliptic integral of the first kind. We need not consider the gap equation for $\overline{\Delta_{\mathbf{k}}}$ because it is the complex conjugate of $\Delta_{\mathbf{k}}$ in the absence of the anti-Hermitian term V^{NH}. To obtain Eq.(3.87), we have substituted the formula:

$$\begin{aligned}\int_0^{2\pi} d\theta_{\mathbf{k}_2} \frac{e^{-i\theta_{\mathbf{k}_2}}}{|\mathbf{k}_1 - \mathbf{k}_2|} &= \frac{2}{k_>} e^{-i\theta_{\mathbf{k}_1}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!\Gamma(n+2)} \left(\frac{k_<}{k_>}\right)^{2n+1} \\ &= \frac{4}{k_>} e^{-i\theta_{\mathbf{k}_1}} G\left(\frac{k_>}{k_<}\right), \end{aligned} \tag{3.89}$$

into Eq.(3.37).

To solve the gap equation, we adopt the same approximation in Sec.3.4. At $k = k_F$, the gap equation is given by

$$\int_0^1 dx \frac{x^2}{\sqrt{(x^2-1)^2 + \Delta^2 x^2}} \left[x - \frac{\alpha'}{4\pi} G(x)\right] + \int_1^\infty dx \frac{1}{x\sqrt{(x^2-1)^2 + \Delta^2/x^2}} \left[1 - \frac{\alpha'}{4\pi} G(x)\right] = \frac{1}{\phi}. \tag{3.90}$$

In Fig.3.3, we show the parameter α dependence of the gap Δ, at $\phi = 2$, which corresponds to the case of $\nu = 1/2$. In the region of $\alpha < \alpha_c$, where $\alpha_c \simeq 7.5$, the gap has a finite value. Therefore, the ground state of the system is the p-wave BCS pairing state in that region. On the other hand, in the region of $\alpha > \alpha_c$ the gap goes to zero. That is, the ground state is the no-pairing state. The answer to the question as to which case must be applied to the real system depends on the value of ϵ_F. In Sec.IV of Ref.[46], Halperin, Lee and Read estimated the effective mass of CFs by the dimensional analysis, which holds in the limit of $\hbar\omega_c \gg e^2/\epsilon\ell_B$, and numerically obtained gaps of several fractional quantum Hall states. When the condition $\hbar\omega_c \gg e^2/\epsilon\ell_B$ is satisfied, the energy scale of the system is $e^2/\epsilon\ell_B$ only. Therefore, we obtain

$$\frac{\hbar^2}{M} = C \cdot \frac{e^2}{\epsilon} \ell_B, \tag{3.91}$$

where C is a constant. On the other hand, in the CF theory the energy gap $E_g^{(\nu)}$ of FQH states at $\nu = p/(2p+1)$, where p is an integer, is given by

$$E_g^{(\nu)} = \frac{2\pi\overline{\rho}}{M|p|} |2\nu|^{\frac{1}{2}}. \tag{3.92}$$

In deriving Eq.(3.92), we assume the simple particle-hole symmetry around at $\nu = 1/2$. Substituting Eq.(3.91) into Eq.(3.92), we obtain

$$E_g^{(\nu)} = \frac{C}{|2p+1|} \frac{e^2}{\epsilon\ell_B}. \tag{3.93}$$

The parameter C is determined by fitting Eq.(3.93) with numerically obtained energy gaps. From their analysis, α is estimated to ~ 6.7. This value of α is lower than α_c. Therefore, the p-wave pairing occurs

at $\nu = 1/2$ when the condition $\hbar\omega_c \gg e^2/\epsilon\ell_B$ is satisfied. The ratio of $e^2/\epsilon\ell_B$ to $\hbar\omega_c$ is determined by $(e^2/\epsilon\ell_B)/\hbar\omega_c \sim 4.9 \times 10^2 \times (m_b/m_e)/(\epsilon\sqrt{B})$ where m_e is the electron mass in the vacuum and the external magnetic field is measured in units of tesla. Therefore, the pairing state of CFs is possible in the sample with a small band mass m_b and a large dielectric constant ϵ and under a strong magnetic field B. On the other hand, in the weak magnetic field limit: $\hbar\omega_c \ll e^2/\epsilon\ell_B$, α has the following form:

$$\alpha \sim \frac{e^2/\epsilon\ell_B}{\hbar\omega_c} \propto 1/\sqrt{B}, \tag{3.94}$$

using the dimensional analysis. In that case, α diverges in the limit of $B \to 0$. Therefore, the pairing state is not stabilized in that limit.

Let us remark on the relevance of our theory to the $\nu = 5/2$ state. Contrary to the $\nu = 1/2$ case, we have filled Landau levels at $\nu = 5/2$. What is the role of these filled Landau levels? One possible scenario is that the Coulomb interaction for electrons in partially filled Landau level is screened by electrons in filled Landau levels. If this is true, we can also apply the above discussion to the $\nu = 5/2$ state.

3.7 Real Spin Effect and Zeeman Energy

Next we discuss an effect of the real spin degrees of freedom and the Zeeman energy. To begin with, we discuss the former in the absence of the latter. The spin unpolarized pairing state is possible as in the case of the bilayer quantum Hall systems (see chap.4). In that case, the ground state is also the p-wave pairing state. However, the expression of the ground state energy is $\langle H \rangle$ in Eq.(3.52) times 2 because of the spin degrees of freedom (see Eq.(4.50)). Furthermore, the ground state energy contains the Fermi wave number as the parameter and $\langle H \rangle$ is proportional to k_F^4. It is understood from Eq.(3.52) because $\langle H \rangle$ is proportional to $\epsilon_F^2 \propto k_F^4$. In the spin unpolarized pairing case, the Fermi wave number k_F is equal to $k_F^p/\sqrt{2}$, where $k_F^p (= 1/\ell_B)$ is the Fermi wave number of the spin polarized case. Putting it all together, the ground state energy of the spin unpolarized pairing state is half of $\langle H \rangle$ estimated in the case of spin polarized case. Being $\langle H \rangle < 0$, the ground state energy of the spin polarized state is lower than that of the spin unpolarized pairing state. As a result, if the pairing of CFs is possible then it is the spin-polarized pairing state. With regard to the effect of the Zeeman energy to the spinless CFs, it is nothing but shifting the chemical potential.

3.8 Summary

In this chapter, we have discussed the possibility of the pairing of CFs at $\nu = 1/m$, where m is an even integer. We have derived the gap equations for the CF pairing state by the pairing approximation. The pairing interaction is derived from the potential V^H. The solution of the gap equations in the absence of V^{NH} and the Coulomb interaction indicates that the p-wave pairing state of CFs has the largest gap. Furthermore, it has the lowest energy. Therefore, the ground state is the p-wave pairing state of CFs. For that state, the ground state wave function has derived. The orbital function has the form: $\sim 1/z$. From the phase dependence of it, we see that the angular momentum of it and the cyclotron motion are in the opposite direction. This fact is consistent with the discussion in Sec.2.4.

Next we take into account the anti-Hermitian term V^{NH}. The interaction term V^{NH} corresponds to an imaginary vector potential. The effect of the imaginary vector potential is known as a delocalization effect in the localization-delocalization phenomena[41]. The delocalization in that problem corresponds to the no-pairing in the CF pairing theory. Therefore, V^{NH} is expected to have a pair-breaking effect. In fact, if we consider the gap equation taking into account V^{NH} in the absence of the Coulomb interaction, the solution is a gapless pairing state. However, the effect of V^{NH} is not enough to cause pair-breaking but the gap of the pairing state goes to zero. Considering the ground state energy it is on an unstable point. Therefore, when we take into account the Coulomb interaction, the ground state changes from the gapless pairing state to the gapful pairing state.

On the other hand, the anti-Hermitian term V^{NH} corresponds to the three-body interaction term in the Chern-Simons gauge theory. In that case, if we assume the condensate of composite bosons, it is shown that

the three-body term is irrelevant. Therefore, the anti-Hermitian term V^{NH} is expected to be irrelevant for the pairing state.

With regard to the effect of the Coulomb interaction, we examine the stability of the pairing state in the presence of the Coulomb interaction. We consider the gap equation taking into account the Coulomb interaction and neglecting V^{NH}. We estimate the region as the ratio of the Coulomb interaction to the Fermi energy where the pairing state is realized. The pairing state is possible when the condition $\hbar\omega_c \gg e^2/\epsilon\ell_B$ holds. This is realized in samples with a small band mass, a large dielectric constant and a strong magnetic field.

In order to discuss the polarization of the real spin, we compare the ground state energy of spin polarized pairing state with that of spin unpolarized pairing state. The ground state energy of spin polarized pairing state is lower than that of spin unpolarized pairing state. Therefore, if the pairing state of CFs is possible then it is the spin-polarized pairing state.

To discuss the $\nu = 5/2$ state, we must take into account the effect of filled Landau levels. The possible scenario is that the Coulomb interaction for electrons in partially filled Landau level is screened by electrons in filled Landau levels.

Chapter 4

Pairing States in Bilayer Qunatum Hall Systems

In this chapter, we discuss the pairing state of CFs in bilayer quantum Hall systems. Applying the pairing approximation, we derive gap equations for the CF pairing state. Contrary to single layer systems, the pseudo-spin degrees of freedom must be taken into account. Not only a pseudo-spin triplet pairing state but also a pseudo-spin singlet pairing state is possible. First, we show that in the absence of an interlayer tunneling the ground state is the p-wave pairing state at $\nu = 1/m$, where m is an integer. With respect to an interlayer tunneling, we consider the case of $\nu = 1/2$ and the case of $\nu = 1$. The importance of the interlayer Coulomb interaction in bilayer quantum Hall systems is also discussed.

4.1 Hamiltonian

In bilayer systems, to control the system we have many parameters, such as the interlayer separation, an interlayer tunneling and a charge imbalance $\overline{\rho}_1 - \overline{\rho}_2$. Here we concentrate on the case of $\overline{\rho}_1 = \overline{\rho}_2$ and $\nu = 1/m$, where m is an integer. The Coulomb interaction for electrons is given by

$$V^C = \frac{1}{2} \sum_{\alpha,\beta=1,1} \int d^2\mathbf{r} d^2\mathbf{r}' V^C_{\alpha\beta} (\mathbf{r} - \mathbf{r}') \, \delta\rho_\alpha (\mathbf{r}) \, \delta\rho_\beta (\mathbf{r}') \,, \tag{4.1}$$

where $\uparrow(\downarrow)$ denotes the index of electrons in upper(lower) layer, $\delta\rho_\alpha (\mathbf{r}) = \psi^\dagger_\alpha (\mathbf{r}) \, \psi_\alpha (\mathbf{r}) - \bar{\rho}_\alpha$ with $\bar{\rho}_\alpha$ being the average particle density in the layer α and

$$V^C_{\alpha\beta} (\mathbf{r}) = \frac{e^2}{\epsilon \sqrt{|\mathbf{r}|^2 + d^2 \left(1 - \delta_{\alpha\beta}\right)}}, \tag{4.2}$$

with ϵ being the dielectric constant and d being the interlayer separation.

The Hamiltonian for the interlayer tunneling for electrons is given by

$$H^t_{\mathrm{el}} = - \int d^2\mathbf{r} \;\; t \left[\psi^\dagger_\uparrow (\mathbf{r}) \, \psi_\downarrow (\mathbf{r}) + \psi^\dagger_\downarrow (\mathbf{r}) \, \psi_\uparrow (\mathbf{r}) \right]. \tag{4.3}$$

Applying the Rajaraman-Sondhi transformation, we obtain the Hamiltonian for the interlayer tunneling for CFs. In that case, if $\phi_1 \neq \phi_2$ holds then we must add an extremal phase factor. However, in the case of $\phi_1 = \phi_2$, the tunneling Hamiltonian for CFs is given by

$$H^t_{\mathrm{CF}} = - \int d^2\mathbf{r} \;\; t \left[\pi_\uparrow (\mathbf{r}) \, \phi_\downarrow (\mathbf{r}) + \phi_\downarrow (\mathbf{r}) \, \pi_\uparrow (\mathbf{r}) \right]. \tag{4.4}$$

For the tunneling Hamiltonian, we adopt Eq.(4.4) and neglect the phase factor caused by mapping from electrons into CFs in the following.

