The Best Achievable Performance of Sampled-Data Control Systems

(サンプル値制御系で達成可能な最良性能)

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To My Parents and Kazue

Abstract

This thesis proposes a framework for sampled-data control systems with a large class of samplers and holds and, based on it, investigates the best achievable performance of sampled-data control systems. This research is motivated by a simulation result that the best achievable performance of sampled-data control systems does not always converge to that of continuous-time control systems even if the sampling period approaches zero.

By introducing the notions of regular samplers and holds, we can treat a large class of samplers and holds. These notions enable us to treat practical samplers and holds in a more convenient way than the conventional notions of generalized samplers and holds. Moreover, using a lifting technique and a matrix representation of an operator, some basic properties of a sampled-data control system are derived. Especially, one property of them expresses a relationship between a sampled-data control system and a continuous-time control system and plays an important role in the subsequent analysis.

Based on the prepared framework for sampled-data control systems, their best achievable performance is studied. Here, the best achievable performance of sampled-data control systems means the best performance obtained by adjustment of a discrete-time controller when a plant, a sampling period, a sampler, and a hold are provided. This best achievable performance can be improved by an appropriate choice of a sampling environment, that is, the triplet of a sampling period, a sampler, and a hold. First, the theoretical bound for this improvement is compared with the best achievable performance of continuous-time control systems. It is shown that these two are not always equal. This means that the best achievable performance of continuous-time control systems may not be recovered by sampled-data control systems. Next, supposing that a sequence of sampling environments is provided, we obtain a necessary and sufficient condition in order that the best achievable performance of sampled-data control systems for each environment converges to the theoretical bound. When the theoretical bound is equal to the best achievable performance of continuous-time control systems, this convergence means that the best achievable performance of sampled-data control systems converges to that of continuous-time control systems. Simplification of the obtained condition is also considered.

Preface

In this thesis, I study the best achievable performance of sampled-data control systems. Special attention is paid on how this best performance depends on a choice of a sampling period, a sampler, and a hold. One reason why I began this study is that I have been interested in a relationship between control and information since I read Amari's monograph [1] and paper [2] and Ohara's thesis [69]. (The contents of [69] was published in English as [70].) In the monograph [1], Amari developed a differential-geometric theory on statistical estimation and test, which he named *information geometry*; in the paper [2], he considered information geometry of a system theory; in the thesis [69], Ohara applied information geometry to a control system design. Because I wrote my master-course thesis on *sampled-data control systems*, it was natural for me to consider how information is related to a sampled-data control system. A sampled-data control system is a system to control a continuous-time plant by means of a discrete-time controller. If we are interested in its behavior only at sampling instants, this system can be regarded as a *discrete-time control system*. Thus, we use the term of a sampled-data control system when we are interested also in its intersample behavior and regard the system as a hybrid one in the sense that it includes both continuous-time signals and discrete-time signals. The time period with which the discrete-time controller works is called a *sampling period*. Now, suppose that, for a provided sampled-data control system, its sampling period is made smaller. Then, because more information can be used during a fixed time period in order to control a plant, it is considered that a control performance can be improved with an appropriate choice of a controller. I expected that I can capture the notion of information in control by noticing this performance improvement. From this idea, I started the present study.

In addition to the interests mentioned above, I had another reason to begin the present research. Since a lifting technique was introduced by Yamamoto [94, 95], a lot of papers have been published on sampled-data control systems. Many of them are based on the following idea. When a sampling period, a sampler, and a hold are chosen *a priori* in a sampled-data control system, a lifting technique enables us to regard this system just as a discrete-time control system taking into account its intersample behavior. Hence, we can apply here well-established methodologies for discrete-time control systems, and this means that we can analyze and synthesize a sampled-data control system considering its intersample behavior. Although this is a great progress, I somehow felt unsatisfied. This is because, if we consider a sampled-data control system only along this approach, a theory on a sampled-data control system is just a translation of an existing theory on a discrete-time control system. I rather wanted to consider a unique characteristic of a sampled-data control system, which is not possessed by either a a continuous-time control system or a discrete-time control system. Then, what characterizes a sampled-data control system? I noticed that in a sampled-data control system we can choose a sampling period, a sampler, and a hold so as to get a good control performance, while we cannot do this in a continuous-time or discrete-time control system. So this is one of characters of a sampled-data control system. Considering in this way, I found it interesting to investigate how the best achievable performance of sampled-data control systems depends on a choice of a sampling period, a sampler, and a hold.

Intuitively, it seems obvious that the best achievable performance of sampled-data control systems approaches that of continuous-time control systems as the sampling period approaches zero. It is considered that a sampled-data controller is widely accepted as a substitute of a continuous-time controller partly because this conjecture is believed to be correct. However, this conjecture is *not* always correct as will be seen in Example 1.3. Considering that this is a fundamental conjecture in the use of a sampled-data controller, we have to clarify why and when this conjecture fails to hold.

This thesis investigates properties of the best achievable performance of sampled-data control systems and gives a necessary and sufficient condition in order that the best achievable performance of sampled-data control systems converges to that of continuous-time control systems. From this result, it is seen that the mentioned convergence to the best continuous-time control performance depends on any of a provided plant, a sampling period, a sampler, and a hold. The obtained condition gives us some insight about what is important to improve sampled-data control performance.

I was helped by many people while I did this research and compiled it into this thesis.

I am particularly thankful to Professor Hidenori Kimura in the University of Tokyo. Since he came to the University of Tokyo in 1995, he has allowed me to work as his research associate. In spite that he himself was always under the pressure of a horrible amount of works, he took care to lighten my work load and let me concentrate on the research. Moreover, he gave me a number of valuable comments on the research and encouraged me when I tended to lose enthusiasm. He also carefully read the draft of this thesis and gave me many suggestions. I can never imagine that I could accomplish this research without his help.

Thanks are due to Professor Seiichi Shin in the University of Tokyo. He was my advisor while I was a doctoral-course student from 1993 to 1995. His unique viewpoints free from any conventional ideas stimulated me a lot. Particularly, he suggested me to doubt whether the best achievable performance of sampled-data control systems always converges to that of

continuous-time control systems.

I am grateful to Professor Shinji Hara in Tokyo Institute of Technology for his valuable comments. Especially, he gave me an idea to try an anti-aliasing filter whose bandwidth depends on the sampling period. In fact, with such a filter, it is often the case that the best achievable performance of sampled-data control systems does not converge to that of continuous-time control systems. Therefore, this comment was really a key point to develop the present theory.

I wish to express my gratitude to Professor Yutaka Yamamoto in Kyoto University. He gave me instructive advice not only on my research but also on my writing style, mathematical reasoning, and manners as a researcher. Furthermore, I am obliged to Professors Hara and Yamamoto for giving me a chance to talk about the present topic at the Second Asian Control Conference in 1997.

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Finally, I would like to thank my parents, Yukio and Yoshiko, who brought me up and gave me a high education. I also wish to thank my wife, Kazue, for her patience and encouragement.

Tokyo, Japan *Yasuaki Oishi*
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Notation

$\square$	The end of a definition, an example, a remark, or a proof
$:=$	Defined as
$\mathbf{0}$	The zero vector
M^{T}	The transpose of a matrix M
$\overline{\sigma}(M)$	The maximum singular value of a matrix M
$\underline{\sigma}(M)$	The minimum singular value of a matrix M
$\widehat{\boldsymbol{a}}(s)$	$:= \int_0^\infty \boldsymbol{a}(t)\mathrm{e}^{-st}\,\mathrm{d}t$. The Laplace transform of a continuous-time signal $\boldsymbol{a}(t)$.
$\tilde{\boldsymbol{a}}$	$:= W_\tau \boldsymbol{a}$. The lifting of a continuous-time signal $\boldsymbol{a}(t)$. Here, $\tilde{\boldsymbol{a}}$ is a functional sequence $\{\tilde{\boldsymbol{a}}[k]\}_{k=0}^\infty$ defined by $\tilde{\boldsymbol{a}}[k](t) := \boldsymbol{a}(k\tau + t)$ for $k = 0, 1, \ldots$ and $0 \le t < \tau$.
$\breve{\boldsymbol{a}}(z)$	$:= \sum_{k=0}^\infty \tilde{\boldsymbol{a}}[k]z^{-k}$. The z-transform of a lifted continuous-time signal $\tilde{\boldsymbol{a}} = W_\tau \boldsymbol{a}$.
$\breve{\boldsymbol{a}}_{\mathrm{d}}(z)$	$:= \sum_{k=0}^\infty \boldsymbol{a}_{\mathrm{d}}[k]z^{-k}$. The z-transform of a discrete-time signal $\boldsymbol{a}_{\mathrm{d}}[k]$.
$\overline{A}$	The complex conjugate of A. When A is a vector or a matrix, conjugate is taken elementwise. When A is a function of a real number t, its conjugate $\overline{A}$ is defined by $\overline{A}(t) := \overline{A(t)}$. When A is a function of a complex number s, $\overline{A}$ is defined by $\overline{A}(s) := \overline{A(\overline{s})}$. When A is an operator, $\overline{A}$ is defined as an operator $\overline{A}\boldsymbol{a} := \overline{A\overline{\boldsymbol{a}}}$.
A^*	The complex conjugate transpose of A when A is a matrix. The adjoint operator of A when A is an operator.
$A^\sim(z)$	$:= A\left(\frac{1}{\overline{z}}\right)^*$
$\widehat{P}(s)$	The continuous-time transfer function of a continuous-time operator P
$\breve{P}_{\mathrm{d}}(z)$	The discrete-time transfer function of a discrete-time operator P_{d}

$\underline{S}(t)$ — The kernel function of a regular sampler S. The operation of $S : \boldsymbol{p} \mapsto \boldsymbol{p}_{\mathrm{d}}$ is specified as $\boldsymbol{p}_{\mathrm{d}}[k] := \int_0^{k\tau} \underline{S}(k\tau - r)\boldsymbol{p}(r)\,\mathrm{d}r$ for $k = 0, 1, \ldots$.

$\underline{H}(t)$ — The kernel function of a regular hold H. The operation of $H : \boldsymbol{q}_{\mathrm{d}} \mapsto \boldsymbol{q}$ is specified as $\boldsymbol{q}(k\tau + t) := \sum_{\ell=0}^{k} \underline{H}(k\tau + t - \ell\tau)\boldsymbol{q}_{\mathrm{d}}[\ell]$ for $k = 0, 1, \ldots$ and $0 \leq t < \tau$.

$\check{P}(z)$, $\check{S}(z)$, $\check{H}(z)$ — The lifting-based transfer functions of a continuous-time operator P, a sampler-type operator S, and a hold-type operator H

$\check{H}^{\mathrm{in}}(z)$ — An inner factor of a real rational function $\check{H}(z)$

$\check{H}^{\mathrm{out}}(z)$ — An outer factor of a real rational function $\check{H}(z)$

$\|\boldsymbol{v}\|_2$ — $:= (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$. The Euclid norm of an n-dimensional vector $\boldsymbol{v} = [v_1 \ v_2 \ \ldots \ v_n]^{\mathrm{T}} \in \mathbb{C}^n$.

$\|\boldsymbol{a}\|_{\mathcal{L}^2}$ — $:= \left(\int_0^\infty \|\boldsymbol{a}(t)\|_2^2\,\mathrm{d}t\right)^{1/2}$ for a function $\boldsymbol{a} \in (\mathcal{L}^2)^n$

$(\boldsymbol{f}, \boldsymbol{g})_{\mathcal{L}^2[0,\tau)}$ — $:= \int_0^\tau \boldsymbol{f}(t)^*\boldsymbol{g}(t)\,\mathrm{d}t$ for functions $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{L}^2[0,\tau)^n$

$\|\boldsymbol{f}\|_{\mathcal{L}^2[0,\tau)}$ — $:= (\boldsymbol{f}, \boldsymbol{f})_{\mathcal{L}^2[0,\tau)}^{1/2} = \left(\int_0^\tau \|\boldsymbol{f}(t)\|_2^2\,\mathrm{d}t\right)^{1/2}$ for a function $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$

$\|\widetilde{\boldsymbol{a}}\|_{\ell^2_{\mathcal{L}^2[0,\tau)}}$ — $:= \sum_{k=0}^{\infty} \|\widetilde{\boldsymbol{a}}[k]\|_{\mathcal{L}^2[0,\tau)}^2$ for an $\mathcal{L}^2[0,\tau)^n$-valued sequence $\widetilde{\boldsymbol{a}} = \{\widetilde{\boldsymbol{a}}[k]\}_{k=0}^{\infty} \in \ell^2_{\mathcal{L}^2[0,\tau)}$

$\|\widehat{\boldsymbol{a}}\|_{\mathcal{H}^2}$ — $:= \left\{\sup_{\sigma>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\widehat{\boldsymbol{a}}(\sigma + \mathrm{i}\omega)\|_2^2\,\mathrm{d}\omega\right\}^{1/2}$ for a function $\widehat{\boldsymbol{a}} \in \mathcal{H}^2$

$\|\check{\boldsymbol{a}}\|_{\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}}$ — $:= \left\{\sup_{r>1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\check{\boldsymbol{a}}(r\mathrm{e}^{\mathrm{i}\omega})\|_{\mathcal{L}^2[0,\tau)}^2\,\mathrm{d}\omega\right\}^{1/2}$ for an $\mathcal{L}^2[0,\tau)^n$-valued function $\check{\boldsymbol{a}} \in \mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$

$\|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}$ — $:= \left(\sum_{k=0}^{\infty} \|\boldsymbol{a}_{\mathrm{d}}[k]\|_2^2\right)^{1/2}$ for a sequence $\boldsymbol{a}_{\mathrm{d}} \in (\ell^2)^n$

$\|\check{\boldsymbol{a}}_{\mathrm{d}}\|_{\mathfrak{H}^2}$ — $:= \left\{\sup_{r>1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\check{\boldsymbol{a}}_{\mathrm{d}}(r\mathrm{e}^{\mathrm{i}\omega})\|_2^2\,\mathrm{d}\omega\right\}^{1/2}$ for a function $\check{\boldsymbol{a}}_{\mathrm{d}} \in \mathfrak{H}^2$

$\|A\|_{X\to Y}$ — $:= \sup_{\substack{\boldsymbol{x}\in X \\ \|\boldsymbol{x}\|_X \neq 0}} \frac{\|A\boldsymbol{x}\|_Y}{\|\boldsymbol{x}\|_X}$. The induced norm of an operator A that maps X to Y.

$\|A\|_{\mathrm{ind}}$ — The appropriate induced norm of an operator A

$\|P\|$ — $:= \|P\|_{\mathcal{L}^2\to\mathcal{L}^2}$. The induced norm from $(\mathcal{L}^2)^n$ to $(\mathcal{L}^2)^\ell$ unless specified in other way.

