博士論文

Accelerating convergence and tractability of multivariate numerical integration when the L1-norms of the higher order derivatives of the integrand grow at most exponentially

(被積分関数の高階偏微分のL1ノルムの増大度が高々指数的である場合の多次元数値積分の加速的な収束と計算容易性)

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Abstract

Quasi-Monte Carlo integration is an equal weight rule for numerical integration. Among other things, Dick proved that QMC rules using good digital nets achieve the rate of convergence $O(n^{-\alpha+\epsilon})$ for every $\epsilon > 0$ for a integrand which has mixed partial derivatives of order α for each variable. Later, Yoshiki, Matsumoto and others proved analogical results for smooth functions using dyadic digital nets. This thesis is devoted to develop these studies for b-adic digital nets for an integer $b \geq 2$, and investigate weighted function spaces of smooth functions which achieve tractability with very fast convergence.

The first contribution of this thesis is to extend the study by Matsumoto, Saito and Matoba. They considered a discretized version of Dick's results for dyadic digital nets. In particular, they defined a practically computable criterion WAFOM for dyadic digital nets. We extend the study to the b-adic case. Moreover, we give upper and lower bounds on WAFOM as a generalization of the works of Matsumoto and Yoshiki. Furthermore, we give a MacWilliams-type identity on weight enumerator polynomials for the metric function we consider, by which we can compute the minimum distance as well as WAFOM.

The second contribution of this thesis is, beyond the existence result given as the first contribution, to give an explicit construction algorithm for low-WAFOM digital nets. We use Niederreiter-Xing sequences and Dick's interlacing construction.

The third contribution of this thesis is to give formulas and bounds for b-adic Walsh coefficients of smooth functions. First we establish a formula in which b-adic Walsh coefficients of smooth functions are expressed in terms of those derivatives. Furthermore, we give bounds on b-adic Walsh coefficients for α times continuously differentiable functions. These results for the dyadic case recover results for smooth functions by Yoshiki. In particular, we obtain a class of infinitely differentiable functions whose Walsh coefficients decay sufficiently fast as WAFOM works well. This part is a joint work with Takehito Yoshiki.

The last contribution of this thesis is to prove accelerating convergence and tractability for a weighted normed space of non-periodic smooth functions whose L^1-norms of the higher order derivatives of the integrand grow at most exponentially. The growth of the L^1-norms of the higher order derivatives is controlled by a weight sequence $\boldsymbol{u}$. First we show that this space achieves accelerating convergence for all s, which is the number of variables, and $\boldsymbol{u}$ considered. Accelerating convergence roughly means that the integration error converges as $O(q^{(\log n)^p})$ for some $q \in (0, 1)$ and $p > 1$. Second we establish the notions of tractability which correspond to accelerating convergence: accelerating convergence with polynomial tractability (AC-PT) and accelerating convergence with strong tractability (AC-ST). Roughly speaking, AC-PT (resp. AC-ST) holds if accelerating convergence holds and the number of function evaluation to guarantee the error depends only polynomially on s (resp. is independent of s). We show that AC-ST holds for the space if weights $\boldsymbol{u}$ decay sufficiently fast.

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Contents

Chapter 1

Introduction

Multivariate integration appears in many applications including finance, physics and computer graphics [18, 19, 26, 30, 31]. In the univariate case, there are many known integration rules such as trapezoidal rule and Simpson's rule. If the number of variables increases, however, the problem generally becomes difficult. For example, if we use the product of univariate integration rules, the computational cost grows exponentially on the number of variables.

Monte Carlo (MC) integration and Quasi-Monte Carlo (QMC) integration are successful methods for multivariate numerical integration. Both rules approximate the integration value by the average of function values. MC integration uses sample points taken independently and randomly, whereas QMC integration uses well-designed sample points. Of course, how to design point sets is a central issue of QMC integration. We restrict ourselves to integration on the s-dimensional unit cube $[0,1)^s$ since functions on general domains can be transformed to functions on the unit cube. There are two construction schemes which mainly have been investigated: lattices, see e.g., [49] and digital nets, see e.g., [38, 14]. In this thesis we focus on QMC integration using digital nets. Hereafter we often identify point sets with the QMC rule using the point sets.

It is well-known that the integration error by MC rules probabilistically converges to zero as $O(1/\sqrt{n})$, where n is the number of function values we use. This rate of convergence is independent of the dimension but is considered to be slow; in order to reduce the error by a factor of 10, we need 100 times as many points. One advantage of QMC integration is that the rate of convergence is faster than $O(1/\sqrt{n})$ for sufficiently smooth integrands. The first success in QMC integration is the Koksma-Hlawka inequality in [27, 24], which states that if a function $f\colon [0,1]^s \to \mathbb{R}$ has bounded variation then the integration error of f is bounded by

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\, d\boldsymbol{x} - |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}) \right| \leq \|f\|_{\mathrm{tot}} D^*(P),$$

where $\|f\|_{\mathrm{tot}}$ is the total variation of f in the sense of Hardy and Krause and

$D^*(P)$ is a measure of disuniformity of P called star-discrepancy. From this inequality, we can adopt the star-discrepancy as a criterion of point sets for QMC integration. There are many known point sets and sequences whose star-discrepancy decays as $D^*(P) \in O(n^{-1}(\log n)^{s-1})$, see [38, Chapter 3] and the references therein. Thus the convergence rate of QMC integration using low-discrepancy point sets is faster than that of MC integration for functions whose variation is finite. Recently, it has been known that we can improve the rate of convergence if we require integrands to have much smoothness. Among other things, Dick introduced a class of digital nets named "higher order digital nets", and proved that QMC rules using good higher order digital nets achieve the rate of convergence $O(n^{-\alpha+\epsilon})$ for a integrand which has mixed partial derivatives of order α for each variable. Later, Yoshiki, Matsumoto and others developed Dick's works for smooth (i.e., infinitely differentiable) functions [34, 35, 58]. This thesis is devoted to develop these studies for b-adic digital nets for an integer $b \geq 2$, and investigate weighted function spaces of smooth functions which achieve tractability with very fast convergence.

Let us recall numerical integration using digital nets. The first construction of digital nets was provided by Sobol' [51] and Faure [16]. Niederreiter [37] gave the notion of (t, m, s)-nets over $\mathbb{Z}_b$, which consist of b^m points in $[0, 1)^s$ and which satisfy some geometrical condition. Here the value t governs the quality of (t, m, s)-nets (it was also proved in [37] that the star-discrepancy of a (t, m, s)-net is roughly bounded by a constant times b^{t-m} and so smaller t is better), and $\mathbb{Z}_b$ is a cyclic group with b elements, which we identify with the set $\{0, 1, \dots, b-1\}$. The general framework of digital nets are defined also in [37] as an explicit construction of (t, m, s)-nets. The construction of digital nets are based on linear algebra over $\mathbb{Z}_b$ and one important property is that digital nets have the structure of a group with respect to the digit-wise summation modulo b. Many constructions of digital nets with small t-values are known, see [14] and the references therein.

Beyond these studies, how to exploit the smoothness of the integrand was shown by Dick [8, 9, 10]. We now recall Dick's results in more detail. Key tools to analyze the integration error of QMC rules using digital nets are Walsh functions and Walsh coefficients, which were first introduced by Walsh [57], see also [17, 5]. Let k be a nonnegative integer whose b-adic expansion is $k = \kappa_1 b^{a_1-1} + \cdots + \kappa_v b^{a_v-1}$ where κ_i and a_i are integers with $0 < \kappa_i \leq b-1$, $a_1 > \cdots > a_v \geq 1$. For $k = 0$ we assume that $v = 0$ and $a_0 = 0$. The b-adic k-th Walsh function $\mathrm{wal}_k(x)$ is defined as

$$\mathrm{wal}_k(x) := \omega_b^{\sum_{i=1}^{v} \kappa_i \xi_{a_i}},$$

for $x \in [0, 1)$ whose b-adic expansion is given by $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$, which is unique in the sense that infinitely many of the digits ξ_i are different from $b-1$. We also consider s-dimensional Walsh functions. For $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \dots, x_s) \in [0, 1)^s$, the b-adic $\boldsymbol{k}$-th Walsh function $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ is defined as

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

For $f\colon [0,1)^s \to \mathbb{C}$, we define the $\boldsymbol{k}$-th Walsh coefficient of f as

$$\widehat{f}(\boldsymbol{k}) := \int_{[0,1)^s} f(\boldsymbol{x})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\, d\boldsymbol{x}.$$

It is well-known that the Walsh system $\{\mathrm{wal}_{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal basis in $L^2[0,1)^s$. Hence we have a Walsh series expansion

$$f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k}\in\mathbb{N}^s} \widehat{f}(\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$$

for any $f \in L^2[0,1]^s$. We can now give the integration error for a digital net P. If f is given by Walsh series (this assumption is satisfied if $f\colon [0,1]^s \to \mathbb{R}$ has continuous mixed partial derivatives up to order 1 for each variable), the integration error of f for a digital net P is given by $\sum_{\boldsymbol{k}\in P^\perp\setminus\{0\}} \widehat{f}(\boldsymbol{k})$, where $P^\perp := \{\boldsymbol{k} \in \mathbb{N}_0^s \mid \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = 1 \text{ for all } \boldsymbol{x} \in P\}$ is the dual net of P. Hence the QMC error of P is bounded by $\sum_{\boldsymbol{k}\in P^\perp\setminus\{0\}} |\widehat{f}(\boldsymbol{k})|$, and thus we would like to know the bound on Walsh coefficients. Analogous to the well-known result that the decay of Fourier coefficients reflects the smoothness of the function, Dick proved that the decay of Walsh coefficients also reflects it. More precisely, he defined a metric function $\mu_\alpha(k) = a_1 + \cdots + a_{\min(\alpha,v)}$ for the one-dimensional case and $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^s \mu_\alpha(k_j)$ for the s-dimensional case. He proved that the $\boldsymbol{k}$-th Walsh coefficient of a function $f\colon [0,1]^s \to \mathbb{R}$ which has square-integrable mixed partial derivatives up to order α for each variable is bounded by $C_{b,s,\alpha}\|f\|_{\alpha,s} b^{-\mu_\alpha(\boldsymbol{k})}$ where $C_{b,s,\alpha}$ is a positive constant which depends on b, α and s and $\|f\|_{\alpha,s}$ is a norm of Sobolev type which uses all mixed partial derivatives up to order α for each variable. The above argument implies the following Koksma-Hlawka type inequality for a digital net P:

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\, d\boldsymbol{x} - |P|^{-1} \sum_{\boldsymbol{x}\in P} f(\boldsymbol{x}) \right| \leq C_{b,s,\alpha}\|f\|_{\alpha,s} \sum_{\boldsymbol{k}\in P^\perp\setminus\{0\}} b^{-\mu_\alpha(\boldsymbol{k})}. \tag{1.1}$$

Since the term $\mathrm{WF}_\alpha(P) := \sum_{\boldsymbol{k}\in P^\perp\setminus\{0\}} b^{-\mu_\alpha(\boldsymbol{k})}$ depends only on P, it can be used as a criterion for the quality of digital nets for numerical integration. Dick introduced the notion of higher order digital nets and gave the construction of them of which the criterion is sufficiently small to achieve "higher order convergence" of order $n^{-\alpha+\epsilon}$.

As a discretized version of Dick's results, Matsumoto, Saito and Matoba introduced the notion of WAFOM [34]. They considered a metric $\mu_\infty(k) = a_1 + \cdots + a_v$ and $\mu_\infty(\boldsymbol{k}) = \sum_{j=1}^s \mu_\infty(k_j)$ instead of μ_α and a criterion named WAFOM $\mathrm{WF}(P) := \sum_{\boldsymbol{k}\in P^\perp\setminus\{0\}} b^{-\mu_\infty(\boldsymbol{k})}$ (more precisely, a truncated summation of $\mathrm{WF}(P)$) instead of $\mathrm{WF}_\alpha(P)$ for dyadic (i.e., $b = 2$) digital nets. The decay of Walsh coefficients of order $O(b^{-\mu_\infty(\boldsymbol{k})})$ was not known at this moment. They showed only a discrete version of (1.1) with some error due to the discretization. One advantage of WAFOM is that it is computable in a reasonable time and thus we can search for good digital nets with respect to WAFOM by computer, see [34, 22, 21] for numerical experiments. One important result is that lowest-WAFOM decays "accelerating" as $O(n^{-C \log n})$ for some positive constant C [35]. The word "accelerating" means that the exponent $\log n$ of n increases as n increases.

More recently, Yoshiki established a result for smooth functions in [58]. He introduced "dyadic difference" of a function and gave a formula in which dyadic Walsh coefficients are given by dyadic differences multiplied by constants. Moreover he established a formula for dyadic Walsh coefficients of smooth functions expressed in terms of those derivatives. In particular, he gave a function space of smooth functions whose Walsh coefficients decay as $|\widehat{f}(\boldsymbol{k})| \leq 2^{s/p} \|f\|_{\mathrm{Y},p} 2^{-\mu_{\infty,\mathrm{Y}}(\boldsymbol{k})}$, where $\|f\|_{\mathrm{Y},p}$ is given by the supremum of all L^p-norms of the mixed partial derivatives, $\mu_{\infty,\mathrm{Y}}(k) = a_1 + \cdots + a_v + v$ and $\mu_{\infty,\mathrm{Y}}(\boldsymbol{k}) = \sum_{j=1}^{s} \mu_{\infty,\mathrm{Y}}(k_j)$. This result gives a Koksma-Hlawka type inequality as

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\, d\boldsymbol{x} - |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}) \right| \leq 2^{s/p} \|f\|_{\mathrm{Y},p} \sum_{\boldsymbol{k} \in P^\perp \setminus \{0\}} 2^{-\mu_{\infty,\mathrm{Y}}(\boldsymbol{k})}$$

for a dyadic digital net P. Considering that we achieve accelerating convergence for the lowest WAFOM-values and that Yoshiki's criterion $\sum_{\boldsymbol{k} \in P^\perp \setminus \{0\}} b^{-\mu_{\infty,\mathrm{Y}}(\boldsymbol{k})}$ is smaller than WAFOM, the space introduced by Yoshiki is a function space of smooth functions which achieve accelerating convergence whereas it is not explicitly written in [58].

We have reviewed about Koksma-Hlawka type inequalities and the rate of convergence so far. Another important issue is the dependence on the number of variables s since s can be hundreds or more in computational applications. This is related to the notion of tractability if we require no exponential dependence on s. Let us briefly recall the notion of tractability (see [41, 42, 43] for more information). Let $n(\varepsilon, s)$ be the information complexity, i.e., the minimal number n of function values which approximate the s-variate integration within ε. An integration problem is said to be tractable if $n(\varepsilon, s)$ does not grow exponentially on ε nor s. In particular, two notions of tractability has been mainly considered: polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1} s^{\tau_2}$, and strong polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1}$ for $\tau_1, \tau_2 \geq 0$. A common way to obtain tractability is to consider weighted function spaces introduced by Sloan and Woźniakowski [50]. Weighted spaces mean that the dependence on the successive variables can be moderated by weights.

Now we are ready to explain the contributions of this thesis. The first contribution of this thesis is to extend the studies in [34] on WAFOM for dyadic digital nets. We extend the notions of the Dick weight and WAFOM over a general finite abelian group G. We give a lower bound on WAFOM of order $N^{-C'_G (\log N)/s}$ and an upper bound on lowest WAFOM of order $N^{-C_G (\log N)/s}$ for given (G, N, s) if $(\log N)/s$ is sufficiently large, where C'_G and C_G are constants depending only on the cardinality of G and N is the cardinality of quadrature rules in $[0, 1)^s$. These bounds generalize the bounds given for $G = \mathbb{F}_2$ in [35, 59]. Furthermore, we give a MacWilliams-type identity on weight enumerator polynomials for the Dick weight, by which we can compute the minimum Dick weight as well as WAFOM. This part is based on [54].

The second contribution of this thesis is to give an explicit construction algorithm for low-WAFOM digital nets. In [35] and its generalization given as the first contribution of this thesis, only the existence of low-WAFOM point sets was proved. We construct low-WAFOM digital nets using Niederreiter-Xing sequences and Dick's interlacing construction. This part is based on [53].

The third contribution of this thesis is to give formulas and bounds for b-adic Walsh coefficients of smooth functions. First we establish a formula in which the b-adic Walsh coefficients of smooth functions are expressed in terms of those derivatives as

$$\widehat{f}(k) = (-1)^v \int_0^1 f^{(v)}(x) W(k)(x)\, dx,$$

where the function $W(k)(\cdot)\colon [0,1) \to \mathbb{C}$ is given by the iterated integral of Walsh functions as in Definition 5.2.1. This formula is a generalization of the formula for the dyadic Walsh coefficients of smooth functions in [58], however our method is different from that in [58]. Our main idea is first separating the interval [0,1) to appropriate intervals on which particular Walsh functions take constant values, and then applying integration by parts iteratively. Furthermore, we give bounds on the b-adic Walsh coefficients for α times continuously differentiable functions. Our bounds for the dyadic case recover results for smooth functions in [58]. Our assumption is somewhat stronger than that of [10]. Instead, we obtain bounds asymptotically better with respect to α than results in [10]. In particular, we obtain a class of infinitely smooth functions whose Walsh coefficients decay as $|\widehat{f}(k)| \in O(b^{-\mu_\infty(\boldsymbol{k})})$. This result gives a Koksma-Hlawka type inequality with respect to b-adic WAFOM. This part is based on [55], a joint work with Takehito Yoshiki.

The last contribution of this thesis is to prove accelerating convergence and tractability for a weighted normed space of non-periodic smooth functions

$$\mathcal{F}_{s,\boldsymbol{u}} := \left\{ f \in C^\infty[0,1]^s \;\middle|\; \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} := \sup_{(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_0^s} \frac{\|f^{(\alpha_1,\dots,\alpha_s)}\|_{L^1}}{\prod_{j=1}^s u_j^{\alpha_j}} < \infty \right\}$$

with a sequence of positive weights $\boldsymbol{u} = \{u_j\}_{j\geq 1}$. Here $f^{(\alpha_1,\dots,\alpha_s)}$ is defined as $f^{(\alpha_1,\dots,\alpha_s)} := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_s)^{\alpha_s} f$. The space $\mathcal{F}_{s,1/2}$ is a space considered by Yoshiki (we note that he considered more general ANOVA-type function spaces in [58]). It is easy to check that all functions in $\mathcal{F}_{s,\boldsymbol{u}}$ are analytic from Taylor's theorem. This space can be regarded as a Sobolev space of infinite order [15]. First we show that $\mathcal{F}_{s,\boldsymbol{u}}$ achieves accelerating convergence for all s and $\boldsymbol{u}$ considered. Accelerating convergence roughly means that the integration error converges as $O(q^{(\log n)^p})$ for some $q \in (0,1)$ and $p > 1$. Note that $q^{(\log n)^p} = n^{-(\log q^{-1})(\log n)^{p-1}}$, hence the exponent $(\log n)^{p-1}$ of n increases as n increases (which is why we call this accelerating convergence). Second we establish the notions of tractability which correspond to accelerating convergence: accelerating convergence with polynomial tractability (AC-PT) and accelerating convergence with strong tractability (AC-ST). Roughly speaking, AC-PT (resp. AC-ST) holds if accelerating convergence holds and $n(\varepsilon, s)$ depends only polynomially on s (resp. is independent of s). We define the Walsh space $\mathcal{W}_{s,\boldsymbol{a},b}$ into which $\mathcal{F}_{s,\boldsymbol{u}}$ is embedded and prove that the notions of AC-PT and AC-PT are equivalent for $\mathcal{W}_{s,\boldsymbol{a},b}$ and that AC-PT holds for $\mathcal{W}_{s,\boldsymbol{a},b}$ iff the weights $\boldsymbol{a}$ grow polynomially fast. These results enable us to show that AC-ST holds for $\mathcal{F}_{s,\boldsymbol{u}}$ if weights $\boldsymbol{u}$ decay sufficiently fast. This part is based on [52].

Finally, we remark that this thesis is based on the following papers:

- [54], see Chapter 3,
- [53], see Chapter 4,
- [55], see Chapter 5,
- [52], see Chapter 6.

Chapter 2

Notation and definitions

Throughout this thesis, we use the following notation. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let b be an integer greater than 1. Let $\mathbb{Z}_b = \mathbb{Z}/b\mathbb{Z}$ be the residue class ring modulo b. We identify $\mathbb{Z}_b$ with the set $\{0, 1, \dots, b-1\} \subset \mathbb{Z}$. Let $\omega_b = \exp(2\pi\sqrt{-1}/b)$. For a set S, we denote by $|S|$ the cardinality of S. For a group or a ring R and positive integers s and n, we denote by $R^{s\times n}$ the set of $s \times n$ matrices with components in R. The operators $\oplus$ and $\ominus$ denote the digitwise addition and subtraction modulo b, respectively. That is, for $k, k' \in \mathbb{N}_0$ whose b-adic expansions are $k = \sum_{i=1}^{\infty} \kappa_i b^{i-1}$ and $k' = \sum_{i=1}^{\infty} \kappa'_i b^{i-1}$ with $\kappa_i, \kappa'_i \in \mathbb{Z}_b$ for all i, $\oplus$ and $\ominus$ are defined as

$$k \oplus k' = \sum_{i=1}^{\infty} \eta_i b^{i-1} \text{ and } k \ominus k' = \sum_{i=1}^{\infty} \eta'_i b^{i-1},$$

where $\eta_i = \kappa_i + \kappa'_i \pmod b$ and $\eta'_i = \kappa_i - \kappa'_i \pmod b$, respectively. In case of vectors in $\mathbb{N}_0^s$, the operators $\oplus$ and $\ominus$ are applied componentwise. We define $f^{(n_1,\dots,n_s)} := \partial^{n_1+\cdots+n_s} f/\partial x_1^{n_1} \cdots \partial x_s^{n_s}$.

In this chapter, we introduce notions including Walsh functions and digital nets and consider QMC integration using digital nets.

2.1 Walsh functions

In this section, we introduce Walsh functions and Walsh coefficients, which are widely used in analyzing QMC integration. More information of the Walsh analysis can be found in the books [45, 20].

We first give the definition of Walsh functions for the one-dimensional case and then generalize it to the higher-dimensional case.

Definition 2.1.1. *Let $b \geq 2$ be a positive integer. We denote the b-adic expansion of $k \in \mathbb{N}_0$ by $k = \kappa_1 + \kappa_2 b + \cdots + \kappa_i b^{i-1}$ with $\kappa_1, \dots, \kappa_i \in \mathbb{Z}_b$. Then the k-th b-adic Walsh function ${}_b\mathrm{wal}_k \colon [0,1) \to \{1, \omega_b, \dots, \omega_b^{b-1}\}$ is defined as*

$${}_b\mathrm{wal}_k(x) := \omega_b^{\kappa_1\xi_1+\cdots+\kappa_i\xi_i},$$

for $x \in [0,1)$ whose b-adic expansion is given by $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$, which is unique in the sense that infinitely many of the ξ_i are different from $b-1$.

Definition 2.1.2. *Let $b \geq 2$ and s be positive integers. Let $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ and $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. Then the $\boldsymbol{k}$-th b-adic Walsh function ${}_b\mathrm{wal}_{\boldsymbol{k}} \colon [0,1)^s \to \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ is defined as*

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

Since we shall always use Walsh functions in a fixed base b, we omit the subscript and simply write wal_k or $\mathrm{wal}_{\boldsymbol{k}}$ in this paper. Some important properties of Walsh functions, used in this paper, are described below, see [14, Appendix A.2] for the proof.

Proposition 2.1.3. *The following holds true:*

1. *For all $\boldsymbol{k} \in \mathbb{N}_0^s$, we have*

$$ \int_0^1 \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\, d\boldsymbol{x} = \begin{cases} 1 & \text{if } \boldsymbol{k} = 0, \\ 0 & \text{otherwise.} \end{cases} $$

2. *For all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$, we have*

$$ \int_{[0,1)^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{x})}\, d\boldsymbol{x} = \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{l}, \\ 0 & \text{otherwise.} \end{cases} $$

3. *For all $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{N}_0^s$ and $\boldsymbol{x} \in [0,1)^s$, we have*

$$ \begin{aligned} \mathrm{wal}_{\boldsymbol{k} \oplus \boldsymbol{k}'}(\boldsymbol{x}) &= \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{wal}_{\boldsymbol{k}'}(\boldsymbol{x}), \\ \mathrm{wal}_{\boldsymbol{k} \ominus \boldsymbol{k}'}(\boldsymbol{x}) &= \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}'}(\boldsymbol{x})}. \end{aligned} $$

4. *The system $\{\mathrm{wal}_{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L^2[0,1)^s$ for any positive integer s.*

We define Walsh coefficients as follows.

Definition 2.1.4. *Let $\boldsymbol{k} \in \mathbb{N}_0^s$ and $f \colon [0,1)^s \to \mathbb{C}$. The $\boldsymbol{k}$-th Walsh coefficient of f is defined as*

$$ \widehat{f}(\boldsymbol{k}) := \int_{[0,1)^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\, d\boldsymbol{x}. $$

The Walsh series of the function f is given by

$$ f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) $$

for any $f \in L^2[0,1)^s$.

We note that the notation $\widehat{f}$ is used as discrete Fourier coefficients and $\mathcal{F}(f)$ is used as Walsh coefficients in Chapter 3.

2.2 Digital nets

In this thesis, we consider the discretized setting as well as the non-discretized setting. By abuse of notation, the words “digital net”, “dual net” and “Dick weight” are used in the two settings. The discretized setting was first consider in [34] for $b = 2$ and will be generalized in Chapter 3 for $b \geq 2$. We also consider the discretized setting in Chapter 4. In this section, we consider the non-discretized setting.

We introduce digital nets in $[0,1)^s$. The definition of digital nets over finite rings is given in [29]. we adopt an equivalent definition of digital nets, which is proposed as digital nets with generating matrices in [13, Definition 4.3].

For a positive integer m and a non-negative integer k with its b-adic expansion $k = \sum_{i=1}^{\infty} \kappa_i b^{i-1}$, we define the m-digit truncated vector $\mathrm{tr}_m(k) \in \mathbb{Z}_b^m$ as $\mathrm{tr}_m(k) = (\kappa_1, \kappa_2, \dots, \kappa_m)^\top$.

Definition 2.2.1. *Let $G_1, \dots, G_s \in \mathbb{Z}_b^{l \times d}$ be $l \times d$ matrices over $\mathbb{Z}_b$ with $d \leq l$. Let $0 \leq k < b^d$. For $1 \leq j \leq s$ and $1 \leq i \leq l$, define $y_{i,k.j} \in \mathbb{Z}_b$ as*

$$(y_{1,k,j}, \dots, y_{l,k,j})^\top = G_j \mathrm{tr}_d(k).$$

Then we define

$$x_{k,j} = \frac{y_{1,k,j}}{b} + \frac{y_{2,k,j}}{b^2} + \cdots + \frac{y_{l,k,j}}{b^l} \in [0,1)$$

for $1 \leq j \leq s$. In this way we obtain the k-th point $\boldsymbol{x}_k = (x_{k,1}, \dots, x_{k,s})$. We define $P = P(G_1, \dots, G_s) := \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^d-1}\}$ (P is considered as a multiset) and call it a d-dimensional digital net over $\mathbb{Z}_b$ with precision l, or simply a digital net.

The *dual net* of a digital net plays an important role in the subsequent analysis, which is defined as follows.

Definition 2.2.2. *For positive integers d, l with $d \leq l$, let $P = P(G_1, \dots, G_s)$ be a d-dimensional digital net over $\mathbb{Z}_b$ with precision l. The dual net of P, denoted by $P^\perp = P^\perp(G_1, \dots, G_s)$, is defined as*

$$P^\perp := \{\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s \mid G_1^\top \mathrm{tr}_l(k_1) + \cdots + G_s^\top \mathrm{tr}_l(k_s) = 0\}.$$

By easy calculation, we have the following.

Lemma 2.2.3. *Let P be a digital net with generating matrices $G_1, \dots, G_s$. Then we have*

$$P^\perp = \{\boldsymbol{k} \in \mathbb{N}_0^s \mid \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = 1 \text{ for all } \boldsymbol{x} \in P\}.$$

The next lemma, which is a slight generalization of [14, Lemma 4.75] to our context, connects a digital net with Walsh functions.