After the extended CF transformations the form of V^C is not changed because $\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) = \pi(\mathbf{r})\phi(\mathbf{r})$. Taking into account the Coulomb interaction (4.1) and the interlayer tunneling (4.4) in Eq.(2.33) and setting all components of $K_{\alpha\beta}$ as even integers, the CF Hamiltonian for bilayer systems is given by

$$H = \sum_{\mathbf{k}\alpha\beta} \xi^{\mathbf{k}}_{\alpha\beta} \pi_{\mathbf{k}\alpha} \phi_{\mathbf{k}\beta} + \frac{1}{2\Omega} \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} \sum_{\alpha\beta} V^{\alpha\beta}_{\mathbf{k}_1\mathbf{k}_2} \pi_{\mathbf{k}_1\alpha} \pi_{-\mathbf{k}_1\beta} \phi_{-\mathbf{k}_2\beta} \phi_{\mathbf{k}_2\alpha}, \tag{4.5}$$

where $\xi^{\mathbf{k}}_{\uparrow\uparrow} = \xi^{\mathbf{k}}_{\downarrow\downarrow} = k^2/2m_b$, $\xi^{\mathbf{k}}_{\uparrow\downarrow} = \xi^{\mathbf{k}}_{\downarrow\uparrow} = -t$ and $V^{\alpha\beta}_{\mathbf{k}_1\mathbf{k}_2} = K_{\alpha\beta} V_{\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_1+\mathbf{k}_2} + V^C_{\alpha\beta\mathbf{k}_1-\mathbf{k}_2}$. The interaction $V^{\alpha\beta}_{\mathbf{k}_1\mathbf{k}_2}$ is the one for CF pairs with zero total momentum. The matrix K is given by

$$K = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_1 \end{bmatrix}, \tag{4.6}$$

where ϕ_1 and ϕ_2 are even integers.

4.2 Pairing Approximation

To discuss the pairing state of CFs, we introduce the pairing approximation in Eq.(4.5). We obtain

$$\begin{aligned} H \simeq \sum_{\mathbf{k}\alpha\beta} \xi^{\mathbf{k}}_{\alpha\beta} \pi_{\mathbf{k}\alpha} \phi_{\mathbf{k}\beta} \quad + \quad \frac{1}{2\Omega} \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} \sum_{\alpha\beta} V^{\alpha\beta}_{\mathbf{k}_1\mathbf{k}_2} \Big\{ \langle \pi_{\mathbf{k}_1\alpha} \pi_{-\mathbf{k}_1\beta} \rangle \phi_{-\mathbf{k}_2\beta} \phi_{\mathbf{k}_2\alpha} \\ + \pi_{\mathbf{k}_1\alpha} \pi_{-\mathbf{k}_1\beta} \langle \phi_{-\mathbf{k}_2\beta} \phi_{\mathbf{k}_2\alpha} \rangle - \langle \pi_{\mathbf{k}_1\alpha} \pi_{-\mathbf{k}_1\beta} \rangle \langle \phi_{-\mathbf{k}_2\beta} \phi_{\mathbf{k}_2\alpha} \rangle \Big\}. \end{aligned} \tag{4.7}$$

Now we introduce the gap functions $\Delta^{\mathbf{k}}_{\alpha\beta}$ and $\overline{\Delta}^{\mathbf{k}}_{\alpha\beta}$:

$$\Delta^{\mathbf{k}}_{\alpha\beta} \equiv -\frac{1}{\Omega} \sum_{\mathbf{k}(\neq\mathbf{k}')} V^{\alpha\beta}_{\mathbf{k}\mathbf{k}'} \langle \phi_{-\mathbf{k}'\beta} \phi_{\mathbf{k}'\alpha} \rangle, \tag{4.8}$$

$$\overline{\Delta}^{\mathbf{k}}_{\alpha\beta} \equiv -\frac{1}{\Omega} \sum_{\mathbf{k}(\neq\mathbf{k}')} V^{\alpha\beta}_{\mathbf{k}'\mathbf{k}} \langle \pi_{\mathbf{k}'\alpha} \pi_{-\mathbf{k}'\beta} \rangle. \tag{4.9}$$

Note that $\overline{\Delta}^{\mathbf{k}}_{\alpha\beta}$ is not necesarilly complex conjugate of $\Delta^{\mathbf{k}}_{\alpha\beta}$ because $\left(V^{\alpha\beta}_{\mathbf{k}\mathbf{k}'}\right)^* \neq V^{\alpha\beta}_{\mathbf{k}'\mathbf{k}}$ in the presence of V^{NH}. Using these fields, the Hamiltonian becomes

$$H = {\sum_{\mathbf{k}}}' \pi^{\mathbf{k}} \mathcal{E}^{\mathbf{k}} \phi^{\mathbf{k}} + \text{const.}, \tag{4.10}$$

where

$$\phi^{\mathbf{k}} \equiv \begin{bmatrix} \phi_{\mathbf{k}\uparrow} \\ \phi_{\mathbf{k}\downarrow} \\ \pi_{-\mathbf{k}\uparrow} \\ \pi_{-\mathbf{k}\downarrow} \end{bmatrix}, \tag{4.11}$$

$$\pi^{\mathbf{k}} \equiv \begin{bmatrix} \pi_{\mathbf{k}\uparrow} & \pi_{\mathbf{k}\downarrow} & \phi_{-\mathbf{k}\uparrow} & \phi_{-\mathbf{k}\downarrow} \end{bmatrix}, \tag{4.12}$$

$$\mathcal{E}^{\mathbf{k}} \equiv \begin{bmatrix} \xi^{\mathbf{k}} & -\Delta^{\mathbf{k}} \\ -\overline{\Delta}^{\mathbf{k}} & -\xi^{\mathbf{k}} \end{bmatrix}. \tag{4.13}$$

From Eqs.(4.8) and (4.9), it is seen that

$$\left(\Delta^{-\mathbf{k}}\right)_{\beta\alpha} = -\left(\Delta^{\mathbf{k}}\right)_{\alpha\beta}, \qquad \left(\overline{\Delta}^{-\mathbf{k}}\right)_{\beta\alpha} = -\left(\overline{\Delta}^{\mathbf{k}}\right)_{\alpha\beta}. \tag{4.14}$$

The next step is the diagonalization of the martix $\mathcal{E}^{\mathbf{k}}$. We must consider it both for the triplet pairing case and the singlet pairing case because of the pseudo-spin degrees of freedom.

4.2.1 Triplet Pairing

In the triplet pairing cases, $\Delta^{\mathbf{k}}$ and $\overline{\Delta}^{\mathbf{k}}$ satisfy the following relations:

$$\left(\Delta^{-\mathbf{k}}\right)_{\alpha\beta} = -\left(\Delta^{\mathbf{k}}\right)_{\alpha\beta}, \qquad \left(\Delta^{\mathbf{k}}\right)_{\beta\alpha} = \left(\Delta^{\mathbf{k}}\right)_{\alpha\beta}, \tag{4.15}$$

$$\left(\overline{\Delta}^{-\mathbf{k}}\right)_{\alpha\beta} = -\left(\overline{\Delta}^{\mathbf{k}}\right)_{\alpha\beta}, \qquad \left(\overline{\Delta}^{\mathbf{k}}\right)_{\beta\alpha} = \left(\overline{\Delta}^{\mathbf{k}}\right)_{\alpha\beta}. \tag{4.16}$$

To simplify the discussion, we concentrate on the case of symmetric layers. That is,

$$\left(\Delta^{\mathbf{k}}\right)_{\uparrow\uparrow} = \left(\Delta^{\mathbf{k}}\right)_{\downarrow\downarrow} \equiv \Delta^1_{\mathbf{k}}, \tag{4.17}$$

$$\left(\Delta^{\mathbf{k}}\right)_{\uparrow\downarrow} = \left(\Delta^{\mathbf{k}}\right)_{\downarrow\uparrow} \equiv \Delta^2_{\mathbf{k}}. \tag{4.18}$$

In this case, $\mathcal{E}^{\mathbf{k}}$ has the following form:

$$\mathcal{E}^{\mathbf{k}} = \begin{bmatrix} \xi_{\mathbf{k}} & -t & -\Delta^1_{\mathbf{k}} & -\Delta^2_{\mathbf{k}} \\ -t & \xi_{\mathbf{k}} & -\Delta^2_{\mathbf{k}} & -\Delta^1_{\mathbf{k}} \\ -\overline{\Delta}^1_{\mathbf{k}} & -\overline{\Delta}^2_{\mathbf{k}} & -\xi_{\mathbf{k}} & t \\ -\overline{\Delta}^2_{\mathbf{k}} & -\overline{\Delta}^1_{\mathbf{k}} & t & -\xi_{\mathbf{k}} \end{bmatrix}. \tag{4.19}$$

The eigenvalues of the matrix $\mathcal{E}^{\mathbf{k}}$ are given by $\pm E^+_{\mathbf{k}}$ and $\pm E^-_{\mathbf{k}}$, where

$$E^+_{\mathbf{k}} = \sqrt{(\xi_k - t)^2 + (\overline{\Delta}^1_{\mathbf{k}} + \overline{\Delta}^2_{\mathbf{k}})(\Delta^1_{\mathbf{k}} + \Delta^2_{\mathbf{k}})}, \tag{4.20}$$

$$E^-_{\mathbf{k}} = \sqrt{(\xi_k + t)^2 + (\overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}})(\Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}})}. \tag{4.21}$$

The eigenvectors of $\mathcal{E}^{\mathbf{k}}$ are given by

$$\mathbf{u}^1_{\mathbf{k}} = \frac{1}{\sqrt{4E^+_{\mathbf{k}}(E^+_{\mathbf{k}} + \xi_{\mathbf{k}} - t)}} \begin{bmatrix} E^+_{\mathbf{k}} + \xi_k - t \\ E^+_{\mathbf{k}} + \xi_k - t \\ -\overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \\ -\overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \end{bmatrix}, \tag{4.22}$$

$$\mathbf{u}^2_{\mathbf{k}} = \frac{1}{\sqrt{4E^-_{\mathbf{k}}(E^-_{\mathbf{k}} - \xi_k - t)}} \begin{bmatrix} -\Delta^1_{\mathbf{k}} + \Delta^2_{\mathbf{k}} \\ \Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \\ E^-_{\mathbf{k}} - \xi_k - t \\ -E^-_{\mathbf{k}} + \xi_k + t \end{bmatrix}, \tag{4.23}$$

$$\mathbf{u}^3_{\mathbf{k}} = \frac{1}{\sqrt{4E^+_{\mathbf{k}}(E^+_{\mathbf{k}} - \xi_{\mathbf{k}} + t)}} \begin{bmatrix} -E^+_{\mathbf{k}} + \xi_k - t \\ -E^+_{\mathbf{k}} + \xi_k - t \\ -\overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \\ -\overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \end{bmatrix}, \tag{4.24}$$

$$\mathbf{u}^4_{\mathbf{k}} = \frac{1}{\sqrt{4E^-_{\mathbf{k}}(E^-_{\mathbf{k}} + \xi_k + t)}} \begin{bmatrix} -\Delta^1_{\mathbf{k}} + \Delta^2_{\mathbf{k}} \\ \Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \\ -E^-_{\mathbf{k}} - \xi_k - t \\ E^-_{\mathbf{k}} + \xi_k + t \end{bmatrix}. \tag{4.25}$$

