博士論文

論文題目　On stability of viscosity solutions under non-Euclidean metrics

（非ユークリッド距離構造の下での粘性解の安定性について）

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Preface

The theory of nonlinear partial differential equations has been developed as a powerful mathematical tool to understand nonlinear phenomena in various fields of science. In recent years problems in the setting of non-Euclidean metrics attracts a great deal of attention. For example, analysis on a complex system is expected to have important applications to study propagation of chemicals in human bodies. Also, a crystalline curvature flow is proposed as a mathematical model of crystal growth that involves anisotropic curvatures and surface facets. These examples are known to have not only non-Euclidean structures but also singularities in metrics. Several equations with singularities are posed to describe these phenomena. In fact, there are already several known results on Hamilton-Jacobi equations on spaces without differential structures such as networks and fractals. On the other hand, to understand the crystalline curvature flow, it is known that we need to solve a very singular diffusion equation whose diffusion coefficient contains a non-local term like the Dirac delta function.

We now point out that the aforementioned works only establish the existence and uniqueness of solutions via the theory of viscosity solutions, but few of them are concerned with stability of the solutions. Note that the stability of solutions is a fundamental property of nonlinear equations as well as comparison principles. Roughly speaking, it claims that a uniform limit of a sequence of solutions to approximate equations is a solution of the original equation. It is worth remarking that stability results play an important role in the study of certain asymptotic problems including, as typical examples, homogenization and large time behavior of solutions.

This thesis gathers our new results on the stability and its applications to viscosity solutions of Hamilton-Jacobi equations on an abstract metric space and very singular diffusion equations. We also consider some other related topics such as a principle of Perron method to construct solutions and approximation of a solution that can be viewed as a dual problem of the stability.

This thesis consists of five main chapters. The first is an overview and the other four are devoted to details on each topic.

During my Ph.D. program I have met a number of researchers who support my mathematical work. My first and greatest gratitude goes to my supervisor, Professor Yoshikazu Giga, who guides me to the field of nonlinear partial differential equations. His erudition and insight about mathematics always inspire me very much. Without his consistent advice and encouragement, this thesis could have never been completed. I also send my gratitude to my colleagues Tokinaga Namba and Tatsuya Miura, with whom I have often discussed the theory of nonlinear partial differential equations and talked abound research problems. In addition, it is my great honer visiting Professor Juan Manfredi, Professor Piotr Rybka and Professor Antonio Siconolfi in short stays, where some of my works were completed. They all spared time hearing my talks and I learned a lot through the discussion with them. I am grateful to Professor Nao Hamamuki, Professor Hiroyoshi Mitake, Professor Norbert Požár, Professor Qing Liu and Professor Hung Tran. These young diligent researchers show their interests in my work and give me a lot of advice to improve my papers. At last, I thank Professor Hitoshi Ishii, Professor Mi-Ho Giga, Professor Jin Feng, Professor Katsuyuki Ishii, Professor Fabio Camilli, Professor Italo Capuzzo Dolcetta, Professor Cyril Imbert, Professor Emmanuelfor Nick Barron, Professor Naoki Yamada and Professor Masahiro Yamamoto for their valuable comments on my work.

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Tokyo, June 2015 *Atsushi Nakayasu*

Contents

Chapter 1

Overview

1.1 Nonlinear partial differential equations under non-Euclidean metrics

Hamilton-Jacobi equations are basic subjects in the theory of nonlinear partial differential equations, especially in viscosity solutions theory. Let us introduce two different forms of Hamilton-Jacobi equations. One is the linear growth Hamilton-Jacobi equation

$$\partial_t u + c(x)|Du| = 0 \quad \text{in } (0,\infty) \times \mathbf{R}^N, \tag{1.1.1}$$

where c is a positive function on $\mathbf{R}^N$. The other one is the quadratic Hamilton-Jacobi equation

$$\partial_t u + \frac{1}{2}|Du|^2 = V(x) \quad \text{in } (0,\infty) \times \mathbf{R}^N, \tag{1.1.2}$$

where V is a real-valued function on $\mathbf{R}^N$. The first equation (1.1.1) appears in surface evolution; we will explain it later. On the other hand, the second equation (1.1.2) is a typical partial differential equation describing a classical mechanics with conservative force; see [3]. We remark that both equations can be written in the common form

$$\partial_t u + H(x, Du) = 0 \quad \text{in } (0,\infty) \times \mathbf{R}^N. \tag{1.1.3}$$

Here, H is a real-valued function on $\mathbf{R}^N \times \mathbf{R}^N$ and it is called a Hamiltonian; in (1.1.1), $H(x,p) = c(x)|p|$ while $H(x,p) = |p|^2/2 - V(x)$ in (1.1.2).

A notion of viscosity solutions was introduced by Crandall and Lions in [20] and [7] as weak solutions of the Hamilton-Jacobi equations. The theory of viscosity solution was then extended to second order differential equations; see [18] and [6].

Now, let us consider an equation of the form

$$\partial_t u - |Du| \operatorname{div} \frac{Du}{|Du|} = 0 \quad \text{in } (0,\infty) \times \mathbf{R}^N, \tag{1.1.4}$$

called a (level set) mean curvature flow equation. We remark that this equation has a singularity at $Du = 0$ and thus the general results in [6] cannot be applied directly. Such equations were first studied independently by Evans-Spruck [8] and by Chen-Giga-Goto [4]. They established a comparison principle by introducing new notions of solutions so that singularity can be handled. Their notions in [8] and [4] are slightly different but it turns out they are equivalent [11].

We remark that the Hamilton-Jacobi equation (1.1.1) and the mean curvature flow equation (1.1.4) share a geometric background concerning surface evolution [11]. Imagine a moving curve Γ_t by the law $V = a\kappa + c$ on Γ_t, where V denotes a normal velocity, κ is the mean curvature of Γ_t and $a \geq 0$ is a given constant. When the curve is given as a level set of a function u, i.e. $\Gamma_t = \{x \in \mathbf{R}^N \mid u(t,x) = 0\}$, the function u satisfies the mean curvature flow equation (1.1.4) if $a = 1$ and the Hamilton-Jacobi equation (1.1.1) if $a = 0$.

In these years nonlinear equations under a singular metrics are proposed. Let us explain two directions which this thesis discusses. One is Hamilton-Jacobi equations on generalized spaces such as topological networks and post-critically finite fractals. It is often presented as a Hamilton-Jacobi equation on a metric space (X, d) and the viscosity-like solution is called a metric viscosity solution. Consider the equation

$$\partial_t u + H(x, |Du|) = 0 \quad \text{in } (0,\infty) \times X. \tag{1.1.5}$$

Here, H is a continuous convex Hamiltonian defined on $X \times \mathbf{R}_+$. A Hamilton-Jacobi equation on a general metric space was first studied in [17] and it inspired the further works [22], [2] and [10].

The other direction is crystalline curvature flow

$$V = \kappa_\gamma + c \quad \text{on } \Gamma_t, \tag{1.1.6}$$

where κ_γ is an anisotropic curvature related to the norm γ with singularity, say γ is the L^∞ norm. This kind of flow was proposed by Angenent, Gurtin and Taylor; see, e.g., [1], [25] and [26]. The corresponding level-set equation of planer crystalline curvature flow becomes

$$\partial_t u - \gamma(Du)[\operatorname{div} \nabla\gamma(Du) - c] = 0 \tag{1.1.7}$$

and moreover when the curve Γ_t is given by the graph of a function h the equation will be

$$\partial_t h - (1 + |h_x|)[2\delta(h_x)h_{xx} + c] = 0. \tag{1.1.8}$$

Here, δ is the Dirac delta function and the equations (1.1.7) and (1.1.8) are classified into very singular diffusion equations. The theory of viscosity solutions of such equations is introduced in the series of papers by Giga-Giga [12], [13], [14] and [16].

1.2 Abstract of each chapter

Chapter 2 is an introduction to a metric viscosity solution: The goal is to construct a proper notion of a solution to the Hamilton-Jacobi equation of the form (1.1.5) on a complete metric space (X, d) in spirit of Giga-Hamamuki-Nakayasu [17], which studies an Eikonal equation. Although the gradient Du of the unknown function u is not well-defined in spaces without tangent vector structure such as networks and fractals, the quantity corresponding to the modulus of gradient $|Du|$ can be characterized by a maximum of directional derivatives

$$|Du|(x,t) := \sup\left\{|w_s(0)| \mid w(s,t) = u(\xi(s),t),\ \xi \in Lip(\mathbf{R}),\ |\xi'| \le 1,\ \xi(0) = x\right\}.$$

This method is available at least to subsolutions and by defining the notion of supersolution based on optimality we establish a unique existence theorem for the Cauchy problem of the equation (1.1.5). In addition we will investigate the relationship between the metric viscosity solution of this chapter and the classical viscosity solution introduced by Crandall-Lions [7] when X is a Euclidean space. This chapter is based on [22].

In Chapter 3 we study the stability of a metric viscosity solution and its application to the Hamilton-Jacobi equation (1.1.5). We consider the notion of metric viscosity solutions by Gangbo-Swiech [9] and [10], which is based on the characterization of the modulus of gradient $|Du|$ by the local slope

$$|\nabla u|(t,x) := \limsup_{y \to x} \frac{|u(t,y) - u(t,x)|}{d(y,x)}.$$

In Section 3.3 we will show that the stability of the Gangbo-Swiech solutions holds under an assumption on maxima of the sequence of solutions. The additional condition always holds when the space X is locally compact and conversely the locally compactness is essential. In Section 3.4 we study the large time behavior of solutions to the convex coercive Hamilton-Jacobi equations following the argument in Namah-Roquejoffre [24] as an application of the stability. Let $u = u(t,x)$ be a solution of the equation (1.1.5) on a compact metric space including Sierpinski gasket. We will show that the function $u(t,x) + ct$ converges to a function $v = v(x)$ uniformly as $t \to \infty$ with $c := \sup_x H(x,0)$ and that v is a solution of the limit stationary equation

$$H(x, |Dv|) = c \quad \text{in } X.$$

In Chapter 4 we consider the (additive) eigenvalue problem of the form

$$H(x, Du) = c \quad \text{in } \mathbf{T}^N$$

for a Hamiltonian $H\colon \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$. This kind of eigenvalue problem appears in studies on the homogenization problem and on the large time behavior of solutions and it is important to measure the eigenvalue from application point of view. It is well-known by Contreras-Iturriaga-Paternain-Paternain [5] that if the Hamiltonian is convex, then eigenvalues have the inf-sup type representation formulas

$$c = \inf_{u \in C^1(\mathbf{T}^N)} \sup_{\nabla u} H = \inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H.$$

Here, ∇u denotes the graph of differential of u in the classical sense. It is known that the problem results in approximation of a Lipschitz continuous viscosity solution u with smooth functions u_n and Friedrichs mollifier justify it in the convex Hamiltonian case. In this work we will show the same result for quasiconvex Hamiltonians by replacing the Jensen's inequality in the proof of previous work with a fundamental inequality for quasiconvex functions. We will also give another new proof in this chapter. As an analogue of the fact that the stability of viscosity solution can be transformed to the problem on the graph of differentials, we establish the approximation of a viscosity solution on the level of the graph, which is a stronger result. Namely, one is able to show that if a sequence of $(x_n, p_n) \in \nabla u_n$ converges, then the limit (x, p) is in the graph of the Clarke's generalized gradients ∂u. This chapter is based on [23].

In Chapter 5 we study a one-dimensional very singular diffusion equation of the form

$$u_t = a(u_x)[(W'(u_x))_x + \sigma(t,x)], \tag{1.2.1}$$

where W is a convex function not necessarily differentiable at some point on $\mathbf{R}$ and σ is a sufficiently smooth function. Note that this equation can be derived when the planer curve moved by crystalline curvature flow and that it requires a technique to study it since the equation contains a non-local quantity $\delta(u_x)$ caused by the lack of the smoothness of W. The idea by Giga-Giga [12] to solve it is to consider the energy functional

$$\Phi[f] = \int (W(f_x) - \sigma f)\,dx$$

and to view the equation (1.2.1) as

$$u_t + F(t, u_x, -\nabla\Phi[u]) = 0$$

with some continuous function F. The authors of [12] introduced a generalized solution of this equation by combining the theories of viscosity solutions and subdifferentials. The case when σ does not depend on the spatial variable x was studied in [12] and [13] while a comparison principle was established in [16] when σ depends on x. In this work we study the stability analysis of a solution of the very singular diffusion equation (1.2.1) with spatially inhomogeneous σ. We point out that the difficulty of this problem lies on the restriction of test functions of the viscosity solutions. In other words, the test functions must be in the domain of the subdifferential of Φ and it must be flat at points on which the slope is singular. Moreover, in the spatially inhomogeneous case, the very singular diffusion term cannot be calculated explicitly. In this work we investigate the corresponding obstacle problem and find an effective region which determines the quantity of the non-local curvature. As a result we construct by the Perron method a solution of the Cauchy problem with a periodic boundary condition. This chapter is based on [15].

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Chapter 2

Metric viscosity solutions for Hamilton-Jacobi equations

2.1 Introduction

This chapter studies Hamilton-Jacobi equations of evolution type defined in a general metric space $(\mathcal{X}, d)$. One of the most simple problem is the fully nonlinear equation of the form

$$\partial_t u + |Du| = f(x) \quad \text{in } \mathcal{X} \times (0, T) \tag{2.1.1}$$

for a given bounded continuous function f and an unknown function $u = u(x,t)$ on $\mathcal{X} \times (0,T)$ with $T > 0$; let $\partial_t u$ denote the derivative with respect to the time variable t and $|Du|$ formally denote the modulus of gradient in the space variable x with $\mathbf{R}_+ = [0, \infty)$ values in the sense of [13]. That is,

$$|Du|(x,t) := \sup_{\xi \in Lip_x^1(\mathcal{X})} \left\{ |w_s(0)| \mid w(s,t) = u(\xi(s), t) \right\}, \tag{2.1.2}$$

where $Lip_x^1(\mathcal{X})$ is the set of all absolutely continuous curves $\xi : \mathbf{R} \to \mathcal{X}$ satisfying

$$|\xi'|(t) := \lim_{s \to t} \frac{d(\xi(s), \xi(t))}{|s-t|} \leq 1 \quad \text{a.e. } t$$

and $\xi(0) = x$. Note that the metric derivative $|\xi'|$ is defined for an absolutely continuous curve ξ in a metric space; see [2, Chapter 1]. However, since a metric space has no tangent space structure in general, the quantity corresponding to the derivative ξ' and the gradient Du is not well-defined.

Hamilton-Jacobi equations are fundamental in various fields of mathematics and physics and there are many works studying the equation including (2.1.1). We point out that the theory of viscosity solutions is successful for Hamilton-Jacobi equations defined on a Euclidean space, which is introduced by Crandall and Lions [17, 4]. This theory is extended to Banach spaces [5, 6, 7, 8]; this extension is expected to be useful for discussing an optimal control problem with respect to partial differential equations; we refer the reader to [9] and itself studies a resolvent problem of Hamilton-Jacobi equations in generalized spaces. Hamilton-Jacobi equations also appear in the optimal transport theory [19, Chapter 7, 22, 30] and the equation in Wasserstein spaces is studied by Gangbo, Nguyen and Tudorascu [10]. We also note that study on Hamilton-Jacobi equations on a space with junctions such as a network helps considering the LWR model of traffic flows; see [16] and [18]. Such a problem on a space with junctions is studied in [15, 14].

In order to handle Hamilton-Jacobi equations in such generalized spaces, Giga, Hamamuki and Nakayasu introduced a notion of a viscosity solution of Eikonal equation in [13]. We study time evolution equations in the present work.

We establish a unique existence theorem for an initial value problem of the Hamilton-Jacobi equation. Consider the value function of an optimal control problem

$$U(x,t) = \inf_{\xi \in Lip_x^1(\mathcal{X})} \left\{ \int_0^t f(\xi(r))\,dr + u_0(\xi(t)) \right\},$$

where u_0 is a bounded uniformly continuous function. Then, this value function U formally solves the Hamilton-Jacobi equation (2.1.1) with an initial condition $U(x,0) = u_0(x)$. It is remarkable that the value function U satisfies

$$U(x,t) = \inf_{\xi \in Lip_x^1(\mathcal{X})} \left\{ \int_0^h f(\xi(r))\,dr + U(\xi(h), t-h) \right\},$$

which is called a dynamic programming principle; see, e.g. [3].

We define a notion of a subsolution and a supersolution based on the dynamic programming principle: A function u is a subsolution if $w(h) := u(\xi(h), t-h)$ satisfies

$$-w'(h) \le f(\xi(h))$$

in the viscosity sense for all curves ξ called admissible while u is a supersolution if there exists an admissible curve ξ such that

$$-w'(h) \ge f(\xi(h))$$

holds in the viscosity sense with some function w approximating $h \mapsto u(\xi(h), t-h)$; see Definition 2.2.7. Then, we have the unique existence theorem easily. However, it is not clear how this definition relates to the original equation (2.1.1). We show that our subsolution is equivalent to a subsolution of (2.1.1) in the sense of [13]. However, it seems to be difficult to show the similar statement for a supersolution.

We point out that closely related topics have studied by Ambrosio and Feng [1] and of Gangbo and Swiech [11, 12]. They study the Hamilton-Jacobi equations including (2.1.1) in a complete geodesic metric space. However, our theory is applicable to any spaces with metric structure.

This chapter is organized as follows. In Section 2.2 we prepare to handle generalized Hamilton-Jacobi equations in a metric space and define a class of admissible curves and a sub- and supersolution based on the dynamic programming. In Section 2.3 we prove some equivalent conditions for a subsolution and a supersolution. The unique existence theorem will be shown in Section 2.4.

2.2 Definition of solutions

Consider the Hamilton-Jacobi equation of the form

$$\partial_t u + H(x, |Du|) = 0 \quad \text{in } \mathcal{X} \times (0, T) \tag{2.2.1}$$

with a function $H = H(x, p) : \mathcal{X} \times \mathbf{R}_+ \to \mathbf{R}$ satisfying:

(A1) $H = H(x, p)$ is a continuous function in $\mathcal{X} \times \mathbf{R}_+$.

(A2) H is convex and nondecreasing with respect to the variable p for each $x \in \mathcal{X}$.

Define the function

$$L(x, v) = \sup_{p \in \mathbf{R}_+} (pv - H(x, p)) \in \mathbf{R} \cup \{\infty\} \quad \text{for } (x, v) \in \mathcal{X} \times \mathbf{R}_+.$$

We then see that

Proposition 2.2.1. *Assume (A1) and (A2). Then, the function $L = L(x, v)$ is lower semicontinuous in $\mathcal{X} \times \mathbf{R}_+$ and it is also convex and nondecreasing with respect to the variable v for each $x \in \mathcal{X}$. In addition, the equations*

$$H(x, p) = \sup_{v \in \mathbf{R}_+} (pv - L(x, v)) \quad \textit{for all } (x, p) \in \mathcal{X} \times \mathbf{R}_+, \tag{2.2.2}$$

$$H(x, |p|) = \sup_{v \in \mathbf{R}} (pv - L(x, |v|)) \quad \textit{for all } (x, p) \in \mathcal{X} \times \mathbf{R} \tag{2.2.3}$$

hold.

Proof. Since (A1) shows that $(x, v) \mapsto pv - H(x, p)$ is continuous for each $p \in \mathbf{R}_+$, we see that the supremum L is lower semicontinuous. We also see that $v \mapsto L(x, v)$ is convex and nondecreasing since $v \mapsto pv - H(x, p)$ is affine and nondecreasing for each $(x, p) \in \mathcal{X} \times \mathbf{R}_+$.

Show the equation (2.2.3). First we easily see that

$$L(x, |v|) = \sup_{p \in \mathbf{R}} (pv - H(x, |p|))$$

and hence

$$H(x, |p|) \geq pv - L(x, |v|) \tag{2.2.4}$$

for all $x \in \mathcal{X}$, $p, v \in \mathbf{R}$. Therefore, it suffices to show that the equality of (2.2.4) holds for some $v \in \mathbf{R}$. Note that $p \mapsto H(x, |p|)$ is convex in $\mathbf{R}$ by (A2). Thus, for each $p \in \mathbf{R}$ there exists $v \in \mathbf{R}$ such that

$$H(x, |q|) \geq v(q - p) + H(x, |p|) \quad \text{for all } q \in \mathbf{R},$$

which implies

$$pv - H(x, |p|) \geq \sup_{q \in \mathbf{R}} (vq - H(x, |q|)) = L(x, |v|).$$

We hence have (2.2.3). The equation (2.2.2) follows from (2.2.3). □

We will also assume:

(A3) $c_H(p) := \sup_{x\in\mathcal{X}} H(x,p) < \infty$ for each $p \in \mathbf{R}_+$.

(A4) $\inf_{x\in\mathcal{X}} H(x,0) > -\infty$ and

$$\liminf_{p\to\infty} \inf_{x\in\mathcal{X}} \frac{H(x,p)}{p} > 0.$$

(A5) L is continuous on $D_L := \{(x,v) \in \mathcal{X} \times \mathbf{R}_+ \mid L(x,v) < \infty\}$. A map $x \mapsto V_L(x) := \sup\{v \in \mathbf{R}_+ \mid (x,v) \in D_L\}$ is lower semicontinuous on $\mathcal{X}$.

Remark 2.2.2. The condition (A3) implies that

(A3)' $L(x,v) \geq -c_H(0) > -\infty$ for all $(x,v) \in \mathcal{X}\times\mathbf{R}_+$ and $\ell(v) := \inf_{x\in\mathcal{X}} L(x,v)$ satisfies

$$\liminf_{v\to\infty} \frac{\ell(v)}{v} = \infty. \tag{2.2.5}$$

Indeed, by definition we have

$$L(x,v) \geq pv - H(x,p) \geq pv - c_H(p) \quad \text{for all } p \in \mathbf{R}_+,$$

which shows $L(x,v) \geq -c_H(0)$ and $\liminf_{v\to\infty} \ell(v)/v \geq p$.

We also see that the condition (A4) implies that

(A4)' there exists $V_L > 0$ such that $\sup_{x\in\mathcal{X}} L(x,V_L) < \infty$

if (A2) holds. Indeed, note that there exist $V > 0$ and $P \in \mathbf{R}_+$ such that $H(x,p) \geq pV$ for all $(x,p) \in \mathcal{X} \times [P,\infty)$ and that $H(x,p) \geq H(x,0) \geq \inf_{x\in\mathcal{X}} H(x,0) =: C$. Hence, we see that

$$\begin{aligned} L(x,v) &= \max\{\sup_{p\leq P}(pv - H(x,p)), \sup_{p\geq P}(pv - H(x,p))\} \\ &\leq \max\{(Pv - C), \sup_{p\leq P} p(v - V)\}, \end{aligned}$$

and so $L(x,V) \leq PV - C < \infty$.

We point out that the assumptions (A3) and (A4) can be replaced by (A3)' and (A4)' since we need only (A3)' and (A4)' on the main part of this chapter.

Example 2.2.3. Consider the Hamiltonian of the form

$$H(x,p) = \sigma(x)h(p) - f(x),$$

where h is a continuous, convex, nondecreasing, nonconstant function on $\mathbf{R}_+$; σ and f are bounded continuous functions on $\mathcal{X}$ with $\inf_x \sigma > 0$. Then the conditions (A1)–(A5) are fulfilled. The Lagrangian L becomes

$$L(x,v) = \sigma(x) l\left(\frac{v}{\sigma(x)}\right) + f(x)$$

with $l(v) = \sup_{p\in\mathbf{R}_+}(pv - h(p))$. Typical examples of such h and l include

$$h(p) = \frac{1}{2}p^2, \quad l(v) = \frac{1}{2}v^2,$$

and

$$h(p) = p, \quad l(v) = \begin{cases} 0 & \text{if } v \leq 1, \\ \infty & \text{if } v > 1. \end{cases}$$

We introduce a class of trajectories.

Definition 2.2.4. Let $AC(I, \mathcal{X})$ denote the set of all absolutely continuous curves in $\mathcal{X}$ defined on an interval I of $\mathbf{R}$.

A curve $\xi \in AC([0,\infty), \mathcal{X})$ is called *admissible* if there are finitely many $0 \leq r_1 \leq \cdots \leq r_n < \infty$ such that $|\xi'| = v_I$ a.e. on I and $r \mapsto L(\xi(r), v_I)$ is continuous on I with some constant $v_I \in \mathbf{R}_+$ for each $I = [0, r_1], [r_1, r_2], \cdots, [r_n, \infty)$.

Let $\mathcal{A}(\mathcal{X})$ be the set of all admissible curves and let $\mathcal{A}_x(\mathcal{X}) = \{\xi \in \mathcal{A}(\mathcal{X}) \mid \xi(0) = x\}$ for $x \in \mathcal{X}$.

Remark 2.2.5.

1. Consider a constant curve $\xi(r) = x$ for a fixed point $x \in \mathcal{X}$. Since $|\xi'| = 0$, we see that $\mathcal{A}_x(\mathcal{X})$ is nonempty for all $x \in \mathcal{X}$.

2. For each $\xi \in AC(\mathbf{R}, \mathcal{X})$ with $\xi(0) = x$ take $\hat{\xi} \in AC((L^-, L^+), \mathcal{X})$ in the next proposition. Set $\tilde{\xi}(r) = \hat{\xi}(V_L r)$ for $0 \leq r \leq L^+/V_L$ and $\tilde{\xi}(r) = \hat{\xi}(L_+)$ for $r \geq L^+/V_L$ with $V_L > 0$ in (A4)'. Then, $|\tilde{\xi}'| = V_L$ a.e. on $[0, L^+/V_L)$ and $|\tilde{\xi}'| = V_L$ on $[L^+/V_L, \infty)$. Since $L(\cdot, V_L)$ and $L(\cdot, 0)$ are continuous by (A4)' and (A5), we see that $\tilde{\xi} \in \mathcal{A}_x(\mathcal{X})$.

Proposition 2.2.6. *For $\xi \in AC(\mathbf{R}, \mathcal{X})$ set*

$$\tau_\xi(h) = \int_0^h |\xi'| dr \quad \text{for } h \in \mathbf{R}. \tag{2.2.6}$$

Then, there exists a curve $\hat{\xi} \in AC((L^-, L^+), \mathcal{X})$ such that

$$\xi = \hat{\xi} \circ \tau_\xi, \quad |\hat{\xi}'| = 1 \text{ a.e. in } (L^-, L^+) \tag{2.2.7}$$

with $L^\pm := \lim_{h \to \pm\infty} \tau_\xi(h)$.

This is a well-known fact on the absolutely continuous curves. We refer the reader to [2, Lemma 1.1.4] for its proof.

In order to define a notion of a solution, we recall a notion of a superdifferential and a subdifferential in the viscosity sense. For a continuous function w defined on an open set W in $\mathbf{R}^N$ define the *superdifferential* $D^+w(x)$ and the *subdifferential* $D^-w(x)$ at $x \in W$ as below:

$$\begin{aligned} D^+w(x) &:= \{D\varphi(x) \mid \varphi \text{ is a } C^1 \text{ supertangent of } w \text{ at } x\}, \\ D^-w(x) &:= \{D\varphi(x) \mid \varphi \text{ is a } C^1 \text{ subtangent of } w \text{ at } x\}, \end{aligned}$$

where we say that φ is a C^1 *supertangent* (resp. *subtangent*) of w at x if there exists a neighborhood $U \subset W$ of x such that $\varphi \in C^1(U)$ and

$$\max_U (w - \varphi) = (w - \varphi)(x). \quad (\text{resp. } \min_U (w - \varphi) = (w - \varphi)(x).)$$

As an analogue we define a suitable set of a superdifferential and a subdifferential for a piecewise continuous function w defined on an interval I in $\mathbf{R}$ at $h \in I$: Set

$$\begin{aligned} D^{+,r}w(h) &:= \{\varphi'(h+0) \mid \varphi \text{ is a piecewise } C^1 \text{ right supertangent of } w \text{ at } h\}, \\ D^{-,r}w(h) &:= \{\varphi'(h+0) \mid \varphi \text{ is a piecewise } C^1 \text{ right subtangent of } w \text{ at } h\}, \end{aligned}$$

where we say that φ is a *piecewise* C^1 *right supertangent* (resp. *subtangent*) of w at h if there exists $r > 0$ such that φ is piecewise C^1 on $[h, h+r)$ and

$$\max_{[h,h+r)}(w-\varphi) = (w-\varphi)(h). \quad \text{(resp. } \min_{[h,h+r)}(w-\varphi) = (w-\varphi)(h).)$$

We define a notion of a subsolution and a supersolution of the equation (2.2.1). Let $\mathcal{Q} := \mathcal{X} \times (0, T)$.

Definition 2.2.7. Let u be an arcwise continuous function in $\mathcal{Q}$; for every $\xi \in AC(\mathbf{R}, \mathcal{X})$ the function $(s,t) \mapsto u(\xi(s), t)$ is continuous in $\mathbf{R} \times (0,T)$.