Lemma 2.2.4. *Let P be a digital net over $\mathbb{Z}_b$ and $P^{\perp}$ its dual net. Then we have*

$$|P|^{-1} \sum_{\boldsymbol{x} \in P} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{k} \in P^{\perp}, \\ 0 & \text{otherwise.} \end{cases}$$

From now on, we consider integration using QMC algorithms over digital nets. Assume that f is given by Walsh series and that P is a digital net. Then we have

$$\begin{aligned} |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}) - I(f) &= |P|^{-1} \sum_{\boldsymbol{x} \in P} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) - I(f) \\ &= \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) |P|^{-1} \sum_{\boldsymbol{x} \in P} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) - I(f) \\ &= \sum_{\boldsymbol{k} \in P^{\perp}} \widehat{f}(\boldsymbol{k}) - \widehat{f}(0) \\ &= \sum_{\boldsymbol{k} \in P^{\perp} \setminus \{0\}} \widehat{f}(\boldsymbol{k}). \end{aligned}$$

Hence we have

$$\left| |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}) - I(f) \right| \leq \sum_{\boldsymbol{k} \in P^{\perp} \setminus \{0\}} |\widehat{f}(\boldsymbol{k})|. \tag{2.1}$$

Chapter 3

WAFOM over abelian groups for quasi-Monte Carlo point sets

3.1 Introduction

A strong analogy between coding theory and QMC point sets is well known (see, e.g., [4, 38, 48]). In coding theory, the minimum Hamming weight is used for a criterion for linear codes. Analogically, Niederreiter-Rosenbloom-Tsfasman (NRT) weight is a criterion for digital nets in QMC theory [36, 44]. More precisely, the minimum NRT weight is essentially equivalent to t-value and gives an upper bound on the star-discrepancy, which are important criteria for QMC point sets. In this chapter, as a generalization of [34], we consider the Dick weight μ on "codes over $\mathbb{Z}_b$" and connect them to a criterion WAFOM of digital nets over $\mathbb{Z}_b$ for QMC integration. Furthermore, we establish a MacWilliams-type identity for the Dick weight, which gives a computable formula of the minimum Dick weight and WAFOM.

As we have seen in Chapter 1, higher order convergence results for digital nets, i.e., $\mathrm{Err}(f;\mathcal{P})$ converges faster than N^{-1}, has been established. For a given integer $\alpha > 1$, Dick gave quadrature rules for α-smooth integrands which achieve $\mathrm{Err}(f;\mathcal{P}) \in O(N^{-\alpha+\varepsilon})$ [9]. He introduced a weight which gives a bound on a criterion $\mathrm{WF}_\alpha(\mathcal{P})$ (he did not give a name and we use the name in [34]) for a digital net $\mathcal{P}$ over a finite field with cardinality b, and proved a Koksma-Hlawka type inequality $\mathrm{Err}(f;\mathcal{P}) \leq C_{b,s,\alpha}\|f\|_{\alpha,s} \cdot \mathrm{WF}_\alpha(\mathcal{P})$, where $\|f\|_{\alpha,s}$ is a norm of f for a Sobolev space and $C_{b,s,\alpha}$ is a constant depend only on b, s, and α. Later he improved the constant factor of the lowest WF_α for digital nets over a finite cyclic group [10].

As a discretized version of Dick's method, Matsumoto, Saito and Matoba introduced the Dick weight μ and a related criterion WAFOM $\mathrm{WF}(P)$ for an $\mathbb{F}_2$-

digital net P [34]. One remarkable merit of WAFOM is that WAFOM is easily computable by the inversion formula [34, (4.2)], which is easier to implement than the formula of WF_α derived from [3, Section 4]. Using this merit, they executed a random search of low-WAFOM point sets and showed that such point sets perform better than some standard low-discrepancy point sets. There are several studies on low-WAFOM point sets. The existence of low-WAFOM point sets was shown by Matsumoto and Yoshiki [35].

In this chapter, as a generalization of [34] we propose the Dick weight and WAFOM for digital nets over $\mathbb{Z}_b$ and for subgroups of $G^{s\times n}$ where G is a finite abelian group. WAFOM over $\mathbb{Z}_b$ is also a discretized version of Dick's method and thus satisfies a Koksma-Hlawka type inequality. Moreover, we give a MacWilliams-type identity of weight enumerator polynomials for the Dick weight. Using this identity we obtain a computable formula of the minimum Dick weight as well as WAFOM, which is a generalization of the inversion formula for WAFOM in the dyadic case. Furthermore, we give generalizations of known properties of WAFOM over $\mathbb{F}_2$ in [35] and [59]. More precisely, we give a lower bound on WAFOM and prove the existence of low-WAFOM point sets. In particular, we improve some of the results in [35]. These results imply that there exist positive constants C, D, D' and F depending only on b and independent of s, n and N such that $N^{-C\log N/s} \leq \min\{\mathrm{WF}(P) \mid P \text{ is a digital net}, |P| \leq N\} \leq FN^{-D(\log N)/s+D'}$, if $(\log N)/s$ is sufficiently large.

These results are similar to the works of Dick, but there is no implication between them. Dick fixed the smoothness α, while our method requires n-smoothness on the function where n is as above. Thus, in our case, the function class is getting smaller for n being increased.

The rest of this chapter is organized as follows. In Section 3.2, we introduce the necessary background and notation, such as the discretization scheme of QMC integration and the discrete Fourier transform. In Section 3.3, we define the Dick weight and WAFOM over a general finite abelian group G, and prove a Koksma-Hlawka type inequality in the case that G is cyclic. In Section 3.4, we define the weight enumerator polynomial, give the MacWilliams-type identity for the Dick weight, and give a computable formula of WAFOM. In Section 3.5, we give a lower bound on WAFOM, prove the existence of low-WAFOM point sets, and study the order of WAFOM.

3.2 Preliminaries

In this chapter we use the following notation. We denote by O the zero matrix. We denote by e the base of the natural logarithm.

3.2.1 Discretized QMC in base b

In this subsection, we explain discretized QMC in base b. This discretization is a straightforward generalization of the $b = 2$ case in [34].

Let s be a positive integer. Let $\mathcal{P} \subset [0,1)^s$ be a point set in an s-dimensional unit cube with finite cardinality $|\mathcal{P}| = N$, and let $f\colon [0,1)^s \to \mathbb{R}$ be an integrable function. Recall that quasi-Monte Carlo integration by $\mathcal{P}$ is an approximation value

$$I_{\mathcal{P}}(f) := \frac{1}{N} \sum_{\boldsymbol{x} \in \mathcal{P}} f(\boldsymbol{x})$$

of the actual integration

$$I(f) := \int_{[0,1)^s} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

The QMC integration error is $\mathrm{Err}(f;\mathcal{P}) := |I_{\mathcal{P}}(f) - I(f)|$.

Here, we fix a positive integer n, which is called the degree of discretization or the precision. We consider an n-digit discrete approximation in base b. We associate a matrix $B := (b_{i,j}) \in \mathbb{Z}_b^{s\times n}$ with a point $\boldsymbol{x}_B = (x_B^1, \ldots, x_B^s) = (\sum_{j=1}^n b_{1,j} b^{-j}, \ldots, \sum_{j=1}^n b_{s,j} b^{-j}) \in [0,1)^s$, and with an s-dimensional cube $\mathbf{I}_B := \prod_{i=1}^s I_i \subset [0,1)^s$, where each edge $I_i := [x_B^i, x_B^i + b^{-n})$ is a half-open interval with length b^{-n}. We define n-digit discrete approximation f_n of f as

$$f_n\colon \mathbb{Z}_b^{s\times n} \to \mathbb{R},\ \ B := (b_{i,j}) \mapsto \frac{1}{\mathrm{Vol}(\mathbf{I}_B)} \int_{\mathbf{I}_B} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Let P be a subset of $\mathbb{Z}_b^{s\times n}$. We define n-th discretized QMC integration of f by P as

$$I_{P,n}(f) := \frac{1}{|P|} \sum_{B \in P} f_n(B)$$

and define the n-th discretized QMC integration error as

$$\mathrm{Err}(f; P, n) := |I_{P,n}(f) - I(f)|.$$

For each $B \in P$, we take the center point of the cube I_B. Let $\mathcal{P} \subset [0,1)^s$ be the set of such center points given by P. By a slight extension of [34, Lemma 2.1], if f is continuous with Lipschitz constant K then we have $|I_{P,n}(f) - I_{\mathcal{P}}(f)| \leq K\sqrt{s}b^{-n}$. We take n large enough so that $K\sqrt{s}b^{-n}$ is negligibly small compared to the order of QMC integration error $|I_{\mathcal{P}}(f) - I(f)|$ by $\mathcal{P}$. Then we may regard the n-th discretized QMC integration error $\mathrm{Err}(f;P,n)$ as an approximation of the QMC integration error $\mathrm{Err}(f;P)$.

As point sets, in this chapter we consider subgroups of $\mathbb{Z}_b^{s\times n}$ as well as digital nets. The definition of digital nets over finite rings is given in [29]. we adopt an equivalent definition of digital nets, which is proposed as digital nets with generating matrices in [13, Definition 4.3].

Definition 3.2.1. *Let $C_1, \ldots, C_s \in \mathbb{Z}_b^{n\times d}$ be matrices and let $X_1, \ldots, X_d \in \mathbb{Z}_b^{s\times n}$ be defined by the j-th row of X_i is the transpose of the i-th column of C_j. Assume that $X_1, \ldots, X_d$ are a free basis of $\mathbb{Z}_b^{s\times n}$ as a $\mathbb{Z}_b$-module. For an integer k with $0 \leq k \leq b^d - 1$, we define a matrix $\boldsymbol{x}_k \in \mathbb{Z}_b^{s\times n}$ as $\boldsymbol{x}_k = \sum_{i=1}^d \kappa_{i-1} X_i$,* *where* $k = \kappa_0 + \kappa_1 b^1 + \cdots + \kappa_{d-1} b^{d-1}$ $(0 \leq \kappa_i \leq b-1)$ *is the b-adic expansion of* k*. We call the set* $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^d-1}\}$ *the digital net generated by the matrices* $C_1, \ldots, C_s$*.*

It is easy to see that digital nets become subgroups of $\mathbb{Z}_b^{s\times n}$.

3.2.2 Discrete Fourier transform

In this subsection, we recall the notion of character groups and the discrete Fourier transform. We refer to [47] for general information on character groups. Let G be a finite abelian group. Let $T := \{z \in \mathbb{C} \mid |z| = 1\}$ be the multiplicative group of complex numbers of absolute value one.

Definition 3.2.2. *We define the character group of* G *by* $G^\vee := \mathrm{Hom}(G, T)$*, namely* $G^\vee$ *is the set of group homomorphisms from* G *to* T*.*

There is a natural pairing $\bullet \colon G^\vee \times G \to T,\ (h, g) \mapsto h \bullet g := h(g)$.

We can see that $\mathbb{Z}_b^\vee$ is isomorphic to $\mathbb{Z}_b$ as an abstract group. Throughout this chapter, we identify $\mathbb{Z}_b^\vee$ with $\mathbb{Z}_b$ through a pairing $\bullet \colon \mathbb{Z}_b \times \mathbb{Z}_b \to T,\ (h, g) \mapsto h \bullet g := \omega_b^{hg}$, where hg is the product in $\mathbb{Z}_b$.

Let R be a commutative ring containing $\mathbb{C}$. Let $f : G \to R$ be a function. We define the discrete Fourier transform of f as below.

Definition 3.2.3. *The discrete Fourier transform of* f *is defined by* $\widehat{f} \colon G^\vee \to R,\ h \mapsto \frac{1}{|G|}\sum_{g\in G} f(g)(h \bullet g)$*. Each value* $\widehat{f}(h)$ *is called a discrete Fourier coefficient.*

We assume that $P \subset G$ is a subgroup. We define $P^\perp := \{h \in G^\vee \mid h \bullet g = 1 \text{ for all } g \in P\}$. Since $P^\perp$ is the kernel of the restriction map $G^\vee \to P^\vee$, we have $|P^\perp| = |G|/|P|$. We recall the orthogonality of characters.

Lemma 3.2.4. *Suppose that* $P \subset G$ *is a subgroup and* $g \in G$*. Then we have*

$$\sum_{h \in P^\perp} h \bullet g = \begin{cases} |P^\perp| & \text{if } g \in P, \\ 0 & \text{if } g \notin P. \end{cases}$$

This lemma implies the Poisson summation formula and the Fourier inversion formula.

Theorem 3.2.5 (Poisson summation formula)**.**

$$\frac{1}{|P|}\sum_{g\in P} f(g) = \sum_{h\in P^\perp} \widehat{f}(h).$$

Proof.

$$\sum_{h\in P^\perp} \widehat{f}(h) = \sum_{h\in P^\perp} \frac{1}{|G|}\sum_{g\in G} f(g)(h \bullet g)$$

$$= \sum_{g \in G} \frac{1}{|G|} f(g) \sum_{h \in P^\perp} h \bullet g$$

$$= \frac{1}{|G|} \sum_{g \in P} f(g) \cdot |P^\perp| \quad (\because \text{Lemma 3.2.4})$$

$$= \frac{1}{|P|} \sum_{g \in P} f(g). \qquad \square$$

Theorem 3.2.6 (Fourier inversion formula)**.** *For a complex-valued function $f\colon G \to \mathbb{C}$, we have $f(g) = \sum_{h \in G^\vee} \widehat{f}(-h)(h \bullet g)$ for any $g \in G$. Moreover, if f is real-valued, we have $f(g) = \sum_{h \in G^\vee} \overline{\widehat{f}(h)}(h \bullet g)$.*

Proof. By Lemma 3.2.4, we have $\sum_{h \in G^\vee} h \bullet g = 0$ if $g \neq 0$ and $\sum_{h \in G^\vee} h \bullet g = |G|$ if $g = 0$. Thus we have

$$\sum_{h \in G^\vee} \widehat{f}(-h)(h \bullet g) = \sum_{h \in G^\vee} \frac{1}{|G|} \sum_{g' \in G} f(g')((-h) \bullet g')(h \bullet g)$$
$$= \frac{1}{|G|} \sum_{g' \in G} f(g') \sum_{h \in G^\vee} (h \bullet (g - g'))$$
$$= f(g),$$

which proves the complex-valued case. If f is real-valued, we have $\widehat{f}(-h) = \overline{\widehat{f}(h)}$, and thus the complex-valued case implies the real-valued case. $\square$

3.2.3 Walsh coefficients and discrete Fourier coefficients

In this subsection, we see the relationship between Walsh coefficients and discrete Fourier coefficients. As a corollary, we prove that the n-digit discrete approximation f_n of f is essentially equal to the appropriate approximation of the Walsh series of f. Let $A = (a_{i,j}) \in \mathbb{Z}_b^{s \times n}$. We define maps $\phi_i\colon \mathbb{Z}_b^{s \times n} \to \mathbb{N}_0$ as $\phi_i(A) = \sum_{j=1}^n a_{i,j} b^{j-1}$ and $\phi\colon \mathbb{Z}_b^{s \times n} \to \mathbb{N}_0^s$ as $\phi(A) = (\phi_1(A), \dots, \phi_s(A))$. Note that $\phi_i(A) < b^n$ holds for all $1 \leq i \leq s$ and $A \in \mathbb{Z}_b^{s \times n}$. In this chapter, we denote by $\mathcal{F}(f)(\boldsymbol{k})$ the $\boldsymbol{k}$-th Walsh coefficient.

Lemma 3.2.7. *Let $f\colon [0,1)^s \to \mathbb{R}$ and $A = (a_{i,j}) \in \mathbb{Z}_b^{s \times n}$. Then we have*

$$\overline{\mathcal{F}(f)(\phi(A))} = \widehat{f_n}(A).$$

Proof. Since $\phi_i(A) < b^n$ holds for all $1 \leq i \leq s$, for all $\boldsymbol{x} = (x_1, \dots, x_s) \in \mathbf{I}_B$ we have

$${}_b\mathrm{wal}_{\phi(A)}(\boldsymbol{x}) = \prod_{i=1}^s {}_b\mathrm{wal}_{\phi_i(A)}(x_i) = \prod_{i=1}^s \omega_b^{a_{i,1}b_{i,1} + \cdots + a_{i,n}b_{i,n}} = B \bullet A.$$

Therefore we have

$$\overline{\mathcal{F}(f)(\phi(A))} = \int_{[0,1)^s} f(\boldsymbol{x})\, {}_b\mathrm{wal}_{\phi(A)}(\boldsymbol{x})\, d\boldsymbol{x}$$

$$
\begin{aligned}
&= \sum_{B \in \mathbb{Z}_b^{s\times n}} \int_{\mathbf{I}_B} f(\boldsymbol{x})_b\mathrm{wal}_{\phi(A)}(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \sum_{B \in \mathbb{Z}_b^{s\times n}} \int_{\mathbf{I}_B} f(\boldsymbol{x})(B \bullet A)\, d\boldsymbol{x} \\
&= \sum_{B \in \mathbb{Z}_b^{s\times n}} (B \bullet A) \int_{\mathbf{I}_B} f(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \sum_{B \in \mathbb{Z}_b^{s\times n}} (B \bullet A) \cdot \mathrm{Vol}(\mathbf{I}_B) f_n(B) \\
&= \sum_{B \in \mathbb{Z}_b^{s\times n}} (B \bullet A) \cdot b^{-sn} f_n(B) = \widehat{f_n}(A),
\end{aligned}
$$

which proves the lemma. $\square$

Let $f \sim \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \mathcal{F}(f)(\boldsymbol{k})_b\mathrm{wal}_k$ be the Walsh expansion of a real valued function $f\colon [0,1)^s \to \mathbb{R}$. Lemma 3.2.7 implies that considering n-digit discrete approximation f_n of f is as same as considering the Walsh polynomial $\sum_{\boldsymbol{k}<b^n} \mathcal{F}(f)(\boldsymbol{k}) \cdot {}_b\mathrm{wal}_k$, where $\boldsymbol{k} = (k_1, \dots, k_s) < b^n$ means that $k_i < b^n$ holds for every $i = 1, \dots, s$, namely we have the following.

Proposition 3.2.8. *Let $f\colon [0,1)^s \to \mathbb{R}$. For $B \in \mathbb{Z}_b^{s\times n}$, we have $f_n(B) = \sum_{\boldsymbol{k}<b^n} \mathcal{F}(f)(\boldsymbol{k})_b\mathrm{wal}_k(\boldsymbol{x}_B)$.*

Proof.

$$
\begin{aligned}
f_n(B) &= \sum_{A \in \mathbb{Z}_b^{s\times n}} \overline{\widehat{f_n}(A)B} \bullet A \ (\because \text{Theorem 3.2.6}) \\
&= \sum_{A \in \mathbb{Z}_b^{s\times n}} \mathcal{F}(f)(\phi(A))_b\mathrm{wal}_{\phi(A)}(\boldsymbol{x}_B) \ (\because \text{Lemma 3.2.7}) \\
&= \sum_{\boldsymbol{k}<b^n} \mathcal{F}(f)(\boldsymbol{k})_b\mathrm{wal}_k(\boldsymbol{x}_B). \qquad \square
\end{aligned}
$$

3.3 WAFOM over a finite abelian group

In this section, we expand the notion of WAFOM defined in [34], more precisely, we define WAFOM over a finite abelian group with b elements.

First, we evaluate the n-th discretized QMC integration error of f with its discrete Fourier coefficients. Let $P \subset \mathbb{Z}_b^{s\times n}$ be a subgroup. We have $I(f) = \widehat{f_n}(O)$ by the definition of the discrete Fourier inversion, and we have $I_{P,n}(f) = \sum_{A\in P^\perp} \widehat{f_n}(A)$ by the Poisson summation formula (Theorem 3.2.5). Hence we have

$$
\mathrm{Err}(f; P, n) = |I_{P,n}(f) - I(f)| = \left| \sum_{A\in P^\perp\setminus\{O\}} \widehat{f_n}(A) \right| \le \sum_{A\in P^\perp\setminus\{O\}} |\widehat{f_n}(A)|,
$$

and thus we would like to bound the value $|\widehat{f_n}(A)|$. Dick gives an upper bound of the $\boldsymbol{k}$-th b-adic Walsh coefficient $\mathcal{F}(f)(\boldsymbol{k})$ for n-smooth function f (for the definition of n-smoothness, see [9] or [14, §14]).

Theorem 3.3.1 ([14], Theorem 14.23)**.** *There is a constant $C_{b,s,n}$ depending only on b, s and n such that for any n-smooth function $f\colon [0,1)^s \to \mathbb{R}$ and any $\boldsymbol{k} \in \mathbb{N}^s$ it holds that*

$$|\mathcal{F}(f)(\boldsymbol{k})| \leq C_{b,s,n}\|f\|_{n,s} \cdot b^{-\mu_n(\boldsymbol{k})},$$

where $\|f\|_{n,s}$ is a norm of f for a Sobolev space and $\mu_n(\boldsymbol{k})$ is the n-weight of $\boldsymbol{k}$, which are defined in [14, (14.6) and Theorem 14.23], see also Chapter 1.

Instead of μ_n, we define the Dick weight μ on dual groups of general finite abelian groups below, which is a generalization of the Dick weight over $\mathbb{F}_2$ defined in [34]. Actually, μ is a special case of $\mu_n \circ \phi$. More precisely, if $G = \mathbb{Z}_b$ and $\alpha \geq n$ hold, then we have $\mu = \mu_\alpha \circ \phi$ as a function from $(\mathbb{Z}_b^\vee)^{s\times n} (\simeq \mathbb{Z}_b^{s\times n})$ to $\mathbb{N}_0$.

Definition 3.3.2. *Let G be a finite abelian group and let $A \in (G^\vee)^{s\times n}$. The Dick weight $\mu\colon (G^\vee)^{s\times n} \to \mathbb{N}_0$ is defined as*

$$\mu(A) := \sum_{i,j} j \times \delta(a_{i,j}),$$

with $\delta(h) = 0$ for $h = 0$ and $\delta(h) = 1$ for $h \neq 0$.

We obtain the next corollary.

Corollary 3.3.3. *There exists a constant $C_{b,s,n}$ depending only on b, s and n such that for any n-smooth function $f\colon [0,1)^s \to \mathbb{R}$ and any $A \in (\mathbb{Z}_b)^{s\times n}$ it holds that*

$$|\widehat{f_n}(A)| \leq C_{b,s,n}\|f\|_n \cdot b^{-\mu(A)}.$$

Proof. This is the direct corollary of Theorem 3.3.1, Lemma 3.2.7, and the equality $\mu(A) = \mu_n \circ \phi(A)$. □

By the above corollary, we have a bound on the n-th discretized QMC integration error

$$\mathrm{Err}(f; P, n) := |I(f) - I_{P,n}(f)| \leq C_{b,s,n}\|f\|_n \times \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)},$$

for a subgroup P of $\mathbb{Z}_b^{s\times n}$.

Hence, as a generalization of [34], we define a kind of figure of merit (the Walsh figure of merit or WAFOM).

Definition 3.3.4. *Let s, n be positive integers. Let G be a finite abelian group with b elements. Let $P \subset G^{s\times n}$ be a subgroup of $G^{s\times n}$. We define the Walsh figure of merit of P by*

$$\mathrm{WF}(P) := \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)}.$$

In order to stress the role of the precision n, we sometimes denote $\mathrm{WF}^n(P)$ instead of $\mathrm{WF}(P)$.

Then, as we have seen, we have the Koksma-Hlawka type inequality

$$\mathrm{Err}(f; P, n) := |I(f) - I_{P,n}(f)| \leq C_{b,s,n}\|f\|_n \times \mathrm{WF}(P)$$

for a subgroup $P \subset \mathbb{Z}_b^{s\times n}$. This shows that $\mathrm{WF}(P)$ is a quality measure of the point set P for quasi-Monte Carlo integration when $G = \mathbb{Z}_b$.

3.4 MacWilliams identity over an abelian group

In this section, we assume that s, n are positive integers. Recall that G is a finite abelian group and $G^\vee$ its character group. We consider an abelian group $G^{s\times n}$. Let $P \subset G^{s\times n}$ be a subgroup.

We are interested in the weight enumerator polynomial of $P^\perp$

$$W_{P^\perp}(x, y) := \sum_{A\in P^\perp} x^{M-\mu(A)} y^{\mu(A)} \in \mathbb{C}[x, y],$$

where $M := n(n + 1)s/2$.

Let $R := \mathbb{C}[x_{i,j}(h)]$, where $x_{i,j}(h)$ is a family of indeterminates for $1 \leq i \leq s$, $1 \leq j \leq n$, and $h \in G^\vee$. We define functions $f_{i,j} \colon G^\vee \to R$ as $f_{i,j}(h) = x_{i,j}(h)$ and $f \colon (G^{s\times n})^\vee = (G^\vee)^{s\times n} \to R$ as

$$f(A) := \prod_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} f_{i,j}(a_{i,j}) = \prod_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} x_{i,j}(a_{i,j}).$$

Now the complete weight enumerator polynomial of $P^\perp$, in a standard sense [32, Chapter 5], is defined by

$$GW_{P^\perp}(x_{i,j}(h)) := \sum_{A\in P^\perp} \prod_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} x_{i,j}(a_{i,j}),$$

and similarly, the complete weight enumerator polynomial of P is defined by

$$GW_P^*(x_{*i,j}(g)) := \sum_{B\in P} \prod_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} x_{*i,j}(b_{i,j})$$

in $R^* := \mathbb{C}[x_{*i,j}(g)]$ where $x_{*i,j}(g)$ is a family of indeterminates for $1 \leq i \leq s$, $1 \leq j \leq n$, and $g \in G$. We note that if we substitute

$$x_{i,j}(0) \leftarrow x^j, \quad x_{i,j}(h) \leftarrow y^j \text{ for } h \neq 0, \tag{3.1}$$

we have an identity

$$GW_{P^\perp}(x_{i,j}(h))|_{\text{above substitution}} = W_{P^\perp}(x, y).$$

A standard formula of the Fourier transform tells that, if $f_1\colon G_1 \to R$, $f_2\colon G_2 \to R$ are functions and $f_1 f_2\colon G_1 \times G_2 \to R$ is their multiplication at the value, then

$$\widehat{f_1 f_2} = \widehat{f_1}\widehat{f_2}$$

holds. This implies that

$$\widehat{f}(B) = \prod_{\substack{1\le i\le s\\ 1\le j\le n}} \widehat{f_{i,j}}(b_{i,j}) = \frac{1}{|G|^{sn}} \prod_{\substack{1\le i\le s\\ 1\le j\le n}} \sum_{h\in G^\vee} f_{i,j}(h)(h \bullet b_{i,j}).$$

Hence, by the Poisson summation formula (Theorem 3.2.5), we have

$$\begin{aligned} GW_{P^\perp}(x_{i,j}(h)) &= \sum_{A\in P^\perp} f(A) \\ &= |P^\perp| \sum_{B\in P} \widehat{f}(B) \\ &= \frac{1}{|P|} \prod_{\substack{1\le i\le s\\ 1\le j\le n}} \sum_{h\in G^\vee} f_{i,j}(h)(h \bullet b_{i,j}). \end{aligned}$$

Thus we have the MacWilliams identity below, which is a variant of Generalized MacWilliams identity [32, Chapter 5 §6]:

Proposition 3.4.1 (MacWilliams identity)**.**

$$GW_{P^\perp}(x_{i,j}(h)) = \frac{1}{|P|} GW^*_P(\textit{substituted}),$$

where in the right hand side every $x_{*i,j}(g)$ *is substituted by*

$$x_{*i,j}(g) \leftarrow \sum_{h\in G^\vee} (h \bullet g) x_{i,j}(h).$$

We consider specializations of this identity. First, we consider a specialization $\overline{GW}_{P^\perp}(x_1, \dots, x_n, y_1, \dots, y_n)$ of $GW_{P^\perp}(x_{i,j}(h))$ obtained by the substitution

$$x_{i,j}(0) \leftarrow x_j, \quad x_{i,j}(h) \leftarrow y_j \text{ for } h \neq 0.$$

We have

$$\begin{aligned} \sum_{h\in G^\vee} (h \bullet g) x_{i,j}(h) \Big|_{\text{above substitution}} &= (0 \bullet g) x_j + \sum_{h\in G^\vee \setminus\{0\}} (h \bullet g) y_j \\ &= x_j - y_j + \sum_{h\in G^\vee} (h \bullet g) y_j \\ &= x_j - y_j + \begin{cases} b y_j & (\text{if } g = 0) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

$$= \begin{cases} x_j + (b-1)y_j & (\text{if } g = 0) \\ x_j - y_j & (\text{otherwise}) \end{cases},$$

where we use Lemma 3.2.4 for the third equality. Thus, we have the following formula.

Corollary 3.4.2.

$$\overline{GW}_{P^\perp}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \le i \le s \\ 1 \le j \le n}} (x_j + \eta(b_{i,j})y_j),$$

where $\eta(b_{i,j}) = b - 1$ *if* $b_{i,j} = 0$ *and* $\eta(b_{i,j}) = -1$ *if* $b_{i,j} \neq 0$.