The eigenvalues of Eqs.(4.22),(4.23), (4.24) and (4.25) are given by $E^+_{\mathbf{k}}$,$E^-_{\mathbf{k}}$,$-E^+_{\mathbf{k}}$ and $-E^-_{\mathbf{k}}$, respectively. The matrix $\mathcal{E}^{\mathbf{k}}$ is diagonalized by matrices $U^{\mathbf{k}}$ and $\overline{U}^{\mathbf{k}}$:

$$U^{\mathbf{k}} = [\ \mathbf{u}^1_{\mathbf{k}}\ \ \mathbf{u}^2_{\mathbf{k}}\ \ \mathbf{u}^3_{\mathbf{k}}\ \ \mathbf{u}^4_{\mathbf{k}}\ ], \tag{4.26}$$

$$\overline{U}^{\mathbf{k}} = {}^{\mathrm{T}}[\ \overline{\mathbf{u}}^1_{\mathbf{k}}\ \ \overline{\mathbf{u}}^2_{\mathbf{k}}\ \ \overline{\mathbf{u}}^3_{\mathbf{k}}\ \ \overline{\mathbf{u}}^4_{\mathbf{k}}\ ], \tag{4.27}$$

where,

$$\overline{\mathbf{u}}^1_{\mathbf{k}} = \frac{1}{\sqrt{4E^+_{\mathbf{k}}(E^+_{\mathbf{k}} + \xi_{\mathbf{k}} - t)}} \begin{bmatrix} E^+_{\mathbf{k}} + \xi_k - t \\ E^+_{\mathbf{k}} + \xi_k - t \\ -\Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \\ -\Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \end{bmatrix}, \tag{4.28}$$

$$\overline{\mathbf{u}}^2_{\mathbf{k}} = \frac{1}{\sqrt{4E^-_{\mathbf{k}}(E^-_{\mathbf{k}} - \xi_k - t)}} \begin{bmatrix} -\overline{\Delta}^1_{\mathbf{k}} + \overline{\Delta}^2_{\mathbf{k}} \\ \overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \\ E^-_{\mathbf{k}} - \xi_k - t \\ -E^-_{\mathbf{k}} + \xi_k + t \end{bmatrix}, \tag{4.29}$$

$$\overline{\mathbf{u}}^3_{\mathbf{k}} = \frac{1}{\sqrt{4E^+_{\mathbf{k}}(E^+_{\mathbf{k}} - \xi_k + t)}} \begin{bmatrix} -E^+_{\mathbf{k}} + \xi_k - t \\ -E^+_{\mathbf{k}} + \xi_k - t \\ -\Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \\ -\Delta^1_{\mathbf{k}} - \Delta^2_{\mathbf{k}} \end{bmatrix}, \tag{4.30}$$

$$\overline{\mathbf{u}}^4_{\mathbf{k}} = \frac{1}{\sqrt{4E^-_{\mathbf{k}}(E^-_{\mathbf{k}} + \xi_k + t)}} \begin{bmatrix} -\overline{\Delta}^1_{\mathbf{k}} + \overline{\Delta}^2_{\mathbf{k}} \\ \overline{\Delta}^1_{\mathbf{k}} - \overline{\Delta}^2_{\mathbf{k}} \\ -E^-_{\mathbf{k}} - \xi_k - t \\ E^-_{\mathbf{k}} + \xi_k + t \end{bmatrix}. \tag{4.31}$$

We introduce the quasiparticle field operators $q^{\mathbf{k}}$ and $p^{\mathbf{k}}$:

$$q^{\mathbf{k}} = \begin{bmatrix} q_{\mathbf{k}\uparrow} \\ q_{\mathbf{k}\downarrow} \\ p_{-\mathbf{k}\uparrow} \\ p_{-\mathbf{k}\downarrow} \end{bmatrix} \equiv \overline{U}^{\mathbf{k}} \phi^{\mathbf{k}}, \tag{4.32}$$

$$p^{\mathbf{k}} = [\ p_{\mathbf{k}\uparrow}\ \ p_{\mathbf{k}\downarrow}\ \ q_{-\mathbf{k}\uparrow}\ \ q_{-\mathbf{k}\downarrow}\ ] \equiv \pi^{\mathbf{k}} U^{\mathbf{k}}. \tag{4.33}$$

The fields $p_{\mathbf{k}\alpha}$ and $q_{\mathbf{k}\alpha}$ satisfy the following anti-commutation relations:

$$\begin{cases} \{q_{\mathbf{k}\alpha}, p_{\mathbf{k}'\beta}\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\beta}, \\ \{q_{\mathbf{k}\alpha}, q_{\mathbf{k}'\beta}\} = 0, \\ \{p_{\mathbf{k}\alpha}, p_{\mathbf{k}'\beta}\} = 0. \end{cases} \tag{4.34}$$

Using these fields, the Hamiltonian becomes

$$\begin{aligned} H &\simeq \sum_{\mathbf{k}}{}' p^{\mathbf{k}} \overline{U}^{\mathbf{k}} \mathcal{E}^{\mathbf{k}} U^{\mathbf{k}} q^{\mathbf{k}} + \text{const.} \\ &= \sum_{\mathbf{k}}{}' p^{\mathbf{k}} \begin{bmatrix} E^+_{\mathbf{k}} & 0 & 0 & 0 \\ 0 & E^-_{\mathbf{k}} & 0 & 0 \\ 0 & 0 & -E^+_{\mathbf{k}} & 0 \\ 0 & 0 & 0 & -E^-_{\mathbf{k}} \end{bmatrix} q^{\mathbf{k}} + \text{const.}. \end{aligned} \tag{4.35}$$

Now we calculate $\langle \phi_{-\mathbf{k}\beta}\phi_{\mathbf{k}\alpha}\rangle$ in the right hand side of Eq.(4.8). As an example, we show the detailed calculation of $\langle \phi_{-\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle$:

$$\begin{aligned}
\langle \phi_{-\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle &= \langle \left(p^{\mathbf{k}}\overline{U}^{\mathbf{k}}\right)_3 \left(U^{\mathbf{k}}q^{\mathbf{k}}\right)_2\rangle \\
&= \sum_{j=1}^{4}\langle p_j^{\mathbf{k}}\overline{U}_{j3}^{\mathbf{k}}U_{2j}^{\mathbf{k}}q_j^{\mathbf{k}}\rangle \\
&= \sum_{j=1}^{4}\overline{U}_{j3}^{\mathbf{k}}U_{2j}^{\mathbf{k}}\langle p_j^{\mathbf{k}}q_j^{\mathbf{k}}\rangle \\
&= \sum_{j=1}^{4}\left(\overline{\mathbf{u}}_j^{\mathbf{k}}\right)_3\left(\mathbf{u}_j^{\mathbf{k}}\right)_2\langle p_j^{\mathbf{k}}q_j^{\mathbf{k}}\rangle \\
&= -\frac{\Delta_{\mathbf{k}}^1+\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}f(E_{\mathbf{k}}^+)+\frac{\Delta_{\mathbf{k}}^1-\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^-}f(E_{\mathbf{k}}^-) \\
&\qquad +\frac{\Delta_{\mathbf{k}}^1+\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}f(-E_{\mathbf{k}}^+)-\frac{\Delta_{\mathbf{k}}^1-\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^-}f(-E_{\mathbf{k}}^-) \\
&= \frac{\Delta_{\mathbf{k}}^1+\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}\tanh\frac{E_{\mathbf{k}}^+}{2k_BT}-\frac{\Delta_{\mathbf{k}}^1-\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}\tanh\frac{E_{\mathbf{k}}^-}{2k_BT}.
\end{aligned} \tag{4.36}$$

Here the function $f(E)=1/(\exp(E/k_BT)+1)$ is the fermion distribution funtion. Other values of $\langle \phi_{-\mathbf{k}\beta}\phi_{\mathbf{k}\alpha}\rangle$ is given by $\langle \phi_{-\mathbf{k}\downarrow}\phi_{\mathbf{k}\uparrow}\rangle = \langle \phi_{-\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle$ and

$$\langle \phi_{-\mathbf{k}\downarrow}\phi_{\mathbf{k}\uparrow}\rangle = \langle \phi_{-\mathbf{k}\downarrow}\phi_{\mathbf{k}\downarrow}\rangle = \frac{\Delta_{\mathbf{k}}^1+\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}\tanh\frac{E_{\mathbf{k}}^+}{2k_BT}+\frac{\Delta_{\mathbf{k}}^1-\Delta_{\mathbf{k}}^2}{4E_{\mathbf{k}}^+}\tanh\frac{E_{\mathbf{k}}^-}{2k_BT}. \tag{4.37}$$

The values of $\langle \pi_{\mathbf{k}\alpha}\pi_{-\mathbf{k}\beta}\rangle$ are given by replacing $\Delta_{\alpha\beta}^{\mathbf{k}}$ with $\overline{\Delta}_{\alpha\beta}^{\mathbf{k}}$ in these equations. Substituting these formula into Eqs.(4.8) and (4.9), we obtain

$$\Delta_{\uparrow\uparrow}^{\mathbf{k}'} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})}V_{\mathbf{k}\mathbf{k}'}^{\uparrow\uparrow}\left[\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}+\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^+}\tanh\frac{E_{\mathbf{k}'}^+}{2k_BT}+\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}-\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^-}\tanh\frac{E_{\mathbf{k}'}^-}{2k_BT}\right], \tag{4.38}$$

$$\Delta_{\uparrow\downarrow}^{\mathbf{k}'} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})}V_{\mathbf{k}\mathbf{k}'}^{\uparrow\downarrow}\left[\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}+\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^+}\tanh\frac{E_{\mathbf{k}'}^+}{2k_BT}-\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}-\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^-}\tanh\frac{E_{\mathbf{k}'}^-}{2k_BT}\right], \tag{4.39}$$

$$\overline{\Delta}_{\uparrow\uparrow}^{\mathbf{k}'} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})}V_{\mathbf{k}'\mathbf{k}}^{\uparrow\uparrow}\left[\frac{\overline{\Delta}_{\uparrow\uparrow}^{\mathbf{k}'}+\overline{\Delta}_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^+}\tanh\frac{E_{\mathbf{k}'}^+}{2k_BT}+\frac{\overline{\Delta}_{\uparrow\uparrow}^{\mathbf{k}'}-\overline{\Delta}_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^-}\tanh\frac{E_{\mathbf{k}'}^-}{2k_BT}\right], \tag{4.40}$$

$$\overline{\Delta}_{\uparrow\downarrow}^{\mathbf{k}'} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})}V_{\mathbf{k}'\mathbf{k}}^{\uparrow\downarrow}\left[\frac{\overline{\Delta}_{\uparrow\downarrow}^{\mathbf{k}'}+\overline{\Delta}_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^+}\tanh\frac{E_{\mathbf{k}'}^+}{2k_BT}-\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}-\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^-}\tanh\frac{E_{\mathbf{k}'}^-}{2k_BT}\right]. \tag{4.41}$$

At zero temperature, these equations has the following forms:

$$\Delta_{\uparrow\downarrow}^{\mathbf{k}'} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})}V_{\mathbf{k}\mathbf{k}'}^{\uparrow\downarrow}\left[\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}+\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^+}+\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}'}-\Delta_{\uparrow\downarrow}^{\mathbf{k}'}}{E_{\mathbf{k}'}^-}\right], \tag{4.42}$$

$$\Delta^{\mathbf{k}'}_{\uparrow\downarrow} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V^{\uparrow\downarrow}_{\mathbf{k}\mathbf{k}'}\left[\frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow}+\Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{+}_{\mathbf{k}'}} - \frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow}-\Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{-}_{\mathbf{k}'}}\right], \tag{4.43}$$

$$\overline{\Delta}^{\mathbf{k}'}_{\uparrow\uparrow} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V^{\uparrow\uparrow}_{\mathbf{k}'\mathbf{k}}\left[\frac{\overline{\Delta}^{\mathbf{k}'}_{\uparrow\uparrow}+\overline{\Delta}^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{+}_{\mathbf{k}'}} + \frac{\overline{\Delta}^{\mathbf{k}'}_{\uparrow\uparrow}-\overline{\Delta}^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{-}_{\mathbf{k}'}}\right], \tag{4.44}$$

$$\overline{\Delta}^{\mathbf{k}'}_{\uparrow\downarrow} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V^{\uparrow\downarrow}_{\mathbf{k}'\mathbf{k}}\left[\frac{\overline{\Delta}^{\mathbf{k}'}_{\uparrow\uparrow}+\overline{\Delta}^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{+}_{\mathbf{k}'}} - \frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow}-\Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^{-}_{\mathbf{k}'}}\right]. \tag{4.45}$$