$\|\widehat{P}\|_{\mathcal{H}^\infty}$ — $:= \sup_{\mathrm{Re}\, s>0} \overline{\sigma}\{\widehat{P}(s)\}$ for $\widehat{P} \in \mathcal{H}^\infty$

$\|\check{P}_{\mathrm{d}}\|_{\mathfrak{H}^\infty}$ — $:= \sup_{z\in\mathbb{D}} \overline{\sigma}\{\check{P}_{\mathrm{d}}(z)\}$ for $\check{P}_{\mathrm{d}} \in \mathfrak{H}^\infty$

$\|L\|_{\mathrm{L}}$ — The induced norm of a large operator L, that is, an operator from $\mathcal{L}^2[0,\tau)^n$ to $\mathcal{L}^2[0,\tau)^\ell$

$\|F\|_{\mathrm{F}}$ — The induced norm of a flat operator F, that is, an operator from $\mathcal{L}^2[0,\tau)^n$ to $\mathbb{C}^\ell$

$\|T\|_{\mathrm{T}}$ — The induced norm of a tall operator T, that is, an operator from $\mathbb{C}^n$ to $\mathcal{L}^2[0,\tau)^\ell$

$\|\check{P}\|_{\mathfrak{H}^\infty_{\mathrm{L}}}$ — $:= \sup_{z\in\mathbb{D}} \|\check{P}(z)\|_{\mathrm{L}}$ for $\check{P} \in \mathfrak{H}^\infty_{\mathrm{L}}$

$\|\check{S}\|_{\mathfrak{H}^\infty_{\mathrm{F}}}$ — $:= \sup_{z\in\mathbb{D}} \|\check{S}(z)\|_{\mathrm{F}}$ for $\check{S} \in \mathfrak{H}^\infty_{\mathrm{F}}$

$\|\check{H}\|_{\mathfrak{H}^\infty_{\mathrm{T}}}$ — $:= \sup_{z\in\mathbb{D}} \|\check{H}(z)\|_{\mathrm{T}}$ for $\check{H} \in \mathfrak{H}^\infty_{\mathrm{T}}$

$\|\cdot\|_{\mathfrak{L}^\infty_{\mathrm{L}}}$, $\|\cdot\|_{\mathfrak{L}^\infty_{\mathrm{F}}}$, $\|\cdot\|_{\mathfrak{L}^\infty_{\mathrm{T}}}$, $\|\cdot\|_{\mathfrak{L}^\infty}$

$\|A\|_{\mathfrak{L}^\infty_{\mathrm{L}}} := \operatorname*{ess\,sup}_{|z|=1} \|A(z)\|_{\mathrm{L}}$. The other norms $\|\cdot\|_{\mathfrak{L}^\infty_{\mathrm{F}}}$, $\|\cdot\|_{\mathfrak{L}^\infty_{\mathrm{T}}}$, and $\|\cdot\|_{\mathfrak{L}^\infty}$ are similarly defined by replacement of $\|\cdot\|_{\mathrm{L}}$ with $\|\cdot\|_{\mathrm{F}}$, $\|\cdot\|_{\mathrm{T}}$, and $\overline{\sigma}\{\cdot\}$, respectively.

$\|\Phi\|_{\mathrm{H}}$ — The Hankel norm of a real rational function $\Phi(z)$

$\mathcal{A}_{\mathrm{R}}$ — The space of all real and matrix-valued functions $\widehat{P}(s)$ that are analytic in $\mathrm{Re}\, s > 0$ and is continuous in $\mathbb{C}_{+\mathrm{e}}$

$\mathbb{C}$ — The field of complex numbers

$\mathbb{C}_{\mathrm{e}}$ — $\mathbb{C} \cup \{\infty\}$

$\mathbb{C}_{+\mathrm{e}}$ — $:= \{s \in \mathbb{C}_{\mathrm{e}} : \mathrm{Re}\, s \geq 0 \text{ or } s = \infty\}$

$\mathbb{C}^n$ — The Euclid space of n-dimensional column vectors composed of complex numbers

$\mathbb{D}_\rho$ — $:= \{z \in \mathbb{C}_{\mathrm{e}} : |z| > \rho \text{ or } z = \infty\}$ for $\rho > 0$

$\mathbb{D}$ — $:= \mathbb{D}_1$

$\mathcal{D}$ — The space of all real functions $a(t)$ such that $\mathrm{e}^{\epsilon t}a(t)$ belongs to $\mathcal{L}^2$ for some positive $\epsilon > 0$

$\det M$ — The determinant of a matrix M

$\grave{E}^s_m$ — A tall operator that maps $\boldsymbol{v} \in \mathbb{C}^n$ to $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$ with $\boldsymbol{f}(t) := \frac{1}{\sqrt{\tau}}\mathrm{e}^{(s+\mathrm{i}2\pi m/\tau)t}\boldsymbol{v}$ for $0 \leq t < \tau$

$\acute{E}^s_m$ — A flat operator that maps $\boldsymbol{g} \in \mathcal{L}^2[0,\tau)^n$ to $\boldsymbol{u} \in \mathbb{C}^n$ with $\boldsymbol{u} := \frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{e}^{-(s+\mathrm{i}2\pi m/\tau)t}\boldsymbol{g}(t)\,\mathrm{d}t$

$\mathcal{F}(G, K)$ — $:= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. The lower fractional transform.

H_τ^{zo} The zero-order hold with the sampling period τ

$\mathcal{H}^2$ The Hardy space consisting of all $\mathbb{C}^n$-valued functions $\widehat{\boldsymbol{a}}(s)$ such that $\widehat{\boldsymbol{a}}(s)$ is analytic in $\mathrm{Re}\, s > 0$ and satisfies $\sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\widehat{\boldsymbol{a}}(\sigma + \mathrm{i}\omega)\|_2^2 \,\mathrm{d}\omega < \infty$

$\mathfrak{H}^2$ The Hardy space consisting of all $\mathbb{C}^n$-valued functions $\breve{\boldsymbol{a}}_{\mathrm{d}}(z)$ such that $\breve{\boldsymbol{a}}_{\mathrm{d}}(z)$ is analytic in $\mathbb{D}$ and satisfies $\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\breve{\boldsymbol{a}}_{\mathrm{d}}(re^{\mathrm{i}\omega})\|_2^2 \,\mathrm{d}\omega < \infty$

$\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$ The Hardy space consisting of all $\mathcal{L}^2[0,\tau)^n$-valued functions $\breve{\tilde{\boldsymbol{a}}}(z)$ such that $\breve{\tilde{\boldsymbol{a}}}(z)$ is analytic in $\mathbb{D}$ and satisfies $\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\breve{\tilde{\boldsymbol{a}}}(re^{\mathrm{i}\omega})\|^2_{\mathcal{L}^2[0,\tau)} \,\mathrm{d}\omega < \infty$

$\mathcal{H}^\infty$ The Hardy space of all matrix-valued functions $\widehat{P}(s)$ that are analytic and bounded in the open half plane $\mathrm{Re}\, s > 0$

$\mathfrak{H}^\infty$ The Hardy space of all matrix-valued functions $\breve{P}_{\mathrm{d}}(z)$ that are analytic and bounded in $\mathbb{D}$

$\mathfrak{H}^\infty_{\mathrm{L}}$ The Hardy space of all large-operator-valued functions that are analytic and bounded in $\mathbb{D}$.

$\mathfrak{H}^\infty_{\mathrm{F}}$ The Hardy space of all flat-operator-valued functions that are analytic and bounded in $\mathbb{D}$.

$\mathfrak{H}^\infty_{\mathrm{T}}$ The Hardy space of all tall-operator-valued functions that are analytic and bounded in $\mathbb{D}$.

i The imaginary unit

I The identity matrix or the identity operator

I_n The $n \times n$-identity matrix

$\mathrm{Im}\, s$ The imaginary part of a complex number s

$\mathcal{K}$ The set of all continuous-time operators that have continuous-time state-space representations and have input- and output-signal dimensions consistent with a provided plant G

$\mathcal{K}_0$ $:= \{K_0 \in \mathcal{K} : \widehat{K}_0(\infty) = O\}$

$\mathcal{K}_{00}$ $:= \{K_{00} \in \mathcal{K} : \widehat{K}_{00}(\infty) = O$ with multiplicity two or more$\}$

$\mathcal{K}_{\mathrm{d}}$ The set of all discrete-time operators that have discrete-time state-space representations and have input- and output-signal dimensions consistent with a provided sampling environment (τ, S, H)

$(\ell^2)^n$ The Hilbert space of one-sided sequences of n-dimensional vectors $\boldsymbol{a}_{\mathrm{d}} = \{\boldsymbol{a}_{\mathrm{d}}[k]\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} \|\boldsymbol{a}_{\mathrm{d}}[k]\|_2^2 < \infty$. When there is no fear of confusion, it is written as ℓ^2.

$\ell^2_{\mathcal{L}^2[0,\tau)}$ The Hilbert space of all functional sequences $\tilde{\boldsymbol{a}} = \{\tilde{\boldsymbol{a}}[k]\}_{k=0}^{\infty}$ such that each $\tilde{\boldsymbol{a}}[k]$ is a function belonging to $\mathcal{L}^2[0,\tau)^n$ and there holds $\sum_{k=0}^{\infty} \|\tilde{\boldsymbol{a}}[k]\|^2_{\mathcal{L}^2[0,\tau)} < \infty$

$(\mathcal{L}^2)^n$ The Hilbert space of Lebesgue-square-integrable n-dimensional-vector-valued functions $\boldsymbol{a}(t)$ that are defined on $[0,\infty)$ and satisfy $\int_0^\infty \|\boldsymbol{a}(t)\|_2^2 \,\mathrm{d}t < \infty$. When there is no fear of confusion, it is written as $\mathcal{L}^2$.

$\mathcal{L}^2[0,\tau)^n$ The Hilbert space of Lebesgue-square-integrable n-dimensional-vector-valued functions $\boldsymbol{a}(t)$ that are defined on $[0,\tau)$ and satisfy $\int_0^\tau \|\boldsymbol{a}(t)\|_2^2 \,\mathrm{d}t < \infty$. Here, $\tau > 0$. When there is no fear of confusion, it is written as $\mathcal{L}^2[0,\tau)$.

O The zero matrix or the zero operator

$\mathbb{R}$ The field of real numbers

R The continuous-time operator whose continuous-time transfer function is $\widehat{R}(s) = \dfrac{1}{s+1} I$

R_τ The continuous-time operator whose continuous-time transfer function is $\widehat{R}_\tau(s) = \dfrac{1}{\tau s+1} I$

$\mathrm{Re}\, s$ The real part of a complex number s

$\mathcal{RH}^\infty$ The set of all real rational matrix-valued functions $\widehat{P}(s)$ that belong to $\mathcal{H}^\infty$

$\mathfrak{RH}^\infty$ The set of all real rational matrix-valued functions $\breve{P}_{\mathrm{d}}(z)$ that belong to $\mathfrak{H}^\infty$

$\mathfrak{RH}^\infty_{\mathrm{L}}$, $\mathfrak{RH}^\infty_{\mathrm{F}}$, $\mathfrak{RH}^\infty_{\mathrm{T}}$

The sets of all real rational operator-valued functions that belong to $\mathfrak{H}^\infty_{\mathrm{L}}$, $\mathfrak{H}^\infty_{\mathrm{F}}$, and $\mathfrak{H}^\infty_{\mathrm{T}}$, respectively

S_τ^{id} The ideal sampler with the sampling period τ

W_τ The lifting operator. It maps a continuous-time signal $\boldsymbol{a}(t)$ to a functional sequence $\tilde{\boldsymbol{a}} = \{\tilde{\boldsymbol{a}}[k]\}_{k=0}^{\infty}$, where each $\tilde{\boldsymbol{a}}[k]$ is a function belonging to $\mathcal{L}^2[0,\tau)$ and is defined as $\tilde{\boldsymbol{a}}[k](t) := \boldsymbol{a}(k\tau + t)$ for $0 \le t < \tau$.

Chapter 1

Introduction

1.1. Background and Objectives

A Continuous-Time Control System and a Sampled-Data Control System

Today's engineering cannot be discussed without mentioning digital techniques such as digital computation and digital signal processing. Control is not an exception. In a theoretical world, researches on continuous-time controllers are still dominating because a system with a continuous-time controller has a simple structure and is easy to be investigated. However, in a practical world, most of controllers are realized as sampled-data ones. This is because a sampled-data controller is constructed with digital techniques and can realize a more complicated control law with higher precision than continuous-time controllers, which are based on analog techniques.

Let us begin by showing an example of a continuous-time control system.

Example 1.1. We consider a system to control an inverted pendulum, which is often used for a laboratory experiment of a control system. This system is presented in Figure 1.1 (a). A stick called a pendulum is connected to a cart by a free joint and the cart is driven by a motor along a straight rail in both directions. By adjusting a voltage given to the motor, we can change the velocity of the cart as we like. Moreover, a sensor attached to the free joint enables us to measure the angle between the vertical line and the pendulum. Our purpose is to keep the pendulum standing up vertically on the free joint by adjusting the voltage to the motor based on the measured angle of the pendulum. This is a desired function of a controller to be designed.

Figure 1.1 (b) shows an abstraction of the system of the inverted pendulum. The symbol $u(t)$ stands for the voltage given to the motor at the time t, whereas $y(t)$ expresses the angle of the pendulum at the time t. The voltage $u(t)$, which is really given to the motor, may be

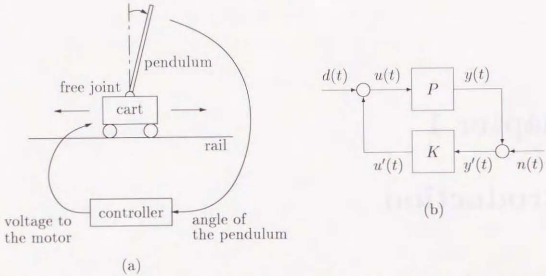


Figure 1.1. A continuous-time control system to control an inverted pendulum: (a) the appearance of the system; (b) its abstraction.

different from the voltage $u'(t)$, which the controller intends to give, because a disturbance $d(t)$ comes in from the outside. The angle $y(t)$ is also contaminated by a sensor noise $n(t)$ and the measured angle $y'(t)$ is somewhat different from the real angle. Suppose that all the signals $u(t), y(t), \ldots$ are functions of a continuous time t. A symbol P stands for a mathematical model that characterizes the dynamics from the actual voltage $u(t)$ to the actual angle $y(t)$. This P is an object to be controlled and is called a *plant*. By neglecting nonlinearities and higher-order dynamics included in the real dynamics, we describe the plant P in linear ordinary differential equations of a finite order. This P can be regarded as an operator that maps a function $u(t)$ to a function $y(t)$.