Second, we consider the specialization (3.1) of $GW_{P^\perp}$. We have already seen that $GW_{P^\perp}|_{(\text{substitution } (3.1))} = W_{P^\perp}(x, y)$ holds. Since

$$W_{P^\perp}(x, y) = \overline{GW}_{P^\perp}(x^1, \ldots, x^n, y^1, \ldots, y^n)$$

follows, Corollary 3.4.2 implies the following formula:

Theorem 3.4.3.

$$W_{P^\perp}(x, y) = \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \le i \le s \\ 1 \le j \le n}} (x^j + \eta(b_{i,j})y^j),$$

where $\eta(b_{i,j}) = b - 1$ *if* $b_{i,j} = 0$ *and* $\eta(b_{i,j}) = -1$ *if* $b_{i,j} \neq 0$.

Using Theorem 3.4.3, we can compute $\mathrm{WF}(P)$ and $\delta_{P^\perp}$, the minimum Dick weight of $P^\perp$. The minimum Dick weight of $P^\perp$ is defined as

$$\delta_{P^\perp} := \min_{B \in P^\perp \setminus \{O\}} \mu(B),$$

which is used for bounding WAFOM (see Section 3.5.3). First, we introduce how to compute $\mathrm{WF}(P)$. The following formula to compute WAFOM is a generalization of [34, Corollary 4.2] ,which treats the case $G = \mathbb{F}_2$.

Corollary 3.4.4. *Let* $P \subset \mathbb{Z}_b^{s \times n}$ *be a subgroup. Then we have*

$$\mathrm{WF}(P) = -1 + \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \le i \le s \\ 1 \le j \le n}} (1 + \eta(b_{i,j})b^{-j}).$$

Proof.

$$\mathrm{WF}(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)}$$

$$
\begin{aligned}
&= -1 + \sum_{A \in P^\perp} b^{-\mu(A)} \\
&= -1 + W_{P^\perp}(1, b^{-1}) \\
&= -1 + \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \le i \le s \\ 1 \le j \le n}} (1 + \eta(b_{i,j}) b^{-j}). \qquad \square
\end{aligned}
$$

The merit of Theorem 3.4.3 and Corollary 3.4.4 is that the number of summation depends only on $|P|$ linearly, not $|P^\perp| = b^{sn}/|P|$. We can calculate weight enumerator polynomials by sn times multiplication between an integer polynomial with a binomial, and $|P|$ times addition of such polynomials of degree $n(n+1)/2$. In the case of computing WAFOM, we need sn times of multiplication of real numbers and $|P|$ times of summation of such real numbers, thus we need $O(sn|P|)$ times of operations of real numbers. On the other hand, to calculate weight enumerator polynomials based on the definition, we need $|P^\perp|$ times of summations of monomials, and to calculate weight WAFOM based on the definition, we need $|P^\perp|$ times of summations of real numbers.

For QMC, the size $|P|$ cannot exceed a reasonable number of computer operations, so $|P^\perp| = b^{sn}/|P|$ can be large if sn is sufficiently large. This implies that the computational complexity of calculating weight enumerator polynomials or WAFOM using Theorem 3.4.3 or Corollary 3.4.4 is smaller if sn is large.

Second, we introduce how to compute $\delta_{P^\perp}$. The minimum Dick weight $\delta_{P^\perp}$ is equal to the degree of leading nonzero term of $-1 + W_{P^\perp}(1, y)$, namely:

Lemma 3.4.5. *Let $W_{P^\perp}(1, y) = 1 + \sum_{i=1}^{\infty} a_i y^i$. Then we have $\delta_{P^\perp} = \min\{i \mid a_i \neq 0\}$.*

Thus we can obtain the minimum Dick weight of $P^\perp$ by calculating the weight enumerator polynomial of $P^\perp$.

Remark 3.4.6. *Because of Lemma 3.5.15 in Section 3.5.5, in order to compute $\delta_{P^\perp}$ it is sufficient to compute $W_{P^\perp}(1, y)$ only up to degree $\delta_{P^\perp} \le d^2/(2s) + 3d/2 + s$.*

3.5 Estimation of WAFOM

The following arguments from Section 3.5.1 to Section 3.5.4 are generalizations of [35] which deals with the case $G = \mathbb{F}_2$, and arguments in Section 3.5.5 are generalizations of [59], which deals with the case $G = \mathbb{F}_2$. The methods for proofs are similar to [35] and [59]. In this section, we suppose that s and n are positive integers and that G is a finite abelian group.

3.5.1 Geometry of the Dick weight

Recall that G is a finite abelian group with $b \geq 2$ elements, $G^\vee$ its character group. The Dick weight $\mu\colon (G^\vee)^{s\times n} \to \mathbb{N}_0$ induces a metric

$$d(A, B) := \mu(A - B) \text{ for } A, B \in (G^\vee)^{s\times n}$$

and thus $(G^\vee)^{s\times n}$ can be regarded as a metric space.

Let $S_{s,n}(m) := |\{A \in (G^\vee)^{s\times n} \mid \mu(A) = m\}|$, namely $S_{s,n}(m)$ is the cardinality of the sphere in $(G^\vee)^{s\times n}$ with center 0 and radius m. A combinatorial interpretation of $S_{s,n}(m)$ is as follows. One has $s \times n$ dice. Each die has b faces. For each value $i = 1, \dots, n$, there exist exactly s dice with value 0 on one face and i on the other $b - 1$ faces. Then, $S_{s,n}(m)$ is the number of ways that the summation of the upper surfaces of $s \times n$ dice is m. This combinatorial interpretation implies the following identity:

$$\prod_{k=1}^{n} (1 + (b-1)x^k)^s = \sum_{m=0}^{\infty} S_{s,n}(m)x^m.$$

You can also see this identity from Theorem 3.4.3 for $P = \{O\}$, $x \leftarrow 1$, and $y \leftarrow x$. Note that the right hand side is a finite sum. It is easy to see that $S_{s,n}(m)$ is monotonically increasing with respect to s and n, and $S_{s,m}(m) = S_{s,m+1}(m) = S_{s,m+2}(m) = \cdots$ holds.

Definition 3.5.1. $S_s(m) := S_{s,m}(m)$.

We have the following identity between formal power series:

$$\prod_{k=1}^{\infty} (1 + (b-1)x^k)^s = \sum_{m=0}^{\infty} S_s(m)x^m. \tag{3.2}$$

For any positive integer M, we define

$$\mathcal{B}_{s,n}(M) := \{A \in (G^\vee)^{s\times n} \mid \mu(A) \leq M\}, \quad \mathrm{vol}_{s,n}(M) := |\mathcal{B}_{s,n}(M)|,$$

namely $\mathcal{B}_{s,n}(M)$ is the ball in $G^{s\times n}$ with center 0 and radius M, and $\mathrm{vol}_{s,n}(M)$ is its cardinality. We have $\mathrm{vol}_{s,n}(M) = \sum_{m=0}^{M} S_{s,n}(m)$, and thus $\mathrm{vol}_{s,n}(M)$ inherits properties of $S_{s,n}(m)$, namely, $\mathrm{vol}_{s,n}(M)$ is also monotonically increasing with respect to s and n, and $\mathrm{vol}_{s,M}(M) = \mathrm{vol}_{s,M+1}(M) = \mathrm{vol}_{s,M+2}(M) = \dots$ holds.

Definition 3.5.2. $\mathrm{vol}_s(M) := \mathrm{vol}_{s,M}(M)$.

3.5.2 Combinatorial inequalities

Lemma 3.5.3.

$$\mathrm{vol}_{s,n}(M) \leq \mathrm{vol}_s(M) \leq \exp(2\sqrt{(b-1)sM}).$$

Proof. We have already seen the first inequality. We prove the next inequality along [33, Exercise 3(b), p.332], which treats only $S = 1$ and $b = 2$ case. If $M = 0$ it is trivial, and so we assume that $M > 0$. Define a polynomial with non-negative integer coefficients by

$$f_{s,M}(x) := \prod_{k=1}^{M} (1 + (b-1)x^k)^s.$$

Since $f_{s,M}(x)$ has only non-negative coefficients, from Identity (3.2) we have $\sum_{m=0}^{M} S_s(m)x^m \leq f_{s,M}(x) \quad (x \in (0,1))$. Hence we have

$$\mathrm{vol}_s(M) = \sum_{m=0}^{M} S_s(m) \leq \sum_{m=0}^{M} S_s(M)x^{m-M} \leq f_{s,M}(x)/x^M \quad (x \in (0,1)).$$

By taking the logarithm of the both sides and using the well-known inequality $\log(1 + X) \leq X$, for all $x \in (0,1)$ we have

$$\begin{aligned}
\mathrm{vol}_{s,n}(M) &\leq s \sum_{k=1}^{M} \log(1 + (b-1)x^k) + M \log(1/x) \\
&< s(b-1) \sum_{k=1}^{M} x^k + M \log\left(1 + \frac{1-x}{x}\right) \\
&< s(b-1) \frac{x}{1-x} + M \frac{1-x}{x}.
\end{aligned}$$

By comparison of the arithmetic mean and the geometric mean, the last expression attains the minimum value $2\sqrt{(b-1)sM}$ when $s(b-1)x/(1-x) = M(1-x)/x$ holds, namely $x = (1 + \sqrt{(b-1)s/M})^{-1} \in (0,1)$. □

Lemma 3.5.4.

$$S_{s,n}(M) \leq S_s(M) \leq \exp(2\sqrt{(b-1)sM}).$$

Proof. It follows from Lemma 3.5.3 and the inequality $S_s(M) \leq \mathrm{vol}_s(M)$. □

3.5.3 Bounding WAFOM by the minimum weight

Definition 3.5.5. *Let $P \subset G^{s \times n}$ be a subgroup. The minimum Dick weight of $P^\perp$ is defined by*

$$\delta_{P^\perp} := \min_{B \in P^\perp \setminus \{O\}} \mu(B)$$

The next lemma bounds $\mathrm{WF}(P)$ by the minimum weight of $P^\perp$.

Lemma 3.5.6. *For a positive integer M, define*

$$C_{s,n}(M) := \sum_{A \in (G^\vee)^{s\times n} \setminus \mathcal{B}_{s,n}(M-1)} b^{-\mu(A)} = \sum_{m=M}^{\infty} S_{s,n}(m) b^{-m}$$

and

$$C_s(M) := \sum_{m=M}^{\infty} S_s(m) b^{-m}.$$

Then we have

$$\mathrm{WF}^n(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \leq C_{s,n}(\delta_{P^\perp}) \leq C_s(\delta_{P^\perp}).$$

Proof. The last inequality is trivial, so it suffices to prove the first inequality. Since $P^\perp \setminus \{O\} \subset (G^\vee)^{s\times n} \setminus \mathcal{B}_{s,n}(\delta_{P^\perp} - 1)$ holds, we have

$$\begin{aligned}\mathrm{WF}^n(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} &\leq \sum_{A \in (G^\vee)^{s\times n} \setminus \mathcal{B}_{s,n}(\delta_{P^\perp} - 1)} b^{-\mu(A)} \\ &= C_{s,n}(\delta_{P^\perp}).\end{aligned}$$

□

We shall estimate $C_s(\lceil M' \rceil)$ (C for the Complement of the ball) for rather general real number M': from Lemma 3.5.4 it follows that

$$\begin{aligned}C_s(\lceil M' \rceil) &= \sum_{m=\lceil M' \rceil}^{\infty} S_s(m) b^{-m} \\ &\leq \sum_{m=\lceil M' \rceil}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\ &= b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} + \sum_{m=\lceil M' \rceil + 1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}}. \qquad (3.3)\end{aligned}$$

First, we estimate the second term of the above. The function

$$\exp(2\sqrt{(b-1)sm}) b^{-m} = \exp(2\sqrt{(b-1)sm} - m \log b)$$

is monotonically decreasing with respect to m if

$$\begin{aligned}\frac{d}{dm}\left(2\sqrt{(b-1)sm} - m\log b\right) \leq 0 &\iff \frac{2(b-1)s}{2\sqrt{(b-1)sm}} - \log b \leq 0 \\ &\iff \sqrt{\frac{(b-1)s}{m}} \leq \log b \\ &\iff m \geq (\log b)^{-2}(b-1)s,\end{aligned}$$

hence we assume that $M' \geq (\log b)^{-2}(b-1)s$. Then, we have

$$\begin{aligned}
&\sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\
&\leq \int_{m=\lceil M' \rceil}^{\infty} e^{-m\log b} e^{2\sqrt{(b-1)sm}}\, dm \\
&= \int_{m=\lceil M' \rceil}^{\infty} \exp\left(-(\log b)\left(\sqrt{m} - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) dm \\
&\leq \int_{m=M'}^{\infty} \exp\left(-(\log b)\left(\sqrt{m} - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) dm \\
&= \int_{x=\sqrt{M'}}^{\infty} \exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) 2x\, dx.
\end{aligned}$$

In order to bound the last integral from above, for a positive number c we assume that $\sqrt{M'} \geq (1+c)\sqrt{(b-1)s}/\log b$ or equivalently $M' \geq (1+c)^2(\log b)^{-2}(b-1)s$. This assumption is stronger than the previous assumption $M' \geq (\log b)^{-2}(b-1)s$. Then, on the domain of integration $x \geq \sqrt{M'} \geq (1+c)\sqrt{(b-1)s}/\log b$, we have $cx \leq (1+c)(x - \sqrt{(b-1)s}/\log b)$. Hence the estimation continues:

$$\begin{aligned}
&\sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\
&\leq \int_{x=\sqrt{M'}}^{\infty} \exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) \\
&\qquad\qquad \times 2\frac{1+c}{c}\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right) dx \\
&= \frac{1+c}{c}\frac{1}{\log b}\left[-\exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right)\right]_{x=\sqrt{M'}}^{\infty} \\
&= \frac{1+c}{c}\frac{1}{\log b}\exp\left(-(\log b)\left(\sqrt{M'} - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) \\
&= \frac{1+c}{c}\frac{1}{\log b}\exp(-(\log b)M' + 2\sqrt{(b-1)sM'}) \\
&= \frac{1+c}{c}\frac{1}{\log b} b^{-M'} e^{2\sqrt{(b-1)sM'}}.
\end{aligned}$$

Second, we consider the first term of (3.3). We have already proved that $\exp(2\sqrt{(b-1)sm})b^{-m}$ is monotonically decreasing if $m \geq (\log b)^{-2}(b-1)s$, and thus the assumption $M' \geq (\log b)^{-2}(b-1)s$ implies

$$b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} \leq b^{-M'} e^{2\sqrt{(b-1)sM'}}.$$

Therefore we have

$$\begin{aligned} C_s(\lceil M' \rceil) &\leq b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} + \sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\ &\leq b^{-M'} e^{2\sqrt{(b-1)sM'}} + \frac{1+c}{c}\frac{1}{\log b} b^{-M'} e^{2\sqrt{(b-1)sM'}} \\ &= \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}}. \end{aligned}$$

Now we proved:

Proposition 3.5.7. *Let c be a positive real number. Let M' be a real number with $M' \geq (1+c)^2(\log b)^{-2}(b-1)s$. Then we have the following bound*

$$C_{s,n}(\lceil M' \rceil) \leq C_s(\lceil M' \rceil) \leq \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}}.$$

3.5.4 Existence of low-WAFOM point sets

We denote the probability of the event A by $\mathrm{Prob}[A]$. Let p_b be the smallest prime factor of b. Let d be a positive integer. Choose d matrices $B_1, \ldots, B_d \in G^{s \times n}$ independently and uniformly at random. Let $P = \langle B_1, \ldots, B_d \rangle \subset G^{s \times n}$ be the G-linear span of $B_1, \ldots, B_d$, namely $P = \{g_1 B_1 + \cdots + g_d B_d \mid g_1, \ldots, g_d \in G\}$ where $g \in G$ acts on $B = (b_{ij})$ by $gB = (gb_{ij})$. Note that $|P| \leq b^d$.

Remark 3.5.8. *If $G = \mathbb{Z}_b$, by the theory of invariant factor decomposition, we can say that there exist matrices $B'_1, \ldots, B'_d$ such that $P' := \langle B'_1, \ldots, B'_d \rangle$ includes P and becomes a free $\mathbb{Z}_b$-module of rank d. Thus if $G = \mathbb{Z}_b$, we can replace "subgroup P" in this subsection with a "digital net P," since in this subsection we consider only the existence of a subgroup which has large minimum Dick weight, and $P \subset P'$ implies that $\delta_{P^\perp} \leq \delta_{P'^\perp}$.*

First, we evaluate $\mathrm{Prob}[\mathrm{perp}_L]$, where we define perp_L as the event that $B_1, \ldots, B_d$ are all perpendicular to $L \in (G^\vee)^{s \times n}$.

Lemma 3.5.9. *Let $L \in (G^\vee)^{s \times n}$ be a nonzero matrix. Then we have* $\mathrm{Prob}[L \perp B] \leq 1/p_b$. *Especially we have* $\mathrm{Prob}[\mathrm{perp}_L] \leq p_b^{-d}$.

Proof. We consider the map $(L\bullet)\colon G^{s \times n} \to \mathbb{C}, B \mapsto L \bullet B$. Then we have the surjective group homomorphism $G^{s \times n} \to \mathrm{Im}(L\bullet)$, and thus $|\mathrm{Im}(L\bullet)|$ divides $G^{s \times n}$. Moreover, since L is nonzero, $|\mathrm{Im}(L\bullet)|$ is larger than 1. Hence we have $|\mathrm{Im}(L\bullet)| \geq p_b$. Therefore we have $\mathrm{Prob}[L \perp B] = |\mathrm{Im}(L\bullet)|^{-1} \leq 1/p_b$, and especially we have $\mathrm{Prob}[\mathrm{perp}_L] = \mathrm{Prob}[L \perp B]^d \leq p_b^{-d}$. □

Let M be a positive integer. We evaluate the probability of the event that $\delta_{P^\perp} \geq M$. We have

$$\begin{aligned}
\mathrm{Prob}[\delta_{P^\perp} \geq M] &= 1 - \mathrm{Prob}[\delta_{P^\perp} \leq M-1] \\
&= 1 - \mathrm{Prob}[\exists L \in \mathcal{B}_{s,n}(M-1)\backslash\{O\} \text{ s.t. } L \in P^\perp] \\
&= 1 - \mathrm{Prob}[\exists L \in \mathcal{B}_{s,n}(M-1)\backslash\{O\} \text{ s.t. } L \perp B_1, \dots, L \perp B_d] \\
&= 1 - \mathrm{Prob}[\cup_{L \in \mathcal{B}_{s,n}(M-1)\backslash\{O\}} \mathrm{perp}_L] \\
&\geq 1 - \sum_{L \in \mathcal{B}_{s,n}(M-1)\backslash\{O\}} \mathrm{Prob}[\mathrm{perp}_L] \\
&\geq 1 - (\mathrm{vol}_{s,n}(M-1) - 1) \cdot {p_b}^{-d} \\
&> 1 - \mathrm{vol}_{s,n}(M-1) \cdot {p_b}^{-d}.
\end{aligned}$$

This shows:

Proposition 3.5.10. *If* $\mathrm{vol}_{s,n}(M-1) \leq {p_b}^d$ *holds, then there exists a subgroup* $P \subset G^{s \times n}$ *with* $|P| \leq b^d$ *satisfying* $\delta_{P^\perp} \geq M$*.*

By Lemma 3.5.3, the condition of this proposition is satisfied if it holds that

$$e^{2\sqrt{(b-1)s(M-1)}} \leq {p_b}^d \iff M \leq \frac{(\log p_b)^2 d^2}{4(b-1)s} + 1. \tag{3.4}$$

Therefore we have the following sufficient condition on the existence of M.

Proposition 3.5.11. *If* $M \leq (\log p_b)^2 d^2/(4(b-1)s) + 1$ *holds, then Inequality (3.4) is satisfied, and hence there exists a subgroup* $P \subset G^{s \times n}$ *with* $|P| \leq b^d$ *satisfying* $\delta_{P^\perp} \geq M$*.*

From now on, we define $\alpha_b := (\log p_b)/2$ and $M' := A^2 d^2/((b-1)s)$ where $A \leq \alpha_b$ and we take M to be $\lfloor M' + 1 \rfloor$ so that P with $|P| \leq b^d$ and $\delta_{P^\perp} \geq M$ exists. Then, by Proposition 3.5.7, we have the following upper bound of $\mathrm{WF}(P)$:

Proposition 3.5.12. *Let* $\alpha_b := (\log p_b)/2$*. Take a real number* A *with* $A \leq \alpha_b$ *and an arbitrary real number* $c > 0$*. Then for any positive integers* s*,* n*, and* $d \geq (1+c)(b-1)s/(A \log b)$*, there exists a subgroup* $P \subset G^{s \times n}$ *with* $|P| \leq b^d$ *satisfying*

$$\mathrm{WF}^n(P) \leq \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-A^2 d^2/((b-1)s)} e^{2Ad}.$$

Proof. Define $M' := A^2 d^2/((b-1)s)$ and $M := \lfloor M'+1 \rfloor$. By Proposition 3.5.11, there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ and $\delta_{P^\perp} \geq M$. For this P, from Lemma 3.5.6 and Proposition 3.5.7 we have

$$\begin{aligned}
\mathrm{WF}(P) &\leq C_s(M) \\
&= C_s(\lceil M' \rceil)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}} \\
&= \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-A^2d^2/((b-1)s)} e^{2Ad},
\end{aligned}
$$

which proves the proposition. □

In particular, take $A = \alpha_b$ and we have the next theorem.

Theorem 3.5.13. *Let $\alpha_b := (\log p_b)/2$ and take an arbitrary real number $c > 0$. Then for any s, n, and $d \geq (1+c)(b-1)s/(\alpha_b \log b)$, there exists a subgroup $P \subset G^{s\times n}$ with $|P| \leq b^d$ satisfying*

$$
\mathrm{WF}(P) \leq \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-\alpha_b^2 d^2/((b-1)s)} e^{2\alpha_b d}.
$$

Applying Theorem 3.5.13 to the case $G = \mathbb{F}_2$, we can improve [35, Theorem 2 and Remark 5].

Corollary 3.5.14. *Let $\alpha := \alpha_2 = (\log 2)/2$ and take an arbitrary real number $c > 0$. Then for any n and $d \geq (1+c)s/(\alpha \log 2)$, there exists a linear subspace $P \subset \mathbb{F}_2^{s\times n}$ with $\dim P \leq d$ satisfying*

$$
\mathrm{WF}(P) \leq \left(1 + \frac{1+c}{c}\frac{1}{\log 2}\right) 2^{-\alpha^2 d^2/s} e^{2\alpha d}.
$$

3.5.5 A lower bound on WAFOM

In this subsection, we show a lower bound on WAFOM(P), as a generalization of [59]. The next lemma gives an upper bound on the minimum Dick weight of $P^\perp$ for given $P \subset G^{s\times n}$, which implies a lower bound of WAFOM(P).

Lemma 3.5.15. *Suppose that s and n are positive integers. Let $P \subset G^{s\times n}$ be a subgroup with $|P| \leq b^d$. Let q, r be nonnegative integers which satisfy $d = qs + r$ and $0 \leq r < s$. Then we have the following:*

1. *$\delta_{P^\perp} \leq sq(q+1)/2 + (q+1)(r+1) \leq d^2/2s + 3d/2 + s$.*
2. *Let C be an arbitrary positive real number greater than $1/2$. If $d/s \geq (\sqrt{C + 1/16} + 3/4)/(C - 1/2)$ holds, then we have $\delta_{P^\perp} \leq Cd^2/s$.*

Proof. We define a subgroup $Q := \{A = (a_{ij}) \in (G^\vee)^{s\times n} \mid a_{ij} = 0 \text{ if } (q+2 \leq j \leq n) \text{ or } (j = q+1 \text{ and } r+2 \leq i \leq s)\}$. We have $|Q| = b^{qs+r+1} = b^{d+1}$. There is a $\mathbb{Z}$-module isomorphism $P^\perp/(P^\perp \cap Q) \simeq (P^\perp + Q)/Q$, and thus we have

$$
|P^\perp \cap Q| = \frac{|P^\perp| \cdot |Q|}{|P^\perp + Q|} \geq \frac{b^{sn-d} \cdot b^{d+1}}{|(G^\vee)^{s\times n}|} = b,
$$

especially there exists a non-zero matrix $A' \in (P^\perp \cap Q)$. Therefore we have

$$
\delta_{P^\perp} \leq \mu(A') \leq \max\{\mu(A) \mid A = (a_{ij}) \in Q\} = sq(q+1)/2 + (q+1)(r+1),
$$

where the last equality holds if the components of A is as follows:

$$\begin{cases} a_{ij} = 0 \text{ if } (q+2 \le j \le n) \text{ or } (j = q+1 \text{ and } r+2 \le i \le s) \\ a_{ij} \neq 0 \text{ if } (1 \le j \le q) \text{ or } (j = q+1 \text{ and } 1 \le i \le r+1) \end{cases}.$$

In particular, since $q \le d/s$ and $r+1 \le s$, we have

$$\begin{aligned} \delta_{P^\perp} &\le sq(q+1)/2 + (q+1)(r+1) \\ &\le \frac{d}{2}\left(\frac{d}{s}+1\right) + \left(\frac{d}{s}+1\right)s = \frac{d^2}{s}\left(\frac{1}{2} + \frac{3s}{2d} + \frac{s^2}{d^2}\right), \end{aligned}$$

which proves the first statement.

Let C be a real number greater than $1/2$ and we assume $d/s \ge (\sqrt{C+1/16} + 3/4)/(C-1/2)$. Then we have $1/2 + 3s/2d + s^2/d^2 \le C$. Thus we obtain

$$\delta_{P^\perp} \le \frac{d^2}{s}\left(\frac{1}{2} + \frac{3s}{2d} + \frac{s^2}{d^2}\right) \le Cd^2/s,$$

which proves the second statement. □

The above lemma gives a lower bound of $\mathrm{WF}(P)$.

Theorem 3.5.16. *Suppose that s and n are positive integers. Let G be a finite abelian group with $b \ge 2$ elements. Let $P \subset G^{s\times n}$ be a subgroup with $|P| \le b^d$. Let C be an arbitrary positive real number greater than $1/2$. If $d/s \ge (\sqrt{C+1/16} + 3/4)/(C-1/2)$ holds, then we have*

$$\mathrm{WF}^n(P) \ge b^{-Cd^2/s}.$$

Proof.

$$\mathrm{WF}^n(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \ge b^{-\delta_{P^\perp}} \ge b^{-Cd^2/s}.$$

□

3.5.6 Order of WAFOM

In this subsection, we consider the order of $\mathrm{WF}(P)$ where P is a subgroup of $G^{s\times n}$ with $|P| = b^d$.

We fix the base b. Let $D := \alpha_b = (\log p_b)/2$. We fix a positive integer E satisfying $E > (b-1)/(D\log b)$. Let c be the real number such that $E = (1+c)(b-1)/(D\log b)$ (by the assumption that $E > (b-1)/(D\log b)$, c is positive). Note that c, D and E depend only on b.

We assume that $d/s \ge E$. Then, by Proposition 3.5.12, there exists a subgroup $P \subset G^{s\times n}$ with $|P| \le b^d$ satisfying

$$\mathrm{WF}^n(P) \le \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) b^{-D^2d^2/((b-1)s)} e^{2Dd}.$$

Moreover, by Theorem 3.5.16, for every P with $|P| \le b^d$ we have $\mathrm{WF}^n(P) \ge b^{-Cd^2/s}$ where $C = (1/2 + 3/(2E) + 1/E^2)$. Thus we have the following lemma.

Lemma 3.5.17. *If $d/s \geq E$, we have*

$$-Cd^2/s \leq \min\{\log_b(\mathrm{WF}^n(P)) \mid P \subset G^{s\times n} \text{ subgroup}, |P| \leq b^d\}$$
$$\leq -D^2d^2/((b-1)s) + 2Dd/\log b + \log_b\left(1 + \frac{1+c}{c}\frac{1}{\log b}\right).$$

Especially, let $N = b^d$ and we have the following.