On the other hand, the ground state energy of the system is given by

$$\langle H\rangle \simeq \sum_{\mathbf{k}\alpha\beta}{}' \xi^{k}_{\alpha\beta}\langle\pi_{\mathbf{k}\alpha}\phi_{\mathbf{k}\beta}\rangle + \frac{1}{2\Omega}\sum_{\mathbf{k}_1\neq\mathbf{k}_2}\sum_{\alpha\beta} V^{\alpha\beta}_{\mathbf{k}_1\mathbf{k}_2}\langle\pi_{\mathbf{k}_1\alpha}\pi_{-\mathbf{k}_1\beta}\rangle\langle\phi_{-\mathbf{k}_2\beta}\phi_{\mathbf{k}_2\alpha}\rangle. \tag{4.46}$$

At zero temperature, we obtain

$$\langle\pi_{\mathbf{k}\uparrow}\phi_{\mathbf{k}\uparrow}\rangle = \langle\pi_{\mathbf{k}\downarrow}\phi_{\mathbf{k}\downarrow}\rangle = \frac{E^{+}_{\mathbf{k}}-\xi_k+t}{4E^{+}_{\mathbf{k}}} + \frac{E^{-}_{\mathbf{k}}-\xi_k-t}{4E^{-}_{\mathbf{k}}}, \tag{4.47}$$

$$\langle\pi_{\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle = \langle\pi_{\mathbf{k}\downarrow}\phi_{\mathbf{k}\uparrow}\rangle = \frac{E^{+}_{\mathbf{k}}-\xi_k+t}{4E^{+}_{\mathbf{k}}} - \frac{E^{-}_{\mathbf{k}}-\xi_k-t}{4E^{-}_{\mathbf{k}}}. \tag{4.48}$$

Substituting these formula and gap equations (4.42) and (4.43) into Eq.(4.46), we obtain

$$\begin{aligned}\langle H\rangle \simeq\ & \frac{1}{2}\sum_{\mathbf{k}}(\xi_k-t)\left(1-\frac{\xi_k-t}{E^{+}_{\mathbf{k}}}\right) - \frac{1}{4}\sum_{\mathbf{k}}\frac{\left(\overline{\Delta}^{\mathbf{k}}_{\uparrow\uparrow}+\overline{\Delta}^{\mathbf{k}}_{\uparrow\downarrow}\right)\left(\Delta^{\mathbf{k}}_{\uparrow\uparrow}+\Delta^{\mathbf{k}}_{\uparrow\downarrow}\right)}{E^{+}_{\mathbf{k}}} \\ & + \frac{1}{2}\sum_{\mathbf{k}}(\xi_k+t)\left(1-\frac{\xi_k+t}{E^{-}_{\mathbf{k}}}\right) - \frac{1}{4}\sum_{\mathbf{k}}\frac{\left(\overline{\Delta}^{\mathbf{k}}_{\uparrow\uparrow}-\overline{\Delta}^{\mathbf{k}}_{\uparrow\downarrow}\right)\left(\Delta^{\mathbf{k}}_{\uparrow\uparrow}-\Delta^{\mathbf{k}}_{\uparrow\downarrow}\right)}{E^{-}_{\mathbf{k}}}.\end{aligned} \tag{4.49}$$

In the absence of the interlayer tunneling, Eq.(4.49) becomes

$$\langle H\rangle \simeq \sum_{\mathbf{k}}\xi_k\left(1-\frac{\xi_k}{E_{\mathbf{k}}}\right) - \frac{1}{2}\sum_{\mathbf{k}}\frac{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}. \tag{4.50}$$

Note that Eq.(4.50) is twice of Eq.(3.46) because of the pseudo-spin degrees of freedom.

4.2.2 Singlet Pairing

In the singlet pairing cases, $\Delta^{\mathbf{k}}$ and $\overline{\Delta}^{\mathbf{k}}$ satisfys the following relaitons:

$$\left(\Delta^{-\mathbf{k}}\right)_{\alpha\beta} = \left(\Delta^{\mathbf{k}}\right)_{\alpha\beta}, \qquad \left(\Delta^{\mathbf{k}}\right)_{\beta\alpha} = -\left(\Delta^{\mathbf{k}}\right)_{\alpha\beta}, \tag{4.51}$$

$$\left(\overline{\Delta}^{-\mathbf{k}}\right)_{\alpha\beta} = \left(\overline{\Delta}^{\mathbf{k}}\right)_{\alpha\beta}, \qquad \left(\overline{\Delta}^{\mathbf{k}}\right)_{\beta\alpha} = -\left(\overline{\Delta}^{\mathbf{k}}\right)_{\alpha\beta}. \tag{4.52}$$

Therefore, we can set $\Delta_{\mathbf{k}}$ and $\overline{\Delta}_{\mathbf{k}}$ as

$$\Delta_{\mathbf{k}} = \begin{bmatrix} 0 & \Delta_{\mathbf{k}} \\ -\Delta_{\mathbf{k}} & 0 \end{bmatrix}, \tag{4.53}$$

$$\overline{\Delta}_{\mathbf{k}} = \begin{bmatrix} 0 & -\overline{\Delta}_{\mathbf{k}} \\ \overline{\Delta}_{\mathbf{k}} & 0 \end{bmatrix}. \tag{4.54}$$

The matrix $\mathcal{E}^{\mathbf{k}}$ is given by

$$\mathcal{E}^{\mathbf{k}} = \begin{bmatrix} \xi_k & -t & 0 & -\Delta_{\mathbf{k}} \\ -t & \xi_k & \Delta_{\mathbf{k}} & 0 \\ 0 & \overline{\Delta}_{\mathbf{k}} & -\xi_k & t \\ -\overline{\Delta}_{\mathbf{k}} & 0 & t & -\xi_k \end{bmatrix}. \tag{4.55}$$

The eigenvalues of $\mathcal{E}^{\mathbf{k}}$ are given by $\pm E_{\mathbf{k}} + t$ and $\pm E_{\mathbf{k}} - t$, where

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + \overline{\Delta}_{\mathbf{k}} \Delta_{\mathbf{k}}}. \tag{4.56}$$

The eigenvectors of $\mathcal{E}^{\mathbf{k}}$ are given by

$$\mathbf{u}_{\mathbf{k}}^1 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} - \xi_k)}} \begin{bmatrix} -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \\ E_{\mathbf{k}} - \xi_k \\ E_{\mathbf{k}} - \xi_k \end{bmatrix}, \tag{4.57}$$

$$\mathbf{u}_{\mathbf{k}}^2 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} + \xi_k)}} \begin{bmatrix} -E_{\mathbf{k}} - \xi_k \\ -E_{\mathbf{k}} - \xi_k \\ -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \end{bmatrix}, \tag{4.58}$$

$$\mathbf{u}_{\mathbf{k}}^3 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} - \xi_k)}} \begin{bmatrix} E_{\mathbf{k}} - \xi_k \\ E_{\mathbf{k}} - \xi_k \\ -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \end{bmatrix}, \tag{4.59}$$

$$\mathbf{u}_{\mathbf{k}}^4 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} + \xi_k)}} \begin{bmatrix} -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \\ -E_{\mathbf{k}} - \xi_k \\ -E_{\mathbf{k}} - \xi_k \end{bmatrix}. \tag{4.60}$$

The eigenvalues of Eqs.(4.57),(4.58), (4.59) and (4.60) are $E_{\mathbf{k}} + t$, $E_{\mathbf{k}} - t$, $-E_{\mathbf{k}} - t$ and $-E_{\mathbf{k}} + t$, respectively. The matrix $\mathcal{E}^{\mathbf{k}}$ is diagonalized by matrices $U^{\mathbf{k}}$ and $\overline{U}^{\mathbf{k}}$:

$$U^{\mathbf{k}} = \begin{bmatrix} \mathbf{u}_{\mathbf{k}}^1 & \mathbf{u}_{\mathbf{k}}^2 & \mathbf{u}_{\mathbf{k}}^3 & \mathbf{u}_{\mathbf{k}}^4 \end{bmatrix}, \tag{4.61}$$

$$\overline{U}^{\mathbf{k}} = {}^{\mathrm{T}}\begin{bmatrix} \overline{\mathbf{u}}_{\mathbf{k}}^1 & \overline{\mathbf{u}}_{\mathbf{k}}^2 & \overline{\mathbf{u}}_{\mathbf{k}}^3 & \overline{\mathbf{u}}_{\mathbf{k}}^4 \end{bmatrix}, \tag{4.62}$$

where,

$$\overline{\mathbf{u}}_{\mathbf{k}}^1 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} - \xi_k)}} \begin{bmatrix} -\overline{\Delta}_{\mathbf{k}} \\ \overline{\Delta}_{\mathbf{k}} \\ E_{\mathbf{k}} - \xi_k \\ E_{\mathbf{k}} - \xi_k \end{bmatrix}, \tag{4.63}$$

$$\overline{\mathbf{u}}_{\mathbf{k}}^2 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} + \xi_k)}} \begin{bmatrix} -E_{\mathbf{k}} - \xi_k \\ -E_{\mathbf{k}} - \xi_k \\ -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \end{bmatrix}, \tag{4.64}$$

$$\overline{\mathbf{u}}_{\mathbf{k}}^3 = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}} - \xi_k)}} \begin{bmatrix} E_{\mathbf{k}} - \xi_k \\ E_{\mathbf{k}} - \xi_k \\ -\Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} \end{bmatrix}, \tag{4.65}$$

$$\overline{\mathbf{u}}^4_{\mathbf{k}} = \frac{1}{\sqrt{4E_{\mathbf{k}}(E_{\mathbf{k}}+\xi_k)}} \begin{bmatrix} -\overline{\Delta}_{\mathbf{k}} \\ \overline{\Delta}_{\mathbf{k}} \\ -E_{\mathbf{k}}-\xi_k \\ -E_{\mathbf{k}}-\xi_k \end{bmatrix}. \tag{4.66}$$

As in the triplet pairing case, we introduce quasiparticle fields by Eq.(4.32) and Eq.(4.33). Anti-commutation relation (4.34) also holds in the singlet pairing cases. Using these quasiparticle fields, we obtain

$$H = \sum_{\mathbf{k}}{}' p^{\mathbf{k}} \begin{bmatrix} E_{\mathbf{k}}+t & 0 & 0 & 0 \\ 0 & E_{\mathbf{k}}-t & 0 & 0 \\ 0 & 0 & -E_{\mathbf{k}}-t & 0 \\ 0 & 0 & 0 & -E_{\mathbf{k}}+t \end{bmatrix} q^{\mathbf{k}} + \text{const.}. \tag{4.67}$$

To obtain the gap equations, we calculate values of $\langle \phi_{-\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle$ and $\langle \pi_{\mathbf{k}\downarrow}\pi_{-\mathbf{k}\uparrow}\rangle$. They are given by

$$\langle \phi_{-\mathbf{k}\uparrow}\phi_{\mathbf{k}\downarrow}\rangle = \frac{\Delta_{\mathbf{k}}}{4E_{\mathbf{k}}}\left[\tanh\frac{E_{\mathbf{k}}+t}{2k_BT} + \tanh\frac{E_{\mathbf{k}}-t}{2k_BT}\right], \tag{4.68}$$

$$\langle \pi_{\mathbf{k}\downarrow}\pi_{-\mathbf{k}\uparrow}\rangle = \frac{\overline{\Delta}_{\mathbf{k}}}{4E_{\mathbf{k}}}\left[\tanh\frac{E_{\mathbf{k}}+t}{2k_BT} + \tanh\frac{E_{\mathbf{k}}-t}{2k_BT}\right]. \tag{4.69}$$