The block K is a controller to be designed. At least, a controller must make the pendulum stand up on the free joint even if a small disturbance $d(t)$ and a noise $n(t)$ come in — in other words, a controller must make the system *stable*. In addition to it, a controller is usually required to make the system to have a *good performance*. For example, it may be desired that the system attenuates the effect of a disturbance $d(t)$ and a sensor noise $n(t)$; It may be desired that the system is stabilized robustly against the dynamics neglected at modeling. Here, suppose that the dynamics of K is described by means of differential equations just as P. This description is appropriate when K is realized in an analog circuit for example. In such a case, K is called a *continuous-time controller* and a constructed system is called a *continuous-time control system*. □

A continuous-time control system has a simple structure in the sense that its two components P and K are both expressed by differential equations. Because of this simplicity, a lot of methods to design a controller K has been proposed. Among them, modern design methodologies like $\mathcal{H}^\infty$ and $\mathcal{H}^2$ are distinguished from the classical ones like PID in the sense that the best

achievable performance can be computed theoretically. This means that an engineer does not have to repeat trial and error in vain; If the provided performance specification is impossible to be achieved, he can say so. A problem here is that, if we try to construct a continuous-time controller in a real world using an analog technique, it is difficult to realize a complicated control law with a high precision. These days, control engineers are asked to solve more and more complicated control problems. For instance, a plant to be controlled often has multiple inputs and multiple outputs; several performance specifications such as disturbance attenuation and robust stabilization are assigned simultaneously. In such a situation, controllers that accomplish the best achievable performance tend to be more complicated than the ones practically realizable as continuous-time controllers. Then, it is questionable whether the best achievable performance is really attained.

A remedy for this is to introduce a sampled-data controller. Since a sampled-data controller is constructed based on digital techniques, it can realize a complicated control law with a high precision.

Example 1.2. Let us consider to control the system of Example 1.1 using a sampled-data controller. The resulting system typically looks like Figure 1.2 (a). Here, a controller is a digital one frequently implemented in a digital computer. Since the input and the output of this controller are digital signals, we need analog-to-digital (A/D) and digital-to-analog (D/A) signal converters in order to connect this controller to our inverted pendulum. Moreover, since analog-to-digital conversion is sensitive to a high-frequency noise, it is usual to cut such a noise using a low-pass filter, which is called an *anti-aliasing filter*.

Figure 1.2 (b) shows an abstraction of the system in (a). Here, an analog signal is modeled as a *function* of a continuous-time t and is called a *continuous-time signal*. A digital signal is regarded as a *sequence* and is called a *discrete-time signal*. In this figure, a continuous-time signal is presented by a solid arrow, whereas a discrete-time signal by a broken arrow. The plant P is the same as in the previous example. The symbols F, S, K_{d}, and H denote the system components corresponding to an anti-aliasing filter, an analog-to-digital converter, a digital controller, and a digital-to-analog converter, respectively. It is assumed that the operator F is described in differential equations and has a low-pass property. The block S is called a *sampler*. There can be various sampling schemes. Typically, S is assumed to work as $y''_{\mathrm{d}}[k] := y''(k\tau)$ with the symbols in the figure. See Figure 1.3 (a) for the operation of this typical S. Here, $\tau > 0$ is a time period chosen in advance and is called a *sampling period*. From now on, this particular sampler is called the *ideal sampler* with the sampling period τ. The block K_{d} is called a *discrete-time controller*. Its operation is assumed to be expressed by *difference* equations while that of a continuous-time controller is by *differential* equations. Finally, the block H is called a *hold*. The most typical hold works as $u'(k\tau + t) := u'_{\mathrm{d}}[k]$ for any $k = 0, 1, \ldots$ and $0 \le t < \tau$. See Figure 1.3 (b). This hold is called the *zero-order hold* with the sampling period τ. Just

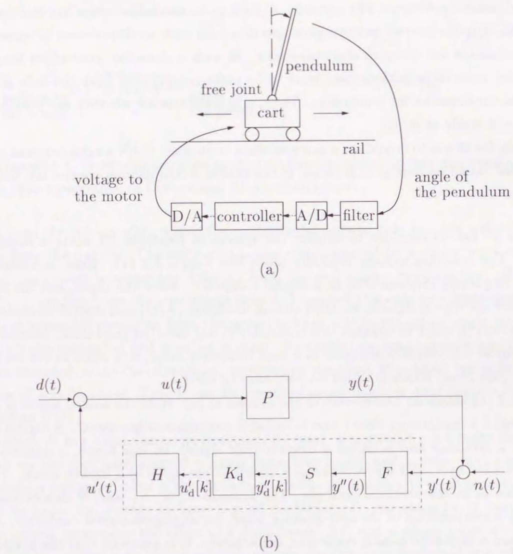


Figure 1.2. A sampled-data control system to control an inverted pendulum: (a) the appearance of the system; (b) its abstraction.

like the sampler case, there can be many other types holds. A train of blocks F, S, K_d, and H is called a *sampled-data controller*. As in Example 1.1, the system components F, S, K_d, and H can be regarded as operators. □

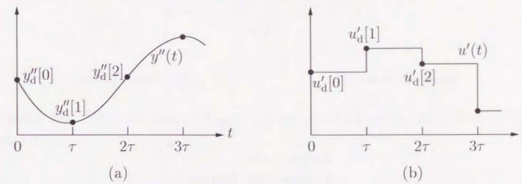


Figure 1.3. The operation of (a) the ideal sampler and (b) the zero-order hold.

Samplers and holds different from the ideal sampler and the zero-order hold are called *generalized samplers and holds*, respectively. If we choose a generalized sampler and a generalized hold appropriately for a provided plant, it is possible to improve a control performance beyond the best performance achievable by the ideal sampler and the zero-order hold [55, 54]. However, realization of these generalized devices is more difficult than the typical ones.

A drawback of a sampled-data controller is that it makes a system more complicated than a continuous-time controller. Particularly in a sampled-data control system, both continuous-time signals and discrete-time signals are included; one component is described in differential equations and another is in difference equations. Because of this *hybrid nature*, synthesis and analysis of a sampled-data control system have been difficult for a long time. Let us see this next.

Synthesis and Analysis of a Sampled-Data Control System

Let us consider to synthesize a sampled-data control system for a provided plant P. We suppose that a sampling period τ, an anti-aliasing filter F, a sampler S, and a hold H are given in some way. What is considered here is to design a discrete-time controller K_d so that the resulting system is stable and has a good performance. We mean this design of K_d by synthesis of a sampled-data control system. Two approaches have been taken to this problem [36, 20], that is,

(i) designing a controller in the continuous-time domain and discretizing it;

(ii) discretizing a plant and designing a controller in the discrete-time domain (see Figure 1.4). Once K_{d} is obtained, we can construct a sampled-data controller by combining it with a provided sampler and hold.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
P \arrow[r, "\substack{\text{design in the}\\ \text{continuous-time domain}}", "\text{(i)}"'] \arrow[d, "\text{discretization}"', "\text{(ii)}"] & K \arrow[d, "\text{(i)}"', "\text{discretization}"] \\
P_{\mathrm{d}} \arrow[r, "\text{(ii)}", "\substack{\text{design in the}\\ \text{discrete-time domain}}"'] & K_{\mathrm{d}}
\end{tikzcd}$$

Figure 1.4. Two conventional approaches for a synthesis of a sampled-data control system: P is a plant to be controlled; P_{d} is a discretized plant; K is a continuous-time controller; K_{d} is a discrete-time controller to be obtained.

In the first approach (i), a continuous-time controller K is designed for a plant P just as a design of a continuous-time control system and then K is discretized into K_{d}. One way for this discretization is to put $K_{\mathrm{d}} := S'KH'$ using some sampler S' and some hold H'. (It is not necessary that S' and H' are identical with S and H, respectively.) Another way is to transform K by a bilinear transformation. In this approach (i), we can use well-established methodologies to design a continuous-time controller K. Moreover, since we carry out a controller design in the continuous-time domain, we can give a performance specification in a natural way using concepts in the continuous-time domain. A problem of this approach is that, even if a continuous-time controller K attains a desirable control performance, actually implemented into the system is a sampled-data controller $HK_{\mathrm{d}}SF$. Therefore, when the dynamics of a sampled-data controller is not close enough to that of the original continuous-time controller, a designed sampled-data control system does not behave as expected; sometimes it even falls unstable although a continuous-time control system made of P and K is stable. It is often said that if the sampling period τ is "small enough," the dynamics of $HK_{\mathrm{d}}SF$ is almost similar to that of K. However, this does not give an answer to a question like "What sampling period is small enough?" or "What shall we do if the sampling period is not small enough?"

In the second approach (ii), we modify the system diagram of Figure 1.2 (b) as in Figure 1.5 and define a discretized plant by $P_{\mathrm{d}} := SFPH$. Then, both P_{d} and K_{d}, which constitute the system, are expressed by difference equations. Let us call this type of system a *discrete-time control system*. The structure of a discrete-time control system is as simple as that of

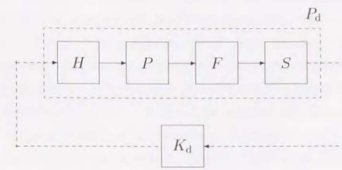


Figure 1.5. Interpretation of a sampled-data control system as a discrete-time control system.

a continuous-time control system, whose components are described in differential equations. Actually, design methodologies for a discrete-time control system have been developed almost in a parallel way to those for a continuous-time control system [20, 40, 99]. Based on them, a discrete-time controller K_{d} can be obtained. As far as the dynamics at the sampling instants $t = 0, \tau, 2\tau, \ldots$ is concerned, no approximation is involved in this approach. Therefore, the resulting sampled-data control system behaves exactly as expected at least at the sampling instants. One disadvantage of this approach is that the intersample behavior of the system is completely neglected. Indeed, it is reported that a sampled-data control system designed along this approach sometimes behaves badly between the sampling instants, though its behavior at sampling instants is good. This phenomenon is called *ripples*. Again, it is said that, if the sampling period is "small enough," such a phenomenon does not occur. However, this cannot be a real answer as we have seen before. Another disadvantage is that it is not easy to translate provided performance specification into terms of the discrete-time domain, where a controller design is carried out. If this translation is not appropriate, it may happen that a designed sampled-data control system does not satisfy the original performance specification given in the continuous-time domain.

So far, we have seen how difficult synthesis of a sampled-data control system is. Similarly, analysis of this system is not easy either. There are two approaches to analyze it. One is to regard a sampled-data controller as an approximation of some continuous-time controller and to analyze the continuous-time control system made of this continuous-time controller and the provided plant. This is analogous to the synthesis approach (i). Another approach is to neglect the intersample behavior of the provided sampled-data control system and to analyze it in the discrete-time domain. This corresponds to the approach (ii) before. It is now obvious that these approaches for analysis have problems. The first approach does not really analyze the provided sampled-data control system. If the approximation by a continuous-time controller is not good, the performance of the continuous-time control system, which is analyzed, is completely different from the real performance of the original sampled-data control system. The second approach for analysis neglects the intersample behavior. Therefore, it is possible to overlook a bad intersample behavior. Moreover, since analysis has to be carried out in the discrete-time domain, it is not obvious what it means in the continuous-time domain.

A sampled-data controller is introduced to realize a complicated control law with a high precision. Although the best achievable performance of continuous-time control systems can be computed theoretically, an actual system is realized as a sampled-data control system in many cases. Here, it is natural to conjecture that the best performance theoretically achievable by continuous-time control systems can be asymptotically attained by sampled-data control systems if the sampling period is chosen "small enough." However, this conjecture is not easy to be proven since synthesis and analysis of a sampled-data control system are not straightforward. Indeed, it is difficult to obtain the best achievable performance of sampled-data control systems. It is also difficult to analyze the performance of a provided sampled-data control system considering intersample behavior without approximation.

Åström *et al.* [7] showed that, when a continuous-time plant having a relative degree larger than two is discretized with a small sampling period, the plant gains an additional unstable zero. This result is interesting because an unstable zero is considered to degrade the control performance and thus it seems to be against our conjecture. Although the effects of unstable zeros can be quantified in terms of integral-type constraints (See [37, 14] and the references therein), direct relationships between unstable zeros and the best achievable performance are not clear when the performance is measured by the $\mathcal{L}^2$-induced norm or the $\mathcal{H}^2$-norm. Hence, it could not be a help to prove whether our conjecture is correct or not.

Lifting-Based Approach to a Sampled-Data Control System

In these two decades, many researchers have tried to directly deal with a hybrid nature of a sampled-data control system. Thompson *et al.* [88, 87] used a conic sector to capture this hybrid nature. Francis and Georgiou [35] investigated stability of a sampled-data control system in detail. Leung *et al.* [64] analyzed this system assuming that a band-limited input is injected into the system. Keller and Anderson [59] tried to handle the intersample behavior of a sampled-data control system by discretizing the system with a smaller period than the controller period.

Difficulty in the treatment of a sampled-data control system was removed to a considerable degree by a so-called *lifting technique*, which was introduced by Yamamoto [94, 95]. In particular, it enables us to synthesize and analyze a sampled-data control system excluding approximation unlike the first approach before and taking into account the intersample behavior of the system unlike the second approach. A basic idea of lifting is to chop a provided continuous-time signal $\boldsymbol{a}(t)$ at each sampling instant $t = k\tau$, $k = 0, 1, \ldots$, and to regard it as a sequence

of functional fragments $\{\tilde{\boldsymbol{a}}[k]\}_{k=1}^{\infty}$, where each $\tilde{\boldsymbol{a}}[k]$ is a function defined as $\tilde{\boldsymbol{a}}[k](t) := \boldsymbol{a}(k\tau + t)$ for $0 \le t < \tau$. See Figure 1.6. Note that this sequence can be considered as a discrete-time signal, which takes a value in a functional space defined on $[0, \tau)$. Doing so, we regard all the continuous-time signals included in a sampled-data control system as discrete-time ones. Then, a hybrid nature of the system is not a problem anymore and its synthesis and analysis become much easier, namely, they can be done by applying techniques for discrete-time control systems.

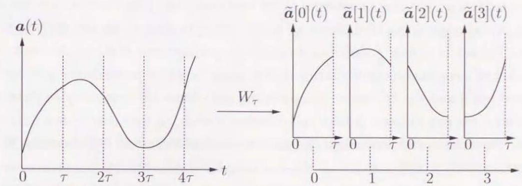


Figure 1.6. Lifting of a continuous-time signal $\boldsymbol{a}(t)$.

