博士論文

論文題目: Approximate Controllability, Non-homogeneous Boundary Value Problems and Inverse Source Problems for Fractional Diffusion Equations

（非整数階拡散方程式に対する近似可制御性、非斉次境界値問題およびソース項決定逆問題）

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Abstract

In the present dissertation, we consider the approximate controllability and inverse source problems for *fractional diffusion equations*. In Chapter 1, we consider the fractional diffusion equation with homogeneous boundary data and prove the approximate controllability via distributed control on an arbitrarily given subdomain. In Chapter 2, we prove the approximate controllability by Dirichlet boundary data. To this end, we also consider the regularity of the solution with non-homogeneous boundary value. The main tool in these two chapters is the transposition method, which is the application of integration by parts. In Chapter 3, we prove the stability of the inverse problem of determining the time-dependent factor in a source term or a coefficient of reaction term from the one-point observations.

The contents in Chapters 1 and 3 are based on the collaborations with Professor Masahiro Yamamoto and Professor Yavar Kian respectively. Especially, Chapter 1 is the author's accepted manuscript of an article published as the version of record in Applicable Analysis ©25 Oct 2013 (http://www.tandfonline.com/doi/full/10.1080/00036811.2013.850492).

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Contents

Chapter 1

Approximate Controllability

1.1 Introduction

In this chapter, we consider the controllability for the *fractional diffusion equation* which evolves in a bounded domain in the Euclidean space.

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. We consider the following initial value/boundary value problem of fractional differential equation:

$$\begin{cases} \partial_t^\alpha u + \mathscr{L}u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.1.1}$$

In (1.1.1), $u = u(x,t)$ is the state to be controlled and $f = f(x,t)$ is the control which is localized in a subdomain ω of Ω. Here the Caputo fractional derivative ∂_t^α is defined by

$$\partial_t^\alpha h(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dh}{d\tau}(\tau) d\tau \tag{1.1.2}$$

for $0 < \alpha < 1$ (see [20] and [28] for example). Moreover $\mathscr{L}$ denotes a symmetric and uniformly elliptic operator, which is specified later and $T > 0$ is a fixed value. If $\partial_t^\alpha u$ is replaced by $\partial_t u$, then (1.1.1) is a classical diffusion equation.

Equation (1.1.1) is called a fractional diffusion equation and regarded as a model of anomalous diffusion in heterogeneous media. Adams and Gelhar [1] pointed out that the field data in a highly heterogeneous aquifer cannot be described well by the classical diffusion equation. Hatano and Hatano [17] applied the continuous-time random walk (CTRW) as a microscopic model of the diffusion of ions in heterogeneous media. From the CTRW model, one can derive equation (1.1.1) as a macroscopic model. For the derivation, see Gorenflo and Mainardi [16], Metzler and Klafter [25] and Roman and Alemany [34] for example.

As for mathematical treatments of fractional diffusion equations and fractional calculus, we can refer to many literature. As monographs of fractional calculus, see Kilbas, Srivastava and Trujillo [20], Podlubny [28] and Samko, Kilbas and Marichev [38] for example. These books mainly deal with basic properties of fractional derivatives and ordinary differential equations of fractional orders. As for mathematical works concerned with partial differential equations with time fractional derivatives, see the following literature and the references therein; Gejji and Jafari [15] solved equation (1.1.1) with $0 < \alpha \le 2$ in a one-dimensional or two-dimensional bounded domain. Agarwal [3] solved equation (1.1.1) in a one-dimensional bounded domain by means of finite sine transform and presented some numerical results for it. Luchko [22, 23] considered a diffusion equation in a multi-dimensional bounded domain and showed the unique existence of the solution to (1.1.1) with $f = 0$ using Fourier's method—constructing the solution by eigenfunction expansion. In the same way, Sakamoto and Yamamoto [36] established the regularity and qualitative properties of solution to (1.1.1), and discussed some inverse problems.

In spite of the importance, there are very few works on control problems, especially the controllability for fractional differential equations. The purpose of this chapter is to discuss the approximate controllability where we are requested to steer a given initial state $u_0 = u_0(x)$ to a prescribed target function $u_1(x)$ in time T by means of the control $f = f(x,t)$ which is distributed on $\omega \Subset \Omega$. We say that equation (1.1.1) is *approximately controllable* if for any $u_1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control $f \in C_0^\infty(\omega \times (0,T))$ such that the solution u to (1.1.1) satisfies

$$\|u(\cdot,T) - u_1\|_{L^2(\Omega)} \le \varepsilon.$$

If for any $u_1 \in L^2(\Omega)$, there exists f such that

$$u(\cdot,T) = u_1,$$

then (1.1.1) is said to be *exactly controllable*. It is known that equation (1.1.1) is approximately controllable for arbitrary $T > 0$ and subdomain $\omega \Subset \Omega$ if $\alpha = 1$ (see Fattorini [11] for example). In this article, assuming that $0 < \alpha < 1$, we will show the approximate controllability of equation (1.1.1) for arbitrarily given $\omega \Subset \Omega$ and $T > 0$. To this end, for the solution u of (1.1.1), the value of $u(\cdot,T)$ should make sense in $L^2(\Omega)$. Therefore, we will show in Section 1.3 that $u \in C([0,T]; L^2(\Omega))$ with an appropriate regularity of f.

Here we refer to the literature of control theory. As for works of control problems, see Coron [9], Micu and Zuazua [26] and Russell [35] for example. Sakawa [37] represented the solution of a classical diffusion equation by the Green function and proved the approximate controllability. Fattorini [12] studied equations in a Hilbert space and considered the approximate controllability for heat equations as an application. As for boundary control for classical diffusion equations, see MacCamy, Mizel and Seidman [24], Russell [35], Schmidt and Weck [39] and the references therein. In order to prove the approximate controllability for (1.1.1), we consider the *dual system* and show a weak type of unique continuation (see Sections 1.3 and 1.4). See also Dolecki and Russell [10] and Triggiani [41] which discuss the relation between controllability and observability.

The remainder of this chapter is composed of four sections. In Section 1.2, we define the solution of initial value/boundary value problem (1.1.1) and state our main results. In

Section 1.3, we study the fundamental properties—the unique existence and regularity of the solution to (1.1.1), and we reduce the proof of the main result to the case of the zero initial value. In Section 1.4, we discuss the dual system and prove a weak type of unique continuation property. Thanks to the non-local property of $\partial_t^\alpha u$, the dual system is rather different from the original system (1.1.1) and we need an independent analysis. In Section 1.5, we complete the proof of the main results.

1.2 Main result

In this section, we define the solution of the fractional diffusion equation and state our main result.

Let us denote by $L^2(\Omega)$ a usual L^2-space equipped with the scalar product $(\cdot,\cdot)$ and by $H^l(\Omega)$ and $H_0^m(\Omega)$, $l,m \in \mathbb{N}$, the Sobolev spaces (see Adams [2] for example). We define the differential operator $\mathscr{L}$ by

$$\mathscr{L}u(x) = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x), \quad x \in \Omega, \tag{1.2.1}$$

where the coefficients satisfy the following:

$$a_{ij} = a_{ji},\ a_{ij} \in C^1(\overline{\Omega}), \quad 1 \le i,j \le d, \quad c \in C(\overline{\Omega}), \qquad c(x) \ge 0, \quad x \in \overline{\Omega}, \tag{1.2.2}$$

and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d. \tag{1.2.3}$$

Henceforth we always regard L as the operator $\mathscr{L}$ in $L^2(\Omega)$ whose domain $\mathcal{D}(L)$ is $H^2(\Omega) \cap H_0^1(\Omega)$. That is, we understand that $u(\cdot,t) \in \mathcal{D}(L)$ means $u(\cdot,t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for $t \ge 0$.

Thus we are now ready to define a solution to (1.1.1).

Definition 1.2.1. We call a function u a solution to (1.1.1) if the following conditions are satisfied:

(a) $\partial_t^\alpha u(\cdot,t) + Lu(\cdot,t) = f(\cdot,t)$ holds in $L^2(\Omega)$ for almost all $t \in (0,T)$.

(b) $u \in C([0,T]; L^2(\Omega))$ and $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0$.

Our main result is stated as follows:

Theorem 1.2.1. *Let $0 < \alpha < 1$. Then equation* (1.1.1) *is approximately controllable for arbitrarily given $T > 0$ and an arbitrary subdomain ω in Ω. That is,*

$$\overline{\{u(\cdot, T);\ f \in C_0^\infty(\omega \times (0, T))\}} = L^2(\Omega),$$

where u is the solution to (1.1.1) *and the closure on the left-hand side is taken in $L^2(\Omega)$.*

By Proposition 1.3.1 in Section 1.3, we know that the solution u exists uniquely and $u(\cdot, T) \in L^2(\Omega)$ and so the statement of the theorem is well-defined.

Fattorini [11] showed that approximate controllability for classical diffusion equations is independent of $T > 0$. As is shown in the above theorem, fractional diffusion equations have the same property. The rest part of the chapter is devoted to the proof of Theorem 1.2.1.

1.3 Regularity of the solution to (1.1.1)

For the proof of Theorem 1.2.1, we first have to show that the assertion in the theorem makes sense, that is, we will show that equation (1.1.1) possesses a unique solution $u \in C([0, T]; L^2(\Omega))$.

In order to state the result, we prepare the notation. Since L is a symmetric and uniformly elliptic operator, the spectrum of L is composed entirely of eigenvalues and we can number them with multiplicities:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots .$$

By $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$, we denote an orthonormal eigenfunction corresponding to λ_n:

$$L\varphi_n = \lambda_n \varphi_n, \quad n = 1, 2, \cdots .$$

The eigenfunction φ_n is uniquely determined up to the factors ± 1. Then it is known that the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$.

Moreover we define the Mittag-Leffler function by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. We can directly verify that $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$ (see [20] and [28] for example).

Henceforth C denotes the generic constant which is independent of f in (1.1.1), but may depend on α and the coefficients of the operator L.

Then we can state the unique existence of the solution to (1.1.1) as follows:

Proposition 1.3.1. *Let $0 < \alpha < 1$ and $u_0 \equiv 0$ in* (1.1.1).

(*i*) *Suppose that* $f \in L^p(0,T;L^2(\Omega))$ *with* $p \geq 2$ *and* $p > 1/\alpha$. *Then there exists a unique solution* $u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ *to* (1.1.1) *such that*

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C\|f\|_{L^p(0,T;L^2(\Omega))}, \tag{1.3.1}$$

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq Ct^{\alpha-1/p}\|f\|_{L^p(0,T;L^2(\Omega))}. \tag{1.3.2}$$

Moreover we represent u *as*

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t (f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau \right) \varphi_n(x). \tag{1.3.3}$$

(*ii*) *Suppose that* $f \in C_0^\infty(\omega \times (0,T))$. *Then the solution* u *given by* (1.3.3) *further belongs to* $C^\infty([0,T];H^2(\Omega) \cap H_0^1(\Omega))$. *Moreover the series in* (1.3.3) *is convergent in* $C^m([0,T];H^2(\Omega))$ *and satisfies*

$$\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} \leq Ct^\alpha\|\partial_t^m f\|_{L^\infty(0,T;H^2(\Omega))} \tag{1.3.4}$$

for any $m = 0,1,2,\ldots$.

Remark 1.3.1. Since we have $C_0^\infty(\omega \times (0,T)) \subset C_0^\infty(\Omega \times (0,T))$ by the zero extension, we apply Proposition 1.3.1 to see the unique existence of the solution $u \in C([0,T];L^2(\Omega))$ to (1.1.1) with $f \in C_0^\infty(\omega \times (0,T))$. By the above proposition, the source term f needs not very smooth. In other words, as space of controls, we can take any function space $\mathcal{X}$ satisfying

$$C_0^\infty(\omega \times (0,T)) \subset \mathcal{X} \subset L^p(0,T;L^2(\Omega)),$$

so that the approximate controllability holds. Indeed, by $\mathcal{X} \subset L^p(0,T;L^2(\Omega))$ and Proposition 1.3.1, the value $u(\cdot,T)$ with $f \in \mathcal{X}$ belongs to $L^2(\Omega)$. Moreover, since $C_0^\infty(\omega \times (0,T)) \subset \mathcal{X}$, we have

$$\{u(\cdot,T); f \in C_0^\infty(\omega \times (0,T))\} \subset \{u(\cdot,T); f \in \mathcal{X}\} \subset L^2(\Omega).$$

Applying Theorem 1.2.1, we find that

$$\overline{\{u(\cdot,T); f \in \mathcal{X}\}} = L^2(\Omega).$$

In order to prove Proposition 1.3.1, we show the following lemmata.

Lemma 1.3.2. *Let* $0 < \alpha < 1$ *and* $\beta \in \mathbb{R}$ *be arbitrary and* μ *satisfy* $\pi\alpha/2 < \mu < \pi\alpha$. *Then there exists a constant* $C = C(\alpha,\beta,\mu) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|} \leq C, \quad \mu \leq |\arg(z)| \leq \pi.$$

In particular, we have

$$|E_{\alpha,\beta}(-\eta)| \le \frac{C}{1+|\eta|}$$

for $\eta \ge 0$. The proof of Lemma 1.3.2 can be found on p. 35 in [28].

Now we are ready to prove Proposition 1.3.1.

Proof of Proposition 1.3.1. (i). Noting that $f \in L^p(0,T;L^2(\Omega)) \subset L^2(\Omega \times (0,T))$, we apply Theorem 2.2 (i) in [36] to see that there exists a unique solution $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1.1) given by (1.3.3) with the estimate

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C\|f\|_{L^2(\Omega\times(0,T))} \le C\|f\|_{L^p(0,T;L^2(\Omega))}.$$

Next we prove that $u \in C([0,T];L^2(\Omega))$ and estimate (1.3.2). A straightforward calculation yields that

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(\Omega)} &= \left\| \sum_{n=1}^{\infty} \left(\int_0^t (f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau \right) \varphi_n \right\|_{L^2(\Omega)} \\
&= \left\| \int_0^t \left(\sum_{n=1}^{\infty} (f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n \right) d\tau \right\|_{L^2(\Omega)} \\
&\le \int_0^t \left\| \sum_{n=1}^{\infty} (f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n \right\|_{L^2(\Omega)} d\tau \\
&= \int_0^t \left(\sum_{n=1}^{\infty} |(f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)|^2 \right)^{1/2} d\tau.
\end{aligned}$$

Noting that $|\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)| \le C\tau^{\alpha-1}$ by Lemma 1.3.2, we have

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\int_0^t \left(\sum_{n=1}^{\infty} |(f(\cdot,t-\tau),\varphi_n)|^2 \right)^{1/2} \tau^{\alpha-1}d\tau = C\int_0^t \|f(\cdot,t-\tau)\|_{L^2(\Omega)}\tau^{\alpha-1}d\tau.$$

Now we take $q \in [1,\infty)$ so that $1/p + 1/q = 1$. Then we see that $t^{\alpha-1} \in L^q(0,T)$ by $p > 1/\alpha$. Therefore by Hölder's inequality, we have

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\|f\|_{L^p(0,t;L^2(\Omega))} \left(\int_0^t \tau^{q(\alpha-1)}d\tau \right)^{1/q} \le Ct^{\alpha-1/p}\|f\|_{L^p(0,T;L^2(\Omega))}.$$

Thus we have proved estimate (1.3.2). Moreover the above calculation also indicates that the series in (1.3.3) is convergent in $C([0,T];L^2(\Omega))$.