Substituting these equations into Eqs.(4.8) and (4.9), we obtain

$$\Delta_{\mathbf{k}} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V^{\uparrow\downarrow}_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}\left[\tanh\frac{E_{\mathbf{k}'}-t}{2k_BT} + \tanh\frac{E_{\mathbf{k}'}+t}{2k_BT}\right], \tag{4.70}$$

$$\overline{\Delta}_{\mathbf{k}} = -\frac{1}{4\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k})} V^{\uparrow\downarrow}_{\mathbf{k}'\mathbf{k}}\frac{\overline{\Delta}_{\mathbf{k}'}}{E_{\mathbf{k}'}}\left[\tanh\frac{E_{\mathbf{k}'}-t}{2k_BT} + \tanh\frac{E_{\mathbf{k}'}+t}{2k_BT}\right]. \tag{4.71}$$

At zero temperature, these equations become

$$\Delta_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k}),E_{\mathbf{k}'}>t} V^{\uparrow\downarrow}_{\mathbf{k}'\mathbf{k}}\frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}, \tag{4.72}$$

$$\overline{\Delta}_{\mathbf{k}} = -\frac{1}{2\Omega}\sum_{\mathbf{k}'(\neq\mathbf{k}),E_{\mathbf{k}'}>t} V^{\uparrow\downarrow}_{\mathbf{k}\mathbf{k}'}\frac{\overline{\Delta}_{\mathbf{k}'}}{E_{\mathbf{k}'}}. \tag{4.73}$$

On the other hand, the ground state energy is given by

$$\langle H\rangle \simeq \frac{1}{2}\sum_{\mathbf{k}}(\xi_k - t)\left(1-\frac{\xi_k}{E_{\mathbf{k}}}\right) + \frac{1}{2}\sum_{\mathbf{k},E_{\mathbf{k}}>t}(\xi_k+t)\left(1-\frac{\xi_k}{E_{\mathbf{k}}}\right) - \frac{1}{2}\sum_{\mathbf{k},E_{\mathbf{k}}>t}\frac{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}. \tag{4.74}$$

In the absence of the interlayer tunneling, Eq.(4.74) corresponds to Eq.(4.50).

4.3 Pure Pairing State

Now we discuss the ground state properties in the absence of the interlayer tunneling and the Coulomb interaction. In that case, the discussion is almost the same as in the single-layer system.

The difference is that we have pseudo-spin degrees of freedom in bilayer systems and must take into account the possibility of various triplet pairings and singlet pairings. As we have done in single-layer systems, we replace the electronic band mass m_b with the effective mass M implicitly in the gap equations to take into account the renormalization effect. M is a parameter in our theory. Furthermore, we neglect V^{NH} term. When we take into account V^{NH} and neglect the Coulomb interaction and the interlayer

tunneling, the gap equations are solved exactly and the resulting state is the gapless pairing state. However, this gapless pairing state is not stabilized because the state with $\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}} \equiv 0$ is the relative maximum of $\langle H \rangle$, which is shown from the analysis of the ground state energy (4.50) (see Sec.3.5). Applying the same discussion in Sec.3.5, V^{NH} is irrelevant for the pairing states of CFs. We discuss the gap equation in the absence of V^{NH}. In that case the gap $\overline{\Delta}_{\mathbf{k}}$ is the complex conjugate of $\Delta_{\mathbf{k}}$. Therefore, we only consider the gap equation for $\Delta_{\mathbf{k}}$.

First, we discuss the gap equation for the triplet pairing state. We assume that each component of $\Delta^{\mathbf{k}}$ has the same angular $\theta_{\mathbf{k}}$ dependence ($\theta_{\mathbf{k}}$ denotes the direction of $\mathbf{k}$). This assumption is introduced for convenience and the result about the pairing of the ground state given below holds without it. In that case, the gap $\Delta^{\mathbf{k}}$ is given by

$$\Delta^{\mathbf{k}} = \Delta_{\mathbf{k}} \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{4.75}$$

where a and b are complex numbers satisfying $|a|^2 + |b|^2 = 1$. Furthermore, we set

$$\Delta_{\mathbf{k}} = e^{-i\ell\theta_{\mathbf{k}}} \Delta_k \tag{4.76}$$

for the ℓ-wave pairing state. As we will see later, the pairing state with $(a, b) = (0, 1)$ corresponds to the Halperin (m, m, n) state, which is, in the absence of the interlayer tunneling, numerically shown to be the ground state of bilayer systems with an appropriate interlayer separation d [15, 26]. Hence we set $(a, b) = (0, 1)$.

The gap equations are almost the same as in the single-layer systems. In fact, the gap equation is given by

$$\Delta_k = \frac{\phi_2}{2M} \int_0^k dk' \frac{k'\Delta_{k'}}{E_{k'}} \left(\frac{k'}{k}\right)^\ell + \frac{\phi_2}{2M} \int_k^\infty dk' \frac{k'\Delta_{k'}}{E_{k'}} \left(\frac{k}{k'}\right)^\ell. \tag{4.77}$$

Note that this equation is obtained by replacing ϕ in Eq.(3.47) with ϕ_2.

For the singlet pairing cases with $\ell \geq 2$, the gap equation is given by the same equation (4.77) if we assume Eq.(4.76). The s-wave pairing is not possible in bilayer systems because of the following reason. As discussed in Sec.3.2, we do not have attractive interaction from V^H because $I^H_{\ell=0}(\lambda) \equiv 0$. Therefore, the possibility of an attractive interaction is caused by V^{NH}. In that case, we must take into account the gap equation of $\overline{\Delta}_{\mathbf{k}}$, and the gap equations are given by

$$\Delta_k = \frac{\phi_2}{2M} \left[\int_0^k dk' \frac{k'\Delta_{k'}}{E_{k'}} - \int_k^\infty dk' \frac{k'\Delta_{k'}}{E_{k'}} \right]. \tag{4.78}$$

$$\Delta_k = \frac{\phi_2}{2M} \left[-\int_0^k dk' \frac{k'\Delta_{k'}}{E_{k'}} + \int_k^\infty dk' \frac{k'\Delta_{k'}}{E_{k'}} \right]. \tag{4.79}$$

Taking similar steps in Sec.3.5, we can solve Eqs.(4.78) and (4.79) exactly and obtain $\Delta_k \equiv 0$ and $\overline{\Delta}_k \equiv 0$. Therefore, the s-wave pairing is not possible in bilayer systems. This fact is understood intuitively because the attractive interaction between CFs is caused by the Lorentz force of Chern-Simons gauge field (see Sec.2.4).

To solve the gap equation (4.77), we introduce the same approximation Eq.(3.50) for Δ_k. With this approximation, the gap equation has the same form as in the case of the single-layer case and is given by

$$\frac{1}{\phi_2} = \int_0^1 dx \frac{x^{2\ell+1}}{\sqrt{(x^2-1)^2 + \Delta^2 x^{2\ell}}} + \int_1^\infty dx \frac{x^{1-2\ell}}{\sqrt{(x^2-1)^2 + \Delta^2 x^{-2\ell}}} \equiv F_\ell(\Delta) \tag{4.80}$$

both for singlet- and triplet-pairings. In Fig.4.1, we show the function $F_\ell(\Delta)$ for various ℓ and the horizontal line corresponds to $1/\phi_2 = 0.5$. Although the form of the gap equation is the same as in the single-layer case, we must take into account the possibility of the singlet pairing cases. However, the gap of the singlet pairing states has lower value than that of the triplet p-wave pairing state. Therefore, the pairing state with the largest gap is the p-wave pairing state.

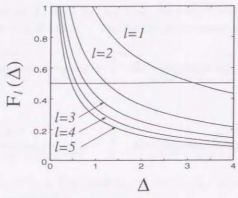

Figure 4.1: The gap dependence of the right hand side of Eq.(4.80).

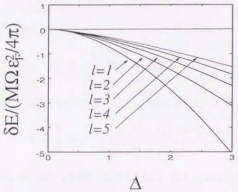

Figure 4.2: The energy difference between the pairing state and the no-pairing state.

On the other hand, the energy difference between the pairing state and the no-pairing state is shown in Fig.4.2, where δE is given by $\delta E = \langle H \rangle_{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}} - \langle H \rangle_{\overline{\Delta}_{\mathbf{k}}\Delta_{\mathbf{k}}=0}$ with $\langle H \rangle$ being given by Eq.(4.50). The value of δE has always negative value and the ground state is the pairing state as long as $\overline{\Delta}\Delta \neq 0$. Furthermore, the value of ℓ which gives the lowest δE is $\ell = 1$. From the discussion of the gap equations, we have found that the pairing state with the largest gap Δ is $\ell = 1$. Putting it all together, the ground state is the triplet p-wave pairing state of CFs.

4.4 Effect of Interlayer Tunneling

Let us discuss the effect of an interlayer tunneling. Though we can apply the discussion below to general cases, we concentrate on the case of $\nu = 1/2$ and $\nu = 1$. As we have seen in the previous section, anti-Hermitian term V^{NH} is irrelevant for pairing states of CFs. Therefore, we neglect it. Furthermore, we neglect the Coulomb interaction. The effect of the Coulomb interaction is discussed later.

In that case, the gap equations are given by

$$\Delta^{\mathbf{k}}_{\uparrow\uparrow} = -\frac{\phi_1}{4\Omega} \sum_{\mathbf{k}'(\neq \mathbf{k})} V^H_{\mathbf{k}\mathbf{k}'} \left[\frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow} + \Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^+_{\mathbf{k}'}} + \frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow} - \Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^-_{\mathbf{k}'}} \right], \tag{4.81}$$

$$\Delta^{\mathbf{k}}_{\uparrow\downarrow} = -\frac{\phi_2}{4\Omega} \sum_{\mathbf{k}'(\neq \mathbf{k})} V^H_{\mathbf{k}\mathbf{k}'} \left[\frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow} + \Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^+_{\mathbf{k}'}} - \frac{\Delta^{\mathbf{k}'}_{\uparrow\uparrow} - \Delta^{\mathbf{k}'}_{\uparrow\downarrow}}{E^-_{\mathbf{k}'}} \right]. \tag{4.82}$$

The gap function $\overline{\Delta}^{\mathbf{k}}_{\alpha\beta}$ is the complex conjugate of $\Delta^{\mathbf{k}}_{\alpha\beta}$ because we neglect the anti-Hermitian term V^{NH}.