Along this line, existing methodologies for synthesis and analysis of control systems have been translated so as to be applicable to sampled-data control systems. Since these new methodologies consider an intersample behavior of systems without any approximation, clearly they are superior to conventional ones. This was done for the $\mathcal{H}^\infty$-control by Bamieh *et al.* [11, 9], Tadmor [86], Toivonen [89], Kabamba and Hara [56], Hayakawa *et al.* [48] and others; for the $\mathcal{H}^2$-control by Chen and Francis [18, 19, 16], Khargonekar and Sivashankar [61], Bamieh and Pearson [10], and Hagiwara and Araki [42] to name a few. The $\mathcal{L}^1$-control for sampled-data control systems was considered by Dullerud and Francis [28] and Bamieh *et al.* [8]. A sampled-data robust stabilization was studied by Sivashankar and Khargonekar [83] and Dullerud and Glover [29, 30, 31, 27]. Although many of these papers assumed the ideal sampler and the zero-order hold for digital/analog signal conversion, a lifting-based approach is effective for a system with a generalized sampler and hold. This was pursued in [45, 86, 53, 56, 5, 66] to name a few. Especially, Tadmor [86] considered not only an optimal design of a discrete-time controller K_d but also that of a sampler and/or a hold. (A closely related problem was considered by Sun *et al.* [84].) Mirkin and Rotstein [66] considered lifting of a sampler and a hold, whereas most of other papers have used lifting of a plant only. A *multirate sampled-data control system* is a generalization of a sampled-data control system and has been investigated for a long time [63, 58, 6, 41, 67, 43, 33]. In a multirate system, each of a controller, a sampler, and a hold may work with its own time period. Lifting can be used in order to analyze this system, too. About this topic, there are the works of Voulgaris and Bamieh [93] and Chen and Qiu [21, 81].

The Best Achievable Performance of Sampled-Data Control Systems

Now let us consider our conjecture about the best achievable performance. From now on, sometimes we use a shorter term, *the best continuous-time control performance*, in place of the best achievable performance of continuous-time control systems. Similarly, we sometimes say *the best sampled-data control performance* meaning the best achievable performance of sampled-data control systems. In these terms, what we are interested in is whether the best sampled-data control performance converges to the best continuous-time control performance as the sampling period approaches zero. Since we have a lifting technique, we are ready to consider this problem. Indeed, it is now possible to measure the performance of a sampled-data control system and that of a continuous-time control system using the same performance indices such as the $\mathcal{L}^2$-induced norm and the $\mathcal{H}^2$-norm. Moreover, we can obtain the best sampled-data control performance with respect to these indices and compare it with the best continuous-time control performance. Trentelmen and Stoorvogel [91] and Osburn and Bernstein [80] chose the $\mathcal{H}^2$-norm as the performance index and proved that the best sampled-data control performance converges to the best continuous-time control performance as the sampling period approaches zero. Hara *et al.* [45] chose the $\mathcal{L}^2$-induced norm (or equivalently the $\mathcal{H}^\infty$-norm) as the performance index and did the same thing in a special case. (Tadmor [86] considered a closely related problem, on which we will comment in Remark 4.6.)

However, the above proofs treated only the situation that the ideal sampler and the zero-order hold are used for a sampler and a hold, respectively, and the same anti-aliasing filter F is used for any sampling period τ. In a more general case, the best sampled-data performance *may not* converge to the best continuous-time control performance. This is seen from the next example.

Example 1.3. Here, we consider a robust stabilization problem both with a continuous-time control scheme and with a sampled-data control scheme. As we saw in Example 1.1, an operator P is a mathematical model of a controlled object and may be slightly different from the actual one. Therefore, a controller needs to stabilize not only P but also a plant slightly different from P. There are several approaches to handle this problem. One way to formulate it in the context of the $\mathcal{H}^\infty$-control is the following.

Consider a system in Figure 1.7 (a), where a controlled object is expressed as the combination of P, Δ, and W. Here, the operator P expresses a nominal plant model and Δ stands for uncertainty of our knowledge about the plant. Specifically, we suppose that it is known that Δ is an element of the set

$$\boldsymbol{\Delta}(\gamma) := \{\Delta : \Delta \text{ is a causal linear operator satisfying } \|\Delta\| < \gamma\}$$

but it is *not* known which element it is. Here, γ is some positive number and $\|\Delta\|$ denotes the $\mathcal{L}^2$-induced norm of the operator Δ (See Section 2.3.1 for its precise definition). The

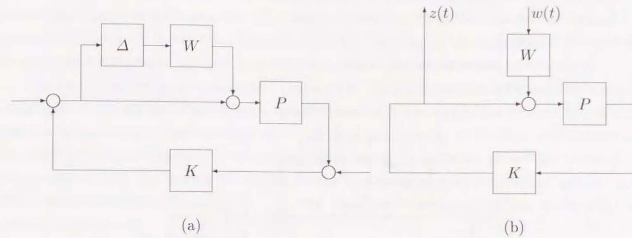


Figure 1.7. Robust stabilization using a continuous-time controller: (a) a continuous-time control system with a uncertain plant; (b) a system whose $\mathcal{L}^2$-induced norm should be made small.

operator W is a known linear operator described in linear differential equations. This W is introduced in order to tailor the uncertainty included in the plant. For example, if the plant is not considered to be well-identified in a high-frequency range, we let W have a high gain there. Then, the system composed of P, Δ, and W has a large uncertainty in this range. As a consequence of the small-gain theorem, it can be shown that this system is stable for any $\Delta \in \boldsymbol{\Delta}(\gamma)$ if and only if $\gamma \|T_{z\leftarrow w}\| \le 1$, where $T_{z\leftarrow w}$ denotes the operator from $w(t)$ to $z(t)$ in Figure 1.7 (b) and $\|T_{z\leftarrow w}\|$ its $\mathcal{L}^2$-induced norm. Therefore, in view of robust stabilization, the best continuous-time control performance can be expressed by the infimum value of $\|T_{z\leftarrow w}\|$ over all continuous-time controllers K that stabilize the system.

The things are almost the same about a sampled-data control system. Here, we consider a system presented in Figure 1.8 (a), where the sampling period τ and the operators P, S, F, H, and W are provided in advance. It is proven that this system is stable for any element of $\boldsymbol{\Delta}(\gamma)$ if and only if $\gamma \|T_{z\leftarrow w}\| \le 1$, where $T_{z\leftarrow w}$ is an operator from $w(t)$ to $z(t)$ in Figure 1.8 (b) [83]. Consequently, the best sampled-data control performance in this setting is the infimum of $\|T_{z\leftarrow w}\|$ over all discrete-time controllers K_d that stabilize the system.

Now, suppose that P is an unstable plant having a continuous-time transfer function $1/(s-1)$. (See Section 2.3.1 for the definition of a transfer function.) Let W have a continuous-time transfer function $s+1$, which has a high gain in a high-frequency range. Then, we can compute the best continuous-time control performance by transforming the problem into a model-matching problem just as explained in Example 6.1.2 of [34]. In order to consider the best sampled-data control performance, we choose S to be the ideal sampler with the sampling

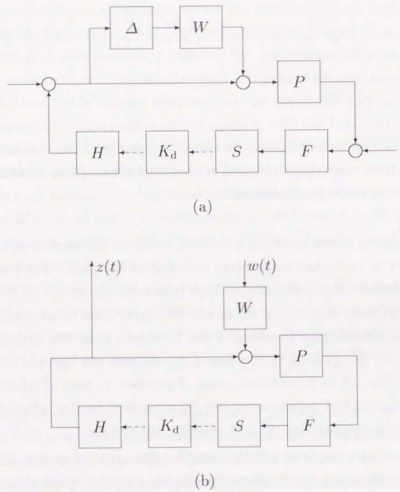


Figure 1.8. Robust stabilization using a sampled-data controller: (a) a sampled-data control system with a uncertain plant; (b) a system whose $\mathcal{L}^2$-induced norm should be made small.

period τ and H to be the zero-order hold with the same sampling period τ. We consider two cases regarding an anti-aliasing filter F. That is, in the first case, the transfer function of F is chosen as $1/(\tau s+1)$; in the other case as $1/(s+1)$. Then, using an algorithm of Bamieh and Pearson [9], for example, the best sampled-data control performance can be computed.

The simulation results are presented in Figure 1.9. The solid line stands for the best sampled-data control performance when F has a transfer function $1/(\tau s+1)$. (The symbol $\widehat{F}(s)$ denotes the continuous-time transfer function of F.) It *does not* converge to the best continuous-time control performance expressed by the dash-dot line even if the sampling period approaches zero. On the other hand, the broken line stands for the best sampled-data control performance with the transfer function of F being $1/(s+1)$. This line *does* converge to the best continuous-time control performance. □

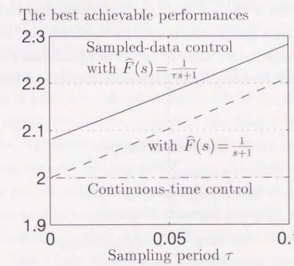


Figure 1.9. The best sampled-data control performance does not always converge to the best continuous-time control performance.

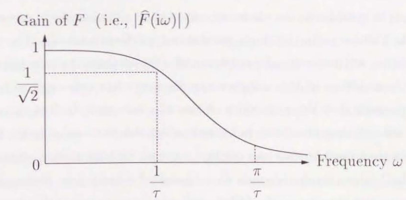


Figure 1.10. A gain plot of the first F over frequencies.

If the continuous-time transfer function of F is $1/(\tau s+1)$, its gain plot over frequencies looks like Figure 1.10. It is a low-pass filter whose bandwidth is proportional to $1/\tau$. This choice of a filter seems quite reasonable considering an aliasing effect, which occurs at the ideal sampler S. In order to see this, consider a continuous-time signal having high-frequency components beyond the Nyquist frequency π/τ and suppose that it is fed into the ideal sampler S. Sampled by the ideal sampler, a distinction between the frequency ω and its side-band frequencies $\omega + 2\pi m/\tau$, $m = \pm 1, \pm 2, \ldots$, disappears. Because of this effect, the high-frequency components are folded onto the low-frequency range and contaminate low-frequency components. Hence, it is often said that high-frequency components exceeding the frequency π/τ should be attenuated before fed into the ideal sampler S. This is consistent with the above choice of the filter because this filter has a low gain beyond the frequency π/τ. However, the simulation result shows that the best sampled-data control performance in this case does not converge to the best continuous-time control performance even if the sampling period approaches zero. This means that our fundamental conjecture about a sampled-data control system is not always correct. Since this conjecture is considered to help the widespread use of sampled-data controllers, it is a pressing need to clarify why such a non-converging phenomenon occurs and how it can be avoided.

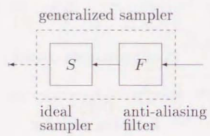


Figure 1.11. Interpreting the pair of the ideal sampler S and an anti-aliasing filter F as a generalized sampler.

Moreover, it is seen from the above example that a choice of an anti-aliasing filter F affects convergence to the best continuous-time control performance. In this thesis, we combine an anti-aliasing filter with the ideal sampler and regard them as one generalized sampler. See Figure 1.11. Considering in this way, we can also say that convergence to the best continuous-time control performance depends on a choice of a sampler. In fact, another simulation result shows that if we use a generalized hold instead of the zero-order hold, the best sampled-data control performance may or may not converge to the best continuous-time control performance. This means that convergence depends on a choice of a hold, too. Hence, in order to investigate this performance convergence issue, first we have to construct a framework to treat sampled-data control systems with a large class of samplers and holds.

Objectives of This Thesis

In this thesis, the following two problems are considered:

(i) Construction of a lifting-based framework for sampled-data control systems with a large class of samplers and holds;

(ii) Investigation of the best sampled-data control performance, especially on its convergence to the best continuous-time control performance.

In Problem (i), our purpose is to construct a general, clear, useful framework for sampled-data control systems. Here, special attention is paid on treatment of samplers and holds, which have not been treated so seriously. A principal reason why we make this framework is to provide a solid basis for the analysis of the best sampled-data control performance. However, this framework is significant in its own right because it is believed to be useful for other advanced problems on sampled-data control systems, too.

Although there are many studies on generalized holds [55, 54, 45, 86, 56, 5, 66], they assumed that the kernel functions of holds are defined on $[0,\tau)$, where τ is the sampling period. In other words, a discrete-time signal received by a hold at the time $t = k\tau$ affects its output only during $k\tau \le t < (k+1)\tau$. Therefore, the first-order hold, which is often quoted as an example of generalized holds, is not covered by their hold classes (Example 3.7). Similarly in the studies on generalized samplers [86, 53, 66], it is assumed that samplers have kernel functions defined on $[0,\tau)$. A problem here is that treatment of the ideal sampler, which is the most important sampler in practice, is difficult in their frameworks because its kernel function turns out to be the delta function. This thesis shows that, by enlarging the domain of kernel functions to $[0,\infty)$, we can resolve these difficulties. Consequently, our framework obtains generality regarding treatment of samplers and holds.

Furthermore, we try to make our framework clear for ease of the subsequent analysis. For this purpose, we consider lifting and lifting-based transfer functions of various system components including samplers and holds, while it has been usual to consider lifting of continuous-time plants only. Besides, we interpret the FR-operators of [4, 3, 42, 5] as matrix representations of the above transfer functions (also see [96]). Since FR-operators are frequency responses of a sampled-data control system in some sense, they are defined only on the unit circle $|z| = 1$. On the other hand, our matrix representations can be defined on $|z| \ge 1$. Hence, our representations give more information on sampled-data control systems. Especially for samplers and holds, relationships between their matrix representations and kernel functions are presented.

Based on this framework, three basic properties of sampled-data control systems are obtained. They play an important role in the analysis of the best sampled-data control performance. This fact shows usefulness of our framework.

In the next part of this thesis, we consider Problem (ii) above, that is, we investigate the best achievable performance of sampled-data control systems in relation with that of continuous-time control systems. System performance is measured by the $\mathcal{L}^2$-induced norm as in Example 1.3. Our main interest is concerned with obtaining a condition in order that the best sampled-data control performance converges to the best continuous-time control performance.

One reason to consider this problem is that, as we saw previously, it is related to a fundamental conjecture about a sampled-data control scheme. That is, it is conjectured that the best continuous-time control performance is recovered by sampled-data controllers, and based on this conjecture we accept a sampled-data controller as a substitute of a continuous-time controller. However, this conjecture is not always correct as we have seen. Accordingly, there is a need to clarify when it is correct and when it is not. Another reason to consider Problem (ii) is that it is expected that by investigating this problem we can see what is important in samplers and holds in order to improve control performance. If a condition for the performance convergence is obtained, it may give an insight about how to choose a sampler and a hold.