We call u a *subsolution* of (2.2.1) if for each $(x,t) \in \mathcal{Q}$ and every $\xi \in \mathcal{A}_x(\mathcal{X})$ the inequality

$$-p \le L(\xi(h), |\xi'|(h+0)) \tag{2.2.8}$$

holds with $w(s,t) := u(\xi(s), t)$ for all $p \in D^{+,r}w(h)$ and all $h \in [0,t)$.

We call u a *supersolution* of (2.2.1) if for each $(x,t) \in \mathcal{Q}$ and $\varepsilon > 0$ there exist $\xi \in \mathcal{A}_x(\mathcal{X})$ and a continuous function w such that

$$w(0) = u(x,t), \quad w(h) \ge u(\xi(h), t-h) - \varepsilon \tag{2.2.9}$$

and the inequality

$$-p \ge L(\xi(h), |\xi'|(h+0)) \tag{2.2.10}$$

holds for all $p \in D^{-,r}w(h)$ and all $h \in [0,t)$.

Remark 2.2.8. The definition of admissible curves ξ nearly means that $|\xi'|$ is piecewise constant and $h \mapsto L(\xi(r), |\xi'|(r))$ is piecewise continuous. This notation is useful in combining two curves; if $\xi \in \mathcal{A}_x(\mathcal{X})$ and $\bar{\xi} \in \mathcal{A}_{\xi(h)}(\mathcal{X})$ with $h \in [0,\infty)$, then the curve $\tilde{\xi}$ defined by $\tilde{\xi}(r) = \xi(r)$ for $0 \le r \le h$ and $\tilde{\xi}(r) = \bar{\xi}(r-h)$ for $r \ge h$ belongs to $\mathcal{A}_x(\mathcal{X})$. Reflecting these piecewise conditions we need test all piecewise C^1 functions in Definition 2.2.7.

2.3 Remarks on the solution

The definition of a subsolution and a supersolution is based on a sub- and superoptimality principle. For simplicity write

$$L[\xi](r) := L(\xi(r), |\xi'|(r)).$$

The following propositions are valid:

Proposition 2.3.1. *For an arcwise continuous function u on $\mathcal{Q}$ the following conditions are equivalent:*

(i) u is a subsolution of (2.2.1).

(ii) u satisfies a suboptimality principle: For each $(x,t) \in \mathcal{Q}$ and each $\xi \in \mathcal{A}_x(\mathcal{X})$ the inequality

$$u(x,t) \le \int_0^h L[\xi]\,dr + u(\xi(h), t-h) \quad \textit{for all } h \in [0,t) \tag{2.3.1}$$

holds.

Proposition 2.3.2. *For an arcwise continuous function u on $\mathcal{Q}$ the following conditions are equivalent:*

(i) u is a supersolution of (2.2.1).

(ii) u satisfies a superoptimality principle: For each $(x,t) \in \mathcal{Q}$ and $\varepsilon > 0$ there exists $\xi \in \mathcal{A}_x(\mathcal{X})$ such that the inequality

$$u(x,t) \geq \int_0^h L[\xi]\,dr + u(\xi(h), t-h) - \varepsilon \quad \text{for all } h \in [0,t) \tag{2.3.2}$$

holds.

Proof of Proposition 2.3.1. First show (ii) $\Rightarrow$ (i). Fix $(x,t) \in \mathcal{Q}$, $\xi \in \mathcal{A}_x(\mathcal{X})$ and a piecewise C^1 right subtangent of $w(h) := u(\xi(h), t-h)$ at $h \in [0,t)$. Note that by considering $\tilde{\xi}(r) = \xi(r+h)$ in the suboptimality we have

$$\begin{aligned} w(h) = u(\xi(h), t-h) &\leq \int_0^\theta L[\tilde{\xi}]\,dr + u(\tilde{\xi}(\theta), t-h-\theta) \\ &= \int_h^{h+\theta} L[\xi]\,dr + w(h+\theta) \end{aligned}$$

for all $\theta \in [0, t-h)$. We hence obtain

$$\varphi(h) - \varphi(h+\theta) \leq w(h) - w(h+\theta) \leq \int_h^{h+\theta} L[\xi]\,dr,$$

which implies (2.2.8) with $p = \varphi'(h+0)$ since $L[\xi]$ is piecewise continuous. Therefore, u is a subsolution.

Next show (i) $\Rightarrow$ (ii). Fix $(x,t) \in \mathcal{Q}$ and $\xi \in \mathcal{A}_x(\mathcal{X})$, and let $w(h) := u(\xi(h), t-h)$. Note that $\int_0^h L[\xi]\,dr$ is piecewise C^1 on $[0,t)$ since $L[\xi]$ is piecewise continuous. Therefore,

$$\ell(h) := w(h) + \int_0^h L[\xi]\,dr$$

satisfies $D^{+,r}\ell(h) \subset [0,\infty)$ for all $h \in [0,t)$ since (2.2.8) holds for all

$$p \in D^{+,r}w(h) = D^{+,r}\ell(h) - L(\xi(h), |\xi'|(h+0)).$$

We now claim that $\ell(0) \leq \ell(h)$ for each $h \in [0,t)$. Suppose, on the contrary, that $\ell(0) > \ell(h)$ at some $h \in (0,t)$. Since $\ell(0) > \ell(0) - c > \ell(h)$ holds for some positive number c, the function

$$\ell(r) + \frac{c}{h} r$$

attains a maximum over $[0,h]$ at some $r^* \in [0,h)$. Hence, $-c/h \in D^{+,r}\ell(r^*)$. In view of $D^{+,r}\ell(h) \subset [0,\infty)$ we have $c \leq 0$, which contradicts to $c > 0$. Therefore, we see that $\ell(0) \leq \ell(h)$ and so (2.3.1) holds for all $h \in [0,t)$. $\square$

Proof of Proposition 2.3.2. First show (ii) $\Rightarrow$ (i). For $(x,t) \in \mathcal{Q}$ and $\varepsilon > 0$ take $\xi \in \mathcal{A}_x(\mathcal{X})$ such that (2.3.2) holds. Set

$$w(h) := u(x,t) - \int_0^h L[\xi]\,dr.$$

so that (2.2.9) holds. Since $L[\xi]$ is piecewise continuous, w is piecewise C^1 and

$$w'(h+0) = -L(\xi(h), |\xi'|(h+0)) \quad \text{for all } h \in [0,t),$$

which shows that u is a supersolution. Indeed, for each piecewise C^1 right subtangent φ of w at some $h \in [0,t)$, we see that

$$\varphi(h) - \varphi(h+\theta) \geq w(h) - w(h+\theta) = \int_h^{h+\theta} L[\xi]\,dr$$

for all $\theta > 0$ small enough, which implies (2.2.10) with $p = \varphi'(h+0)$. Therefore, u is a supersolution.

Next show (i) $\Rightarrow$ (ii). For each $(x,t) \in \mathcal{Q}$ and $\varepsilon > 0$ take $\xi \in \mathcal{A}_x(\mathcal{X})$ and a continuous function w such that (2.2.9) and (2.2.10) hold. Note that $\int_0^h L[\xi]\,dr$ is piecewise C^1 on $[0,t)$ since $L[\xi]$ is piecewise continuous. Therefore,

$$\ell(h) := w(h) + \int_0^h L[\xi]\,dr$$

satisfies $D^{-,r}\ell(h) \subset (-\infty, 0]$ for all $h \in [0,t)$ since (2.2.10) holds for all

$$p \in D^{-,r}w(h) = D^{-,r}\ell(h) - L(\xi(h), |\xi'|(h+0))$$

We now claim that $\ell(0) \geq \ell(h)$ for each $h \in [0,t)$. Suppose, on the contrary, that $\ell(0) < \ell(h)$ at some $h \in (0,t)$. Since $\ell(0) < \ell(0) + c < \ell(h)$ holds for some positive number c, the function

$$\ell(r) - \frac{c}{h}r$$

attains a minimum over $[0,h]$ at some $r^* \in [0,h)$. Hence, $c/h \in D^{-,r}\ell(r^*)$. In view of $D^{-,r}\ell(h) \subset (-\infty, 0]$ we have $c \leq 0$, which contradicts to $c > 0$. Therefore, we have $\ell(0) \geq \ell(h)$ so that

$$w(0) \geq w(h) + \int_0^h L[\xi]\,dr,$$

which yields (2.3.2) by (2.2.9) for all $h \in [0,t)$. □

Next let us consider relationship between a solution by Definition 2.2.7 and another one based on the characterization of the modulus of gradient (2.1.2). Set

$$LC_x^1(\mathcal{X}) = \{\xi \in AC([0,\infty), \mathcal{X}) \mid |\xi'| \leq 1,\ \xi(0) = x,\ |\xi'| \text{ is piecewise constant}\}.$$

Definition 2.3.3 (Metric viscosity solution)**.** Let u be an arcwise continuous function on $\mathcal{Q}$.

We call u a *metric viscosity subsolution* of (2.2.1) if for each $(x,t) \in \mathcal{Q}$ and every $\xi \in LC^1_x(\mathcal{X})$ the inequality

$$q + H(x, |p|) \le 0 \tag{2.3.3}$$

holds for all $(p,q) \in D^+_{s,t} w(0,t)$ with $w(s,t) = u(\xi(s),t)$, where

$$D^+_{s,t} w(s,t) := \{(\varphi_s, \varphi_t)(s,t) \mid \varphi \text{ is a } C^1 \text{ supertangent of } w \text{ at } (s,t)\}.$$

We then have

Proposition 2.3.4. *Assume (A1)–(A5) and let u be an arcwise continuous function on $\mathcal{Q}$. Then, the following conditions are equivalent:*

(i) u is a subsolution of (2.2.1).

(ii) u is a metric viscosity subsolution of (2.2.1).

(iii) u satisfies a suboptimality principle.

Proof. The statement (i) $\Leftrightarrow$ (iii) has already shown in Proposition 2.3.1.

Show (iii) $\Rightarrow$ (ii). Fix $(x,t) \in \mathcal{Q}$, $\xi \in LC^1_x(\mathbf{R}, \mathcal{X})$ and a C^1 supertangent φ of $w(s,t) = u(\xi(s),t)$ at $(0,t)$. In order to prove (2.3.3) we should show

$$q + |p|v - L(x,v) \le 0 \tag{2.3.4}$$

for all $v \in \mathbf{R}_+$. Note that (2.3.4) is trivial for $v > V_L(x)$, i.e. $L(x,v) = \infty$ and we only need to show (2.3.4) for all $v < V_L(x)$ since letting $v \to V_L(x)$ yields (2.3.4) at $v = V_L(x)$. Take $\sigma(r) = \pm vr$ so that $\xi \circ \sigma$ is v-Lipschitz. We now observe that

$$\begin{aligned}
\varphi(0,t) - \varphi(\sigma(h), t-h) &\le u(x,t) - u(\xi(\sigma(h)), t-h) \\
&\le \int_0^h L[\xi \circ \sigma]\, dr \\
&\le \int_0^h L(\xi(\sigma(r)), v)\, dr
\end{aligned}$$

for all $h \in [0,t)$ small enough. Since $v < V_L(x)$ it follows from (A5) that $r \in [0,t) \mapsto L(\xi(\sigma(r)), v)$ is continuous at 0. Therefore, we have

$$-p\sigma'(0) + q \le L(x,v)$$

which yields (2.3.4) and so (2.3.3) holds.

Next show (ii) $\Rightarrow$ (iii). First note that for any $\hat{\xi} \in LC^1_x(\mathcal{X})$ the function $w(s,t) = u(\hat{\xi}(s),t)$ satisfies

$$q + H(\xi(s), |p|) = \sup_{v \in \mathbf{R}} \{q + pv - L(\xi(s), |v|)\} \le 0 \tag{2.3.5}$$

for all $(p,q) \in D^+_{s,t} w(s,t)$ and all $(s,t) \in \mathbf{R} \times (0,T)$. Fix $(\hat{x}, \hat{t}) \in \mathcal{Q}$ and $\xi \in \mathcal{A}_{\hat{x}}(\mathcal{X})$. Take $\hat{\xi} \in LC^1_{\hat{x}}(\mathcal{X})$ and $\tau = \tau_\xi$ satisfying (2.2.6), (2.2.7). We show that

$$w(0,\hat{t}) \le w(\tau(h), \hat{t}-h) + \int_0^h L(\hat{\xi}(\tau(r)), \tau'(r))\, dr \quad \text{for all } h \in [0,\hat{t}) \tag{2.3.6}$$

Note that τ' is piecewise constant and $L[\xi \circ \tau]$ is piecewise continuous. Let us take an interval $I = (a, b)$ with $[a, b] \subset [0, \hat{t})$ on which τ' is constant $v \in \mathbf{R}_+$ and $r \mapsto L(\hat{\xi}(\tau(r)), v)$ is continuous. By (2.3.5)

$$q + pv - L(\hat{\xi}(s), v) \leq 0 \quad \text{for all } (p, q) \in D^+_{s,t} w(s, t),\ (s, t) \in J \times (0, T),$$

where $J = \tau(I)$ if $v > 0$ or $J = \mathbf{R}$ if $v = 0$. By a classical result on viscosity solutions,

$$w(s, t) \leq w(\lambda(h)) + \int_0^h L(\hat{\xi}(s + vr), v)\, dr$$

for all $(s, t) \in J \times (0, T)$ and $0 \leq h < h^* := \sup\{h \in [0, \infty) \mid \lambda(h) \in J \times (0, T)\}$ with $\lambda(h) = (s + vh, t - h)$. Note that $h^* \geq (b - a) \wedge t$. Letting $s \to \tau(a)$, $t = \hat{t} - a$, $h \to (b - a) \wedge t = b - a$, we have

$$w(\tau(a), \hat{t} - a) \leq w(\tau(b), \hat{t} - b) + \int_a^b L(\hat{\xi}(\tau(r)), v)\, dr.$$

Combining such an inequality shows (2.3.6), which implies (2.3.1). □

Remark 2.3.5. Unfortunately, we have no idea of a notion corresponding to "metric viscosity supersolutions" which is equivalent to the supersolution of Definition 2.2.7 at the present stage. It would contain "for each point (x, t)", "existence of a curve ξ" and "existence of an approximation w of $u(\xi(\cdot), \cdot)$". However, it is difficult to find suitable dependence. Any arguments about the equivalence does not work well for any choice of the dependence.

Remark 2.3.6. It is a natural question, from a partial differential equations point of view, to ask how our notion of solutions for the evolutionary equations relates to the solutions for stationary equations introduced in the previous work [13, Definition 2.1]. For simplicity, consider the stationary equation $|Dv| = f(x)$ in $\mathcal{X}$ with an unknown function v and a given non-negative continuous function f. Then, $u(x, t) := v(x) - ct$ is expected to be a solution of the evolutionary equation $\partial_t u + |Du| = f(x) - c$ in $\mathcal{X} \times (0, T)$. Indeed, Proposition 2.3.4 immediately shows that if v is a metric viscosity subsolution of the stationary equation in the sense of [13], then u is a metric viscosity subsolution. On the other hand, the assertion is still open that if $f(a) = 0$ at some $a \in \mathcal{X}$, and v is a metric viscosity supersolution of $|Dv| = f(x)$ in $\mathcal{X} \setminus \{a\}$, then u is a supersolution of the evolutionary equation.

At the end of this section we study the relationship between our definition of solutions and the classical notion of viscosity solutions introduced by Crandall and Lions [4]. Hereafter, let $\mathcal{X}$ be the Euclidean space $\mathbf{R}^N$ with the standard Euclidean norm.

Proposition 2.3.7. *Assume (A1)–(A5). Then, being a subsolution of* (2.2.1) *is equivalent to being a viscosity subsolution of* (2.2.1) *in Crandall-Lions sense and a supersolution of* (2.2.1) *is a viscosity supersolution of* (2.2.1) *in Crandall-Lions sense.*

This proposition can be proved by a similar argument to in [3, Section III.3], which considers optimal control problems.

Proof. First note that $\mathcal{A}_x(\mathcal{X})$ in the statements of Proposition 2.3.1 and 2.3.2 can be replaced by the set of all broken-lines starting from x; say $\mathcal{L}_x(\mathcal{X})$. Indeed, for each $\xi \in \mathcal{A}_x(\mathcal{X})$ there exists a sequence of $\xi_n \in \mathcal{L}_x(\mathcal{X})$ such that $|\xi_n' - \xi'| \leq 1/n$ on some closed set E_n with $|[0,t] \setminus E_n| \leq 1/n$ in view of Lusin's theorem. Here, $|E|$ is the Lebesgue measure of E. Note moreover that we may assume $|\xi_n'| = |\xi'|$ a.e. since $|\xi'|$ is piecewise constant. We then easily see that ξ_n converges to ξ uniformly on $[0,t]$ and

$$\int_0^t L[\xi_n]\,dr = \int_0^t L(\xi_n(r), |\xi'|(r))\,dr \to \int_0^t L[\xi]\,dr.$$

Therefore, our notion of sub- and supersolutions equivalent to the sub- and superoptimality for $\mathcal{L}_x(\mathcal{X})$, respectively.

Now, [3, Proposition II.5.18] implies that satisfying the suboptimality is equivalent to that u solves $\partial_t u + Du \cdot v \leq L(x, |v|)$ in Crandall-Lions sense for all $v \in \mathbf{R}^N$ and so u is a viscosity subsolution of (2.2.1) since $H(x, |p|) = \sup_{v \in \mathbf{R}^N}(p \cdot v - L(x, |v|))$ for all $p \in \mathbf{R}^N$.

In order to show the relationship between the notions of subsolutions, let u satisfy the superoptimality principle. Fix a C^1 subtangent φ of u at $(x,t) \in \mathcal{Q}$ and suppose that $\varphi_t + H(x, |D\varphi|) =: -\theta < 0$ at (x,t) with $\theta > 0$. We may assume that $(u - \varphi)(x,t) = \max_{\overline{B}(x,r_0) \times [t-r_0, t+r_0]}(u - \varphi)$ and $\varphi_t + H(x, |D\varphi|) \leq -\theta/2$ in $B(x, r_0) \times (t - r_0, t + r_0)$ by taking $r_0 > 0$ small enough. Here, $B(x,r)$ is the open ball with center x and radius $r > 0$ and $\overline{B}(x,r)$ is its closure. For each $\varepsilon > 0$ take the curve $\xi_\varepsilon \in \mathcal{L}_x(\mathcal{X})$ satisfying (2.3.2). Let h_ε be the first exit time of ξ_ε from the ball $B(x, r_0)$. We then see that

$$\varphi(x,t) \geq \int_0^h L[\xi_\varepsilon]\,dr + \varphi(\xi_\varepsilon(h), t-h) - \varepsilon \quad \text{for all } h \in [0, h_\varepsilon].$$

By putting $h = h_\varepsilon$, we have

$$\begin{aligned}\int_0^{h_\varepsilon} \ell(|\xi_\varepsilon'|)\,dr \leq \int_0^{h_\varepsilon} L[\xi_\varepsilon]\,dr &\leq \varphi(x,t) - \varphi(\xi_\varepsilon(h_\varepsilon), t - h_\varepsilon) + \varepsilon \\ &\leq C(|\xi_\varepsilon(h_\varepsilon) - x| + h_\varepsilon) + \varepsilon \leq 2Cr_0 + \varepsilon.\end{aligned}$$

Here, C is the Lipschitz constant of φ. Also,

$$\int_0^{h_\varepsilon} |\xi_\varepsilon'|\,dr \geq |\xi_\varepsilon(h_\varepsilon) - x| = r_0 > 0.$$

Therefore, we see by Lemma 2.4.4 that $\liminf_{\varepsilon \to 0} h_\varepsilon > 0$. Now, since φ satisfies $\partial_t \varphi + D\varphi \cdot v \leq L(x, |v|) - \theta/2$ for all $v \in \mathbf{R}^N$, [3, Proposition II.5.18] shows

$$\varphi(x,t) \leq \int_0^h L[\xi_\varepsilon]\,dr + \varphi(\xi_\varepsilon(h), t-h) - \theta h/2 \quad \text{for all } h \in [0, h_\varepsilon].$$

Hence, we have $0 \leq \varepsilon - \theta h_\varepsilon/2$. In particular, $h_\varepsilon \to 0$ as $\varepsilon \to 0$ but this contradicts to $\liminf_{\varepsilon \to 0} h_\varepsilon > 0$. □

On the other hand, it is unclear whether a Crandall-Lions supersolution satisfies the superoptimality in general settings. We need additional regularity condition for the solution. Indeed, if a bounded Lipschitz continuous function u on $\overline{\mathcal{Q}}$ solve (2.2.1) in Crandall-Lions sense then u satisfies our definition of solutions since u is equal to the value function starting from $u(\cdot, 0)$, which is the unique solution of (2.2.1).

2.4 Unique existence theorem

We study the initial value problem of (2.2.1) with

$$u|_{t=0} = u_0 \quad \text{on } \mathcal{X}, \tag{2.4.1}$$

where we assume that

(A6) u_0 is bounded and uniformly continuous on $\mathcal{X}$; there exists a modulus ω_{u_0} such that

$$|u_0(x) - u_0(y)| \le \omega_{u_0}(d(x,y)) \quad \text{for all } x, y \in \mathcal{X}.$$

Here, a modulus ω is a function of the class $C(\mathbf{R}_+, \mathbf{R}_+)$ with $\omega(0) = 0$.

The main purpose of this section is to establish a unique existence theorem for (2.2.1) and (2.4.1).

Definition 2.4.1. An arcwise continuous function u on $\mathcal{X} \times [0,T)$ is called a *solution* of the initial value problem (2.2.1) and (2.4.1) if u is both a subsolution and a supersolution of (2.2.1), and satisfies

$$u(0,x) = u_0(x) \quad \text{for all } x \in \mathcal{X}. \tag{2.4.2}$$

Consider the *cost functional* to the initial value problem (2.2.1) and (2.4.1);

$$C_t[\xi] := \int_0^t L[\xi]\,dr + u_0(\xi(t))$$

for $t \in [0,T]$ and $\xi \in \mathcal{A}_x(\mathcal{X})$. Define the *value function* U by

$$U(x,t) = \inf_{\xi \in \mathcal{A}_x(\mathcal{X})} C_t[\xi] \quad \text{for } (x,t) \in \mathcal{X} \times [0,T].$$

We will show that the value function is a unique solution of (2.2.1) and (2.4.1).

We first show a regularity of the value function.

Lemma 2.4.2. *Assume (A1)–(A6). Then the value function U satisfies*

$$-c_H(0)t + \inf u_0 \le U(x,t) \le tL(x,0) + u_0(x) \tag{2.4.3}$$

for all $(x,t) \in \mathcal{X} \times [0,T]$.

Proof. Let $(x,t) \in \mathcal{X} \times [0,T]$. Since $\xi_0(r) = x$ is of $\mathcal{A}_x(\mathcal{X})$, we have

$$C_t[\xi_0] = tL(x,0) + u_0(x) < \infty.$$

For each $\xi \in \mathcal{A}_x(\mathcal{X})$, the assumptions imply that

$$C_t[\xi] = \int_0^t L[\xi]\,dr + u_0(\xi(h)) \ge -c_H(0)t + \inf u_0 > -\infty.$$

Therefore, we have (2.4.3). ☐

Proposition 2.4.3. *Assume (A1)–(A6). Then the value function U is bounded and arcwise uniformly continuous on $\mathcal{X} \times [0,T]$, i.e. for each $\xi \in Lip^1(\mathcal{X})$ the function $w(s,t) = u(\xi(s),t)$ is uniformly continuous in $\mathbf{R} \times [0,T]$.*

Proof. First note that (2.4.3) implies

$$-|c_H(0)|T + \inf u_0 \le U(x,t) \le T|\sup L(x,0)| + \sup u_0. \tag{2.4.4}$$

In particular, U is a bounded function.

Fix $\xi \in AC(\mathcal{X})$. In order to show continuity of $(s,t) \mapsto U(\xi(s),t)$ let us estimate $U(\xi(s),t) - U(\xi(\bar{s}),\bar{t})$ for $s,\bar{s} \in \mathbf{R}$, $t,\bar{t} \in [0,T]$. Let $\varepsilon > 0$. By the definition of $U(\xi(\bar{s}),\bar{t})$ there exists a curve $\bar{\xi} \in \mathcal{A}_{\xi(\bar{s})}(\mathcal{X})$ such that

$$U(\xi(\bar{s}),\bar{t}) \ge \int_0^{\bar{t}} L[\bar{\xi}]\,dr + u_0(\bar{\xi}(\bar{t})) - \varepsilon. \tag{2.4.5}$$

We now construct a curve $\tilde{\xi} \in \mathcal{A}_{\xi(s)}(\mathcal{X})$ such that

$$\tilde{\xi}(r) = \bar{\xi}(r - r_0) \quad \text{for } r \ge r_0$$

with some $r_0 \ge 0$. Note that for such a curve

$$\begin{aligned} U(\xi(s),t) &\le \int_0^t L[\tilde{\xi}]\,dr + u_0(\tilde{\xi}(t)) \\ &= \int_0^{r_0} L[\tilde{\xi}]\,dr + \int_0^{t-r_0} L[\bar{\xi}]\,dr + u_0(\bar{\xi}(t-r_0)). \end{aligned} \tag{2.4.6}$$

Set

$$\tilde{\xi}(r) = \begin{cases} \hat{\xi}(\tau(s) + V_L \frac{\tau(\bar{s})-\tau(s)}{|\tau(\bar{s})-\tau(s)|} r) & \text{for } 0 \le r \le |\tau(\bar{s}) - \tau(s)|/V_L =: r_1 \\ \xi(\bar{s}) & \text{for } r_1 \le r \le r_1 + |\bar{t} - t| =: r_0 \end{cases}$$

with $\hat{\xi}$ and $\tau = \tau_\xi$ taken by Proposition 2.2.6. Then, noting that $|\tilde{\xi}'| = V_L$ on $[0,r_1]$ and $|\tilde{\xi}'| = 0$ on $[r_1, r_0]$, we have

$$R_1 := \int_0^{r_0} L[\tilde{\xi}]\,dr \le \int_0^{r_0} L(\tilde{\xi}(r), V_L)\,dr \le r_0 \sup_x L(x, V_L) \tag{2.4.7}$$

The inequalities (2.4.5)–(2.4.7) yields

$$U(\xi(s),t) - U(\xi(\bar{s}),\bar{t}) \le R_1 - R_2 + R_3 + \varepsilon, \tag{2.4.8}$$

where

$$R_2 := \int_{t-r_0}^{\bar{t}} L[\bar{\xi}]\,dr, \quad R_3 := u_0(\bar{\xi}(t-r_0)) - u_0(\bar{\xi}(\bar{t})).$$

Now noting that

$$t - r_0 = t - |\tau(\bar{s}) - \tau(s)|/V_L - |\bar{t} - t| \le \bar{t},$$

we hence see that

$$R_2 \ge -c_H(0)(\bar{t} - t + r_0). \tag{2.4.9}$$

We also have

$$R_3 \le \omega_{u_0}\left(\int_{t-r_0}^{\bar{t}} |\bar{\xi}'|\,dr\right). \tag{2.4.10}$$

holds in any cases. Combining (2.4.7)–(2.4.10), we have

$$U(\xi(s),t)-U(\xi(\bar{s}),\bar{t}) \leq r_0 \sup_x L(x,V_L)+c_H(0)(\bar{t}-t+r_0) + \omega_{u_0}\left(\int_{t-r_0}^{\bar{t}} |\bar{\xi}'|\,dr\right)+\varepsilon.$$

Note that

$$r_1 = |\tau(\bar{s})-\tau(s)|/V_L = \frac{1}{V_L}\left|\int_s^{\bar{s}} |\xi'|\,dr\right| \leq \frac{|\bar{s}-s|}{V_L} \to 0$$

as $|\bar{s}-s| \to 0$. Therefore, if we show

$$\lim_{\delta\downarrow 0} \sup_{|\bar{s}-s|+|\bar{t}-t|\leq\delta} \int_{t-r_0}^{\bar{t}} |\bar{\xi}'|\,dr = 0, \tag{2.4.11}$$

the proof is completed.