First, we consider the case of $\phi_1 = \phi_2 = \phi$. In this case, the total Landau level filling fraction is given by $\nu = 1/\phi$. It includes the case of $\nu = 1/2, 1/4, \cdots$. Taking the same steps in the last section, we obtain

$$\int_0^1 dx \frac{x^{2\ell+1}}{\sqrt{(x^2-1-\tau)^2+|\Delta_+|^2 x^{2\ell}}} + \int_1^\infty dx \frac{x^{1-2\ell}}{\sqrt{(x^2-1-\tau)^2+|\Delta_+|^2 x^{-2\ell}}} = \frac{1}{\phi}, \tag{4.83}$$

for $a+b \neq 0$ and

$$\int_0^1 dx \frac{x^{2\ell+1}}{\sqrt{(x^2-1+\tau)^2+|\Delta_-|^2 x^{2\ell}}} + \int_1^\infty dx \frac{x^{1-2\ell}}{\sqrt{(x^2-1+\tau)^2+|\Delta_-|^2 x^{-2\ell}}} = \frac{1}{\phi}, \tag{4.84}$$

for $a-b \neq 0$. Here $\tau \equiv t/\epsilon_F$, $\Delta_+ \equiv (a+b)\Delta$, and $\Delta_- \equiv (a-b)\Delta$. To observe the change of the ground state properties, we introduce an angular θ by

$$\theta = \tan^{-1}\frac{a}{b}. \tag{4.85}$$

The gap for the ground state is defined by

$$\Delta = \sqrt{\frac{|\Delta_+|^2+|\Delta_-|^2}{2}}. \tag{4.86}$$

Solving Eqs.(4.83) and (4.84) numerically, we obtain $|\Delta_+|$ and $|\Delta_-|$. The θ is calculated by

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{|\Delta_+|^2-|\Delta_-|^2}{|\Delta_+|^2+|\Delta_-|^2}\right). \tag{4.87}$$

On the other hand, from Eqs.(4.49), (4.75), (4.76) and (3.50), the ground state energy of the triplet pairing state is given by

$$\begin{aligned}
\langle H\rangle \;=\; & -\frac{M\Omega}{4\pi}\ell_F^2\overline{\Delta}\Delta\left[\int_0^1 dx\, \frac{x^{2\ell+1}}{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{2\ell}}}\cdot\frac{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{2\ell}}-x^2+1+\tau}{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{2\ell}}+x^2+1+\tau}\right. \\
& + \int_1^\infty dx\, \frac{x^{-2\ell+1}}{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{-2\ell}}}\cdot\frac{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{-2\ell}}-x^2+1+\tau}{\sqrt{(x^2-1-\tau)^2+\overline{\Delta^+}\Delta^+x^{-2\ell}}+x^2-1-\tau} \\
& + \int_0^1 dx\, \frac{x^{2\ell+1}}{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{2\ell}}}\cdot\frac{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{2\ell}}-x^2+1-\tau}{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{2\ell}}+x^2+1-\tau} \\
& \left. + \int_1^\infty dx\, \frac{x^{-2\ell+1}}{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{-2\ell}}}\cdot\frac{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{-2\ell}}-x^2+1-\tau}{\sqrt{(x^2-1+\tau)^2+\overline{\Delta^-}\Delta^-x^{-2\ell}}+x^2-1+\tau}\right].
\end{aligned} \tag{4.88}$$

Hereafter we concentrate on the case of the p-wave pairing case and set $\ell = 1$ because we have shown in the last section that the ground state is the p-wave pairing state in the absence of an interlayer tunneling. In Fig.4.3, we show the tunneling dependence of θ at $\nu = 1/2$ ($\phi = 2$). We see that the $(3,3,1)$ state, which is the p-wave pairing state with $(a,b) = (0,1)$ ($\theta = 0$), evolves toward the Pfaffian state, which is the p-wave pairing state with $(a,b) = (1/\sqrt{2}, 1/\sqrt{2})$ ($\theta = \pi/4$). In the region $\tau \geq 2$, we do not have the solution of Eq.(4.84). Therefore, $a \equiv b$. As a result, the ground state is the Pfaffian state in the region $\tau \geq 2$.

Now we discuss whether the $(3,3,1)$ state and the Pfaffian state belong to the different phases or not. Ho addressed that the $(3,3,1)$ state and the Pfaffian state are continuously connected each other [25]. In Fig.4.4, we show the tunneling dependence of the gap Δ. As shown in Fig.4.4, there is a cusp at $\tau = 2$. There is also a cusp in the ground state energy. In Fig.4.5, we show the τ dependence of the energy $E = \langle H\rangle$, which is calculated from Eq.(4.88). We see that there is a cusp at $\tau = 2$. This is an indication that the

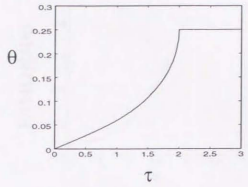

Figure 4.3: The tunneling dependence of θ at $\nu = 1/2$.

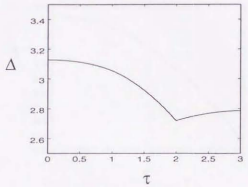

Figure 4.4: The tunneling dependence of the gap Δ at $\nu = 1/2$.

$(3,3,1)$ state and the Pfaffian state are different phases. In Ref.[47], the ground state degeneracy on a Torus of the $(3,3,1)$ state and the Pfaffian state was discussed. It was shown that the ground state degeneracy of the $(3,3,1)$ state and that of the Pfaffian state are different. Therefore, the $(3,3,1)$ state and the Pfaffian state are different phases.

Let us turn to the case of $\phi_1 = 0$ and $\phi_2 = 2 \neq 0$, that is, $\nu = 1$. In this case, (a,b) is uniquely determined as $(0,1)$ because $\phi_1 = 0$ and the ground state is uniquely given by the state continuously connected with the $(1,1,1)$ state. Then, the gap equation is given by

$$\int_0^1 dx \left[\frac{x^{2\ell+1}}{(x^2-1-t)^2+\Delta^2 x^{2\ell}} + \frac{x^{2\ell+1}}{(x^2-1+t)^2+\Delta^2 x^{2\ell}} \right.$$
$$\left. \frac{x^{2\ell-1}}{\sqrt{((1+t)x^2-1)^2+\Delta^2 x^{2\ell+4}}} + \frac{x^{2\ell-1}}{\sqrt{((1-t)x^2-1)^2+\Delta^2 x^{2\ell+4}}} \right] = 1. \tag{4.89}$$

In Fig.4.6, we show the tunneling dependence of the gap Δ for the p-wave pairing state. We do not have any cusp. Therefore, the ground state is continuously connected with the $(1,1,1)$ state.

4.5 Wave Function

Now we discuss the ground state wave function. When we set the ground state wave function as $|GS\rangle$, the ket $|GS\rangle$ satisfy the following equations:

$$q_{\mathbf{k}_1}|GS\rangle = 0, \qquad q_{\mathbf{k}_1}|GS\rangle = 0, \tag{4.90}$$

$$q_{-\mathbf{k}_1}|GS\rangle = 0, \qquad q_{-\mathbf{k}_1}|GS\rangle = 0. \tag{4.91}$$

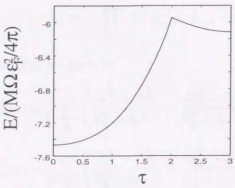

Figure 4.5: The tunneling dependence of E at $\nu = 1/2$. There is a cusp at $\tau = 2$.

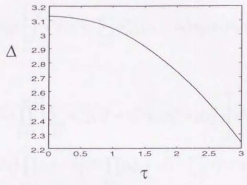

Figure 4.6: The tunneling dependence of the gap Δ at $\nu = 1$.

From Eqs.(4.32) and (4.33), these equations are described by

$$\left[\left(\overline{\mathbf{u}}^j_{\mathbf{k}}\right)_1 \phi_{\mathbf{k}\uparrow} + \left(\overline{\mathbf{u}}^j_{\mathbf{k}}\right)_2 \phi_{\mathbf{k}\downarrow} + \left(\overline{\mathbf{u}}^j_{\mathbf{k}}\right)_3 \pi_{-\mathbf{k}\uparrow} + \left(\overline{\mathbf{u}}^j_{\mathbf{k}}\right)_4 \pi_{-\mathbf{k}\downarrow}\right] |GS\rangle = 0, \quad \text{for} \quad j = 1, 2 \tag{4.92}$$

$$\left[\left(\mathbf{u}^j_{\mathbf{k}}\right)_1 \pi_{\mathbf{k}\uparrow} + \left(\mathbf{u}^j_{\mathbf{k}}\right)_2 \pi_{\mathbf{k}\downarrow} + \left(\mathbf{u}^j_{\mathbf{k}}\right)_3 \phi_{-\mathbf{k}\uparrow} + \left(\mathbf{u}^j_{\mathbf{k}}\right)_4 \phi_{-\mathbf{k}\downarrow}\right] |GS\rangle = 0. \quad \text{for} \quad j = 3, 4 \tag{4.93}$$

The Grassmannian operator $\pi_{\mathbf{k}\alpha}$ and $\phi_{\mathbf{k}\alpha}$ satisfy the anti-commutation relation by replacing $\phi_{\mathbf{k}\alpha}$ with $\partial/\partial\pi_{\mathbf{k}\alpha}$. Hence Eqs.(4.92) and (4.93) are the differential equations with respect to the Grassmannian variables $\pi_{\mathbf{k}\alpha}$ and $\pi_{-\mathbf{k}\alpha}$. Solving these differential equations, we obtain

$$\begin{aligned}
|GS\rangle &= \exp\left(\sum_{\mathbf{k}}{}' \frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} + \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^+_{\mathbf{k}} + \xi_k - t} \left(\pi_{\mathbf{k}\uparrow} + \pi_{\mathbf{k}\downarrow}\right)\left(\pi_{-\mathbf{k}\uparrow} + \pi_{-\mathbf{k}\downarrow}\right)\right. \\
&\quad \left. - \sum_{\mathbf{k}}{}' \frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} - \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^-_{\mathbf{k}} + \xi_k + t} \left(\pi_{\mathbf{k}\uparrow} - \pi_{\mathbf{k}\downarrow}\right)\left(\pi_{-\mathbf{k}\uparrow} - \pi_{-\mathbf{k}\downarrow}\right)\right)|0\rangle \\
&= \exp\left[\sum_{\mathbf{k}}{}' \left\{\left(\frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} + \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^+_{\mathbf{k}} + \xi_k - t} - \frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} - \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^-_{\mathbf{k}} + \xi_k + t}\right)\left(\pi_{\mathbf{k}\uparrow}\pi_{-\mathbf{k}\uparrow} + \pi_{\mathbf{k}\downarrow}\pi_{-\mathbf{k}\downarrow}\right)\right.\right. \\
&\quad \left.\left. + \left(\frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} + \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^+_{\mathbf{k}} + \xi_k - t} + \frac{\Delta^{\mathbf{k}}_{\uparrow\uparrow} - \Delta^{\mathbf{k}}_{\uparrow\downarrow}}{E^-_{\mathbf{k}} + \xi_k + t}\right)\left(\pi_{\mathbf{k}\uparrow}\pi_{-\mathbf{k}\downarrow} + \pi_{\mathbf{k}\downarrow}\pi_{-\mathbf{k}\uparrow}\right)\right\}\right]|0\rangle.
\end{aligned} \tag{4.94}$$

From Eq.(4.94), the first quantized wave function of $|GS\rangle$ for $2N$ CFs in each layer is given by

$$\Psi_{CF}(z^{\uparrow}_1, z^{\uparrow}_2, \cdots, z^{\uparrow}_{2N}; z^{\downarrow}_1, z^{\downarrow}_2, \cdots, z^{\downarrow}_{2N}) = \langle 0|\phi_{orb}(z^{\downarrow}_{2N}) \cdots \phi_{orb}(z^{\downarrow}_1)\phi_{orb}(z^{\uparrow}_{2N}) \cdots \phi_{orb}(z^{\uparrow}_1)|GS\rangle$$

$$= \prod_{\alpha,\beta=\uparrow,\downarrow} \mathrm{Pf}\, \phi_{orb}^{\alpha\beta}\left(z_{2j-1}^{\alpha} - z_{2j}^{\beta}\right), \tag{4.95}$$

where the orbital wave functions: $\phi_{orb}^{\alpha\beta}(z)$ are given by

$$\phi_{orb}^{\uparrow\uparrow}(z) = \phi_{orb}^{\downarrow\downarrow}(z) = \frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}} + \Delta_{\uparrow\downarrow}^{\mathbf{k}}}{E_{\mathbf{k}}^{+} + \xi_k - t} - \frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}} - \Delta_{\uparrow\downarrow}^{\mathbf{k}}}{E_{\mathbf{k}}^{-} + \xi_k + t}\right) e^{i\mathbf{k}\,\mathbf{r}}, \tag{4.96}$$

$$\phi_{orb}^{\uparrow\downarrow}(z) = \phi_{orb}^{\downarrow\uparrow}(z) = \frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}} + \Delta_{\uparrow\downarrow}^{\mathbf{k}}}{E_{\mathbf{k}}^{+} + \xi_k - t} + \frac{\Delta_{\uparrow\uparrow}^{\mathbf{k}} - \Delta_{\uparrow\downarrow}^{\mathbf{k}}}{E_{\mathbf{k}}^{-} + \xi_k + t}\right) e^{i\mathbf{k}\,\mathbf{r}}. \tag{4.97}$$