Investigation is carried out by two steps. At the first step, we consider how much the best sampled-data control performance can be improved by adjustment of a sampling environment. Here, the *best sampled-data control performance* means the optimal performance obtained by choosing an appropriate discrete-time controller when a controlled plant and a sampling environment is provided. Moreover, a *sampling environment* is the triplet of a sampling period, a sampler, and a hold. With this terminology, it is seen that the best sampled-data control performance can be improved by appropriate adjustment of a sampling environment. Then, does the best sampled-data control performance reach the best continuous-time control performance by this adjustment? Against our intuition, it is shown that the former does not always reach the latter. This is because a sampled-data controller cannot instantaneously respond to its input, while a continuous-time controller possessing a nonzero direct feedthrough term can respond. Therefore, when the best continuous-time control performance can be achieved only with a continuous-time controller having nonzero direct feedthrough term, there exists a gap between the best sampled-data control performance and the best continuous-time control performance. At the second step, we consider the theoretical bound that the best sampled-data control performance can reach by adjustment of a sampling environment and investigate what environment we should choose to attain this theoretical bound. In particular, we suppose that a sequence of sampling environments is provided and obtain a necessary and sufficient condition in order that the best sampled-data control performance corresponding to each environment converges to the theoretical bound. If we notice the plants that do not have a gap observed at the first step, this condition is necessary and sufficient in order that the best sampled-data control performance converges to the best continuous-time control performance. Not only a general case but also special cases, where a provided sampling environment has some special structure, are considered. In these special cases, the necessary and sufficient condition above can be modified so as to be tested easily.

1.2. Construction of This Thesis

This thesis is constructed as follows.

Chapter 2

The purpose of this chapter is to introduce notation and terminology utilized throughout this thesis and to present useful facts. Although the contents of Sections 2.1–2.5 are used throughout the subsequent chapters, Section 2.6 is given as a preparation for Section 4.4 particularly. Most of the results presented in this chapter are more or less known in the control community. However, the contents of Sections 2.4.3, 2.4.4, and 2.6 are new.

Section 2.1 gives mathematical notation used in this thesis.

In Section 2.2, notions of continuous-time and discrete-time signals are presented and their frequency-domain representations, i.e., the Laplace transform and the z-transform, respectively, are introduced.

A sampled-data control system is constructed by combination of four different types of operators: a continuous-time operator, a discrete-time operator, a sampler-type operator, and a hold-type operator. Section 2.3 explains these operators in turn. Subsection 2.3.1 discusses continuous-time operators. After basic notions such as their linearity, causality, time-invariance, and boundedness are defined, their transfer functions and state-space representations are introduced. Moreover, important spaces of transfer functions are defined, that is, $\mathcal{H}^\infty$ and $\mathcal{RH}^\infty$. Subsection 2.3.2 talks about discrete-time operators almost in the same manner as the previous subsection. Their basic notions, transfer functions, state-space representations, and important spaces of their transfer functions are presented. A brief subsection 2.3.3 defines sampler-type and hold-type operators.

Section 2.4 introduces the notion of lifting, which is a key idea for recent studies on sampled-data control systems. This notion plays a fundamental role in Chapters 3 and 4. First, lifting of a continuous-time signal is defined in Subsection 2.4.1. Using this concept, lifting of a continuous-time operator is considered in Subsection 2.4.2. Especially, a lifting-based transfer function, a lifting-based state-space representation, and important spaces of transfer functions are in turn explained with respect to continuous-time operators. Subsection 2.4.3 does a similar thing regarding sampler-type and hold-type operators. Subsection 2.4.4 introduces the notion of matrix representations of operators. Since lifting-based transfer functions are operator-valued, their matrix representations are useful for their analysis.

Although the purpose of this thesis is to investigate sampled-data control systems, they are frequently discussed in comparison with continuous-time control systems. In Section 2.5, a continuous-time control system is investigated. After its standard configuration is given, parametrization of all stabilizing controllers (Youla parametrization) is presented with the help of the notions of unimodularity and coprimeness. Moreover, we consider the best performances achievable by two limited classes of continuous-time control systems, respectively.

The last section 2.6 is given as a preparation for Section 4.4. It talks about a model-matching problem including continuous-time, sampler-type, and hold-type operators. By generalizing techniques for a usual model-matching problem, which includes continuous-time operators only, we can consider the above problem. In particular, inner-outer factorization, Hankel norms, and Nehari's theorem are generalized so as to be applicable on operator-valued functions.

Chapter 3

This chapter gives a framework for sampled-data control systems with a large class of generalized samplers and holds. The contents here are important not only as a preparation for the analysis in Chapter 4 but also in their own right. Since the framework given here is considered to be more general and clearer than the other existing frameworks, it would be useful for other sampled-data control problems than the one considered in this thesis.

In the introductory section 3.1, problems of the existing frameworks for sampled-data control systems are discussed.

Section 3.2 introduces regular samplers and holds and investigates their properties. They are defined as special types of sampler-type and hold-type operators, respectively, and are more general than the conventional generalized samplers and holds. Properties of regular samplers and holds are obtained especially on their transfer functions, state-space representations, and matrix representations.

In Section 3.3, the notion of a sampled-data control system is introduced. After its standard structure is explained, the notions of a sampling environment, input-output stability of the system, and the best achievable performance of the system are presented.

Section 3.4 is devoted to derivation of basic properties of sampled-data control systems. Three theorems are stated with respect to the systems. In particular, the last theorem establishes a new relationship between a sampled-data control system and a corresponding continuous-time control system. This theorem plays an essential role in the next chapter. The proof of this theorem is given in Section 3.5. Section 3.6 is a conclusion of this chapter.

Chapter 4

This is a main chapter of this thesis and deals with the properties of the best sampled-data control performance.

Section 4.1 reviews the experimental result in Example 1.3 and restate the result using the notions introduced in the preceding chapters. Then, the importance of this study is discussed.

Section 4.2 gives a theoretical bound about how much the best sampled-data control performance can be improved by adjustment of an sampling environment, which is the triplet of a sampling period, a sampler, and a hold. It is remarkable that this bound is not necessarily equal to the best continuous-time control performance.

In Section 4.3, we obtain a necessary and sufficient condition in order that a provided sequence of sampling environments guarantees that the best sampled-data control performance converges to its theoretical bound for all plants. Moreover, we notice a class of plants with which the best continuous-time control performance is equal to the theoretical bound of the best sampled-data control performance. Then, the condition above is shown to be necessary and sufficient in order that the best sampled-data control performance converges to the best continuous-time control performance for all plants in this class.

In Section 4.4, the condition obtained in the previous section is equivalently modified into a couple of simpler conditions. One condition means the Hankel norm of some function approaches zero; the other condition means that side-band-frequency components of a sampler and a hold decrease in some precisely defined sense. This simplification is carried out by applying techniques for a model-matching problem, which are introduced in Section 2.6. Moreover, it is shown that these conditions can be further simplified when a provided sampler and hold have special structures.

Finally, this chapter is concluded by Section 4.5.

Chapter 5

For both of Chapters 3 and 4, related problems that require further research are described. Significance of the problems and possible approaches are stated, too. This chapter concludes this thesis.

Chapter 2

Preliminaries

This chapter prepares concepts which are utilized throughout this thesis and summarizes useful results. Sections 2.1–2.5 work as a basis of the whole thesis. However, Section 2.6 is a preparation for Section 4.4 in particular. Therefore, it is possible to skip it first and come back at need. Although most of the results are known in the control community, contents of Sections 2.4.3, 2.4.4, and 2.6 are new contributions.

In Section 2.1, mathematical notation is given. In Section 2.2, continuous-time and discrete-time signals are defined together with their frequency-domain representations. Section 2.3 introduces four types of operators that compose a sampled-data control system altogether; namely, a continuous-time operator, a discrete-time operator, a sampler-type operator, and a hold-type operator. Here, especially about a continuous-time operator and a discrete-time operator, their transfer functions, state-space representations, and associated Hardy spaces are presented. Section 2.4 introduces a lifting technique as a preparation for Chapters 3 and 4. Lifting is a key notion in the recent studies on sampled-data control systems. In relation to lifting, the notions of lifting-based transfer functions, lifting-based state-space representations, and matrix representations of operators are given. Section 2.5 talks about continuous-time control systems, which are frequently compared with sampled-data control systems in this thesis. The contents of Section 2.6 are utilized in Section 4.4. Here, a model-matching problem on continuous-time, sampler-type, and hold-type operators is considered. Inner-outer factorization, Hankel norms, and Nehari's theorem are generalized so as to be applicable to this problem.

2.1. Mathematical Concepts

The following mathematical notions are used in this thesis.

The imaginary unit is denoted by i. The symbols $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real numbers and complex numbers, respectively. Let $\mathbb{C}_e$ denote $\mathbb{C} \cup \{\infty\}$. Introduce topology into $\mathbb{C}_e$ by letting fundamental neighborhoods of ∞ be the sets of the form $\{s \in \mathbb{C}_e : |s| > \rho \text{ or } s = \infty\}$

for $\rho > 0$. The set $\mathbb{C}_{+e}$ is defined to be $\{s \in \mathbb{C}_e : \mathrm{Re}\, s \geq 0 \text{ or } s = \infty\}$. Let $\mathbb{D}_\rho$ be the set $\{z \in \mathbb{C}_e : |z| > \rho \text{ or } z = \infty\}$ for $\rho > 0$. Especially, $\mathbb{D}_1$ is simply written as $\mathbb{D}$.

For a positive integer n, the symbol $\mathbb{C}^n$ denotes the Euclid space of n-dimensional column vectors composed of complex numbers. The Euclid norm is expressed by $\|\cdot\|_2$. The zero vector is denoted by $\mathbf{0}$. The zero matrix and the zero operator is expressed by O. The identity matrix or the identity operator is expressed by I. When there is a need to show its size, I_n is used to mean the $n \times n$-identity matrix.

Suppose A is a vector, a matrix, a function, or an operator. Then, the complex conjugate of A is denoted by $\overline{A}$. When A is a vector or a matrix, conjugate is taken componentwise. When A is a function of a real number t, its conjugate $\overline{A}$ is defined by $\overline{A}(t) := \overline{A(t)}$. When A is a function of a complex number s, $\overline{A}$ is defined by $\overline{A}(s) := \overline{A(\overline{s})}$. When A is an operator and the complex conjugate is defined in its domain and range, $\overline{A}$ is defined as an operator $\overline{A}\boldsymbol{a} := \overline{A\overline{\boldsymbol{a}}}$. In each case, A is called **real** if $A = \overline{A}$. This definition applies also when A is an operator-valued function.

The asterisk (*) stands for a complex-conjugate-transpose matrix or an adjoint operator depending on the context. The transpose of a matrix M is expressed as M^{T}. The maximum and minimum singular values of a matrix M are written as $\overline{\sigma}(M)$ and $\underline{\sigma}(M)$, respectively. Here, the maximum singular value of M, i.e., $\overline{\sigma}(M)$, is defined as the nonnegative square root of the maximum eigenvalue of the semi-positive definite matrix M^*M; the minimum singular value of M, i.e., $\underline{\sigma}(M)$, is the nonnegative square root of the minimum eigenvalue of M^*M. When a matrix M is provided, the operation to multiply M is defined. With a slight abuse of notation, this operator is denoted by the same symbol M. Then, the induced norm of a multiplication operator M is equal to $\overline{\sigma}(M)$.

In general, the norm of a normed space X is written as $\|\cdot\|_X$. For an operator P mapping X to a normed space Y, its induced norm is given by

$$\|P\|_{X\to Y} := \sup_{\substack{x\in X\\ \|x\|_X \neq 0}} \frac{\|Px\|_Y}{\|x\|_X}.$$

If $\|P\|_{X\to Y}$ is finite, P is called **bounded**. When the spaces X and Y are clear from the context, we sometimes use a simplified symbol $\|\cdot\|_{\mathrm{ind}}$ to express an induced norm.

Now, important spaces $(\ell^2)^n$, $(\mathcal{L}^2)^n$, and $\mathcal{L}^2[0,\tau)^n$ are introduced in turn. Here, n is a positive integer.

The symbol $(\ell^2)^n$ denotes the Hilbert space of one-sided square-summable sequences each term of which belongs to $\mathbb{C}^n$. That is, a sequence $\boldsymbol{a}_{\mathrm{d}} = \{\boldsymbol{a}_{\mathrm{d}}[k]\}_{k=0}^{\infty}$ belongs to $(\ell^2)^n$ if $\boldsymbol{a}_{\mathrm{d}}[k] \in \mathbb{C}^n$ for $k = 0, 1, \ldots$ and

$$\sum_{k=0}^{\infty} \|\boldsymbol{a}_{\mathrm{d}}[k]\|_2^2 < \infty.$$

The square root of the left-hand side is the norm of $\boldsymbol{a}_{\mathrm{d}}$ and is written as $\|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}$.

The space $(\mathcal{L}^2)^n$ is the Hilbert space of Lebesgue-square-integrable functions mapping $[0,\infty)$ to $\mathbb{C}^n$. Namely, a function $\boldsymbol{a}(t)$ belongs to $(\mathcal{L}^2)^n$ if $\boldsymbol{a}(t) \in \mathbb{C}^n$ for $0 \leq t < \infty$ and

$$\int_0^{\infty} \|\boldsymbol{a}(t)\|_2^2 \,\mathrm{d}t < \infty.$$

The square root of the left-hand side above gives the norm of $\boldsymbol{a}$ and is denoted by $\|\boldsymbol{a}\|_{\mathcal{L}^2}$.

Finally, $\mathcal{L}^2[0,\tau)^n$ is the Hilbert space of Lebesgue-square-integrable functions mapping $[0,\tau)$ to $\mathbb{C}^n$. Here, τ is some positive number. We present its inner product explicitly for the later use:

$$(\boldsymbol{f},\boldsymbol{g})_{\mathcal{L}^2[0,\tau)} := \int_0^{\tau} \boldsymbol{f}(t)^*\boldsymbol{g}(t)\,\mathrm{d}t \quad \text{for } \boldsymbol{f},\boldsymbol{g} \in \mathcal{L}^2[0,\tau)^n.$$

The norm in $\mathcal{L}^2[0,\tau)^n$ is defined as $\|\boldsymbol{f}\|_{\mathcal{L}^2[0,\tau)} := \sqrt{(\boldsymbol{f},\boldsymbol{f})_{\mathcal{L}^2[0,\tau)}}$.

Unless there is a fear of confusion, we write the above spaces simply as ℓ^2, $\mathcal{L}^2$, and $\mathcal{L}^2[0,\tau)$, respectively, dropping the dimension n.

2.2. Continuous-Time and Discrete-Time Signals

This section introduces mathematical notions about continuous-time and discrete-time signals. The formulation presented here is standard in the literature on the $\mathcal{H}^\infty$-control. Further details are found in [34, 26, 40, 99, 62] for example.