(ii). Since $f \in C_0^\infty(\omega \times (0,T))$, we also have

$$f \in C^\infty([0,T];H^2(\Omega) \cap H_0^1(\Omega)) = C^\infty([0,T];\mathcal{D}(L)).$$

Therefore we have

$$\begin{aligned}
\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} &\le C\|\partial_t^m u(\cdot,t)\|_{\mathcal{D}(L)} \\
&= C\left\|\frac{\partial^m}{\partial t^m}\sum_{n=1}^{\infty}\left(\int_0^t (f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n\right\|_{\mathcal{D}(L)} \\
&= C\left\|\sum_{n=1}^{\infty}\left(\int_0^t (\partial_t^m f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n\right\|_{\mathcal{D}(L)} \\
&= C\left\|\int_0^t\left(\sum_{n=1}^{\infty}(\partial_t^m f(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n\right)d\tau\right\|_{\mathcal{D}(L)} \\
&\le C\int_0^t\left\|\sum_{n=1}^{\infty}(\partial_t^m f_n(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n\right\|_{\mathcal{D}(L)} d\tau \\
&= C\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^2|(\partial_t^m f(\cdot,t-\tau),\varphi_n)|^2|\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)|^2\right)^{1/2} d\tau \\
&\le C\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^2|(\partial_t^m f(\cdot,t-\tau),\varphi_n)|^2\right)^{1/2}\tau^{\alpha-1}d\tau \\
&= C\int_0^t\|\partial_t^m f(\cdot,t-\tau)\|_{\mathcal{D}(L)}\tau^{\alpha-1}d\tau \le C\|\partial_t^m f\|_{L^\infty(0,T;\mathcal{D}(L))}\left(\int_0^t\tau^{\alpha-1}d\tau\right) \\
&\le Ct^\alpha\|\partial_t^m f\|_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}$$

Similarly to (i), we have proved estimate (1.3.4) and the convergence of the series in (1.3.3) in $C^m([0,T];H^2(\Omega))$. $\square$

Remark 1.3.2. We conclude this section with the reduction of Theorem 1.2.1 to the case of $u_0=0$. Let $u(f,u_0)$ be the solution of (1.1.1) and assume that

$$\overline{\{u(f,0)(\cdot,T);\ f\in C_0^\infty(\omega\times(0,T))\}}=L^2(\Omega). \tag{1.3.5}$$

Let $u_0,u_1\in L^2(\Omega)$ be arbitrary. By (1.3.5), noting $u(0,u_0)(\cdot,T)\in L^2(\Omega)$ by Theorem 2.1 in [36], for any $\varepsilon>0$ we can choose $f_\varepsilon\in C_0^\infty(\omega\times(0,T))$ such that

$$\|u(f_\varepsilon,0)(\cdot,T)-(u_1-u(0,u_0)(\cdot,T))\|_{L^2(\Omega)}<\varepsilon.$$

Noting that $u(f_\varepsilon,u_0)=u(0,u_0)+u(f_\varepsilon,0)$ by the linearity, we have

$$\|u(f_\varepsilon,u_0)(\cdot,T)-u_1\|_{L^2(\Omega)}<\varepsilon,$$

Thus for the proof of Theorem 1.2.1, it suffices to assume that $u_0=0$.

1.4 Solution of the Dual System

In this section, for the proof of the theorem we study the dual system for (1.1.1).

Let us consider the following initial value/boundary value problem:

$$\begin{cases} D_t^\alpha v + \mathscr{L} v = 0 & \text{in} \quad \Omega \times (0,T), \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ I_{T-}^{1-\alpha} v(\cdot, T) = v_0 & \text{in} \quad \Omega. \end{cases} \tag{1.4.1}$$

Here D_t^α and I_{T-}^ν denote the backward Riemann-Liouville derivative and integral, which are defined by

$$D_t^\alpha v(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau - t)^{-\alpha} v(\tau) d\tau \tag{1.4.2}$$

for $\alpha \in (0,1)$ (see pp. 69-71 in [20] for example) and

$$I_{T-}^\nu v(t) := \frac{1}{\Gamma(\nu)} \int_t^T (\tau - t)^{\nu-1} v(\tau) d\tau$$

for $\nu > 0$ respectively. Note that in particular, if $0 < \alpha < 1$, then we can rewrite $D_t^\alpha v(t)$ by

$$D_t^\alpha v(t) = -\frac{d}{dt} I_{T-}^{1-\alpha} v(t). \tag{1.4.3}$$

The third equation in (1.4.1) means that

$$I_{T-}^{1-\alpha} v(x,T) := \lim_{t \to T} \frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau - t)^{-\alpha} v(x,\tau) d\tau = v_0(x), \quad 0 < \alpha < 1.$$

We define the solution to (1.4.1) similarly to (1.1.1).

Definition 1.4.1. We call v a solution to (1.4.1) if

(a') $D_t^\alpha v(\cdot, t) + Lv(\cdot, t) = 0$ holds in $L^2(\Omega)$ for almost all $t \in (0,T)$.

(b') $I_{T-}^{1-\alpha} v \in C([0,T]; L^2(\Omega))$ and $\lim_{t \to T} \|I_{T-}^{1-\alpha} v(\cdot, t) - v_0\|_{L^2(\Omega)} = 0$.

We first show fundamental results for (1.4.1).

Proposition 1.4.1. *Let $v_0 \in L^2(\Omega)$. Then (1.4.1) possesses a unique solution v and v is represented by*

$$v(x,t) = \sum_{n=1}^\infty (v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha) \varphi_n(x) \tag{1.4.4}$$

and there exists a constant $C > 0$ such that

$$\|D_t^{\alpha-1} v\|_{C([0,T];L^2(\Omega))} \le C \|v_0\|_{L^2(\Omega)}. \tag{1.4.5}$$

Moreover

(*i*) $v \in C([0,T); H^2(\Omega) \cap H_0^1(\Omega))$ *and* $D_t^\alpha v \in C([0,T); L^2(\Omega))$, *and*

$$\|v(\cdot,t)\|_{H^2(\Omega)} + \|D_t^\alpha v(\cdot,t)\|_{L^2(\Omega)} \le C(T-t)^{-1}\|v_0\|_{L^2(\Omega)}. \tag{1.4.6}$$

(*ii*) *Let* $q \in \mathbb{R}$ *satisfy* $1 < q < 1/(1-\alpha)$. *Then* $v \in L^q(0,T; L^2(\Omega))$ *and there exists a constant* $C > 0$ *such that*

$$\|v\|_{L^q(0,T;L^2(\Omega))} \le C\|v_0\|_{L^2(\Omega)}. \tag{1.4.7}$$

If we further assume $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *then* $v \in L^q(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ *and satisfies*

$$\|D_t^\alpha v\|_{L^q(0,T;L^2(\Omega))} + \|v\|_{L^q(0,T;H^2(\Omega))} \le C\|v_0\|_{H^2(\Omega)}. \tag{1.4.8}$$

(*iii*) $v : [0,T) \to L^2(\Omega)$ *is analytically extended to* $S_T := \{z \in \mathbb{C}; \mathrm{Re}\, z < T\}$.

Remark 1.4.1. We note that (1.4.1) has a character of a backward problem in time, that is, a value at $t = T$ is given. Therefore the regularity of the solution is worse at $t = T$ and the analytic extension is impossible over T.

As is seen in the next section, the following proposition plays an essential role in the proof of Theorem 1.2.1.

Proposition 1.4.2. *Let* $v_0 \in L^2(\Omega)$ *and let* $\omega \Subset \Omega$ *be an arbitrary subdomain. If a solution* $v \in C([0,T); H^2(\Omega) \cap H_0^1(\Omega))$ *to* (1.4.1) *vanishes in* $\omega \times (0,T)$, *then* $v = 0$ *in* $\Omega \times (0,T)$.

For the proof of the above propositions, we state the following lemma.

Lemma 1.4.3. *For* $\lambda, \alpha > 0$ *and positive integer* $m \in \mathbb{N}$, *we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0. \tag{1.4.9}$$

Proof. Since $E_{\alpha,\beta}(z)$ is an entire function of z, equation (1.4.9) can be obtained by termwise differentiation. $\square$

Proof of Proposition 1.4.1. The proof is composed of five steps.

Step 1. We first show the uniqueness of the solution to (1.4.1) within the class given in Definition 1.4.1. It is sufficient to prove that system (1.4.1) has only a trivial solution under the initial condition $v_0 = 0$.

Let v be a solution to (1.4.1) with $v_0 = 0$. By taking the inner product $(\cdot, \cdot)$ of (1.4.1) with φ_n and by setting $v_n(t) = (v(\cdot, t), \varphi_n)$, we obtain

$$D_t^\alpha v_n(t) = -\lambda_n v_n(t), \quad \text{a.e. } t \in (0, T). \tag{1.4.10}$$

Since $D_t^{\alpha-1} v \in C([0, T]; L^2(\Omega))$, we see that $I_{T-}^{1-\alpha} v_n(t) = (I_{T-}^{1-\alpha} v(\cdot, t), \varphi_n)$ is continuous in $t \in [0, T]$. Moreover,

$$|I_{T-}^{1-\alpha} v_n(t)|^2 \le \sum_{n=1}^{\infty} |I_{T-}^{1-\alpha} v_n(t)|^2 = \left\| I_{T-}^{1-\alpha} v(\cdot, t) \right\|_{L^2(\Omega)}^2 \to 0 \quad \text{as } t \to T.$$

Therefore we have

$$I_{T-}^{1-\alpha} v_n(T) = 0. \tag{1.4.11}$$

Due to the existence and uniqueness of the ordinary fractional differential equation (see p.122 in [28] for example), (1.4.10) and (1.4.11) yield that

$$v_n(t) \equiv 0, \quad n = 1, 2, \cdots.$$

Since $\{\varphi_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$, we have

$$v = 0 \quad \text{in } \Omega \times (0, T).$$

Thus we have proved the uniqueness of the solution to (1.4.1).

In the rest four steps, we will show that v given by (1.4.4) satisfies the assertions of Proposition 1.4.1.

Step 2. Second, we prove that condition (b') in Definition 1.4.1 and estimate (1.4.5) hold. We set

$$v_N(x, t) = \sum_{n=1}^{N} (v_0, \varphi_n)(T - t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - t)^\alpha) \varphi_n(x).$$

By termwise integration, we have

$$\begin{aligned}
I_{T-}^{1-\alpha} v_N(\cdot, t) &= \frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau - t)^{-\alpha} v_N(\cdot, \tau) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau - t)^{-\alpha} \left(\sum_{n=1}^{N} (v_0, \varphi_n)(T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) \varphi_n \right) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{N} (v_0, \varphi_n) \left(\int_t^T (\tau - t)^{-\alpha} (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) d\tau \right) \varphi_n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} (v_0, \varphi_n) I_{T-}^{1-\alpha} \left((T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha) \right) \varphi_n \\
&= \sum_{n=1}^{N} (v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha) \varphi_n
\end{aligned}
$$

in $L^2(\Omega)$ (see p.78 in [20] for example). Moreover, for any $t \in [0,T]$ and $M, N \in \mathbb{N}$ with $M > N$, by Lemma 1.3.2, we have

$$
\begin{aligned}
\left\| I_{T-}^{1-\alpha} v_N(\cdot,t) - I_{T-}^{1-\alpha} v_M(\cdot,t) \right\|_{L^2(\Omega)}^2 &= \left\| \sum_{n=N+1}^{M} (v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha) \varphi_n \right\|_{L^2(\Omega)}^2 \\
&= \sum_{n=N+1}^{M} |(v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha)|^2 \\
&\le C^2 \sum_{n=N+1}^{M} |(v_0, \varphi_n)|^2 \to 0 \quad \text{as } N, M \to \infty.
\end{aligned}
$$

That is, $\sum_{n=1}^{\infty} (v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha) \varphi_n$ converges to $I_{T-}^{1-\alpha} v(\cdot,t)$ in $L^2(\Omega)$ uniformly in $t \in [0,T]$. Therefore

$$
I_{T-}^{1-\alpha} v \in C([0,T]; L^2(\Omega)).
$$

Similarly, by Lemma 1.3.2, we have

$$
\| I_{T-}^{1-\alpha} v(\cdot,t) \|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |(v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha)|^2 \le C^2 \sum_{n=1}^{\infty} |(v_0, \varphi_n)|^2 = C^2 \|v_0\|_{L^2(\Omega)}^2,
$$

that is,

$$
\| I_{T-}^{1-\alpha} v \|_{C([0,T]; L^2(\Omega))} \le C \|v_0\|_{L^2(\Omega)}.
$$

Furthermore we have

$$
\begin{aligned}
&\| I_{T-}^{1-\alpha} v(\cdot,t) - v_0 \|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |(v_0, \varphi_n)|^2 (E_{\alpha,1}(-\lambda_n (T-t)^\alpha) - 1)^2, \\
&\lim_{t \to T} (E_{\alpha,1}(-\lambda_n (T-t)^\alpha) - 1) = 0, \quad n \in \mathbb{N}, \\
&\sum_{n=1}^{\infty} |(v_0, \varphi_n)|^2 (E_{\alpha,1}(-\lambda_n (T-t)^\alpha) - 1)^2 \le (C+1)^2 \sum_{n=1}^{\infty} |(v_0, \varphi_n)|^2 < \infty, \quad 0 \le t \le T.
\end{aligned}
$$

Therefore the Lebesgue theorem yields

$$
\lim_{t \to T} \left\| I_{T-}^{1-\alpha} v(\cdot,t) - v_0 \right\|_{L^2(\Omega)} = 0.
$$

Step 3. Third, we prove that condition (a') in Definition 1.4.1 and (1.4.6) hold. For simplicity, we set

$$
I_{T-}^{1-\alpha} v(\cdot,t) = \sum_{n=1}^{\infty} (v_0, \varphi_n) E_{\alpha,1}(-\lambda_n (T-t)^\alpha) \varphi_n =: \sum_{n=1}^{\infty} h_n(t).
$$

Then each h_n is continuously differentiable in $[0, T)$. For any $t \in [0, T-\delta]$ with arbitrarily fixed $\delta > 0$, by Lemma 1.4.3, we have

$$\begin{aligned}
\left\| \sum_{n=N+1}^{\infty} \frac{dh_n}{dt}(t) \right\|_{L^2(\Omega)}^2 &= \left\| \sum_{n=N+1}^{\infty} \lambda_n (v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha)\varphi_n \right\|_{L^2(\Omega)}^2 \\
&= \sum_{n=N+1}^{\infty} |\lambda_n (v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha)|^2 \\
&\le \sum_{n=N+1}^{\infty} |(v_0, \varphi_n)|^2 \lambda_n^2 (T-t)^{2\alpha-2} \left(\frac{C}{1+\lambda_n (T-t)^\alpha} \right)^2 \\
&= C^2 (T-t)^{-2} \sum_{n=N+1}^{\infty} |(v_0, \varphi_n)|^2 \left(\frac{\lambda_n (T-t)^\alpha}{1+\lambda_n (T-t)^\alpha} \right)^2 \qquad (1.4.12) \\
&\le C^2 \delta^{-2} \sum_{n=N+1}^{\infty} |(v_0, \varphi_n)|^2 \to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} \frac{dh_n}{dt}(t)$ converges in $L^2(\Omega)$ uniformly in $t \in [0, T-\delta]$. By (1.4.3) and Lemma 1.4.3, we have

$$\begin{aligned}
D_t^\alpha v(\cdot, t) &= -\frac{d}{dt} I_{T-}^{1-\alpha} v(\cdot, t) = -\frac{d}{dt} \sum_{n=1}^{\infty} h_n(t) = -\sum_{n=1}^{\infty} \frac{dh_n}{dt}(t) \\
&= -\sum_{n=1}^{\infty} \lambda_n (v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha)\varphi_n = -Lv(\cdot, t)
\end{aligned}$$

in $L^2(\Omega)$ and

$$D_t^\alpha v = -Lv \in C([0, T); L^2(\Omega)),$$

which yields

$$v \in C([0, T); H^2(\Omega) \cap H_0^1(\Omega)).$$

Similarly to (1.4.12), we have

$$\|D_t^\alpha v(\cdot, t)\|_{L^2(\Omega)} = \|Lv(\cdot, t)\|_{L^2(\Omega)} = \left\| -\sum_{n=1}^{\infty} \frac{dh_n}{dt}(t) \right\|_{L^2(\Omega)} \le C(T-t)^{-1} \|v_0\|_{L^2(\Omega)}$$

for $0 \le t < T$, which implies estimate (1.4.6).

Step 4. Fourth, we prove (1.4.7) and (1.4.8). Direct calculations yield

$$\|v(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |(v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-t)^\alpha)|^2 \le C^2 (T-t)^{2(\alpha-1)} \|v_0\|_{L^2(\Omega)}^2,$$

which implies

$$\|v(\cdot, t)\|_{L^2(\Omega)}^q \le C^q (T-t)^{q(\alpha-1)} \|v_0\|_{L^2(\Omega)}^q.$$

Moreover since $q < 1/(1-\alpha)$, we have

$$\|v\|^q_{L^q(0,T;L^2(\Omega))} = \int_0^T \|v(\cdot,t)\|^q_{L^2(\Omega)} dt \le C^q \|v_0\|^q_{L^2(\Omega)} \int_0^T (T-t)^{q(\alpha-1)} dt \le C^q C \|v_0\|^q_{L^2(\Omega)}.$$

Hence we have

$$\|v\|_{L^q(0,T;L^2(\Omega))} \le C\|v_0\|_{L^2(\Omega)}.$$

If v_0 further belongs to $H^2(\Omega) \cap H^1_0(\Omega)$, then we have

$$\begin{aligned}\|D^\alpha_t v(\cdot,t)\|^2_{L^2(\Omega)} = \|Lv(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty \lambda_n^2 |(v_0,\varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)|^2 \\ &\le C^2 (T-t)^{2(\alpha-1)} \sum_{n=1}^\infty \lambda_n^2 |(v_0,\varphi_n)|^2 \le C(T-t)^{2(\alpha-1)} \|v_0\|^2_{H^2(\Omega)}.\end{aligned}$$

Therefore we can show (1.4.8) similarly to (1.4.7).