Performing the Fourier transformation and substituting Eqs.(4.96) and (4.97) into Eq.(4.94), we obtain

$$|GS\rangle = \exp\left[\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \sum_{\alpha\beta} \phi_{orb}^{\alpha\beta}(z_1 - z_2)\pi_\alpha(\mathbf{r}_1)\pi_\beta(\mathbf{r}_2)\right]|0\rangle. \tag{4.98}$$

For $2N$ particles, we obtain

$$\begin{aligned}
|GS\rangle_{2N} &= \int\prod_{j=1}^{N} d^2z_j^{\uparrow} d^2z_j^{\downarrow} \prod_{j=1}^{N}\left[\sum_{\alpha\beta}\phi_{orb}^{\alpha\beta}(z_j^{\alpha} - z_j^{\beta})\pi_\alpha(z_j^{\alpha})\pi_\beta(z_j^{\beta})\right]|0\rangle \\
&= \int\prod_{j=1}^{N} d^2z_j^{\uparrow}\, d^2z_j^{\downarrow} \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^{\phi_1}\prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})^{\phi_1}\prod_{i,j}(z_i^{\downarrow} - z_j^{\uparrow})^{\phi_2} e^{-\frac{1}{4\ell_B^2}\sum_{j=1}^{2N}(|z_j^{\uparrow}|^2 + |z_j^{\downarrow}|^2)} \\
&\quad\times\prod_{j=1}^{N}\left[\sum_{\alpha\beta}\phi_{orb}^{\alpha\beta}(z_j^{\alpha} - z_j^{\beta})\psi_\alpha^{\dagger}(z_j^{\alpha})\psi_\beta^{\dagger}(z_j^{\beta})\right]|0\rangle.
\end{aligned} \tag{4.99}$$

The orbital wave function $\phi_{orb}^{\alpha\beta}(z)$ is calculated as in Sec.3.4.2. In the absence of the interlayer tunneling, we obtain $\phi_{orb}^{\uparrow\downarrow}(z) = \phi_{orb}^{\downarrow\uparrow}(z) \sim 1/z$ and $\phi_{orb}^{\uparrow\uparrow}(z) = \phi_{orb}^{\downarrow\downarrow}(z) = 0$. Therefore, in that case the ground state wave function is the $(\phi_1 + 1, \phi_1 + 1, \phi_2 - 1)$ state because

$$\mathrm{Pf}\left(\frac{1}{z_{2j-1}^{\downarrow} - z_{2j}^{\downarrow}}\right) = -\prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^1\prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})^1\prod_{i,j}(z_i^{\uparrow} - z_j^{\downarrow})^{-1}. \tag{4.100}$$

This equation is shown by using the Cauchy identity[25].

4.6 Stability of Pairing States

Next we discuss the Coulomb interaction effect in the absence of the interlayer tunneling. As was pointed out by Haldane and Rezayi [20] and Ho [25], the p-wave fermion BCS pairing state with $(a, b) = (0, 1)$ is equivalent to the $(1, 1, -1)$ state. Therefore, the p-wave pairing state of CF's with (ϕ_1, ϕ_2) corresponds to the Halperin $(\phi_1 + 1, \phi_1 + 1, \phi_2 - 1)$ state. It is supported numerically [15, 26] that this state is the ground state of the bilayer quantum Hall systems in the absence of the interlayer tunneling and with an appropriate inter layer separation d. Hence in order to discuss whether the pairing state is stabilized with respect to the Coulomb interaction or not, we examine the stability of the Halperin (m, m, n) state. Since the Halperin (m, m, n) wave function is of Jastrow-type, we can get deep physical insight from it. As in the ^{4}He system, this Jastrow-factors are factorized into two parts: a short-range component and a long-range component [48]. The former is determined by the two-body problem and the latter by the phonon effect. However, phonon modes are pushed up to high energy modes because of the incompressible nature of quantum Hall systems. Hence the Jastrow factor is completely determined from the consideration of the

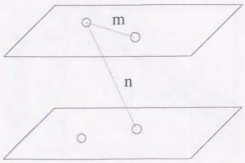

Figure 4.7: We assume that the relative angular momentum for each pair of electrons in the same layer is m and the relative angular momentum for each pair of electrons in the opposite layer is n.

two-body problem. In that sense, the index $m(n)$ is identified as the relative angular momentum of pair of electrons in the same(opposite)-layer. Therefore, to understand the effect of the Coulomb interaction, we focus on the short-range two-body correlation.

The appropriate basis for the two-body electron correlation is given by

$$\langle z_1, z_2|\psi_m\rangle = \frac{1}{\sqrt{4^{m+1}\ell_B^{2m+4}\pi^2 m!}}(z_1 - z_2)^m \, \mathrm{e}^{-\frac{1}{4\ell_B^2}(|z_1|^2+|z_2|^2)}. \tag{4.101}$$

This is the lowest Landau level wave function for an electron pair with the relative angular momentum m and the angular momentum of the central motion being zero. Suppose N electrons in each layer. We assume the relative angular momentum m for each pair of electrons in the same layer and the relative angular momentum n for each pair of electrons in opposite layer (see Fig.4.7). The total Coulomb energy $E_C^{(2)}(m,n)$ estimated by the first order perturbation is given by

$$E_C^{(2)}(m,n) = \frac{N(N-1)}{2} \times \epsilon(m, d=0) \times 2 + N^2 \times \epsilon(n,d) \tag{4.102}$$

where

$$\begin{aligned} \epsilon(m,d) &= \langle\psi_m|\frac{e^2}{\epsilon\sqrt{r^2+d^2}}|\psi_m\rangle \\ &= \frac{e^2}{\epsilon\ell_B} \times \frac{1}{m!}\int_0^\infty dx \frac{x^{2m+1}}{\sqrt{x^2+\lambda^2}} e^{-x^2} \end{aligned} \tag{4.103}$$

with $\lambda \equiv d/2\ell_B$. Taking the thermodynamic limit $N \to \infty$ in Eq.(4.102), we obtain

$$E_C^{(2)}(m,n)/N^2 = \epsilon(m, d=0) + \epsilon(n,d). \tag{4.104}$$

It is our purpose to find the pair (m,n), which gives the lowest $E_C^{(2)}(m,n)$. However, we cannot choose arbitrary pair of (m,n). There is a constraint for the choice of m and n. It is shown that the angular momentum of the electron at the edge of the sample is equal to $(N-1)\times m + N \times n(\equiv M)$ from the Halperin (m,m,n) wave function. Since the wave function of this electron is proportional to $z^M e^{-r^2/4\ell_B^2}$, the density of it has its maximum at $r = \sqrt{2M}\ell_B(\equiv R)$. Of course πR^2 is the area of the system. Taking the thermodynamic limit $N \to \infty$, we obtain

$$2\pi\ell_B^2 \times N \times (m+n) = \Omega. \tag{4.105}$$

Substituting $N/\Omega = \rho/2$ and $\nu = \rho/(B/\phi_0)$ into Eq.(4.105), we obtain

$$m + n = 2/\nu. \tag{4.106}$$

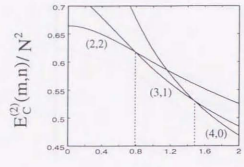

Figure 4.8: The energy $E_C^{(2)}(m,n)$ for $(m,n) = (4,0), (3,1)$, and $(2,2)$ in units of $e^2/\epsilon\ell_B$. In the region $0.789 < d/2\ell_B < 1.480$, the choice of $(m,n) = (3,1)$ has the lowest energy. Therefore, the Halperin $(3,3,1)$ state is stabilized in this region. In other regions, the pairing states are not stabilized.

Now we seek the (m,n) which gives the lowest $E_C^{(2)}(m,n)$ under the condition (4.106). The analytical form of the function $\epsilon(m,d)$ is given by differentiating the function $\int_0^\infty dx \ \frac{x^m}{\sqrt{x^2+\lambda^2}} e^{-\alpha x^2} = \alpha^{-1/2} e^{\alpha\lambda} \mathrm{Erfc}(x)$ with respect to α and setting $\alpha = 1$. Here the function $\mathrm{Efrc}(x) = \int_x^\infty dt \ e^{-t^2}$ is the error function. The explicit forms of function $\epsilon(m,d)$ for various m are given by

$$\epsilon(m=0,d)/(e^2/\epsilon\ell_B) = e^{\lambda^2}\mathrm{Erfc}(\lambda), \tag{4.107}$$

$$\epsilon(m=1,d)/(e^2/\epsilon\ell_B) = \frac{1}{2}\left[(1-2\lambda^2)e^{\lambda^2}\mathrm{Erfc}(\lambda)+\lambda\right], \tag{4.108}$$

$$\epsilon(m=2,d)/(e^2/\epsilon\ell_B) = \frac{1}{8}\left[(3-4\lambda^2+4\lambda^4)e^{\lambda^2}\mathrm{Erfc}(\lambda)+(3\lambda-2\lambda^3)\right], \tag{4.109}$$

$$\epsilon(m=3,d)/(e^2/\epsilon\ell_B) = \frac{1}{48}\left[(15-18\lambda^2+12\lambda^4-8\lambda^6)e^{\lambda^2}\mathrm{Erfc}(\lambda)+(15\lambda-8\lambda^3+4\lambda^5)\right], \tag{4.110}$$

$$\epsilon(m=4,d)/(e^2/\epsilon\ell_B) = \frac{1}{384}\left[(105-120\lambda^2+72\lambda^4-32\lambda^6+16\lambda^8)e^{\lambda^2}\mathrm{Erfc}(\lambda) + (105\lambda-50\lambda^3+20\lambda^5-8\lambda^7)\right]. \tag{4.111}$$

The function $\epsilon(m,d)$ has the simple form at $d=0$:

$$\epsilon(m,d=0)/(e^2/\epsilon\ell_B) = \frac{\Gamma(m+\frac{1}{2})}{2n!} = \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}. \tag{4.112}$$

Let us estimate $E_C^{(2)}(m,n)$ for the case of $\nu = 1/2$ and $\nu = 1$. For the case of $\nu = 1/2$, the constraint (4.106) is $m+n = 4$. Therefore, the possible choice for (m,n) is, $(4,0)$, $(3,1)$ and $(2,2)$. The pair with $m < n$ always has larger energy than that with $m \geq n$. In Fig.4.8, we show the energy $E_C^{(2)}(m,n)/N^2$ for $(m,n) = (4,0), (3,1)$, and $(2,2)$. The region where the choice of $(m,n) = (3,1)$ gives the lowest energy $E_C^{(2)}(m,n)$ is $0.789 < d/2\ell_B < 1.480$. The Halperin $(3,3,1)$ state is stabilized in this region. In other regions, the pairing states are not stabilized. When we start with CFs with $(\phi_1,\phi_2) = (2,2)$, the p-wave pairing occurs in the region $0.789 < d/2\ell_B < 1.480$. However, the pairing state dose not occur in the region $d/2\ell_B < 0.789$ and $d/2\ell_B > 1.480$ and the ground state is a compressible state. [1]

For the case of $\nu = 1$, the constraint (4.106) is $m+n = 2$. Therefore, the possible choice for (m,n) is $(2,0)$ and $(1,1)$. In Fig.4.9, we show the energy $E_C^{(2)}(m,n)/N^2$ for $(2,0)$ and $(1,1)$. The region where the choice of $(m,n) = (1,1)$ gives the lowest energy $E_C^{(2)}(m,n)$ is $0 < d/2\ell_B < 0.703 (\equiv d_c)$. The Halperin $(1,1,1)$ state is stabilized in this region. In another region, the pairing states are not stabilized. The Halperin $(1,1,1)$

[1] The choice of $(\phi_1,\phi_2) = (2,2)$ is not appropriate in the region $d/2\ell_B > 1.480$. If we deal with the state in this region by CF pairing theory, it is natural to choose $(\phi_1,\phi_2) = (4,0)$. However, it is unclear whether the pairing state of such CFs corresponds to the Halperin $(3,3,1)$ state or not.