In order to introduce the notion of a continuous-time signal, we need the **continuous-time truncation operator** Π_T, where $T > 0$. Let n be a positive integer. For a function $\boldsymbol{a}(t)$ that maps $[0,\infty)$ to $\mathbb{C}^n$, a function $\Pi_T\boldsymbol{a}$ is defined by $(\Pi_T\boldsymbol{a})(t) := \boldsymbol{a}(t)$ for $0 \leq t < T$ and $(\Pi_T\boldsymbol{a})(t) := \mathbf{0}$ for $T \leq t$.

A function $\boldsymbol{a}(t)$ is called a **continuous-time signal** if it maps $[0,\infty)$ to $\mathbb{C}^n$ and its truncation $\Pi_T\boldsymbol{a}$ is Lebesgue square integrable for each $T > 0$, that is,

$$\int_0^{T} \|\boldsymbol{a}(t)\|_2^2 \mathrm{d}t < \infty.$$

To write a continuous-time signal, we use t for its independent variable, which is called the **time**, and express dependence on t by parentheses. Here, we allowed a continuous-time signal to take a complex value for simplicity of mathematical treatment. However, in a practical system, a continuous-time signal takes real values only.

Especially, a continuous-time signal that belongs to the space $\mathcal{L}^2$ is important in connection with the Laplace transform introduced next.

For a continuous-time signal $\boldsymbol{a}(t)$, its **Laplace transform** $\widehat{\boldsymbol{a}}(s)$ is defined as

$$\widehat{\boldsymbol{a}}(s) := \int_0^{\infty} \boldsymbol{a}(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

For s where this integral converges, $\widehat{\boldsymbol{a}}(s)$ becomes a function of s. This $\widehat{\boldsymbol{a}}(s)$ is sometimes called a **frequency-domain** representation of the signal $\boldsymbol{a}(t)$. In contrast, the original $\boldsymbol{a}(t)$ is a **time-domain** representation. In the following, we usually use s for the independent variables of frequency-domain functions.

Here, the Hardy space $\mathcal{H}^2$ becomes important. A $\mathbb{C}^n$-valued function $\widehat{\boldsymbol{a}}(s)$ belongs to $\mathcal{H}^2$ if $\widehat{\boldsymbol{a}}(s)$ is analytic in $\operatorname{Re} s > 0$ and fulfills

$$\sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\widehat{\boldsymbol{a}}(\sigma + \mathrm{i}\omega)\|_2^2 \mathrm{d}\omega < \infty.$$

The norm of $\widehat{\boldsymbol{a}}$ in this space, which is written as $\|\widehat{\boldsymbol{a}}\|_{\mathcal{H}^2}$, is defined as the square root of the left-hand side of the above formula. The following fact known as the Paley-Wiener theorem is significant for our use [50, p. 131] [32, Theorem 11.9] [82, Theorem 19.2].

Proposition 2.1. *A function belongs to $\mathcal{H}^2$ if and only if this function is the Laplace transform of a function in $\mathcal{L}^2$. Moreover, for a function $\boldsymbol{a}(t)$ that belongs to $\mathcal{L}^2$, there holds $\|\boldsymbol{a}\|_{\mathcal{L}^2} = \|\widehat{\boldsymbol{a}}\|_{\mathcal{H}^2}$.*

Therefore, the Laplace transform is an isometric isomorphism that maps $\mathcal{L}^2$ onto $\mathcal{H}^2$.

Next, we introduce notions on discrete-time signals. The discussion goes almost in parallel with the case of continuous-time signals. First, the **discrete-time truncation operator** π_K, where K is a positive integer, is required. For a one-sided $\mathbb{C}^n$-valued sequence $\boldsymbol{a}_\mathrm{d} = \{\boldsymbol{a}_\mathrm{d}[k]\}_{k=0}^{\infty}$, we define $\pi_K \boldsymbol{a}_\mathrm{d}$ to be a sequence $(\pi_K \boldsymbol{a}_\mathrm{d})[k] = \boldsymbol{a}_\mathrm{d}[k]$ for $0 \leq k < K$ and $(\pi_K \boldsymbol{a}_\mathrm{d})[k] = \boldsymbol{0}$ for $K \leq k$.

A one-sided $\mathbb{C}^n$-valued sequence $\boldsymbol{a}_\mathrm{d} = \{\boldsymbol{a}_\mathrm{d}[k]\}_{k=0}^{\infty}$ is called a **discrete-time signal** if $\pi_K \boldsymbol{a}_\mathrm{d}$ is square-summable for each $K = 1, 2, \ldots$, that is,

$$\sum_{k=0}^{K-1} \|\boldsymbol{a}_\mathrm{d}[k]\|_2^2 < \infty.$$

In order to differentiate a discrete-time signal from a continuous-time one, we put a suffix "d" and use square brackets to express dependence on the independent variable k. The variable k runs over all nonnegative integers $0, 1, \ldots$ and it is related to the time t by $t = k\tau$. Here, τ is a fixed positive number called a **sampling period**. Therefore, a discrete-time signal is a signal that has values only at discrete time points $t = 0, \tau, 2\tau, \ldots$.

Similarly to the case of a continuous-time signal, a discrete-time signal belonging to the space ℓ^2 is important when we consider its frequency-domain representation.

For a discrete-time signal $\boldsymbol{a}_\mathrm{d} = \{\boldsymbol{a}_\mathrm{d}[k]\}_{k=0}^{\infty}$, its ***z*-transform** $\breve{\boldsymbol{a}}(z)$ is a formal series

$$\breve{\boldsymbol{a}}_\mathrm{d}(z) := \sum_{k=0}^{\infty} \boldsymbol{a}_\mathrm{d}[k] z^{-k}.$$

In the case of discrete-time signals, this $\breve{\boldsymbol{a}}_\mathrm{d}(z)$ is considered to be a frequency-domain representation of a signal. For a clear contrast with the continuous-time signal case, we use "$\breve{\ }$" in place of "$\widehat{\ }$" to express the z-transform and use the variable z in place of s.

A discrete-time counterpart to $\mathcal{H}^2$ is the Hardy space $\mathfrak{H}^2$. A function $\breve{\boldsymbol{a}}_\mathrm{d}(z)$ belongs to $\mathfrak{H}^2$ if $\breve{\boldsymbol{a}}_\mathrm{d}(z)$ is analytic at any $z \in \mathbb{D}$ and satisfies

$$\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\breve{\boldsymbol{a}}_\mathrm{d}(r\mathrm{e}^{\mathrm{i}\omega})\|_2^2 \mathrm{d}\omega < \infty.$$

The square root of the left-hand side above is adopted as the norm of $\mathfrak{H}^2$. The next fact is a source of significance of this space $\mathfrak{H}^2$ [32, p. 8] [82, Theorem 17.12].

Proposition 2.2. *A function belongs to $\mathfrak{H}^2$ if and only if it is the z-transform of some sequence in ℓ^2. Moreover, for any sequence in ℓ^2, say $\boldsymbol{a}_\mathrm{d}$, there holds, $\|\boldsymbol{a}_\mathrm{d}\|_{\ell^2} = \|\breve{\boldsymbol{a}}_\mathrm{d}\|_{\mathfrak{H}^2}$.*

This proposition means that the z-transform is an isometric isomorphism mapping ℓ^2 onto $\mathfrak{H}^2$.

We close this section by giving properties of $\mathcal{H}^2$, which induce characteristics of kernel functions of samplers and holds in the next chapter. Here, we consider the scalar case.

Proposition 2.3. *Let $\widehat{a}(s)$ be any scalar function that belongs to $\mathcal{H}^2$ and let δ and τ be any positive numbers. Then, the following properties hold.*

(a) In the half plane $\operatorname{Re} s \geq \delta$, the function value $\widehat{a}(s)$ converges to zero uniformly as $|s|$ approaches infinity.

(b) Let B be any bounded closed set contained in the open half plane $\operatorname{Re} s > 0$. Then, the infinite series $\sum_{m=-\infty}^{\infty} |\widehat{a}(s + \mathrm{i}2\pi m/\tau)|^2$ converges uniformly for all $s \in B$.

For the proof of Property (a), see Corollary 2 of Theorem 11.3 in [32]. Property (b) is proven in Appendix A.

In Table 2.1, we summarize the notions introduced in this section with comparing the continuous-time signal case and the discrete-time signal case.

Table 2.1. Notions about continuous-time and discrete-time signals: (a) Their typical time-domain representations; (b) The Hilbert spaces in the time domain; (c) Their typical frequency-domain representations; (d) The Hardy spaces in the frequency domain.

Signals	(a)	(b)	(c)	(d)
continuous-time signal	$\boldsymbol{a}(t)$	$\mathcal{L}^2$	$\widehat{\boldsymbol{a}}(s)$	$\mathcal{H}^2$
discrete-time signal	$\boldsymbol{a}_\mathrm{d}[k]$	ℓ^2	$\breve{\boldsymbol{a}}_\mathrm{d}(z)$	$\mathfrak{H}^2$

2.3. System Component Operators

Having introduced notions on signals, we next present notions on operators that stand for system components. Since a sampled-data control system includes both continuous-time signals and discrete-time signals, this system is constructed of four types of system components:

(i) a component whose input and output are continuous-time signals;

(ii) a component whose input and output are discrete-time signals;

(iii) a component whose input is a continuous-time signal and whose output is a discrete-time signal;

(iv) a component whose input is a discrete-time signal and whose output is a continuous-time signal.

In order to express these components, we prepare the following four types of operators:

(i) a continuous-time operator;

(ii) a discrete-time operator;

(iii) a sampler-type operator;

(iv) a hold-type operator.

In this section, these types of operators are introduced in turn and, at the same time, spaces and norms related to them are defined. The formulation for continuous-time operators and discrete-time operators are standard again in the $\mathcal{H}^\infty$-control theory. For example, see [34, 26, 40, 99, 62].

2.3.1. Continuous-Time Operators

An operator mapping a continuous-time signal to a continuous-time signal is called a **continuous-time operator**. A continuous-time operator P is called **linear** if there hold $P(\alpha\boldsymbol{a}) = \alpha(P\boldsymbol{a})$ and $P(\boldsymbol{a}+\boldsymbol{b}) = P\boldsymbol{a}+P\boldsymbol{b}$ for any continuous-time signals $\boldsymbol{a}$ and $\boldsymbol{b}$ and any complex number α. A continuous-time operator P is said to be **causal** if $\Pi_T(P\boldsymbol{a}) = P(\Pi_T\boldsymbol{a})$ for any continuous-time signal $\boldsymbol{a}$ and any $T > 0$; in a word, if the output of P in the interval $[0, T)$ only depends on its input in $[0, T)$ for each $T > 0$. Furthermore, for a continuous-time signal $\boldsymbol{a}$, consider its translation $\boldsymbol{a}_T$, which is defined as $\boldsymbol{a}_T(t) := \boldsymbol{a}(t-T)$ for $t \geq T$ and $\boldsymbol{a}_T(t) := \boldsymbol{0}$ otherwise. With this notation, suppose that a continuous-time operator P satisfies $\boldsymbol{b}_T = P\boldsymbol{a}_T$ whenever $\boldsymbol{b} = P\boldsymbol{a}$ and $T > 0$. Then, this P is called **continuous-time time-invariant**. If the above equality holds for $T = \ell\tau$, where ℓ is any positive integer and τ is some fixed positive number, P is called **$\boldsymbol{\tau}$-periodic**.

Suppose that P maps $\mathcal{L}^2$, which is a space of special continuous-time signals, into $\mathcal{L}^2$. Then, the **$\mathcal{L}^2$-induced norm** of P can be defined as

$$\|P\| := \sup_{\substack{\boldsymbol{a}\in\mathcal{L}^2 \\ \|\boldsymbol{a}\|_{\mathcal{L}^2}\neq 0}} \frac{\|P\boldsymbol{a}\|_{\mathcal{L}^2}}{\|\boldsymbol{a}\|_{\mathcal{L}^2}}.$$

If P has a finite $\mathcal{L}^2$-induced norm, it is called **$\mathcal{L}^2$-bounded** or simply **bounded**. The $\mathcal{L}^2$-induced norm plays a central role in the $\mathcal{H}^\infty$-control theory because $\|P\|$ works as an index of the system performance when P represents a controlled system. Usually, a small value of $\|P\|$ means a good system performance. Therefore, we let the symbol $\|\cdot\|$, which has no subscript, imply the $\mathcal{L}^2$-induced norm unless it is specified in other way.

Next, we introduce the notion of a transfer function, which is a frequency-domain representation of a continuous-time operator.

Let us consider a continuous-time operator P whose operation is described as

$$(P\boldsymbol{a})(t) = D\boldsymbol{a}(t) + \int_0^t \underline{P}(t-r)\boldsymbol{a}(r)\mathrm{d}r, \quad t \geq 0, \tag{2.1}$$

with some matrix D and a function $\underline{P}(t)$. If

$$\widehat{P}(s) := D + \int_0^\infty \underline{P}(t)\mathrm{e}^{-st}\mathrm{d}t \tag{2.2}$$

is well-defined for some $s \in \mathbb{C}_{\mathrm{e}}$, this function $\widehat{P}(s)$ is called the **continuous-time transfer function** of P. From a property of the Laplace transform, we can derive $\widehat{\boldsymbol{b}} = \widehat{P}\widehat{\boldsymbol{a}}$ when $\boldsymbol{b} = P\boldsymbol{a}$. Moreover, the transfer function of an operator PQ, which means successive operation of Q and then P, is equal to the function product $\widehat{P}(s)\widehat{Q}(s)$. When $\boldsymbol{b}(t)$ is the derivative of $\boldsymbol{a}(t)$ with respect to t, there holds $\widehat{\boldsymbol{b}}(s) = s\widehat{\boldsymbol{a}}(s) - \boldsymbol{a}(+0)$. Because of this, it is considered that s itself corresponds to the derivation with respect to t.