Step 5. Finally, we prove the assertion (iii). It follows that $(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)$ is analytic in S_T because $E_{\alpha,\alpha}(-\lambda_n z)$ is an entire function (see Section 1.8 in [20] and [28] for example). Therefore $\sum_{n=1}^N (v_0,\varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\varphi_n$ is analytic in S_T. If we fix $\delta > 0$ arbitrarily, then for $z \in \mathbb{C}$ with $\operatorname{Re} z \le T - \delta$, we have

$$\begin{aligned}&\left\| \sum_{n=N+1}^\infty (v_0,\varphi_n)(T-z)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-z)^\alpha)\varphi_n \right\|^2_{L^2(\Omega)} \\ &= \sum_{n=N+1}^\infty |(v_0,\varphi_n)(T-z)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-z)^\alpha)|^2 \\ &\le \sum_{n=N+1}^\infty |(v_0,\varphi_n)|^2 |T-z|^{2\alpha-2} \left(\frac{C}{1+\lambda_n|T-z|^\alpha}\right)^2 \\ &\le C^2 \delta^{2\alpha-2} \sum_{n=N+1}^\infty |(v_0,\varphi_n)|^2 \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

That is, $v(\cdot,z) = \sum_{n=1}^\infty (v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^\alpha)\varphi_n$ is uniformly convergent in any compact subset of S_T. Hence v is also analytic in S_T. □

Proof of Proposition 1.4.2. Since $v(x,t) = 0$ in $\omega \times (0,T)$ and $v : [0,T) \to L^2(\Omega)$ can be analytically extended to S_T, we have

$$v(x,t) = \sum_{n=1}^\infty (v_0,\varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\varphi_n(x) = 0, \quad x \in \omega,\ t \in (-\infty,T). \tag{1.4.13}$$

Let $\{\mu_k\}_{k\in\mathbb{N}}$ be the set of all the eigenvalues of L. We note that $\{\mu_k\}_{k\in\mathbb{N}}$ is numbered without multiplicities. By $\{\varphi_{kj}\}_{1\le j\le m_k}$ we denote an orthonormal basis of $\ker(\mu_k - L)$. Then we can rewrite (1.4.13) by

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}(x)\right)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)=0,\quad x\in\omega,\ t\in(-\infty,T). \tag{1.4.14}$$

Moreover, for any $z\in\mathbb{C}$ with $\operatorname{Re} z=\xi>0$ and $N\in\mathbb{N}$, noting that φ_{kj}, $1\le j\le m_k$, $1\le k\le N$ are orthonormal, we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|^2_{L^2(\Omega)}\\
&=\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}|(v_0,\varphi_{kj})|^2\right)e^{2\xi(t-T)}(T-t)^{2\alpha-2}|E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)|^2\\
&\le C^2e^{2\xi(t-T)}(T-t)^{2\alpha-2}\|v_0\|^2_{L^2(\Omega)},
\end{aligned}$$

that is,

$$\begin{aligned}
&\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{L^2(\Omega)}\\
&\le Ce^{\xi(t-T)}(T-t)^{\alpha-1}\|v_0\|_{L^2(\Omega)}.
\end{aligned}$$

The right-hand side of the above is integrable over $t\in(-\infty,T)$;

$$\int_{-\infty}^{T}e^{\xi(t-T)}(T-t)^{\alpha-1}dt=\int_0^\infty e^{-\xi\eta}\eta^{\alpha-1}d\eta=\frac{\Gamma(\alpha)}{\xi^\alpha}.$$

Hence the Lebesgue dominant convergence theorem yields that

$$\begin{aligned}
&\int_{-\infty}^{T}e^{z(t-T)}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}(x)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right)dt\\
=&\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}(x)\left(\int_{-\infty}^{T}e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)dt\right)\\
=&\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\frac{(v_0,\varphi_{kj})}{z^\alpha+\mu_k}\varphi_{kj}(x),\quad x\in\Omega,\ \operatorname{Re} z>0.
\end{aligned}\tag{1.4.15}$$

For the calculation on (1.4.15), see p.21 in [28]. By (1.4.14) and (1.4.15), we have

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\frac{(v_0,\varphi_{kj})}{z^\alpha+\mu_k}\varphi_{kj}(x)=0,\quad x\in\omega,\ \operatorname{Re} z>0,$$

that is,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\frac{(v_0,\varphi_{kj})}{\eta+\mu_k}\varphi_{kj}(x)=0, \quad x\in\omega,\ \mathrm{Re}\,\eta>0. \tag{1.4.16}$$

By using analytic continuation in η, equality (1.4.16) holds for $\eta \in \mathbb{C}\setminus\{-\mu_k\}_{k\in\mathbb{N}}$. Then we can take a suitable circle which includes $-\mu_\ell$ and does not include $\{-\mu_k\}_{k\neq\ell}$. By integrating (1.4.16) on the circle, we have

$$v_\ell(x) := \sum_{j=1}^{m_\ell}(v_0,\varphi_{\ell j})\varphi_{\ell j}(x)=0, \quad x\in\omega.$$

Since $(L-\mu_\ell)v_\ell = 0$ in Ω and $v_\ell = 0$ in ω, the unique continuation result for an elliptic operator (see Isakov [19], Nirenberg [27] and Protter [30] for example) implies $v_\ell = 0$ in Ω for each $\ell\in\mathbb{N}$. Since $\{\varphi_{\ell j}\}_{1\le j\le m_\ell}$ is linearly independent in Ω, we see that $(v_0,\varphi_{\ell j})=0$ for $1\le j\le m_\ell,\ \ell\in\mathbb{N}$. This implies $v=0$ in $\Omega\times(0,T)$. □

1.5 Proof of Main Result

In this section, we prove Theorem 1.2.1 using Propositions 1.4.1 and 1.4.2. For convenience of calculation, we introduce the notation of fractional integrals for $\alpha>0$;

$$I_{0+}^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,$$
$$I_{T-}^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^T (\tau-t)^{\alpha-1}f(\tau)d\tau.$$

In particular, we have

$$\partial_t^{\alpha}f(t) = I_{0+}^{1-\alpha}f'(t) \quad \text{and} \quad D_t^{\alpha}f(t) = -\frac{d}{dt}I_{T-}^{1-\alpha}f(t)$$

if $0<\alpha<1$. The following lemma holds.

Lemma 1.5.1. *Let $\alpha>0$ and $1<p,q<\infty$ satisfy $1/p+1/q\le 1+\alpha$. If $f\in L^p(0,T)$ and $g\in L^q(0,T)$, then*

$$\int_0^T I_{0+}^{\alpha}f(t)g(t)dt = \int_0^T f(t)I_{T-}^{\alpha}g(t)dt.$$

For the above lemma, see p. 34 in [38] for example.

Now we are ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Let u_f be the solution to (1.1.1) for $f \in C_0^\infty(\omega \times (0,T))$ and $u_0 = 0$, and v be the solution to (1.4.1) for $v_0 \in L^2(\Omega)$. We first prove that

$$\int_\Omega u_f(\cdot, T) v_0 dx = \int_0^T \int_\omega f v dx dt. \tag{1.5.1}$$

holds for every $f \in C_0^\infty(\omega \times (0,T))$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$.

Since the first equation in (1.1.1) holds in $C^\infty([0,T]; L^2(\Omega))$ by Proposition 1.3.1 and v belongs to $L^q(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ with some $q > 1$ by Proposition 1.4.1 (ii), we can see that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (\partial_t^\alpha u_f + \mathscr{L} u_f - f) v dx dt \\ &= \int_0^T \int_\Omega (\partial_t^\alpha u_f) v dx dt + \int_0^T \int_\Omega (\mathscr{L} u_f) v dx dt - \int_0^T \int_\Omega f v dx dt. \end{aligned} \tag{1.5.2}$$

In terms of Lemma 1.5.1, we calculate the first term on (1.5.2) as follows;

$$\begin{aligned} \int_0^T \int_\Omega (\partial_t^\alpha u_f) v dx dt &= \int_0^T \int_\Omega I_{0+}^{1-\alpha} \frac{\partial u_f}{\partial t} \cdot v dx dt = \int_0^T \int_\Omega \frac{\partial u_f}{\partial t} I_{T-}^{1-\alpha} v dx dt \\ &= \int_\Omega u_f \cdot I_{T-}^{1-\alpha} v dx \Big|_{t=0}^{t=T} - \int_0^T \int_\Omega u_f \cdot \frac{\partial}{\partial t} I_{T-}^{1-\alpha} v dx dt \\ &= \int_\Omega u_f(\cdot, T) v_0 dx + \int_0^T \int_\Omega u_f (D_t^\alpha v) dx dt. \end{aligned} \tag{1.5.3}$$

Here we have used the integration in t by parts and initial conditions in (1.1.1) and (1.4.1). In terms of $u_f \in C^\infty([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ and $v \in L^q(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ by Propositions 1.3.1 and 1.4.1, we apply the Green formula to the second term on (1.5.2) to have

$$\begin{aligned} \int_0^T \int_\Omega (\mathscr{L} u_f) v dx dt &= \int_0^T \int_\Omega u_f (\mathscr{L} v) dx dt + \int_0^T \int_{\partial\Omega} \left(u_f \frac{\partial v}{\partial \nu_L} - \frac{\partial u_f}{\partial \nu_L} v \right) d\sigma dt \\ &= \int_0^T \int_\Omega u_f (\mathscr{L} v) dx dt, \end{aligned} \tag{1.5.4}$$

where

$$\frac{\partial u}{\partial \nu_L}(x) := \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \nu_j(x)$$

and $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\partial\Omega$ at x. In the above calculation, we have used boundary conditions in (1.1.1) and (1.4.1). We substitute (1.5.3) and (1.5.4) into (1.5.2) and have

$$\begin{aligned} 0 &= \left(\int_\Omega u_f(\cdot, T) v_0 dx + \int_0^T \int_\Omega u_f (D_t^\alpha v) dx dt \right) + \int_0^T \int_\Omega u_f (\mathscr{L} v) dx dt - \int_0^T \int_\omega f v dx dt \\ &= \int_\Omega u_f(\cdot, T) v_0 dx + \int_0^T \int_\Omega u_f \left(D_t^\alpha v + L v \right) dx dt - \int_0^T \int_\omega f v dx dt \end{aligned}$$

$$= \int_\Omega u_f(\cdot, T) v_0 dx - \int_0^T \int_\omega f v dx dt.$$

Thus (1.5.1) holds for $f \in C_0^\infty(\omega \times (0,T))$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$.

Let $f \in C_0^\infty(\omega \times (0,T))$ be fixed. Then the mapping

$$v_0 \mapsto \int_\Omega u_f(\cdot, T) v_0 dx - \int_0^T \int_\omega f v dx dt$$

is a linear and bounded functional on $L^2(\Omega)$ by (1.4.7). Hence the density argument implies that (1.5.1) holds for any $v_0 \in L^2(\Omega)$.

In order to prove the density of $\{u_f(\cdot, T); f \in C_0^\infty(\omega \times (0,T))\}$ in $L^2(\Omega)$, we have to show that if $v_0 \in L^2(\Omega)$ satisfies

$$\int_\Omega u_f(\cdot, T) v_0 dx = 0 \tag{1.5.5}$$

for any $f \in C_0^\infty(\omega \times (0,T))$, then $v_0 = 0$. This can be shown as follows. By (1.5.5) and (1.5.1), we have

$$\int_0^T \int_\omega f v dx dt = 0$$

for any $f \in C_0^\infty(\omega \times (0,T))$. Then by the fundamental lemma of the calculus of variations, we have

$$v(x,t) = 0, \quad (x,t) \in \omega \times (0,T).$$

By Proposition 1.4.2, we have

$$v(x,t) = 0, \quad (x,t) \in \Omega \times (0,T).$$

By the uniqueness of the solution to (1.4.1),

$$v_0(x) = 0, \quad x \in \Omega.$$

Thus we have completed the proof. $\square$

Chapter 2

Non-homogeneous Boundary Value Problem

2.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$ with smooth boundary $\Gamma = \partial\Omega$. We consider the following initial value/boundary value problem of partial differential equation with non-homogeneous boundary value:

$$\begin{cases} \partial_t^\alpha u + \mathscr{L}u = f & \text{in} \quad \Omega \times (0,T), \\ u = g & \text{on} \quad \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{2.1.1}$$

In (2.1.1), $u = u(x,t)$ is the state to be controlled and $g = g(x,t)$ is the control which is localized on a subboundary Γ_0 of Γ. The functions $f = f(x,t)$ and $u_0 = u_0(x)$ are given in $\Omega \times (0,T)$ and Ω respectively. Here $\mathscr{L}$ is given by (1.2.1) with the coefficients satisfying (1.2.2) and (1.2.3) and ∂_t^α denotes the Caputo derivative (see (1.1.2)).

The aim of this chapter is to study the boundary control problem for fractional diffusion equations. We say that equation (2.1.1) is *approximately controllable for T and Γ_0* if for any $u_1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control g supported in $\Gamma_0 \times (0,T)$ such that the solution u of (2.1.1) satisfies

$$\|u(\cdot,T) - u_1\|_{L^2(\Omega)} \le \varepsilon.$$

We can refer to [9] and [35] for the general theory of control problems for partial differential equations. These works deal with controllability of equations with integer order and the relations with other concepts—observability, stabilizability, pole assignability, etc. There are various works about control problems for equations with integer orders. In particular, for the boundary control of heat equations, see MacCamy, Mizel and Seidman [24], Sakawa [37], Schmidt and Weck [39], Washburn [42] and the references therein. As for the control problems of fractional diffusion equations by interior control, we can refer to Fujishiro and Yamamoto [13]. However, to the author's best knowledge, there are few works on the control problems for fractional diffusion equations, especially the controllability by the boundary control.

The remainder of this chapter is composed of four sections. In Section 2.2, we state the main result. In Section 2.3, we represent the solution by eigenfunction expansion to show its unique existence and regularity for smooth g. In Section 2.4, we study the dual system of (2.1.1) and prove their properties—regularity, analyticity and weak type of unique continuation. In particular, unique continuation property plays an essential role in the proof of our main result. In Section 2.5, we complete the proof of our main result.

2.2 Main result

In this section, we prepare the settings and state our main results.

We first note that by the linearity, we can assume $u_0 \equiv 0$ and $f \equiv 0$ without loss of generality (see Remark 1.3.2):

$$\begin{cases} \partial_t^\alpha u + \mathscr{L} u = 0 & \text{in} \quad \Omega \times (0,T), \\ u = g & \text{on} \quad \Gamma \times (0,T), \\ u(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.2.1}$$

Henceforth we mainly consider (2.2.1) instead of (2.1.1).

Next we prepare the notations. Let $L^2(\Gamma)$ be the usual L^2-space with the scalar product $\langle \cdot, \cdot \rangle$ and $H^s(\Gamma)$, $s \in \mathbb{R}$, be the Sobolev spaces on Γ. As in the previous chapter, let $L : L^2(\Omega) \to L^2(\Omega)$ be the differential operator $\mathscr{L}$ with its domain $H^2(\Omega) \cap H_0^1(\Omega)$ and $\{(\lambda_n, \varphi_n)\}_{n \in \mathbb{N}}$ be the eigen system for L (see Section 1.3). The operator $\partial_{\nu_L} : H^s(\Omega) \to H^{s-3/2}(\Gamma)$, $s > 3/2$, is defined as

$$\partial_{\nu_L} u(x) := \frac{\partial u}{\partial \nu_L}(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \nu_j(x),$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to Γ at x. In particular, $\partial_{\nu_L} \varphi_n$ belongs to $L^2(\Gamma)$ since $\varphi_n \in H^2(\Omega)$. Now we are ready to state the following result;

Theorem 2.2.1. *Let $0 < \alpha < 1$ and $0 < \theta < 1/4$. If $g \in C_0^\infty(\Gamma \times (0,T))$, then there exists a unique solution $u \in C^\infty([0,T]; H^2(\Omega))$ to (2.2.1) such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C t^{\alpha\theta} \|g\|_{L^\infty(0,T;L^2(\Gamma))}, \tag{2.2.2}$$

$$\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} \le C \left(t^{\alpha(\theta-1)+1} \|\partial_t^{m+1} g\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t^m g(\cdot,t)\|_{H^{3/2}(\Gamma)} \right) \tag{2.2.3}$$

for $m = 0, 1, 2, \ldots$. Moreover we represent u as

$$u(x,t) = -\sum_{n=1}^{\infty} \left(\int_0^t \langle g(\cdot, t-\tau), \partial_{\nu_L} \varphi_n \rangle \, \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x), \tag{2.2.4}$$

and the series is convergent in $C^m([0,T]; H^2(\Omega))$ for any $m \in \mathbb{N}$.