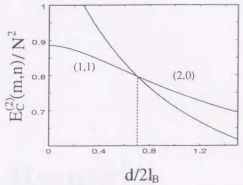

Figure 4.9: The energy $E_C^{(2)}(m,n)$ for $(m,n) = (2,0)$, and $(1,1)$ in units of $e^2/\epsilon\ell_B$. In the region $0 < d/2\ell_B < 0.703$, the choice of $(m,n) = (1,1)$ has the lowest energy. Therefore, the Halperin $(1,1,1)$ state is stabilized in this region. In another region, the pairing states are not stabilized.

state corresponds to the p-wave pairing state of CFs with $(\phi_1, \phi_2) = (0,2)$. Though the above estimation is crude, the critical value d_c for $\nu = 1$ is close to $2\ell_B$ which was obtained by Murphy *et al.* experimentally [49].

In general, the estimation of $E_C^{(2)}(m,n)$ shows that the pair (m,n) giving the lowest $E_C^{(2)}(m,n)$ is $(2/\nu, 0)$ for $d \gg \ell_B$. With decreasing d, it changes as $(2/\nu, 0) \to (2/\nu - 1, 1) \to \cdots \to (1/\nu, 1/\nu)$. In this sequence, there appear the regions where $(m,n) = (\text{even}, \text{even})$. In these regions, the p-wave pairing state is not stabilized. The Jastrow-factor with $(m,n) = (\text{even}, \text{even})$ must be multiplied by some wave function of a no-pairing state of CFs instead of the p-wave pairing state of CFs. On the other hand, in the region of d with $(m,n) = (\text{odd}, \text{odd})$, the ground state is the p-wave pairing state, that is, the quantum Hall state.

4.7 Summary

In this chapter, we have considered the pairing state of CFs in bilayer systems. In general conditions we have derived the gap equations at the total Landau level filling $\nu = 1/m$, where m is an integer.

In bilayer systems, we must take into account not only the possibility of the triplet pairing states but also the possibility of the singlet pairing states. However, they are higher energy state than the p-wave pairing state. Furthermore, the possibility of the s-wave pairing state has been excluded. In the absence of the interlayer tunneling, we have shown that the ground state is the p-wave pairing state of CFs. Therefore, the p-wave pairing state is possible not only at $\nu = 1/2$ but also at more general filling $\nu = 1/m$, where m is an integer.

With regard to the effect of the interlayer tunneling, we consider the $\nu = 1/2$ case and the $\nu = 1$ case. At $\nu = 1/2$, the ground state evolves from the Halperin $(3,3,1)$ state toward the Pfaffian state. However, there is a cusp at the transition point. It indicates that the $(3,3,1)$ state and the Pfaffian state are different phase. On the other hand, at $\nu = 1$ the ground state is uniquely determined independent of the strength of an interlayer tunneling. The ground state is continuously connected with the $(1,1,1)$ state.

The effect of the Coulomb interaction and the stability of the pairing state are also discussed. The p-wave pairing state of CFs corresponds to the Halperin (m,m,n) state. The wave function of the Halperin (m,m,n) state is of Jastrow type. However, because of the incompressible nature of quantum Hall liquids only the short range correlation effect is relevant. The most fundamental short range correlation in the quantum Hall systems is the motion of pair with non-zero relative angular momentum. With regard to the two-body correlation, we have calculated the total energy of the Coulomb interaction by the first order perturbation and estimated the region of the interlayer separation d, where the pairing state is stabilized.

Chapter 5

Concluding Remarks

5.1 Conclusions

In conclusion, we have discussed the pairing state of CFs in the single-layer systems at $\nu = 1/m$, where m is an even integer, and in the bilayer systems at $\nu = 1/n$, where n is an integer.

The Hamiltonian which correctly describe the pairing state has been derived by performing a non-unitary transformation, which is an extended version of the Chern-Simons gauge transformation. By this transformation, we have two two-body potentials. One is an Hermitian term and another is an anti-Hermitian term. An attractive interaction between CFs is caused by the Hermitian term, which have the form of the minimal coupling between the CF current and the Chern-Simons gauge field. Therefore, it has a form of the Lorentz force. The pairing motion of CF pairs are derived by this attractive interaction. The angular momentum of it and that of the cyclotron motion are in the opposite direction. On the other hand, the anti-Hermitian term has no classical meaning.

First, we have discussed the possibility of the pairing state in the single-layer systems. Applying the pairing approximation, the gap equations have been derived at $\nu = 1/m$, where m is an even integer. In the absence of the Coulomb interaction and the anti-Hermitian term, we have shown that the ground state of the system is the p-wave pairing state of CFs. With regard to the anti-Hermitian term, we have discussed that it has a pair-breaking effect. The anti-Hermitian term corresponds to an imaginary vector potential. The effect of an imaginary vector potential was considered in the localization-delocalization phenomena [41]. The role of it is the delocalization effect, which corresponds to the pair-breaking effect in the CF pairing theory. To examine this point, we have taken it into account in the gap equations and solve them exactly. The solution is the gapless pairing state. However, this state is on an unstable point. If we take into account the Coulomb interaction, the ground state of the system changes from the gapless pairing state to the gapful pairing state.

On the other hand, the anti-Hermitian term corresponds to the three-body term in the Chern-Simons gauge theory and it was shown that it is irrelevant in the bosonic Chern-Simons gauge theory [44]. The irrelevance of it is also shown by taking the same steps in Ref.[44] because we can deal with the superconducting state of fermions as the superconducting state of bosons by the Ginzburg-Landau theory. Therefore, we can neglect it in the discussion of the pairing states of CFs.

When we solve the gap equation taking into account the Coulomb interaction, the gap decreases with increasing the ratio of Coulomb interaction to the Fermi energy of CFs. However, we have shown that the gap still have a finite value if the system is in the strong magnetic field limit. Hence the pairing state, which results in the quantum Hall effect, is possible in that situation. We have also discussed the effect of the real spin degrees of freedom and the Zeeman energy. From the analysis of the ground state energy, we have shown that the ground state energy of the spin unpolarized pairing state is higher than that of the spin polarized pairing state. Hence the pairing state is the spin polarized pairing state. The effect of the Zeeman energy to it is nothing but shifting the chemical potential of CFs.

Second, we have discussed the pairing state of CFs in the bilayer systems at $\nu = 1/m$, where m is an integer. We have derived the gap equations for both the triplet pairing states and the singlet pairing states.

In the absence of the Coulomb interaction and the interlayer tunneling, the ground state of the system is the p-wave pairing state. Hence not only at $\nu = 1/2$ but also at other filling fractions $\nu = 1/m$ the ground state of the system is the p-wave pairing state of CFs. With regard to the interlayer tunneling, we have discussed the $\nu = 1/2$ case and the $\nu = 1$ case. At $\nu = 1/2$, the ground state evolves from the $(3, 3, 1)$ state toward the Pfaffian state. However, at the transition point there is a cusp. It indicates that the $(3, 3, 1)$ state and the Pfaffian state are the different phases. On the other hand, at $\nu = 1$, the ground state is uniquely determined and independent of the interlayer tunneling. The ground state is continuously connected with the $(1, 1, 1)$ state.

In the absence of the interlayer tunneling, the pairing state of CFs corresponds to the Halperin (m, m, n) state. We have discussed the stability of the pairing state with regard to the Coulomb interaction by using this fact. The Halperin (m, m, n) function is of Jastrow type wave function, and is completely determined by the two-body correlation effect because of the incompressible nature of the quantum Hall liquids. Calculating the Coulomb interaction for two-body correlations by the first order perturbation, we have estimated the regions where the pairing state is stabilized.

5.2 Open problems

This thesis offers a starting point of the composite fermion pairing theory in the quantum Hall systems. The possible extensions of it are listed below.

- **How we can detect the p-wave pairing state?**
  The pairing picture of the quantum Hall systems has no experimental foundation. We must invent an experiment to investigate the p-wave pairing state. This problem is related to the nature of the quasiparticles in CF pairing liquids and/or the edge states.

- **What is the nature of quasiparticles?**
  It is well-known that quasiparticles in quantum Hall systems at $\nu = 1/m$, where m is an odd integer, have fractional charges and obey a fractional statistics. How about the quasiparticles of CF pairing states? To discuss the nature of quasiparticles, we need the Ginzburg-Landau (GL) theory of the pairing state of CFs. Contrary to the usual BCS pairing case, we do not have a cutoff in momentum space. This complicates the derivation of the GL theory.

- **What is the relevance to the composite boson theory in bilayer systems?**
  In the composite boson theory, the existence of an bulk gapless mode was predicted at $\nu = 1$ [16]. We must examine it by the CF pairing theory. Furthermore, we do not know the relation between meron excitations in the composite boson theory [18] and quasiparticles in the CF pairing theory.

- **Edge states?**
  Edge states in the fractional quantum Hall systems are believed to be the chiral Luttinger liquid [50]. How about edge states of the CF pairing state? From edge states, we can expect to extract some important information about the nature of the bulk state.

- **What is the effect of the impurity?**
  To discuss the stability of the quantum Hall effect by CF pairings, we have to take into account the effect of impurities. Impurities cause some pair-breaking effect in the CF pairing state. The important question that why the FQHE at even denominator is fragile has something to do with it.

- **How can we get the phase diagram in bilayer systems?**
  Experimentally the phase diagram at $\nu = 1$ in bilayer systems was obtained by Murphy *et al.* [49]. Though we have presented the condition for the interlayer separation in Sec.4.6, the whole condition which includes also the interlayer tunneling has not obtained. However, we have derived the ground state wave functions. If we calculate the ground state energy then the phase diagram will be obtained.

Recently, the $\nu = 5/2$ state has been reconsidered in Refs.[51, 52, 53]. All these numerical works concluded that the $\nu = 5/2$ state is not a singlet d-wave pairing but a p-wave pairing state. The p-wave pairing state

discussed in chap.3 is related to this problem. If the effect of the filled Landau levels and that of the tilted magnetic field [54] are understood and we take account of them in our theory, the long standing mystery of the $\nu = 5/2$ will be solved.

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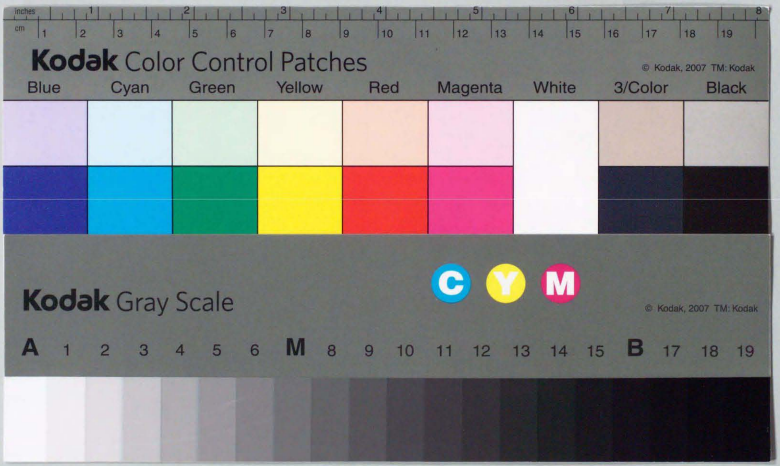
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