By (2.4.4) and (2.4.8) we observe that

$$\begin{aligned}
&\int_{t-r_0}^{\bar{t}} L(\bar{\xi}(r),|\bar{\xi}'|(r))\,dr \\
&\leq r_0 \sup_x L(x,V_L)+R_3+\varepsilon-U(\xi(s),t)+U(\xi(\bar{s}),\bar{t}) \\
&\leq 4\sup|u_0|+1+(\mathrm{Lip}[\xi]/V_L+1)|\sup_x L(x,V_L)|+T|\sup_x L(x,0)|+|c_H(0)|T,
\end{aligned}$$

for $\varepsilon<1$, $|\bar{s}-s|+|\bar{t}-t|<1$ and hence

$$\int_{t-r_0}^{\bar{t}} \ell(|\bar{\xi}'|)\,dr \leq C < \infty$$

holds with some constant C independent of $\bar{s}, s, \bar{t}, t$. Therefore, the next lemma shows (2.4.11). □

Lemma 2.4.4. *Let $\{v_n\}$ be a sequence of nonnegative, Lebesgue measurable functions v_n defined on $[0,s_n]$ with $s_n \downarrow 0$ such that*

$$\int_0^{s_n} \ell(v_n(r))\,dr \leq C \quad \textit{for } n \in \mathbf{N}$$

holds for some constant C. Then,

$$\int_0^{s_n} v_n(r)\,dr \to 0 \quad \textit{as } n\to\infty.$$

Proof. Note that (2.2.5) implies that for every large $M>0$ there exists $V_M \geq 0$ such that $\ell(v) \geq Mv$ holds for all $v \leq V_M$. We now observe that

$$\begin{aligned}
\int_0^{s_n} \ell(v_n(r))\,dr &= \int_{[0,s_n]\cap\{v_n\geq V_M\}} \ell(v_n(r))\,dr + \int_{[0,s_n]\cap\{v_n< V_M\}} \ell(v_n(r))\,dr \\
&\geq M\int_{[0,s_n]\cap\{v_n\geq V_M\}} v_n(r)\,dr - c_H(0)s_n.
\end{aligned}$$

We also see by this that

$$
\begin{aligned}
\int_0^{s_n} v_n(r)\,dr &= \int_{[0,s_n]\cap\{v_n\geq V_M\}} v_n(r)\,dr + \int_{[0,s_n]\cap\{v_n<V_M\}} v_n(r)\,dr \\
&\leq \frac{1}{M}\int_{[0,s_n]\cap\{v_n\geq V_M\}} \ell(v_n(r))\,dr + \frac{c_H(0)}{M}s_n + V_M s_n \\
&\leq \frac{C}{M} + \frac{c_H(0)}{M}s_n + V_M s_n.
\end{aligned}
$$

Therefore,

$$
\limsup_{n\to\infty}\int_0^{s_n} v_n(r)\,dr \leq \frac{C}{M}
$$

for all $M>0$, and so letting $M\to\infty$ implies the conclusion. ☐

Theorem 2.4.5. *Assume (A1)–(A6). Then the value function U is a solution of* (2.2.1)*,* (2.4.1)*.*

Proof. First show that U is a subsolution. Fix $(x,t)\in\mathcal{Q}$ and $\xi\in\mathcal{A}_x(\mathcal{X})$. Since there exists $\bar{\xi}\in\mathcal{A}_{\xi(h)}(\mathcal{X})$ for $h\in[0,t]$ and $\varepsilon>0$ such that

$$
U(\xi(h),t-h)\geq\int_0^{t-h} L[\bar{\xi}]\,dr + u_0(\bar{\xi}(t-h))-\varepsilon,
$$

taking $\tilde{\xi}(r)=\xi(r)$ for $0\leq r\leq h$, $\tilde{\xi}(r)=\bar{\xi}(r-h)$ for $r\geq h$ implies that

$$
\begin{aligned}
U(x,t) &\leq \int_0^t L[\tilde{\xi}]\,dr + u_0(\tilde{\xi}(t)) \\
&= \int_0^h L[\xi]\,dr + \int_0^{t-h} L[\bar{\xi}]\,dr + u_0(\bar{\xi}(t-h)).
\end{aligned}
$$

Combining these two inequalities yields

$$
U(x,t)\leq\int_0^h L[\xi]\,dr + U(\xi(h),t-h)+\varepsilon.
$$

Since ε is arbitrary, U satisfies a suboptimality and hence U is a subsolution by Proposition 2.3.1.

To prove that U is a supersolution, for $(x,t)\in\mathcal{Q}$ and $\varepsilon>0$ take $\xi\in\mathcal{A}_x(\mathcal{X})$ such that

$$
U(x,t)\geq\int_0^t L[\xi]\,dr + u_0(\xi(t))-\varepsilon.
$$

Since $\tilde{\xi}(r)=\xi(r+h)$ belongs to $\mathcal{A}_{\xi(h)}(\mathcal{X})$ for $h\in[0,t]$, we have

$$
U(\xi(h),t-h)\leq\int_h^t L[\xi]\,dr + u_0(\xi(t)).
$$

Combining these two inequalities yields

$$
U(x,t)\geq\int_0^h L[\xi]\,dr + U(\xi(h),t-h)-\varepsilon.
$$

Therefore, U satisfies a superoptimality and hence U is a supersolution by Proposition 2.3.2.

Since it is clear that U satisfies (2.4.2) by definition, we see that U is a solution. ☐

Remark 2.4.6. This proof also shows that a dynamic programming principle is valid for the value function:

$$U(x,t) = \inf_{\xi \in \mathcal{A}_x(\mathcal{X})} \left\{ \int_0^h L[\xi]\,dr + U(\xi(h), t-h) \right\} \quad \text{for all } h \in [0,t].$$

This condition also indicates that the value function satisfies a semigroup property.

We show a comparison theorem.

Theorem 2.4.7. *Assume (A1)–(A6). Assume that arcwise continuous functions u and v on $\mathcal{X} \times [0,T)$ are a subsolution and a supersolution, respectively. Then, the inequality*

$$\sup_{\mathcal{Q}} (u - v) \le \sup_{x \in \mathcal{X}} (u(x,0) - v(x,0)) \tag{2.4.12}$$

holds.

Proof. Fix $(x,t) \in \mathcal{Q}$, $\varepsilon > 0$ and $\xi \in \mathcal{A}_x(\mathcal{X})$ such that

$$v(x,t) \ge \int_0^h L[\xi]\,dr + v(\xi(h), t-h) - \varepsilon \quad \text{for all } h \in [0,t].$$

Note that $L[\xi]$ is piecewise continuous on $[0,t]$. Since $(s,t) \mapsto v(\xi(s),t)$ is continuous on $[0,t] \times [0,T)$ by the arcwise continuity of v, letting $h \to t$ we have

$$v(x,t) \ge \int_0^t L[\xi]\,dr + v(\xi(t), 0) - \varepsilon$$

We also see that

$$u(x,t) \le \int_0^h L[\xi]\,dr + u(\xi(h), t-h) \quad \text{for all } h \in [0,t),$$

which implies

$$u(x,t) \le \int_0^t L[\xi]\,dr + u(\xi(t), 0).$$

Combining the inequalities, we obtain

$$\begin{aligned} u(x,t) - v(x,t) &\le u(\xi(t),0) - v(\xi(t),0) + \varepsilon \\ &\le \sup_{x \in \mathcal{X}} (u(x,0) - v(x,0)) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, the proof is complete. □

In view of Theorem 2.4.5 and Theorem 2.4.7, the value function U is a unique solution of (2.2.1) and (2.4.1).

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Chapter 3

On asymptotic behaviors of metric viscosity solutions

3.1 Introduction

In this chapter we study stability of a solution of Hamilton-Jacobi equations on a generalized space. Let (X, d) be a general complete geodesic metric space and let H be a continuous function on $X \times \mathbf{R}_+$ called a *Hamiltonian*. Consider a Cauchy problem of a Hamilton-Jacobi equation of the form

$$\partial_t u + H(x, |Du|) = 0 \quad \text{in } (0, \infty) \times X \tag{3.1.1}$$

with the initial condition $u|_{t=0} = u_0$ and a corresponding stationary equation of the form

$$H(x, |Dv|) = c \quad \text{in } X \tag{3.1.2}$$

with some $c \in \mathbf{R}$.

The theory of Hamilton-Jacobi equations on generalized spaces have been developing in these years. For example, [17] and [3] study a stationary equations on topological networks and post-critically finite fractals including the Sierpinski gasket. In order to cover them a metric viscosity solution is posed, which means a theory of viscosity solutions on a general metric space. A notion of metric viscosity solution was first introduced by Giga-Hamamuki-Nakayasu [11] to the stationary equation (3.1.2) in a spirit of [17]. It was attempted to apply this idea to the evolutionary equation (3.1.1) in [15]. Afterwards some different notions of metric viscosity solutions were proposed by several authors; see, e.g., [1], [8] [9]. In particular, the metric viscosity solution by Gangbo-Swiech [8], [9] is apparently compatible with stability argument. In fact, the authors of [8] construct a solution of (3.1.1) by Perron method while the other materials show a representation formula of a metric viscosity solution.

The main aim of the present work is to establish a general stability result for the Gangbo-Swiech solutions. Roughly speaking, the stability is the proposition claiming that the semilimit of a family of viscosity solutions is a viscosity solution; see [2, Theorem A.2]. At least in the classical theory of viscosity solutions the stability is a fundamental property to derive some asymptotic behavior of the solution. Large time asymptotics of a viscosity solution of Hamilton-Jacobi equations is studied by Namah-Roquejoffre [16] and Fathi [6] independently. Another aim of this chapter is to establish, as a consequence of the stability, a large time asymptotic behavior of the solution on a singular space such as the Sierpinski gasket. Based on the argument in [16] we will show that the solution $u(t,x)+ct$ of (3.1.1) goes to a function v as $t \to \infty$ and v is a solution of the stationary problem (3.1.2) with some constant $c \in \mathbf{R}$.

We restrict ourselves to the case when the metric space X is compact but the Sierpinski gasket can be handled. Let us extend H to $X \times \mathbf{R}$ as an even function $H(x,p) = H(x,|p|)$. The basic assumptions on the Hamiltonian are:

(A1) H is continuous.

(A2) H is convex in the second variable.

(A3) H is coercive in the sense of

$$\lim_{p\to\infty} \inf_x H(x,p) = \infty.$$

(A4) $\sup_x H(x,0) < \infty$.

Set $c := \sup_x H(x,0)$; otherwise the stationary equation (3.1.2) has no solution. We first show that the stationary equation has at least one solution. Then, a standard barrier method implies that there exist upper and lower semilimits $\overline{u}(x)$ and $\underline{u}(x)$ of $u(t,x)+ct$ as $t \to \infty$ as real-valued functions. As a result the stability argument yields that $\overline{u}$ and $\underline{u}$ are a subsolution and a supersolution of the limit equation. Next note that for each $x \in A := \{x \in X \mid H(x,0) = \sup_x H(x,0)\}$ the solution $u(t,x)+ct$ is non-increasing since $u_t + c \le 0$ and so $\overline{u} = \underline{u}$ on A by Dini's theorem. We see that $\overline{u} = \underline{u}$ by a comparison principle for the stationary equation. This means that $u(t,x)+ct$ converges to a solution $\overline{u} = \underline{u}$ of the stationary equation locally uniformly.

In order to justify this argument we will establish solvability of (3.1.2) and a comparison principle for Gangbo-Swiech solutions of (3.1.2), which are new. The authors of [16] invoke a result by Lions-Papanicolaou-Varadhan [14]. The argument is based on the ergodic theory but in this work we will follow a direct approach via Perron method by Fathi-Siconolfi [7].

3.2 Definition of Gangbo-Swiech solutions

In this section we review the definition of metric viscosity solutions proposed by Gangbo and Swiech; see [8] and [9]. Let (X,d) be a complete geodesic metric space.

For a real-valued function u on an open subset Q of the spacetime $\mathbf{R} \times X$ define the *upper local slope* and *lower local slope*

$$|\nabla^+ u|(t,x) := \limsup_{y\to x} \frac{[u(t,y)-u(t,x)]_+}{d(y,x)},$$
$$|\nabla^- u|(t,x) := \limsup_{y\to x} \frac{[u(t,y)-u(t,x)]_-}{d(y,x)}$$

and the *local slope*

$$|\nabla u|(t,x) := \limsup_{y\to x} \frac{|u(t,y)-u(t,x)|}{d(y,x)}.$$

It is easy to see that $|\nabla^- u| = |\nabla^+(-u)|$.

We next introduce smoothness classes for functions on a metric space.

Definition 3.2.1. We denote by $\mathcal{C}(Q)$ the set of all functions u on Q such that u is locally Lipschitz continuous on Q and $\partial_t u$ is continuous on Q. We also set

$$\begin{aligned}
\overline{\mathcal{C}}^1(Q) &:= \{u \in \mathcal{C}(Q) \mid |\nabla^+ u| = |\nabla u| \text{ and they are continuous}\},\\
\underline{\mathcal{C}}^1(Q) &:= \{u \in \mathcal{C}(Q) \mid |\nabla^- u| = |\nabla u| \text{ and they are continuous}\}.
\end{aligned}$$

Lemma 3.2.2. *Let $u(t,x) := a(t)\phi(d(x,y)^2) + b(t)$ with $y \in X$, $\phi \in C^1(\mathbf{R}_+)$, $\phi' \geq 0$, $a, b \in C^1(\mathbf{R})$. Then, $u \in \underline{\mathcal{C}}^1(\mathbf{R}\times X)$ and moreover*

$$|\nabla^- u|(t,x) = |\nabla u|(t,x) = 2a(t)\phi'(d(x,y)^2)d(x,y).$$

See [1] or [8] for the proof.

We consider a Hamilton-Jacobi equation of the form

$$F(z, |Du|, \partial_t u) = 0 \quad \text{in } Q. \tag{3.2.1}$$

Here, $z = (t,x)$, and $F = F(z,p,q) \in C(Q\times\mathbf{R}\times\mathbf{R})$ is even and convex in p and strictly increasing in q. Set

$$F_r(z,p,q) := \begin{cases} \sup_{p'\in B} F(z, p+rp', q) & \text{if } r \geq 0 \\ \inf_{p'\in B} F(z,p+rp',q) & \text{if } r \leq 0 \end{cases}$$

for $r \in \mathbf{R}$, where $B := [-1,1]$. Note that $(z,p,q,r) \mapsto F_r(z,p,q)$ is continuous since $(z,p,q,r,p') \mapsto F(z,p+rp',q)$ is continuous and B is compact. Also, it is easy to check $r \mapsto F_r(z,p,q)$ is non-decreasing.

For a function u defined on Q with values in the extended real numbers $\bar{\mathbf{R}} := \mathbf{R}\cup\{\pm\infty\}$, we take its upper and lower semicontinuous envelope u^* and u_*.

Definition 3.2.3 (Metric viscosity solutions of (3.2.1))**.** Let u be an $\bar{\mathbf{R}}$-valued function on Q.

We say that u is a *metric viscosity subsolution* (resp. *supersolution*) of (3.2.1) when for every $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \underline{\mathcal{C}}^1(Q)$ (resp. $\psi_1 \in \overline{\mathcal{C}}^1(Q)$) and $\psi_2 \in \mathcal{C}(Q)$, if $u^* - \psi$ (resp. $u_* - \psi$) attains a zero local maximum (resp. minimum) at a point $z = (t,x) \in Q$, i.e. $(u^*-\psi)(z) = \max_{B_R(z)}(u^*-\psi) = 0$ (resp. $(u_*-\psi)(z) = \min_{B_R(z)}(u_*-\psi) = 0$) for some $R > 0$, then

$$F_{-|\nabla\psi_2|^*(z)}(z, |\nabla\psi_1|(z), \partial_t\psi(z)) \leq 0 \text{ (resp. } F_{|\nabla\psi_2|^*(z)}(z, |\nabla\psi_1|(z), \partial_t\psi(z)) \geq 0.)$$

We say that u is a *metric viscosity solution* of (3.2.1) if u is both a metric viscosity subsolution and a metric viscosity supersolution of (3.2.1).

By a similar way we also define a notion of metric viscosity solutions for a stationary equation of the form

$$H(x, |Dv|) = 0 \quad \text{in } U \tag{3.2.2}$$

with $U \subset X$ open. Here, $H = H(x,p) \in C(U \times \mathbf{R})$ is even and convex in p. Note that one is able to define the local slopes $|\nabla^- v|$, $|\nabla^+ v|$, $|\nabla v|$ and smoothness $\mathcal{C}(U)$, $\overline{\mathcal{C}}^1(U)$, $\underline{\mathcal{C}}^1(U)$ for a function v on U.

Definition 3.2.4 (Metric viscosity solutions of (3.2.2))**.** Let v be an $\bar{\mathbf{R}}$-valued function on U.

We say that v is a *metric viscosity subsolution* (resp. *supersolution*) of (3.2.2) when for every $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \underline{\mathcal{C}}^1(U)$ (resp. $\psi_1 \in \overline{\mathcal{C}}^1(U)$) and $\psi_2 \in \mathcal{C}(Q)$, if $v^* - \psi$ (resp. $v_* - \psi$) attains a zero local maximum (resp. minimum) at a point $x \in U$, then

$$H_{-|\nabla\psi_2|^*(x)}(x, |\nabla\psi_1|(x)) \leq 0 \text{ (resp. } H_{|\nabla\psi_2|^*(x)}(x, |\nabla\psi_1|(x)) \geq 0.)$$

We say that v is a *metric viscosity solution* of (3.2.2) if v is both a metric viscosity subsolution and a metric viscosity supersolution of (3.2.2).

These notions satisfies the following natural propositions.

Proposition 3.2.5 (Consistency)**.** *If u is a metric viscosity subsolution of* (3.2.1) *in $I \times U$ with an open interval I and u is of the form $u(t,x) = v(x)$, then v is a metric viscosity subsolution of* (3.2.2) *in U with $H(x,p) := F(0,x,p,0)$.*

Conversely, if v is a metric viscosity subsolution of (3.2.2) *in U, then $u(t,x) := v(x)$ is a metric viscosity subsolution of* (3.2.1) *in $\mathbf{R} \times U$ with $F(t,x,p,a) := H(x,p)$.*

Proposition 3.2.6 (Transitive relation)**.** *Assume that $F = F(z,p,q)$, $G = G(z,p,q)$ satisfy $G \leq F$ and let u be a metric viscosity subsolution of* (3.2.1)*. Then, u is a metric viscosity subsolution of $G(z, |Du|, \partial_t u) = 0$ in Q.*

Proposition 3.2.7 (Locality)**.** *Let Q_1 and Q_2 be two open subsets of $(0,\infty) \times X$. If u is a metric viscosity subsolution of* (3.2.1) *in Q_1 and is a metric viscosity subsolution of* (3.2.1) *in Q_2, then u is a metric viscosity subsolution of* (3.2.1) *in $Q = Q_1 \cup Q_2$*

Proposition 3.2.8 (Change of variable)**.** *Let ϕ be a C^1 diffeomorphism from an interval I to an interval J. If u is a metric viscosity subsolution of* (3.2.1) *in $I \times U$, then $v(s,x) := u(\phi^{-1}(s), x)$ is a metric viscosity subsolution of*

$$F(x, |Dv|, \phi' v_s) = 0 \quad \text{in } J \times U.$$

Proposition 3.2.9 (Composition)**.** *Let a be a non-zero constant and $b = b(t)$ be a C^1 function on an interval I. If u is a metric viscosity subsolution of* (3.2.1) *in $I \times U$, then $v(t,x) := au(t,x) + b(t)$ is a metric viscosity subsolution of*

$$F(x, \frac{|Dv|}{a}, \frac{v_t - b'(t)}{a}) = 0 \quad \text{in } I \times U.$$

Proposition 3.2.10 (Strong solutions)**.** *Let* $\psi_1 \in \overline{\mathcal{C}}^1(Q)$ *and* $\psi_2 \in \mathcal{C}(Q)$*. If* $\psi = \psi_1 + \psi_2$ *satisfies*

$$F_{|\nabla \psi_2|^*(z)}(z, |\nabla \psi_1|(z), \partial_t \psi(z)) \leq 0 \quad \textit{for all } z \in Q,$$

then ψ *is a metric viscosity subsolution of* (3.2.1)*.*

The proofs are straightforward so we omit them. For the proof of Proposition 3.2.10 see [9, Lemma 2.8].

3.3 Stability results

Let A be a topological space. For a family of functions $\{u(\cdot; a)\}_{a \in A}$ defined on Q take its upper and lower semicontinuous envelopes

$$u^*(z; a) := \limsup_{(z', a') \to (z, a)} u(z'; a'), \quad u_*(z; a) := \liminf_{(z', a') \to (z, a)} u(z'; a').$$

The functions $u^*(\cdot; a)$ and $u^*(\cdot; a)$ are respectively called the upper and lower semilimit of $\{u(\cdot; a)\}$ at $a \in A$. Also note that $u^*(\cdot; a)$ is upper semicontinuous and that for each (z, a) there exists a sequence (z_j, a_j) such that

$$(z_j, a_j, u(z_j; a_j)) \to (z, a, u^*(z; a)).$$

One of the main results of this section is:

Lemma 3.3.1 (Stability)**.** *Let* $F = F(z, p, q; a) \in C(Q \times \mathbf{R} \times \mathbf{R} \times A)$ *and let* $u = u(\cdot; a)$ *be a family of metric viscosity subsolutions (resp. supersolution) of* (3.2.1) *with* $F = F(\cdot; a)$*. Assume that* $a \in A$ *satisfies for each* $z \in Q$

$$\limsup_{a' \to a} \sup_{B_r(z)} u(\cdot; a') \leq \sup_{B_r(z)} u^*(\cdot; a) \ \textit{(resp.} \liminf_{a' \to a} \inf_{B_r(z)} u(\cdot; a') \leq \inf_{B_r(z)} u_*(\cdot; a)\textit{)} \tag{3.3.1}$$

for all $r > 0$ *small enough. Then, the upper (resp. lower) semilimit* $\overline{u} := u^*(\cdot, a)$ *(resp.* $\underline{u} := u^*(\cdot, a)$*) is a metric viscosity subsolution (resp. supersolution) of* (3.2.1) *with* $F = F(\cdot; a)$*.*

Remark 3.3.2. An sufficient condition of the assumption (3.3.1) is that the metric space X is locally compact. Indeed, since $B := \overline{B}_r(z)$ is compact for small r, we are able to take a sequence of maximum points $z_{a'}$ of $u(\cdot; a')$ and assume that $z_{a'}$ converges to some $\bar{z} \in B$ as $a' \to a$ by taking a subsequence if necessary. Then,

$$\limsup_{a' \to a} \sup_B u(\cdot; a') = \limsup_{a' \to a} u(z_{a'}; a') \leq u^*(\bar{z}; a) \leq \sup_B u^*(\cdot; a).$$

We also point out that if the assumption is removed, then the lemma may be false in general; see [11].

A direct consequence of Lemma 3.3.1 is:

Corollary 3.3.3 (Stability under extremum)**.** *Let* $F = F(z, p, q) \in C(Q \times \mathbf{R} \times \mathbf{R})$*. Let* S *be a family of metric viscosity subsolutions (resp. supersolutions) of* (3.2.1)*. Then* $\overline{u}(z) := \sup_{v \in \mathcal{S}} v(z)$ *(resp.* $\underline{u}(z) := \inf_{v \in \mathcal{S}} v(z)$*) is a metric viscosity subsolution (resp. supersolutions) of* (3.2.1)*.*

Proof. Set $A = S$ with the indiscrete topology and trivial families $\{F\}_{v\in S}$ and $\{U(\cdot;v) = v\}_{v\in S}$. Note that $U^*(z;v) = u^*(z)$ and $\limsup_{v'\to v} \sup_{B_r(z)} U(\cdot;v') = \sup_{B_r(z)} U^*(\cdot;v) = \sup_{B_r(z)} u^*$. Therefore, by applying Lemma 3.3.1 we see that u is a metric viscosity subsolution of (3.2.1). □

Our proof of Lemma 3.3.1 is inspired by [8]. First recall Ekeland's variational principle of a classical version [4], [5].

Lemma 3.3.4 (Ekeland's variational principle)**.** *Let (X,d) be a complete metric space and let $F\colon X\to \bar{\mathbf{R}}$ be a upper semicontinuous function bounded from above (resp. below) satisfying $D(F) := \{F > -\infty\}$ (resp. $D(F) := \{F < +\infty\}$) is not empty. Then, for each $\hat{x} \in D(F)$, there exists $\bar{x} \in X$ such that $d(\hat{x},\bar{x}) \le 1$, $F(\bar{x}) \ge F(\hat{x})$ (resp. $F(\bar{x}) \le F(\hat{x})$) and $x \to F(x) - md(\bar{x},x)$ attains a strict maximum (resp. minimum) at $\bar{x}$ with $m := \sup F - F(\hat{x})$ (resp. $m := \inf F - F(\hat{x})$).*

See [5] for the proof.

Proof of Lemma 3.3.1. Fix $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \underline{\mathcal{C}}^1(Q)$ and $\psi_2 \in \mathcal{C}(Q)$ such that $\overline{u} - \psi$ attains a zero maximum at $\hat{z} = (\hat{t},\hat{x})$ over $\overline{B}_R(\hat{z}) \subset Q$ with some $R > 0$, i.e.

$$(\overline{u} - \psi)(\hat{z}) = \sup_{\overline{B}_R(\hat{z})} (\overline{u} - \psi) = 0. \tag{3.3.2}$$

Set $\tilde{\psi}_2(z) := \psi_2(z) + d(\hat{x},x)^2 + (t-\hat{t})^2$ with $z = (t,x)$ and $\tilde{\psi} = \psi_1 + \tilde{\psi}_2$.

Take a subsequence $a_j \to a$ and a sequence of points $z_j = (t_j,x_j) \in \overline{B}_R(\hat{z})$ such that $z_j \to \hat{z}$ and $u_j(z_j) = u(z_j;a_j) \to \overline{u}(\hat{z}) = u^*(\hat{z};a)$, where $u_j := u(\cdot;a_j)$. We see by Ekeland's variational principle (Lemma 3.3.4) that there exists $w_j = (s_j,y_j) \in \overline{B}_R(\hat{z})$ such that $z = (t,x) \mapsto ((u_j)^* - \tilde{\psi})(z) - m_j d(y_j,x)$ attains maximum at w_j over $\overline{B}_R(\hat{z})$ with

$$m_j := \sup_{\overline{B}_R(\hat{z})} ((u_j)^* - \tilde{\psi}) - ((u_j)^* - \tilde{\psi})(z_j) \ge 0$$

Note that

$$\limsup_{j\to\infty} m_j = \limsup_{j\to\infty} \sup_{\overline{B}_R(\hat{z})} ((u_j)^* - \tilde{\psi}) - (\overline{u} - \tilde{\psi})(\hat{z})$$

and so $m_j \to 0$ by the assumptions (3.3.1) and (3.3.2). We also observe that

$$((u_j)^* - \tilde{\psi})(w_j) \ge (u_j - \tilde{\psi})(z_j) - m_j d(y_j,x_j) \ge (u_j - \tilde{\psi})(z_j) - 2Rm_j$$

and that the last term converges to $(\overline{u} - \tilde{\psi})(\hat{z})$ as $j \to \infty$. Therefore,

$$\begin{aligned}\limsup_{j\to\infty} d(\hat{x},y_j)^2 + (s_j - \hat{t})^2 &\le \limsup_{j\to\infty} ((u_j)^* - \psi)(w_j) - (\overline{u} - \tilde{\psi})(\hat{z}) \\ &\le \limsup_{j\to\infty} \sup_{\overline{B}_R(\hat{z})} ((u_j)^* - \psi) - (\overline{u} - \tilde{\psi})(\hat{z})\end{aligned}$$

and it follows from (3.3.1) and (3.3.2) that $w_j = (s_j,y_j) \to \hat{z} = (\hat{t},\hat{x})$.

Now, since u_j is a metric viscosity subsolution,

$$F_{-r_j}(w_j, |\nabla\psi_1|(w_j), \partial_t\tilde{\psi}(w_j); a_j) \le 0.$$

Here, r_j is some non-negative number such that

$$r_j \le |\nabla\psi_2|^*(w_j) + m_j + 2d(\hat{x}, y_j) + 2|s_j - \hat{t}|$$

and so $\limsup r_j \le |\nabla\psi_2|^*(\hat{z})$. Since $(z, p, q, r; a) \to F_r(z, p, q; a)$ is continuous and $r \to F_r(z, p, q; a)$ is non-decreasing, we see that

$$F_{-|\nabla\psi_2|^*(\hat{z})}(\hat{z}, |\nabla\psi_1|(\hat{z}), \partial_t\psi(\hat{z}); a) \le 0.$$

Therefore, u is a subsolution. $\square$

Another goal of this section is a principle to construct a metric viscosity solution by the Perron method.