Now, the set of transfer functions is introduced. The Hardy space $\mathcal{H}^\infty$ is the space of $n \times \ell$-matrix-valued functions that are analytic and bounded in $\mathrm{Re}\, s > 0$. We do not explicitly write the dimensions n and ℓ because they are usually clear from the context. The norm in the space $\mathcal{H}^\infty$ is defined by

$$\|\widehat{P}\|_{\mathcal{H}^\infty} := \sup_{\mathrm{Re}\, s>0} \overline{\sigma}\{\widehat{P}(s)\}.$$

Although an element of $\mathcal{H}^\infty$ is not always expressed as in (2.2), it is possible to associate an operator on $\mathcal{L}^2$ with each element of $\mathcal{H}^\infty$ as follows. Suppose that an $\mathcal{H}^\infty$-function $\widehat{P}(s)$ is provided. If $\boldsymbol{a}(t)$ belongs to $\mathcal{L}^2$, its Laplace transform $\widehat{\boldsymbol{a}}(s)$ belongs to $\mathcal{H}^2$ by Proposition 2.1, which implies that the function product $\widehat{P}(s)\widehat{\boldsymbol{a}}(s)$ is an element of $\mathcal{H}^2$. Again, by Proposition 2.1, there is a unique $\boldsymbol{b} \in \mathcal{L}^2$ such that $\widehat{\boldsymbol{b}}(s) = \widehat{P}(s)\widehat{\boldsymbol{a}}(s)$. Now, this correspondence between $\boldsymbol{a}$ and $\boldsymbol{b}$ induces an operator on $\mathcal{L}^2$. Also in this case, the $\mathcal{H}^\infty$-function $\widehat{P}(s)$ is called the **continuous-time transfer function** of this newly defined operator. This terminology is consistent with the original definition of this word because the operator defined from $\widehat{P}(s)$ is identical to the P in (2.1) especially when $\widehat{P}(s)$ can be represented as in (2.2). An operator defined from an $\mathcal{H}^\infty$-function as above is shown to be linear, causal, time-invariant, and bounded. Furthermore, from the isomorphism between $\mathcal{L}^2$ and $\mathcal{H}^2$ (Proposition 2.1), the next relationship is derived [26, pp. 22-23] [99, Theorem 4.4 and Remark 4.2].

Proposition 2.4. *Consider a function $\widehat{P}(s)$ in $\mathcal{H}^\infty$ and write as P the corresponding operator on $\mathcal{L}^2$. Then, there holds*

$$\|P\| = \|\widehat{P}\|_{\mathcal{H}^\infty}.$$

Next, some subsets of $\mathcal{H}^\infty$ are introduced.

The first subset is $\mathcal{RH}^\infty$. It is defined as the subset of $\mathcal{H}^\infty$ that consists of real rational functions only. The set $\mathcal{RH}^\infty$ is closely related to a state-space representation of a continuous-time operator.

In general, a **continuous-time state-space representation** of a continuous-time operator $P : \boldsymbol{a} \mapsto \boldsymbol{b}$ is

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{a}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{0}, \\ \boldsymbol{b}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{a}(t), \end{aligned}$$

where A, B, C, and D are real matrices and $\boldsymbol{x}(t)$ is a continuous-time signal. Sometimes the above representation is briefly written as (A, B, C, D). A continuous-time state-space representation is called **strictly proper** if its "D"-matrix is equal to the zero matrix. For one operator P, its state-space representation is not unique. A state-space representation of P is called **minimal** if the dimension of $\boldsymbol{x}(t)$ is the achievable minimum. In general, if a continuous-time operator P has a continuous-time state-space representation, it is linear, causal, time-invariant and real. Moreover, the transfer function of P is $\widehat{P}(s) = D + C(sI - A)^{-1}B$. Now, there holds the following property.

Proposition 2.5. *A continuous-time operator P has a continuous-time state-space representation if and only if it has a transfer function $\widehat{P}(s)$, which is real and rational.*

Proof. The "only if" part is easy from the above expression of a transfer function. In order to show the "if" part, construct a control or an observer canonical form [57, Section 6.1] for a provided real rational function. □

If a transfer function $\widehat{P}(s)$ has a strictly proper state-space representation, it is also called **strictly proper**. It is seen that $\widehat{P}(s)$ is strictly proper if and only if $\widehat{P}(s)$ is real and rational and satisfies $\widehat{P}(\infty) = O$. The next proposition shows a relationship between a state-space representation and $\mathcal{RH}^\infty$. It is proven by slight modification of the proof of the previous proposition.

Proposition 2.6. *A continuous-time operator P is bounded and has a continuous-time state-space representation, if and only if it has a continuous-time transfer function $\widehat{P}(s)$ belonging to $\mathcal{RH}^\infty$.*

Proof. [if] By the previous proposition, a continuous-time operator P has a continuous-time state-space representation. Moreover, if $\widehat{P} \in \mathcal{RH}^\infty$, P has to be bounded, as we have seen before Proposition 2.4.

[only if] By the previous proposition again, $\widehat{P}$ is a real rational function. Suppose it has a pole in $\mathrm{Re}\, s \geq 0$ or at $s = \infty$. Then, there exists a function $\widehat{\boldsymbol{a}} \in \mathcal{H}^2$ such that $\widehat{P}\widehat{\boldsymbol{a}}$ does not belong to $\mathcal{H}^2$. This contradicts with the assumption that P is bounded. Hence, $\widehat{P}$ cannot have a pole in $\mathrm{Re}\, s \geq 0$ or at $s = \infty$, which means $\widehat{P} \in \mathcal{RH}^\infty$. □

Next, consider a real function $Q(s)$ that is analytic in $\mathrm{Re}\, s > 0$ and is continuous in $\mathbb{C}_{+e}$. We write the set of such functions as $\mathcal{A}_{\mathrm{R}}$. By definition, $\mathcal{RH}^\infty \subsetneq \mathcal{A}_{\mathrm{R}} \subsetneq \mathcal{H}^\infty$. For any function $Q(s)$ in $\mathcal{A}_{\mathrm{R}}$, its $\mathcal{H}^\infty$-norm is attained on the imaginary axis, that is,

$$\|Q\|_{\mathcal{H}^\infty} = \sup_{\omega \in \mathbb{R}} \overline{\sigma}\{Q(\mathrm{i}\omega)\}.$$

It is proven as a consequence of the maximum modulus theorem.

This set $\mathcal{A}_{\mathrm{R}}$ is a variant of the disc algebra, which is the set of functions analytic in $|z| < 1$ and continuous in $|z| \leq 1$. In the work of Dullerud [29, 27], which derived a nonconservative robust stability condition for a sampled-data control system, the set $\mathcal{A}_{\mathrm{R}}$ played a key role. A property of $\mathcal{A}_{\mathrm{R}}$ which is important in this thesis is as follows.

Proposition 2.7. *A function $Q(s)$ defined in $\mathbb{C}_{+e}$ belongs to $\mathcal{A}_{\mathrm{R}}$, if and only if there exists a functional sequence $\{Q_j\}_{j=1}^\infty$, $Q_j \in \mathcal{RH}^\infty$, such that $\|Q - Q_j\|_{\mathcal{H}^\infty} \to 0$ as $j \to \infty$.*

It is proven by a slight modification of the proof of Proposition 20.4.3 of [23]. Another proof is presented below.

Proof. The "if" part is easily proven by definition. In order to prove the "only if" part, define the functions

$$Q'(w) := Q(s)\Big|_{s=(1-w)/(1+w)}, \qquad Q'_\rho(w) := Q'(\rho w),$$

where $0 < \rho < 1$. As ρ approaches unity, $Q'_\rho(w)$ converges to $Q'(w)$ uniformly on $|w| = 1$, which implies that so does in $|w| \leq 1$ due to the maximum modulus theorem. Expanding $Q'_\rho(w)$ to the Taylor series around the origin, we can see that each $Q'_\rho(w)$ can be approximated by real rational functions uniformly in $|w| \leq 1$ and, hence, so can be $Q'(w)$. Transforming them back to the s-domain, we see the claim holds. □

2.3.2. Discrete-Time Operators

We can discuss discrete-time operators analogously to continuous-time operators.

An operator that maps a discrete-time signal to a discrete-time signal is called a **discrete-time operator**. A discrete-time operator is expressed by a capital letter with a suffix "d," like P_{d}, in this thesis. Linearity of a discrete-time operator is defined similarly to the case of continuous-time operators. A discrete-time operator P_{d} is called **causal** if $\pi_K(P_{\mathrm{d}}\boldsymbol{a}_{\mathrm{d}}) = P_{\mathrm{d}}(\pi_K \boldsymbol{a}_{\mathrm{d}})$ for any discrete-time signal $\boldsymbol{a}_{\mathrm{d}}$ and any positive integer K. With K being a positive integer, let $\boldsymbol{a}_{\mathrm{d},K}$ be a discrete-time signal such that $\boldsymbol{a}_{\mathrm{d},K}[k] = \boldsymbol{a}_{\mathrm{d}}[k-K]$ for $k \geq K$ and $\boldsymbol{a}_{\mathrm{d},K}[k] = \boldsymbol{0}$ for $0 \leq k < K$. Then, a discrete-time operator P_{d} is said to be **discrete-time time-invariant** if $\boldsymbol{b}_{\mathrm{d}} = P_{\mathrm{d}}\boldsymbol{a}_{\mathrm{d}}$ implies $\boldsymbol{b}_{\mathrm{d},K} = P_{\mathrm{d}}\boldsymbol{a}_{\mathrm{d},K}$ for any $K = 1, 2, \ldots$.

Recall that ℓ^2 is the space of certain special discrete-time signals. Suppose that a discrete-time operator P_{d} maps ℓ^2 into ℓ^2. In this case, we can define the **ℓ^2-induced norm** of P_{d} as

$$\|P_{\mathrm{d}}\|_{\ell^2 \to \ell^2} := \sup_{\substack{\boldsymbol{a}_{\mathrm{d}} \in \ell^2 \\ \|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2} \neq 0}} \frac{\|P_{\mathrm{d}}\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}}{\|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}}.$$

If P_{d} has a finite induced norm, it is called **ℓ^2-bounded** or just **bounded**.

Next, we consider a frequency-domain representation of a discrete-time operator. While the Laplace transform was used in the continuous-time case, the z-transform is used this time. Let P_{d} be a linear causal time-invariant discrete-time operator. Then, there exists a matrix sequence $\{P_{\mathrm{d},k}\}_{k=0}^{\infty}$ such that there holds $\boldsymbol{b}_{\mathrm{d}} = P_{\mathrm{d}}\boldsymbol{a}_{\mathrm{d}}$ if and only if

$$\boldsymbol{b}_{\mathrm{d}}[k] = \sum_{\ell=0}^{k} P_{\mathrm{d},k-\ell}\boldsymbol{a}_{\mathrm{d}}[\ell] \quad \text{for } k = 0, 1, \ldots. \tag{2.3}$$

Here, consider a formal series with an indeterminate z

$$\check{P}_{\mathrm{d}}(z) := \sum_{k=0}^{\infty} P_{\mathrm{d},k} z^{-k}.$$

Then, we can see that (2.3) is equivalent to $\check{\boldsymbol{b}}_{\mathrm{d}} = \check{P}_{\mathrm{d}}\check{\boldsymbol{a}}_{\mathrm{d}}$, where $\check{\boldsymbol{a}}_{\mathrm{d}}$ and $\check{\boldsymbol{b}}_{\mathrm{d}}$ are the z-transforms of $\boldsymbol{a}_{\mathrm{d}}$ and $\boldsymbol{b}_{\mathrm{d}}$, respectively. When the series $\check{P}_{\mathrm{d}}(z)$ converges for some $z \in \mathbb{C}_{\mathrm{e}}$, this function $\check{P}_{\mathrm{d}}(z)$ is called the **discrete-time transfer function** of P_{d}. Although the same symbol "$\check{\ }$" is used to express the z-transform, distinction should be clear from the context. Furthermore, note that we use different symbols "$\check{\ }$" and "$\hat{\ }$" to distinguish between a discrete-time transfer function and a continuous-time transfer function. Similarly to the continuous-time operator case, a transfer function of an operator composition $P_{\mathrm{d}}Q_{\mathrm{d}}$, where both P_{d} and Q_{d} are discrete-time operators, is equal to $\check{P}_{\mathrm{d}}(z)\check{Q}_{\mathrm{d}}(z)$.

Finally, a Hardy space of transfer functions is introduced. Let $\mathfrak{H}^{\infty}$ be the Hardy space of $n \times \ell$-matrix-valued functions that are analytic and bounded in $\mathbb{D}$. The dimensions n and ℓ are suppressed in this symbol because they are often obvious from the context. The norm of $\mathfrak{H}^{\infty}$ is defined as

$$\|\check{P}_{\mathrm{d}}\|_{\mathfrak{H}^{\infty}} := \sup_{z \in \mathbb{D}} \bar{\sigma}\{\check{P}_{\mathrm{d}}(z)\}.$$

Especially when $\check{P}_{\mathrm{d}}(z)$ is continuous in $\mathbb{D} \cup \{|z| = 1\}$, the maximum modulus theorem implies

$$\|\check{P}_{\mathrm{d}}\|_{\mathfrak{H}^{\infty}} := \sup_{|z|=1} \bar{\sigma}\{\check{P}_{\mathrm{d}}(z)\}.$$

Corresponding to Proposition 2.4, there holds the next proposition. For its proof, Proposition 2.2 is used [20, Lemma 4.3.3].

Proposition 2.8. *Suppose that a discrete-time operator P_{d} has a discrete-time transfer function belonging to $\mathfrak{H}^{\infty}$. Then, there holds*

$$\|P_{\mathrm{d}}\|_{\ell^2 \to \ell^2} = \|\check{P}_{\mathrm{d}}\|_{\mathfrak{H}^{\infty}}.$$

Define $\mathfrak{R}\mathfrak{H}^{\infty}$ as the subset of $\mathfrak{H}^{\infty}$ that consists of real rational functions only. It has a close relationship to a state-space representation of a discrete-time operator.

In general, suppose that a discrete-time operator $P_{\mathrm{d}} : \boldsymbol{a}_{\mathrm{d}} \mapsto \boldsymbol{b}_{\mathrm{d}}$ can be represented as

$$\begin{aligned} \boldsymbol{x}_{\mathrm{d}}[k+1] &= A\boldsymbol{x}_{\mathrm{d}}[k] + B\boldsymbol{a}_{\mathrm{d}}[k], \qquad \boldsymbol{x}_{\mathrm{d}}[0] = \boldsymbol{0}, \\ \boldsymbol{b}_{\mathrm{d}}[k] &= C\boldsymbol{x}_{\mathrm{d}}[k] + D\boldsymbol{a}_{\mathrm{d}}[k] \end{aligned}$$

with real matrices A, B, C, and D and a discrete-time signal $\boldsymbol{x}_{\mathrm{d}}[k]$. Then, this is called a **discrete-time state-space representation** of P_{d}. The above state-space representation is sometimes denoted by (A, B, C, D). If the dimension of $\boldsymbol{x}_{\mathrm{d}}$ is the achievable minimum, the representation is called **minimal**. If the "D"-matrix is equal to zero, the representation is called **strictly proper**. It is easy to see $\check{P}_{\mathrm{d}}(z) = D + C(zI - A)^{-1}B$. Moreover, there holds the next property. Its proof is similar to the continuous-time case.

Proposition 2.9. *A discrete-time operator P_{d} has a discrete-time state-space representation if and only if it has a discrete-time transfer function $\check{P}_{\mathrm{d}}(z)$, which is real and rational.*

A discrete-time transfer function $\check{P}_{\mathrm{d}}(z)$ is called **strictly proper**, if it has a strictly proper state-space representation. Now the relationship between $\mathfrak{R}\mathfrak{H}^{\infty}$ and a state-space representation is presented.