By the above theorem, equation (2.2.1) has a unique solution $u \in C^\infty([0,T]; H^2(\Omega))$ for any $g \in C_0^\infty(\Gamma_0 \times (0,T))$ (g is regarded as a function on $\Gamma \times (0,T)$ by the zero extension). In particular, the value $u(\cdot, T)$ at time $t = T$ makes sense in $L^2(\Omega)$ and we are ready to state the following result;

Theorem 2.2.2. *Let* $0 < \alpha < 1$. *Then equation* (2.2.1) *is approximately controllable for arbitrarily given* $T > 0$ *and an arbitrary relatively open subset* Γ_0 *of* Γ. *That is,*

$$\overline{\{u_g(\cdot,T);\ g \in C_0^\infty(\Gamma_0 \times (0,T))\}} = L^2(\Omega),$$

where u_g *is the solution to* (2.2.1) *and the closure on the left-hand side is taken in* $L^2(\Omega)$.

2.3 Proof of Theorem 2.2.1

For the representation of the solution to (2.2.1), we study the following elliptic boundary value problem;

$$\begin{cases} \mathscr{L}u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \Gamma, \end{cases} \tag{2.3.1}$$

where g is given on Γ.

In order to describe the regularity of the solution of (2.3.1), we first consider the fractional power of the operator L, which is represented as follows;

$$\mathcal{D}(L^\theta) = \left\{ u \in L^2(\Omega);\ \sum_{n=1}^\infty \lambda_n^{2\theta} |(u, \varphi_n)|^2 < \infty \right\},$$

$$L^\theta u = \sum_{n=1}^\infty \lambda_n^\theta (u, \varphi_n)\varphi_n, \quad u \in \mathcal{D}(L^\theta),$$

where $\theta > 0$. Then $\mathcal{D}(L^\theta)$ is a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{D}(L^\theta)}$ defined by

$$\|u\|_{\mathcal{D}(L^\theta)} := \|L^\theta u\|_{L^2(\Omega)} = \left(\sum_{n=1}^\infty \lambda_n^{2\theta} |(u, \varphi_n)|^2 \right)^{1/2}, \quad u \in \mathcal{D}(L^\theta).$$

The domain $\mathcal{D}(L^\theta)$ with $0 \le \theta \le 1$ is expressed by using the Sobolev spaces with norm equivalence;

$$\mathcal{D}(L^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 \le \theta < 1/4, \\ H_D^{2\theta}(\Omega), & 1/4 < \theta \le 1, \end{cases}$$

$$C^{-1}\|u\|_{H^{2\theta}} \le \|u\|_{\mathcal{D}(L^\theta)} \le C\|u\|_{H^{2\theta}}, \quad u \in \mathcal{D}(L^\theta), \tag{2.3.2}$$

where $H_D^s(\Omega) := \{u \in H^s(\Omega) \mid \gamma_0 u = 0\}$ and the operator $\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ maps a function u to its restriction $u|_\Gamma$ to the boundary Γ for $s > 1/2$. Note that henceforth C

denotes the generic constant which may depend on α and the coefficients of the operator L. For the details of $\mathcal{D}(L^\theta)$ and the Sobolev spaces with fractional powers, see Fujiwara [14] and Yagi [43] for example.

For $g \in H^{3/2}(\Gamma)$, by using the trace theorem and lifting and applying the well known results for the elliptic boundary value problems with homogeneous data (see Theorems 8.1 and 9.8 in Agmon [4] for example), we see that (2.3.1) has a unique solution $u \in H^2(\Omega)$ satisfying

$$\|u\|_{H^2(\Omega)} \le C\|g\|_{H^{3/2}(\Gamma)}.$$

In the following, we will discuss (2.3.1) for non-smooth g by the *transposition method*. To this end, we consider the *dual system*;

$$\begin{cases} \mathscr{L}v = f & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \Gamma, \end{cases} \tag{2.3.3}$$

where f is given in Ω. It is well known that for any $f \in L^2(\Omega)$, (2.3.3) possesses a unique solution $v \in H^2(\Omega)$ satisfying

$$\|v\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.3.4}$$

In particular, $\partial_{\nu_L} v$ belongs to $H^{1/2}(\Gamma)$. By ${}_{-s}\langle \cdot, \cdot \rangle_s$, $s \ge 0$, we denote the duality paring in $H^{-s}(\Gamma)$ and $H^s(\Gamma)$. Now we can define the solution of (2.3.1) in a weaker sense.

Definition 2.3.1. A function u is called a *weak solution* of (2.3.1) if

$$(u, f) + {}_{-1/2}\langle g, \partial_{\nu_L} v_f \rangle_{1/2} = 0 \tag{2.3.5}$$

holds for any $f \in L^2(\Omega)$, where v_f is the unique solution of (2.3.3).

According to Chapter 2 of Lions and Magenes [21], we see that for $g \in L^2(\Gamma)$, (2.3.1) has a unique weak solution $u \in H^{1/2}(\Omega)$ satisfying

$$\|u\|_{H^{1/2}(\Omega)} \le C\|g\|_{L^2(\Gamma)}.$$

Let $\Lambda : L^2(\Gamma) \to H^{1/2}(\Omega)$ be the linear operator which maps g to the unique weak solution u of (2.3.1). Then we have

$$\|\Lambda g\|_{H^{1/2}(\Omega)} \le C\|g\|_{L^2(\Gamma)}.$$

In particular, for any $0 \le \theta < 1/4$, Λg belongs to $\mathcal{D}(L^\theta)$ and satisfies

$$\|\Lambda g\|_{\mathcal{D}(L^\theta)} \le C\|\Lambda g\|_{H^{2\theta}(\Omega)} \le C\|\Lambda g\|_{H^{1/2}(\Omega)} \le C\|g\|_{L^2(\Gamma)}. \tag{2.3.6}$$

where we have used (2.3.2). By substituting $f = \lambda_n \varphi_n$ and $u = \Lambda g$ in (2.3.5), we obtain

$$\lambda_n(\Lambda g, \varphi_n) = -\langle g, \partial_{\nu_L}\varphi_n \rangle, \quad n = 1, 2, \ldots. \tag{2.3.7}$$

For the proof of Theorem 2.2.1, we also recall the notation of fractional integrals and state some formulae. A straightforward calculation yields

$$I_{0+}^{\alpha}\left[t^{\nu}\right] = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha}$$

for $\nu > -1$ and $\alpha > 0$. By the analyticity of Mittag-Leffler functions, we have

$$I_{0+}^{1-\alpha}\Big(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})\Big) = E_{\alpha,1}(-\lambda t^{\alpha}), \quad t > 0. \tag{2.3.8}$$

for $0 < \alpha < 1$, which is a particular case of (1.100) in [28]. Moreover from Lemma 1.5.1, it follows that

$$\int_0^t I_{0+}^{\alpha} f(t-\tau) g(\tau) d\tau = \int_0^t f(t-\tau)(I_{0+}^{\alpha} g)(\tau) d\tau. \tag{2.3.9}$$

Now we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Step 1. First we show that u in (2.2.4) is a unique solution of (2.2.1). Since the uniqueness can be shown similarly to Theorem 2.1 in [36], it is sufficient to confirm that equation (2.2.1) is satisfied.

By (2.3.7) and Lemma 1.4.3, we have

$$\begin{aligned} u(x,t) &= -\sum_{n=1}^{\infty}\left(\int_0^t \langle g(\cdot,t-\tau), \partial_{\nu_L}\varphi_n\rangle\, \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^{\alpha}) d\tau\right)\varphi_n(x) \\ &= \sum_{n=1}^{\infty}\left(\int_0^t (\Lambda g(\cdot,t-\tau), \varphi_n)\lambda_n \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^{\alpha}) d\tau\right)\varphi_n(x) \\ &= -\sum_{n=1}^{\infty}\left(\int_0^t (\Lambda g(\cdot,t-\tau), \varphi_n)\cdot \frac{\partial}{\partial\tau}\Big(E_{\alpha,1}(-\lambda_n \tau^{\alpha})\Big) d\tau\right)\varphi_n(x). \end{aligned} \tag{2.3.10}$$

Since $g \in C_0^{\infty}(\Gamma\times(0,T))$, the integration by parts yields

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left((\Lambda g(\cdot,t),\varphi_n) + \int_0^t \frac{\partial}{\partial\tau}(\Lambda g(\cdot,t-\tau),\varphi_n)\cdot E_{\alpha,1}(-\lambda_n\tau^{\alpha})d\tau\right)\varphi_n(x) \\ &= \Lambda g(x,t) - \sum_{n=1}^{\infty}\left(\int_0^t \big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n\tau^{\alpha})d\tau\right)\varphi_n(x). \end{aligned}$$

We set

$$\begin{aligned} w(x,t) &:= u(x,t) - \Lambda g(x,t) \\ &= -\sum_{n=1}^{\infty}\left(\int_0^t \big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n\tau^{\alpha})d\tau\right)\varphi_n(x). \end{aligned} \tag{2.3.11}$$

Then by (2.3.8) and (2.3.9), we have

$$w(x,t) = -\sum_{n=1}^{\infty}\left(\int_0^t \big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)\cdot I_{0+}^{1-\alpha}\Big(\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^{\alpha})\Big)d\tau\right)\varphi_n(x)$$

$$= -\sum_{n=1}^{\infty}\left(\int_0^t \left((I_{0+}^{1-\alpha}\partial_t \Lambda g)(\cdot, t-\tau), \varphi_n\right)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n(x)$$
$$= -\sum_{n=1}^{\infty}\left(\int_0^t \left((\partial_t^\alpha \Lambda g)(\cdot, t-\tau), \varphi_n\right)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n(x).$$

By Theorem 2.2 in [36] (or Proposition 3.1 in [13]), w solves

$$\begin{cases} \partial_t^\alpha w + \mathscr{L}w = -\partial_t^\alpha \Lambda g & \text{in} \quad \Omega\times(0,T), \\ w = 0 & \text{on} \quad \Gamma\times(0,T), \\ w(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

By substituting $w = u - \Lambda g$, we see that $u(\cdot,0) = 0$ and

$$\partial_t^\alpha u(\cdot,t) + \mathscr{L}u(\cdot,t) = 0$$

holds in $L^2(\Omega)$ for almost every $t\in(0,T)$. Moreover since $w\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, we have

$$\gamma_0 u = \gamma_0(w+\Lambda g) = \gamma_0(\Lambda g) = g.$$

Step 2. Next we prove that the function u given by (2.2.4) satisfies estimates (2.2.2)-(2.2.3). Using representation (2.3.10),

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(\Omega)} &= \left\|\sum_{n=1}^{\infty}\left(\int_0^t \lambda_n(\Lambda g(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n\right\|_{L^2(\Omega)} \\
&= \left\|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n(\Lambda g(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n\right)d\tau\right\|_{L^2(\Omega)} \\
&\le \int_0^t\left\|\sum_{n=1}^{\infty}\lambda_n(\Lambda g(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\varphi_n\right\|_{L^2(\Omega)} d\tau \\
&= \int_0^t\left(\sum_{n=1}^{\infty}\left|\lambda_n(\Lambda g(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\right|^2\right)^{1/2} d\tau \\
&= \int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\theta}|(\Lambda g(\cdot,t-\tau),\varphi_n)|^2\cdot\left|\lambda_n^{1-\theta}\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\right|^2\right)^{1/2} d\tau. \quad (2.3.12)
\end{aligned}$$

By Lemma 1.3.2, we have

$$|\lambda_n^{1-\theta}\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)| \le \lambda_n^{1-\theta}\tau^{\alpha-1}\cdot\frac{C}{1+\lambda_n\tau^\alpha} = C\cdot\frac{(\lambda_n\tau^\alpha)^{1-\theta}}{1+\lambda_n\tau^\alpha}\cdot\tau^{\alpha\theta-1} \le C\tau^{\alpha\theta-1}. \quad (2.3.13)$$

Applying (2.3.6) and (2.3.13) to (2.3.12), we obtain

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\theta}|(\Lambda g(\cdot,t-\tau),\varphi_n)|^2\right)^{1/2}\tau^{\alpha\theta-1}d\tau$$

$$= C\int_0^t \|\Lambda g(\cdot,t-\tau)\|_{\mathcal{D}(L^\theta)}\tau^{\alpha\theta-1}d\tau \le C\int_0^t \|g(\cdot,t-\tau)\|_{L^2(\Gamma)}\tau^{\alpha\theta-1}d\tau$$
$$\le C\left(\int_0^t \tau^{\alpha\theta-1}d\tau\right)\|g\|_{L^\infty(0,T;L^2(\Gamma))} \le Ct^{\alpha\theta}\|g\|_{L^\infty(0,T;L^2(\Gamma))}.$$

Thus we have proved estimate (2.2.2).

In order to show (2.2.3), we estimate $w = u - \Lambda g$. By (2.3.11),

$$\begin{aligned}
\|Lw(\cdot,t)\|_{L^2(\Omega)} &= \left\|\sum_{n=1}^\infty \lambda_n \left(\int_0^t \big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n\tau^\alpha)d\tau\right)\varphi_n\right\|_{L^2(\Omega)} \\
&= \left\|\int_0^t \left(\sum_{n=1}^\infty \lambda_n\big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n\tau^\alpha)\varphi_n\right)d\tau\right\|_{L^2(\Omega)} \\
&\le \int_0^t \left\|\sum_{n=1}^\infty \lambda_n\big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n\tau^\alpha)\varphi_n\right\|_{L^2(\Omega)} d\tau \\
&\le \int_0^t \left(\sum_{n=1}^\infty \lambda_n^{2\theta}\big|\big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)\big|^2\cdot|\lambda_n^{1-\theta}E_{\alpha,1}(-\lambda_n\tau^\alpha)|^2\right)^{1/2} d\tau. \quad (2.3.14)
\end{aligned}$$

Similarly to (2.3.13), we have

$$|\lambda_n^{1-\theta}E_{\alpha,1}(-\lambda_n\tau^\alpha)| \le \lambda_n^{1-\theta}\cdot\frac{C}{1+\lambda_n\tau^\alpha} = C\cdot\frac{(\lambda_n\tau^\alpha)^{1-\theta}}{1+\lambda_n\tau^\alpha}\cdot\tau^{\alpha(\theta-1)} \le C\tau^{\alpha(\theta-1)}. \quad (2.3.15)$$

Applying (2.3.6) and (2.3.15) to (2.3.14), we obtain

$$\begin{aligned}
\|Lw(\cdot,t)\|_{L^2(\Omega)} &\le C\int_0^t \left(\sum_{n=1}^\infty \lambda_n^{2\theta}\big|\big((\partial_t\Lambda g)(\cdot,t-\tau),\varphi_n\big)\big|^2\right)^{1/2}\tau^{\alpha(\theta-1)}d\tau \\
&= C\int_0^t \|(\partial_t\Lambda g)(\cdot,t-\tau)\|_{\mathcal{D}(L^\theta)}\tau^{\alpha(\theta-1)}d\tau \\
&\le C\int_0^t \|(\partial_t g)(\cdot,t-\tau)\|_{L^2(\Gamma)}\tau^{\alpha(\theta-1)}d\tau \le C\left(\int_0^t \tau^{\alpha(\theta-1)}d\tau\right)\|\partial_t g\|_{L^\infty(0,T;L^2(\Gamma))} \\
&\le Ct^{\alpha(\theta-1)+1}\|\partial_t g\|_{L^\infty(0,T;L^2(\Gamma))}.
\end{aligned}$$

Since $u = w + \Lambda g$, we have

$$\begin{aligned}
\|u(\cdot,t)\|_{H^2(\Omega)} &\le \|w(\cdot,t)\|_{H^2(\Omega)} + \|\Lambda g(\cdot,t)\|_{H^2(\Omega)} \le C\|Lw(\cdot,t)\|_{L^2(\Omega)} + C\|g(\cdot,t)\|_{H^{3/2}(\Gamma)} \\
&\le C\left(t^{\alpha(\theta-1)+1}\|\partial_t g\|_{L^\infty(0,T;L^2(\Gamma))} + \|g(\cdot,t)\|_{H^{3/2}(\Gamma)}\right).
\end{aligned}$$

Similarly we can also show

$$\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} \le C\left(t^{\alpha(\theta-1)+1}\|\partial_t^{m+1} g\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t^m g(\cdot,t)\|_{H^{3/2}(\Gamma)}\right)$$

for any $m \in \mathbb{N}$.