Proposition 3.3.5 (Perron method)**.** *Let $F = F(z, p, q) \in C(Q \times \mathbf{R} \times \mathbf{R})$ and let g be an $\bar{\mathbf{R}}$-valued function on ∂Q. Let $\mathcal{S}$ denote the set of all metric viscosity subsolutions (resp. supersolution) v of* (3.2.1) *satisfying $v^* \le g$ (resp. $v_* \le g$) on ∂Q. Then, $u(z) := \sup_{v\in\mathcal{S}} v(z)$ (resp. $u(z) := \inf_{v\in\mathcal{S}} v(z)$) is a metric viscosity solution of* (3.2.1)*.*

Perron method for construction of a viscosity solution to Hamilton-Jacobi equations was first presented by H. Ishii [13]. Actually, the authors of [8] have already established a similar result for metric viscosity solutions ([8, Theorem 7.6]). However, let us give a proof since we have slightly improved the result to apply it directly to construction of a solution of the limit equation (3.1.2). We remark that the function

$$\overline{u}(x) := \begin{cases} +\infty & \text{if } x \in Q \\ g(x) & \text{if } x \in \partial Q \end{cases}$$

is a supersolution of (3.2.1).

Proof. We only show that u is a supersolution since being a subsolution is due to Corollary 3.3.3. Fix $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \overline{\mathcal{C}}^1(Q)$ and $\psi_2 \in \mathcal{C}(Q)$ such that $u_* - \psi$ attains a zero minimum at $\hat{z} := (\hat{t}, \hat{x})$ over $\overline{B}_R(\hat{z}) \subset Q$ with some $R > 0$. Set $\tilde{\psi}_2(z) := \psi_2(z) - d(\hat{z}, z)^2$ and $\tilde{\psi} = \psi_1 + \tilde{\psi}_2$. Suppose by contradiction that

$$F_{|\nabla\psi_2|^*(\hat{z})}(\hat{z}, |\nabla\psi_1|(\hat{z}), \partial_t\psi(\hat{z})) < 0.$$

Since $(z, p, q, r) \mapsto F_r(z, p, q)$ is continuous and $r \mapsto F_r(z, p, q)$ is non-decreasing, we may assume that $\tilde{\psi} = \psi_1 + \tilde{\psi}_2$ is a subsolution of

$$F_{|\nabla\tilde{\psi}_2|^*(z)}(z, |\nabla\psi_1|(z), \partial_t\tilde{\psi}(z)) \le 0 \quad \text{for all } z \in B_R(\hat{z})$$

by taking R small enough. Recalling Proposition 3.2.10, we see that $\tilde{\psi}$ is a metric viscosity subsolution of (3.2.1) in $B_R(\hat{z})$. Now observe that

$$(u - \tilde{\psi})(z) \ge (u_* - \tilde{\psi})(z) \ge d(\hat{z}, z)^2 \ge \frac{R^2}{4} =: m > 0$$

for all $z \in \overline{B}_R(\hat{z}) \setminus B_{R/2}(\hat{z})$. Construct a new function

$$w(z) = \begin{cases} \max\{\tilde{\psi}(z) + m/2, u(z)\} & \text{if } z \in B_R(\hat{z}), \\ u(z) & \text{otherwise.} \end{cases}$$

Then, w is equal to u on $Q \setminus B_{R/2}(\hat{z})$ and so it is a subsolution of (3.2.1) in $Q \setminus B_{R/2}(\hat{z})$. It follows from Proposition 3.2.9 and Corollary 3.3.3 that w is a subsolution of (3.2.1) in $B_R(\hat{z})$. Therefore, Proposition 3.2.7 shows that w is a subsolution of (3.2.1) in Q and so $w \in \mathcal{S}$. In particular, $u(\hat{z}) \geq w(\hat{z})$ but $w(\hat{z}) = \psi(\hat{z}) + m/2 = u(\hat{z}) + m/2$. Since $m > 0$, we obtain a contradiction and conclude that u is a supersolution. $\square$

3.4 Application to large time behavior

We study large time asymptotic behaviors of solutions of the Hamilton-Jacobi equation (3.1.1) with a Hamiltonian H satisfying (A1)–(A4) on a compact geodesic metric space (X, d). First note uniqueness of the constant c such that (3.1.2) admits a solution.

Proposition 3.4.1. *Assume (A1), (A2) and that X is compact. Let $c \in \mathbf{R}$ be a constant such that* (3.1.2) *admits a real-valued continuous solution. Then,*

$$c = \max_{x \in X} H(x, 0). \tag{3.4.1}$$

Proof. By the assumption (A2) it is enough to show that

$$\sup_{x \in X} \inf_{p \in \mathbf{R}_+} H(x, p) \leq c \leq \sup_{x \in X} H(x, 0). \tag{3.4.2}$$

It is easy to show the second inequality of (3.4.2). Indeed, since a solution v attains a minimum at some point $\hat{x} \in X$, we have $H_0(\hat{x}, 0) = H(\hat{x}, 0) \geq c$, which implies $\sup_{x \in X} H(x, 0) \geq c$. In order to prove the first inequality of (3.4.2), fix $\hat{x} \in X$. Let us consider the function $v(x) - nd(x, \hat{x})^2/2$ and take its maximum point x_n for each $n = 0, \cdots$. Now, since $v(x_n) - nd(x_n, \hat{x})^2/2 \geq v(\hat{x})$, we have $d(x_n, \hat{x})^2 \leq 2(\max v - v(\hat{x}))/n$. Therefore, $x_n \to \hat{x}$ as $n \to \infty$. Since u is a subsolution, $H_0(x_n, nd(x_n, \hat{x})) = H(x_n, nd(x_n, \hat{x})) \leq c$. Hence, $\inf_{p \in \mathbf{R}_+} H(x_n, p) \leq c$ and sending $n \to \infty$ yields $\inf_{p \in \mathbf{R}_+} H(\hat{x}, p) \leq c$. We now obtained the inequalities (3.4.2). $\square$

Remark 3.4.2. It is a problem whether the inequalities (3.4.2) holds even if we remove the compactness assumption. One can show them by a similar argument to the proofs in Section 3.3 using Ekeland's variational principle provided $p \mapsto \sup_{x \in X} H(x, p)$ is continuous.

In view of this proposition we hereafter define c by (3.4.1). Now we are able to state the main theorem of large time behavior.

Theorem 3.4.3 (Large time behavior)**.** *Assume (A1)–(A4), $u_0 \in Lip(X)$ and that X is compact. Let u be a Lipschitz continuous solution of* (3.1.1) *with $u|_{t=0} = u_0$ on $[0, \infty) \times X$. Then, $u(t, x) + ct$ converges to a function v locally uniformly as $t \to \infty$ in X and v is a solution of* (3.1.2).

In order to prove this theorem we first establish regularity, existence and comparison results for the stationary equation (3.1.2). Set

$$A := \{x \in X \mid H(x, 0) = c\}.$$

Proposition 3.4.4 (Lipschitz continuity of solutions of (3.1.2)). *Assume (A1), (A3) and (A4). Then, real-valued continuous solutions of* (3.1.2) *are equi-Lipschitz continuous.*

Proof. Note that there exists a constant $L \in \mathbf{R}_+$ such that $H(x,p) \geq c$ for all $x \in X$ and $p \geq L$ by (A3). Fix a real-valued continuous solution v. Consider the function $v(x)-v(y)-2L\sqrt{d(x,y)^2+\varepsilon^2}$ for $x, y \in X$ and take its maximum point x_ε with respect to x for each $\varepsilon > 0$. Note that $x \mapsto v(y) + 2L\sqrt{d(x,y)^2+\varepsilon^2}$ is of $\underline{\mathcal{C}}^1(X)$ by Lemma 3.2.2. Hence, we see that

$$H_0\left(x_\varepsilon, \frac{2Ld(x_\varepsilon,y)}{\sqrt{d(x_\varepsilon,y)^2+\varepsilon^2}}\right) = H\left(x_\varepsilon, \frac{2Ld(x_\varepsilon,y)}{\sqrt{d(x_\varepsilon,y)^2+\varepsilon^2}}\right) \leq c.$$

Therefore, we see that $2Ld(x_\varepsilon,y)/\sqrt{d(x_\varepsilon,y)^2+\varepsilon^2} \leq L$ and so $x_\varepsilon \to y$ as $\varepsilon \to 0$. Now, for each $x, y \in X$, we have

$$v(x) - v(y) - 2L\sqrt{d(x,y)^2+\varepsilon^2} \leq v(x_\varepsilon) - v(y) - 2L\sqrt{d(x_\varepsilon,y)^2+\varepsilon^2}$$

Sending $\varepsilon \to 0$ yields

$$v(x) - v(y) - 2Ld(x,y) \leq 0,$$

which means that all subsolutions of (3.1.2) is $2L$-Lipschitz continuous. □

Theorem 3.4.5 (Existence of a solution of (3.1.2)). *Assume (A1), (A3) and (A4). Then, there exists at least one Lipschitz continuous solution of* (3.1.2) *whenever A is non-empty.*

Proof. Define

$$S(x,y) := \sup\{w(x) \mid w \in C(X) \text{ is a subsolution of (3.1.2) with } w(y) = 0\}.$$

Note that the constant $w \equiv 0$ is a subsolution of (3.1.2). Also Proposition 3.4.4 ensures that the solutions of (3.1.2) are equi-Lipschitz continuous and hence $v := S(\cdot, y)$ is a Lipschitz continuous function on X. Now, Corollary 3.3.3 implies that v is a subsolution of (3.1.2) in X while Proposition 3.3.5 shows that v is a supersolution of (3.1.2) in $X \setminus \{y\}$. Since $H(x,p) \geq H(x,0) = c$ for $x \in A$, we see that $v = S(\cdot, y)$ is a solution for every $y \in A \neq \emptyset$. □

Theorem 3.4.6 (Comparison principle for (3.2.2)). *Let U be an open subset of X such that $\overline{U}$ is compact. Assume (A1), (A2) and that $H(x,0) < 0$ for all $x \in U$. Let u be a subsolution and v be a supersolution of* (3.2.2) *such that $u^* < +\infty$ and $v_* > -\infty$. If $u^* \leq v_*$ on ∂U, then $u^* \leq v_*$ in U.*

Proof. First note that we may assume $u^*(x_0) \neq -\infty$ and $v_*(x_0) \neq +\infty$ at some $x_0 \in U$; otherwise the conclusion holds. Fix $\theta \in (0,1)$ and consider the upper semicontinuous function defined by

$$\Phi(x,y) := \theta u^*(x) - v_*(y) - \frac{1}{2\varepsilon}d(x,y)^2$$

for $\varepsilon > 0$. Thanks to the compactness of $\overline{U}$, we are able to take a maximum point $(x_\varepsilon, y_\varepsilon) \in \overline{U} \times \overline{U}$ of Φ. It follows from $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(x_0, x_0)$ that

$$\begin{aligned}\frac{1}{2\varepsilon}d(x_\varepsilon,y_\varepsilon)^2 &\leq \theta u^*(x_\varepsilon) - v_*(y_\varepsilon) - \theta u^*(x_0) + v_*(x_0)\\ &\leq \theta \sup u^* - \inf u_* - \theta u^*(x_0) + v_*(x_0) < +\infty.\end{aligned}$$

Hence, $d(x_\varepsilon, y_\varepsilon) \to 0$ and so we may assume that x_ε and y_ε converge to a same point $\bar{x} \in \overline{U}$ by taking a subsequence. Let us consider the case when $\bar{x} \in U$. Then, since u and v are a subsolution and a supersolution,

$$H(x_\varepsilon, \frac{1}{\theta\varepsilon} d(x_\varepsilon, y_\varepsilon)) \le 0,$$
$$H(y_\varepsilon, \frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon)) \ge 0.$$

By the convexity of H the second inequality yields

$$(1-\theta)H(y_\varepsilon, 0) + \theta H(y_\varepsilon, \frac{1}{\theta\varepsilon} d(x_\varepsilon, y_\varepsilon)) \ge 0$$

Hence,

$$(1-\theta)H(y_\varepsilon, 0) + \theta H(y_\varepsilon, \frac{1}{\theta\varepsilon} d(x_\varepsilon, y_\varepsilon)) - \theta H(x_\varepsilon, \frac{1}{\theta\varepsilon} d(x_\varepsilon, y_\varepsilon)) \ge 0.$$

Sending $\varepsilon \to 0$ yields $(1-\theta)H(\bar{x}, 0) \ge 0$. Since $H(\bar{x}, 0) < 0$ and $\theta < 1$, we obtain a contradiction. Therefore, $\bar{x} \in \partial U$. We now observe that

$$\theta u^*(x_\varepsilon) - v_*(y_\varepsilon) \ge \Phi(x_\varepsilon, y_\varepsilon) \ge \sup_{x \in U} \Phi(x, x) = \sup_U (\theta u^* - v_*).$$

Hence, we see that $\sup_U(\theta u^* - v_*) \le (\theta u^* - v_*)(\bar{x}) \le \sup_{\partial U}(\theta u^* - v_*)$. Sending $k \to 1$ implies $\sup_U(u^* - v_*) \le \sup_{\partial U}(u^* - v_*)$. □

Corollary 3.4.7 (Comparison principle for (3.1.2))**.** *Assume that X is compact. Let u be a subsolution and v be a supersolution of* (3.1.2) *such that $u^* < +\infty$ and $v_* > -\infty$. If $u^* \le v_*$ on A, then $u^* \le v_*$ on X.*

Proof. It follows from the definition of A that $H(x, 0) - c < 0$ for all $x \in U := X \setminus A$. Therefore, Theorem 3.4.6 implies $u^* \le v_*$ in $X \setminus A$. □

We will also require a comparison principle for the evolution equation (3.1.1).

Theorem 3.4.8 (Comparison principle for (3.1.1))**.** *Assume (A1) and that X is compact. Let u be a subsolution and v be a supersolution of* (3.1.1) *such that $u^* < +\infty$ and $v_* > -\infty$. If $u^*|_{t=0} \le v_*|_{t=0}$, then $u^* \le v_*$ on $(0, \infty) \times X$.*

One is able to prove this theorem with the same idea as in [9, Proof of Proposition 3.3] and so we omit the proof.

Before starting the proof of Theorem 3.4.3, let us explain that the initial value problem (3.1.1) and $u|_{t=0} = u_0$ admits a unique Lipschitz continuous solution. We will construct a solution by Perron method while the uniqueness is a direct consequence of the comparison principle (Theorem 3.4.8). Let $\mathrm{Lip}[u_0]$ denote the Lipschitz constant of u_0 and set $K = \max_{x \in X} |H(x, \mathrm{Lip}[u_0])|$. First note that $\overline{u}(t, x) := u_0(x) + Kt$ and $\underline{u}(t, x) := u_0(x) - Kt$ are a Lipschitz continuous supersolution and subsolution on $[0, \infty) \times X$, respectively. We then can construct a continuous solution u such that $\underline{u} \le \overline{u}$ by using Proposition 3.3.5 and Theorem 3.4.8. Take a constant $L \in \mathbf{R}_+$ such that $H(x, p) \ge c$ for all $x \in X$ and $p \ge L$. Then, we see that $|u(t, x) - u(s, y)| \le K|t - s| + Ld(x, y)$ by a similar argument to the proof of Propositiont:liplhj. Actually, this is a standard argument and we refer the reader to [10].

We are now able to prove the main theorem stated at the top of this section.

Proof of Theorem 3.4.3. Take the solution v_0 of (3.1.2) in Theorem 3.4.5. Noting that u_0 and v_0 are bounded since X is compact, we are also able to see that $v_0 - M \le u_0 \le v_0 + M$ for some large $M > 0$. Recall Propositions 3.2.5 and 3.2.9, which imply that $v_0 - ct \pm M$ are solutions of (3.1.1). We then see by a comparison principle for (3.1.1) (Theorem 3.4.8) that $v_0 - ct - M \le u \le v_0 - ct + M$. Thus, the upper and lower semi-limits

$$\overline{v}(x) := \sup_{(t_j,x_j)\to(\infty,x)} \limsup_j \{u(t_j,x_j) + ct_j\},$$
$$\underline{v}(x) := \inf_{(t_j,x_j)\to(\infty,x)} \liminf_j \{u(t_j,x_j) + ct_j\}$$

can be defined as a bounded function on X since $v_0 - M \le \underline{v} \le \overline{v} \le v_0 + M$.

We next note that Propositions 3.2.8 and 3.2.9 show the function

$$w^\lambda(t,x) := u\left(\frac{t}{\lambda}, x\right) + c\frac{t}{\lambda}$$

is a solution of

$$\lambda \partial_t w^\lambda + H(x, |Dw^\lambda|) = c \quad \text{in } (0,\infty) \times X$$

for each $\lambda > 0$. Since

$$\overline{v}(x) = \sup_{(t_j,x_j,\lambda_j)\to(t,x,0)} \limsup_j w^{\lambda_j}(t_j,x_j),$$
$$\underline{v}(x) = \inf_{(t_j,x_j,\lambda_j)\to(t,x,0)} \liminf_j w^{\lambda_j}(t_j,x_j)$$

for all $t > 0$ and $x \in X$, i.e. $\overline{v}$ and $\underline{v}$ are respectively nothing but the upper and lower semilimit of w^λ as $\lambda \to 0$, the stability result (Proposition 3.3.1) and Proposition 3.2.5 shows that $\overline{v}$ and $\underline{v}$ are a subsolution and a supersolution of (3.1.2), respectively.

We next claim that $\overline{v} = \underline{v}$ on the set A. Indeed, for each $x \in A$, $u(t,x) + ct$ converges to some $v(x)$ since $\partial_t u + c \le 0$ and so it is a decreasing sequence. We also obtain that u is equi-Lipschitz continuous. By connecting these two facts, we see that $\overline{v} \le v \le \underline{v}$ on A.

Finally, the comparison principle (Theorem 3.4.7) shows that $\overline{v} \le \underline{v}$ on the whole space X. Thus, we can conclude that $u(t,x) + ct$ converges to some function $v = \overline{v} = \underline{v}$ which is a solution of (3.1.2). □

Remark 3.4.9. The convexity assumption (A2) is used only to guarantee a comparison principle holds for the stationary equation (3.1.2). It is possible to weaken the condition. For instance, let us consider the specific Hamiltonian $H(x,p) = \sqrt{|p|}$, which is not convex. One easily see that the equation (3.1.2) is equivalent to $|Dv| = c^2$. Since a comparison principle for the convex Hamiltonian $|Dv| = c^2$ implies a comparison principle $\sqrt{|Dv|} = c$, the same behavior of the solution must occur to the Hamiltonian $H(x,p) = \sqrt{|p|}$. This scheme works for quasiconvex Hamiltonians $H(x,p) = h(|p|) + f(x)$ with $h\colon \mathbf{R}_+ \to \mathbf{R}$ such that $h(p) - \lambda p$ is non-decreasing for some $\lambda > 0$.

Let us introduce the functions $\phi_-, \phi_\infty \in C(X)$ by

$$\phi_-(x) := \inf_{t>0} (u(x,t) + ct),$$
$$\phi_\infty(x) := \min\{S(x,y) + \phi_-(y) \mid y \in A\},$$

where the function S is defined in the proof of Theorem 3.4.5. Note by Theorem 3.4.5 and Corollary 3.3.3 the function ϕ_∞ is a solution of (3.1.2).

We then have

Theorem 3.4.10 (Asymptotic profile)**.** $\lim_{t\to\infty}(u(x,t)+ct)=\phi_\infty(x)$ *for all* $x \in X$.

The proof can be done similar to [12], so we omit it.

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Chapter 4

Minimax formula of the additive eigenvalue for quasiconvex Hamiltonians

4.1 Introduction

It is well-known that the additive eigenvalue for a Hamilton-Jacobi equation has an inf-sup type representation formula if the Hamiltonian is continuous, convex and coercive. In this article we will introduce two approaches to this problem. One is similar to known arguments using Jensen's inequality directly to the Hamiltonian while the other one invokes Clarke's generalized gradient. Both of these two approaches will derive the representation formula under a weaker assumption on the Hamiltonian. We now stress that the latter approach is rather new as far as the author knows and using a crucial lemma on convergence of mollifications of Lipschitz continuous functions (Lemma 4.1.2), whose proof will be given in Section 4.4.

For simplicity, we consider first-order Hamilton-Jacobi equations in the periodic setting of the form

$$H(x, Du) = a \quad \text{in } \mathbf{T}^N := \mathbf{R}^N/\mathbf{Z}^N \tag{4.1.1}$$

with a parameter $a \in \mathbf{R}$. Here, $H = H(x,p)\colon \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ is a function called a Hamiltonian satisfying the following conditions:

(A1) (Continuity) H is continuous on $\mathbf{T}^N \times \mathbf{R}^N$.

(A2) (Convexity) H is convex in the variable $p \in \mathbf{R}^N$ for each $x \in \mathbf{T}^N$.

An (additive) eigenvalue is a unique constant $a \in \mathbf{R}$ such that (4.1.1) admits a viscosity solution u ([12]) with Lipschitz continuity; Du denotes a gradient of the unknown function $u = u(x)$. Then, the eigenvalue $a = c$, if exists, will satisfy the representation formulas

$$c = \inf_{u \in C^1(\mathbf{T}^N)} \sup_{\nabla u} H, \tag{4.1.2}$$

$$c = \inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H, \tag{4.1.3}$$

where ∇u is the graph of the classical gradients (also denoted by ∇u) of u, i.e.

$$\nabla u := \{(x,p) \in \mathbf{T}^N \times \mathbf{R}^N \mid u(y) = u(x) + p \cdot (y-x) + o(|y-x|) \text{ as } y \to x\}.$$

Note that Lipschitz continuous functions $u \in \mathrm{Lip}(\mathbf{T}^N)$ are differentiable almost everywhere by Rademacher's theorem and $\nabla u \in L^\infty(\mathbf{T}^N)$.

This kind of expression (the right-hand side of (4.1.2)) was found as a variational formula of Mañé's critical value with respect to the corresponding Lagrangian by Contreras-Iturriaga-Paternain-Paternain [13]. On the other hand, this is a pure partial differential equations problem. In view of this, the above minimax formula was established by Fathi in the context of weak KAM (Kolmogorov-Arnold-Moser) theory powered by the viscosity solution theory; see [15, Section 6]. We remark that the additive eigenvalue problem (4.1.1) also appears in solving homogenization problems [20] and long time behaviors [22]. It is also known that the minimax formula is useful for computing the additive eigenvalue numerically [18]. This work will provide natural extensions for these theories.

In this article we extend the representation formula for general quasiconvex Hamiltonians (see (A2') below) instead of the convexity assumption (A2) with two different proofs.

(A2') (Quasiconvexity) H is quasiconvex in the variable $p \in \mathbf{R}^N$ for each $x \in \mathbf{T}^N$, i.e. $H(x, \theta p + (1-\theta)q) \le \max\{H(x,p), H(x,q)\}$ for all $p, q \in \mathbf{R}^N$, $0 \le \theta \le 1$ and $x \in \mathbf{T}^N$.

We remark that the quasiconvexity is sometimes called level-set convexity since (A2') is equivalent to the condition that the sublevel sets $\{p \in \mathbf{R}^N \mid H(x,p) \le a\}$ are convex for all $a \in \mathbf{R}$ and $x \in \mathbf{T}^N$.

Recently several authors study homogenization problems with quasiconvex Hamiltonians; see [16] and [1]. In fact, the authors of [1] mention some relation between the eigenvalue and the minimax expression and [1, Proposition 6.2] will immediately show one of the representations (4.1.3). Indeed, we can show (4.1.3) easily in view of Propositions 4.2.1 and 4.4.1. On the other hand, to show (4.1.2) need more advanced calculations such as Lemmas 4.1.1 and 4.1.2 below, and there seem to be no results on it as far as the author knows. We also point out that the authors of [24] posed a Hamiltonian of the form

$$H(x,p) = H_\varepsilon^2(x,p) = \sigma\left(\frac{x}{\varepsilon}\right)\frac{p}{p_s}\tanh\left(\frac{p_s}{p}\right)$$

with a positive continuous function σ, a constant $p_s > 0$ and a parameter $\varepsilon > 0$. This Hamiltonian is quasiconvex (A2') as well as non-coercive. Long time behavior and homogenization for this Hamiltonian have been studied in [17] and [19].

In order to explain the main idea of one of the proofs, let us review the known proof under the assumptions (A1) and (A2). This proof is inspired by [6], [18, Proposition 2.2] and [21, Subsection 4.2]. First, it is easy to show the inequalities $\inf_{u\in\mathrm{Lip}} \sup_{\nabla u} H \le c \le \inf_{u\in C^1} \sup_{\nabla u} H$. We hence claim $\inf_{u\in C^1} \sup_{\nabla u} H \le \inf_{u\in\mathrm{Lip}} \sup_{\nabla u} H$. Now, for $u \in \mathrm{Lip}(\mathbf{T}^N)$ take mollifications $u_n := u * \eta_n \in C^\infty(\mathbf{T}^N)$ with the standard Friedrichs mollifier η_n. Then, we observe that

$$
\begin{aligned}
H(x, \nabla u_n(x)) &= H\left(x, \int_{\mathbf{T}^N} \nabla u(x-y)\eta_n(y)dy\right) \\
&\leq \int_{\mathbf{T}^N} H(x, \nabla u(x-y))\eta_n(y)dy \\
&\leq \int_{\mathbf{T}^N} H(x-y, \nabla u(x-y))\eta_n(y)dy + \alpha_n \leq \sup_{\nabla u} H + \alpha_n
\end{aligned}
$$

for all $x \in \mathbf{T}^N$ and therefore we will have the desired inequality. Here, we have invoked the convexity (A2) so that Jensen's inequality yields the first inequality; the second equality follows from the continuity (A1) with some error term $\alpha_n > 0$ such that $\alpha_n \to 0$.

Our idea of the proof is to use another Jensen-like inequality for quasiconvex functions stated below.

Lemma 4.1.1 (Fundamental inequality for quasiconvex functions)**.** *Let f be a lower semicontinuous function defined on $\mathbf{R}^N$. Then, f is quasiconvex on $\mathbf{R}^N$ if and only if*

$$
f\left(\int_{\Omega} X d\mu\right) \leq \operatorname*{ess\,sup}_{\Omega} f \circ X
$$

for all measure spaces (Ω, μ) with $\mu(\Omega) = 1$ and all $\mathbf{R}^N$-valued integrable functions X on Ω.

In view of this inequality, we can improve the proof for the representation formula. In fact, this lemma has already been proved by Barron, Jensen, Liu and Wang; see [7] (in one-dimensional setting) and [8]. We will give a short proof in Section 4.3. We also point out that a discrete version of Lemma 4.1.1 is studied in [14].

The other proof is one using the *generalized gradients* of Lipschitz functions u defined by

$$
\partial u := \overline{\mathrm{co}}_p \, \overline{\nabla u},
$$

i.e. $\partial u \subset \{(x,p) \in \mathbf{T}^N \times \mathbf{R}^N\}$ is the closed convex hull with respect to p of the closure of the classical gradients ∇u. This is nothing but Clarke's gradients; see [9] and [10]. Also note that ∂u is compact since $\nabla u \in L^\infty(\mathbf{T}^N)$. Now, the quasiconvexity of H implies that $\inf_{u \in \mathrm{Lip}} \sup_{\nabla u} H = \inf_{u \in \mathrm{Lip}} \sup_{\partial u} H$. The remaining inequality $\inf_{u \in C^1} \sup_{\nabla u} H \leq \inf_{u \in \mathrm{Lip}} \sup_{\partial u} H$ can be shown by a graph convergence of the standard mollifications of Lipschitz functions stated below. The proof will be given in Section 4.4.

Lemma 4.1.2 (Convergence of mollifications)**.** *Let $u \in \mathrm{Lip}(\mathbf{T}^N)$ and let $u_n \in C^\infty(\mathbf{T}^N)$ be the standard mollification $u * \eta_n$. If a sequence $(x_n, p_n) \in \nabla u_n$ converges to $(x,p) \in \mathbf{T}^N \times \mathbf{R}^N$, then $(x,p) \in \partial u$.*

In our arguments, the quasiconvexity (A2') is essential. We point out that the authors of [2] and [3] obtain partial results on homogenization for Hamiltonians without convexity such as

$$
H(x,p) = (|p|^2 - 1)^2 - V(x)
$$

with a bounded function V. Representation formula for such Hamiltonians is an open problem.

This chapter is organized as follows. In Section 4.2 we give a complete statement of our main result on the minimax formula. We prove it in Section 4.3 by using the fundamental inequality for quasiconvex functions (Lemma 4.1.1) while we give another proof in Section 4.4 with the generalized gradient and Lemma 4.1.2. The contexts of Sections 4.3 and 4.4 are independent so the reader can skip Section 4.3.