Theorem 3.5.18. *Let G be a finite abelian group with $|G| = b$. Let $P \subset G^{s\times n}$ be a subgroup with $|P| \leq N$. Let c, C, D, and E are constants as Lemma 3.5.17, which depend only on b. Suppose that $(\log N)/s \geq E$. Then we have*

$$N^{-C(\log N)/s} \leq \min\{\mathrm{WF}^n(P) \mid P \subset G^{s\times n} \text{ subgroup}, |P| \leq N\}$$
$$\leq \left(1 + \frac{1+c}{c}\frac{1}{\log b}\right) N^{-D^2(\log N)/((\log b)(b-1)s)+2D/\log b}.$$

Chapter 4

An explicit construction of point sets with large minimum Dick weight

4.1 Introduction

In the previous chapter, as a generalization of [35] we proved the existence of digital nets whose minimum Dick weight is large. However, the proof was not constructive. Throughout this chapter we assume that $b = p$ is prime. In this chapter, we give a construction algorithm of digital nets over $\mathbb{F}_p = \mathbb{Z}_b$ whose minimum Dick weight is large.

We use the same notation $P \subset \mathbb{Z}_b^{s\times n}$, $P^\perp$ and $\delta_{P^\perp}$ as in Chapter 3. In this chapter, using Niederreiter-Xing sequences and Dick's construction, we explicitly construct a linear subspace $P \subset \mathbb{Z}_b^{s\times n}$ of dimension m which achieves $\delta_{P^\perp} \geq \lfloor m/11s \rfloor (m/2 + 8\sqrt{(s\lfloor m/11s \rfloor - 2)/3} + s/2 + 8) + 1$ when $s\lfloor m/11s \rfloor \geq 2$ for each m. This is the same order as m^2/s. This implies that we can explicitly construct point sets with low WAFOM.

The rest of this chapter is organized as follows. In Section 4.2.1, we recall the definition of higher order digital nets and Dick's construction. In Section 4.2.2, we recall results on Niederreiter-Xing sequences. In Section 4.3, we show our main results using Dick's construction and Niederreiter-Xing sequences.

4.2 Preliminaries

In this section, we recall existing definitions and theorems necessary to prove our results. Let s, n, m be positive integers. Let $\mathbb{F}_p^m$ denote the set of row vectors of dimension m over $\mathbb{F}_p$.

4.2.1 Higher order digital nets

To define higher order digital nets, we recall digital nets with generating matrices $C_1, \ldots, C_s \in \mathbb{F}_p^{n \times m}$. We define a map $\Phi(C_1, \ldots, C_s) \colon \mathbb{F}_p^m \to \mathbb{F}_p^{s \times n}$ by $\mathbf{c} \mapsto (C_1\mathbf{c}^T, \ldots, C_s\mathbf{c}^T)^T$, namely the i th row of $\Phi(C_1, \ldots, C_s)(\mathbf{c})$ is $\mathbf{c}C_i^T$. We define $P(C_1, \ldots, C_s) \subset \mathbb{F}_p^{s \times n}$ as the image of the map $\Phi(C_1, \ldots, C_s)$. $P(C_1, \ldots, C_s)$ is called a digital net with generating matrices $C_1, \ldots, C_s$, or shortly, a digital net.

Definition 4.2.1 (Higher order digital nets)**.** *[9][14, Definition 15.2]. Let $s, \alpha, n, m \in \mathbb{N}$, let $0 < \beta \leq \min(1, \alpha m/n)$ be a real number and let $0 \leq t \leq \beta n$ be an integer. Let $C_1, \ldots, C_s \in \mathbb{F}_p^{n \times m}$. We define $\mathbf{c}_j^{(i)} \in \mathbb{F}_p^m$ as the j th row vector of the matrix C_i for $1 \leq j \leq n$ and $1 \leq i \leq s$. If, for all $1 \leq d_{i,v_1} < \cdots < d_{i,1} \leq n$, where $0 \leq v_i \leq m$ and $1 \leq i \leq s$, with*

$$\sum_{i=1}^{s} \sum_{j=1}^{\min(v_i,\alpha)} d_{i,j} \leq \beta n - t$$

the vectors

$$\mathbf{c}_{d_{1,v_1}}^{(1)}, \ldots, \mathbf{c}_{d_{1,1}}^{(1)}, \ldots, \mathbf{c}_{d_{s,v_s}}^{(s)}, \ldots, \mathbf{c}_{d_{s,1}}^{(s)}$$

are linearly independent over $\mathbb{F}_p$, then the digital net with generating matrices $C_1, \ldots, C_s$ is called a higher order digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_p$ or for short, a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_p$.

For $\alpha = \beta = 1$ and $n = m$, we obtain a (classical) digital (t, m, s)-net in base p, which is compatible with Definition 4.2.7.

We state an equivalent definition in terms of the dual space using the Dick α-weight with precision n [11, §2].

Definition 4.2.2 (Dick α-weight with precision n)**.** *Let p be a prime and s, n be positive integers. Let $\alpha \in \mathbb{N} \cup \{\infty\}$. We define $\mu_{\alpha,n} \colon \mathbb{F}_p^n \to \mathbb{Z}$ by*

$$\mu_{\alpha,n}(\mathbf{a}) = \begin{cases} 0 & (\textit{if } \mathbf{a} = \mathbf{0}) \\ i_1 + \cdots + i_{\min(\alpha,v)} & (\textit{otherwise}) \end{cases},$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_p^n$ and $i_1, \ldots, i_v$ are defined as follows: The non-zero components of $\mathbf{a}$ are $a_{i_1}, a_{i_2}, \ldots, a_{i_v}$, with $n \geq i_1 > i_2 > \cdots > i_v \geq 1$.

Let $A \in \mathbb{F}_p^{s \times n}$. Let $\mathbf{a}^{(i)} \in \mathbb{F}_p^n$ be the i th row of the matrix A for $1 \leq i \leq s$. We define the Dick α-weight with precision n of A by

$$\mu_{\alpha,n}(A) := \sum_{i=1}^{s} \mu_{\alpha,n}(\mathbf{a}^{(i)}).$$

For any non-zero linear subspace P of $\mathbb{F}_p^{s \times n}$, we define the minimum distance

$$\delta_{\alpha,n}(P) := \min_{A \in P \setminus \{\mathbf{0}\}} \mu_{\alpha,n}(A).$$

In particular, we are interested in the case $\alpha = \infty$ and we define $\delta_P := \delta_{\infty,n}(P)$.

The next theorem characterizes higher order digital nets.

Theorem 4.2.3. *[11, Theorem 3]. Given matrices $C_1, \dots, C_s \in \mathbb{F}_p^{n\times m}$ generate a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{F}_p$ if and only if*

$$\delta_{\alpha,n}(P(C_1, \dots, C_s)^{\perp}) \geq \beta n - t + 1.$$

To prove our results, we need the following construction by Dick.

Definition 4.2.4 (Dick's construction)**.** *[14, 15.2]. Let $d \in \mathbb{N}$, let $C_1, \dots, C_{sd}$ be the generating matrices of a digital (t', m, sd)-net over $\mathbb{F}_p$ (in the sense of (t, m, s)-net, see Definition 4.2.1 and the comment below Definition 4.2.1). Let $\mathbf{c}_j^{(i)}$ be the j-th row vector of C_i for $1 \leq j \leq m$ and $1 \leq i \leq sd$. We define the matrices $D_i \in \mathbb{F}_p^{dm\times m}$ for $1 \leq i \leq s$ as below: Let $\mathbf{d}_j^{(i)}$ be the j-th row vector of D_i for $1 \leq j \leq dm$ and $1 \leq i \leq s$. We define $\mathbf{d}_l^{(i)} = \mathbf{c}_u^{(v)}$ whenever $l = (u - i)d + v$ for $1 \leq l \leq dm$ with $(i - 1)d + 1 \leq v \leq id$ and $1 \leq u \leq m$; that is, the row vectors of D_i from top to bottom are*

$$\mathbf{c}_1^{((i-1)d+1)}, \dots, \mathbf{c}_1^{(id)}, \mathbf{c}_2^{((i-1)d+1)}, \dots, \mathbf{c}_2^{(id)}, \dots, \mathbf{c}_m^{((i-1)d+1)}, \dots, \mathbf{c}_m^{(id)}.$$

Theorem 4.2.5. *[14, Theorem 15.7]. Let $d \in \mathbb{N}$, let $\alpha \in \mathbb{N} \cup \{\infty\}$, and let $C_1, \dots, C_{sd}$ be the generating matrices of a digital (t', m, sd)-net over $\mathbb{F}_p$. Then the matrices $D_1, \dots, D_s$ defined as above are generating matrices of a higher order digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{F}_p$ with*

$$t \leq \min(d, \alpha) \cdot \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

Remark 4.2.6. *Theorem 4.2.3 and Theorem 4.2.5 are proved for finite α in the references. The case $\alpha = \infty$ reduces to the finite case, since $\alpha \geq n$ implies $\mu_{\alpha,n} = \mu_{n,n}$.*

4.2.2 (t, m, s)-nets and (t, s)-sequences

For our construction, we need good (t, m, s)-nets. In this section, we recall the definition of (t, m, s)-nets and (t, s)-sequences and known theorems (see the recent survey [39] for details).

Definition 4.2.7 ((t, m, s)-nets)**.** *Let $b \geq 2$, $m \geq 1$, $0 \leq t \leq m$, and $s \geq 1$ be integers. A point set $P = \{\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}\} \subset [0, 1)^s$ is called a (t, m, s)-net in base b if for all nonnegative integers $d_1, \dots, d_s$ with $d_1 + \cdots + d_s = m - t$ the elementary intervals*

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}}\right)$$

contain exactly b^t points for all choices of $0 \leq a_i < b^{d_i}$ ($a_i \in \mathbb{Z}$) for $1 \leq i \leq s$.

If a given (t, m, s)-net is a digital net, we call it a digital (t, m, s)-net. (t, s)-sequences are analogs of (t, m, s)-nets.

Definition 4.2.8 ((t,s)-sequences)**.** *Let $b \geq 2$, $t \geq 0$, and $s \geq 1$ be integers. A sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in $[0,1)^s$ is a (t,s)-sequence in base b if for all integers $k \geq 0$ and $m > t$ the points $\mathbf{x}_n$ with $kb^m \leq n < (k+1)b^m$ form a (t,m,s)-net in base b.*

Theorem 4.2.9. *[40, Lemma 8.2.13]. Let q be a prime power and $s \geq 1$ be an integer. If there exists a digital (t,s)-sequence over $\mathbb{F}_q$, then for every integer $m \geq \max(t,1)$ there exists a digital $(t,m,s+1)$-net over $\mathbb{F}_q$.*

Let $d_q(s)$ be the least value of t for which there exists a digital (t,s)-sequence over $\mathbb{F}_q$.

Theorem 4.2.10. *[40, Theorem 8.4.4]. For any prime power q and any dimension $s \geq 1$, we have*

$$d_q(s) \leq \frac{3q-1}{q-1}(s-1) - \frac{(2q+4)(s-1)^{1/2}}{(q^2-1)^{1/2}} + 2.$$

Remark 4.2.11. *The proof of this theorem uses algebraic function fields with many rational places which are constructed by explicit extensions of algebraic function fields, and thus constructive [40, §8.4].*

Remark 4.2.12. *There are tables of the value $d_q(s)$ provided by the database at `http://mint.sbg.ac.at` launched by Schürer and Schmid [46], which lists some better values of $d_q(s)$ than those given by Theorem 4.2.10 for some (s,q).*

4.3 Main result

Theorem 4.3.1. *Let s, m be positive integers and c be a positive real number. We put $d := \lfloor \frac{m}{(2c+1)s} \rfloor$ and $s' := sd - 1$. We assume that $s' \geq 1$. If there exists a (t', s')-sequence with a nonnegative integer t', then there exists a linear subspace $P \subset \mathbb{F}_p^{s \times dm}$ of dimension m satisfying*

$$\delta_{P^\perp} \geq d\left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1.$$

Proof. By assumption and Theorem 4.2.9, there exists a $(\min(t',m), m, s'+1)$-net (namely a $(\min(t',m), m, sd)$-net) over $\mathbb{F}_p$.

By Theorem 4.2.5, from this digital net we can construct a higher order digital $(t, \infty, 1, dm \times m, s)$-net P over $\mathbb{F}_p$ with

$$\begin{aligned} t &\leq d \min(m, \min(t', m) + \lfloor s(d-1)/2 \rfloor) \\ &\leq d(t' + s(d-1)/2). \end{aligned}$$

Therefore, by Theorem 4.2.3, we have

$$\delta_{P^\perp} \geq 1 \cdot dm - t + 1$$

$$
\begin{aligned}
&\geq d(m - t' - s(d-1)/2) + 1 \\
&\geq d(m - t' - m/(4c+2) + s/2) + 1 \qquad \left(\because sd \leq s \cdot \frac{m}{(2c+1)s} = \frac{m}{2c+1}\right) \\
&= d\left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1. \qquad \square
\end{aligned}
$$

Remark 4.3.2. *$d = \lfloor \frac{m}{(2c+1)s} \rfloor$ implies $m \geq (2c+1)sd$. Moreover, the assumption $s' = sd - 1 \geq 1$ implies that the integer d must be positive, thus we have $m \geq (2c+1)s$.*

The next lemma extends Theorem 4.3.1 to arbitrary precision.

Lemma 4.3.3. *Let s, m, n be positive integers. Then we have*

$$\max\{\delta_{P^\perp} \mid P \subset \mathbb{F}_p^{s \times n}, \dim P = m\} \geq \max\{\delta_{P^\perp} \mid P \subset \mathbb{F}_p^{s \times (n+1)}, \dim P = m\}.$$

Proof. There is a linear subspace $Q \subset \mathbb{F}_p^{s \times (n+1)}$ of dimension m such that

$$\delta_{Q^\perp} = \max\{\delta_{P^\perp} | P \subset \mathbb{F}_p^{s \times (n+1)}, \dim P = m\}.$$

We define a map $\mathrm{pr} : \mathbb{F}_p^{s \times (n+1)} \to \mathbb{F}_p^{s \times n}$ by cutting off the right-end column. We have $\dim Q^\perp = s(n+1) - m$ and $\dim \mathrm{pr}(Q^\perp) \geq sn - m$, and hence we have $\dim \mathrm{pr}(Q^\perp)^\perp \leq m$. Therefore

$$
\begin{aligned}
\max\{\delta_{P^\perp} \mid P \subset \mathbb{F}_p^{s \times n}, \dim P = m\} &\geq \delta_{\mathrm{pr}(Q^\perp)} \\
&\geq \delta_{Q^\perp} \\
&= \max\{\delta_{P^\perp} \mid P \subset \mathbb{F}_p^{s \times (n+1)}, \dim P = m\}. \square
\end{aligned}
$$

The next corollary is an extension of Theorem 4.3.1 to arbitrary precision n.

Corollary 4.3.4. *Let s, m, n be positive integers and c be a positive real number. We put $d := \lfloor \frac{m}{(2c+1)s} \rfloor$ and $s' := sd - 1$. We assume that $s' \geq 1$. If there exists a (t', s')-sequence with a nonnegative integer t', then there exists a linear subspace $P \subset \mathbb{F}_p^{s \times n}$ of dimension m satisfying*

$$\delta_{P^\perp} \geq \left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1.$$

Proof. We may assume $n > dm$, since this corollary follows directly from Theorem 4.3.1 and Lemma 4.3.3 if $n \leq dm$. We consider P of Theorem 4.3.1. Let P' be its image by the composition of two inclusions $P \subset \mathbb{F}_p^{s \times dm} \subset \mathbb{F}_p^{s \times n}$, where the latter inclusion is given by supplementing 0 column vectors on the right side ($n - dm$ of 0 column vectors). Then we have

$$
\begin{aligned}
\delta_{P'^\perp} &\geq \min(\delta_{P^\perp}, dm + 1) \\
&= \min\left(d\left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1, dm + 1\right)
\end{aligned}
$$

$$\geq d\left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1,$$

where the last inequality holds because

$$\begin{aligned}
&(dm+1) - \left(d\left(\frac{4c+1}{4c+2}m - t' + s/2\right) + 1\right) \\
&\quad = d(m/(4c+2) + t' - s/2) \\
&\quad \geq d((2c+1)s/(4c+2) + t' - s/2) \quad (\because \text{Remark 4.3.2}) \\
&\quad = dt' \\
&\quad \geq 0.
\end{aligned}$$

□

We use the explicit bound on (t, s)-sequences of Theorem 4.2.10 and we obtain the following explicit bound on $\delta_{P^\perp}$.

Corollary 4.3.5. *Let s, m, n be positive integers. We put $c_p := (3p-1)/(p-1)$ and $d := \lfloor \frac{m}{(2c_p+1)s} \rfloor$. If $sd \geq 2$ and $m \leq sn$, then there exists a linear subspace $P \subset \mathbb{F}_p^{s\times n}$ of dimension m satisfying*

$$\delta_{P^\perp} \geq d\left(m/2 + \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} + s/2 + 2c_p - 2\right) + 1.$$

Proof. By Theorem 4.2.10, there exists a $(t', sd-1)$-sequence with

$$t' \leq c_p(sd-2) - \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} + 2.$$

Therefore, by Corollary 4.3.4, there exists a linear subspace $P \subset \mathbb{F}_p^{s\times n}$ of dimension m satisfying

$$\begin{aligned}
\delta_{P^\perp} &\geq d\left(\frac{4c_p+1}{4c_p+2}m - t' + s/2\right) + 1 \\
&\geq d\left(\frac{4c_p+1}{4c_p+2}m - \left(c_p(sd-2) - \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} + 2\right) + s/2\right) + 1 \\
&= d\left(\frac{4c_p+1}{4c_p+2}m - c_p sd + 2c_p + \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} - 2 + s/2\right) + 1 \\
&\geq d\left(\frac{4c_p+1}{4c_p+2}m - c_p \cdot \frac{m}{2c_p+1} + 2c_p + \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} - 2 + s/2\right) + 1 \\
&\geq d\left(m/2 + \frac{(2p+4)(sd-2)^{1/2}}{(p^2-1)^{1/2}} + s/2 + 2c_p - 2\right) + 1.
\end{aligned}$$

□

In particular, in the case $p = 2$ we obtain the next corollary.

Corollary 4.3.6. *Let s, m be positive integers. We put $d := \lfloor m/11s \rfloor$. If $sd \geq 2$ and $m \leq sn$, then there exists a linear subspace $P \subset \mathbb{F}_2^{s\times n}$ of dimension m satisfying*

$$\delta_{P^\perp} \geq d\left(m/2 + 8\sqrt{(sd-2)/3} + s/2 + 8\right) + 1.$$

Note that the right hand side is of order m^2/s when m increases faster than s, namely, if $m/s \to \infty$.

We shall give an upper bound on $\mathrm{WF}(P)$ for P of Corollary 4.3.6, by using an upper-bound formula of $\mathrm{WF}(P)$ by $\delta_{P^\perp}$ given by Matsumoto and Yoshiki [35]. We recall:

Proposition 4.3.7. *[35, Proposition 1]. Let M be a positive integer, c a positive real number. Assume $M \geq (1+c)^2(\log 2)^{-2}s$. Then we have the following bound*

$$C_{s,n}(M) := \sum_{\substack{A \in \mathbb{F}_2^{s\times n} \\ \mu(A) \geq M}} 2^{-\mu(A)} < \frac{1+c}{c}\frac{1}{\log 2}2^{-M}e^{2\sqrt{sM}}.$$

We consider P in Corollary 4.3.6, namely P is an m-dimensional subspace $P \subset \mathbb{F}_2^{s\times n}$ which satisfies $\delta_{P^\perp} \geq d(m/2 + 8\sqrt{(sd-2)/3} + s/2 + 8) + 1$ where $d := \lfloor m/11s \rfloor$. We put $M := d(m/2 + 8\sqrt{(sd-2)/3} + s/2 + 8) + 1$ and $c := (\log 2)(M/s)^{1/2} - 1$, then the assumption of Proposition 4.3.7 is satisfied. Therefore we have

$$\begin{aligned}
\mathrm{WF}(P) &= \sum_{A \in P^\perp \setminus \{\mathbf{0}\}} 2^{-\mu(A)} \\
&\leq C_{s,n}(M) \quad (\because P^\perp \setminus \{\mathbf{0}\} \subset \{A \in \mathbb{F}_2^{s\times n} | \mu(A) \geq M\}) \\
&< \frac{1+c}{c}\frac{1}{\log 2}2^{-M}e^{2\sqrt{sM}}.
\end{aligned}$$

This shows:

Corollary 4.3.8. *Let s, m be positive integers. We put $d := \lfloor m/11s \rfloor$. If $s\lfloor m/11s \rfloor \geq 2$, then P of Corollary 4.3.6 satisfies*

$$\mathrm{WF}(P) < \frac{1+c}{c}\frac{1}{\log 2}2^{-M}e^{2\sqrt{sM}},$$

where $M := d(m/2 + 8\sqrt{(sd-2)/3} + s/2 + 8) + 1$ and $c := (\log 2)(M/s)^{1/2} - 1$.

We remark on the asymptotic behavior of this bound when $m/s \to \infty$. Then M is of order m^2/s, and c is of order $(M/s)^{1/2}$ and thus is of order m/s. Thus, the coefficient of the right hand side $\frac{1+c}{c}$ monotonously decreases and converges to 1. This implies that $\mathrm{WF}(P)$ is of order $\mathcal{O}(2^{-M}e^{2\sqrt{sM}})$ for M being of order m^2/s, which is comparable with the nonconstructive bound given in [35, Theorem 1]. and the previous chapter.

Chapter 5

Formulas for the Walsh coefficients of smooth functions and their application to bounds on the Walsh coefficients

5.1 Introduction

Throughout this chapter we use the following notation: We assume that k is a nonnegative integer whose b-adic expansion is $k = \kappa_1 b^{a_1-1} + \cdots + \kappa_v b^{a_v-1}$ where κ_i and a_i are integers with $0 < \kappa_i \leq b-1$, $a_1 > \cdots > a_v \geq 1$. For $k = 0$ we assume that $v = 0$ and $a_0 = 0$.

In this chapter, we focus on the decay of the Walsh coefficients of smooth functions. There are several studies for the decay of the Walsh coefficients. Fine considered the Walsh coefficients of functions which satisfy a Hölder condition in [17]. Dick proved the decay of the Walsh coefficients of functions of smoothness $\alpha \geq 1$ in [8, 9] and studied it in more detail in [10]: It was proved that if a function f has $\alpha - 1$ derivatives for which $f^{(\alpha-1)}$ satisfies a Lipschitz condition, then $|\widehat{f}(k)| \in O(b^{-a_1-\cdots-a_{\min(\alpha,v)}})$ [9]. Dick also proved that this order is the best possible. That is, for f of smoothness α, if there exists $1 \leq r \leq \alpha$ such that $\widehat{f}(k)$ decays faster than $b^{-a_1-\cdots-a_r}$ for all $k \in \mathbb{N}_0$ and $v \geq r$, then f is a polynomial of degree at most $r-1$ [10, Theorem 20].

Recently, Yoshiki gave a method to analyze the dyadic (i.e., 2-adic) Walsh coefficients in [58]. He introduced dyadic differences of (maybe discontinuous) functions and gave a formula in which the dyadic Walsh coefficients are given by dyadic differences multiplied by constants. Dyadic differences of a smooth function are expressed in terms of derivatives of the function. This enabled him to establish a formula for the dyadic Walsh coefficients of smooth functions expressed in terms of those derivatives. From this formula, he obtained a bound on the dyadic Walsh coefficients for α times continuously differentiable functions for $\alpha \geq 1$.

In this chapter, we establish a formula in which the b-adic Walsh coefficients of smooth functions are expressed in terms of those derivatives as

$$\widehat{f}(k) = (-1)^v \int_0^1 f^{(v)}(x) W(k)(x)\, dx,$$

where the function $W(k)(\cdot)\colon [0,1) \to \mathbb{C}$ is given by the iterated integral of Walsh functions as in Definition 5.2.1. This formula is a generalization of the formula for the dyadic Walsh coefficients of smooth functions in [58], however our method is different from that in [58]. Our main idea is first separating the interval [0,1) to appropriate intervals on which particular Walsh functions take constant values, and then applying integration by parts iteratively. We also establish another formula for the Walsh coefficients to use all of the smoothness of functions.

Furthermore, we give bounds on the b-adic Walsh coefficients for α times continuously differentiable functions. Our assumption is somewhat stronger than that of [10]. Instead, we obtain bounds asymptotically better with respect to α than results in [10]. Our bounds for the dyadic case recover results for smooth functions in [58]. Moreover, we obtain a class of infinitely smooth functions whose Walsh coefficients decay as $|\widehat{f}(k)| \in O(b^{-a_1-\cdots-a_v})$. We also obtain improved bounds on the Walsh coefficients for functions in periodic and non-periodic reproducing kernel Hilbert spaces which are considered in [10].

The rest of this chapter is organized as follows. We give two formulas for the Walsh coefficients of smooth functions in Sections 5.2 and 5.4. Bounds on the Walsh coefficients of smooth functions and Bernoulli polynomials are given in Sections 5.3 and 5.5, respectively. In Section 5.6 (resp. Section 5.7), we give a bound on the Walsh coefficients of functions in non-periodic (resp. periodic) reproducing kernel Hilbert spaces.

5.2 Integral formula for the Walsh coefficients of smooth functions

We introduce further notation which is used throughout this chapter. For $k > 0$, let $k' = k - \kappa_v b^{a_v - 1}$. Let $v(k) := v$ be the number of non-zero digits of k.

In this section, we define the function $W(k)(\cdot)$ and establish a formula in which the Walsh coefficients of smooth functions are expressed in terms of $W(k)(\cdot)$ and derivatives of the functions.

Definition 5.2.1. *For $k \in \mathbb{N}_0$, we define functions $W(k)(\cdot)\colon [0,1] \to \mathbb{C}$ recursively as*

$$W(0)(x) := 1,$$

$$W(k)(x) := \int_0^x \overline{\mathrm{wal}_{\kappa_v b^{a_v - 1}}(y)} W(k')(y)\, dy,$$

and the integral value of $W(k)(\cdot)$ as

$$I(k) := \int_0^1 W(k)(x)\, dx.$$

By definition, $W(k)(x)$ is continuous for all $k \in \mathbb{N}_0$. Note that we have

$$W(k)(x) = \int_0^x W(k')(y)\, dy \qquad \text{for } x \in [0, b^{-a_v}]$$

since we have $\mathrm{wal}_{\kappa_v b^{a_v - 1}}(y) = 1$ for all $y \in [0, b^{-a_v})$. We show the periodicity of $W(k)(\cdot)$ in the next lemma.

Lemma 5.2.2. *Let $k \in \mathbb{N}_0$. Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$, where $0 \leq c < b^{a_v}$ is an integer and $0 \leq x' < b^{-a_v}$ is a real number. Then we have*

$$W(k)(x) = \frac{1 - \overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W(k)(x').$$

In particular, $W(k)(\cdot)$ is a periodic function with period b^{-a_v+1} if $v > 0$.

Proof. We prove the lemma by induction on v. If $v = 0$, trivially the result holds. Hence we now assume that the claim holds for $v - 1$. Then $W(k')(\cdot)$ is periodic with period $b^{-a_{v-1}+1}$ and in particular with period b^{-a_v} if $v > 1$, and $W(k')(\cdot)$ is constant if $v = 1$. Hence we have

$$\begin{aligned} W(k)(x) &= \sum_{i=0}^{c-1} \int_{ib^{-a_v}}^{(i+1)b^{-a_v}} \overline{\mathrm{wal}_{\kappa_v b^{a_v - 1}}(y)} W(k')(y)\, dy \\ &\quad + \int_{cb^{-a_v}}^{cb^{-a_v}+x'} \overline{\mathrm{wal}_{\kappa_v b^{a_v - 1}}(y)} W(k')(y)\, dy \\ &= \sum_{i=0}^{c-1} \overline{\omega}_b^{i\kappa_v} \int_0^{b^{-a_v}} W(k')(y)\, dy + \overline{\omega}_b^{c\kappa_v} \int_0^{x'} W(k')(y)\, dy \\ &= \frac{1 - \overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W(k)(x'). \end{aligned}$$

□

Now we are ready to show a formula for the Walsh coefficients. For $n \in \mathbb{N}_0$, we define two symbols $k^n_>$ and $k^n_\leq$ as $k^n_> := \sum_{i=n+1}^{v} \kappa_i b^{a_i - 1}$ and $k^n_\leq := \sum_{i=1}^{\min(n,v)} \kappa_i b^{a_i - 1}$, respectively. Note that $k^n_\leq + k^n_> = k$.