Moreover the above estimates also indicate the convergence of the series in (2.3.11) in $C^m([0,T];H^2(\Omega))$. Hence we see that (2.2.4) is convergent in $C^m([0,T];H^2(\Omega))$ and consequently $u\in C^m([0,T];H^2(\Omega))$ for $m=0,1,2,\dots$. Thus the proof of Theorem 2.2.1 is completed. $\square$

2.4 Dual System

In this section, we prove some properties of the solution to (1.4.1), which we have studied also in the previous chapter. In order to prove Theorem 2.2.2, we also need more results for it—especially the unique continuation property from the subboundary Γ_0 (Proposition 2.4.2). To this end, we also prove the analyticity of $\partial_{\nu_L}v$ in Proposition 2.4.1.

Proposition 2.4.1. *Let $0<\alpha<1$, $0<\theta<1/4$ and $r\in(1,\infty)$ satisfy $r(1-\alpha\theta)<1$. Let $v\in C([0,T);H^2(\Omega)\cap H_0^1(\Omega))$ be the solution of (1.4.1) for $v_0\in L^2(\Omega)$. Then $\partial_{\nu_L}v$ belongs to $L^r(0,T;L^2(\Gamma))$ with the estimate;*

$$\|\partial_{\nu_L}v\|_{L^r(0,T;L^2(\Gamma))}\le C\|v_0\|_{L^2(\Omega)} \tag{2.4.1}$$

Moreover $\partial_{\nu_L}v:[0,T)\to L^2(\Gamma)$ is analytically extended to $S_T:=\{z\in\mathbb{C};\operatorname{Re}z<T\}$.

Proposition 2.4.2. *Let Γ_0 be open in Γ and $v\in C([0,T);H^2(\Omega)\cap H_0^1(\Omega))$ be the solution of (1.4.1) corresponding to $v_0\in L^2(\Omega)$. If $\partial_{\nu_L}v=0$ on $\Gamma_0\times(0,T)$, then $v=0$ in $\Omega\times(0,T)$.*

Proof of Proposition 2.4.1. We first prove $\partial_{\nu_L}v\in L^r(0,T;L^2(\Gamma))$ and estimate (2.4.1). By $0<\theta<1/4$ and the boundedness of the operator $\partial_{\nu_L}:H^s(\Omega)\to H^{s-3/2}(\Gamma)$, $s>3/2$, we have

$$\begin{aligned}\|\partial_{\nu_L}v(\cdot,t)\|^2_{L^2(\Gamma)}&\le\|\partial_{\nu_L}v(\cdot,t)\|^2_{H^{1/2-2\theta}(\Gamma)}\le C\|v(\cdot,t)\|^2_{H^{2-2\theta}(\Omega)}\le C\|v(\cdot,t)\|^2_{\mathcal{D}(L^{1-\theta})}\\&=C\sum_{n=1}^\infty\lambda_n^{2-2\theta}|(v_0,\varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)|^2\\&=C\sum_{n=1}^\infty|(v_0,\varphi_n)|^2\cdot|\lambda_n^{1-\theta}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)|^2.\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}|\lambda_n^{1-\theta}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)|&\le\lambda_n^{1-\theta}(T-t)^{\alpha-1}\cdot\frac{C}{1+\lambda_n(T-t)^\alpha}\\&\le C\cdot\frac{(\lambda_n(T-t)^\alpha)^{1-\theta}}{1+\lambda_n(T-t)^\alpha}\cdot(T-t)^{\alpha\theta-1}\le C(T-t)^{\alpha\theta-1}.\end{aligned}$$

Therefore we have

$$\|\partial_{\nu_L} v(\cdot,t)\|_{L^2(\Gamma)} \le C(T-t)^{\alpha\theta-1}\left(\sum_{n=1}^{\infty}|(v_0,\varphi_n)|^2\right)^{1/2} = C(T-t)^{\alpha\theta-1}\|v_0\|_{L^2(\Omega)}.$$

Hence we obtain

$$\|\partial_{\nu_L} v\|_{L^r(0,T;L^2(\Gamma))} \le C\|v_0\|_{L^2(\Omega)}.$$

Next we prove the analyticity of $\partial_{\nu_L} v(\cdot,t)$ in $t \in S_T$. Since $\partial_{\nu_L} : H^2(\Omega) \to L^2(\Gamma)$ is bounded, we have

$$\partial_{\nu_L} v(\cdot,t) = \sum_{n=1}^{\infty}(v_0,\varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})\partial_{\nu_L}\varphi_n \tag{2.4.2}$$

and the series in (2.4.2) is convergent in $L^2(\Gamma)$. We note that $E_{\alpha,\alpha}(-\lambda_n z)$ is an entire function (see Section 1.8 in [20] for example). Therefore each $(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})$ is analytic in $z \in S_T$ and so is their linear combination $\sum_{n=1}^{N}(v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\partial_{\nu_L}\varphi_n$. If we fix $\delta > 0$ arbitrarily, then for $z \in \mathbb{C}$ with $\operatorname{Re} z \le T-\delta$, we have

$$\begin{aligned}
&\left\|\sum_{n=M}^{N}(v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\partial_{\nu_L}\varphi_n\right\|_{L^2(\Gamma)}^2 \\
\le{}& C\left\|\sum_{n=M}^{N}(v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\varphi_n\right\|_{H^2(\Omega)}^2 \\
\le{}& C\left\|\sum_{n=M}^{N}(v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\varphi_n\right\|_{\mathcal{D}(L)}^2 \\
={}& C\sum_{n=M}^{N}|\lambda_n(v_0,\varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})|^2 \\
\le{}& C\sum_{n=M}^{N}|(v_0,\varphi_n)|^2|T-z|^{-2}\left(\frac{\lambda_n|T-z|^{\alpha}}{1+\lambda_n|T-z|^{\alpha}}\right)^2 \\
\le{}& C\delta^{-2}\sum_{n=M}^{N}|(v_0,\varphi_n)|^2 \to 0 \quad \text{as } M,N\to\infty.
\end{aligned}$$

That is, (2.4.2) is uniformly convergent in $\{z \in \mathbb{C}; \operatorname{Re} z \le T-\delta\}$. Hence $\partial_{\nu_L} v$ is also analytic in S_T. □

Proof of Proposition 2.4.2. Since $\partial_{\nu_L} v = 0$ in $\Gamma_0 \times (0,T)$ and $\partial_{\nu_L} v : [0,T) \to L^2(\Omega)$ can be analytically extended to S_T, we have

$$\partial_{\nu_L} v(x,t) = \sum_{n=1}^{\infty}(v_0,\varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})\partial_{\nu_L}\varphi_n(x) = 0, \quad x\in\Gamma_0,\ t\in(-\infty,T). \tag{2.4.3}$$

Let $\{\mu_k\}_{k\in\mathbb{N}}$ be all spectra of L without multiplicities and we denote by $\{\varphi_{kj}\}_{1\le j\le m_k}$ an orthonormal basis of $\mathrm{Ker}(\mu_k - L)$. By using these notations, we can rewrite (2.4.3) by

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\partial_{\nu_L}\varphi_{kj}(x)\right)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)=0, \quad x\in\Gamma_0,\ t\in(-\infty,T). \tag{2.4.4}$$

Let $0<\theta<1/4$ be fixed. Then for any $z\in\mathbb{C}$ with $\mathrm{Re}\,z=\xi>0$ and $N\in\mathbb{N}$, we repeat the similar calculation as in the previous proof and obtain

$$\begin{aligned}
&\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\partial_{\nu_L}\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{L^2(\Gamma)}^2\\
\le&\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\partial_{\nu_L}\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{H^{1/2-2\theta}(\Gamma)}^2\\
\le&\, C\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{H^{2-2\theta}(\Omega)}^2\\
\le&\, C\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{\mathcal{D}(L^{1-\theta})}^2\\
=&\, C\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}|(v_0,\varphi_{kj})|^2\right)e^{2\xi(t-T)}\left|\mu_k^{1-\theta}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right|^2.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
\left|\mu_k^{1-\theta}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right| &\le \mu_k^{1-\theta}(T-t)^{\alpha-1}\cdot\frac{C}{1+\mu_k(T-t)^\alpha}\\
&\le C\cdot\frac{(\mu_k(T-t)^\alpha)^{1-\theta}}{1+\mu_k(T-t)^\alpha}\cdot(T-t)^{\alpha\theta-1}\\
&\le C(T-t)^{\alpha\theta-1}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k}(v_0,\varphi_{kj})\partial_{\nu_L}\varphi_{kj}\right)e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\mu_k(T-t)^\alpha)\right\|_{L^2(\Gamma)}\\
&\le Ce^{\xi(t-T)}(T-t)^{\alpha\theta-1}\|v_0\|_{L^2(\Omega)}.
\end{aligned}$$

The right-hand side of the above is integrable on $(-\infty,T)$;

$$\int_{-\infty}^{T}e^{\xi(t-T)}(T-t)^{\alpha\theta-1}dt=\frac{\Gamma(\alpha\theta)}{\xi^{\alpha\theta}}.$$

Hence the Lebesgue theorem yields that

$$\begin{aligned}
&\int_{-\infty}^{T} e^{z(t-T)} \left(\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \partial_{\nu_L} \varphi_{kj}(x) (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k (T-t)^{\alpha}) \right) dt \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \partial_{\nu_L} \varphi_{kj}(x) \left(\int_{-\infty}^{T} e^{z(t-T)} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k (T-t)^{\alpha}) dt \right) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{(v_0, \varphi_{kj})}{z^{\alpha} + \mu_k} \partial_{\nu_L} \varphi_{kj}(x), \quad \text{a. e. } x \in \Gamma, \ \mathrm{Re}\, z > 0. \qquad (2.4.5)
\end{aligned}$$

By (2.4.4) and (2.4.5), we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{(v_0, \varphi_{kj})}{z^{\alpha} + \mu_k} \partial_{\nu_L} \varphi_{kj}(x) = 0, \quad \text{a. e. } x \in \Gamma_0, \ \mathrm{Re}\, z > 0,$$

that is,

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{(v_0, \varphi_{kj})}{\eta + \mu_k} \partial_{\nu_L} \varphi_{kj}(x) = 0, \quad \text{a. e. } x \in \Gamma_0, \ \mathrm{Re}\, \eta > 0. \qquad (2.4.6)$$

By using analytic continuation in η, we may assume (2.4.6) holds for $\eta \in \mathbb{C} \setminus \{-\mu_k\}_{k \in \mathbb{N}}$. Then we can take a suitable disk which includes $-\mu_\ell$ and does not include $\{-\mu_k\}_{k \neq \ell}$. By integrating (2.4.6) in the disk, we have

$$\sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \partial_{\nu_L} \varphi_{\ell j}(x) = 0, \quad \text{a. e. } x \in \Gamma_0.$$

By setting $\widetilde{v}_\ell := \sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \varphi_{\ell j}$, we have

$$(L - \mu_\ell) \widetilde{v}_\ell = 0 \quad \text{in } \Omega \quad \text{and} \quad \partial_{\nu_L} \widetilde{v}_\ell = 0 \quad \text{on } \Gamma_0.$$

Therefore the unique continuation result for eigenvalue problem of elliptic operator (see Corollary 2.2 in [39] for example) implies

$$\widetilde{v}_\ell(x) = \sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \varphi_{\ell j}(x) = 0, \quad x \in \Omega$$

for each $\ell \in \mathbb{N}$. Since $\{\varphi_{\ell j}\}_{1 \le j \le m_\ell}$ is linearly independent in Ω, we see that

$$(v_0, \varphi_{\ell j}) = 0, \quad 1 \le j \le m_\ell, \ \ell \in \mathbb{N}.$$

This implies $v = 0$ in $\Omega \times (0, T)$. $\square$

2.5 Proof of Theorem 2.2.2

In this section, we complete the proof of Theorem 2.2.2 by using the results which we have proved in Sections 2.3 and 2.4.

Let u_g be the solution to (2.2.1) for $g \in C_0^\infty(\Gamma_0 \times (0,T))$, and v be the solution to (1.4.1) for $v_0 \in L^2(\Omega)$. We first prove that

$$\int_\Omega u_g(\cdot,T)v_0 dx + \int_0^T \int_{\Gamma_0} g\frac{\partial v}{\partial \nu_L} d\sigma dt = 0 \tag{2.5.1}$$

holds for any $g \in C_0^\infty(\Gamma_0 \times (0,T))$ and $v_0 \in H^2(\Omega)\cap H_0^1(\Omega)$.

Since the first equation in (2.2.1) holds in $C^\infty([0,T];L^2(\Omega))$ and $v \in L^q(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ by Theorem 2.2.1 and Proposition 2.4.1 respectively, we can see that

$$0 = \int_0^T \int_\Omega (\partial_t^\alpha u_g + \mathscr{L}u_g)v dxdt = \int_0^T \int_\Omega (\partial_t^\alpha u_g)v dxdt + \int_0^T \int_\Omega (\mathscr{L}u_g)v dxdt. \tag{2.5.2}$$

We calculate the first term on the right-hand side of (2.5.2) as follows;

$$\begin{aligned}\int_0^T \int_\Omega (\partial_t^\alpha u_g)v dxdt &= \int_0^T \int_\Omega I_{0+}^{1-\alpha}\frac{\partial u_g}{\partial t}\cdot v dtdx = \int_0^T \int_\Omega \frac{\partial u_g}{\partial t}\cdot I_{T-}^{1-\alpha} v dtdx \\ &= \int_\Omega u_g \cdot I_{T-}^{1-\alpha} v dx\bigg|_{t=0}^{t=T} - \int_\Omega \int_0^T u_g\cdot \frac{\partial}{\partial t} I_{T-}^{1-\alpha} v dtdx \\ &= \int_\Omega u_g(\cdot,T)v_0 dx + \int_0^T \int_\Omega u_g(D_t^\alpha v) dxdt.\end{aligned}$$

Here we have used the integration in t by parts and the initial condition in (2.1.1). In terms of $u_g \in C^\infty([0,T];H^2(\Omega))$ and $Lv \in L^q(0,T;L^2(\Omega))$ by Theorem 2.2.1 and Proposition 2.4.1, we apply the Green formula to the second term on the right-hand side of (2.5.2) and have

$$\begin{aligned}\int_0^T \int_\Omega (\mathscr{L}u_g)v dxdt &= \int_0^T \int_\Omega u_g(\mathscr{L}v) dxdt + \int_0^T \int_\Gamma \left(u_g\frac{\partial v}{\partial \nu_L} - \frac{\partial u_g}{\partial \nu_L}v\right) d\sigma dt \\ &= \int_0^T \int_\Omega u_g(\mathscr{L}v) dxdt + \int_0^T \int_{\Gamma_0} g\frac{\partial v}{\partial \nu_L} d\sigma dt.\end{aligned}$$

In the above calculation, we have used boundary conditions in (2.2.1) and (1.4.1). Therefore we have

$$\begin{aligned}0 &= \left\{\int_\Omega u_g(\cdot,T)v_0 dx + \int_0^T \int_\Omega u_g(D_t^\alpha v) dxdt\right\} + \left\{\int_0^T \int_\Omega u_g(\mathscr{L}v) dxdt + \int_0^T \int_{\Gamma_0} g\frac{\partial v}{\partial \nu_L} d\sigma dt\right\} \\ &= \int_\Omega u_g(\cdot,T)v_0 dx + \int_0^T \int_\Omega u_g(D_t^\alpha v + \mathscr{L}v) dxdt + \int_0^T \int_{\Gamma_0} g\frac{\partial v}{\partial \nu_L} d\sigma dt \\ &= \int_\Omega u_g(\cdot,T)v_0 dx + \int_0^T \int_{\Gamma_0} g\frac{\partial v}{\partial \nu_L} d\sigma dt.\end{aligned}$$

Thus (2.5.1) holds for $g \in C_0^\infty(\Gamma_0 \times (0,T))$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$.

Let $g \in C_0^\infty(\Gamma_0 \times (0,T))$ be fixed. Then the mapping

$$v_0 \mapsto \int_\Omega u_g(\cdot,T) v_0 dx + \int_0^T \int_{\Gamma_0} g \frac{\partial v}{\partial \nu_L} d\sigma dt$$

is a linear and bounded functional by (2.4.1). Hence the density argument implies that (2.5.1) holds for any $v_0 \in L^2(\Omega)$.

In order to prove the density of $\{u_g(\cdot,T);\ g \in C_0^\infty(\Gamma_0 \times (0,T))\}$ in $L^2(\Omega)$, we will show that

$$\{u_g(\cdot,T);\ g \in C_0^\infty(\Gamma_0 \times (0,T))\}^\perp = \{0\}. \tag{2.5.3}$$

This can be shown as follows.

Let v_0 belong to the left-hand side of (2.5.3), then (2.5.1) yields

$$\int_0^T \int_{\Gamma_0} g \frac{\partial v}{\partial \nu_L} d\sigma dt = -\int_\Omega u_g(\cdot,T) v_0 dx = 0$$

for any $g \in C_0^\infty(\Gamma_0 \times (0,T))$. By the fundamental lemma of the calculus of variations, we have

$$\frac{\partial v}{\partial \nu_L}(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,T).$$

By Proposition 2.4.2, we have

$$v_0(x) = 0, \quad x \in \Omega.$$

Thus we have shown (2.5.3) and the proof of Theorem 2.2.2 is completed.