4.2 Minimax formula

In this section we give a rigorous definition of the viscosity solutions and the eigenvalues of the Hamilton-Jacobi equations (4.1.1) and a complete statement of the main theorem on the minimax formula. First, define the graphs of *superdifferentials* D^+u and *subdifferentials* D^-u for a function u by

$$D^+u := \{(x,p) \in \mathbf{T}^N \times \mathbf{R}^N \mid u(y) \le u(x) + p\cdot(y-x) + o(|y-x|) \text{ as } y \to x\},$$
$$D^-u := \{(x,p) \in \mathbf{T}^N \times \mathbf{R}^N \mid u(y) \ge u(x) + p\cdot(y-x) + o(|y-x|) \text{ as } y \to x\}.$$

Note that the superdifferentials and the subdifferentials can be characterized by smooth functions touching u from above or below; see [11, Section 2]. A function $u \in \mathrm{Lip}(\mathbf{T}^N)$ is called a *viscosity subsolution*, a *viscosity supersolution* or a *viscosity solution* of the Hamilton-Jacobi equation (4.1.1) with $a \in \mathbf{R}$ if

$$\sup_{D^+u} H \le a, \quad \inf_{D^-u} H \ge a, \quad \sup_{D^+u} H \le a \le \inf_{D^-u} H,$$

respectively. A *subeigenvalue*, a *supereigenvalue* or an *eigenvalue* of the additive eigenvalue problem (4.1.1) is a constant $a \in \mathbf{R}$ such that there exists at least one viscosity subsolution, supersolution or solution of (4.1.1), respectively. We now define the *upper critical value* and *lower critical value* $c^\pm \in \mathbf{R} \cup \{\pm\infty\}$ by

$$c^+ = c^+(H) := \inf\{a \in \mathbf{R} \mid a \text{ is a subeigenvalue of (4.1.1)}\},$$
$$c^- = c^-(H) := \sup\{a \in \mathbf{R} \mid a \text{ is a supereigenvalue of (4.1.1)}\}.$$

For later convenience we prepare several notations: Let $B(x,r)$ denote the open ball with center x and radius $r > 0$ and let $\overline{B}(x,r)$ denote its closure. For the graphs $G = \nabla u, \partial u, D^\pm u \subset \mathbf{T}^N \times \mathbf{R}^N$ and a point $x \in \mathbf{T}^N$, set $G(x) := \{p \in \mathbf{R}^N \mid (x,p) \in G\}$. A modulus is a non-negative function ω defined on $[0,\infty)$ with $\lim_{r\to 0}\omega(r) = 0$.

The following propositions give basic properties of the critical values.

Proposition 4.2.1 (Characterization and rough estimates)**.**

$$c^+(H) = \inf_{u\in\mathrm{Lip}(\mathbf{T}^N)} \sup_{D^+u} H, \quad c^-(H) = \sup_{u\in\mathrm{Lip}(\mathbf{T}^N)} \inf_{D^-u} H, \tag{4.2.1}$$

$$\min_{x\in\mathbf{T}^N} H(x,0) \le c^\pm(H) \le \max_{x\in\mathbf{T}^N} H(x,0). \tag{4.2.2}$$

Proof. We only show the equation and inequalities for the upper critical value $c^+(H)$ since a symmetric argument shows a proof for the lower critical value $c^-(H)$. The proof is not so difficult; for a subeigenvalue $a \in \mathbf{R}$, since there exists a Lipschitz subsolution, $\inf_{u\in\mathrm{Lip}(\mathbf{T}^N)} \sup_{D^+u} H \le a$. We also see that

Lipschitz functions $u \in \mathrm{Lip}(\mathbf{T}^N)$ themselves are a subsolution of the equation (4.1.1) with $a = \sup_{D^+u} H$. Therefore, (4.2.1) holds. Moreover, $u = 0$ is a subsolution of (4.1.1) with $a = \max_{x\in\mathbf{T}^N} H(x, 0)$. For a subeigenvalue $a \in \mathbf{R}$ and the subsolution $u \in \mathrm{Lip}(\mathbf{T}^N)$ of (4.1.1), since $(x, 0) \in D^+u$ at a maximum point $x \in \mathbf{T}^N$ of u, we have $\min_{x\in\mathbf{T}^N} H(x, 0) \le a$. We have shown (4.2.2). □

Proposition 4.2.2 (Monotonicity of critical values)**.** *Let H_1 and H_2 be two Hamiltonians such that $H_1 \le H_2$ on $\mathbf{T}^N \times \mathbf{R}^N$. Then, $c^\pm(H_1) \le c^\pm(H_2)$, respectively.*

The proof is trivial so we omit it.

Proposition 4.2.3 (Upper and lower critical values)**.** *Assume that H satisfies (A1). Then, $c^-(H) \le c^+(H)$. Moreover, if*

(A3) (Coercivity) H is coercive in the variable $p \in \mathbf{R}^N$ uniformly in $x \in \mathbf{T}^N$, i.e.

$$\liminf_{|p|\to\infty} \inf_{x\in\mathbf{T}^N} H(x, p) = +\infty,$$

then $c^-(H) = c^+(H)$ and they are a unique eigenvalue of (4.1.1)*.*

This is a well-known fact; we refer the reader to [20], [16] and [19]. Under the assumptions (A1) and (A3) the unique eigenvalue $c = c(H) := c^+(H) = c^-(H)$ is called *critical value* of (4.1.1).

The generalized effective Hamiltonian introduced in the author's previous work [19] is nothing but the upper critical value c^+:

Proposition 4.2.4. *Assume that H satisfies (A1) and let H_n be a sequence of Hamiltonians satisfying (A1) and (A3). If H_n converges to H in the sense of*

$$\liminf_n \inf_{\mathbf{T}^N\times\mathbf{R}^N} (H_n - H) \ge 0, \ \limsup_n \sup_{\mathbf{T}^N\times B(0,R)} (H_n - H) \le 0 \textit{ for all } R > 0,$$

then $c(H_n) \to c^+(H)$.

Proof. Consider the specific approximation $H_n(x, p) = H(x, p) + |p|/n$ for $n = 1, \cdots$. Since $H_n \ge H$, we see by Proposition 4.2.2 that $c(H_n) = c^+(H_n) \ge c^+(H)$, which immediately yields $\liminf_n c(H_n) \ge c^+(H)$. In order to the opposite inequality, fix a subeigenvalue a and take the Lipschitz continuous subsolution u of (4.1.1). Note that the closure of D^+u is compact by the Lipschitz continuity. Hence, H_n becomes coincident to H on D^+u for sufficiently large n. Therefore, $c(H_n) = c^+(H_n) \le a$, which shows $\limsup_n c(H_n) \le c^+(H)$. For general approximations one can show by the same arguments as in [19, Theorem 4.1] that $c(H_n)$ is a convergent sequence and that the limit does not depend on the choice of the approximations. Finally, we have $\lim_n c(H_n) = c^+(H)$. □

We state our main result.

Theorem 4.2.5 (Minimax formulas)**.** *Assume (A1) and (A2') (not (A2)). Then,*

$$c^+(H) = \inf_{u\in\mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H = \inf_{u\in C^\infty(\mathbf{T}^N)} \sup_{\nabla u} H.$$

In particular, if (A3) holds, then they are nothing but the critical value $c(H)$ (the unique eigenvalue of (4.1.1)*).*

Some inequalities hold unconditionally.

Proposition 4.2.6.

$$\inf_{u\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H \le \inf_{u\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{D^+u} H = c^+(H) \le \inf_{u\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\partial u} H$$
$$\le \inf_{u\in C^\infty(\mathbf{T}^N)} \sup_{\nabla u} H.$$

Proof. These inequalities follow from the well-known orders $\nabla u \subset D^+u \subset \partial u$ for $u \in \mathrm{Lip}(\mathbf{T}^N)$ and $\nabla u = D^+u = \partial u$ for $u \in C^1(\mathbf{T}^N)$; see [5, Lemma II.1.8 and Subsection II.4.1]. □

4.3 Proof with fundamental inequality for quasiconvex functions

Lemma 4.1.1 will result in the fundamental property of convex sets with probability measures by the level-set convexity of f:

Lemma 4.3.1 (Fundamental inclusion for convex sets)**.** *Let C be a closed subset of $\mathbf{R}^N$. Then, C is convex if and only if*

$$e := \int_\Omega X\,d\mu \in C$$

for all measure spaces (Ω, μ) with $\mu(\Omega) = 1$ and all $\mathbf{R}^N$-valued integrable functions X on Ω satisfying $X \in C$ μ-a.e. on Ω.

Proof. The "if" part is easy; for $x, y \in C$ and $0 \le \theta \le 1$, set $\Omega = \{\pm 1\}$, $\nu(\{-1\}) = \theta$, $\nu(\{1\}) = 1-\theta$, $X(-1) = x$, $X(1) = y$. Then, since $\int_\Omega X\,d\mu = \theta x + (1-\theta)y$, we have $\theta x + (1-\theta)y \in C$.

We show the "only if" part. Suppose conversely that $e \notin C$. Then, by the hyperplane separation theorem (see, e.g., [23, Theorem 11.4]) one is able to find a vector $v \in \mathbf{R}^N$ such that

$$v\cdot x \le a < v \cdot e \quad \text{for all } x \in C$$

with some $a \in \mathbf{R}$. Since $X \in C$ μ-a.e.,

$$v\cdot e = v\cdot \int_\Omega X\,d\mu = \int_\Omega v\cdot X\,d\mu \le \int_\Omega a\,d\mu = a,$$

which is contradicts to $v \cdot e > a$. Therefore, $e \in C$. □

Proof of Lemma 4.1.1. The "if" part is easy as Lemma 4.3.1. We show the "only if" part. First note that we may assume that $\operatorname{ess\,sup} f\circ X = \sup f \circ X$ since $\tilde\Omega := \{f\circ X \le \operatorname{ess\,sup}_\Omega f\circ X\}$ satisfies $\operatorname{ess\,sup}_\Omega f\circ X = \sup_{\tilde\Omega} f\circ X$ and $\mu(\Omega\setminus\tilde\Omega) = 0$. Set $E := X(\tilde\Omega)$ and take its closed convex hull $\overline{\mathrm{co}}\,E$. Then, Lemma 4.3.1 shows that $\int_{\tilde\Omega} X\,d\mu \in \overline{\mathrm{co}}\,E$ and therefore

$$f\left(\int_\Omega X\,d\mu\right) = f\left(\int_{\tilde\Omega} X\,d\mu\right) \le \sup_{\overline{\mathrm{co}}\,E} f = \sup_E f = \sup_{\tilde\Omega} f\circ X = \operatorname*{ess\,sup}_\Omega f\circ X$$

Here, the middle equation follows from the quasiconvexity assumption of f. □

Remark 4.3.2. We can easily prove the standard Jensen's inequality by applying Lemma 4.3.1 to the closed convex set $\{(x,y) \mid y \geq f(x)\}$ for a convex function f.

We are now able to show Theorem 4.2.5.

Proof of Theorem 4.2.5 using Lemma 4.1.1. It is enough to show

$$\inf_{u \in C^\infty(\mathbf{T}^N)} \sup_{\nabla u} H \leq \inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H. \tag{4.3.1}$$

Fix $u \in \mathrm{Lip}(\mathbf{T}^N)$ and take the standard mollifications $u_n := u * \eta_n \in C^\infty(\mathbf{T}^N)$. Note that H is uniformly continuous on $\mathbf{T}^N \times \overline{B}(0,R)$ with $R := \operatorname{ess\,sup}_{\mathbf{T}^N} |\nabla u|$; there is a modulus ω such that $|H(x,p) - H(y,q)| \leq \omega(|x-y| + |p-q|)$ for all $x, y \in \mathbf{T}^N$ and $p, q \in \overline{B}(0,R)$. Fix $(x,p) \in \nabla u_n$. Then, we can calculate that

$$\begin{aligned} H(x,p) = H(x, \nabla u_n(x)) &= H\left(x, \int_{\mathbf{T}^N} \nabla u(x-y)\eta_n(y)dy\right) \\ &\leq \operatorname*{ess\,sup}_{y \in \mathrm{spt}(\eta_n)} H(x, \nabla u(x-y)) \\ &\leq \sup_{\nabla u} H + \operatorname*{ess\,sup}_{y \in \mathrm{spt}(\eta_n)} \omega(|y|). \end{aligned}$$

Here, we have used the quasiconvexity (A2') and Lemma 4.1.1 in order to obtain the first inequality. Taking a limit with respect to n, we have $\sup_{\nabla u_n} H(x,p) \leq \sup_{\nabla u} H$, which implies (4.3.1). We have obtained all inequalities to show Theorem 4.2.5. □

This proof also shows approximation of viscosity solutions, whose convex versions have been established in [6] and [5, Section II.5].

Proposition 4.3.3 (Approximation of viscosity solutions)**.** *Assume (A1) and (A2'). Let $u \in \mathrm{Lip}(\mathbf{T}^N)$ and let $u_n \in C^\infty(\mathbf{T}^N)$ be the standard mollification $u * \eta_n$. If u is a viscosity subsolution of* (4.1.1)*, then u_n are a viscosity subsolution of $H(x, Du_n) = a + \omega(1/n)$ in $\mathbf{T}^N$ with some modulus ω.*

4.4 Proof with generalized gradients

We begin with:

Proposition 4.4.1. *Assume (A1) and (A2'). Then,*

$$\inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\nabla u} H = \inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{D^+ u} H = \inf_{u \in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\partial u} H.$$

Proof. This is true since

$$\sup_{\nabla u} H = \sup_{\overline{\nabla u}} H = \sup_{\overline{\mathrm{co}}_p\, \overline{\nabla u}} H = \sup_{\partial u} H \tag{4.4.1}$$

for all $u \in \mathrm{Lip}(\mathbf{T}^N)$. In order to obtain the second equality, we need the quasiconvexity assumption (A2'). □

Remark 4.4.2. This proof also implies [1, Lemma 2.1].

We prove Lemma 4.1.2 in order to show the remaining inequality in Theorem 4.2.5

$$\inf_{u\in C^\infty(\mathbf{T}^N)} \sup_{\nabla u} H \le \inf_{u\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{\partial u} H. \tag{4.4.2}$$

The proof, which uses Jensen's inequality to distance functions from convex sets, is due to A. Siconolfi. A similar technique appears in [16]. We first prepare:

Lemma 4.4.3 (Continuity of generalized gradients)**.** *Let $u \in \mathrm{Lip}(\mathbf{T}^N)$. Then, for each $x \in \mathbf{T}^N$ there exists a modulus ω_x such that*

$$d(\partial u(x), p) \le \omega_x(|y-x|) \quad \textit{for all } (y,p)\in \partial u. \tag{4.4.3}$$

This lemma means that the the generalized gradients ∂u is upper semicontinuous as a set-valued function. The proof is easy since ∂u is compact (see, e.g., [10, Proposition 2.1.5] and [4, Proposition 1.4.8]) but we prove it for completeness.

Proof. Fix arbitrary $\varepsilon > 0$. Since ∂u and $\{x\}\times\{p \mid d(\partial u(x), p) = \varepsilon\}$ are disjoint compact sets, ∂u and $B(x,\delta)\times\{p \mid d(\partial u(x),p) \ge \varepsilon-\delta\}$ have empty intersections for some small $\delta > 0$. Therefore, every $(y,p)\in\partial u$ with $|y-x| < \delta$ satisfies $d(\partial u(x),p) < \varepsilon - \delta < \varepsilon$. □

Proof of Lemma 4.1.2. First note that the set $\partial u(x)$ is non-empty closed convex and hence $d(\partial u(x),\cdot)$ is a (Lipschitz) continuous convex function on $\mathbf{R}^N$. We observe by Jensen's inequality that

$$\begin{aligned} d(\partial u(x), q) = d(\partial u(x), \nabla u_n(y)) &= d\left(\partial u(x), \int_{\mathbf{T}^N} \nabla u(y-z)\eta_n(z)dz\right) \\ &\le \int_{\mathbf{T}^N} d(\partial u(x), \nabla u(y-z))\eta_n(z)dz \end{aligned}$$

for all $(y,q)\in\nabla u_n$. By Lemma 4.4.3 we have

$$\begin{aligned} d(\partial u(x), p_n) &\le \int_{\mathbf{T}^N} d(\partial u(x), \nabla u(x_n - z))\eta_n(z)dz \\ &\le \int_{\mathbf{T}^N} \omega_x(|x_n - z - x|)\eta_n(z)dz \le \sup_{z\in \operatorname{spt}\eta_n} \omega_x(|x_n - z - x|). \end{aligned}$$

This shows that $d(\partial u(x), p_n) \to 0$ and therefore $p \in \partial u(x)$. □

Lemma 4.1.2 yields another proof of Theorem 4.2.5.

Proof of Theorem 4.2.5 using Lemma 4.1.2. It is enough to show (4.4.2). Fix $u\in\mathrm{Lip}(\mathbf{T}^N)$ and take the standard mollifications $u_n := u*\eta_n \in C^\infty(\mathbf{T}^N)$. Also take a maximum point $(x_n,p_n)\in\nabla u_n$ of H so that $H(x_n,p_n) = \sup_{\nabla u_n} H$. Now, note that the sequence (x_n,p_n) has an accumulation point $(x,p)\in\mathbf{T}^N\times \overline{B}(0, \operatorname{ess\,sup}_{\mathbf{T}^N}|\nabla u|)$ since u is Lipschitz continuous. We then see by Lemma 4.1.2 that $(x,p)\in\partial u$ and therefore

$$\inf_{u\in C^\infty(\mathbf{T}^N)} \sup_{\nabla u} H \le \sup_{\nabla u_n} H = H(x_n,p_n) \to H(x,p) \le \sup_{\partial u} H.$$

Since $u\in\mathrm{Lip}(\mathbf{T}^N)$ is arbitrary, we can obtain the desired inequality (4.4.2). We now have obtained all the equations in Theorem 4.2.5. □

Remark 4.4.4. This proof is a bit longer than the proof in Section 4.3 but may give a deeper observation. For example, there is a question that if $u_n \in C^\infty$ converges to $u \in \mathrm{Lip}$ uniformly, then a sequence $(x_n, p_n) \in \nabla u_n$ has an accumulation point belonging to ∂u. This is an open problem concerned with stability of viscosity solutions. We also remark that one is able to prove Proposition 4.3.3 by combining Lemma 4.1.2 and the equation (4.4.1).

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Chapter 5

Viscosity solutions for one-dimensional singular diffusion equations

5.1 Introduction

In this chapter we study a one-dimensional nonlinear degenerate parabolic equation whose diffusion effect is very strong at particular slopes of unknown functions. We are in particular interested in an equation, where the driving force term is spatially inhomogeneous. A typical example is a quasilinear equation

$$u_t = a(u_x)[(W'(u_x))_x + \sigma(t,x)], \tag{5.1.1}$$

where W is a given convex function on $\mathbf{R}$ but may not be of class $C^1(\mathbf{R})$ so that its derivative W' may have jump discontinuities and σ is a given Lipschitz function depending on the space variable x as well as the time variable t; here a is a given nonnegative continuous function, and u_t and u_x denote the time and the space derivative of an unknown function $u = u(t,x)$.

In order to explain the motivation of this work, let us consider an evolution law of a curve $\Gamma_t \subset \mathbf{R}^2$ moved by an anisotropic curvature flow

$$V = M_0(\mathbf{n})(\kappa_{\gamma_0} + \sigma) \quad \text{on } \Gamma_t, \tag{5.1.2}$$

where V is the normal velocity of the evolving curve in the direction of the normal vector $\mathbf{n}$ and let the mobility M_0 and the surface energy density γ_0 be positive functions on the unit circle; the term κ_{γ_0} called a nonlocal curvature is the first variation of surface energy. We note that if γ_0 is the constant 1, then κ_{γ_0} is nothing but usual curvature κ; the quantity κ_{γ_0} formally equals $((\gamma_0)_{\theta\theta} + \gamma_0)\kappa$ if one writes γ_0 as a function of the argument θ of $\mathbf{n} = (\cos\theta, \sin\theta)$. The equation (5.1.2) appears in crystal growth as an equation to describe the interface of two phases; see, e.g., [2].

If the curve Γ_t is given as a graph of a function $u = u(t,x)$, the equation

(5.1.2) then becomes of the form (5.1.1) with

$$a(p) = M(p, -1), \qquad M(p,q) = \sqrt{p^2+q^2}\, M_0\left(\frac{(p,q)}{\sqrt{p^2+q^2}}\right),$$
$$W(p) = \gamma(p, -1), \qquad \gamma(p,q) = \sqrt{p^2+q^2}\, \gamma_0\left(\frac{(p,q)}{\sqrt{p^2+q^2}}\right).$$

Assume that the Frank diagram $F := \big\{(p,q) \in \mathbf{R}^2 \mid \gamma(p,q) \le 1\big\}$ is convex so that W is a convex function. If F has a smooth (C^2) boundary ∂F, the theory of (5.1.2) is well developed [6], [9], [16]. Indeed, since W is $C^2(\mathbf{R})$, we are able to apply the classical theory of viscosity solutions [7] to the equation (5.1.1). We are concerned with the case that ∂F is of class C^2 except finitely many points. A typical example of F is a polygon so that W is a piecewise linear function. For examples if γ is a crystalline energy of the form

$$\gamma(p,q) = |p| + |q|,$$

then $W''(p)$ is twice the Dirac delta function δ and so the equation (5.1.1) formally becomes

$$u_t = a(u_x)[2\delta(u_x)u_{xx} + \sigma],$$

which is not a classical partial differential equation.

Admissible curves such as polygons moving by a crystalline energy with no driving force have been studied by Taylor [19, 20] and by Angenent and Gurtin [1]. For the evolution law of graphs (5.1.1) a notion of solutions is introduced by adapting the subdifferential theory [10] ($\sigma = 0$) and [11]. Elliott, Gardiner and Schätzle [8] study relationship between the solutions in the sense of [10] and admissible curves. When σ is independent of x, the theory of viscosity solutions to (5.1.1) and (5.1.2) is established in a series of papers [12], [13], [14].

The goal of the present work is to establish a global-in-time existence theorem of a viscosity solution for a class of equations including (5.1.1) with a given continuous periodic initial condition. Our result is a generalization of [12, Section 8, 9] to the equation with spatially inhomogeneous driving force. Notion of viscosity solutions to (5.1.1) with σ depending on x is introduced in [15], where a comparison theorem is established. The authors of [15] also show some existence results by showing that a special semi-explicit variational solution studied in [17] is a viscosity solution but their initial data is very restrictive. We also point out that in a recent paper by Chambolle and Novaga [4] the authors establish short-time existence for (5.1.2) by time-discrete implicit scheme, which is introduced in [5], [3]. Our argument based on the theory of viscosity solutions is completely different from theirs and can be applied to a fully nonlinear equation.

Following [15], let us consider an energy functional which formally equals

$$\Phi[f] = \int_{\mathbf{T}} (W(f_x) - \sigma f)\, dx$$

for a smooth function f; we assumed a periodic boundary condition so that $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$ with $\omega > 0$. Let $\partial^0\Phi[f]$ be the canonical restriction of the subdifferential $\partial\Phi[f]$ in the Hilbert space $H := L^2(\mathbf{T})$, i.e.

$$\partial^0\Phi[f] = \arg\min \left\{\|\lambda\|_H \mid \lambda \in \partial\Phi[f]\right\}.$$

As mentioned in [11], the above minimizing problem is equivalent to an obstacle problem: The condition $\lambda \in -\partial\Phi[f]$ holds if and only if λ is of the form $\lambda = \xi'$ such that $\xi \in \partial W(f_x) + Z$ a.e. on $\mathbf{T}$, where Z is a primitive function of σ, i.e. $Z_x = \sigma$. Therefore, we minimize

$$\left\{ \int_{\mathbf{T}} |\xi'|^2 dx \mid \xi \in \partial W(f_x) + Z \text{ a.e. on } \mathbf{T} \right\}. \tag{5.1.3}$$

There might be a chance that there is no such ξ satisfying $\xi \in \partial W(f_x) + Z$ a.e. on $\mathbf{T}$. We need to require special structure to guarantee the existence of such ξ. A sufficient condition is that f is flat (facet) on a nontrivial interval (called a faceted region) containing each fixed point x whenever ∂W has a jump at $f_x(x)$. Such a function f is called a faceted function and we see that (5.1.3) admits a unique minimizer $\bar{\xi}$ for a faceted function f since the problem is convex. It is natural to guess that $\bar{\xi}'$ gives a candidate for the value of the nonlocal curvature

$$\Lambda_W^\sigma(f)(x) = (W'(f_x))_x + \sigma(x).$$

Based on this observation we establish a notion of viscosity solutions to (5.1.1).

We prove the existence theorem by Perron's method, which is standard in the theory of viscosity solutions for regular equations; we refer the reader to [18], [7]. In our problem, however, it is necessary to modify a smooth faceted test function keeping its property. In the previous work [12] it suffices to modify the test function outside the faceted region. However, this method heavily relies on the fact that the nonlocal curvature $\Lambda_W^\sigma(f)$ is constant on a faceted region when σ is independent of x.

The main idea to solve this problem is to find a small effective region which determines the quantity of the nonlocal curvature. We construct a modification as in the previous work [12] using the effective region instead of the faceted region. Then the argument works well for our setting with the spatially inhomogeneous driving force term σ.

This chapter is organized as follows. In Section 5.2 we recall the definition of faceted functions and the nonlocal curvature Λ_W^σ and define generalized solutions for the equations. In Section 5.3 we describe how to construct an effective region and modifications for test functions. In Section 5.4 we prove Perron type existence theorems and Section 5.5 is devoted to proving the existence theorem for periodic initial data.

5.2 Definition of solutions

In this section we recall some notions of functions and the nonlocal curvature Λ_W^σ introduced in [15, Section 2] and define generalized solutions for fully nonlinear equations of the form

$$u_t + F(t, u_x, \Lambda_W^\sigma(u)) = 0 \quad \text{in } Q := (0,T) \times U, \tag{5.2.1}$$

where $T > 0$ and U is an open set in $\mathbf{R}$. We assume the following conditions throughout this paper.

(W) Assume W is a convex function on $\mathbf{R}$ with values in $\mathbf{R}$ of class C^2 outside a closed discrete set P and that its second derivative W'' is bounded in any compact set except all points in P.

(S) The continuous function $\sigma = \sigma(t,x)$ on $[0,T] \times U$ is Lipschitz continuous in x uniformly with respect to t, i.e. there exists a constant L such that

$$|\sigma(t,x) - \sigma(t,y)| \leq L|x-y| \quad \text{for all } t \in [0,T),\ x,y \in U.$$

(F1) F is continuous on $[0,T] \times \mathbf{R} \times \mathbf{R}$ with values in $\mathbf{R}$.

(F2) $F(t,p,X) \leq F(t,p,Y)$ for all $t \in [0,T]$, $p \in \mathbf{R}$, $X \geq Y$.

The discrete set P in (W) is either a finite set or a countable set having no accumulation point in $\mathbf{R}$. If P is nonempty, P is of form $\{p_j\}_{j=1}^{m}$, $\{p_j\}_{j=1}^{\infty}$, $\{p_j\}_{j=-1}^{-\infty}$ or $\{p_j\}_{j=-\infty}^{\infty}$, where $\{p_j\}$ is a strictly increasing sequence $p_j < p_{j+1}$ with $\lim_{j\to\infty} p_j = \infty$ and $\lim_{j\to-\infty} p_j = -\infty$, and m is a positive integer. We often let $\sigma(t)$ denote the function $\sigma(t)(x) = \sigma(t,x)$ for $t \in [0,T)$. We say that a family of a functions σ_t on U is equi-Lipschitz continuous if there exists a constant L such that $|\sigma_t(x) - \sigma_t(y)| \leq L|x-y|$ for all t and $x,y \in U$. Our assumption (S) is equivalent to saying that $\sigma(t)$ on U is equi-Lipschitz continuous.

5.2.1 Faceted functions

We first define a notion of a faceted function.

Definition 5.2.1 (Faceted function)**.** A function $f \in C^1(U)$ is *faceted* at a point $\hat{x} \in U$ with *slope* $p \in \mathbf{R}$ (or *p-faceted* at $\hat{x}$) if there exists a closed nontrivial finite interval $I = [c_l, c_r] \subset U$ containing $\hat{x}$ (i.e. $c_l, c_r \in U$ satisfy $c_l < c_r$ and $c_l \leq \hat{x} \leq c_r$) such that

$$\begin{aligned} f'(x) &= p \quad \text{for all } x \in I, \\ f'(x) &\neq p \quad \text{for all } x \in J \setminus I \end{aligned}$$

with some neighborhood $J = (b_l, b_r) \subset U$ of I. The closed interval I is called a *faceted region* of f containing $\hat{x}$. We say that a function f is P-faceted at $\hat{x}$ if f is p-faceted at $\hat{x}$ for some $p \in P$ and let

$$C_P^2(U) := \left\{ f \in C^2(U) \mid f \text{ is } P\text{-faceted at } \hat{x} \text{ whenever } f'(\hat{x}) \in P \right\}.$$

We also define the *left transition number* $\chi_l = \chi_l(f,\hat{x})$ and the *right transition number* $\chi_r = \chi_r(f,\hat{x})$ for a p-faceted function f at $\hat{x}$ by

$$\chi_l = \begin{cases} +1 & \text{if } f' < p \text{ on } (b_l, c_l), \\ -1 & \text{if } f' > p \text{ on } (b_l, c_l), \end{cases}$$

$$\chi_r = \begin{cases} +1 & \text{if } f' > p \text{ on } (c_r, b_r), \\ -1 & \text{if } f' < p \text{ on } (c_r, b_r). \end{cases}$$

Let $R(f,\hat{x}) = [c_l, c_r]$ denote a maximal closed interval containing $\hat{x}$ on which f' is constant, i.e.

$$\begin{aligned} c_l &:= \inf\{x \in U \mid f'(y) = f'(\hat{x}) \text{ for all } y \in [x, \hat{x}]\}, \\ c_r &:= \sup\{x \in U \mid f'(y) = f'(\hat{x}) \text{ for all } y \in [\hat{x}, x]\}. \end{aligned}$$

The interval $R(f,\hat{x})$ is nothing but the faceted region if f is a P-faceted function at $\hat{x}$.