Theorem 5.2.3. *Let $k \in \mathbb{N}_0$. Assume that $f \in C^\alpha[0,1]$ for a positive integer α. Then for an integer $0 \leq n \leq \min(\alpha, v)$ we have*

$$\widehat{f}(k) = (-1)^n \int_0^1 f^{(n)}(x) \overline{\mathrm{wal}_{k^n_>}(x)} W(k^n_\leq)(x)\, dx.$$

Proof. We prove the formula by induction on n. For $n=0$, the result holds by the definition of the Walsh coefficients. Hence assume now that $n>0$ and that the result holds for $n-1$. We have $\mathrm{wal}_{k^{n-1}_{>}}(x) = \mathrm{wal}_{k^{n}_{>}}(x)\mathrm{wal}_{\kappa_n b^{a_n-1}}(x)$ for all $x \in [0,1)$ and

$$\mathrm{wal}_{k^{n}_{>}}(x) = \mathrm{wal}_{k^{n}_{>}}(ib^{-a_n+1}) \qquad \text{for } x \in [ib^{-a_n+1}, (i+1)b^{-a_n+1})$$

for each integer $0 \le i < b^{a_n-1}$. Hence we have

$$\begin{aligned}
\widehat{f}(k) &= (-1)^{n-1} \int_0^1 f^{(n-1)}(x)\overline{\mathrm{wal}_{k^{n-1}_{>}}(x)}W(k^{n-1}_{\le})(x)\,dx \\
&= (-1)^{n-1} \sum_{i=0}^{b^{a_n-1}-1} \overline{\mathrm{wal}_{k^{n}_{>}}(ib^{-a_n+1})} \times \\
&\qquad \int_{ib^{-a_n+1}}^{(i+1)b^{-a_n+1}} f^{(n-1)}(x)\overline{\mathrm{wal}_{\kappa_n b^{a_n-1}}(x)}W(k^{n-1}_{\le})(x)\,dx \\
&= (-1)^{n-1} \sum_{i=0}^{b^{a_n-1}-1} \overline{\mathrm{wal}_{k^{n}_{>}}(ib^{-a_n+1})} \left(\left[f^{(n-1)}(x)W(k^{n}_{\le})(x) \right]_{ib^{-a_n+1}}^{(i+1)b^{-a_n+1}} \right. \\
&\qquad \left. - \int_{ib^{-a_n+1}}^{(i+1)b^{-a_n+1}} f^{(n)}(x)W(k^{n}_{\le})(x)\,dx \right) \\
&= (-1)^{n} \sum_{i=0}^{b^{a_n-1}-1} \overline{\mathrm{wal}_{k^{n}_{>}}(ib^{-a_n+1})} \int_{ib^{-a_n+1}}^{(i+1)b^{-a_n+1}} f^{(n)}(x)W(k^{n}_{\le})(x)\,dx \\
&= (-1)^{n} \int_0^1 f^{(n)}(x)\overline{\mathrm{wal}_{k^{n}_{>}}(x)}W(k^{n}_{\le})(x)\,dx,
\end{aligned}$$

where we use the induction assumption for $n-1$ for the first equality and $W(k^{n}_{\le})(ib^{-a_n+1}) = W(k^{n}_{\le})((i+1)b^{-a_n+1}) = 0$ by Lemma 5.2.2 for the fourth equality, respectively. This proves the result for n. □

Now we consider the s-variate case. For a function $f\colon [0,1)^s \to \mathbb{R}$, let $f^{(n_1,\dots,n_s)} := (\partial/\partial x_1)^{n_1} \cdots (\partial/\partial x_s)^{n_s} f$ be the $(n_1,\dots,n_s)$-th derivative of f. Considering coordinate-wise integration, we have the following.

Theorem 5.2.4. *Let $\boldsymbol{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$. Assume that $f\colon [0,1]^s \to \mathbb{R}$ has continuous mixed partial derivatives up to order α_j in each variable x_j. Let n_j be integers with $0 \le n_j \le \min(\alpha_j, v(k_j))$ for $1 \le j \le s$. Then we have*

$$\widehat{f}(\boldsymbol{k}) = (-1)^{n_1+\cdots+n_s} \int_{[0,1)^s} f^{(n_1,\dots,n_s)}(\boldsymbol{x}) \prod_{j=1}^{s} \overline{\mathrm{wal}_{k_j{}^{n_j}_{>}}(x_j)}W(k_j{}^{n_j}_{\le})(x_j)d\boldsymbol{x}.$$

5.3 The Walsh coefficients of smooth functions

Let $f \in C^\alpha[0,1]$ and $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. By Theorem 5.2.3 for $n = \min(\alpha, v)$ and Hölder's inequality, we have

$$\begin{aligned} |\widehat{f}(k)| &\leq \int_0^1 \left| f^{(\min(\alpha,v))}(x) \overline{\mathrm{wal}_{k_>^{\min(\alpha,v)}}(x)} W(k_\leq^{\min(\alpha,v)})(x) \right| dx \\ &\leq \|f^{(\min(\alpha,v))}\|_{L^p} \|W(k_\leq^\alpha)(\cdot)\|_{L^q}. \end{aligned} \tag{5.1}$$

Thus, it suffices to bound $\|W(k_\leq^\alpha)(\cdot)\|_{L^q}$ to bound $|\widehat{f}(k)|$. We give bounds on $\|W(k_\leq^\alpha)(\cdot)\|_{L^\infty}$ for the non-dyadic case, $\|W(k_\leq^\alpha)(\cdot)\|_{L^q}$ for the dyadic case and $|\widehat{f}(k)|$ in Sections 5.3.1, 5.3.2 and 5.3.3, respectively.

We introduce a function μ as follows. For $k \in \mathbb{N}_0$, we define

$$\mu(k) := \begin{cases} 0 & \text{for } k = 0, \\ a_1 + \cdots + a_v & \text{for } k \neq 0. \end{cases} \tag{5.2}$$

For $\boldsymbol{k} = (k_1, \cdots, k_s) \in \mathbb{N}_0^s$, we define $\mu(\boldsymbol{k}) := \sum_{j=1}^s \mu(k_j)$.

For subsequent analysis, we give the exact values of $I(k)$ and $W(k)(b^{-a_v})$ in the next lemma.

Lemma 5.3.1. *For $k \in \mathbb{N}_0$, we have the following.*

(i) $I(k) = \dfrac{b^{-\mu(k)}}{\prod_{i=1}^v (1 - \overline{\omega}_b^{\kappa_i})}$,

(ii) $W(k)(b^{-a_v}) = \dfrac{b^{-\mu(k)}}{\prod_{i=1}^{v-1} (1 - \overline{\omega}_b^{\kappa_i})}$.

(iii) *Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$ where $0 \leq c < b^{a_v}$ is an integer and $0 \leq x' < b^{-a_v}$ is a real number. Then we have*

$$W(k)(x) = (1 - \overline{\omega}_b^{c\kappa_v}) I(k) + \overline{\omega}_b^{c\kappa_v} W(k)(x').$$

Here, the empty products $\prod_{i=1}^0$ and $\prod_{i=1}^{-1}$ are defined to be 1.

Proof. By Lemma 5.2.2 we have

$$\begin{aligned} I(k) &= \sum_{i=0}^{b^{a_v}-1} \int_{ib^{-a_v}}^{(i+1)b^{-a_v}} W(k)(x)\, dx \\ &= \sum_{i=0}^{b^{a_v}-1} \int_0^{b^{-a_v}} \left(\frac{1 - \overline{\omega}_b^{i\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W(k)(b^{-a_v}) + \overline{\omega}_b^{i\kappa_v} W(k)(x) \right) dx \\ &= \frac{W(k)(b^{-a_v})}{1 - \overline{\omega}_b^{\kappa_v}} b^{-a_v} \sum_{i=0}^{b^{a_v}-1} (1 - \overline{\omega}_b^{i\kappa_v}) + \sum_{i=0}^{b^{a_v}-1} \overline{\omega}_b^{i\kappa_v} \int_0^{b^{-a_v}} W(k)(x)\, dx \end{aligned}$$

$$= \frac{W(k)(b^{-a_v})}{1-\overline{\omega}_b^{\kappa_v}}. \tag{5.3}$$

Furthermore, $W(k)(b^{-a_v})$ is computed as

$$\begin{aligned} W(k)(b^{-a_v}) &= \int_0^{b^{-a_v}} W(k')(x)\,dx \\ &= b^{-a_v} I(k'), \end{aligned} \tag{5.4}$$

where we use the fact that $W(k')(\cdot)$ is periodic with period b^{-a_v}, which follows from Lemma 5.2.2, in the last equality. Using equations (5.3) and (5.4) iteratively, we have (i) and (ii). Combining (5.3) and Lemma 5.2.2, we have (iii). □

In the following, we consider two cases in order to bound $\|W(k)(\cdot)\|_{L^\infty}$: the non-dyadic case and the dyadic case. We define two positive constants m_b and M_b as

$$m_b := \min_{c=1,2,\dots,b-1} |1-\overline{\omega}_b^c| = 2\sin(\pi/b),$$

$$M_b := \max_{c=1,2,\dots,b-1} |1-\overline{\omega}_b^c| = \begin{cases} 2 & \text{if } b \text{ is even,} \\ 2\sin((b+1)\pi/2b) & \text{if } b \text{ is odd.} \end{cases}$$

5.3.1 Non-dyadic case

The following lemmas are needed to bound $\sup_{x'\in[0,b^{-a_v}]} |W(k)(x')|$.

Lemma 5.3.2. *Let A, B be complex numbers and r be a positive real number. Then we have* $\sup_{x\in[0,r]} |Ax+B| = \max(|B|, |rA+B|)$.

Proof. We have

$$\sup_{x\in[0,r]} |Ax+B| = \sqrt{\sup_{x\in[0,r]} |Ax+B|^2} = \sqrt{\sup_{x\in[0,r]} (|A|^2x^2 + 2\mathrm{Re}(A\overline{B})x + |B|^2)}.$$

Since $|A|^2x^2 + 2\mathrm{Re}(A\overline{B})x + |B|^2$ is a convex function on $[0,r]$, its maximum value occurs at its endpoints. □

Lemma 5.3.3. *Let a and $1 \le \kappa \le b-1$ be positive integers. Then we have*

$$\sup_{c'=0,1,\dots,ab} \left| \sum_{i=0}^{c'-1} (1-\overline{\omega}_b^{i\kappa}) \right| \le ab.$$

Proof. Since $\sum_{i=0}^{ab-1} \overline{\omega}_b^{i\kappa} = 0$, we have

$$\sup_{c'=0,1,\dots,ab} \left| \sum_{i=0}^{c'-1} (1-\overline{\omega}_b^{i\kappa}) \right| = \sup_{c'=0,1,\dots,ab} \left| c' + \sum_{i=c'}^{ab-1} \overline{\omega}_b^{i\kappa} \right|$$

$$\leq \sup_{c'=0,1,\ldots,ab} \left(c' + \sum_{i=c'}^{ab-1} \left| \overline{\omega}_b^{i\kappa} \right| \right) = ab. \qquad \square$$

We now have an upper bound on $\sup_{x' \in [0, b^{-a_v}]} |W(k)(x')|$.

Lemma 5.3.4. *Let k be a positive integer. If $b > 2$, then we have*

$$\sup_{x' \in [0, b^{-a_v}]} |W(k)(x')| \leq \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{b}{b - M_b} \left(1 - \left(\frac{M_b}{b} \right)^v \right).$$

Proof. We prove the lemma by induction on v. If $v = 1$, we have

$$\begin{aligned} \sup_{x' \in [0, b^{-a_1}]} |W(k)(x')| &= \sup_{x' \in [0, b^{-a_1}]} \left| \int_0^{x'} W(0)(y)\, dy \right| \\ &= \sup_{x' \in [0, b^{-a_1}]} |x'| = b^{-a_1} = b^{-\mu(k)}. \end{aligned}$$

Hence the lemma holds for $v = 1$.

Thus assume now that $v > 1$ and that the result holds for $v - 1$. Let $x' \in [0, b^{-a_v}]$ be a real number and $x' = c' b^{-a_{v-1}} + x''$ where $0 \leq c' < b^{-a_v + a_{v-1}}$ is an integer and $0 \leq x'' < b^{-a_{v-1}}$ is a real number. Then by Lemma 5.3.1 (iii) we have

$$\begin{aligned} |W(k)(x')| &= \left| \int_0^{x'} W(k')(y)\, dy \right| \\ &= \left| \sum_{i=0}^{c'-1} \int_0^{b^{-a_{v-1}}} \left((1 - \overline{\omega}_b^{i\kappa_{v-1}}) I(k') + \overline{\omega}_b^{i\kappa_{v-1}} W(k')(y) \right) dy \right. \\ &\qquad \left. + \int_0^{x''} \left((1 - \overline{\omega}_b^{c'\kappa_{v-1}}) I(k') + \overline{\omega}_b^{c'\kappa_{v-1}} W(k')(y) \right) dy \right| \\ &\leq \left| b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1 - \overline{\omega}_b^{i\kappa_{v-1}}) I(k') + x''(1 - \overline{\omega}_b^{c'\kappa_{v-1}}) I(k') \right| \\ &\qquad + \left| \sum_{i=0}^{c'-1} \overline{\omega}_b^{i\kappa_{v-1}} \int_0^{b^{-a_{v-1}}} W(k')(y)\, dy + \overline{\omega}_b^{c'\kappa_{v-1}} \int_0^{x''} W(k')(y)\, dy \right|. \end{aligned} \tag{5.5}$$

We evaluate the supremum of the first term of (5.5). Note that the first term of (5.5) is equal to $|b^{-a_{v-1}} \sum_{i=0}^{c'} (1 - \overline{\omega}_b^{i\kappa_{v-1}}) I(k')|$ if $x'' = b^{-a_{v-1}}$. By Lemmas 5.3.2, 5.3.3 and 5.3.1 (i), we have

$$\sup_{\substack{0 \leq c' < b^{-a_v + a_{v-1}} \\ x'' \in [0, b^{-a_{v-1}}]}} \left| b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1 - \overline{\omega}_b^{i\kappa_{v-1}}) I(k') + x''(1 - \overline{\omega}_b^{c'\kappa_{v-1}}) I(k') \right|$$

$$
\begin{aligned}
&= \sup_{0\le c'\le b^{-a_v+a_{v-1}}} \left| b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1-\overline{\omega}_b^{i\kappa_{v-1}}) I(k') \right| \\
&\le b^{-a_{v-1}} \frac{b^{-\mu(k')}}{m_b^{v-1}} b^{-a_v+a_{v-1}} \\
&= \frac{b^{-\mu(k)}}{m_b^{v-1}}.
\end{aligned}
$$

We move on to the evaluation of the second term of (5.5). We have

$$
\begin{aligned}
&\sup_{c',x''} \left| \sum_{i=0}^{c'-1} \overline{\omega}_b^{i\kappa_{v-1}} \int_0^{b^{-a_{v-1}}} W(k')(y)\,dy + \overline{\omega}_b^{c'\kappa_{v-1}} \int_0^{x''} W(k')(y)\,dy \right| \\
&= \sup_{c',x''} \left| \sum_{i=0}^{c'-1} \overline{\omega}_b^{i\kappa_{v-1}} \int_{x''}^{b^{-a_{v-1}}} W(k')(y)\,dy + \sum_{i=0}^{c'} \overline{\omega}_b^{i\kappa_{v-1}} \int_0^{x''} W(k')(y)\,dy \right| \\
&= \sup_{c',x''} \left| \frac{1-\overline{\omega}_b^{c'\kappa_{v-1}}}{1-\overline{\omega}_b^{\kappa_{v-1}}} \int_{x''}^{b^{-a_{v-1}}} W(k')(y)\,dy + \frac{1-\overline{\omega}_b^{(c'+1)\kappa_{v-1}}}{1-\overline{\omega}_b^{\kappa_{v-1}}} \int_0^{x''} W(k')(y)\,dy \right| \\
&\le \sup_{x''} \left| \frac{M_b}{m_b}(b^{-a_{v-1}} - x'') + \frac{M_b}{m_b} x'' \right| \cdot \sup_{y\in[0,b^{-a_{v-1}}]} |W(k')(y)| \\
&\le \frac{M_b}{m_b} b^{-a_{v-1}} \cdot \frac{b^{-\mu(k')}}{m_b^{v-2}} \frac{b}{b-M_b} \left(1 - \left(\frac{M_b}{b}\right)^{v-1}\right) \\
&\le \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{M_b}{b-M_b} \left(1 - \left(\frac{M_b}{b}\right)^{v-1}\right),
\end{aligned}
$$

where we use the induction assumption for $v-1$ in the forth inequality and $b \cdot b^{-a_{v-1}} \le b^{-a_v}$ in the last inequality.

By summing up the bounds obtained on each term of (5.5), we have

$$
\begin{aligned}
\sup_{x'\in[0,b^{-a_v}]} |W(k)(x')| &\le \frac{b^{-\mu(k)}}{m_b^{v-1}} + \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{M_b}{b-M_b} \left(1 - \left(\frac{M_b}{b}\right)^{v-1}\right) \\
&= \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{b}{b-M_b} \left(1 - \left(\frac{M_b}{b}\right)^{v}\right). \qquad \square
\end{aligned}
$$

Using the above lemma, we obtain an upper bound on $\|W(k)(\cdot)\|_{L^\infty}$.

Proposition 5.3.5. *Let $k \in \mathbb{N}_0$. If $b > 2$, we have*

$$
\|W(k)(\cdot)\|_{L^\infty} \le \frac{b^{-\mu(k)}}{m_b^v} \left(M_b + \frac{b m_b}{b-M_b} \left(1 - \left(\frac{M_b}{b}\right)^v\right)\right)^{\min(1,v)}.
$$

Proof. The case $k=0$ is obvious. We assume that $k>0$. Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$, where $0 \le c < b^{a_v}$ is an integer and $0 \le x' < b^{-a_v}$ is a real number. By Lemmas 5.3.1 and 5.3.4, we have

$$\begin{aligned}
|W(k)(x)| &= |(1-\overline{\omega}_b^{c\kappa_v})I(k) + \overline{\omega}_b^{c\kappa_v} W(k)(x')| \\
&\le M_b|I(k)| + \sup_{x' \in [0, b^{-a_v}]} |W(k)(x')| \\
&\le \frac{b^{-\mu(k)}}{m_b^v} \left(M_b + \frac{b m_b}{b - M_b} \left(1 - \left(\frac{M_b}{b} \right)^v \right) \right)^{\min(1,v)},
\end{aligned}$$

which proves the proposition. □

5.3.2 Dyadic case

In this subsection, we assume that $b = 2$. In the dyadic case, we can obtain the exact values of $\|W(k)(\cdot)\|_{L^1}$ and $\|W(k)(\cdot)\|_{L^\infty}$. First we show properties of $W(k)(x)$ for the dyadic case.

Lemma 5.3.6. *Let $k \in \mathbb{N}_0$. Assume that $b = 2$ and $x_1, x_2 \in [0, 1)$. Then we have the following.*

(i) *Assume that $x_1 + x_2$ is a multiple of 2^{-a_v+1}. Then we have $W(k)(x_1) = W(k)(x_2)$.*

(ii) *Assume that $x_1 + x_2$ is a multiple of 2^{-a_v} and not a multiple of 2^{-a_v+1}. If $k \neq 0$, then we have $W(k)(x_1) + W(k)(x_2) = W(k)(2^{-a_v})$.*

(iii) *The function $W(k)(x)$ is nonnegative.*

Proof. We prove the lemma by induction on v. The results hold for $v = 0$ since $W(0)(x) = 1$ for all $x \in [0, 1)$. Hence assume now that $v > 0$ and that the results hold for $v - 1$.

First we assume that $x_1 + x_2$ is the multiple of 2^{-a_v+1}. Since $W(k)(\cdot)$ has a period 2^{-a_v+1} by Lemma 5.2.2, we can assume that $x_1, x_2 \in [0, 2^{-a_v+1}]$. Then we can assume that $x_1 \in [0, 2^{-a_v}]$ and that $x_2 = 2^{-a_v+1} - x_1$. Now we prove that $W(k)(x_1) = W(k)(x_2)$. By the induction assumption of (i) for $v - 1$, we have $W(k')(y) = W(k')(2^{-a_v+1} - y)$ for all $y \in [0, 2^{-a_v+1}]$. Hence we have

$$\begin{aligned}
W(k)(x_2) &= W(k)(2^{-a_v+1}) - \int_{x_2}^{2^{-a_v+1}} \overline{\mathrm{wal}_{2^{a_v-1}}(y)} W(k')(y)\, dy \\
&= 0 - \int_{x_2}^{2^{-a_v+1}} (-1) W(k')(2^{-a_v+1} - y)\, dy \\
&= \int_0^{x_1} W(k')(y)\, dy \\
&= W(k)(x_1),
\end{aligned}$$

which proves (i) for v.

Second we assume that $x_1 + x_2$ is a multiple of 2^{-a_v} and not a multiple of 2^{-a_v+1}. Similar to the first case, we can assume that $x_1, x_2 \in [0, 2^{-a_v}]$ and that $x_2 = 2^{-a_v} - x_1$. By the induction assumption of (i) for $v-1$, we have $W(k')(y) = W(k')(2^{-a_v} - y)$ for all $y \in [0, 2^{-a_v}]$. Hence we have

$$\begin{aligned}
W(k)(x_1) + W(k)(x_2) &= \int_0^{x_1} W(k')(y)\,dy + \int_0^{x_2} W(k')(y)\,dy \\
&= \int_0^{x_1} W(k')(y)\,dy + \int_0^{x_2} W(k')(2^{-a_v} - y)\,dy \\
&= \int_0^{x_1} W(k')(y)\,dy + \int_{2^{-a_v}-x_2}^{2^{-a_v}} W(k')(y)\,dy \\
&= \int_0^{2^{-a_v}} W(k')(y)\,dy \\
&= W(k)(2^{-a_v}),
\end{aligned}$$

which proves (ii) for v.

Finally we prove that $W(k)(x)$ is nonnegative. By the induction assumption of (iii) for $v-1$, $W(k')(x)$ is nonnegative. For $x \in [0, 2^{-a_v}]$, we have $W(k)(x) = \int_0^x W(k')(y)dy$, and thus $W(k)(x)$ is nonnegative for $x \in [0, 2^{-a_v}]$. Hence by (i) for v and Lemma 5.2.2, $W(k)(x)$ is nonnegative for $x \in [0, 1)$. □

Now we are ready to consider $\|W(k)(\cdot)\|_{L^q}$ for $1 \le q \le \infty$.

First we consider $\|W(k)(\cdot)\|_{L^1}$. By Lemmas 5.3.1 (i) and 5.3.6 (iii), we have

$$\|W(k)(\cdot)\|_{L^1} = \int_0^1 |W(k)(x)|\,dx = \int_0^1 W(k)(x)\,dx = 2^{-\mu(k)-v}.$$

Second we consider $\|W(k)(\cdot)\|_{L^\infty}$. If $k = 0$, we have $\|W(k)(\cdot)\|_{L^\infty} = 1$. We assume that $k > 0$. Considering the symmetry and the non-negativity of $W(k)(x)$ given by Lemma 5.3.6, we have

$$\begin{aligned}
\|W(k)(\cdot)\|_{L^\infty} &= \sup_{x \in [0, 2^{-a_v}]} |W(k)(x)|\,dx \\
&= \sup_{x \in [0, 2^{-a_v}]} \left| \int_0^x W(k')(y)\,dy \right| \\
&= \int_0^{2^{-a_v}} W(k')(y)\,dy \\
&= W(k)(2^{-a_v}) = 2^{-\mu(k)-v+1}.
\end{aligned}$$

Thus we have $\|W(k)(\cdot)\|_{L^\infty} \le 2^{-\mu(k)-v+\min(1,v)}$ for all $k \in \mathbb{N}_0$.

Finally we consider $\|W(k)(\cdot)\|_{L^q}$. By Hölder's inequality, we have

$$\begin{aligned}
\|W(k)(\cdot)\|_{L^q} &= \left(\int_{[0,1)^s} |W(k)(x)| \cdot |W(k)(x)|^{q-1}\,dx \right)^{1/q} \\
&\le (\|W(k)(\cdot)\|_{L^1} \|W(k)(\cdot)\|_{L^\infty}^{q-1})^{1/q}
\end{aligned}$$

$$\leq 2^{-\mu(k)-v+(1-1/q)\min(1,v)}.$$

We have shown the following proposition.

Proposition 5.3.7. *Let $b = 2$. For $k \in \mathbb{N}_0$ and $1 \leq q \leq \infty$, we have*

$$\|W(k)(\cdot)\|_{L^q} \leq 2^{-\mu(k)-v+(1-1/q)\min(1,v)},$$

and the equality holds if $q = 1$ or $q = \infty$.

5.3.3 Bounds on the Walsh coefficients of smooth functions

For a positive integer α and $k \in \mathbb{N}_0$, we define

$$\mu_\alpha(k) := \mu(k^\alpha_\leq) = \begin{cases} 0 & \text{for } k = 0, \\ a_1 + \cdots + a_v & \text{for } 1 \leq v \leq \alpha, \\ a_1 + \cdots + a_\alpha & \text{for } v \geq \alpha, \end{cases} \tag{5.6}$$

as in [10]. By (5.1), Proposition 5.3.5 and Proposition 5.3.7, we obtain the following bound on the Walsh coefficients of smooth functions.

Theorem 5.3.8. *Let $f \in C^\alpha[0,1]$ and $k \in \mathbb{N}_0$. If $b > 2$, we have*

$$|\widehat{f}(k)| \leq \|f^{(\min(\alpha,v))}\|_{L^1} \frac{b^{-\mu_\alpha(k)}}{m_b^{\min(\alpha,v)}} \times \left(M_b + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^{\min(\alpha,v)}\right)\right)^{\min(1,v)}.$$

If $b = 2$, for $1 \leq p \leq \infty$ we have

$$|\widehat{f}(k)| \leq \|f^{(\min(\alpha,v))}\|_{L^p} \cdot 2^{-\mu_\alpha(k)-\min(\alpha,v)+\min(1,v)/p}.$$

The s-variate case follows in the same way as the univariate case.

Theorem 5.3.9. *Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. Assume that $f\colon [0,1]^s \to \mathbb{R}$ has continuous mixed partial derivatives up to order α_j in each variable x_j. Let $n_j := \min(\alpha_j, v(k_j))$ for $1 \leq j \leq s$. Then, if $b > 2$, we have*

$$|\widehat{f}(\boldsymbol{k})| \leq \|f^{(n_1,\ldots,n_s)}\|_{L^1} \prod_{j=1}^{s} \frac{b^{-\mu_{\alpha_j}(k_j)}}{m_b^{n_j}} \times \left(M_b + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^{n_j}\right)\right)^{\min(1,v(k_j))}.$$

If $b = 2$, for $1 \leq p \leq \infty$ we have

$$|\widehat{f}(\boldsymbol{k})| \leq \|f^{(n_1,\ldots,n_s)}\|_{L^p} \times \prod_{j=1}^{s} 2^{-\mu_{\alpha_j}(k_j)-n_j+\min(1,v(k_j))/p}.$$

As a corollary, we give a sufficient condition for a infinitely smooth function that its Walsh coefficients decay with order $O(b^{-\mu(\boldsymbol{k})})$.

Corollary 5.3.10. *Let $f \in C^\infty[0,1]^s$ and $r_j > 0$ be positive real numbers for $1 \leq j \leq s$. Assume that there exists a positive real number D such that*

$$\|f^{(n_1,\dots,n_s)}\|_{L^1} \leq D \prod_{j=1}^{s} r_j^{n_j}$$

holds for all $n_1, \dots, n_s \in \mathbb{N}_0$. Then for all $\boldsymbol{k} \in \mathbb{N}_0^s$ we have

$$|\widehat{f}(\boldsymbol{k})| \leq D b^{-\mu(\boldsymbol{k})} \prod_{j=1}^{s} (r_j m_b^{-1})^{v(k_j)} C_b^{\min(1, v(k_j))},$$

where C_b is a constant defined as

$$C_b = \begin{cases} 2 & \text{for } b = 2, \\ M_b + \dfrac{b m_b}{b - M_b} & \text{for } b \neq 2. \end{cases}$$

In particular, if $r_j = m_b$ holds for all $1 \leq j \leq s$, then $|\widehat{f}(\boldsymbol{k})| \in O(b^{-\mu(\boldsymbol{k})})$ holds.

5.4 Another formula for the Walsh coefficients

In this section, we give another formula for the Walsh coefficients. For this purpose, we introduce functions $W_j(k)(\cdot)$ and their integration values $I_j(k)$ for $j, k \in \mathbb{N}_0$.

Definition 5.4.1. *For $j, k \in \mathbb{N}_0$, we define functions $W_j(k)(\cdot)\colon [0,1] \to \mathbb{C}$ and complex numbers $I_j(k)$ recursively as*

$$\begin{aligned} W_0(k)(x) &:= W(k)(x), \\ I_j(k) &:= \int_0^1 W_j(k)(x), \\ W_{j+1}(k)(x) &:= \int_0^x (W_j(k)(x) - I_j(k))\, dy. \end{aligned}$$

We note that $W_j(k)(0) = W_j(k)(1) = 0$ for all $j, k \in \mathbb{N}_0$ with $(j,k) \neq (0,0)$.