Chapter 3

Inverse Source Problem

3.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$, with C^2 boundary $\partial\Omega$. We set $\Sigma = \partial\Omega \times (0, T)$ and $Q = \Omega \times (0, T)$. We consider the following two initial-boundary value problem (IBVP in short) for the fractional diffusion equation

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) = f(t)R(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = 0, & x \in \Omega \end{cases} \tag{3.1.1}$$

and

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + f(t)q(x,t)v(x,t) = 0, & (x,t) \in Q, \\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma, \\ v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{3.1.2}$$

with $0 < \alpha < 1$. Here we denote by ∂_t^α the Caputo fractional derivative with respect to t (see (1.1.2)). The differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}u(x,t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x,t)\right),$$

where the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \le i, j \le d, \quad \text{and} \quad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d$$

for some $\mu > 0$. Moreover $\mathcal{B}_\sigma$ is defined as

$$\mathcal{B}_\sigma u(x) = (1 - \sigma(x))u(x) + \sigma(x)\partial_{\nu_A}u(x), \quad x \in \partial\Omega,$$

where

$$\partial_{\nu_A}u(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\nu_j(x)$$

and $\nu = (\nu_1, \ldots, \nu_d)$ is the outward normal unit vector to $\partial\Omega$. Here σ is a C^2 function on $\partial\Omega$ satisfying

$$0 \le \sigma(x) \le 1, \quad x \in \partial\Omega.$$

For the regularity of a_{ij}, we assume

$$\begin{cases} a_{ij} \in C^1(\overline{\Omega}) & \text{if } \sigma \equiv 0, \\ a_{ij} \in C^2(\overline{\Omega}) & \text{if } \sigma \not\equiv 0. \end{cases}$$

Note that the regularity for a_{ij} depends on whether $\sigma \equiv 0$ or not, which is due to condition (3.2.3) in the next section.

In this chapter, we consider the inverse problem which consists of determining the function $\{f(t)\}_{t\in(0,T)}$ in (3.1.1) and (3.1.2) from the observation of the solution at a point $x_0 \in \overline{\Omega}$ for all $t \in (0,T)$.

The partial differential equations with time fractional derivatives such as (3.1.1) and (3.1.2) are proposed as new models describing the anomalous diffusion phenomena. In particular, the fractional diffusion equation can be used as a model for the diffusion of contaminants in a soil. Therefore the inverse problem considered in this chapter means the determination of the time evolution of pollution source in (3.1.1) and reaction rate of pollutants in (3.1.2) respectively. In this chapter, we consider such problems assuming the boundedness of the time-dependent parameter $\{f(t)\}_{t\in(0,T)}$ (see (3.2.1)).

The remainder of this chapter is composed of four sections. In Section 3.2, we state our main results. In Section 3.3, we study the forward problem for the IBVPs (3.1.1) and (3.1.2) and prove the unique existence and regularity of the solutions. In Sections 3.4 and 3.5, we complete the proof of our main results—Theorems 3.2.1 and 3.2.2 respectively.

3.2 Main results

By $L^2(\Omega)$, we denote the usual L^2-space equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$. Moreover $H^s(\Omega)$, $s \in \mathbb{R}$, and $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $1 \le p \le \infty$, are the Sobolev spaces (see Adams [2] for example).

For the time dependent parameter $\{f(t)\}_{t\in(0,T)}$, we always assume

$$f \in L^\infty(0,T). \tag{3.2.1}$$

For other given functions in (3.1.1), we suppose

$$R \in L^p(0,T;H^2(\Omega)), \ \frac{8}{\alpha} < p \le \infty \quad \text{and} \quad \mathcal{B}_\sigma R = 0 \quad \text{on } \Sigma. \tag{3.2.2}$$

On the other hand, in the IBVP (3.1.2), we suppose

$$\begin{cases} q \in L^\infty(0,T;H^2(\Omega)) \quad (\text{and } \partial_\nu q = 0 \text{ on } \Sigma \text{ if } \sigma \not\equiv 0), \\ v_0 \in H^4(\Omega) \quad \text{and} \quad \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A}v_0) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.2.3}$$

Assuming these conditions, we prove in Section 3.3 that the IBVPs (3.1.1) and (3.1.2) admit unique solutions $u, v \in C([0,T]; H^2(\Omega))$ with $\partial_t^\alpha u \in L^p(0,T; H^s(\Omega))$ and $\partial_t^\alpha v \in L^\infty(0,T; H^s(\Omega))$ for some $s > d/2$. Therefore, using the Sobolev embedding theorem (see Theorem 9.8 in Chapter 1 of [21] for example), for any $x_0 \in \overline{\Omega}$, we see that

$$\partial_t^\alpha u(x_0, \cdot) \in L^p(0,T) \quad \text{and} \quad \partial_t^\alpha v(x_0, \cdot) \in L^\infty(0,T).$$

Then our main results can be stated as follows;

Theorem 3.2.1. *Let condition* (3.2.2) *be fulfilled and* u_i *be the solution of* (3.1.1) *for* $f = f_i \in L^\infty(0,T)$, $(i = 1,2)$. *We assume that there exist* $x_0 \in \overline{\Omega}$ *and* $\delta > 0$ *such that*

$$|R(x_0,t)| \geq \delta, \quad a.e.\ t \in (0,T). \tag{3.2.4}$$

Then there exists a constant $C > 0$ *depending on* p, T, Ω, δ *and* $\|R\|_{L^p(0,T;H^2(\Omega))}$ *such that*

$$\|f_1 - f_2\|_{L^p(0,T)} \leq C\|\partial_t^\alpha u_1(x_0,\cdot) - \partial_t^\alpha u_2(x_0,\cdot)\|_{L^p(0,T)}, \tag{3.2.5}$$

$$\|\partial_t^\alpha u_1(x_0,\cdot) - \partial_t^\alpha u_2(x_0,\cdot)\|_{L^p(0,T)} \leq C\|f_1 - f_2\|_{L^\infty(0,T)}. \tag{3.2.6}$$

In particular, if we take $p = \infty$ *in* (3.2.2), *then*

$$\begin{aligned} C^{-1}\|\partial_t^\alpha u_1(x_0,\cdot) - \partial_t^\alpha u_2(x_0,\cdot)\|_{L^\infty(0,T)} &\leq \|f_1 - f_2\|_{L^\infty(0,T)} \\ &\leq C\|\partial_t^\alpha u_1(x_0,\cdot) - \partial_t^\alpha u_2(x_0,\cdot)\|_{L^\infty(0,T)}. \end{aligned}$$

Theorem 3.2.2. *Let condition* (3.2.3) *be fulfilled and* v_i *be the solution of* (3.1.2) *for* $f = f_i \in L^\infty(0,T)$ *with* $\|f_i\|_{L^\infty(0,T)} \leq M$ $(i = 1,2)$. *We assume that there exist* $x_0 \in \overline{\Omega}$ *and* $\delta > 0$ *such that*

$$|q(x_0,t)v_2(x_0,t)| \geq \delta, \quad a.e.\ t \in (0,T). \tag{3.2.7}$$

Then there exists a constant $C > 0$ *depending on* M, T, Ω, δ *and* $\|q\|_{L^\infty(0,T;H^2(\Omega))}$ *such that*

$$\begin{aligned} C^{-1}\|\partial_t^\alpha v_1(x_0,\cdot) - \partial_t^\alpha v_2(x_0,\cdot)\|_{L^\infty(0,T)} &\leq \|f_1 - f_2\|_{L^\infty(0,T)} \\ &\leq C\|\partial_t^\alpha v_1(x_0,\cdot) - \partial_t^\alpha v_2(x_0,\cdot)\|_{L^\infty(0,T)}. \end{aligned} \tag{3.2.8}$$

In Theorem 4.4 of Sakamoto and Yamamoto [36], a similar problem to Theorem 3.2.1 is considered, but our result is more applicable in the point of view that the factor $R(x,t)$ is also allowed to depend on t. Moreover, we may assume less regularity for R in Theorem 3.2.1. The arguments of Theorem 3.2.2 can also be applied to parabolic equations in order to consider the result of Theorem 1.1 in [8] with observations of the solution at a point $x_0 \in \Omega$ when $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$.

For such inverse problems with $\alpha = 1$, we can also refer to Section 1.5 of Prilepko, Orlovsky and Vasin [29], Cannon and Esteva [7] and Saitoh, Tuan and Yamamoto [31, 32], for example. In our main results, we assume conditions (3.2.4) and (3.2.7), which means that the observation point cannot be far from the source. On the other hand, in [7] and [31, 32], the determination of time dependent factor in the source term is considered without assuming such conditions and the logarithmic type and Hölder type estimates are proved respectively. However, the results for fractional diffusion equations without these conditions have not been obtained yet. Here we restrict ourselves to the case with assumptions (3.2.4) and (3.2.7), and show the Lipschitz type stability.

Let us remark that the results of this chapter can be extended to the case $d \geq 4$. For this purpose additional conditions such as more regularity for a_{ij} and $\partial\Omega$ are required. In order to avoid technical difficulties, we only treat the case $d \leq 3$.

3.3 Forward problem

This section is devoted to the proof of unique existence and regularity of the solution of the IBVPs (3.1.1) and (3.1.2).

Proposition 3.3.1. *Let conditions* (3.2.1) *and* (3.2.2) *be fulfilled. Then the IBVP* (3.1.1) *admits a unique solution* $u \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}u \in C([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^p(0,T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1 - 1/(p\alpha)$. *Moreover we have*

$$\|\mathcal{A}u\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} \leq C\|fR\|_{L^p(0,T;H^2(\Omega))}. \tag{3.3.1}$$

with $C > 0$ *depending on* Ω, T *and* γ

Proposition 3.3.2. *Let conditions* (3.2.1) *and* (3.2.3) *be fulfilled. Then the IBVP* (3.1.2) *admits a unique solution* $v \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}v \in C([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1$. *Moreover, we have*

$$\|\mathcal{A}v\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \leq C\|v_0\|_{H^4(\Omega)} \tag{3.3.2}$$

with C *depending on* Ω, T, $\|f\|_{L^\infty(0,T)}$, $\|q\|_{L^\infty(0,T;H^2(\Omega))}$ *and* γ.

If all coefficients are independent of time variable t, then we can apply eigenfunction expansion and the problems can be reduced to ordinary differential equations of fractional order (e.g. [36]). However, since we consider the determination of the time dependent factor of coefficients, we apply fixed point theorem to show the unique existence of the solutions to (3.1.1) and (3.1.2) as in Beckers and Yamamoto [5].

In order to prove these results, we consider the IBVPs with more general data in the next subsections.

3.3.1 Intermediate results

We introduce the following IBVPs

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) = F(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.3.3}$$

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + p(x,t)v(x,t) = F(x,t), & (x,t) \in Q, \\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma, \\ v(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.3.4}$$

and

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) + p(x,t)v(x,t) = 0, & (x,t) \in Q, \\ \mathcal{B}_\sigma v(x,t) = 0, & (x,t) \in \Sigma, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.3.5}$$

We also consider the following conditions

$$F \in L^p(0,T; H^2(\Omega)), \ \frac{8}{\alpha} < p \le \infty \quad \text{and} \quad \mathcal{B}_\sigma F = 0 \quad \text{on } \Sigma \tag{3.3.6}$$

and

$$\begin{cases} 1) \ p \in L^\infty(0,T; H^2(\Omega)) \quad (\text{and } \partial_\nu p = 0 \text{ on } \Sigma \text{ if } \sigma \not\equiv 0), \\ 2) \ v_0 \in H^4(\Omega) \quad \text{and} \quad \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A}v_0) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.3.7}$$

Note that if we set $F(x,t) = f(t)R(x,t)$, then conditions (3.2.1) and (3.2.2) are equivalent to (3.3.6). Similarly, if we assume $p(x,t) = f(t)q(x,t)$, then conditions (3.2.1) and (3.2.3) are equivalent to (3.3.7). Now let us consider the following intermediate results.

Lemma 3.3.3. *Let condition* (3.3.6) *be fulfilled. Then the IBVP* (3.3.3) *admits a unique solution* $u \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}u \in C([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^p(0,T; H^{2\gamma}(\Omega))$$

for all $0 \le \gamma < 1 - 1/(p\alpha)$. *Moreover we have*

$$\|\mathcal{A}u\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} \le C\|F\|_{L^p(0,T;H^2(\Omega))} \tag{3.3.8}$$

with $C > 0$ *depending on* Ω, T *and* γ.

Lemma 3.3.4. *Let* $F \in L^\infty(0,T; H^2(\Omega))$ *satisfy* $\mathcal{B}_\sigma F = 0$ *and condition 1) of* (3.3.7) *be fulfilled. Then the IBVP* (3.3.4) *admits a unique solution* $v \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}v \in C([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T; H^{2\gamma}(\Omega))$$

for all $0 \le \gamma < 1$. *Moreover, we have*

$$\|\mathcal{A}v\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))} \tag{3.3.9}$$

with C *depending on* Ω, T, $\|p\|_{L^\infty(0,T;H^2(\Omega))}$ *and* γ.

Lemma 3.3.5. *Let condition* (3.3.7) *be fulfilled. Then the IBVP* (3.3.5) *admits a unique solution* $v \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\mathcal{A}v \in C([0,T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T; H^{2\gamma}(\Omega))$$

for all $0 \le \gamma < 1$. *Moreover, we have*

$$\|\mathcal{A}v\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|v_0\|_{H^4(\Omega)} \tag{3.3.10}$$

with C *depending on* Ω, T, $\|p\|_{L^\infty(0,T;H^2(\Omega))}$ *and* γ.

From these three lemmata we deduce easily Propositions 3.3.1 and 3.3.2.

3.3.2 Preliminary

We define the operator A as $\mathcal{A}+1$ in $L^2(\Omega)$ equipped with the boundary condition $\mathcal{B}_\sigma h = 0$;

$$\begin{cases} D(A) := \{h \in H^2(\Omega);\ \mathcal{B}_\sigma h = 0 \text{ on } \partial\Omega\}, \\ Ah := \mathcal{A}h + h, \quad h \in D(A). \end{cases} \tag{3.3.11}$$

Then A is a selfadjoint and strictly positive operator with an orthonormal basis of eigenfunctions $(\phi_n)_{n\ge1}$ of finite order associated to an increasing sequence of eigenvalues $(\lambda_n)_{n\ge1}$. We can define the fractional power A^γ, $\gamma \ge 0$, of A by

$$\begin{aligned} &D(A^\gamma) := \left\{ h \in L^2(\Omega);\ \sum_{n=1}^\infty \lambda_n^{2\gamma} |(h,\phi_n)|^2 < \infty \right\}, \\ &A^\gamma h := \sum_{n=1}^\infty \lambda_n^\gamma (h,\phi_n)\phi_n, \quad h \in D(A^\gamma). \end{aligned} \tag{3.3.12}$$

Then $D(A^\gamma)$ is a Hilbert space with the norm $\|\cdot\|_{D(A^\gamma)}$ defined by $\|h\|_{D(A^\gamma)} := \|A^\gamma h\|$. Since $D(A)$ is continuously embedded into $H^2(\Omega)$ with norm equivalence (see Theorem 5.4 in Chapter 2 of [21] for example), we see by interpolation that

$$\begin{gathered} D(A^\gamma) \subset H^{2\gamma}(\Omega), \\ C^{-1}\|h\|_{H^{2\gamma}(\Omega)} \le \|h\|_{D(A^\gamma)} \le C\|h\|_{H^{2\gamma}(\Omega)}, \quad h \in D(A^\gamma) \end{gathered}$$

for $0 \le \gamma \le 1$.