Remark 5.2.2. We note that a p-faceted function f at $\hat{x}$ agrees with an affine function

$$\ell_p(x) := p(x - \hat{x}) + f(\hat{x})$$

on $I = R(f, \hat{x})$ and that

$$\chi_l = \begin{cases} +1 & \text{if } f > \ell_p \text{ on } (b_l, c_l), \\ -1 & \text{if } f < \ell_p \text{ on } (b_l, c_l), \end{cases}$$

$$\chi_r = \begin{cases} +1 & \text{if } f > \ell_p \text{ on } (c_r, b_r), \\ -1 & \text{if } f < \ell_p \text{ on } (c_r, b_r). \end{cases}$$

5.2.2 Nonlocal curvature with a nonuniform driving force

We next recall the definition of the nonlocal curvature for a smooth faceted function. Assume (W) and that

$$\sigma \text{ is a Lipschitz function on } U. \tag{5.2.2}$$

For $f \in C^2_P(U)$ and $\hat{x} \in U$ define the *nonlocal curvature* $\Lambda^\sigma_W(f)(\hat{x})$ as below.

On one hand, if $f'(\hat{x}) \notin P$, we set

$$\Lambda^\sigma_W(f)(\hat{x}) = W''(f'(\hat{x}))f''(\hat{x}) + \sigma(\hat{x})$$

as expected. On the other hand, if $p := f'(\hat{x}) \in P$, i.e. f is p-faceted at $\hat{x}$, the definition is more involved since it is based on the obstacle problem (5.1.3).

Let Z be a primitive function of σ and let

$$\Delta = |\partial W(p)| = \lim_{q \downarrow p} W'(q) - \lim_{q \uparrow p} W'(q).$$

We also take the faceted region $I = R(f, \hat{x}) = [c_l, c_r]$ and the transition numbers $\chi_l = \chi_l(f, \hat{x})$, $\chi_r = \chi_r(f, \hat{x})$. We note that

$$Z \in C^{1,1}(I),\ \Delta > 0,\ I \text{ is a nontrivial closed interval and } \chi_l, \chi_r \in [-1, 1]. \tag{5.2.3}$$

For later convenience we have defined K for χ_l, χ_r whose values are in $[-1, 1]$ not necessarily in $\{\pm 1\}$. Let $K = K^{Z,\Delta,I}_{\chi_l \chi_r}$ be the set of all $\xi \in H^1(I)$ satisfying an obstacle condition

$$Z(x) - \Delta/2 \le \xi(x) \le Z(x) + \Delta/2 \quad \text{for all } x \in I$$

and a boundary condition

$$\xi(c_l) = Z(c_l) - \chi_l \Delta/2, \quad \xi(c_r) = Z(c_r) + \chi_r \Delta/2.$$

We now consider the functional $J = J^{Z,\Delta,I}_{\chi_l \chi_r}$ on $L^2(I)$ defined by

$$J[\xi] = \begin{cases} \int_I |\xi'(x)|^2 \, dx & \text{if } \xi \in K, \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to see that K is a closed convex set with respect to H^1 norm and thus J admits a unique minimizer denoted by $\bar{\xi} = \xi^{Z,\Delta,I}_{\chi_l \chi_r}$.

An equivalent condition to being a minimizer of the obstacle problem is known. Assume (5.2.3). For $\xi \in K$ define the *upper coincidence set* D_+ and the *lower coincidence set* D_- by

$$D_\pm = D_\pm(\xi) = \{x \in I \mid \xi(x) = Z(x) \pm \Delta/2\}.$$

We say that ξ satisfies *concave-convex condition* if ξ is concave outside the upper coincidence set D_+ and convex outside the lower coincidence set D_-.

Proposition 5.2.3 (Characterization of minimizer)**.** *A function $\xi \in K$ is the minimizer of J if and only if ξ satisfies the concave-convex condition.*

This proposition is proved in the same way as in [15, Proposition 2.2], which shows the equivalence with the assumption $\chi_l, \chi_r = \pm 1$, and so we omit it. Noting that Proposition 5.2.3 in particular implies that the minimizer of the obstacle problem $\bar{\xi}$ belongs to $C^{1,1}(I)$, so we define

$$\Lambda^{Z'}_{\chi_l \chi_r}(x; I, \Delta) = \bar{\xi}'(x) \quad \text{for } x \in I.$$

The reason we write Z' instead of Z is that the derivative $\bar{\xi}'$ depends on Z only through its derivative. Proposition 5.2.3 also shows that restriction of $\bar{\xi}$ is also a minimizer of an obstacle problem on the restricted domain:

Corollary 5.2.4. *Let $M = [c_l, c_r] \subset I$ be a nontrivial closed interval. Then,*

$$\xi^{Z,\Delta,I}_{\chi_l \chi_r} = \xi^{Z,\Delta,M}_{\chi'_l \chi'_r} \quad \textit{on } M.$$

with

$$\chi'_l = 2(\bar{\xi}(c_l) - Z(c_l))/\Delta, \quad \chi'_r = 2(\bar{\xi}(c_r) - Z(c_r))/\Delta.$$

Definition 5.2.5 (Nonlocal curvature)**.** Assume (W) and (5.2.2). Let $f \in C^2_P(U)$ and $\hat{x} \in U$.

(i) If $f'(\hat{x}) \notin P$, then define

$$\Lambda^\sigma_W(f)(\hat{x}) = W''(f'(\hat{x}))f''(\hat{x}) + \sigma(\hat{x}).$$

(ii) If f is P-faceted at $\hat{x}$, then define

$$\Lambda^\sigma_W(f)(\hat{x}) = \Lambda^\sigma_{\chi_l \chi_r}(\hat{x}; I, \Delta)$$

with $\Delta = |\partial W(p)|$, $I = R(f, \hat{x})$, $\chi_l = \chi_l(f, \hat{x})$, $\chi_r = \chi_r(f, \hat{x})$.

We prepare several propositions on the nonlocal curvature.

Proposition 5.2.6 (Comparison)**.** *Assume (W) and* (5.2.2)*. Let $f, g \in C^2_P(U)$ and $\hat{x} \in U$. If $\max_U(f - g) = (f - g)(\hat{x})$, then*

$$\Lambda^\sigma_W(f)(\hat{x}) \le \Lambda^\sigma_W(g)(\hat{x}).$$

Proposition 5.2.7 (Continuity with respect to σ and x)**.** *Assume (W) and let $f \in C^2_P(U)$ and $\hat{x} \in U$. Let $y, y_k \in R(f, \hat{x})$ and equi-Lipschitz continuous functions σ, σ_k on U satisfy $y_k \to y$ and $\sigma_k \to \sigma$ uniformly. Then*

$$\Lambda^{\sigma_k}_W(f)(y_k) \to \Lambda^\sigma_W(f)(y).$$

Proposition 5.2.8 (Continuity with respect to I). *Assume* (5.2.2), $\chi_l, \chi_r = \pm 1$, $\Delta > 0$. *Let nontrivial intervals* $I = [c_l, c_r]$, $I^k = [c_l^k, c_r^k]$ *of* U *satisfy* $I^k \to I$, *i.e.* $c_l^k \to c_l$ *and* $c_r^k \to c_r$, *and let* $y \in I$, $y_k \in I_k$ *satisfy* $y_k \to y$. *Then*

$$\Lambda^{\sigma}_{\chi_l \chi_r}(y_k; I_k, \Delta) \to \Lambda^{\sigma}_{\chi_l \chi_r}(y; I, \Delta).$$

Proposition 5.2.6–5.2.8 are immediate consequence of [15, Theorem 2.8, 2.9, 2.12].

5.2.3 Admissible functions and definition of a generalized solution

We recall a natural class of test function.

Definition 5.2.9 (Admissible function). Let I and J be open intervals in $\mathbf{R}$. An *admissible* function on $Q := J \times I$ is a function φ of the form

$$\varphi(t, x) = f(x) + g(t) \quad \text{on } Q \tag{5.2.4}$$

with some functions $f \in C_P^2(I)$ and $g \in C^1(J)$. Let $A_P(Q)$ be the set of all admissible functions on Q.

We are now able to define a generalized solution in the viscosity sense for the singular parabolic equation (5.2.1). For a real-valued function u recall the *upper semicontinuous envelope* and the *lower semicontinuous envelope*

$$\begin{aligned} u^*(t, x) &:= \lim_{\varepsilon \downarrow 0} \sup\{u(s, y) \mid (s, y) \in Q,\ |s - t| + |y - x| < \varepsilon\}, \\ u_*(t, x) &:= \lim_{\varepsilon \downarrow 0} \inf\{u(s, y) \mid (s, y) \in Q,\ |s - t| + |y - x| < \varepsilon\} \end{aligned}$$

for $(t, x) \in \overline{Q}$.

Definition 5.2.10 (Viscosity solution). A real-valued function u on Q is a *viscosity subsolution* of (5.2.1) in Q if $u^* < \infty$ in $[0, T) \times \overline{U}$ and

$$\varphi_t(\hat{t}, \hat{x}) + F(\hat{t}, \varphi_x(\hat{t}, \hat{x}), \Lambda_W^{\sigma(\hat{t})}(\varphi(\hat{t}, \cdot))(\hat{x})) \leq 0 \tag{5.2.5}$$

whenever $(\hat{t}, \hat{x}) \in Q$ and $\varphi \in A_P(Q)$ satisfy

$$\max_Q (u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x}). \tag{5.2.6}$$

A real-valued function u on Q is a *viscosity supersolution* of (5.2.1) in Q if $u_* > -\infty$ in $[0, T) \times \overline{U}$ and

$$\varphi_t(\hat{t}, \hat{x}) + F(\hat{t}, \varphi_x(\hat{t}, \hat{x}), \Lambda_W^{\sigma(\hat{t})}(\varphi(\hat{t}, \cdot))(\hat{x})) \geq 0 \tag{5.2.7}$$

whenever $(\hat{t}, \hat{x}) \in Q$ and $\varphi \in A_P(Q)$ satisfy

$$\min_Q (u_* - \varphi) = (u_* - \varphi)(\hat{t}, \hat{x}). \tag{5.2.8}$$

If u is both a subsolution and a supersolution, u is called a *viscosity solution.*

Hereafter we suppress the word "viscosity". A function φ satisfying (5.2.6) or (5.2.8) is called a *test function* of u at $(\hat{t}, \hat{x})$.

The following propositions are easily derived.

Proposition 5.2.11 (Smooth solution and viscosity solution)**.** *We assume (W), (S), (F2). If $\varphi \in A_P(Q)$ of the form* (5.2.4) *with $f \in C^2_P(U)$ and $g \in C^1(0,T)$ satisfies* (5.2.5) *(resp.* (5.2.7)*) for each $(\hat{t}, \hat{x}) \in Q$, then φ is a subsolution (resp. supersolution) of* (5.2.1) *in Q.*

Proof. We only show that φ is a subsolution. Fix $\psi \in A_P(Q)$ of the form

$$\psi(t,x) = \tilde{f}(x) + \tilde{g}(t) \quad \text{on } Q$$

with $\tilde{f} \in C^2_P(U)$ and $\tilde{g} \in C^1(0,T)$, and suppose that

$$\varphi(t,x) - \psi(t,x) = f(x) - \tilde{f}(x) + g(t) - \tilde{g}(t)$$

attains a maximum at a point $(\hat{x}, \hat{t}) \in Q$. We then see that $f'(\hat{x}) = \tilde{f}'(\hat{x})$ and $g'(\hat{t}) = \tilde{g}'(\hat{t})$. Moreover, Proposition 5.2.6 yields

$$\Lambda^{\sigma(\hat{t})}_W(f)(\hat{x}) \le \Lambda^{\sigma(\hat{t})}_W(\tilde{f})(\hat{x}).$$

Therefore, we have

$$\tilde{g}'(\hat{t}) + F(\hat{t}, \tilde{f}'(\hat{x}), \Lambda^{\sigma(\hat{t})}_W(\tilde{f})(\hat{x})) \le g'(\hat{t}) + F(\hat{t}, f'(\hat{x}), \Lambda^{\sigma(\hat{t})}_W(f)(\hat{x})) \le 0$$

by (F2) and (5.2.5). $\square$

Proposition 5.2.12 (Addition by affine functions)**.** *Let u be a subsolution (resp. supersolution) of* (5.2.1) *in Q and $a, b \in \mathbf{R}$. Then $v(t,x) = u(t,x) - ax - b$ is a subsolution (resp. supersolution) of*

$$v_t + F(t, v_x + a, \Lambda^{\sigma}_{W_a}(v)) = 0 \textit{ in } Q,$$

where $W_a(p) = W(p+a)$.

In order to show the existence of a solution by Perron's method we define a local version of the notion of solutions. We say that a function $\varphi \in C(Q)$ is *locally admissible* at a point $(\hat{t}, \hat{x}) \in Q$ if φ is admissible on $J \times I$ with some bounded open intervals I and J such that $\hat{t} \in J \subset (0,T)$ and $\hat{x} \in I \subset U$.

Definition 5.2.13. A real-valued function u on Q is a *subsolution in the local sense* of (5.2.1) in Q if $u^* < \infty$ in $[0,T) \times \overline{U}$ and (5.2.5) holds for all locally admissible $\varphi \in C(Q)$ at $(\hat{t}, \hat{x}) \in Q$ satisfying (5.2.6). A *supersolution in the local sense* is defined by replacing $u^* < \infty$ by $u_* > -\infty$, the inequality (5.2.5) by (5.2.7) and the equality (5.2.6) by (5.2.8) as before.

Lemma 5.2.14. *A real-valued function u on Q is a subsolution (resp. supersolution) of* (5.2.1) *in Q if and only if u is a subsolution (resp. supersolution) in the local sense of* (5.2.1) *in Q.*

These facts can be shown by the same argument as in [12, Section 6].

5.3 Effective region and canonical modification

In this section we construct an upper and lower modification $f^{\#,\varepsilon}$ and $f_{\#,\varepsilon}$ for a faceted function f and a small number $\varepsilon > 0$. These modifications play an important role in order to prove a Perron type existence theorem in the next section.

Definition 5.3.1. Let $f \in C(U) \cap C_P^2(U_1)$ satisfy $f'(\hat{x}) = 0$ with an open interval $U_1 = (a_l, a_r) \subset U$ and $\hat{x} \in U_1$. Let

$$\begin{aligned} p_1 &= \sup\{p \in P \cup \{-\infty\} \mid p < 0\} \in [-\infty, 0), \\ p_2 &= \inf\{p \in P \cup \{\infty\} \mid p > 0\} \in (0, \infty]. \end{aligned}$$

Consider the case (i) $f'(\hat{x}) = 0 \notin P$. We then define $M = [d_l, d_r]$ by

$$d_l = d_r = \hat{x}, \quad \text{i.e. } M = \{\hat{x}\}$$

and set

$$f^{\#,\varepsilon}(x) = f^{\#}(x) = f(x) + (x - \hat{x})^4 \quad \text{for } x \in U.$$

Let us note that there exists an open neighborhood $U_2 = (b_l, b_r) \subset U_1$ of $\hat{x}$ such that

$$\frac{p_1}{2} < f'(x) < \frac{p_2}{2} \quad \text{for all } x \in U_2, \tag{5.3.1}$$

$$d_l + \frac{\sqrt[3]{p_1}}{2} \leq b_l < d_l, \quad d_r < b_r \leq d_r + \frac{\sqrt[3]{p_2}}{2}. \tag{5.3.2}$$

Consider the case (ii) $f'(\hat{x}) = 0 \in P$, i.e. f is P-faceted at $\hat{x}$. Take the faceted region $[c_l, c_r] = R(f, \hat{x})$ and the minimizer of the obstacle problem ξ. Define $M = [d_l, d_r]$ by

$$\begin{aligned} d_l &= \max\{x \leq \hat{x} \mid x \in D_-(\xi) \cup \{c_l\}\}, \\ d_r &= \min\{x \geq \hat{x} \mid x \in D_+(\xi) \cup \{c_r\}\}. \end{aligned}$$

Take an open interval $U_2 = (b_l, b_r) \subset U_1 \cap J$ such that (5.3.1) and (5.3.2) hold, where J is the neighborhood of $R(f, \hat{x})$ appearing in Definition 5.2.1. Define $f^{\#,\varepsilon}$ for each $\varepsilon > 0$ as below: First set

$$f^{\#,\varepsilon}(x) = f(x) = f(\hat{x}) \quad \text{for } x \in M = [d_l, d_r].$$

If $d_l \in D_-(\xi)$, set

$$f^{\#,\varepsilon}(x) = f(x) + (x - d_l)^4 \quad \text{for } x \in U,\ x \leq d_l.$$

If $d_l \notin D_-(\xi)$, that is $d_l = c_l$ and $d_l \in D_+(\xi)$, set

$$f^{\#,\varepsilon}(x) = \begin{cases} f(d_l) = f(\hat{x}) & \text{for } x \in [d_l - \varepsilon, d_l], \\ f(x + \varepsilon) & \text{for } x \in [b_l, d_l - \varepsilon], \\ f(x) + f(b_l + \varepsilon) - f(b_l) & \text{for } x \in U,\ x \leq b_l. \end{cases}$$

If $d_r \in D_+(\xi)$, set

$$f^{\#,\varepsilon}(x) = f(x) + (x - d_r)^4 \quad \text{for } x \in U,\ x \geq d_r.$$

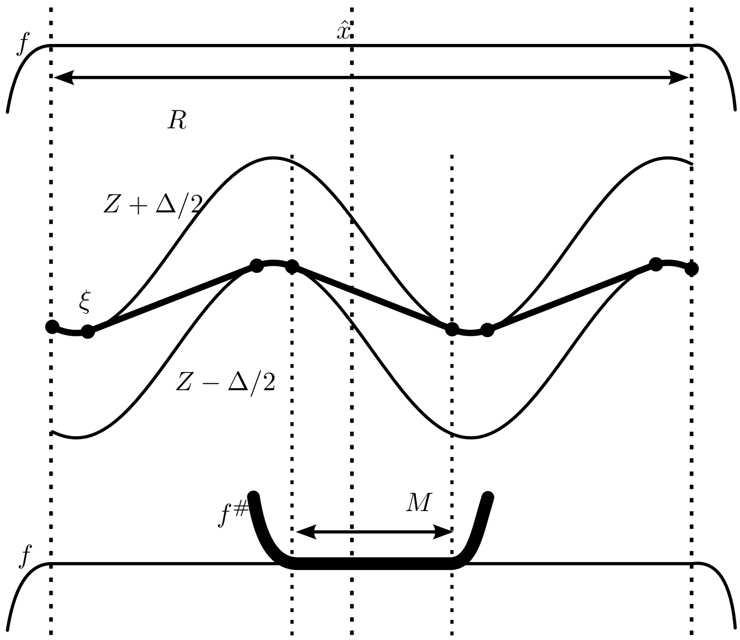

Figure 5.1: Construction of M and $f^{\#} = f^{\#,\varepsilon}$ (case $d_l \in D_-(\xi)$ and $d_r \in D_+(\xi)$)

If $d_r \notin D_+(\xi)$, that is $d_r = c_r$ and $d_r \in D_-(\xi)$, set

$$f^{\#,\varepsilon}(x) = \begin{cases} f(d_r) = f(\hat{x}) & \text{for } x \in [d_r, d_r + \varepsilon], \\ f(x - \varepsilon) & \text{for } x \in [d_r + \varepsilon, b_r], \\ f(x) + f(b_r - \varepsilon) - f(b_r) & \text{for } x \in U,\, x \geq b_r. \end{cases}$$

We call the function $f^{\#,\varepsilon}$ an *upper canonical modification* of f at $\hat{x}$ with an *effective region* M and a *canonical neighborhood* U_2. By a similar way we are able to construct a *lower canonical modification* $f_{\#,\varepsilon}$ with an *effective region* M and a *canonical neighborhood* U_2: Let $-f_{\#,\varepsilon}$ be an upper canonical modification of $-f$ at $\hat{x}$.

The figures below illustrate how to construct the effective region M and the upper canonical modification $f^{\#} = f^{\#,\varepsilon}$ when f is P-faceted at $\hat{x}$ and $\chi_l = \chi_r = -1$. While Figure 5.1 indicates the case $d_l \in D_-(\xi)$ and $d_r \in D_+(\xi)$, Figure 5.2 shows the cases $d_l \in D_-(\xi)$ and $d_r \notin D_+(\xi)$.

The upper and lower canonical modification fulfills

Proposition 5.3.2. *Assume (W). Let $U_1 = (a_l, a_r) \subset U$ be an open interval. For $f \in C(U) \cap C_P^2(U_1)$ and $\hat{x} \in U_1$ satisfying $f'(\hat{x}) = 0$, let f^ε be an upper canonical modification $f^{\#,\varepsilon}$ (resp. lower canonical modification $f^{\#,\varepsilon}$) with effective region $M = [d_l, d_r]$ and a canonical neighborhood $U_2 = (b_l, b_r)$ and let $s = 1$ (resp. $s = -1$). Let $y, y_\varepsilon \in M$, $y_k \in U$ and equi-Lipschitz functions σ, σ_k satisfy $y_\varepsilon \to y$, $y_k \to y$ and $\sigma_k \to \sigma$ uniformly. Then the conditions*

$$f^\varepsilon \in C(U) \cap C_P^2(U_2), \tag{5.3.3}$$

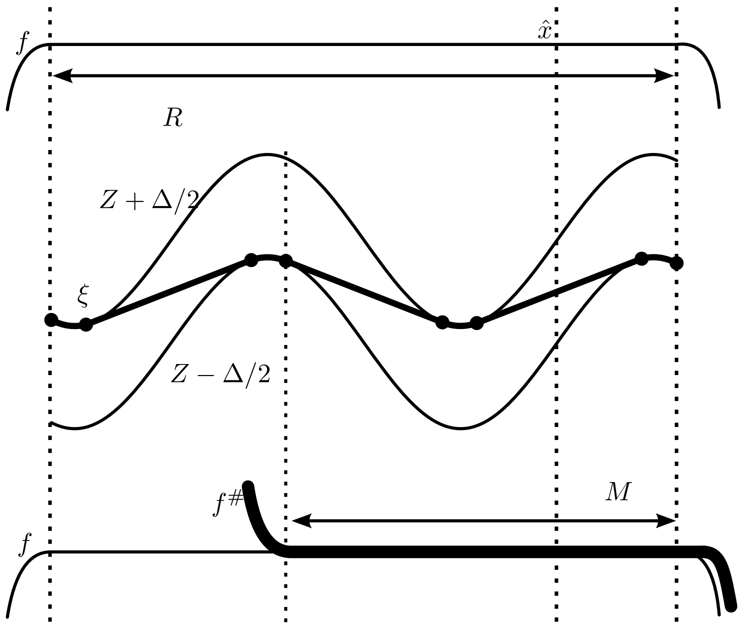

Figure 5.2: Construction of M and $f^{\#} = f^{\#,\varepsilon}$ (case $d_l \in D_-(\xi)$ and $d_r \notin D_+(\xi)$)

$$sf^\varepsilon > sf \quad \text{on } U \setminus M, \tag{5.3.4}$$

$$\inf_{U\setminus U_2} s(f^\varepsilon - f) > 0, \tag{5.3.5}$$

$$f^\varepsilon(y) = f(y) = f(\hat{x}), \tag{5.3.6}$$

$$\lim_k (f^\varepsilon)'(y_k) = (f^\varepsilon)'(y), \tag{5.3.7}$$

$$(f^\varepsilon)'(y) = f'(y) = f'(\hat{x}) = 0, \tag{5.3.8}$$

$$\limsup_k s\Lambda_W^{\sigma_k}(f^\varepsilon)(y_k) \le s\Lambda_W^\sigma(f^\varepsilon)(y) \tag{5.3.9}$$

hold for all $\varepsilon > 0$ *small enough, and*

$$\Lambda_W^\sigma(f^\varepsilon)(y_\varepsilon) \to \Lambda_W^\sigma(f)(y) \quad \text{as } \varepsilon \to 0, \tag{5.3.10}$$

$$s\Lambda_W^\sigma(f)(y) \le s\Lambda_W^\sigma(f)(\hat{x}) \tag{5.3.11}$$

hold.

Proof. We only consider the case $f^\varepsilon = f^{\#,\varepsilon}$ and $s = 1$. Since it is easy to verify the conditions (5.3.3)–(5.3.11) in the case (i) $f'(\hat{x}) = 0 \notin P$, we only consider the case (ii) f is P-faceted at $\hat{x}$. The conditions (5.3.3)–(5.3.8) are shown by the definition of the canonical modification.

Show (5.3.9). Take a subsequence k_j such that

$$\Lambda_W^{\sigma_{k_j}}(f^{\#,\varepsilon})(y_{k_j}) \to \limsup_k \Lambda_W^{\sigma_k}(f^{\#,\varepsilon})(y_k).$$

Since Proposition 5.2.7 implies

$$\Lambda_W^{\sigma_{k_j}}(f^{\#,\varepsilon})(y_{k_j}) \to \Lambda_W^{\sigma}(f^{\#,\varepsilon})(y)$$

provided that $y_{k_j} \in R^\varepsilon = [c_l^\varepsilon, c_r^\varepsilon] := R(f^{\#,\varepsilon}, \hat{x})$ for each j, we may assume that $y_{k_j} \notin R^\varepsilon$. Also it is enough to consider the case $y_{k_j} < c_l^\varepsilon$. Hence,

$$\begin{aligned}\Lambda_W^{\sigma_{k_j}}(f^{\#,\varepsilon})(y_{k_j}) &= W''((f^{\#,\varepsilon})'(y_{k_j}))(f^{\#,\varepsilon})''(y_{k_j}) + \sigma_{k_j}(y_{k_j})\\ &\to \sigma(y).\end{aligned}$$

Since $y_{k_j} \to y \in M \subset R^\varepsilon$, we observe that

$$y = c_l^\varepsilon = d_l \in D_-(\xi^\varepsilon),$$

where ξ^ε is the minimizer of the obstacle problem $\xi_{\chi_l'\chi_r'}^{Z,\Delta,R^\varepsilon}$ with a primitive Z of σ, $\Delta = |\partial W(0)|$, $R^\varepsilon =$, $\chi_l' = \chi_l(f^{\#,\varepsilon}, \hat{x})$, $\chi_r' = \chi_r(f^{\#,\varepsilon}, \hat{x})$. Noting that $\xi^\varepsilon - Z + \Delta/2$ attains zero minimum at y, we have

$$\sigma(y) \le \Lambda_W^{\sigma}(f^{\#,\varepsilon})(y),$$

and hence

$$\limsup_k \Lambda_W^{\sigma_k}(f^\varepsilon)(y_k) = \lim_j \Lambda_W^{\sigma_{k_j}}(f^\varepsilon)(y_{k_j}) \le \Lambda_W^{\sigma}(f^{\#,\varepsilon})(y).$$

Show (5.3.10). Write $R = R(f, \hat{x})$, $\chi_l = \chi_l(f, \hat{x})$, $\chi_r = \chi_r(f, \hat{x})$ so that $\xi = \xi_{\chi_l\chi_r}^{Z,\Delta,R}$. Also note that χ_l' and χ_r' are independent of ε. It follows from Corollary 5.2.4 that

$$\Lambda_W^{\sigma}(f)(y) = \Lambda_{\chi_l\chi_r}^{\sigma}(y; R, \Delta) = \Lambda_{\chi_l'\chi_r'}^{\sigma}(y; M, \Delta).$$

Since $R^\varepsilon \to M$ as $\varepsilon \to 0$, we see by Proposition 5.2.8 that

$$\Lambda_W^{\sigma}(f^{\#,\varepsilon})(y_\varepsilon) = \Lambda_{\chi_l'\chi_r'}^{\sigma}(y_\varepsilon; R^\varepsilon, \Delta) \to \Lambda_{\chi_l'\chi_r'}^{\sigma}(y; M, \Delta) = \Lambda_W^{\sigma}(f)(y).$$

Since $y \notin D_-(\xi)$ for all $y \in [\hat{x}, d_r)$, we see that ξ is concave on $[\hat{x}, d_r]$ by the concave-convex condition of ξ. By a similar argument we see that ξ is convex on $[d_l, \hat{x}]$. Therefore we obtain (5.3.11) for all $y \in M$. □

5.4 Stability results

In this section we show Perron type existence theorem. Let U be an open set in $\mathbf{R}$ and $Q = (0, T) \times U$.