We now establish another formula for the Walsh coefficients of smooth functions.

Theorem 5.4.2. *Let $k, r \in \mathbb{N}_0$ and $f \in C^{v+r}[0,1]$. Then we have*

$$\begin{aligned} \widehat{f}(k) = &\sum_{i=0}^{r} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x)\, dx \\ &+ (-1)^{v+r} \int_0^1 f^{(v+r)}(x)(W_r(k)(x) - I_r(k))\, dx. \end{aligned}$$

Proof. We prove the theorem by induction on r. We have already proved the case $r = 0$ in Theorem 5.2.3. Thus assume now that $r \geq 1$ and that the result holds for $r - 1$. By the induction assumption for $v - 1$, we have

$$\begin{aligned}
\widehat{f}(k) &= \sum_{i=0}^{r-1} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x)\, dx \\
&\quad + (-1)^{v+r-1} \int_0^1 f^{(v+r-1)}(x)(W_{r-1}(k)(x) - I_{r-1}(k))\, dx \\
&= \sum_{i=0}^{r-1} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x)\, dx \\
&\quad + (-1)^{v+r-1} \left([f^{(v+r-1)}(x) W_r(k)(x)]_0^1 - \int_0^1 f^{(v+r)}(x) W_r(k)(x)\, dx \right) \\
&= \sum_{i=0}^{r} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x)\, dx \\
&\quad + (-1)^{v+r} \int_0^1 f^{(v+r)}(x)(W_r(k)(x) - I_r(k))\, dx,
\end{aligned}$$

where we use $W_r(k)(0) = W_r(k)(1) = 0$ in the third equality. This proves the result for r. $\square$

5.5 The Walsh coefficients of Bernoulli polynomials

In this section, we analyze the decay of the Walsh coefficients of Bernoulli polynomials.

For $r \geq 0$, we denote $B_r(x)$ the Bernoulli polynomial of degree r and $b_r(x) = B_r(x)/r!$. For example, we have $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ and so on. Those polynomials have the following properties: For all $r \geq 1$ we have

$$b_r'(x) = b_{r-1}(x) \quad \text{and} \quad \int_0^1 b_r(x)\, dx = 0, \tag{5.7}$$

and for all $r \in \mathbb{N}_0$ we have

$$b_r(1 - x) = (-1)^r b_r(x), \tag{5.8}$$

see [1, Chapter 23]. We clearly have $b_0'(x) = 0$ and $\int_0^1 b_0(x) = 1$.

The Walsh coefficients of Bernoulli polynomials are given as follows. If $r < v$, then by Theorem 5.2.3 and (5.7) we have

$$\widehat{b_r}(k) = (-1)^v \int_0^1 b_r^{(v)}(x) W(k)(x)\, dx = 0.$$

If $r \geq v$, then by Theorem 5.4.2 and (5.7) we have

$$\begin{aligned}
\widehat{b_r}(k) &= \sum_{i=0}^{r-v} (-1)^{v+i} I_i(k) \int_0^1 b_r^{(v+i)}(x)\, dx \\
&\quad + (-1)^r \int_0^1 b_r{}^{(r)}(x)(W_{r-v}(k)(x) - I_{r-v}(k))\, dx \\
&= (-1)^r I_{r-v}(k).
\end{aligned}$$

Now we proved:

Lemma 5.5.1. *For positive integers k and r, we have*

$$\widehat{b_r}(k) = \begin{cases} 0 & \text{if } r < v, \\ (-1)^r I_{r-v}(k) & \text{if } r \geq v. \end{cases}$$

In the following, we give upper bounds on $\|W_j(k)(\cdot) - I_j(k)\|_{L^\infty}$, $|I_j(k)|$ and $\|W_j(k)(\cdot)\|_{L^\infty}$, which give bounds on the Walsh coefficients of Bernoulli polynomials and smooth functions. First we compute $W_j(k)(\cdot)$ and $I_j(k)$.

Lemma 5.5.2. *Let $k, j \in \mathbb{N}_0$. Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$ with $c \in \mathbb{N}_0$ and $x' \in [0, b^{-a_v})$. Then we have*

(i) $W_j(k)(x) = \dfrac{1 - \overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_j(k)(x')$,

(ii) $I_j(k) = \dfrac{W_j(k)(b^{-a_v})}{1 - \overline{\omega}_b^{\kappa_v}}$.

Proof. We prove the lemma by induction on j. We have already proved the case $j = 0$ in Lemmas 5.2.2 and 5.3.1. Thus assume now that $j \geq 1$ and that the result holds for $j - 1$. Then we have

$$\begin{aligned}
W_j(k)(x) &= \int_0^x (W_{j-1}(k)(y) - I_{j-1}(k))\, dy \\
&= \sum_{i=0}^{c-1} \int_0^{b^{-a_v}} \left(\frac{-\overline{\omega}_b^{i\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_{j-1}(k)(b^{-a_v}) + \overline{\omega}_b^{i\kappa_v} W_{j-1}(k)(y) \right) dy \\
&\quad + \int_0^{x'} \left(\frac{-\overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_{j-1}(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_{j-1}(k)(y) \right) dy \\
&= \sum_{i=0}^{c-1} \overline{\omega}_b^{i\kappa_v} W_j(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_j(k)(x') \\
&= \frac{1 - \overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_j(k)(x'),
\end{aligned}$$

where we use the induction assumption for $j-1$ in the second and third equalities and the definition of $W_j(k)(\cdot)$ in the third equality. This proves (i) for j.

Now we compute $I_j(k)$. Replacing $W(k)(x)$ to $W_j(k)(x)$ in (5.3), we have $I_j(k) = W_j(k)(b^{-a_v})/(1 - \overline{\omega}_b^{\kappa_v})$, which proves (ii) for j. □

The following lemmas give bounds on $\|W_j(k)(\cdot) - I_j(k)\|_{L^\infty}$, $|I_j(k)|$ and $\|W_j(k)(\cdot)\|_{L^\infty}$ for the non-dyadic case.

Lemma 5.5.3. *Let $j \in \mathbb{N}_0$. If $b \neq 2$, for any positive integer k we have*

$$\|W_j(k)(\cdot) - I_j(k)\|_{L^\infty} \le \frac{b^{-\mu(k)-ja_v}}{m_b^{v+j}} \left(1 + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right).$$

Proof. Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$ with $c \in \mathbb{N}_0$ and $x' \in [0, b^{-a_v})$. First assume that $j = 0$. Then it follows from Lemmas 5.3.1 and 5.3.4 that

$$\begin{aligned}
|W_0(k)(x) - I_0(k)| &= |-\overline{\omega}_b^{c\kappa_v} I(k) + \overline{\omega}_b^{c\kappa_v} W(k)(x')| \\
&\le |I(k)| + \sup_{x' \in [0, b^{-a_v}]} |W(k)(x')| \\
&\le \frac{b^{-\mu(k)}}{m_b^v}\left(1 + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right),
\end{aligned}$$

which proves the case $j = 0$.

Now we assume that $j > 0$. Then it follows from Lemma 5.5.2 that

$$\begin{aligned}
|W_j(k)(x) - I_j(k)| &= \left|\frac{1 - \overline{\omega}_b^{c\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_j(k)(x') - \frac{W_j(k)(b^{-a_v})}{1 - \overline{\omega}_b^{\kappa_v}}\right| \\
&= \left|\frac{-1}{1 - \overline{\omega}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + W_j(k)(x')\right| \\
&= \left|\frac{-1}{1 - \overline{\omega}_b^{\kappa_v}} (W_j(k)(b^{-a_v}) - W_j(k)(x')) - \frac{\overline{\omega}_b^{\kappa_v}}{1 - \overline{\omega}_b^{\kappa_v}} W_j(k)(x')\right| \\
&\le \frac{1}{m_b}\left|\int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k))\, dy \right. \\
&\qquad \left. + \overline{\omega}_b^{\kappa_v} \int_0^{x'} (W_{j-1}(k)(y) - I_{j-1}(k))\, dy\right| \\
&\le \frac{1}{m_b}(b^{-a_v} - x') \sup_{y \in [0, b^{-a_v}]} |W_{j-1}(k)(y) - I_{j-1}(k)| \\
&\quad + \frac{1}{m_b} x' \sup_{y \in [0, b^{-a_v}]} |W_{j-1}(k)(y) - I_{j-1}(k)| \\
&\le \frac{b^{-a_v}}{m_b} \|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty}.
\end{aligned}$$

Using the case $j = 0$ and this evaluation inductively, we have the case $j > 0$. □

Lemma 5.5.4. *Let j and k be positive integers. If $b > 2$, then we have*

$$|I_j(k)| \le \frac{b^{-\mu(k)-ja_v}}{m_b^{v+j}} \left(1 + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right).$$

Proof. By Lemmas 5.5.2 and 5.5.3, we have

$$\begin{aligned}
|I_j(k)| &= |W_j(k)(b^{-a_v})/(1-\overline{\omega}_b^{\kappa_v})| \\
&\le \frac{1}{m_b}\int_0^{b^{-a_v}} |W_{j-1}(k)(y) - I_{j-1}(k)|\,dy \\
&\le \frac{b^{-a_v}}{m_b}\|W_{j-1}(k)(y) - I_{j-1}(k)\|_{L^\infty} \\
&\le \frac{b^{-\mu(k)-ja_v}}{m_b^{v+j}}\left(1+\frac{bm_b}{b-M_b}\left(1-\left(\frac{M_b}{b}\right)^v\right)\right). \qquad \square
\end{aligned}$$

Lemma 5.5.5. *Let j and k be positive integers. If $b > 2$, then we have*

$$\|W_j(k)(\cdot)\|_{L^\infty} \le \frac{b^{-\mu(k)-ja_v}}{m_b^{v+j}} M_b\left(1+\frac{bm_b}{b-M_b}\left(1-\left(\frac{M_b}{b}\right)^v\right)\right).$$

Proof. Let $x \in [0,1)$ and $x = cb^{-a_v} + x'$, where $0 \le c < b^{a_v}$ is an integer and $0 \le x' < b^{-a_v}$ is a real number. Then we have

$$\begin{aligned}
W_j(k)(x) &= \frac{1-\overline{\omega}_b^{c\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + \overline{\omega}_b^{c\kappa_v} W_j(k)(x') \\
&= \frac{1-\overline{\omega}_b^{c\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}}(W_j(k)(b^{-a_v}) - W_j(k)(x')) + \frac{1-\overline{\omega}_b^{(c+1)\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} W_j(k)(x') \\
&= \frac{1-\overline{\omega}_b^{c\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} \int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k))\,dy \\
&\quad + \frac{1-\overline{\omega}_b^{(c+1)\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} \int_0^{x'} (W_{j-1}(k)(y) - I_{j-1}(k))\,dy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|W_j(k)(x)| &\le \left|\frac{1-\overline{\omega}_b^{c\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} \int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k))\,dy\right| \\
&\quad + \left|\frac{1-\overline{\omega}_b^{(c+1)\kappa_v}}{1-\overline{\omega}_b^{\kappa_v}} \int_0^{x'} (W_{j-1}(k)(y) - I_{j-1}(k))\,dy\right| \\
&= \frac{M_b}{m_b} b^{-a_v}\|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty} \\
&\le \frac{b^{-\mu(k)-ja_v}}{m_b^{v+j}} M_b\left(1+\frac{bm_b}{b-M_b}\left(1-\left(\frac{M_b}{b}\right)^v\right)\right). \qquad \square
\end{aligned}$$

We also consider the dyadic case.

Lemma 5.5.6. *Let k be a positive integer and $j \in \mathbb{N}_0$. If $b = 2$, then we have the following.*

(i) $\|W_j(k)(x) - I_j(k)\|_{L^\infty} \le 2^{-j(a_v+1)-\mu(k)-v}$,

(ii) $|I_j(k)| \le 2^{-j(a_v+1)-\mu(k)-v}$,

(iii) $\|W_j(k)(\cdot)\|_{L^\infty} \le 2^{-j(a_v+1)-\mu(k)-v+1}$.

(iv) *If j is odd, then $I_j(k) = 0$.*

Proof. Lemma 5.3.1 and Proposition 5.3.7 imply (ii) and (iii) for $j = 0$.

Since $W_0(k)(x)$ and $I_0(k)$ are nonnegative, we have

$$\begin{aligned}\|W_0(k)(x) - I_0(k)\|_{L^\infty} &\le \max\left(|\|W_0(k)(\cdot)\|_{L^\infty} - I_0(k)|, |0 - I_0(k)|\right)\\ &\le 2^{-\mu(k)-v},\end{aligned}$$

and thus (i) for $j = 0$ holds.

For the proof for the case $j > 0$, we note that parts of proofs of Lemmas 5.5.3, 5.5.4 and 5.5.5 are valid even in the dyadic case: For $b = 2$ we have

$$\begin{aligned}|W_j(k)(x) - I_j(k)| &\le \frac{b^{-a_v}}{m_b}\|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty},\\ |I_j(k)| &\le \frac{b^{-a_v}}{m_b}\|W_{j-1}(k)(y) - I_{j-1}(k)\|_{L^\infty},\\ |W_j(k)(x)| &\le \frac{M_b}{m_b}b^{-a_v}\|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty}.\end{aligned}$$

Combining these inequalities and the case $j = 0$, we have (i), (ii) and (iii) for $j > 0$.

Now we assume that j is odd and prove $I_j(k) = 0$. By Lemma 5.5.1, we have

$$\widehat{b_{v+j}}(k) = (-1)^{v+j} I_j(k).$$

Hence it suffices to show $\widehat{b_{v+j}}(k) = 0$. Since j is odd, by (5.8) we have $b_{v+j}(x) = (-1)^{v+1} b_{v+j}(1-x)$. Furthermore, $\mathrm{wal}_k(x) = (-1)^v \mathrm{wal}_k(1-x)$ holds for all but finitely many $x \in [0, 1)$, since we have $\mathrm{wal}_{2^{a_i-1}}(x) = -\mathrm{wal}_{2^{a_i-1}}(1-x)$ for $x \in [0,1)\backslash\{l/2^{a_i} \mid 0 \le l < 2^{a_i}\}$ and $\mathrm{wal}_k(x) = \prod_{i=1}^v \mathrm{wal}_{2^{a_i-1}}(x)$. Hence we have

$$\begin{aligned}\widehat{b_{v+j}}(k) &= \int_0^{\frac{1}{2}} b_{v+j}(x)\mathrm{wal}_k(x)\,dx + \int_{\frac{1}{2}}^1 b_{v+j}(x)\mathrm{wal}_k(x)\,dx\\ &= \int_0^{\frac{1}{2}} b_{v+j}(x)\mathrm{wal}_k(x)\,dx + \int_0^{\frac{1}{2}} b_{v+j}(1-x)\mathrm{wal}_k(1-x)\,dx\\ &= \int_0^{\frac{1}{2}} b_{v+j}(x)\mathrm{wal}_k(x)\,dx - \int_0^{\frac{1}{2}} b_{v+j}(x)\mathrm{wal}_k(x)\,dx\\ &= 0.\end{aligned}$$

□

Now we are ready to analyze the decay of the Walsh coefficients of Bernoulli polynomials. For a positive integer α and $k \in \mathbb{N}_0$, we define

$$\mu_{\alpha,\mathrm{per}}(k) = \begin{cases} 0 & \text{for } k = 0,\\ a_1 + \cdots + a_v + (\alpha - v)a_v & \text{for } 1 \le v \le \alpha,\\ a_1 + \cdots + a_\alpha & \text{for } v \ge \alpha,\end{cases} \tag{5.9}$$

as in [10]. By Lemmas 5.5.1, 5.5.4 and 5.5.6, we have the following bound on the Walsh coefficients of Bernoulli polynomials.

Theorem 5.5.7. *For positive integers k and r, we have*

$$|\widehat{b_r}(k)| \begin{cases} = 0 & \text{if } r < v, \\ = 0 & \text{if } r \geq v,\ r - v \text{ is odd and } b = 2, \\ \leq 2^{-\mu_{r,\mathrm{per}}(k) - r} & \text{if } r \geq v,\ r - v \text{ is even and } b = 2, \\ \leq \dfrac{b^{-\mu_{r,\mathrm{per}}(k)}}{m_b^r} c_{b,v} & \text{if } r \geq v \text{ and } b \neq 2, \end{cases}$$

where $c_{b,v} := 1 + \dfrac{bm_b}{b - M_b}\left(1 - \left(\dfrac{M_b}{b}\right)^v\right)$.

5.6 Walsh coefficients of functions in Sobolev spaces

In this section, we consider functions in the Sobolev space

$$\mathcal{H}_\alpha := \{f\colon [0,1] \to \mathbb{R} \mid f^{(i)}\colon \text{abs. conti. for } i = 0, \cdots, \alpha - 1,\ f^{(\alpha)} \in L^2[0,1]\}$$

for which $\alpha \geq 1$ as in [10]. The inner product is given by

$$\langle f, g \rangle_\alpha = \sum_{i=0}^{\alpha-1} \int_0^1 f^{(i)}(x)\, dx \int_0^1 g^{(i)}(x)\, dx + \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x)\, dx.$$

and the corresponding norm in $\mathcal{H}_\alpha$ is given by $\|f\|_{\mathrm{Sob},\alpha} := \sqrt{\langle f, f \rangle_\alpha}$. The space $\mathcal{H}_\alpha$ becomes a reproducing kernel Hilbert space (see [2] for general information of reproducing kernel Hilbert space). The reproducing kernel for this space is given by

$$\mathcal{K}(x, y) = \sum_{i=0}^{\alpha} b_i(x) b_i(y) - (-1)^\alpha \widetilde{b}_{2\alpha}(x - y),$$

where

$$\widetilde{b}_\alpha(x - y) := \begin{cases} b_\alpha(|x - y|), & \alpha : \mathit{even}, \\ (-1)^{1_{x<y}} b_\alpha(|x - y|), & \alpha : \mathit{odd}, \end{cases}$$

where we define $1_{x<y}$ is 1 for $x < y$ and 0 otherwise, see [6, Lemma 2.1]. We have

$$\begin{aligned} f(y) &= \langle f, \mathcal{K}(\cdot, y) \rangle_\alpha \\ &= \sum_{i=0}^{\alpha} \int_0^1 f^{(i)}(x)\, dx\, b_i(y) - (-1)^\alpha \int_0^1 f^{(\alpha)}(x) \widetilde{b}_\alpha(x - y)\, dx, \end{aligned} \tag{5.10}$$

which implies that

$$\widehat{f}(k) = \sum_{i=0}^{\alpha} \int_0^1 f^{(i)}(x)\,dx\,\widehat{b_i}(k) - (-1)^{\alpha} \int_0^1 f^{(\alpha)}(x) \int_0^1 \widetilde{b}_{\alpha}(x-y)\overline{\mathrm{wal}_k(y)}\,dy\,dx. \tag{5.11}$$

However, we have already proved two formulas for the Walsh coefficients: For $f \in C^{\alpha}[0,1]$, in the case $\alpha \geq v$ we have Theorem 5.4.2 for $r = \alpha - v$, which is written as

$$\begin{aligned}\widehat{f}(k) = &\sum_{i=v}^{\alpha}(-1)^i I_{i-v}(k) \int_0^1 f^{(i)}(x)\,dx \\ &+ (-1)^{\alpha} \int_0^1 f^{(\alpha)}(x)(W_{\alpha-v}(k)(x) - I_{\alpha-v}(k))\,dx,\end{aligned} \tag{5.12}$$

and in the case $\alpha < v$ we have Theorem 5.2.3 for $n = \alpha$, which is written as

$$\widehat{f}(k) = (-1)^{\alpha} \int_0^1 f^{(\alpha)}(x)\overline{\mathrm{wal}_{k^{\alpha}_{>}}(x)} W(k^{\alpha}_{\leq})(x)\,dx. \tag{5.13}$$

In this section, we show that Formulas (5.12) and (5.13) are also valid for $f \in \mathcal{H}_{\alpha}$ and give an upper bound for the Walsh coefficients of functions in $\mathcal{H}_{\alpha}$.

5.6.1 Formula for the Walsh coefficients of functions in Sobolev spaces

First we consider the case $\alpha \geq v$. The following lemma is needed to show that (5.12) is also valid for $f \in \mathcal{H}_{\alpha}$.

Lemma 5.6.1. *Assume $\alpha \geq v$. Define functions $h_1, h_2 \colon [0,1] \to \mathbb{C}$ as*

$$\begin{aligned} h_1(x) &:= -\int_0^1 \widetilde{b}_{\alpha}(x-y)\overline{\mathrm{wal}_k(y)}\,dy, \\ h_2(x) &:= W_{\alpha-v}(k)(x) - I_{\alpha-v}(k). \end{aligned}$$

Then $h_1(x) = h_2(x)$ holds for all $x \in [0,1]$.

Proof. For $f \in C^{\alpha}[0,1]$ both formulas (5.11) and (5.12) hold. Furthermore, by Lemma 5.5.1, the first term of each formula is equal. Hence we have

$$\int_0^1 f^{(\alpha)}(x)h_1(x)\,dx = \int_0^1 f^{(\alpha)}(x)h_2(x)\,dx$$

for all $f \in C^{\alpha}[0,1]$. It is well known that if $h \colon [0,1] \to \mathbb{C}$ is continuous and $\int_0^1 g(x)h(x) = 0$ holds for all continuous functions $g \in C^0[0,1]$, then $h(x) = 0$ holds. Thus it suffices to show that h_1 and h_2 are continuous.

By definition, h_2 is continuous. Now we prove that h_1 is continuous. Fix $\epsilon > 0$. Since $b_{\alpha}(z)$ is uniformly continuous on $z \in [0,1]$, there exists δ_1

such that $|b_\alpha(z) - b_\alpha(z')| < \epsilon/2$ for all $z, z' \in [0,1]$ with $|z - z'| < \delta_1$. Let $\delta_2 = \min\left(4^{-1}\epsilon(\max_{z\in[0,1]} |b_\alpha(z)|)^{-1}, \delta_1\right)$. We fix $x \in [0,1]$ and prove $|h_1(x) - h_1(x')| \leq \epsilon$ for all $x' \in [0,1]$ with $|x - x'| < \delta_2$. Without loss of generality, we can assume that $x < x'$. Then we have

$$
\begin{aligned}
&\left|\int_0^1 \widetilde{b}_\alpha(x-y)\overline{\mathrm{wal}_k(y)}\,dy - \int_0^1 \widetilde{b}_\alpha(x'-y)\overline{\mathrm{wal}_k(y)}\,dy\right| \\
&\leq \left(\int_0^x + \int_x^{x'} + \int_{x'}^1\right) \left|\widetilde{b}_\alpha(x-y) - \widetilde{b}_\alpha(x'-y)\right| dy \\
&\leq x \max_{y\in[0,x]} |b_\alpha(x-y) - b_\alpha(x'-y)| + (x'-x) \max_{y\in[x,x']} (|b_\alpha(y-x)| + |b_\alpha(x'-y)|) \\
&\qquad + (1-x') \max_{y\in[x',1]} |b_\alpha(y-x) - b_\alpha(y-x')| \\
&< x\epsilon/2 + 2\delta_2 \max_{z\in[0,1]} |b_\alpha(z)| + (1-x')\epsilon/2 \\
&< \epsilon,
\end{aligned}
$$

which implies the continuity of h_1. □

The following result follows now from the above lemma, Lemma 5.5.1 and (5.11).

Proposition 5.6.2. *Assume $\alpha \geq v$. Then for $f \in \mathcal{H}_\alpha$ we have*

$$
\begin{aligned}
\widehat{f}(k) = &\sum_{i=v}^{\alpha} (-1)^i I_{i-v}(k) \int_0^1 f^{(i)}(x)\,dx \\
&+ (-1)^\alpha \int_0^1 f^{(\alpha)}(x)(W_{\alpha-v}(k)(x) - I_{\alpha-v}(k))\,dx.
\end{aligned}
$$

Now we treat the case $\alpha < v$. Note that $\overline{\mathrm{wal}_{k^\alpha_>}(x)}W(k^\alpha_\leq)(x)$ is continuous since $W(k^\alpha_\leq)(x)$ equals 0 on the set where $\mathrm{wal}_{k^\alpha_>}(x)$ is not continuous. In the same way as the case $\alpha \geq v$, we have the following.

Proposition 5.6.3. *Assume $\alpha < v$. Then we have*

$$
-\int_0^1 \widetilde{b}_\alpha(x-y)\overline{\mathrm{wal}_k(y)}\,dy = \overline{\mathrm{wal}_{k^\alpha_>}(x)}W(k^\alpha_\leq)(x).
$$

In particular, for $f \in \mathcal{H}_\alpha$ we have

$$
\widehat{f}(k) = (-1)^\alpha \int_0^1 f^{(\alpha)}(x)\overline{\mathrm{wal}_{k^\alpha_>}(x)}W(k^\alpha_\leq)(x)\,dx.
$$

5.6.2 Upper bound on the Walsh coefficients of functions in Sobolev spaces

In this subsection, we give a bound on the Walsh coefficients of functions in $\mathcal{H}_\alpha$.

By Propositions 5.6.2 and 5.6.3, for $f \in \mathcal{H}_\alpha$ we have

$$|\widehat{f}(k)| \leq \sum_{i=v}^{\alpha} |I_{i-v}(k)| \left| \int_0^1 f^{(i)}(x)\, dx \right| + N_\alpha \int_0^1 |f^{(\alpha)}(x)|\, dx,$$

where $N_\alpha = \|W_{\alpha-v}(k)(\cdot) - I_{\alpha-v}(k)\|_{L^\infty}$ if $\alpha \geq v$ and $N_\alpha = \|W(k^\alpha_\leq)(\cdot)\|_{L^\infty}$ otherwise. Thus, by Propositions 5.3.5 and 5.3.7 and Lemmas 5.5.3, 5.5.4 and 5.5.6, we have the following.

Theorem 5.6.4. *Let α and k be positive integers. Assume $f \in \mathcal{H}_\alpha$. If $b > 2$, we have*

$$\begin{aligned} |\widehat{f}(k)| \leq \sum_{i=v}^{\alpha} \left| \int_0^1 f^{(i)}(x)\, dx \right| \frac{b^{-\mu_{i,\mathrm{per}}(k)}}{m_b^i} \left(1 + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right) \\ + \int_0^1 |f^{(\alpha)}(x)|\, dx \frac{b^{-\mu_{\alpha,\mathrm{per}}(k)}}{m_b^\alpha} \left(M_b + \frac{bm_b}{b - M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right), \end{aligned}$$

and if $b = 2$, we have

$$|\widehat{f}(k)| \leq \sum_{\substack{v \leq i \leq \alpha \\ i = v \bmod 2}} \left| \int_0^1 f^{(i)}(x)\, dx \right| \frac{b^{-\mu_{i,\mathrm{per}}(k)}}{2^i} + \int_0^1 |f^{(\alpha)}(x)|\, dx \frac{b^{-\mu_{\alpha,\mathrm{per}}(k)}}{2^{\alpha-1}},$$

where for $v > \alpha$ the empty sum $\sum_{i=v}^{\alpha}$ is defined to be 0.

For an integer i with $v \leq i \leq \alpha$, $\mu_{i,\mathrm{per}}(k) \geq \mu_\alpha(k)$ holds for all $k \in \mathbb{N}_0$ by the definitions of $\mu_{i,\mathrm{per}}(k)$ and $\mu_\alpha(k)$. Thus, applying Hölder's inequality to Theorem 5.6.4, we obtain the following corollary.

Corollary 5.6.5. *Let α and k be positive integers. Then, for all $f \in \mathcal{H}_\alpha$, we have*

$$|\widehat{f}(k)| \leq b^{-\mu_\alpha(k)} C_{b,\alpha,q} \|f\|_{p,\alpha},$$

where $\|f\|_{p,\alpha} := \left(\sum_{i=0}^{\alpha} \left|\int_0^1 f^{(i)}(x)\, dx\right|^p + \int_0^1 |f^{(\alpha)}(x)|^p\, dx\right)^{1/p}$, $1/p + 1/q = 1$, *and*

$$C_{b,\alpha,q} := \left(\sum_{i=1}^{\alpha} \frac{1}{m_b^{iq}} \left(1 + \frac{bm_b}{b - M_b}\right)^q + \frac{1}{m_b^{\alpha q}} \left(M_b + \frac{bm_b}{b - M_b}\right)^q\right)^{1/q}$$

for $b > 2$ and $C_{2,\alpha,q} := (\sum_{i=1}^{\alpha} 2^{-iq} + 2^{-(\alpha-1)q})^{1/q}$ for $b = 2$.