In order to prepare for the arguments used in this chapter, we consider the following Cauchy problem in $L^2(\Omega)$;

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = F(t), \quad t \in (0,T), \\ u(0) = 0. \end{cases} \tag{3.3.13}$$

We define the operator valued function $\{S_A(t)\}_{t\geq 0}$ by

$$S_A(t)h = \sum_{n=1}^{\infty}(h, \phi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n, \quad h \in L^2(\Omega),\ t \geq 0,$$

with $E_{\alpha,\beta}$, $\alpha > 0, \beta \in \mathbb{R}$, the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Recall that $S_A(t) \in W^{1,1}(0,T;\mathcal{B}(L^2(\Omega)))$ (e.g. [5] and [36]). Moreover, similarly to Theorem 2.2 in [36], for $F \in L^\infty(0,T;L^2(\Omega))$, problem (3.3.13) admits a unique solution given by

$$u(t) = \int_0^t A^{-1}S'_A(t-s)F(s)ds. \tag{3.3.14}$$

This solution is lying in $L^\infty(0,T;D(A^\gamma))$ for $0 \leq \gamma < 1$, and, in view of Theorem 1 in [5], we have

$$\|A^{\gamma-1}S'_A(t)h\| \leq Ct^{\alpha(1-\gamma)-1}\|h\|, \quad h \in L^2(\Omega),\ t > 0. \tag{3.3.15}$$

In particular, the mapping $t \mapsto A^{-1}S'_A(t)$ belongs to $L^q(0,T;\mathcal{B}(L^2(\Omega)))$ if $q \in (1,\infty)$ satisfy $q(1-\alpha) < 1$. Now we apply the following Young's inequality to (3.3.14);

Lemma 3.3.6. *Let $f \in L^p(0,T)$ and $g \in L^q(0,T)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then the function $f * g$ defined by*

$$f * g(t) := \int_0^t f(t-s)g(s)ds$$

belongs to $C[0,T]$ and satisfies

$$|f * g(t)| \leq \|f\|_{L^p(0,t)}\|g\|_{L^q(0,t)}, \quad t \in [0,T].$$

Proof. Let $\tilde{f}$ and $\tilde{g}$ be defined by

$$\tilde{f}(t) := \begin{cases} f(t), & t \in (0,T), \\ 0, & t \notin (0,T), \end{cases} \quad \text{and} \quad \tilde{g}(t) := \begin{cases} g(t), & t \in (0,T), \\ 0, & t \notin (0,T). \end{cases}$$

Then applying the Young's inequality for functions on $\mathbb{R}$ (see Exercise 4.30 in Brezis [6] or Appendix A in Stein [40] for example), we obtain the desired result. □

Let $p \in (1,\infty]$ be as in (3.3.6). Noting that $\mathcal{A}$ and $A^{-1}S'_A(t)$ commute, we see that for $F \in L^p(0,T;D(A))$,

$$\mathcal{A}u(t) = \int_0^t A^{-1}S'_A(t-s)\mathcal{A}F(s)ds.$$

By $p > 1/\alpha$ and (3.3.15), the mapping $t \mapsto A^{-1}S_A'(t)$ belongs to $L^q(0,T;\mathcal{B}(L^2(\Omega)))$ where $q \in [1,\infty)$ satisfies $1/p + 1/q = 1$. Therefore by Lemma 3.3.6, u belongs to $C([0,T];D(A))$ and satisfies

$$\|\mathcal{A}u(t)\| \le \int_0^t \|A^{-1}S_A'(t-s)\|\|\mathcal{A}F(s)\|ds \le C\int_0^t (t-s)^{\alpha-1}\|F(s)\|_{D(A)}ds \tag{3.3.16}$$

$$\le C\left(\int_0^t s^{(\alpha-1)q}ds\right)^{1/q}\|F\|_{L^p(0,t;D(A))} \le Ct^{\alpha-1/p}\|F\|_{L^p(0,T;D(A))}. \tag{3.3.17}$$

Thus we can define the map $\mathcal{H} : L^p(0,T;D(A)) \to C([0,T];D(A))$ by

$$\mathcal{H}(w)(t) := \int_0^t A^{-1}S_A'(t-s)w(s)ds, \quad w \in L^p(0,T;D(A)). \tag{3.3.18}$$

By using these estimates, we will show the unique existence of the solution applying the fixed point theorem.

3.3.3 Proof of Lemmata 3.3.3-3.3.5

Proof of Lemma 3.3.3. Let A be the operator defined by (3.3.11), then the IBVP (3.3.3) can be rewritten as

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = u(t) + F(t), & t \in (0,T), \\ u(0) = 0, \end{cases} \tag{3.3.19}$$

where $u(t) := u(\cdot,t)$ and $F(t) := F(\cdot,t)$. Noting that $F \in L^p(0,T;D(A))$ by (3.3.6), we see from (3.3.14) that the solution u of (3.3.19) satisfies

$$u(t) = \mathcal{H}(u)(t) + \mathcal{H}(F)(t), \quad t \in (0,T),$$

where the map $\mathcal{H}$ is defined by (3.3.18). Therefore we will look for a fixed point of the map $\mathcal{G} : C([0,T];D(A)) \to C([0,T];D(A))$ defined by

$$\mathcal{G}(w)(t) := \mathcal{H}(w)(t) + \mathcal{H}(F)(t), \quad w \in C([0,T];D(A)),\ t \in (0,T). \tag{3.3.20}$$

By (3.3.16), for $w \in C([0,T];D(A))$, we have

$$\begin{aligned}\|\mathcal{H}(w)(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|w(s)\|_{D(A)}ds \le C\left(\int_0^t (t-s)^{\alpha-1}ds\right)\|w\|_{C([0,T];D(A))} \\ &= \frac{Ct^\alpha}{\alpha}\|w\|_{C([0,T];D(A))}.\end{aligned}$$

Repeating the similar calculation, we get

$$\|\mathcal{H}^2w(t)\|_{D(A)} \le C\int_0^t (t-s)^{\alpha-1}\|\mathcal{H}w(s)\|_{D(A)}ds \le \frac{C}{\alpha}\left(\int_0^t (t-s)^{\alpha-1}s^\alpha ds\right)\|w\|_{C([0,T];D(A))}$$

$$= \frac{C(\Gamma(\alpha)t^\alpha)^2}{\Gamma(2\alpha+1)}\|w\|_{C([0,T];D(A))}.$$

By induction, we have

$$\|\mathcal{H}^n w(t)\|_{D(A)} \le \frac{C\,(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha+1)}\|w\|_{C([0,T];D(A))}, \quad w \in C([0,T];D(A)). \tag{3.3.21}$$

Therefore we obtain

$$\begin{aligned}\|\mathcal{G}^n(w_1) - \mathcal{G}^n(w_2)\|_{C([0,T];D(A))} &= \|\mathcal{H}^n(w_1 - w_2)\|_{C([0,T];D(A))}\\ &\le \frac{C\,(\Gamma(\alpha)T^\alpha)^n}{\Gamma(n\alpha+1)}\|w_1 - w_2\|_{C([0,T];D(A))}\end{aligned}$$

for $w_1, w_2 \in C([0,T];D(A))$. Since $\mathcal{G}^n$ is a contraction for sufficiently large $n \in \mathbb{N}$, $\mathcal{G}$ admits a unique fixed point $u \in C([0,T];D(A)) \subset C([0,T];H^2(\Omega))$. Moreover we have

$$u = \mathcal{G}(u) = \mathcal{G}^n(u) = \mathcal{H}^n(u) + \sum_{k=1}^{n}\mathcal{H}^k(F)$$

for any $n \in \mathbb{N}$. Now we estimate each $\mathcal{H}^k(F)$. First, by (3.3.17), we have

$$\|\mathcal{H}(F)(t)\|_{D(A)} \le Ct^{\alpha-1/p}\|F\|_{L^p(0,T;D(A))}.$$

Next we apply (3.3.16) to have

$$\begin{aligned}\|\mathcal{H}^2(F)(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|\mathcal{H}(F)(s)\|_{D(A)}ds\\ &\le C\left(\int_0^t (t-s)^{\alpha-1}s^{\alpha-1/p}ds\right)\|F\|_{L^p(0,T;D(A))}\\ &= \frac{C\Gamma(\alpha)\Gamma(\alpha+1-1/p)}{\Gamma(2\alpha+1-1/p)}t^{2\alpha-1/p}\|F\|_{L^p(0,T;D(A))}\\ &\le \frac{C\Gamma(\alpha)t^{2\alpha-1/p}}{\Gamma(2\alpha+1-1/p)}\|F\|_{L^p(0,T;D(A))}.\end{aligned}$$

Repeating the similar calculation,

$$\begin{aligned}\|\mathcal{H}^3(F)(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|\mathcal{H}^2(F)(s)\|_{D(A)}ds\\ &\le \frac{C\Gamma(\alpha)}{\Gamma(2\alpha+1-1/p)}\left(\int_0^t (t-s)^{\alpha-1}s^{2\alpha-1/p}ds\right)\|F\|_{L^p(0,T;D(A))}\\ &= \frac{C\Gamma(\alpha)^2 t^{3\alpha-1/p}}{\Gamma(3\alpha+1-1/p)}\|F\|_{L^p(0,T;D(A))}.\end{aligned}$$

By induction, we obtain

$$\|\mathcal{H}^k(F)\|_{C([0,T];D(A))} \le \frac{C\Gamma(\alpha)^{k-1}T^{k\alpha-1/p}}{\Gamma(k\alpha+1-1/p)}\|F\|_{L^p(0,T;D(A))}.$$

Therefore

$$
\begin{aligned}
\|u\|_{C([0,T];D(A))} &\le \|\mathcal{H}^n(u)\|_{C([0,T];D(A))} + \sum_{k=1}^{n} \|\mathcal{H}^k(F)\|_{C([0,T];D(A))} \\
&\le \frac{C\left(\Gamma(\alpha)T^\alpha\right)^n}{\Gamma(n\alpha+1)}\|u\|_{C([0,T];D(A))} + \sum_{k=1}^{n} \frac{C\Gamma(\alpha)^{k-1}T^{k\alpha-1/p}}{\Gamma(k\alpha+1-1/p)}\|F\|_{L^p(0,T;D(A))}
\end{aligned}
$$

and by taking sufficiently large $n \in \mathbb{N}$, we obtain

$$
\|u\|_{C([0,T];D(A))} \le C\|F\|_{L^p(0,T;D(A))} \tag{3.3.22}
$$

with C depending on T and Ω.

Now fix $0 \le \gamma < 1 - 1/(p\alpha)$. Then for all $t \in (0,T)$, we have $\mathcal{A}u(t) \in D(A^\gamma)$ with

$$
A^\gamma(\mathcal{A}u)(t) = \int_0^t A^{\gamma-1}S_A'(t-s)(\mathcal{A}u(s) + \mathcal{A}F(s))ds
$$

and by (3.3.15), we have

$$
\|A^{\gamma-1}S_A'(t)\|_{\mathcal{B}(L^2(\Omega))} \le Ct^{\mu-1},
$$

where $\mu := \alpha(1-\gamma)$. Since $\mu > 1/p$, the mapping $t \mapsto A^{\gamma-1}S_A'(t)$ belongs to $L^q(0,T;\mathcal{B}(L^2(\Omega))$ where $q \in [1,\infty)$ satisfies $1/p + 1/q = 1$. Therefore $\mathcal{A}u$ belongs to $C([0,T];D(A^\gamma))$ and

$$
\begin{aligned}
\|\mathcal{A}u(t)\|_{D(A^\gamma)} &= \left\| \int_0^t A^{\gamma-1}S_A'(t-s)\left(\mathcal{A}u(s) + \mathcal{A}F(s)\right)ds \right\| \\
&\le C\int_0^t (t-s)^{\mu-1}\|u(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\mu-1}\|F(s)\|_{D(A)}ds \\
&\le C\left(\int_0^t (t-s)^{\mu-1}ds\right)\|u\|_{C([0,T];D(A))} + C\left(\int_0^t s^{q(\mu-1)}ds\right)^{1/q}\|F\|_{L^p(0,t;D(A))} \\
&\le CT^\mu\|u\|_{C([0,T];D(A))} + CT^{\mu-1/p}\|F\|_{L^p(0,T;D(A))}
\end{aligned} \tag{3.3.23}
$$

Combining this with (3.3.22), we have

$$
\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C\|F\|_{L^p(0,T;D(A))} \le C\|F\|_{L^p(0,T;H^2(\Omega))}.
$$

Hence we deduce that $\mathcal{A}u \in C([0,T];H^{2\gamma}(\Omega))$ and

$$
\|\mathcal{A}u\|_{C([0,T];H^{2\gamma}(\Omega))} \le C\|F\|_{L^p(0,T;H^2(\Omega))}.
$$

By the original equation $\partial_t^\alpha u = -\mathcal{A}u + F$, we see that $\partial_t^\alpha u$ belongs to $L^p(0,T;H^{2\gamma}(\Omega))$ with the estimate;

$$
\begin{aligned}
\|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} &\le C\|\mathcal{A}u\|_{L^p(0,T;H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^{2\gamma}(\Omega))} \\
&\le C\|\mathcal{A}u\|_{C([0,T];H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^2(\Omega))} \\
&\le C\|F\|_{L^p(0,T;H^2(\Omega))},
\end{aligned}
$$

which implies (3.3.8). Thus we have completed the proof. □

For the proof of Lemma 3.3.4, we prepare the following fact;

Lemma 3.3.7. *Let $u, v \in H^2(\Omega)$ and $d \le 3$, then $uv \in H^2(\Omega)$ with the estimate*

$$\|uv\|_{H^2(\Omega)} \le C\|v\|_{H^2(\Omega)}$$

with C depending on $\|u\|_{H^2(\Omega)}$.

For this lemma, see Theorem 2.1 in Chapter II of Strichartz [33].

Proof of Lemma 3.3.4. Similarly to Lemma 3.3.3, the IBVP (3.3.4) can be rewritten as

$$\begin{cases} \partial_t^\alpha v(t) + Av(t) = (1-p(t))v(t) + F(t), \\ v(0) = 0, \end{cases} \tag{3.3.24}$$

where $v(t) := v(\cdot, t)$ and $F(t) := F(\cdot, t)$. Moreover $p(t)$ denotes the multiplication operator by $p(x,t)$. Then we can see that the solution v of (3.3.24) is a fixed point of the map $\mathcal{K} : C([0,T]; D(A)) \to C([0,T]; D(A))$ defined by

$$\mathcal{K}(w)(t) := (\mathcal{H}(1-p(t))w)(t) + \mathcal{H}(F)(t), \quad w \in C([0,T]; D(A)),\ t \in (0,T).$$

Indeed, Lemma 3.3.7 and 1) of (3.3.7) yields that $(1-p)w$ belongs to $L^\infty(0,T; D(A))$ and satisfies

$$\|(1-p(t))w(t)\|_{D(A)} \le C\|w(t)\|_{D(A)}$$

with C depending on $\|p\|_{L^\infty(0,T;H^2(\Omega))}$. Therefore we can see that $\mathcal{K}$ maps $C([0,T]; D(A))$. Moreover, by the similar calculation to (3.3.21), we have

$$\|(\mathcal{H}(1-p))^n(w)\|_{C([0,T];D(A))} \le \frac{C(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha+1)}\|w\|_{C([0,T];D(A))}, \quad w \in C([0,T]; D(A)) \tag{3.3.25}$$

and

$$\|(\mathcal{H}(1-p))^{n-1}(\mathcal{H}F)\|_{C([0,T];D(A))} \le \frac{C(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha+1)}\|F\|_{L^\infty(0,T;D(A))}, \quad F \in L^\infty(0,T; D(A)). \tag{3.3.26}$$

By (3.3.25), we find

$$\|\mathcal{K}^n(w_1) - \mathcal{K}^n(w_2)\|_{C([0,T];D(A))} \le \frac{C(\Gamma(\alpha)T^\alpha)^n}{\Gamma(n\alpha+1)}\|w_1 - w_2\|_{C([0,T];D(A))}, \qquad w_1, w_2 \in C([0,T]; D(A)),$$

which implies that $\mathcal{K}$ admits a unique fixed point $v \in C([0,T]; D(A)) \subset C([0,T]; H^2(\Omega))$. Then we have

$$v = \mathcal{K}(v) = \mathcal{K}^n(v) = (\mathcal{H}(1-p(t)))^n(v) + \sum_{k=1}^n (\mathcal{H}(1-p(t)))^{k-1}(\mathcal{H}F). \tag{3.3.27}$$

Repeating the argument in the proof of Lemma 3.3.3, we deduce from (3.3.25), (3.3.26) and (3.3.27) that

$$\|v\|_{C([0,T];D(A))} \le C\|F\|_{L^\infty(0,T;D(A))} \tag{3.3.28}$$

with C depending on T, Ω and $\|p\|_{L^\infty(0,T;H^2(\Omega))}$.