Theorem 5.4.1 (Perron type existence)**.** *Assume (W), (S), (F1), (F2). Let u^- and u^+ respectively be a subsolution and a supersolution of* (5.2.1) *satisfying*

$$u^- \le u^+ \text{ in } Q, \quad (u^-)_* > -\infty,\ (u^+)^* < \infty \text{ on } [0, T) \times U. \tag{5.4.1}$$

(1) Then, there exists a solution u of (5.2.1) *satisfying*

$$u^- \le u \le u^+ \quad \text{in } Q. \tag{5.4.2}$$

(2) Moreover, if

$$\sigma(t, x + \omega) = \sigma(t, x) \tag{5.4.3}$$

$$u^-(t, x + \omega) = u^-(t, x), \quad u^+(t, x + \omega) = u^+(t, x) \tag{5.4.4}$$

for all $(t, x) \in Q$ with $\omega > 0$ and $U = \mathbf{R}$, then there exists a solution u of (5.2.1) *satisfying* (5.4.2) *and*

$$u(t, x + \omega) = u(t, x) \quad \text{for all } (t, x) \in Q. \tag{5.4.5}$$

We divide the main part of the proof into two lemmas.

Lemma 5.4.2. *Assume (W), (S), (F1), (F2). Let $\mathcal{S}$ be a nonempty family of subsolutions (resp. supersolutions) of* (5.2.1). *Define*

$$u(t, x) = \sup\{v(t, x) \mid v \in \mathcal{S}\} \ (\text{resp. } v(t, x) = \inf\{v(t, x) \mid v \in \mathcal{S}\})$$

for $(t, x) \in Q$. Assume that $u^* < \infty$ (resp. $v_* > -\infty$) in $[0, T) \times \overline{U}$. Then u is a subsolution (resp. supersolution) of (5.2.1).

Lemma 5.4.3. *Assume (W), (S), (F1), (F2). Let $\mathcal{S}$ be the set of all subsolutions u of* (5.2.1) *satisfying $v \le u^+$ in Q with a supersolution u^+ of* (5.2.1). *If $u \in \mathcal{S}$ is not a supersolution of* (5.2.1) *and satisfies $u_* > -\infty$ in $[0, \infty) \times U$, then there exist a function $v \in \mathcal{S}$ and a point $(s, y) \in Q$ such that $u(s, y) < v(s, y)$.*

We first show the Perron type existence theorems under the assumption that Lemma 5.4.2 and 5.4.3 hold.

Proof of Theorem 5.4.1. we shall show the part (1). Let $\mathcal{S}$ be the set of all subsolutions v of (5.2.1) satisfying $v \le u^+$ in Q. Note that $\mathcal{S}$ is not empty since $u^- \in \mathcal{S}$. Define

$$u(t, x) = \sup\{v(t, x) \mid v \in \mathcal{S}\} \quad \text{for } (t, x) \in Q.$$

We then have $u^- \le u \le u^+$ in Q, which implies $u_* \ge (u^-)_* > -\infty$ and $u^* \le (u^+)^* < \infty$ on $[0, T) \times U$. We hence see that u is a subsolution by Proposition 5.4.2. We next claim that u is a supersolution. If u were not a supersolution, Proposition 5.4.3 would imply that there exist $v \in \mathcal{S}$ and $(s, y) \in Q$ such that $u(s, y) < v(s, y)$, which contradicts the maximality of u. Therefore, we conclude that u is a solution.

It remains to show (5.4.5). Note that for $v \in \mathcal{S}$ the periodicity conditions (5.4.3) and (5.4.4) imply that $\tilde{v}(t, x) = v(t, x \pm \omega) \in \mathcal{S}$. Hence, we see that

$$u(t, x + \omega) = \sup\{v(t, x + \omega) \mid v \in \mathcal{S}\} = \sup\{v(t, x) \mid v \in \mathcal{S}\} = u(t, x).$$

The proof is now complete. □

We next show the lemmas. We note that being a subsolution is equivalent to being a subsolution in the local sense by Lemma 5.2.14.

Proof of Lemma 5.4.2. We only show that u is a subsolution (in the local sense). Fix a point $(\hat{t}, \hat{x}) \in Q$ and a locally admissible test function $\varphi \in C(Q)$ at $(\hat{t}, \hat{x})$ such that (5.2.6) holds. Our goal is to show (5.2.5). Since φ is locally admissible, there exist $f \in C_P^2(U_1)$ and $g \in C^1(I)$ with open intervals U_1 and J such that

$$\begin{aligned} &\varphi(t, x) = f(x) + g(t) \quad \text{on } Q_1 := J \times U_1, \\ &R(f, \hat{x}) \subset U_1 \subset U, \quad \hat{t} \in J \subset (0, T). \end{aligned} \tag{5.4.6}$$

We may assume that

$$(u^* - \varphi)(\hat{t}, \hat{x}) = 0, \quad \varphi_x(\hat{t}, \hat{x}) = 0$$

Therefore, the desired inequality (5.2.5) becomes

$$g'(\hat{t}) + F(\hat{t}, 0, \Lambda_W^{\sigma(\hat{t})}(f)(\hat{x})) \leq 0, \tag{5.4.7}$$

which we should show.

We now let $\psi \in C(Q)$ be an $A_P(Q_2)$ function such that

$$\psi = \varphi \text{ on } K, \quad \psi > \varphi \text{ on } Q \setminus K, \quad \inf_{Q \setminus Q_2} (\psi - \varphi) > 0 \tag{5.4.8}$$

with a closed set K and an open set Q_2 satisfying

$$(\hat{x}, \hat{t}) \in K \subset Q_2 \subset Q.$$

The function ψ is to be determined later. By the definition of the upper semi-continuous envelope there exists a sequence $\{(t_k, s_k)\}_{k \in \mathbf{N}} \subset Q_2$ such that

$$(t_k, x_k, u(t_k, x_k)) \to (\hat{t}, \hat{x}, u^*(\hat{t}, \hat{x})) \quad \text{as } k \to \infty.$$

By the definition of u there exists $\{v_k\}_{k \in \mathbf{N}} \subset \mathcal{S}$ such that

$$v_k(t_k, x_k) > u(t_k, x_k) - 1/k$$

and so

$$v_k(t_k, x_k) \to u^*(\hat{t}, \hat{x}) \quad \text{as } k \to \infty.$$

Taking a maximizer (s_k, y_k) of $v_k^* - \psi$ on $\overline{Q_2}$, we observe that

$$((v_k)^* - \psi)(t_k, x_k) \leq ((v_k)^* - \psi)(s_k, y_k) \leq (u^* - \psi)(s_k, y_k)$$

for each k. Sending $k \to \infty$ yields

$$(u^* - \psi)(\hat{t}, \hat{x}) \leq (u^* - \psi)(\bar{s}, \bar{y}),$$

where

$$(\bar{s}, \bar{y}) = \lim_{k \to \infty} (s_k, y_k) \in \overline{Q_2}$$

by taking a subsequence if necessary. We see that $(\bar{s}, \bar{y}) \in K$ and $(s_k, y_k) \in Q_2$ for sufficiently large k. We also note that

$$\max_Q ((v_k)^* - \psi) = ((v_k)^* - \psi)(s_k, y_k),$$

i.e. ψ is a test function of v_k at (s_k, y_k) by the last inequality of (5.4.8).

Let $f^{\#,\varepsilon}$ be an upper canonical modification of f at $\hat{x}$ with effective region M and canonical neighborhood $U_2 \subset U_1$ for $\varepsilon > 0$. We then see that

$$\psi(x,t) = f^{\#,\varepsilon}(x) + g(t) + (t - \hat{t})^2$$

is an admissible function on a set $Q_2 = J \times U_2 \subset Q_1$ and that (5.4.8) holds with $K = \{\hat{t}\} \times M$ by Proposition 5.3.2. By the above argument we have $v_k^\varepsilon \in \mathcal{S}$ and $(s_k^\varepsilon, y_k^\varepsilon) \in Q_2$ such that

$$(s_k^\varepsilon, y_k^\varepsilon) \to (\hat{t}, y^\varepsilon) \in \{\hat{t}\} \times M \quad \text{as } k \to \infty$$

and ψ is a test function of v_k^ε at $(s_k^\varepsilon, y_k^\varepsilon)$. Since v_k^ε is a subsolution, we have

$$g'(s_k^\varepsilon) + 2(s_k^\varepsilon - \hat{t}) + F(s_k^\varepsilon, (f^{\#,\varepsilon})'(y_k^\varepsilon), \Lambda_W^{\sigma(s_k^\varepsilon)}(f^{\#,\varepsilon})(y_k^\varepsilon)) \leq 0. \tag{5.4.9}$$

Proposition 5.3.2 implies that

$$\lim_{k\to\infty} (f^{\#,\varepsilon})'(y_k^\varepsilon) = f'(\hat{x}),$$
$$\lim_{\varepsilon\to 0} \limsup_{k\to\infty} \Lambda_W^{\sigma(s_k^\varepsilon)}(f^{\#,\varepsilon})(y_k^\varepsilon) \leq \Lambda_W^{\sigma(\hat{t})}(f)(\hat{x}).$$

Therefore, it follows from (5.4.9) that (5.4.7) holds by (F1) and (F2).

We conclude that u is a subsolution. □

Proof of Lemma 5.4.3. Since u is not a supersolution, there exist $(\hat{x}, \hat{t}) \in Q$ and a locally admissible test function $\varphi \in C(Q)$ at $(\hat{t}, \hat{x})$ such that (5.2.6) and

$$\varphi_t(\hat{t}, \hat{x}) + F(\hat{t}, \varphi_x(\hat{t}, \hat{x}), \Lambda_W^{\sigma(\hat{t})}(\varphi(\hat{t}, \cdot))(\hat{x})) < 0 \tag{5.4.10}$$

hold. Since φ is locally admissible, there exist $f \in C_P^2(U_1)$ and $g \in C^1(J)$ with open intervals U_1 and J such that (5.4.6) holds with $Q_1 := J \times U_1$. We may assume that

$$(u_* - \varphi)(\hat{t}, \hat{x}) = 0, \quad \varphi_x(\hat{t}, \hat{x}) = 0$$

by Proposition 5.2.12 with $a = \varphi_x(\hat{t}, \hat{x}) = f'(\hat{x})$ and $b = u_*(\hat{x}, \hat{t}) - f'(\hat{x})\hat{x}$. Therefore, the inequality (5.4.10) becomes

$$g'(\hat{t}) + F(\hat{t}, 0, \Lambda_W^{\sigma(\hat{t})}(f)(\hat{x})) < 0. \tag{5.4.11}$$

Take a lower canonical modification $f_{\#,\varepsilon}$ of f at $\hat{x}$ for $\varepsilon > 0$ with effective region M and canonical neighborhood $U_2 \subset U_1$. Set

$$\psi(x,t) = f_{\#,\varepsilon}(x) + g(t) - (t - \hat{t})^2.$$

We now claim that

$$\psi_t(t,x) + F(t, \psi_x(t,x), \Lambda_W^{\sigma(t)}(\varphi(t,\cdot))(x)) < 0, \tag{5.4.12}$$

i.e.

$$g'(t) - 2(t - \hat{t}) + F(t, (f_{\#,\varepsilon})'(x), \Lambda_W^{\sigma(t)}(f_{\#,\varepsilon})(x)) < 0 \tag{5.4.13}$$

for all (t,x) in some neighborhood of $K := \{\hat{t}\} \times M$ choosing ε small enough. Indeed, since Proposition 5.3.2 implies that

$$\lambda^\varepsilon(t,x) := \Lambda_W^{\sigma(t)}(f_{\#,\varepsilon})(x)$$

is lower semicontinuous at each point of the compact set K, we see that for every $m > 0$ there exists an open set $Q_3 \supset K$ on which the inequality

$$\lambda^\varepsilon(t,x) > \min_K \lambda^\varepsilon - m$$

holds. Choose $y_\varepsilon \in M$ such that $(\hat{t}, y_\varepsilon)$ is a minimum point of λ^ε on $K = \{\hat{t}\} \times M$. Proposition 5.3.2 implies that

$$\Lambda_W^{\sigma(t)}(f_{\#,\varepsilon})(x) > \Lambda_W^{\sigma(\hat{t})}(f)(\hat{x}) - m$$

for all $(t,x) \in Q_3$ with small ε and m. Since Proposition 5.3.2 also implies that

$$|(f_{\#,\varepsilon})'(x)| < m,$$

it follows from (5.4.9) that (5.4.13) and so (5.4.12) holds on Q_3 by (F1) and (F2).

We next claim that $\psi < (u^+)_*$ in Q_3. First note that $\psi \le \varphi \le u \le u^+$ and so $\psi \le (u^+)_*$. If $\psi(t,x) = (u^+)_*(t,x)$ at some point $(t,x) \in Q_3$, then ψ would be a test function of the supersolution u^+ at (t,x). Hence,

$$\psi_t(t,x) + F(t, \psi_x(t,x), \Lambda_W^{\sigma(t)}(\psi(t,\cdot))(x)) \ge 0,$$

which contradicts to (5.4.12).

Take a bounded open set Q_4 such that $K \subset Q_4$ and $\overline{Q_4} \subset Q_3$. Letting $\sigma_1 = \inf_{Q_4}((u^+)_* - \psi) > 0$, we have

$$\psi + \sigma_1 \le (u^+)_* \quad \text{in } Q_4.$$

Since $f_{\#,\varepsilon} < f$ on $U_2 \setminus M$ by Proposition 5.3.2, we also have

$$\psi + \sigma_2 \le u_* \quad \text{in } Q_3 \setminus Q_4$$

with $\sigma_2 = \inf_{Q_3 \setminus Q_4}(u_* - \psi) > 0$. Define a function v by

$$v(t,x) = \begin{cases} \max\{\psi(t,x) + \sigma, u(t,x)\} & \text{for } (t,x) \in Q_3, \\ u(t,x) & \text{for } (t,x) \notin Q_3. \end{cases}$$

with $\sigma = \min\{\sigma_1, \sigma_2\}$. We show that this function v is a desirable function in the statement of this lemma.

Note that $v \ge u$. In addition, since $(u_* - \psi)(\hat{t}, \hat{x}) = 0$, there exists $(s,y) \in Q_4$ such that $(u - \psi)(s,y) < \sigma$, which implies

$$v(s,y) > u(s,y).$$

Since

$$\psi(t,x) + \sigma \le \begin{cases} (u^+)_*(t,x) & \text{if } (t,x) \in Q_4, \\ u_*(t,x) & \text{if } (t,x) \in Q_3 \setminus Q_4, \end{cases}$$

and $u \le u^+$ in Q, we have

$$v = u \quad \text{on } Q \setminus Q_4,$$

$$v \le u^+ \quad \text{in } Q.$$

Noting that ψ is a subsolution of (5.2.1) in Q_3, we see that v is a subsolution of (5.2.1) in Q_3 by Lemma 5.2.12 and 5.4.2. Therefore, if we admit the next lemma, we have $w \in \mathcal{S}$ and so the proof is finished. $\square$

Lemma 5.4.4. *Assume (W), (S), (F1), (F2). Let u be a subsolution of* (5.2.1) *in Q. Let w be a function defined on Q such that $w \geq u$ in Q, $w = u$ in $Q \setminus \overline{N_2}$, and w is a subsolution of* (5.2.1) *in N_1 with open rectangle sets $N_1 = J_1 \times I_1$, $N_2 = J_2 \times I_2$ satisfying $\overline{N_2} \subset N_1$, $\overline{N_1} \subset Q$. Then w is a subsolution of* (5.2.1) *in Q.*

In the classical setting, say $P = \emptyset$, this is easy to prove; if a function is a solution in two domains, then it is a solution in their union. However, this assertion on locality of solutions is not true for our equation (5.2.1) in general.

Proof. Fix a point $(\hat{t}, \hat{x}) \in Q$ and a locally admissible test function $\varphi \in C(Q)$ of w at $(\hat{t}, \hat{x})$, i.e. $\max_Q(w^* - \varphi) = (w^* - \varphi)(\hat{t}, \hat{x})$. Since φ is locally admissible, there exist $f \in C^2_P(U_1)$ and $g \in C^1(J)$ with open intervals U_1 and J such that (5.4.6) holds. We may assume that

$$(w^* - \varphi)(\hat{t}, \hat{x}) = 0, \quad \varphi_x(\hat{t}, \hat{x}) = 0$$

by Proposition 5.2.12 with $a = \varphi_x(\hat{t}, \hat{x}) = f'(\hat{x})$ and $b = w^*(\hat{x}, \hat{t}) - f'(\hat{x})\hat{x}$. We should show (5.4.7).

It is enough to consider the case

$$(\hat{t}, \hat{x}) \in N_2, \quad \varphi(\hat{t}, \hat{x}) > u(\hat{t}, \hat{x});$$

otherwise, φ is a test of the subsolution u and so we have (5.4.7). We may also assume that f is P-faceted at $\hat{x}$ and $R(f, \hat{x})$ is not contained by I_1; otherwise, (5.4.7) holds since w is a subsolution in N_1.

Let $f^{\#} = f^{\#,\varepsilon}$ be a upper canonical modification of f at $\hat{x}$ with effective region M and canonical neighborhood $U_2 \subset U_1$. Set

$$\psi(x, t) = f^{\#}(x) + g(t) + (t - \hat{t})^2.$$

We then observe that

$$\psi > \varphi \geq w^* \geq u^* \quad \text{in } Q \setminus \{\hat{t}\} \times M.$$

Let us assume for the moment that $\psi(\hat{t}, x_0) = u^*(\hat{t}, x_0)$ at some $x_0 \in M$. Then, since ψ is a test function of the subsolution u at $(\hat{t}, x_0)$, we have

$$g'(\hat{t}) + F(\hat{t}, (f^{\#})'(x_0), \Lambda_W^{\sigma(\hat{t})}(f^{\#})(x_0)) \leq 0.$$

Proposition 5.3.2 yields (5.4.7) by (F2). Therefore, we have

$$\psi > u^* \quad \text{in } Q \tag{5.4.14}$$

We now take a faceted function whose faceted region is contained in I_1; set

$$\tilde{f}^{\#}(x) = \begin{cases} f^{\#}(x) + k|x - c_l|^3(|x - c_l| - 1) & \text{for } x \in U,\ x \leq c_l \\ f^{\#}(x) & \text{for } x \in [c_l, c_r] \\ f^{\#}(x) + k|x - c_r|^3(|x - c_r| - 1) & \text{for } x \in U,\ x \geq c_r, \end{cases}$$

where $I_2 \subset [c_l, c_r] \subset I_1$ and $k > 0$. Note that

$$\tilde{\psi}(t, x) := \tilde{f}^{\#}(x) + g(t) + (t - \hat{t})^2$$

is locally admissible in N_1. Taking k small enough, we have

$$\tilde{\psi} > u^* \quad \text{for } (t,x) \in Q$$

by (5.4.14). Noting that

$$w = u \text{ in } Q \setminus \overline{N_2}, \quad \tilde{\psi} = \psi \text{ in } I_1 \times [c_l, c_r] \supset \overline{N_2},$$

we see that $\max_Q(w^* - \tilde{\psi}) = (w^* - \tilde{\psi})(\hat{t}, \hat{x})$. Since $\tilde{\psi}$ is a test function,

$$g'(\hat{t}) + F(\hat{t}, (\tilde{f}^{\#})'(\hat{x}), \Lambda_W^{\sigma(\hat{t})}(\tilde{f}^{\#})(\hat{x})) \leq 0.$$

Note that Proposition 5.2.6 yields

$$\Lambda_W^{\sigma(\hat{t})}(\tilde{f}^{\#})(\hat{x}) \leq \Lambda_W^{\sigma(\hat{t})}(f^{\#})(\hat{x}).$$

Therefore, we have

$$g'(\hat{t}) + F(\hat{t}, (f^{\#})'(\hat{x}), \Lambda_W^{\sigma(\hat{t})}(f^{\#})(\hat{x})) \leq 0,$$

which gives (5.4.7). □

5.5 Existence theorem for periodic initial data

In this section we prove an existence theorem for the equation (5.2.1) with periodic boundary condition and initial condition. In order to utilize the Perron type existence theorem (Theorem 5.4.1) we construct a subsolution u^- and a supersolution u^+ with given initial data; for a general strategy; see [16].

Lemma 5.5.1 (Existence of sub- and supersolutions)**.** *Assume (W), (S), (F1), (F2) with $U = \mathbf{R}$. Also assume that u_0 is a bounded and uniformly continuous function on $\mathbf{R}$ and σ is bounded. Then, there exist an upper semicontinuous function u^+ and an lower semicontinuous function u^- on $\overline{Q}$ such that u^+ and u^- respectively are a supersolution and a subsolution of* (5.2.1) *in Q and*

$$u^-(0,x) = u_0(x) = u^+(0,x), \quad u^-(t,x) \leq u_0(x) \leq u^+(t,x)$$

holds for all $(t,x) \in Q$. Moreover, if

$$u_0(x + \omega) = u_0(x), \tag{5.5.1}$$

then $u^\pm$ can be taken so that it is spatially periodic with period ω, i.e. (5.4.4) *holds.*

We show this existence theorem as in [12, Section 9].

Lemma 5.5.2 ([12, Lemma 9.5])**.** *For each $\delta \in (0, 1/2)$ and $M > 0$ there exists $V = V_{\delta,M} \in C_P^2(\mathbf{R})$ such that*

$$V \geq 0, \quad V'' \geq 0 \textit{ in } \mathbf{R}, \quad V(0) = 0, \quad V(x) \geq M \textit{ for } |x| > \delta, \tag{5.5.2}$$

$$V'(x) = \begin{cases} q & \textit{for } x \leq -1, \\ q' & \textit{for } x \geq 1 \end{cases} \tag{5.5.3}$$

with some $q, q' \notin P$.

We need to show

Lemma 5.5.3. *Let* $V \in C^2_P(\mathbf{R})$ *be such that* $V'' \geq 0$ *and* (5.5.3) *holds with some* $q, q' \notin P$. *Then for* $B \in \mathbf{R}$ *large enough*

$$V^+(t,x) = Bt + V(x) \tag{5.5.4}$$

is a supersolution of (5.2.1) *in* $(0,T) \times \mathbf{R}$.

Proof. We first claim that

$$C := \sup\left\{|\Lambda^{\sigma(t)}_W(V)(x)| \mid (t,x) \in Q\right\} < \infty. \tag{5.5.5}$$

Note that $V'(x) \in [q, q']$ for $x \in \mathbf{R}$ and

$$\sup_{\mathbf{R}} |V''| = \sup_{[-1,1]} |V''| < \infty.$$

Moreover, we have

$$\sup_{p \in [q,q'] \setminus P} |W''(p)| < \infty, \quad \sup_Q |\sigma| < \infty.$$

Therefore, for each $(t,x) \in Q$ with $V'(x) \notin P$ we observe that

$$\begin{aligned} |\Lambda^{\sigma(t)}_W(V)(x)| &\leq |W''(V'(x))||V''(x)| + |\sigma(t,x)| \\ &\leq \sup_{p \in [q,q'] \setminus P} |W''(p)| \sup_{\mathbf{R}} |V''| + \sup_Q |\sigma| < \infty. \end{aligned} \tag{5.5.6}$$

We shall show that

$$c_p := \sup\left\{|\Lambda^{\sigma(t)}_W(V)(x)| \mid (t,x) \in Q,\ V'(x) = p\right\} < \infty$$

for each $p \in P$. Indeed, since a faceted region $R = \{x \in \mathbf{R} \mid V'(x) = p\}$ is a bounded closed interval, Proposition 5.2.7 implies that $(t,x) \mapsto \Lambda^{\sigma(t)}_W(V)(x)$ is continuous on $[0,T] \times R$, and so $c_p < \infty$. We note that the number of faceted regions of V is finite, i.e. $P \cap [q, q']$ is finite by (W). Hence we have

$$\sup\left\{|\Lambda^{\sigma(t)}_W(V)(x)| \mid (t,x) \in Q,\ V'(x) \in P\right\} = \sup_{p \in P \cap [q,q']} c_p < \infty. \tag{5.5.7}$$

Combining (5.5.6) and (5.5.7), we obtain (5.5.5). Moreover, we see that

$$F(t, V'(x), \Lambda^{\sigma(t)}_W(V)(x)) \geq \inf_{[0,T] \times [q,q'] \times [-C,C]} F =: -B_0 > -\infty.$$

Therefore, V^+ in (5.5.4) is a supersolution of (5.2.1) for $B \geq B_0$. $\square$

Proof of Lemma 5.5.1. Let δ be a modulus of continuity of u_0; δ is a continuous nondecreasing function on $[0, \infty)$ with $\delta(0) = 0$ such that

$$|u_0(x) - u_0(y)| \leq \delta(|x - y|) \quad \text{for } x, y \in \mathbf{R}.$$

By Lemma 5.5.2 and 5.5.3 take $V_\delta = V_{\delta,M} \in C^2_P(\mathbf{R})$ and $B_\delta \geq 0$ for small δ and $M = \max u_0 - \min u_0$ satisfying (5.5.2) and that $V^+_\delta(t,x) = B_\delta t + V_\delta(x)$ is a supersolution of (5.2.1). Define

$$u^+_{\varepsilon,\xi}(t,x) := V^+_{\delta(\varepsilon)}(t, x - \xi) + u_0(\xi) + \varepsilon$$

for small $\varepsilon > 0$ and $\xi \in \mathbf{R}$. Note that $u^+_{\varepsilon,\xi}$ is a supersolution of (5.2.1) and

$$u^+_{\varepsilon,\xi}(t,x) \geq V_{\delta(\varepsilon)}(x-\xi) + u_0(\xi) + \varepsilon.$$

On the case $|\xi - x| \leq \delta(\varepsilon)$ we observe that

$$u^+_{\varepsilon,\xi}(t,x) \geq u_0(\xi) + \varepsilon \geq u_0(x);$$

on the other case

$$u^+_{\varepsilon,\xi}(t,x) \geq M + u_0(\xi) \geq u_0(x).$$

Therefore, Lemma 5.4.2 implies that

$$u^+(t,x) := \inf_{\varepsilon>0,\xi\in\mathbf{R}} u^+_{\varepsilon,\xi}(t,x)$$

is an upper semicontinuous supersolution of (5.2.1) satisfying $u^+ \geq u_0$. Moreover, since

$$u^+_{\varepsilon,x}(0,x) = u_0(x) + \varepsilon \to u_0(x) \quad \text{as } \varepsilon \to 0,$$

we have $u^+(0,x) = u_0(x)$ for all $x \in \mathbf{R}$. Under the assumption that u_0 is periodic we see that

$$\begin{aligned} u^+(t,x+\omega) &= \inf_{\varepsilon>0,\xi\in\mathbf{R}} (V^+_{\delta(\varepsilon)}(t,x+\omega-\xi) + u_0(\xi) + \varepsilon) \\ &= \inf_{\varepsilon>0,\xi\in\mathbf{R}} (V^+_{\delta(\varepsilon)}(t,x-\xi) + u_0(\xi+\omega) + \varepsilon) = u^+(t,x). \end{aligned}$$

The same proof is valid for existence of a subsolution u^-. □

Combining Theorem 5.4.1 and 5.5.1 we have

Theorem 5.5.4 (Existence theorem for periodic initial data)**.** *Assume (W), (S), (F1), (F2) and* (5.4.3) *with* $U = \mathbf{R}$ *and* $\omega > 0$*. Let* u_0 *be a continuous function satisfying* (5.5.1)*. Then there exists a solution* u *of* (5.2.1) *satisfying* (5.4.5) *and*

$$u(0,x) = u_0(x) \quad \textit{for all } x \in \mathbf{R}.$$

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Appendix A

Introduction to the theory of viscosity solutions

A.1 Extended real numbers

In this section we introduce a notion of so-called extended real numbers. It is a central concept of calculus to take a limit of real-valued functions. However, the limit may become very large or small. In order to describe this situation we invoke a notion of infinity and the extended real numbers. A point is that the order and topological properties of the real numbers can be naturally extended to the extended real numbers. We refer the readers to Bourbaki's textbook [2] for detail.