Remark 5.6.6. *This corollary can be generalized to tensor product spaces, for which the reproducing kernel is just the product of the one-dimensional kernel, as [14, Section 14.6].*

5.7 The Walsh coefficients of smooth periodic functions

As [10], we consider a subset of the previous reproducing kernel Hilbert space, namely, let $\mathcal{H}_{\alpha,\mathrm{per}}$ be the space of all functions $f \in \mathcal{H}_\alpha$ which satisfy the condition $\int_0^1 f^{(i)}(x)\,dx = 0$ for $0 \le i < \alpha$. This space also has a reproducing kernel, which is given by

$$\mathcal{K}_{\alpha,\mathrm{per}}(x,y) = b_\alpha(x)b_\alpha(y) + (-1)^{\alpha+1}\widetilde{b}_{2\alpha}(x-y),$$

and the inner product is given by

$$\langle f, g\rangle_{\alpha,\mathrm{per}} = \int_0^1 f^{(\alpha)}(x)g^{(\alpha)}(x)\,dx,$$

see [56, (10.2.4)]. We also have the representation

$$\begin{aligned} f(y) &= \langle f, \mathcal{K}_{\alpha,\mathrm{per}}(\cdot, y)\rangle_{\alpha,\mathrm{per}} \\ &= \int_0^1 f^{(\alpha)}(x)\,dx\, b_\alpha(y) + (-1)^{\alpha+1}\int_0^1 f^{(\alpha)}(x)\widetilde{b}_\alpha(x-y)\,dx \end{aligned}$$

and

$$\widehat{f}(k) = \int_0^1 f^{(\alpha)}(x)\,dx\,\widehat{b_\alpha}(k) + (-1)^{\alpha+1}\int_0^1 f^{(\alpha)}(x)\int_0^1 \widetilde{b}_\alpha(x-y)\overline{\mathrm{wal}_k(y)}\,dx\,dy.$$

By the condition $\int_0^1 f^{(i)}(x)\,dx = 0$ for $0 \le i < \alpha$ and Propositions 5.6.2 and 5.6.3, we have the following.

Lemma 5.7.1. *Let α and k be positive integers. Assume $f \in \mathcal{H}_{\alpha,\mathrm{per}}$. If $\alpha \ge v$, then we have*

$$\widehat{f}(k) = (-1)^\alpha \int_0^1 f^{(\alpha)}(x)W_{\alpha-v}(k)(x)\,dx.$$

If $\alpha < v$, then we have

$$\widehat{f}(k) = (-1)^\alpha \int_0^1 f^{(\alpha)}(x)\overline{\mathrm{wal}_{k^\alpha_>}(x)}W(k^\alpha_\le)(x)\,dx.$$

This lemma, Propositions 5.3.5 and 5.3.7 and Lemmas 5.5.4 and 5.5.6 imply the following bound.

Theorem 5.7.2. *Let α and k be positive integers. Assume $f \in \mathcal{H}_{\alpha,\mathrm{per}}$. If $b > 2$, then we have*

$$|\widehat{f}(k)| \le \int_0^1 |f^{(\alpha)}(x)|\,dx \frac{b^{-\mu_{\alpha,\mathrm{per}}(k)}}{m_b^\alpha} M_b \left(1 + \frac{bm_b}{b-M_b}\left(1 - \left(\frac{M_b}{b}\right)^v\right)\right).$$

If $b = 2$, then we have

$$|\widehat{f}(k)| \le \int_0^1 |f^{(\alpha)}(x)|\,dx \frac{b^{-\mu_{\alpha,\mathrm{per}}(k)}}{2^{\alpha-1}}.$$

Chapter 6

Accelerating convergence and tractability of multivariate integration for infinitely differentiable functions

6.1 Introduction

In this chapter we approximate the integral on an s-dimensional unit cube

$$\int_{[0,1)^s} f(\boldsymbol{x})\, d\boldsymbol{x}$$

by the algorithm which uses n function values of the form

$$A_{n,s}(f) := \sum_{i=1}^{n} w_i f(\boldsymbol{t}_i) \qquad \text{for } w_i \in \mathbb{R},\ \boldsymbol{t}_i \in [0,1)^s.$$

One classical issue is the optimal rate of convergence with respect to n. Another important issue is the dependence on the number of variables s, since s can be hundreds or more in computational applications. The latter issue is related to the notion of tractability if we require no exponential dependence on s.

A large number of studies have been devoted to numerical integration on the unit cube for various function spaces. One typical case is that functions are only finitely many times differentiable, e.g., functions with bounded variation, periodic functions in the Korobov space and non-periodic functions in the Sobolev space, see [38, 49, 42, 14] and references therein. For these cases, it is known that the rate of convergence is $O(n^{-\alpha})$ for some $\alpha > 0$ and thus we have polynomial convergence. Another interesting case is that functions are smooth, i.e., infinitely differentiable. Dick [7] gave reproducing kernel Hilbert spaces based on Taylor series which achieve a convergence of $O(n^{-\alpha})$ with $\alpha > 0$ arbitrarily large and the spaces were later generalized in [60]. It was proved in [12, 28] that exponential convergence holds for the Korobov space of periodic functions whose Fourier coefficients decay exponentially fast. Exponential convergence means that the integration error converges as $O(q^{n^p})$ for some $q \in (0,1)$, $p > 0$. Note that exponential convergence was also shown for Hermite spaces on $\mathbb{R}^s$ with exponentially fast decaying Hermite coefficients [25].

In this chapter we focus on a weighted normed space of non-periodic smooth functions

$$\mathcal{F}_{s,\boldsymbol{u}} := \left\{ f \in C^\infty[0,1]^s \;\middle|\; \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} := \sup_{(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_0^s} \frac{\|f^{(\alpha_1,\dots,\alpha_s)}\|_{L^1}}{\prod_{j=1}^s u_j^{\alpha_j}} < \infty \right\} \tag{6.1}$$

with a sequence of positive weights $\boldsymbol{u} = \{u_j\}_{j\geq 1}$. It is easy to check that all functions in $\mathcal{F}_{s,\boldsymbol{u}}$ are analytic from Taylor's theorem. This space is motivated by the results by Yoshiki [58] and can be regarded as a Sobolev space of infinite order [15]. The space $\mathcal{F}_{s,\boldsymbol{u}}$ is closely related to the notion of WAFOM. We observe that generalized WAFOM works well for the space $\mathcal{F}_{s,\boldsymbol{u}}$, see Remark 6.6.1.

The first purpose of this chapter is to show that $\mathcal{F}_{s,\boldsymbol{u}}$ achieves accelerating convergence for all s and $\boldsymbol{u}$ considered. Accelerating convergence roughly means that the integration error converges as $O(q^{(\log n)^p})$ for some $q \in (0,1)$ and $p > 1$. Note that $q^{(\log n)^p} = n^{-(\log q^{-1})(\log n)^{p-1}}$, hence the exponent $(\log n)^{p-1}$ of n increases as n increases (which is why we call this accelerating convergence). We remark that accelerating convergence was first observed in [35] as the decay of the lowest-WAFOM value and that [35] and [58] imply the accelerating convergence result for $\mathcal{F}_{s,1/2}$.

We also consider tractability for $\mathcal{F}_{s,\boldsymbol{u}}$. Let us briefly recall the notion of tractability (see [41, 42, 43] for more information). Let $n(\varepsilon, s)$ be the information complexity, i.e., the minimal number n of function values which approximate the s-variate integration within ε. An integration problem is said to be tractable if $n(\varepsilon, s)$ does not grow exponentially on ε nor s. In particular, two notions of tractability has been mainly considered: polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1} s^{\tau_2}$, and strong polynomial tractability, i.e., $n(\varepsilon, s) \leq C\varepsilon^{-\tau_1}$ for $\tau_1, \tau_2 \geq 0$. A common way to obtain tractability is to consider weighted function spaces introduced by Sloan and Woźniakowski [50]. Weighted spaces mean that the dependence on the successive variables can be moderated by weights. Our weights $\boldsymbol{u}$ play the same role. For tractability results for spaces of smooth functions, see also [23].

The second purpose of this chapter is to establish the notions of tractability which correspond to accelerating convergence: accelerating convergence with polynomial tractability (AC-PT) and accelerating convergence with strong tractability (AC-ST). Roughly speaking, AC-PT (resp. AC-ST) holds if accelerating convergence holds and $n(\varepsilon, s)$ depends only polynomially on s (resp. is independent of s). We define the Walsh space $\mathcal{W}_{s,\boldsymbol{a},b}$ into which $\mathcal{F}_{s,\boldsymbol{u}}$ is embedded and prove that the notions of AC-PT and AC-PT are equivalent for $\mathcal{W}_{s,\boldsymbol{a},b}$ and that AC-PT holds for $\mathcal{W}_{s,\boldsymbol{a},b}$ iff the weights $\boldsymbol{a}$ grow polynomially fast. These results enable us to show that AC-ST holds for $\mathcal{F}_{s,\boldsymbol{u}}$ if weights $\boldsymbol{u}$ decay sufficiently fast.

The rest of this chapter is organized as follows. In Section 6.2, we give necessary background including the Dick weight, definitions of our function spaces and embeddings among them. In Section 6.3, we give precise definitions of the notions of accelerating error convergence and tractability used in this chapter. In Section 6.4, we present Theorem 6.4.1 and Corollary 6.4.2, which are the summary of all results in this chapter. Necessary and sufficient conditions for Theorem 6.4.1 are given in Sections 6.5 and 6.6, respectively.

6.2 Function spaces and embeddings

In this subsection, we introduce function spaces $\mathcal{F}_{s,\boldsymbol{u}}$, $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ considered in this chapter and give embeddings from $\mathcal{F}_{s,\boldsymbol{u}}$ to $\mathcal{W}_{s,\boldsymbol{a},b}$.

The space of smooth functions $\mathcal{F}_{s,\boldsymbol{u}}$ is defined as in (6.1). Throughout this chapter, we always assume that

$$u_1 \geq u_2 \geq \cdots > 0. \tag{6.2}$$

In the previous chapter, we have proved that Walsh coefficients of functions in $\mathcal{F}_{s,\boldsymbol{u}}$ decay sufficiently fast. In order to introduce the result, we define the generalized Dick weight $\mu(\boldsymbol{a};\boldsymbol{k})$ for $\boldsymbol{a} \in \mathbb{R}^s$ and $\boldsymbol{k} \in \mathbb{N}_0^s$ and a weight $v(k)$ for $k \in \mathbb{N}$. We also define the modified Dick weight $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})$, which is modified not to take negative values. Note that the Dick weight is originally defined as the case of $\boldsymbol{a} = 0$ in [34].

Definition 6.2.1. *Let $\boldsymbol{a} = (a_1, \dots, a_s) \in \mathbb{R}^s$ and $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$. We denote the b-adic expansion of k_j by $k_j = \sum_{i=1}^{\infty} \kappa_{j,i} b^{i-1}$ with $\kappa_{j,i} \in \mathbb{Z}_b$ (this is actually a finite sum). We define the generalized Dick weight $\mu(\boldsymbol{a};\boldsymbol{k})$ and the modified Dick weight $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})$ as*

$$\mu(\boldsymbol{a};\boldsymbol{k}) := \sum_{j=1}^{s} \sum_{i=1}^{\infty} (i + a_j) h(\kappa_{j,i}),$$

$$\tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) := \sum_{j=1}^{s} \sum_{i=1}^{\infty} \max(i + a_j, 1) h(\kappa_{j,i}),$$

where $h(\kappa) = 0$ for $\kappa = 0$ and $h(\kappa) = 1$ for $\kappa \neq 0$. A weight $v(k_j)$ is defined as

$$v(k_j) := \sum_{i=1}^{\infty} h(\kappa_{j,i}).$$

We now modify the decay of Walsh coefficients given in Corollary 5.3.10 as follows.

Theorem 6.2.2. *Put $m_b := 2\sin(\pi/b)$ and $M_b := 2\sin(\lfloor b/2 \rfloor \pi/b)$. Assume $f \in \mathcal{F}_{s,\boldsymbol{u}}$. Then it follows that*

$$|\widehat{f}(\boldsymbol{k})| \leq \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} b^{-\mu(0;\boldsymbol{k})} \prod_{j=1}^{s} (m_b^{-1} u_j)^{v(k_j)} D_b^{\min(1,v(k_j))},$$

where $D_b = 2$ for $b = 2$ and $D_b = M_b + bm_b/(b - M_b)$ otherwise.

This decay motivates us to define Walsh spaces $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ of Walsh series whose Walsh coefficients are controlled by the generalized (resp. modified) Dick weight. Let $\boldsymbol{a} = (a_j)_{j \geq 1}$ be a sequence of real-valued weights. Throughout this chapter, we assume

$$a_1 \leq a_2 \leq a_3 \leq \cdots , \tag{6.3}$$

which corresponds to (6.2). We first define $\mathcal{W}_{s,\boldsymbol{a},b}$ as

$$\mathcal{W}_{s,\boldsymbol{a},b} := \left\{ f \colon [0,1)^s \to \mathbb{R} \;\middle|\; f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \text{ and } \|f\|_{\mathcal{W}_{s,\boldsymbol{a},b}} < \infty \right\}$$

equipped with the norm

$$\|f\|_{\mathcal{W}_{s,\boldsymbol{a},b}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k}) b^{\mu(\boldsymbol{a};\boldsymbol{k})}|$$

and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ as

$$\tilde{\mathcal{W}}_{s,\boldsymbol{a},b} := \left\{ f \in \mathcal{W}_{s,\boldsymbol{a},b} \;\middle|\; \|f\|_{\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k}) b^{\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})}| < \infty \right\}.$$

Note that all Walsh series in $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ converge. Indeed, for all $X \in (-1,1)$ and a positive integer l, we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ k_j < b^l \, \forall j}} X^{\mu(\boldsymbol{a};\boldsymbol{k})} &= \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ k_j < b^l \, \forall j}} \prod_{j=1}^{s} \prod_{i=1}^{l} X^{(i+a_j)h(\kappa_{j,i})} \\
&= \prod_{j=1}^{s} \prod_{i=1}^{l} \sum_{\kappa_{j,i}=0}^{b-1} X^{(i+a_j)h(\kappa_{j,i})} \\
&= \prod_{j=1}^{s} \prod_{i=1}^{l} (1 + (b-1) X^{i+a_j}),
\end{aligned}$$

and the rightmost product converges for $l \to \infty$ if $|X| < 1$. This is also true for the modified Dick weight with $\mu(\boldsymbol{a};\boldsymbol{k})$ and $i + a_j$ replaced by $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})$ and $\max(i + a_j, 1)$. Hence we have

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^s} X^{\mu(\boldsymbol{a};\boldsymbol{k})} = \prod_{j=1}^{s} \prod_{i=1}^{\infty} (1 + (b-1) X^{i+a_j}) \qquad \text{for all} \quad |X| < 1, \tag{6.4}$$

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} X^{\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})} = \prod_{j=1}^{s}\prod_{i=1}^{\infty}(1+(b-1)X^{\max(i+a_j,1)}) \qquad \text{for all} \quad |X|<1. \tag{6.5}$$

Thus all functions in $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ converge.

We now give embeddings from $\mathcal{F}_{s,\boldsymbol{u}}$ to $\mathcal{W}_{s,\boldsymbol{a},b}$. From Theorem 6.2.2 we have

$$\begin{aligned}|\widehat{f}(\boldsymbol{k})| &\le \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}}\prod_{j=1}^{s} D_b^{\min(1,v(k_j))} b^{-\mu(-\log_b(m_b^{-1}u_j);\boldsymbol{k})} \\ &\le \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}}\prod_{j=1}^{s} b^{-\mu(-\log_b(D_b m_b^{-1}u_j);\boldsymbol{k})}.\end{aligned}$$

Thus we obtain continuous embeddings

$$\mathcal{F}_{s,\boldsymbol{u}} \subset \mathcal{W}_{s,\boldsymbol{u}'',b} \qquad \text{with} \qquad \|f\|_{\mathcal{W}_{s,\boldsymbol{u}'',b}} \le \prod_{j=1}^{s} D_b^{\min(1,v(k_j))}\|f\|_{\mathcal{F}_{s,\boldsymbol{u}}}, \tag{6.6}$$

$$\mathcal{F}_{s,\boldsymbol{u}} \subset \mathcal{W}_{s,\boldsymbol{u}',b} \qquad \text{with} \qquad \|f\|_{\mathcal{W}_{s,\boldsymbol{u}',b}} \le \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}}, \tag{6.7}$$

where $\boldsymbol{u}' = (-\log_b(D_b m_b^{-1}u_j))_{j\ge1}$ and $\boldsymbol{u}'' = (-\log_b(m_b^{-1}u_j))_{j\ge1}$. Note that all functions in $\mathcal{F}_{s,\boldsymbol{u}}$ are equal to their Walsh expansions, see [9, Section 3.3] or [14, Theorem A.20]. Embedding (6.7) implies that good algorithms for $\mathcal{W}_{s,\boldsymbol{u}',b}$ are also good for $\mathcal{F}_{s,\boldsymbol{u}}$. Thus we mainly consider $\mathcal{W}_{s,\boldsymbol{a},b}$ in the following sections.

The Walsh space $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ is considered instead of $\mathcal{W}_{s,\boldsymbol{a},b}$ in Section 6.6, since the modified Dick weight does not take negative values and thus easier to treat. Actually, $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ equals to $\mathcal{W}_{s,\boldsymbol{a},b}$ set-theoretically. Indeed, we have

$$\mu(\boldsymbol{a};\boldsymbol{k}) \le \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \le \mu(\boldsymbol{a};\boldsymbol{k}) + \sum_{j=1}^{s}\sum_{i\in\mathcal{N}_j}(1-(i+a_j))$$

for all $\boldsymbol{k}\in\mathbb{N}_0^s$, where $\mathcal{N}_j$ is defined as $\mathcal{N}_j := \{i\in\mathbb{N} \mid i+a_j\le 1\}$. Thus we have $\mathcal{W}_{s,\boldsymbol{a},b} = \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ set-theoretically and

$$\|f\|_{\mathcal{W}_{s,\boldsymbol{a},b}} \le \|f\|_{\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}} \le b^{\sum_{j=1}^{s}\sum_{i\in\mathcal{N}_j}(1-(i+a_j))}\|f\|_{\mathcal{W}_{s,\boldsymbol{a},b}}. \tag{6.8}$$

This inequality means that we can consider $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ instead of $\mathcal{W}_{s,\boldsymbol{a},b}$ for accelerating convergence results. Furthermore, in Section 6.6.2, where we consider tractability results, we shall assume some condition of weights which implies that the constant factor of (6.8) is bounded independent of s.

6.3 Integration

Let $\mathcal{H} = \mathcal{F}_{s,\boldsymbol{u}}$, $\mathcal{W}_{s,\boldsymbol{a},b}$ or $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$. We consider multivariate integration

$$I(f) = \int_{[0,1)^s} f(\boldsymbol{x})\,d\boldsymbol{x} \qquad \text{for all} \quad f\in\mathcal{H}.$$

Without loss of generality (see, e.g., [41, Section 4.2]), we can restrict ourselves to approximating $I(f)$ by linear algorithms $A_{n,s}(f)$ of the form

$$A_{n,s}(f) = \sum_{i=1}^{n} w_i f(\boldsymbol{t}_i)$$

where $w_i \in \mathbb{R}$ and $\boldsymbol{t}_i \in [0,1)^s$. For $w_i = n^{-1}$, we obtain quasi-Monte Carlo (QMC) algorithms. They are stable and easy to implement and thus often used in practical computations. The *worst-case error* of the algorithm $A_{n,s}$ is defined by

$$e^{\mathrm{wor}}(A_{n,s}, \mathcal{H}) = \sup_{\substack{f \in \mathcal{H} \\ \|f\|_{\mathcal{H}} \le 1}} |I(f) - A_{n,s}(f)|.$$

Let $e(n,s,\mathcal{H})$ be the *n-th minimal worst-case error*,

$$e(n,s) = e(n,s,\mathcal{H}) = \inf_{A_{n,s}\colon \text{linear algorithm}} e^{\mathrm{wor}}(A_{n,s}, \mathcal{H}),$$

where the infimum is extended over all linear algorithms using n function values. For $n = 0$, the zero algorithm is the best, and thus we have $e(0,s,\mathcal{H}) = 1$. Hence the integration problem is well normalized for all s.

We say that we achieve *accelerating convergence* for $e(n,s)$ if there exist a constant $q \in (0,1)$ and functions $C, C_1 \colon \mathbb{N} \to (0,\infty)$ and $p \colon \mathbb{N} \to (1,\infty)$ such that

$$e(n,s) \le C(s) q^{(\log n / C_1(s))^{p(s)}} \qquad \text{for all} \quad n, s \in \mathbb{N}. \tag{6.9}$$

The right-hand side of (6.9) can be modified as $C(s) n^{-(\log q^{-1}/C_1(s)^{p(s)})(\log n)^{p(s)-1}}$, hence the exponent $(\log n)^{p(s)-1}$ of n increases as n increases (which is why we call this accelerating convergence).

We say that we achieve *uniform accelerating convergence* (U-AC) for $e(n,s)$ if the function $p(s)$ in (6.9) can be taken as a constant, i.e., $p(s) = p > 0$ for all s.

For $\varepsilon \in (0,1)$, we define the *information complexity* of integration

$$n(\varepsilon, s) = n(\varepsilon, s, \mathcal{H}) = \min\{n \in \mathbb{N} \mid e(n,s,\mathcal{H}) \le \varepsilon\}$$

as the minimal number of function values needed to obtain an ε-approximation.

We note that if (6.9) holds, then for all $s \in \mathbb{N}$ and $\varepsilon \in (0,1)$ we have

$$n(\varepsilon, s) \le \left\lceil \exp\left(C_1(s) \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)_+^{1/p(s)} \right) \right\rceil, \tag{6.10}$$

where $(X)_+ := \max(X, 0)$ for $X \in \mathbb{R}$. Furthermore, if (6.10) holds, then for all $s, n \in \mathbb{N}$ we have

$$e(n+1, s) \le C(s) q^{(\log n / C_1(s))^{p(s)}}.$$

This means that (6.9) and (6.10) are essentially equivalent. Accelerating convergence implies that asymptotically $n(\varepsilon, s)$ increases of order $\exp(O((\log \varepsilon^{-1})^{1/p(s)}))$

with respect to ε. However, how does $n(\varepsilon, s)$ depend on s? This, of course, depends on $C(s)$, $C_1(s)$ and $p(s)$ and is the subject of tractability. Tractability means that we control the behavior of $C(s)$, $C_1(s)$ and $p(s)$ and rule out the cases for which $n(\varepsilon, s)$ depends exponentially on s. In this chapter, we consider following two notions of tractability.

We say that we have *accelerating convergence with polynomial tractability* (AC-PT) if there exist real numbers $w > 1$, $A, \tau_2 \geq 0$ and $\tau_1 \in (0, 1)$ such that

$$n(\varepsilon, s) \leq A w^{(\log \varepsilon^{-1})^{\tau_1}} s^{\tau_2} \qquad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0, 1). \tag{6.11}$$

We say that we have *accelerating convergence with strong tractability* (AC-ST) if we have AC-PT with $\tau_2 = 0$. We note that the notion of AC-PT and AC-ST with $\tau_1 = t\,(0 < t < 1)$ coincides with the notion of T-tractability with $T(x, y) = \exp((\log x)^t)y$, see [41, Section 8].

We give relations between these notions. First we note that the right-hand side of (6.11) equals $\exp((\log w)(\log \varepsilon^{-1})^{\tau_1} + \log(As^{\tau_2}))$. Applying the inequality $2^{\tau_1-1}(X^{\tau_1} + Y^{\tau_1}) \leq (X+Y)^{\tau_1} \leq X^{\tau_1} + Y^{\tau_1}$ for $X, Y \geq 0$, we obtain that (6.11) is equivalent to the fact that there exists $B > 0$ such that

$$n(\varepsilon, s) \leq \exp\left(B\left(\log \varepsilon^{-1} + \left(\frac{(\log(As^{\tau_2}))_+}{\log w}\right)^{1/\tau_1}\right)^{\tau_1}\right). \tag{6.12}$$

Comparing (6.12) with (6.10), we obtain the following lemma.

Lemma 6.3.1.

1. *Assume that* (6.9) *holds for* $C(s) = C \geq 0$, $C_1(s) = C_1 > 0$ *and* $p(s) = p > 1$. *Then AC-ST holds with* $\tau_1 = 1/p$.
2. *Assume that AC-PT holds. Then U-AC holds with* $p = 1/\tau_1$, $\log C(s) \in o(s)$ *and* $C_1(s) = 1$.

6.4 Main results

In this section, we present the main results of this chapter. The following theorem gives necessary and sufficient conditions on the weight sequence $\boldsymbol{a}$ for the notions of U-AC, AC-PT and AC-ST for $\mathcal{W}_{s,\boldsymbol{a},b}$.

Theorem 6.4.1. *Consider integration defined over the Walsh space* $\mathcal{W}_{s,\boldsymbol{a},b}$ *with a weight sequence* $\boldsymbol{a}$ *satisfying* (6.3). *Then we have the following.*

1. *U-AC with* $p = 2$ *holds for all* $\boldsymbol{a}$ *considered, and U-AC with* $p > 2$ *does not hold for any* $\boldsymbol{a}$ *considered.*
2. *AC-PT with* $\tau_1 \leq 1/2$ *does not hold for any* $\boldsymbol{a}$ *considered.*
3. *Let* $1/2 < t < 1$ *be a real number. The following are equivalent:*
   (a) *The sequence* $\boldsymbol{a}$ *satisfies* $\liminf_{j\to\infty} a_j/j^{(1-t)/(2t-1)} > 0$,
   (b) *we have AC-PT with* $\tau_1 = t$,

(c) we have AC-ST with $\tau_1 = t$.

This theorem and (6.7) imply the following U-AC and AC-ST results for $\mathcal{F}_{s,\boldsymbol{u}}$.

Corollary 6.4.2. *Consider integration defined over $\mathcal{F}_{s,\boldsymbol{u}}$ with a weight sequence $\boldsymbol{u}$ satisfying* (6.2). *Then we have the following.*

1. *U-AC with $p = 2$ holds for all $\boldsymbol{u}$ considered.*
2. *Let $1/2 < t < 1$ be a real number. If the weight sequence $\boldsymbol{u}$ satisfies $\liminf_{j\to\infty} \log(u_j^{-1})/j^{(1-t)/(2t-1)} > 0$, then we have AC-ST with $\tau = t$.*

The proof of Theorem 6.4.1 will be done as follows. First, Item 3 (iii) clearly implies Item 3 (ii). In Section 6.5, we prove necessary conditions for Theorem 6.4.1. More precisely, we prove the second part of Item 1, Item 2, and that Item 3 (ii) implies Item 3 (i) in Theorems 6.5.3, 6.5.4 and 6.5.5, respectively. In section 6.6, we give sufficient conditions for Theorem 6.4.1 by proving the existence of good QMC algorithms on digital nets. Corollaries 6.6.7 and 6.6.12 imply the first part of Item 1 and that Item 3 (i) implies Item 3 (iii), respectively.

6.5 Lower bounds

We prove the following lower bound on $e(n, s, \mathcal{W}_{s,\boldsymbol{a},b})$ along [12, Theorem 1], which treats the Korobov space.