Next we fix $0 \le \gamma < 1$. Similarly to (3.3.23), we have

$$\mathcal{A}v(t) \in D(A^\gamma), \quad t \in (0,T)$$

and

$$\begin{aligned}
\|\mathcal{A}v(t)\|_{D(A^\gamma)} &\le C\left\|\int_0^t A^{\gamma-1}S_A'(t-s)\left((\mathcal{A}(1-p(s))v)(s)+\mathcal{A}F(s)\right)ds\right\| \\
&\le C\int_0^t (t-s)^{\mu-1}\left(\|(1-p(s))v(s)\|_{D(A)}+\|F(s)\|_{D(A)}\right)ds \\
&\le C\int_0^t (t-s)^{\mu-1}\left(\|v(s)\|_{D(A)}+\|F(s)\|_{D(A)}\right)ds
\end{aligned}$$

with $\mu = \alpha(1-\gamma)$. Therefore $\mathcal{A}v$ belongs to $C([0,T];H^{2\gamma}(\Omega))$ and satisfies

$$\begin{aligned}
\|\mathcal{A}v\|_{C([0,T];H^{2\gamma}(\Omega))} &\le \|\mathcal{A}v\|_{C([0,T];D(A^\gamma))} \le CT^\mu\left(\|v\|_{C([0,T];D(A))}+\|F\|_{L^\infty(0,T;D(A))}\right) \\
&\le C\|F\|_{L^\infty(0,T;D(A))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))},
\end{aligned}$$

where we have used (3.3.28). Moreover, combining this with the original equation, we also have $\partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$ and (3.3.9). □

Proof of Lemma 3.3.5. We split the solution v of (3.3.5) into two terms $v = w + v_0$ where w solves

$$\begin{cases}
\partial_t^\alpha w(x,t)+\mathcal{A}w(x,t)+p(x,t)w(x,t)=F(x,t), & (x,t)\in Q,\\
\mathcal{B}_\sigma w(x,t)=0, & (x,t)\in\Sigma,\\
w(x,0)=0, & x\in\Omega
\end{cases} \tag{3.3.29}$$

with $F(x,t) := -(\mathcal{A}+p(x,t))v_0(x)$. Then (3.3.7) implies $F \in L^\infty(0,T;D(A))$ with the estimate

$$\|F\|_{L^\infty(0,T;H^2(\Omega))} \le C\|v_0\|_{H^4(\Omega)}.$$

By Lemma 3.3.4, the IBVP (3.3.29) admits a unique solution $w \in C([0,T];H^2(\Omega))$ satisfying

$$\mathcal{A}w \in C([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w \in L^\infty(0,T;H^{2\gamma}(\Omega)).$$

Moreover

$$\|\mathcal{A}w\|_{C([0,T];H^{2\gamma}(\Omega))}+\|\partial_t^\alpha w\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|F\|_{L^\infty(0,T;H^2(\Omega))} \le C\|v_0\|_{H^4(\Omega)}.$$

Therefore the IBVP (3.3.5) admits a unique solution $v \in C([0,T];H^2(\Omega))$ satisfying

$$\mathcal{A}v \in C([0,T];H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega)).$$

From the above estimate, we deduce (3.3.10). □

3.4 Proof of Theorem 3.2.1

In this section, we prove Theorem 3.2.1. To this end, we prepare the following lemmata with Gronwall type inequalities;

Lemma 3.4.1. *Let $C, \alpha > 0$ and $u, d \in L^1(0,T)$ be nonnegative functions satisfying*

$$u(t) \leq Cd(t) + C\int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in (0,T),$$

then we have

$$u(t) \leq Cd(t) + C\int_0^t (t-s)^{\alpha-1} d(s) ds, \quad t \in (0,T).$$

For the proof, see Lemma 7.1.1 p.188 of [18].

Lemma 3.4.2. *We take $2 \leq p \leq \infty$ and $\mu > 2/p$. Let $f \in L^\infty(0,T)$ and $u, R \in L^p(0,T)$ be non-negative functions satisfying the integral inequality*

$$f(t) \leq u(t) + \int_0^t (t-s)^{\mu-1} f(s) R(s) ds, \quad a.e.\ t \in (0,T). \tag{3.4.1}$$

Then we have

$$\|f\|_{L^p(0,T)} \leq C\|u\|_{L^p(0,T)}, \tag{3.4.2}$$

where the constant C depends on p, μ, T and $\|R\|_{L^p(0,T)}$.

Proof. We set $d(t) := \|f\|^p_{L^p(0,t)}$. From equation (3.4.1), we have

$$|f(s)|^p \leq C|u(s)|^p + C\left|\int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi\right|^p,$$

which implies

$$d(t) \leq C\|u\|^p_{L^p(0,T)} + C\int_0^t \left|\int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi\right|^p ds. \tag{3.4.3}$$

Now we estimate the right-hand side of the above. By the Cauchy-Schwarz inequality,

$$\int_0^s |f(\xi)R(\xi)|^{p/2} d\xi = \int_0^s |f(\xi)|^{p/2} \cdot |R(\xi)|^{p/2} d\xi \leq \left(\int_0^s |f(\xi)|^p d\xi\right)^{1/2} \left(\int_0^s |R(\xi)|^p d\xi\right)^{1/2},$$

that is,

$$\|fR\|_{L^{p/2}(0,s)} \leq \|f\|_{L^p(0,s)} \|R\|_{L^p(0,s)}.$$

Therefore if $p > 2$, then Lemma 3.3.6 yields that

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)R(\xi)d\xi\right| \le \left(\int_0^s \xi^{r(\mu-1)}ds\right)^{1/r} \|fR\|_{L^{p/2}(0,s)} \le C\|f\|_{L^p(0,s)}\|R\|_{L^p(0,s)},$$

where $r \in [1,\infty)$ satisfies $2/p + 1/r = 1$. For $p = 2$, we also have

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)R(\xi)d\xi\right| \le s^{\mu-1}\|fR\|_{L^1(0,s)} \le C\|f\|_{L^2(0,s)}\|R\|_{L^2(0,s)}.$$

Thus for any $p \ge 2$, we have

$$\left|\int_0^s (s-\xi)^{\mu-1} f(\xi)R(\xi)d\xi\right| \le Cd(s), \tag{3.4.4}$$

where C depends on T, p, μ and $\|R\|_{L^p(0,T)}$. By (3.4.3) and (3.4.4), we have

$$d(t) \le C\|u\|^p_{L^p(0,T)} + C\int_0^t d(s)ds, \quad t \in (0,T).$$

Hence by the Gronwall inequality, we have

$$d(t) \le C\|u\|^p_{L^p(0,T)}, \quad t \in (0,T)$$

with C depending on p, μ, T and $\|R\|_{L^p(0,T)}$. Thus we have proved (3.4.2). □

Now we are ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let u_i be the solutions to (3.1.1) corresponding to f_i ($i = 1,2$) and set $u := u_1 - u_2$ and $f := f_1 - f_2$. Then u solves (3.1.1) and is given by

$$u(t) = \int_0^t A^{-1}S_A'(t-s)u(s) + \int_0^t A^{-1}S_A'(t-s)f(s)R(s)ds,$$

where $u(t) := u(\cdot,t)$ and $R(t) := R(\cdot,t)$.

First we estimate $\|u(t)\|_{D(A)}$. Similarly to the calculation in (3.3.16), we have

$$\begin{aligned}\|u(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds \\ &= C\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{D(A)}ds + Cd(t),\end{aligned} \tag{3.4.5}$$

where we have set

$$d(t) := \int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds.$$

Applying Lemma 3.4.1 to (3.4.5), we have

$$\|u(t)\|_{D(A)} \le Cd(t) + C\int_0^t (t-s)^{\alpha-1} d(s)ds, \quad 0 < t < T. \tag{3.4.6}$$

Here for $\nu > 0$, we note

$$\begin{aligned}
\int_0^t (t-s)^{\nu-1} d(s)ds &= \int_0^t (t-s)^{\nu-1} \left(\int_0^s (s-\xi)^{\alpha-1} |f(\xi)| \|R(\xi)\|_{D(A)} d\xi \right) ds \\
&= \int_0^t \left(\int_\xi^t (t-s)^{\nu-1} (s-\xi)^{\alpha-1} ds \right) |f(\xi)| \|R(\xi)\|_{D(A)} d\xi \\
&= B(\nu, \alpha) \int_0^t (t-\xi)^{\nu+\alpha-1} |f(\xi)| \|R(\xi)\|_{D(A)} d\xi
\end{aligned} \tag{3.4.7}$$

where $B(\cdot, \cdot)$ is the Beta function. In particular, for $\nu = \alpha$, we have

$$\begin{aligned}
\int_0^t (t-s)^{\alpha-1} d(s)ds &= B(\alpha, \alpha) \int_0^t (t-s)^{2\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le T^\alpha B(\alpha, \alpha) \int_0^t (t-s)^{\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le Cd(t).
\end{aligned}$$

Hence the following estimate follows from (3.4.6);

$$\|u(t)\|_{D(A)} \le Cd(t), \quad 0 < t < T. \tag{3.4.8}$$

Next we estimate $\|\mathcal{A}u(t)\|_{D(A^\gamma)}$ for $d/4 < \gamma < 1 - 2/(p\alpha)$. Repeating the calculation in (3.3.23), we find

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\mu-1} \left(\|u(s)\|_{D(A)} + |f(s)| \|R(s)\|_{D(A)} \right) ds, \quad \text{a.e. } t \in (0, T)$$

where $\mu = \alpha(1-\gamma)$. By (3.4.7) with $\nu = \mu$ and (3.4.8), we obtain

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{D(A^\gamma)} &\le C\int_0^t (t-s)^{\mu-1} d(s)ds + C\int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&= CB(\mu, \alpha) \int_0^t (t-s)^{\mu+\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C\int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le CT^\alpha B(\mu, \alpha) \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C\int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\le C\int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds.
\end{aligned}$$

Finally we estimate $|\mathcal{A}u(x_0,t)|$ and complete the proof. Since $\gamma > d/4$, the Sobolev embedding theorem yields

$$|\mathcal{A}u(x_0,t)| \le C\|\mathcal{A}u(\cdot,t)\|_{H^{2\gamma}(\Omega)} \le C\|\mathcal{A}u(t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\mu-1}|f(s)|\|R(s)\|_{D(A)}ds. \tag{3.4.9}$$

From the original equation, we get

$$f(t)R(x_0,t) = \partial_t^\alpha u(x_0,t) + \mathcal{A}u(x_0,t), \quad \text{a.e. } t\in(0,T). \tag{3.4.10}$$

Combining this with (3.2.4) and (3.4.9), we get

$$\begin{aligned}|f(t)| &\le \frac{1}{\delta}\left(|\partial_t^\alpha u(x_0,t)| + |\mathcal{A}u(x_0,t)|\right)\\ &\le C|\partial_t^\alpha u(x_0,t)| + C\int_0^t (t-s)^{\mu-1}|f(s)|\|R(s)\|_{D(A)}ds, \quad \text{a.e. } t\in(0,T)\end{aligned} \tag{3.4.11}$$

with C depending on δ, Ω and T. By Lemma 3.4.2, we see that

$$\|f\|_{L^p(0,T)} \le C\|\partial_t^\alpha u(x_0,\cdot)\|_{L^p(0,T)},$$

which implies (3.2.5). Moreover, by (3.4.9) and (3.4.10), we have

$$\begin{aligned}|\partial_t^\alpha u(x_0,t)| &\le |f(t)R(x_0,t)| + |\mathcal{A}u(x_0,t)|\\ &\le C|f(t)|\|R(\cdot,t)\|_{H^2(\Omega)} + C\int_0^t (t-s)^{\mu-1}|f(s)|\|R(s)\|_{D(A)}ds\\ &\le C\|f\|_{L^\infty(0,T)}\|R(t)\|_{D(A)} + C\|f\|_{L^\infty(0,T)}\int_0^t (t-s)^{\mu-1}\|R(s)\|_{D(A)}ds\\ &\le C\|f\|_{L^\infty(0,T)}\|R(t)\|_{D(A)} + C\|f\|_{L^\infty(0,T)}T^{\mu-1/p}\|R\|_{L^p(0,T;D(A))}, \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\partial_t^\alpha u(x_0,\cdot)\|_{L^p(0,T)} &\le C\|f\|_{L^\infty(0,T)}\|R\|_{L^p(0,T;D(A))} + C\|f\|_{L^\infty(0,T)}T^\mu\|R\|_{L^p(0,T;D(A))}\\ &\le C\|f\|_{L^\infty(0,T)}.\end{aligned}$$

Thus we have proved (3.2.6). □

3.5 Proof of Theorem 3.2.2

In this section, we prove Theorem 3.2.2. We first prepare the following generalized Gronwall's inequality;

Lemma 3.5.1. *Let $\mu, a, b > 0$ and $f \in L^1(0,T)$ be nonnegative function satisfying the integral inequality*

$$f(t) \le a + b\int_0^t (t-s)^{\mu-1} f(s)ds, \quad a.e.\ t \in (0,T).$$

Then we have

$$f(t) \le aE_{\mu,1}\left((b\Gamma(\mu))^{1/\mu} t^{\mu}\right), \quad a.e.\ t \in (0,T).$$

For the proof, see Lemma 7.1.2 on p.189 of [18]. Now we are ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. Let v_i be the solutions to (3.1.2) corresponding to f_i ($i = 1, 2$) and set $v := v_1 - v_2$ and $f := f_2 - f_1$. Then v solves (3.3.4) with $p(x,t) = f_1(t)q(x,t)$ and $F(x,t) = f(t)q(x,t)v_2(x,t)$. Recall that v is given by

$$v(t) = \int_0^t A^{-1}S_A'(t-s)((1-p(t))v)(s) + \int_0^t f(s)A^{-1}S_A'(t-s)R(s)ds,$$

where we have set $v(t) := v(\cdot,t)$ and $R(t) := q(\cdot,t)v_2(\cdot,t)$. Moreover, $p(t)$ denotes the multiplication operator by $p(x,t) := f_1(t)q(x,t)$.

First we estimate $\|v(t)\|_{D(A)}$. Since $(1-p(t))v(t), R(t) \in D(A)$ by (3.2.3), we repeat the calculation in (3.3.23) to have

$$\begin{aligned}\|v(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha-1}\|(1-p(t))v(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|\|R(s)\|_{D(A)}ds \\ &\le C\int_0^t (t-s)^{\alpha-1}\|v(s)\|_{D(A)}ds + C\int_0^t (t-s)^{\alpha-1}|f(s)|ds.\end{aligned}$$

with C depending on Ω, M and $\|q\|_{L^\infty(0,T;H^2(\Omega))}$. Then repeating the arguments used in Theorem 3.2.1, we obtain

$$\|v(t)\|_{D(A)} \le C\int_0^t (t-s)^{\alpha-1}|f(s)|ds, \quad 0 < t < T.$$

and from this estimate we also deduce that for any $0 \le \gamma < 1$,

$$\|\mathcal{A}v(t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\mu-1}|f(s)|ds, \quad 0 < t < T,$$

where $\mu := \alpha(1-\gamma)$. Therefore by taking $\gamma \in (d/4, 1)$, we have

$$\begin{aligned}|\mathcal{A}v(x_0,t) + p(x_0,t)v(x_0,t)| &\le C\|\mathcal{A}v(\cdot,t) + p(\cdot,t)v(\cdot,t)\|_{H^{2\gamma}(\Omega)} \\ &\le C\|\mathcal{A}v(\cdot,t)\|_{H^{2\gamma}(\Omega)} + C\|v(\cdot,t)\|_{H^{2\gamma}(\Omega)} \\ &\le C\|\mathcal{A}v(t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\mu-1}|f(s)|ds. \qquad (3.5.1)\end{aligned}$$

From the original equation, we have

$$f(t)R(x_0,t) = \partial_t^\alpha v(x_0,t) + \mathcal{A}v(x_0,t) + p(x_0,t)v(x_0,t), \quad \text{a.e. } t \in (0,T). \tag{3.5.2}$$

On the other hand, from (3.2.7), we deduce that

$$|R(x_0,t)| \geq c > 0, \quad \text{a.e. } t \in (0,T)$$

with c depending on δ, Ω and T. Therefore, combining this with (3.5.1) and (3.5.2), we obtain

$$\begin{aligned}|f(t)| &\leq C|\partial_t^\alpha v(x_0,t)| + C|\mathcal{A}v(x_0,t) + p(x_0,t)v(x_0,t)| \\ &\leq C\|\partial_t^\alpha v(x_0,\cdot)\|_{L^\infty(0,T)} + C\int_0^t (t-s)^{\mu-1}|f(s)|ds, \quad \text{a.e. } t \in (0,T).\end{aligned}$$

Applying Lemma 3.5.1, we see that

$$|f(t)| \leq C\|\partial_t^\alpha v(x_0,\cdot)\|_{L^\infty(0,T)}.$$

Thus we have proved the second inequality in (3.2.8). Moreover, by (3.5.2), we have

$$\begin{aligned}|\partial_t^\alpha v(x_0,t)| &\leq |f(t)R(x_0,t)| + |\mathcal{A}v(x_0,t) + p(x_0,t)v(x_0,t)| \\ &\leq |f(t)|\|R(\cdot,t)\|_{D(A)} + C\int_0^t (t-s)^{\mu-1}|f(s)|ds \\ &\leq C\left(\|R\|_{L^\infty(0,T;D(A))} + \frac{T^\mu}{\mu}\right)\|f\|_{L^\infty(0,T)}.\end{aligned}$$

Thus we have proved the first inequality in (3.2.8). $\square$

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