Let $\mathbf{R}$ denote the set of all real numbers and let $\pm\infty$ be the positive and negative infinity. All real numbers and $\pm\infty$ are called *extended real numbers*. Define

$$\bar{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}.$$

We extend the order relation $\leq$ to the extended real numbers $\bar{\mathbf{R}}$ by

$$a \leq +\infty,\ -\infty \leq b \text{ for all } a, b \in \bar{\mathbf{R}}.$$

Intervals of $\bar{\mathbf{R}}$ are

$$\begin{aligned}
(a, b) &:= \{c \in \bar{\mathbf{R}} \mid a < c < b\},\\
(a, b] &:= \{c \in \bar{\mathbf{R}} \mid a < c \leq b\},\\
[a, b) &:= \{c \in \bar{\mathbf{R}} \mid a \leq c < b\},\\
[a, b] &:= \{c \in \bar{\mathbf{R}} \mid a \leq c \leq b\}
\end{aligned}$$

for $a, b \in \bar{\mathbf{R}}$. Note that $\bar{\mathbf{R}}$ itself can be written as $[-\infty, \infty]$ while $\mathbf{R} = (-\infty, \infty)$. We also define the induced order topology, i.e. all $(a, +\infty]$ and $[-\infty, b)$ are the basis of the family of open sets in $\bar{\mathbf{R}}$.

The set of the extended real numbers has rich topological structures.

Proposition A.1.1 (Compactness of $\bar{\mathbf{R}}$)**.** *The extended real numbers construct a compact space* $\bar{\mathbf{R}}$.

Consequently, each closed subset of $\bar{\mathbf{R}}$ is compact, and it attains its supremum and infimum as shown later.

Proposition A.1.2. *Let a_n and b_n be two sequences in $\bar{\mathbf{R}}$ such that $a_n \to a \in \bar{\mathbf{R}}$ and $b_n \to b \in \bar{\mathbf{R}}$. If $a_n \leq b_n$ for all n, then $a \leq b$.*

A.2 Extreme values

In this section we study extreme values of a subset of the (extended) real numbers $\bar{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$. The extreme values consist of maximums, minimums, supremums and infimums.

Definition A.2.1 (Maximum and minimum)**.** Let A be a subset of $\bar{\mathbf{R}}$ and let $c \in \bar{\mathbf{R}}$. We say that the number c is a *maximum* (resp. *minimum*) of A if $c \in A$ and $a \leq c$ (resp. $a \geq c$) for all $a \in A$.

Remark A.2.2. The maximum is always unique for a subset A of $\bar{\mathbf{R}}$. Indeed, if c and $\tilde{c}$ are two maximums of A, we see by the definition of maximums that $\tilde{c} \leq c$ and $c \leq \tilde{c}$ so that $c = \tilde{c}$. The uniqueness of the minimum can be verified by a symmetric argument.

However, existence of maximum and minimum may be false for some subsets. In particular, the empty set $\emptyset$ does not attains its maximum and minimum.

We write $\max A$ (resp. $\min A$) to represent the unique maximum (resp. minimum) of a subset A when it exists.

The next proposition gives a sufficient condition for existence of a maximum and a minimum.

Proposition A.2.3. *Let A be a subset of $\bar{\mathbf{R}}$. If A is non-empty and compact, then A attains its maximum and minimum. In particular, a non-empty closed subset A of $\bar{\mathbf{R}}$ attains its maximum and minimum.*

We remark that the maximum and minimum may be the infinity.

In order to define a notion of supremums and infimums we prepare:

Proposition A.2.4. *Let A be a subset of $\bar{\mathbf{R}}$ and let U_A (resp. L_A) denote the set of all $b \in \bar{\mathbf{R}}$ such that $a \leq b$ (resp. $a \geq b$) for all $a \in A$. Then, U_A (resp. L_A) attains its minimum (resp. maximum).*

Proof. In view of Proposition A.2.3, it is enough to show that U_A is non-empty and closed. It is easy to check $U_A \neq \emptyset$ since the infinity $+\infty$ always belongs to U_A. In order to prove U_A is closed, fix a sequence $b_n \in U_A$ such that $b_n \to b$. Since $a \leq b_n$ for all $a \in A$, we have $a \leq b$, i.e. $b \in U_A$. Therefore, U_A is non-empty closed and so it attains its minimum. □

We remark that $U_A = [\min U_A, +\infty]$ and $L_A = [-\infty, \max L_A]$.

Definition A.2.5 (Supremum and infimum)**.** Let A be a subset of $\bar{\mathbf{R}}$. The minimum of U_A (resp. maximum of L_A), which exists uniquely, is called a *supremum* (resp. *infimum*) of A. The supremum (resp. infimum) of A denoted by $\sup A$ (resp. $\inf A$).

We remark that one is able to define the supremum and infimum for any subset of $\bar{\mathbf{R}}$. The next proposition gives a principle to obtain the value of the supremum or infimum.

Proposition A.2.6. *Let A be a subset of* $\bar{\mathbf{R}}$.

- *If A is empty, then* $\sup A = -\infty$ *and* $\inf A = +\infty$.
- *If A is not empty, then* $\sup A = \max \overline{A}$ *and* $\inf A = \min \overline{A}$. *In particular, there exits sequences $a_n \in A$ and $b_n \in A$ such $a_n \to \sup A$ and $b_n \to \inf A$*

We also define extreme values for an $\bar{\mathbf{R}}$-valued function f on a set X (and for a family of the extended real numbers) by its *image* $f(X) := \{f(x) \mid x \in X\}$. Write

$$\max_X f = \max_{x\in X} f(x) := \max f(X), \quad \min_X f = \min_{x\in X} f(x) := \min f(X),$$
$$\sup_X f = \sup_{x\in X} f(x) := \sup f(X), \quad \inf_X f = \inf_{x\in X} f(x) := \inf f(X).$$

Note that the inequality $\inf_X f \le f(x) \le \sup_X f$ always holds for all $x \in X$.

Proposition A.2.7. *Let f and g be two $\bar{\mathbf{R}}$-valued functions on a set X. If $f(x) \le g(x)$ for all $x \in X$, then $\sup_X f \le \sup_X g$ and $\inf_X f \le \inf_X g$.*

The next proposition will be convenient when one change the order of the supremum and infimum operators.

Proposition A.2.8 (Max-min inequality)**.** *Let X and Y are two sets and let f be an $\bar{\mathbf{R}}$-valued function f on the direct product $X \times Y$. Then,*

- $\sup_{x\in X} \inf_{y\in Y} f(x,y) \le \inf_{y\in Y} \sup_{x\in X} f(x,y)$.
- $\sup_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{X\times Y} f$.
- $\inf_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{X\times Y} f$.

A.3 Semicontinuity

In this section we study some limits and semicontinuity of functions. This enables us to have a deep argument about real-valued functions. Let X be a topological space and $\bar{\mathbf{R}}$ denote $\mathbf{R} \cup \{\pm\infty\}$.

The *upper (resp. lower) limit* of a function $f\colon X \to \bar{\mathbf{R}}$ at a point $x \in X$ is defined by

$$\limsup_{x'\to x} f(x') := \inf_{U\in\mathcal{U}_x} \sup_{x'\in U} f(x') \text{ (resp. } \liminf_{x'\to x} f(x') := \sup_{U\in\mathcal{U}_x} \inf_{x'\in U} f(x')\text{)}.$$

Here, $\mathcal{U}_x$ represents the set of all open neighborhood of x.

Definition A.3.1 (Semicontinuity)**.** A function $f\colon X \to \bar{\mathbf{R}}$ is *upper (resp. lower) semicontinuous* at a point $x \in X$ if

$$\limsup_{x'\to x} f(x') \le f(x) \text{ (resp. } \liminf_{x'\to x} f(x') \ge f(x)\text{)},$$

i.e. there exists a sequence of open neighborhood U_n of x such that

$$\lim_{n\to\infty} \sup_{x'\in U_n} f(x') \le f(x) \text{ (resp. } \lim_{n\to\infty} \inf_{x'\in U_n} f(x') \ge f(x)\text{)}.$$

The function f is *upper (resp. lower) semicontinuous* if f is upper (resp. lower) semicontinuous for all $x \in X$.

We also define a sequential notion of semicontinuity.

Definition A.3.2 (Sequentially semicontinuity)**.** A function $f\colon X \to \bar{\mathbf{R}}$ is *sequentially upper (resp. lower) semicontinuous* at a point $x \in X$ if

$$\sup_{x_n \to x} \limsup_{n\to\infty} f(x_n) \le f(x) \text{ (resp. } \inf_{x_n \to x} \liminf_{n\to\infty} f(x_n) \ge f(x)).$$

The function f is *sequentially upper (resp. lower) semicontinuous* if f is *sequentially upper (resp. lower) semicontinuous* for all $x \in X$.

Proposition A.3.3. *If a function $f\colon X \to \bar{\mathbf{R}}$ is upper (resp. lower) semicontinuous at a point $x \in X$, then f is sequentially upper (resp. lower) semicontinuous at $x \in X$.*

If the space X is a metric space, then the converse holds: If a function $f\colon X \to \bar{\mathbf{R}}$ is sequentially upper (resp. lower) semicontinuous at a point $x \in X$, then f is upper (resp. lower) semicontinuous at $x \in X$.

The definition of semicontinuity is two divided parts of continuity.

Proposition A.3.4. *A function $f\colon X \to \bar{\mathbf{R}}$ is both upper and lower semicontinuous at a point $x \in X$ if and only if f is continuous at $x \in X$.*

A function $f\colon X \to \bar{\mathbf{R}}$ is both sequentially upper and lower semicontinuous at a point $x \in X$ if and only if f is sequentially continuous at $x \in X$.

Several properties of continuity can be extended to semicontinuous functions.

Proposition A.3.5. *A function $f\colon X \to \bar{\mathbf{R}}$ is upper (resp. lower) semicontinuous if and only if the subset $\{f < a\}$ (resp. $\{f > a\}$) of X is open for every $a \in \bar{\mathbf{R}}$.*

The extreme value theorem can be extended as follows.

Theorem A.3.6 (Extreme value theorem)**.** *If the space X is compact, then an upper (resp. lower) semicontinuous function $f\colon X \to \bar{\mathbf{R}}$ attains its maximum (resp. minimum) over X.*

If the space X is sequentially compact, then an sequentially upper (resp. lower) semicontinuous function $f\colon X \to \bar{\mathbf{R}}$ attains its maximum (resp. minimum) over X.

Another important property of semicontinuity is that extremum operator keeps the semicontinuity.

Proposition A.3.7. *Let F be a family of upper (resp. lower) semicontinuous functions $g\colon X \to \bar{\mathbf{R}}$ at a point $x \in X$. Then, the function $f\colon X \to \bar{\mathbf{R}}$ given by $f(x) := \inf_{g\in F} g(x)$ (resp. $f(x) := \sup_{g\in F} g(x)$) is upper (resp. lower) semicontinuous at $x \in X$.*

Proof. One is able to observe by Proposition A.2.8 that

$$\limsup_{x'\to x} f(x') = \inf_{U\in\mathcal{U}_x} \sup_{x'\in U} \inf_{g\in F} g(x') \le \inf_{g\in F} \inf_{U\in\mathcal{U}_x} \sup_{x'\in U} g(x').$$

Since each $g \in F$ is upper semicontinuous at x, we have

$$\limsup_{x'\to x} f(x') \le \inf_{g\in F} g(x) = f(x).$$

Therefore, f is upper semicontinuous at x. $\square$

Remark A.3.8. On the other hand, the supremum of upper semicontinuous functions may not be upper semicontinuous in general. For instance, let $f_n : \mathbf{R} \to \bar{\mathbf{R}}$ be such that $f_n(x) = 1$ if $x \geq 1/n$ and $f_n(x) = 0$ otherwise for $n = 1, \cdots$. Then, it is easy to see that the supremum $f(x) := \sup_{n=1,\cdots} f_n(x)$ is given by $f_n(x) = 1$ if $x > 0$ and $f_n(x) = 0$ otherwise. However, this is not upper semicontinuous at $x = 0$. The following proposition gives a sufficient conditions to be upper semicontinuous.

Proposition A.3.9. *Let A be a compact set and let $F = F(x, a) : X \times A \to \bar{\mathbf{R}}$ be an upper (resp. lower) semicontinuous function. Then, the function $f : X \to \bar{\mathbf{R}}$ given by $f(x) := \sup_{a\in A} F(x; a)$ (resp. $f(x) := \inf_{a\in A} F(x; a)$) is upper (resp. lower) semicontinuous.*

Remark A.3.10. According to Theorem A.3.6, the supremum in Proposition A.3.9 is nothing but maximum.

Proof. Fix $x \in X$ and take a sequence $x_n \in X$ such that $x_n \to x$ and $f(x_n) \to \limsup_{x'\to x} f(x')$. For each n one is able to take $a_n \in A$ such that $f(x_n) = F(x_n; a_n)$. Since A is compact, there exists a subsequence n_j such that a_{n_j} converges to some point a. We then see by the upper semicontinuity of F that

$$\limsup_j F(x_{n_j}; a_{n_j}) \leq F(x, a) \leq f(x)$$

and hence $\limsup_j f(x_{n_j}) = \limsup_{x'\to x} f(x') \leq f(x)$. Therefore, f is upper semicontinuous at x. □

We are able to construct corresponding semicontinuous functions for any functions in view of Proposition A.3.7.

Definition A.3.11 (Semicontinuous envelope)**.** For a function $f : X \to \bar{\mathbf{R}}$, let U_f (resp. L_f) denote the set of all upper (resp. lower) semicontinuous functions $g : X \to \bar{\mathbf{R}}$ such that $f \leq g$ (resp. $f \geq g$) on X. Define the *upper (resp. lower) semicontinuous envelope* f^* (resp. f_*) of f by $f^*(x) := \inf_{g\in U_f} g(x)$ (resp. $f_*(x) := \sup_{g\in L_f} g(x)$).

Remark A.3.12. Even for a function f defined only on a subset $F \subset X$, by extending f to X with the value $-\infty$ (resp. $+\infty$) we can construct the upper (resp. lower) semicontinuous envelope f^* (resp. f_*) defined on the whole X.

Proposition A.3.13. *For a function $f : X \to \bar{\mathbf{R}}$, we have $f^*(x) = \limsup_{x'\to x} f(x')$ (resp. $f_*(x) = \liminf_{x'\to x} f(x')$) for all $x \in X$. In particular, there exits a sequence $x_n \in X$ such that $(x_n, f(x_n)) \to (x, f^*(x))$ (resp. $(x_n, f(x_n)) \to (x, f_*(x))$).*

A.4 Definition of viscosity solutions

In this section we give a definition of viscosity solutions for a partial differential equation of the generalized form

$$F(x, u, Du, D^2u) = 0 \quad \text{in } U. \tag{A.4.1}$$

Here, U is an open subset of $\mathbf{R}^N$ and let $F : U \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N \to \bar{\mathbf{R}}$.

Let $u\colon U \to \bar{\mathbf{R}}$ and $x \in U$. A function ϕ is called a *upper (resp. lower) test function* of u at x if there exists an open neighborhood B of x such that $\phi \in C^2(B)$ and $u^* - \phi$ (resp. $u_* - \phi$) attains a zero local maximum (resp. minimum) at x, i.e.

$$(u^* - \phi)(x) = \sup_B (u^* - \phi) = 0 \text{ (resp. } (u_* - \phi)(x) = \inf_B (u_* - \phi) = 0).$$

Let $T^+(u,x)$ (resp. $T^+(u,x)$) denotes the set of all upper (resp. lower) test functions ϕ of $u\colon U \to \bar{\mathbf{R}}$ at $x \in U$. For an $\bar{\mathbf{R}}$-valued function u on U we define the graph of super- and subdifferentials by

$$G^+u := \{(x, \phi(x), D\phi(x), D^2\phi(x)) \mid x \in U, \phi \in T^+(u,x)\},$$
$$G^-u := \{(x, \phi(x), D\phi(x), D^2\phi(x)) \mid x \in U, \phi \in T^-(u,x)\}.$$

Such notions of graphs shall be useful to argue properties of viscosity solutions in view of set-valued analysis [1].

Definition A.4.1 (Viscosity solution)**.** We say that an $\bar{\mathbf{R}}$-valued function u on U is a *viscosity subsolution (resp. supersolution)* of (A.4.1) and solves $F[u] \le 0$ (resp. $F[u] \le 0$) in U in the *viscosity sense* if

$$\sup_{G^+u} F_* \le 0 \text{ (resp. } \inf_{G^-u} F^* \ge 0),$$

i.e. for each $x \in U$ and $\phi \in C^2(x)$ the inequality

$$F_*(x, \phi(x), D\phi(x), D^2\phi(x)) \le 0 \text{ (resp. } F^*(x, \phi(x), D\phi(x), D^2\phi(x)) \ge 0)$$

holds if $u^* - \phi$ (resp. $u_* - \phi$) attains a zero local maximum (resp. minimum) at x. We say that an $\bar{\mathbf{R}}$-valued function u on U is a *viscosity solution* of (A.4.1) and solves $F[u] = 0$ in U in the *viscosity sense* if u is both a viscosity subsolution and a viscosity supersolution of (A.4.1), i.e. $\sup_{G^+u} F_* \le 0 \le \inf_{G^-u} F^*$.

Remark A.4.2. Our definition allows the non-proper functions $u \equiv \pm\infty$ to be always a viscosity solution of (A.4.1). This notation is convenient to state the stability lemmas.

The viscosity solution is a weaker notion of solution than classical solution with enough smoothness.

Proposition A.4.3 (Smooth solution)**.** *Assume that* $F\colon U \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N \to \bar{\mathbf{R}}$ *satisfies the elliptic condition, i.e.*

$$F(x, v, p, X) \ge F(x, v, p, Y) \quad \text{for all } X \le Y.$$

If a smooth function $u \in C^2(U)$ *satisfies*

$$F^*(x, u(x), Du(x), D^2u(x)) \le 0 \text{ (resp. } F_*(x, u(x), Du(x), D^2u(x)) \ge 0)$$

for all $x \in U$*, then* u *is a viscosity subsolution (resp. supersolution) of* (A.4.1).

Lemma A.4.4. $G^+u = \{(x, u(x), Du(x), X) \mid x \in U, X \ge D^2u(x)\}$.

The notion of viscosity solutions also has properties on locality.

Proposition A.4.5 (Locality)**.** *Let* U_1 *and* U_2 *be two open subsets of* $\mathbf{R}^N$*. If a function* $u\colon U_1 \cup U_2 \to \bar{\mathbf{R}}$ *is a viscosity subsolution of* (A.4.1) *in both* U_1 *and* U_2*, then* u *is a viscosity subsolution of* (A.4.1) *in* $U_1 \cup U_2$.

A.5 Stability of viscosity solutions

The main purpose of this section is to establish some properties concerning stability of a family of the viscosity solutions. Let A be a topological set of indexes.

For a function $f\colon X \times A \to \bar{\mathbf{R}}$ with a topological space X, we sometimes let f itself denote the family $\{f(\cdot;a)\}_{a\in A}$ for simplicity.

Definition A.5.1 (Family of viscosity solutions)**.** Let $F\colon U \times \mathbf{R} \times \mathbf{R}^N \times \mathbb{S}^N \times A \to \bar{\mathbf{R}}$ and $u\colon U \times A \to \bar{\mathbf{R}}$. We say that u is a *family of viscosity subsolutions (resp. supersolutions or solutions)* of (A.4.1) and solves $F[u] \leq 0$ (resp. $F[u] \geq 0$ or $F[u] = 0$) in U in the *viscosity sense* if for each $a \in A$ the function $u_a = u(\cdot;a)$ solves $F_a[u_a] \leq 0$ (resp. $F_a[u_a] \geq 0$ or $F_a[u_a] = 0$) in U in the viscosity sense with $F_a = F(\cdot;a)$.

Consider the upper (resp. lower) semicontinuous envelope $\overline{f} := f^*$ (resp. $\underline{f} := f_*$) on $X \times A$. The function $\overline{f}(\cdot;a)$ is called an *upper (resp. lower) semilimit* of the family f at a. Note that the upper (resp. lower) semilimits are always upper (resp. lower) semicontinuous and that $(f(\cdot;a))^* \leq \overline{f}(\cdot;a)$ (resp. $(f(\cdot;a))_* \geq \underline{f}(\cdot;a)$) for all $a \in A$.

Proposition A.5.2 (Stability)**.** *Let $F\colon U \times \mathbf{R} \times \mathbf{R}^N \times \mathbb{S}^N \times A \to \bar{\mathbf{R}}$ and $u\colon U \times A \to \bar{\mathbf{R}}$. If u solves $F[u] \leq 0$ in U in the viscosity sense, then the family $\overline{u}$ solves $\underline{F}[\overline{u}] \leq 0$ in U in the viscosity sense.*

The statement of this proposition is very general and directly yields stability results under uniform convergence and extremum which are standard statements in the classical materials, e.g. [4].

Corollary A.5.3 (Stability under limit)**.** *Let F_n and u_n be a sequence of continuous functions respectively defined on $U \times \mathbf{R} \times \mathbf{R}^N \times \mathbb{S}^N$ and U, and assume that F_n and u_n converges to functions F and u uniformly. If each u_n solves $F_n[u_n] \leq 0$ (resp. $F_n[u_n] \geq 0$) in U in the viscosity sense, then u is a viscosity subsolution (resp. supersolution) of* (A.4.1).

Corollary A.5.4 (Stability under extremum)**.** *Let S be a family of viscosity subsolution (resp. supersolution) of* (A.4.1) *in U. Then, the supremum (resp. infimum) $u(x) := \sup_{v\in S} v(x)$ (resp. $u(x) := \inf_{v\in S} v(x)$) is a viscosity subsolution (resp. supersolution) of* (A.4.1).

Proposition A.5.5 (Perron method)**.** *Let g be an $\bar{\mathbf{R}}$-valued function on ∂U. Let S be the set of all viscosity subsolutions (resp. supersolutions) v of* (A.4.1) *with $v^* \leq g$ (resp. $v_* \geq g$) on ∂U. Then, $u(x) := \sup_{v\in S} v(x)$ (resp. $u(x) := \inf_{v\in S} v(x)$) is a viscosity solution of* (A.4.1).

We first prove the stability with respect to the semilimit of a sequence of viscosity solutions (Proposition A.5.2). It is enough to prove $G^{\pm}\overline{u}$ converges to $G^{\pm}u$ as follows. Now, for a family $u\colon U \times A \to \bar{\mathbf{R}}$ we define the graphs

$$G^{\pm}u = \{(x,v,p,X,a) \mid a \in A,\ (x,v,p,X) \in G^{\pm}u(\cdot;a)\}.$$

Lemma A.5.6. *Let $u\colon U \times A \to \bar{\mathbf{R}}$. Then, for each $(\hat{x},\hat{v},\hat{p},\hat{X},\hat{a}) \in G^{+}\overline{u}$ there exists a sequence $(x_j,v_j,p_j,X_j,a_j) \in G^{+}u$ such that $(x_j,v_j,p_j,X_j,a_j) \to (\hat{x},\hat{v},\hat{p},\hat{X},\hat{a})$.*

Proof. Write $u_\infty := \overline{u}(\cdot;\hat{a})$. Let $\phi \in T^+(u_\infty,\hat{x})$ be an upper text function such that $(\hat{v},\hat{p},\hat{X}) = (\phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x}))$. We now have $\phi \in C^2(B_R(\hat{x}))$ and $u_\infty - \phi$ attains a zero maximum at $\hat{x}$ over $B_R(\hat{x}) \subset U$ with some $R > 0$, i.e. $(u_\infty - \phi)(\hat{x}) = \sup_{B_R(\hat{x})}(u_\infty - \phi) = 0$. Set $\tilde{\phi}(x) := \phi(x) + |x-\hat{x}|^2$. Now note that there exist a subsequence $a_j \to \hat{a}$ and a sequence $x_j \in U$ such that $x_j \to \hat{x}$ and $u(x_j;a_j) \to u^*(\hat{x};\hat{a}) = u_\infty(\hat{x})$. Set $u_j = (u(\cdot;a_j))^*$; then the above convergence implies $u_j(x_j) \to u_\infty(\hat{x})$ since $u_j \le u^*(\cdot;a_j)$. Take a maximum point $y_j \in \overline{B}_R(\hat{x})$ of $u_j - \tilde{\phi}$ over $\overline{B}_R(\hat{x})$ for each j. We then observe that

$$(u_j - \tilde{\phi})(y_j) \ge (u_j - \tilde{\phi})(x_j) \to (u_\infty - \phi)(\hat{x}). \tag{A.5.1}$$

Hence,

$$\begin{aligned}\limsup_j |y_j - \hat{x}|^2 &\le \limsup_j (u_j - \phi)(y_j) - (u_\infty - \phi)(\hat{x}) \\ &\le \sup_{\overline{B}_R(\hat{x})} (u_\infty - \phi) - (u_\infty - \phi)(\hat{x}) = 0\end{aligned}$$

and therefore $y_j \to \hat{x}$. Moreover, $u_j(y_j) \to u_\infty(\hat{x})$ since $\liminf_j u_j(y_j) \ge u_\infty(\hat{x})$ in view of (A.5.1). We now note that $u_\infty(\hat{x}) \ne \pm\infty$ and so $u_j(y_j) \ne \pm\infty$ for j large enough. Since $u_j - \tilde{\phi} - (u_j - \tilde{\phi})(y_j)$ attains the zero maximum at y_j, we see that $(y_j, u_j(y_j), D\tilde{\phi}(y_j), D^2\tilde{\phi}(y_j)) \in G^+u_j = G^+u(\cdot;a_j)$. Also

$$(y_j, u_j(y_j), D\tilde{\phi}(y_j), D^2\tilde{\phi}(y_j)) \to (\hat{x}, u_\infty(\hat{x}), D\tilde{\phi}(\hat{x}), D^2\tilde{\phi}(\hat{x})) = (\hat{x},\hat{v},\hat{p},\hat{X}).$$

The proof is complete. □

Proof of Proposition A.5.2. Fix $a \in A$ and $(x,v,p,X) \in G^+\overline{u}(\cdot;a)$. Then, invoking Lemma A.5.6 one is able to take sequences $a_j \in A$ and $(x_j,v_j,p_j,X_j) \in G^+u(\cdot;a_j)$ such that $(x_j,v_j,p_j,X_j,a_j) \to (x,v,p,X,a)$. Since $\overline{u}(\cdot;a)$ is a viscosity subsolution, $F_j(x_j,v_j,p_j,X_j) \le 0$ with $F_j = (F(\cdot;a_j))_*$. By the definition of $\underline{F}$ we have

$$\underline{F}(x,v,p,X) \le \liminf_j F_j(x_j,v_j,p_j,X_j) \le 0$$

and therefore $\overline{u}$ is a family of viscosity subsolutions. □

Proof of Corollary A.5.3. Just set $A = \mathbf{N}$ in Proposition A.5.2. □

Proof of Corollary A.5.4. Set $A = S$ with the indiscrete topology and trivial families $\{F\}_{v\in S}$ and $\{U(\cdot;v) = v(\cdot)\}_{v\in S}$. Note that $U^*(x;v) = u^*(x)$. Therefore, by applying Proposition A.5.2 we see that u is a viscosity subsolution of (A.4.1). □

Proof of Proposition A.5.5. We only show that u is a supersolution since being a subsolution is due to Corollary A.5.4. Fix $\phi \in T^-(u,\hat{x})$, i.e. $\phi \in C^2(B_R(\hat{x})$ and $u_* - \phi$ attains a zero maximum at $\hat{x}$ over $B_R(\hat{x}) \subset U$ with some $R > 0$. Set $\tilde{\phi}(x) = \phi(x) - |x-\hat{x}|^2$. Suppose by contradiction that

$$F^*(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) < 0.$$

Since F^* is upper semicontinuous and $\tilde{\phi}$ is enough smooth, we may see that $\tilde{\phi}$ is a subsolution of (A.4.1) in $B_R(\hat{x})$ by taking R small enough. Note that $\tilde{\phi}$ is a viscosity subsolution in view of A.4.3. Now observe that

$$(u - \tilde{\phi})(x) \ge (u_* - \tilde{\phi})(x) \ge |x-\hat{x}|^2 \ge \frac{R^2}{4} =: m > 0$$

for all $x \in \overline{B}_R(\hat{x}) \setminus B_{R/2}(\hat{x})$. Construct a new function

$$v(x) := \begin{cases} \max\{\tilde{\phi}(x) + m/2, u(x)\} & \text{if } x \in B_R(\hat{x}), \\ u(x) & \text{otherwise.} \end{cases}$$

Then, v is equal to u on $U \setminus B_{R/2}(\hat{x})$ and so it is a subsolution of (A.4.1) in $U \setminus B_{R/2}(\hat{x})$. It follows from Corollary A.5.4 that v is a subsolution of (A.4.1) in $B_R(\hat{x})$. Therefore, Proposition A.4.5 shows that v is a subsolution of (A.4.1) in U and so $u \in S$. In particular, $u \geq v$. However, since $\tilde{\phi}(\hat{x}) + m/2 = u_*(\hat{x}) + m/2 > u_*(\hat{x})$, we have $v > u$ at some point. Therefore, we obtain a contradiction and conclude that u is a viscosity supersolution of (A.4.1). $\square$

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