Lemma 6.5.1. *Let $\mathcal{A}$ be a finite subset of $\mathbb{N}_0^s$. Then for all $n < |\mathcal{A}|$ we have*

$$e(n, s, \mathcal{W}_{s,\boldsymbol{a},b}) \geq \left(\max_{\boldsymbol{k},\boldsymbol{k}^* \in \mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)} \right)^{-1}.$$

Proof. Take an arbitrary algorithm $A_{n,s}(f) = \sum_{i=1}^n w_i f(\boldsymbol{t}_i)$. Define $g_1(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{A}} c_{\boldsymbol{k}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ for $c_{\boldsymbol{k}} \in \mathbb{C}$ such that $g_1(\boldsymbol{t}_i) = 0$ for all $i = 1, 2, \dots, n$. Since we have n homogeneous linear equations and $|\mathcal{A}| > n$ unknowns $c_{\boldsymbol{k}}$, there exists a nonzero vector of such $c_{\boldsymbol{k}}$'s, and we can normalize the $c_{\boldsymbol{k}}$'s by assuming that

$$\max_{\boldsymbol{k}\in\mathcal{A}} |c_{\boldsymbol{k}}| = c_{\boldsymbol{k}^*} = 1 \quad \text{for some} \quad \boldsymbol{k}^* \in \mathcal{A}.$$

Define the function

$$g_2(\boldsymbol{x}) := C g_1(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}^*}(\boldsymbol{x})} = C \sum_{\boldsymbol{k}\in\mathcal{A}} c_{\boldsymbol{k}} \mathrm{wal}_{\boldsymbol{k}\ominus\boldsymbol{k}^*}(\boldsymbol{x}),$$

where $C = (\max_{\boldsymbol{k},\boldsymbol{k}^*\in\mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)})^{-1}$, and $\overline{g_2}(\boldsymbol{x}) := \overline{g_2(\boldsymbol{x})}$. Then we have

$$\begin{aligned} \|g_2\|_{\mathcal{W}_{s,\boldsymbol{a},b}} &= C \max_{\boldsymbol{k}\in\mathcal{A}} |c_{\boldsymbol{k}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)}| \\ &\leq C \max_{\boldsymbol{k}\in\mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)} \leq C \max_{\boldsymbol{k},\boldsymbol{k}^*\in\mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)} = 1, \end{aligned}$$

where $\|\cdot\|_{\mathcal{W}_{s,\boldsymbol{a},b}}$ is naturally extended to complex-valued Walsh series. Note that $\|\overline{g_2}\|_{\mathcal{W}_{s,\boldsymbol{a},b}} = \|g_2\|_{\mathcal{W}_{s,\boldsymbol{a},b}}$ since $\mu(\boldsymbol{a}; \boldsymbol{k} \ominus \boldsymbol{k}^*) = \mu(\boldsymbol{a}; \boldsymbol{k}^* \ominus \boldsymbol{k})$ for all $\boldsymbol{k}$.

We now define a real-valued function $f(\boldsymbol{x}) := (g_2(\boldsymbol{x}) + \overline{g_2(\boldsymbol{x})})/2$. The norm of f is bounded by

$$\|f\|_{\mathcal{W}_{s,\boldsymbol{a},b}} \le (\|g_2\|_{\mathcal{W}_{s,\boldsymbol{a},b}} + \|\overline{g_2}\|_{\mathcal{W}_{s,\boldsymbol{a},b}})/2 = \|g_2\|_{\mathcal{W}_{s,\boldsymbol{a},b}} \le 1.$$

Note that $A_{n,s}(f) = 0$ since $f(\boldsymbol{t}_i) = 0$ for all i. Furthermore, $I(f) = Cc_{\boldsymbol{k}^*} = C$. Hence,

$$e(n, s, \mathcal{W}_{s,\boldsymbol{a},b}) \ge |I(f) - A_{n,s}(f)| = I(f) = C.$$

Since this holds for all w_i and $\boldsymbol{t}_i$, we conclude that $e(n, s) \ge C$, as claimed. □

For a non-negative integer d, we now define

$$\mathcal{A}_{s,d} = \{\boldsymbol{k} \in \mathbb{N}_0^s \mid k_j < b^d \quad \text{for all} \quad j = 1, 2, \dots, s\}.$$

The cardinality of the set $|\mathcal{A}_{s,d}|$ is clearly b^{sd}. If $a_j \ge 0$ holds for all j, then

$$\begin{aligned}\left(\max_{\boldsymbol{k},\boldsymbol{k}^* \in \mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k}\ominus\boldsymbol{k}^*)}\right)^{-1} &= \left(\max_{\boldsymbol{k}\in\mathcal{A}} b^{\mu(\boldsymbol{a};\boldsymbol{k})}\right)^{-1} \\ &= b^{-\sum_{j=1}^s \sum_{i=1}^d (i+a_j)} = b^{-\sum_{j=1}^s (d(d+1)/2 + a_j d)},\end{aligned}$$

where we use $\boldsymbol{k} \ominus \boldsymbol{k}^* \in \mathcal{A}_{s,d}$ for all $\boldsymbol{k}, \boldsymbol{k}^* \in \mathcal{A}_{s,d}$ for the first equality. This implies the following corollary.

Corollary 6.5.2. *Let $d \in \mathbb{N}$ and assume $a_j \ge 0$ for all j. Then we have*

$$e(n, s, \mathcal{W}_{s,\boldsymbol{a},b}) \ge b^{-\sum_{j=1}^s (d^2/2 + (a_j+1/2)d)} \qquad \text{for all } n < b^{sd}.$$

We prove necessary conditions for U-AC and AC-PT for $\mathcal{W}_{s,\boldsymbol{a},b}$ in the following three theorems. We can assume $a_j \ge 0$ for all j without loss of generality and so we do.

Theorem 6.5.3. *For any $\boldsymbol{a}$ considered, U-AC with $p > 2$ for $\mathcal{W}_{s,\boldsymbol{a},b}$ does not hold.*

Proof. We will argue by contradiction. Suppose that U-AC with $p > 2$ holds. Then (6.9) holds with $p(s) = p$. Taking $s = 1$, from Corollary 6.5.2 we obtain

$$b^{-(d^2/2+(a_1+1/2)d)} \le e(b^d - 1, 1) \le e(b^{d-1}, 1) \le C(1)q^{((\log b)(d-1)/C_1(1))^p}$$

for all positive integer d. Taking the logarithms we have

$$(\log b)(d^2/2 + (a_1 + 1/2)d) \ge -\log C(1) + \log q^{-1}(C_1(1)^{-1} \log b)^p (d-1)^p.$$

However, this inequality does not hold for sufficiently large positive integer d since $p > 2$. This is a contradiction. □

Theorem 6.5.4. *For any* $\boldsymbol{a}$ *considered, AC-PT with* $\tau_1 = 1/2$ *for* $\mathcal{W}_{s,\boldsymbol{a},b}$ *does not hold.*

Proof. We will argue by contradiction. Suppose that AC-PT with $\tau_1 = 1/2$ holds. Then it follows from Lemma 6.3.1 that (6.9) holds with $p(s) = 2$, $\log C(s) \in o(s)$ and $C_1(s) = 1$. Let s and d be positive integers. Then it follows from Corollary 6.5.2 that

$$b^{-(sd^2/2+d\sum_{j=1}^{s}(a_j+1/2))} \leq e(b^{sd}-1, s) \leq e(b^{(s-1)d}, s) \leq C(s)q^{((\log b)(s-1)d)^2}.$$

Taking the logarithms we have

$$\begin{aligned}-\log C(s) + \log q^{-1}(\log b)^2(s-1)^2d^2 &\leq (\log b)\left(sd^2/2 + d\sum_{j=1}^{s}(a_j+1/2)\right)\\ &\leq (\log b)\left(sd^2/2 + sd(a_s+1/2)\right).\end{aligned}$$

Considering the order of d, for all positive integer s we have

$$(\log b)s/2 \geq \log q^{-1}(\log b)^2(s-1)^2.$$

However, this inequality does not hold for sufficiently large positive integer s. This is a contradiction. □

Theorem 6.5.5. *Consider integration defined over* $\mathcal{W}_{s,\boldsymbol{a},b}$. *Assume* (6.3) *and that AC-PT with* $1/2 < \tau_1 < 1$ *holds. Put* $r := (1-\tau_1)/(2\tau_1 - 1)$. *Then we have*

$$\liminf_{j\to\infty} \frac{a_j}{j^r} > 0.$$

Proof. Similar to the proof of Theorem 6.5.4, we have

$$(\log q^{-1})((\log b)(s-1)d)^{1/\tau_1} + o(s) \leq (\log b)(sd^2/2 + sd(a_s+1/2))$$

for all positive integers d and s, since AC-PT with $1/2 < \tau_1 < 1$ holds. Let N be a positive integer and take $d := \lceil s^r/N \rceil$. Then we obtain

$$(\log q^{-1})(\log b)^{1/\tau_1 - 1}N^{-1/\tau_1}s^{2r+1} + o(s^{2r+1}) \leq N^{-2}s^{2r+1}/2 + N^{-1}s^{r+1}a_s + sa_s.$$

Now we will argue by contradiction. Suppose that $\liminf_{j\to\infty} a_j/j^r = 0$. Then there exists arbitrary large s such that $a_s \leq s^r/N$ holds. For such s, we have

$$(\log q^{-1})(\log b)^{1/\tau_1 - 1}N^{-1/\tau_1}s^{2r+1} + o(s^{2r+1}) \leq 3N^{-2}s^{2r+1}/2,$$

and thus

$$(\log q^{-1})(\log b)^{1/\tau_1 - 1}N^{-1/\tau_1} \leq 3N^{-2}/2.$$

We have thus proved that this inequality holds for any positive integer N, but this contradicts the assumption $\tau_1 > 1/2$. □

6.6 Upper bounds

In this section, motivated by [35] and its generalization given in Chapter 3, we prove the existence of good QMC algorithms which achieve U-AC and AC-ST in Sections 6.6.1 and 6.6.2, respectively. Such QMC algorithms are given by digital nets.

From now on, we consider integration defined over $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ using QMC algorithms over digital nets (in the sense of Definition 2.2.1). That is, for a digital net P, we use $P(f) := |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x})$, where we identify the digital net P and the QMC algorithm on P. Applying (2.1) to our setting, we have the following bound on the integration error:

$$\left| |P|^{-1} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}) - I(f) \right| \leq \sum_{\boldsymbol{k} \in P^\perp \setminus \{0\}} |\widehat{f}(\boldsymbol{k})| \leq \|f\|_{\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}} \sum_{\boldsymbol{k} \in P^\perp \setminus \{0\}} b^{-\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})}. \tag{6.13}$$

Remark 6.6.1. *WAFOM is defined as a truncated version of the sum on the rightmost side of* (6.13) *for* $\boldsymbol{a} = 0$ *in Chapter 3 and for* $\boldsymbol{a} = 1$ *in [58, 21]. Note that* $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) = \mu(\boldsymbol{a};\boldsymbol{k})$ *in these cases. Thus the sum (and the sum with* $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})$ *replaced by* $\mu(\boldsymbol{a};\boldsymbol{k})$*) can be regarded as a non-discretized version of WAFOM generalized by weights* $\boldsymbol{a}$ *and we can say that* $\mathcal{F}_{s,\boldsymbol{u}}$*,* $\mathcal{W}_{s,\boldsymbol{a},b}$ *and* $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ *are function spaces for which WAFOM works well.*

We now define the minimal weight of $P^\perp$ by

$$\delta_{P^\perp} := \inf_{\boldsymbol{k} \in P^\perp \setminus \{0\}} \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}).$$

Then the rightmost side of (6.13) is bounded by $\|f\|_{\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}} \sum_{\boldsymbol{k}} b^{-\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})}$, where the sum is extended over all $\boldsymbol{k} \in \mathbb{N}_0^s$ with $\tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \geq \delta_{P^\perp}$. This argument implies the following lemma.

Lemma 6.6.2. *Let P be a digital net. Then we have*

$$e^{\mathrm{wor}}(P, \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}) \leq \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \geq \delta_{P^\perp}}} b^{-\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})}. \tag{6.14}$$

The right-hand side of (6.14) will be evaluated in the following sections.

We now prove a lemma which gives the existence of digital nets whose minimal weight is large. First we define

$$\mathrm{vol}_{s,\boldsymbol{a}}(M) := |\{\boldsymbol{k} \in \mathbb{N}_0^s \mid \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \leq M\}|.$$

Lemma 6.6.3. *Let M be a real number and p_b the smallest prime factor of b. Let d and $l \geq M - a_1 - 1$ be positive integers. If* $\mathrm{vol}_{s,\boldsymbol{a}}(M) \leq {p_b}^d$ *holds, then there exists a d-dimensional digital net P over $\mathbb{Z}_b$ with precision l satisfying* $\delta_{P^\perp} \geq M$.

Proof. Let $G_1, \dots G_s \in \mathbb{Z}_b^{l\times d}$ be matrices. Recall

$$\boldsymbol{k} \in P^{\perp}(G_1, \dots, G_s) \iff G_1^{\top} \operatorname{tr}_l(k_1) + \cdots + G_s^{\top} \operatorname{tr}_l(k_s) = 0.$$

Thus it follows that

$$|\{(G_1, \dots, G_s) \in (\mathbb{Z}_b^{l\times d})^s \mid \boldsymbol{k} \in P^{\perp}(G_1, \dots, G_s)\backslash\{0\}, k_j < b^l \ \forall j\}| \leq b^{sdl}/{p_b}^d,$$

where p_b is the smallest prime factor of b. Hence we have

$$\begin{aligned}
&|\{(G_1, \dots, G_s) \in (\mathbb{Z}_b^{l\times d})^s \mid \min_{\substack{\boldsymbol{k}\in P^{\perp}(G_1,\dots,G_s)\backslash\{0\} \\ k_j<b^l\ \forall j}} \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) > M\}| \\
&= b^{sdl} - |\{(G_1, \dots, G_s) \in (\mathbb{Z}_b^{l\times d})^s \mid \min_{\substack{\boldsymbol{k}\in P^{\perp}(G_1,\dots,G_s)\backslash\{0\} \\ k_j<b^l\ \forall j}} \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \leq M\}| \\
&\geq b^{sdl} - \sum_{0\neq\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})\leq M} |\{(G_1, \dots, G_s) \in (\mathbb{Z}_b^{l\times d})^s \mid \boldsymbol{k} \in P^{\perp}(G_1, \dots, G_s)\backslash\{0\}, k_j < b^l \ \forall j\}| \\
&> b^{sdl} - \operatorname{vol}_{s,\boldsymbol{a}}(M) b^{sdl}/{p_b}^d.
\end{aligned}$$

Thus, if $\operatorname{vol}_{s,\boldsymbol{a}}(M) \leq {p_b}^d$ holds, there exists $(G_1, \dots, G_s) \in (\mathbb{Z}_b^{l\times d})^s$ with

$$\inf_{\substack{\boldsymbol{k}\in P^{\perp}(G_1,\dots,G_s)\backslash\{0\} \\ k_j<b^l\ \forall j}} \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \geq M. \tag{6.15}$$

Furthermore, from the the assumption $l \geq M - a_1 - 1$ we have

$$\min\{\tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \mid \boldsymbol{k} \in \mathbb{N}_0^s, k_j \geq b^l \ \exists j\} \geq \max(1, a_1 + l + 1) \geq M. \tag{6.16}$$

Combining (6.15) and (6.16), we obtain the result. □

6.6.1 Accelerating convergence results

In this subsection, we prove accelerating convergence for $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$. Taking account of (6.8), we have only to consider $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$.

First we prove a bound on $\operatorname{vol}_{s,\boldsymbol{a}}(M)$ along [33, Exercise 3(b), p.332] and its modifications in [35] and Chapter 3, which treat the case of $\boldsymbol{a} = 0$. Since $\operatorname{vol}_{s,\boldsymbol{a}}(M) \leq 1$ holds if $M < 1$, we assume that $M \geq 1$. We have

$$\operatorname{vol}_{s,\boldsymbol{a}}(M) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ \tilde{\mu}(\boldsymbol{a};\boldsymbol{k})\leq M}} 1 \leq \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s \\ \tilde{\mu}(\boldsymbol{a};\boldsymbol{k})\leq M}} X^{\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})-M} \leq \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} X^{\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})-M}$$

for all $X \in (0,1)$, and the mostright hand side is equal to $\prod_{j=1}^{s}\prod_{i=1}^{\infty}(1 + (b-1)X^{\max(i+a_j,1)})/X^M$ from (6.4). By taking the logarithm of the both sides and using the well-known inequality $\log(1+x) \leq x$, for all $X \in (0,1)$ we have

$$\log \operatorname{vol}_{s,\boldsymbol{a}}(M) \leq \sum_{j=1}^{s}\sum_{i=1}^{\infty}(b-1)X^{\max(i+a_j,1)} + M \log X^{-1}. \tag{6.17}$$

We proceed to bound $\sum_{i=1}^{\infty} X^{\max(i+a_j,1)}$. If $a_j \geq 0$, it is equal to $X^{a_j+1}/(1-X)$. Otherwise, we have

$$\begin{aligned}\sum_{i=1}^{\infty} X^{\max(i+a_j,1)} &= \sum_{i\colon i+a_j\leq 1} X^1 + \sum_{i\colon i+a_j>1} X^{i+a_j}\\ &\leq \sum_{i\colon i+a_j\leq 1} 1 + \sum_{i'=1}^{\infty} X^{i'}\\ &= n_j + X/(1-X),\end{aligned}$$

where $n_j := |\mathcal{N}_j| = |\{i \in \mathbb{N} \mid i + a_j \leq 1\}|$. Thus, in both cases, we obtain

$$\sum_{i=1}^{\infty} X^{\max(i+a_j,1)} \leq n_j + \frac{X}{1-X}\min(X^{a_j}, 1).$$

Applying this inequality to (6.17), we have

$$\begin{aligned}\log \mathrm{vol}_{s,\boldsymbol{a}}(M) &\leq (b-1)\sum_{j=1}^{s}\left(n_j + \frac{X}{1-X}\min(X^{a_j},1)\right) + M\log X^{-1}\\ &\leq (b-1)\sum_{j=1}^{s}\big(n_j + (\log X^{-1})^{-1}\min(X^{a_j},1)\big) + M\log X^{-1}.\end{aligned} \tag{6.18}$$

Putting $X = 1/\exp(\sqrt{(b-1)s/M})$ and using $\min(X^{a_j},1)\leq 1$, we obtain

$$\log \mathrm{vol}_{s,\boldsymbol{a}}(M) \leq N_s + 2\sqrt{(b-1)sM},$$

where we define $N_s := (b-1)\sum_{j=1}^{s} n_j$. We have thus proved the following.

Lemma 6.6.4. *For all $M \geq 0$ we have*

$$\mathrm{vol}_{s,\boldsymbol{a}}(M) \leq \exp\left(N_s + 2\sqrt{(b-1)sM}\right).$$

We note that Lemma 6.6.4 and the fact that $\mathrm{vol}_{s,\boldsymbol{a}}(M) \leq 1$ if $M < 1$ implies

$$\mathrm{vol}_{s,\boldsymbol{a}}(M) \leq \exp\left(\left(N_s + 2\sqrt{(b-1)s}\right)\sqrt{M}\right). \tag{6.19}$$

Now we give a bound on the right-hand side of (6.14). From Lemma 6.6.4 we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ \tilde{\mu}(\boldsymbol{a};\boldsymbol{k})\geq M}} b^{-\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})} &\leq \sum_{i=0}^{\infty}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ M+i\leq\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})<M+i+1}} b^{-(M+i)}\\ &\leq \sum_{i=0}^{\infty}\mathrm{vol}_{s,\boldsymbol{a}}(M+i+1)b^{-(M+i)}\end{aligned}$$

$$\leq \sum_{i=0}^{\infty} \exp\left(N_s + 2\sqrt{(b-1)s(M+i+1)}\right) b^{-(M+i)} \qquad (6.20)$$

for all $M \geq 0$. We can easily check $\sqrt{x} \leq x/(2\sqrt{B}) + \sqrt{B}/2$ for all $x, B \geq 0$. Applying this inequality, the right-hand side of (6.20) is bounded by

$$\begin{aligned}
&\sum_{i=0}^{\infty} \exp\left(N_s + \sqrt{(b-1)s/B}(M+i+1) + \sqrt{(b-1)sB}\right) b^{-(M+i)} \\
&= b \exp\left(N_s + \sqrt{(b-1)sB}\right) \sum_{i=0}^{\infty} \exp\left(\left(\sqrt{(b-1)s/B} - \log b\right)(M+i+1)\right).
\end{aligned}$$

Taking B as $\sqrt{(b-1)s/B} = (\log b)/2$, we obtain a bound on the right-hand side of (6.14) by

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \tilde{\mu}(\boldsymbol{a};\boldsymbol{k}) \geq M}} b^{-\tilde{\mu}(\boldsymbol{a};\boldsymbol{k})} \leq C_s \exp(-(\log b)M/2),$$

where the positive constant C_s is defined by

$$C_s = \exp\left(N_s + (\log b)/2 + 2(b-1)s/\log b\right)\left(1 - \exp(-\log b/2)\right)^{-1}.$$

Hence Lemma 6.6.2 implies the following lemma.

Lemma 6.6.5. *Let P be a digital net. Then we have*

$$e^{\mathrm{wor}}(P, \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}) \leq C_s \exp(-\delta_{P^\perp}(\log b)/2).$$

Put $C'_s := N_s + 2\sqrt{(b-1)s}$. It follows from (6.19) that the condition of Lemma 6.6.3 is satisfied if $\exp(C'_s\sqrt{M}) \leq p_b^d$, which is equivalent to $M \leq (d \log p_b / C'_s)^2$. Therefore the following bound on the worst-case error follows from Lemmas 6.6.3 and 6.6.4.

Theorem 6.6.6. *Let d be a positive integer. Then there exists a d-dimensional digital net P over $\mathbb{Z}_b$ with precision l with $l \geq (\log p_b/C'_s)^2 d^2 - 1 - a_1$ such that*

$$e^{\mathrm{wor}}(P, \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}) \leq C_s \exp\left(-\frac{(\log p_b)^2 \log b}{2{C'_s}^2} d^2\right). \qquad (6.21)$$

In particular, $e(b^d, s)$ is bounded by the right-hand side of (6.21). Thus we have the following convergence result.

Corollary 6.6.7. *Spaces $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$ achieve U-AC with $p = 2$ for all $\boldsymbol{a}$ considered.*

6.6.2 Tractability results

We have proved accelerating convergence for $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$, but this convergence heavily depends on s. In this subsection, we prove a tractability result under the assumption of a sufficient condition of Theorem 6.4.1. That is, let $r > 0$ and we assume that the sequence $\boldsymbol{a}$ satisfies $\liminf_{j\to\infty} a_j/j^r > 0$. This implies that there exist a positive real number a and a non-negative integer A such that

$$a_j \geq aj^r \qquad \text{for all} \quad j > A. \tag{6.22}$$

Hence hereafter we assume (6.22). Under this assumption, $\mathcal{N}_j$ is empty for sufficiently large j. Hence the constant factor of (6.8) is independent of s and thus we have only to consider $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$. We also note that $(b-1)\sum_{j=1}^{\infty} n_j$ is finite and we denote it by N. The following arguments are parallel to those in Section 6.6.1.

First we prove a bound on $\mathrm{vol}_{s,\boldsymbol{a}}(M)$ under the assumption (6.22). We need the following lemma to bound $\sum_{j=1}^{s} X^{a_j}$.

Lemma 6.6.8. *For all* $0 < X < 1$, *we have*

$$\sum_{j=1}^{s} X^{aj^r} \leq r^{-1}\Gamma(1/r)(a \log X^{-1})^{-1/r},$$

where $\Gamma(z) := \int_0^\infty t^{z-1}\exp(-t)\,dt$ *is the Gamma function.*

Proof. Since X^{ax^r} is a monotonically decreasing function of x, we have

$$\sum_{j=1}^{s} X^{aj^r} \leq \int_0^s X^{ax^r}\,dx \leq \int_0^\infty \exp(-ax^r \log X^{-1})\,dx.$$

Substituting $ax^r \log X^{-1} = z$, which implies $dx = r^{-1}(a\log X^{-1})^{-1/r} z^{(1-r)/r}\,dz$, we have

$$\begin{aligned}\int_0^\infty \exp(-ax^r \log X^{-1})\,dx &= r^{-1}(a\log X^{-1})^{-1/r}\int_0^\infty z^{(1-r)/r}\exp(-z)\,dz \\ &= r^{-1}\Gamma(1/r)(a\log X^{-1})^{-1/r},\end{aligned}$$

which proves the lemma. $\square$

Combining (6.18) and Lemma 6.6.8, for all $X \in (0,1)$ we have

$$\begin{aligned}\log \mathrm{vol}_{s,\boldsymbol{a}}(M) &\leq (b-1)\left(\sum_{j=1}^{A}\left(n_j + \frac{1}{\log X^{-1}}\right) + \sum_{j=A+1}^{s}\frac{X^{ar^j}}{\log X^{-1}}\right) + M\log X^{-1} \\ &\leq (b-1)\left(\frac{A}{\log X^{-1}} + \frac{r^{-1}\Gamma(1/r)a^{-1/r}}{(\log X^{-1})^{1+1/r}}\right) + N + M\log X^{-1}.\end{aligned}$$

Putting $X = 1/\exp(M^{-r/(2r+1)})$ and using $M \geq 1$, we obtain

$$\log \mathrm{vol}_{s,\boldsymbol{a}}(M) \leq c_1 M^{(r+1)/(2r+1)},$$

where $c_1 = (b-1)(A + r^{-1}\Gamma(1/r)a^{-1/r}) + N + 1$. We have thus proved the following lemma.

Lemma 6.6.9. *Assume* (6.22). *Then for all $M \geq 0$ we have*

$$\mathrm{vol}_{s,\boldsymbol{a}}(M) \leq \exp\left(c_1 M^{(r+1)/(2r+1)}\right).$$

Note that the bound on $\mathrm{vol}_{s,\boldsymbol{a}}(M)$ from this lemma is weaker than Lemma 6.6.4 with respect to M but independent of s instead.

In the following, we bound the the right-hand side of (6.14) along Section 6.6.1. For $M \geq 0$ we have

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \mu(\boldsymbol{a};\boldsymbol{k}) \geq M}} b^{-\mu(\boldsymbol{a};\boldsymbol{k})} \leq \sum_{i=0}^{\infty} \exp(c_1(M+i+1)^{(r+1)/(2r+1)}) b^{-(M+i)}. \tag{6.23}$$

We can easily check an inequality

$$x^{(r+1)/(2r+1)} \leq \frac{r+1}{2r+1} B^{-r} x + \frac{r}{2r+1} B^{r+1} \qquad \text{for all } x, B \geq 0.$$

Applying this inequality, the right-hand side of (6.23) is bounded by

$$\begin{aligned} &\sum_{i=0}^{\infty} \exp\left(c_1 \frac{r+1}{2r+1} B^{-r}(M+i+1) + c_1 \frac{r}{2r+1} B^{r+1}\right) b^{-(M+i)} \\ &= b \exp\left(c_1 \frac{r}{2r+1} B^{r+1}\right) \sum_{i=0}^{\infty} \exp\left(\left(c_1 \frac{r+1}{2r+1} B^{-r} - \log b\right)(M+i+1)\right). \end{aligned}$$

Now we take B as

$$c_1 \frac{r+1}{2r+1} B^{-r} = \frac{\log b}{2}.$$

Thus we have a bound on the right-hand side of (6.14) as

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \mu(\boldsymbol{a};\boldsymbol{k}) \geq M}} b^{-\mu(\boldsymbol{a};\boldsymbol{k})} \leq c_2 \exp(-M(\log b)/2),$$

where the positive constant c_2 is defined as

$$c_2 = \exp\left(\frac{\log b}{2} + \frac{c_1 r}{2r+1}\left(\frac{2c_1(r+1)}{(2r+1)\log b}\right)^{(r+1)/r}\right) \frac{1}{1 - \exp(-(\log b)/2)}.$$

Hence Lemma 6.6.2 implies the following lemma.

Lemma 6.6.10. *Assume* (6.22)*. If P is a digital net, we have*

$$e^{\mathrm{wor}}(P, \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}) \leq c_2 \exp(-\delta_{P^\perp}(\log b)/2).$$

Now we prove the existence of good digital nets. By Lemma 6.6.9, the condition of Lemma 6.6.3 is satisfied if $\exp(c_1 M^{(r+1)/(2r+1)}) \leq {p_b}^d$, which is equivalent to $M \leq (d \log p_b/c_1)^{(2r+1)/(r+1)}$. Therefore we have the following bound on the worst-case error independent of s.

Theorem 6.6.11. *Let d be a positive integer and put $c_3 = (\log p_b/c_1)^{(2r+1)/(r+1)}$. Assume* (6.22)*. Then there exists a d-dimensional digital net P over $\mathbb{Z}_b$ with precision l with $l \geq c_3 d^{(2r+1)/(r+1)} - 1 - a_1$ such that*

$$e^{\mathrm{wor}}(P, \tilde{\mathcal{W}}_{s,\boldsymbol{a},b}) \leq c_2 \exp\left(-\frac{c_3 \log b}{2} d^{(2r+1)/(r+1)}\right). \tag{6.24}$$

In particular, $e(b^d, s)$ is bounded by the right-hand side of (6.24). Thus we have the following tractability result.

Corollary 6.6.12. *Assume* (6.22)*. Then AC-ST with $\tau_1 = (r+1)/(2r+1)$ holds for $\mathcal{W}_{s,\boldsymbol{a},b}$ and $\tilde{\mathcal{W}}_{s,\boldsymbol{a},b}$.*

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