Proposition 2.10. *A discrete-time operator P_{d} is bounded and has a discrete-time state-space representation, if and only if it has a discrete-time transfer function $\check{P}_{\mathrm{d}}(z)$ belonging to $\mathfrak{R}\mathfrak{H}^{\infty}$.*

2.3.3. Sampler-Type and Hold-Type Operators

An operator that maps a continuous-time signal to a discrete-time signal is called a **sampler-type operator**. When a sampled-type operator S maps $\mathcal{L}^2$ into ℓ^2, we can think of its induced norm

$$\|S\|_{\mathcal{L}^2\to\ell^2} := \sup_{\substack{\boldsymbol{a}\in\mathcal{L}^2 \\ \|\boldsymbol{a}\|_{\mathcal{L}^2}\neq 0}} \frac{\|S\boldsymbol{a}\|_{\ell^2}}{\|\boldsymbol{a}\|_{\mathcal{L}^2}}.$$

If $\|S\|_{\mathcal{L}^2\to\ell^2}$ is finite, we say S is **bounded**.

Similarly, an operator that maps a discrete-time signal to a continuous-time signal is called a **hold-type operator**. When a hold-type operator H maps ℓ^2 into $\mathcal{L}^2$, the induced norm of H is defined as

$$\|H\|_{\ell^2\to\mathcal{L}^2} := \sup_{\substack{\boldsymbol{a}_{\mathrm{d}}\in\ell^2 \\ \|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}\neq 0}} \frac{\|H\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}}{\|\boldsymbol{a}_{\mathrm{d}}\|_{\ell^2}}.$$

If this value is finite, H is said to be **bounded**.

At this moment, we cannot consider a transfer function for a sampler-type operator or a hold-type operator. This is one of the tasks of the next section.

Here, we summarize the notation introduced in this section.

Table 2.2. The four types of operators and related symbols: (a) Their induced norms; (b) Their transfer functions; (c) The Hardy spaces that their transfer functions belong to; (d) The real rational subsets of the Hardy spaces.

Operator types	(a)	(b)	(c)	(d)
continuous-time operator (c.-t. sig. $\mapsto$ c.-t. sig.)	$\|P\|$	$\widehat{P}(s)$	$\mathcal{H}^\infty$	$\mathcal{RH}^\infty$
sampler-type operator (c.-t. sig. $\mapsto$ d.-t. sig.)	$\|S\|_{\mathcal{L}^2\to\ell^2}$	—	—	—
hold-type operator (d.-t. sig. $\mapsto$ c.-t. sig.)	$\|H\|_{\ell^2\to\mathcal{L}^2}$	—	—	—
discrete-type operator (d.-t. sig. $\mapsto$ d.-t. sig.)	$\|P_{\mathrm{d}}\|_{\ell^2\to\ell^2}$	$\check{P}_{\mathrm{d}}(z)$	$\mathfrak{H}^\infty$	$\mathfrak{RH}^\infty$

2.4. Lifting of Signals and System Component Operators

In Section 2.3, we have seen four types of operators. A sampled-data control system is composed of these four types of operators; namely, a controlled plant is a continuous-time operator, a

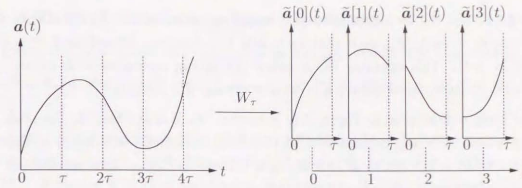


Figure 2.1. Lifting of a continuous-time signal $\boldsymbol{a}(t)$.

discrete-time controller is a discrete-time operator, a sampler is a sampler-type operator, and a hold is a hold-type operator. Because of this hybrid nature of a sampled-data control system, it has been difficult to analyze and synthesize this system considering its intersample behavior.

A lifting technique was introduced by Yamamoto [94, 95] and was developed by Bamieh and Pearson [9] as a remedy for this situation. Its basic idea is to chop a continuous-time signal at each sampling time and regard it as a discrete-time signal whose value is a function on $[0,\tau)$. See Figure 2.1, which is the same as Figure 1.6. In this way, we can treat all the four types of operators above as discrete-time operators by regarding their inputs and outputs as discrete-time signals. Then, analysis and synthesis of a sampled-data control system can be done easily with techniques for discrete-time systems.

In this section, lifting of a continuous-time signal is discussed first and, then, lifting of the four types of operators is considered in succession. These notions are essential for discussions in Chapters 3 and 4. On lifting of continuous-time signals and continuous-time operators, we basically follow [9]. More information can be obtained from recent books [20, 27].

2.4.1. Lifting of Continuous-Time Signals

We need some preparation first. Let τ be a positive number and let n be a positive integer. Recall that $\mathcal{L}^2[0,\tau)$ is the space of all functions that map $[0,\tau)$ to $\mathbb{C}^n$ and are Lebesgue square integrable. Here, consider a functional sequence $\tilde{\boldsymbol{a}} = \{\tilde{\boldsymbol{a}}[k]\}_{k=0}^{\infty}$ such that each $\tilde{\boldsymbol{a}}[k]$ is a function belonging to $\mathcal{L}^2[0,\tau)$ and there holds

$$\sum_{k=0}^{\infty} \|\tilde{\boldsymbol{a}}[k]\|^2_{\mathcal{L}^2[0,\tau)} < \infty.$$

The set of all such sequences is written as $\ell^2_{\mathcal{L}^2[0,\tau)}$. It is actually a Hilbert space. The square root of the left-hand side of the above inequality is adopted as the norm of $\tilde{\boldsymbol{a}}$ in this space and is written as $\|\tilde{\boldsymbol{a}}\|^2_{\ell^2_{\mathcal{L}^2[0,\tau)}}$.

Now, lifting of a continuous-time signal is defined together with the lifting operator W_τ.

Definition 2.11. Let W_τ be an operator that maps a continuous-time signal $\boldsymbol{a}(t)$ to a functional sequence $\tilde{\boldsymbol{a}} = \{\tilde{\boldsymbol{a}}[k]\}_{k=0}^{\infty}$ such that each $\tilde{\boldsymbol{a}}[k]$ is a function defined on $0 \le t < \tau$ by $\tilde{\boldsymbol{a}}[k](t) := \boldsymbol{a}(k\tau + t)$. This operator W_τ is called the **lifting operator** with respect to the sampling period τ. Here, the sequence $\tilde{\boldsymbol{a}} = W_\tau \boldsymbol{a}$ is **lifting** of a signal $\boldsymbol{a}$. □

The idea of lifting is illustrated in Figure 2.1. Note that W_τ is invertible. On the other hand, it is easy to see that each $\tilde{\boldsymbol{a}}[k]$ belongs to $\mathcal{L}^2[0,\tau)$ for any continuous-time signal $\boldsymbol{a}$. Especially, when and only when $\boldsymbol{a}$ belongs to $\mathcal{L}^2$, its lifting $\tilde{\boldsymbol{a}}$ belongs to $\ell^2_{\mathcal{L}^2[0,\tau)}$. It is not difficult to see that this correspondence is isometric, that is, $\|\boldsymbol{a}\|_{\mathcal{L}^2} = \|\tilde{\boldsymbol{a}}\|_{\ell^2_{\mathcal{L}^2[0,\tau)}}$.

Once a continuous-time signal is lifted, it can be regarded as a discrete-time signal. Therefore, it is possible to consider its z-transform just like a normal discrete-time signal.

Definition 2.12. Consider a continuous-time signal $\boldsymbol{a}$ and its lifting $\tilde{\boldsymbol{a}}$. Then, the **z-transform** of $\tilde{\boldsymbol{a}}$ is defined to be

$$\breve{\boldsymbol{a}}(z) := \sum_{k=0}^{\infty} \tilde{\boldsymbol{a}}[k] z^{-k}.$$

□

It is reasonable to use the symbol "$\breve{\ }$" because this $\breve{\boldsymbol{a}}(z)$ is defined in the same way as the z-transform of normal discrete-time signals. However, note that, for each complex number z with which this series converges, this $\breve{\boldsymbol{a}}(z)$ gives a function in $\mathcal{L}^2[0,\tau)$ instead of a finite-dimensional vector.

Just like the case of normal discrete-time signals, we consider a Hardy space of z-transforms. Let us say an $\mathcal{L}^2[0,\tau)$-valued function to be **analytic** at $z_0 \in \mathbb{C}_e$ if a scalar function $(\boldsymbol{f}, \breve{\boldsymbol{a}}(z))_{\mathcal{L}^2[0,\tau)}$ is analytic at $z = z_0$ for any $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)$. (Analyticity and related properties of operator-valued functions are discussed in [49, pp. 92–97] [85, pp. 183–189].) Here, suppose that $\breve{\boldsymbol{a}}(z)$ is an $\mathcal{L}^2[0,\tau)$-valued function that is analytic in $\mathbb{D}$ and satisfies

$$\sup_{r>1} \frac{1}{2\pi} \oint \|\breve{\boldsymbol{a}}(re^{i\omega})\|^2_{\mathcal{L}^2[0,\tau)} d\omega < \infty.$$

The set of all such functions is denoted by $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$. The norm in the space $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$ is defined to be the square root of the left-hand side of the above inequality and is expressed as $\|\breve{\boldsymbol{a}}\|_{\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}}$. Then, in fact, the z-transform is isometric isomorphism from $\ell^2_{\mathcal{L}^2[0,\tau)}$ to $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$. This result is found in [85, pp. 184–185] [9, Theorem 2 (i)] [27, Proposition 2.9].

In summary, we have the next.

Proposition 2.13. *The lifting operator W_τ is an isometric isomorphism that maps $\mathcal{L}^2$ onto $\ell^2_{\mathcal{L}^2[0,\tau)}$. The z-transform is an isometric isomorphism mapping $\ell^2_{\mathcal{L}^2[0,\tau)}$ onto $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$. Namely, for any $\boldsymbol{a} \in \mathcal{L}^2$, there holds*

$$\|\boldsymbol{a}\|_{\mathcal{L}^2} = \|\tilde{\boldsymbol{a}}\|_{\ell^2_{\mathcal{L}^2[0,\tau)}} = \|\breve{\boldsymbol{a}}\|_{\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}},$$

where $\tilde{\boldsymbol{a}} := W_\tau \boldsymbol{a}$ and $\breve{\boldsymbol{a}}(z)$ is the z-transform of $\tilde{\boldsymbol{a}}$.

2.4.2. Lifting of Continuous-Time Operators

Next, lifting of a continuous-time operator is introduced. Again, τ is a positive number.

Definition 2.14. For a continuous-time operator P, its **lifting** is an operator composition $W_\tau P W_\tau^{-1}$. □

Suppose that P is a continuous-time operator and satisfies $\boldsymbol{b} = P\boldsymbol{a}$ for continuous-time signals $\boldsymbol{a}$ and $\boldsymbol{b}$. Let n_a and n_b be the dimensions of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, we have $\tilde{\boldsymbol{b}} = (W_\tau P W_\tau^{-1})\tilde{\boldsymbol{a}}$ writing $\tilde{\boldsymbol{a}} := W_\tau \boldsymbol{a}$ and $\tilde{\boldsymbol{b}} := W_\tau \boldsymbol{b}$. Therefore, $W_\tau P W_\tau^{-1}$ maps an $\mathcal{L}^2[0,\tau)^{n_a}$-valued sequence to an $\mathcal{L}^2[0,\tau)^{n_b}$-valued sequence and it resembles a discrete-time operator in the sense that it maps a sequence to a sequence. A difference from a normal discrete-time operator is that both input signal and output signal take their values in functional spaces instead of finite-dimensional vector spaces.

Now, suppose that P is linear, causal, and τ-periodic. Then, $\tilde{\boldsymbol{b}} = (W_\tau P W_\tau^{-1})\tilde{\boldsymbol{a}}$ can be expressed as

$$\tilde{\boldsymbol{b}}[k] = \sum_{\ell=0}^{k} P_{k-\ell} \tilde{\boldsymbol{a}}[\ell]$$

using an appropriate sequence of operators $\{P_k\}_{k=0}^{\infty}$, where each P_k is an operator from $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathcal{L}^2[0,\tau)^{n_b}$.

Definition 2.15. For a τ-periodic continuous-time operator P, its **lifting-based transfer function** is the formal series

$$\breve{P}(z) := \sum_{k=0}^{\infty} P_k z^{-k},$$

where $\{P_k\}_{k=0}^{\infty}$ is an operator sequence defined as above. □

Note that this is completely parallel to the definition of transfer functions of normal discrete-time operators. Just like the case of discrete-time operators, $\breve{\boldsymbol{b}}(z) = \breve{P}(z)\breve{\boldsymbol{a}}(z)$ whenever $\boldsymbol{b} = P\boldsymbol{a}$.

From now on, we call a bounded linear operator from $\mathcal{L}^2[0,\tau)^n$ to $\mathcal{L}^2[0,\tau)^\ell$ a **large operator** in short, where n and ℓ are positive integers. This name comes from the fact that such an operator is represented as a matrix with infinite numbers of rows and columns when certain bases are taken in $\mathcal{L}^2[0,\tau)^n$ and $\mathcal{L}^2[0,\tau)^\ell$, respectively. Using this terminology, a lifting-based transfer function $\breve{P}(z)$ is a large-operator-valued function. The magnitude of a large operator is measured by the induced norm from $\mathcal{L}^2[0,\tau)^n$ to $\mathcal{L}^2[0,\tau)^\ell$. This norm $\|\cdot\|_{\mathcal{L}^2[0,\tau)\to\mathcal{L}^2[0,\tau)}$ is simply written as $\|\cdot\|_{\mathrm{L}}$ henceforth. Moreover, the Hardy space $\mathfrak{H}^\infty_{\mathrm{L}}$ is defined to be the space of large-operator-valued functions that are analytic and uniformly bounded in $\mathbb{D}$. Here, a large-operator-valued function $\breve{P}(z)$ is said to be **analytic** at $z = z_0$ if a scalar function $(\boldsymbol{g}, \breve{P}(z)\boldsymbol{f})_{\mathcal{L}^2[0,\tau)}$ is analytic at $z = z_0$ for any $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$ and any $\boldsymbol{g} \in \mathcal{L}^2[0,\tau)^\ell$. The norm of $\mathfrak{H}^\infty_{\mathrm{L}}$ is defined by

$$\|\breve{P}\|_{\mathfrak{H}^\infty_{\mathrm{L}}} := \sup_{z\in\mathbb{D}} \|\breve{P}(z)\|_{\mathrm{L}}.$$

Especially when $\check{P} \in \mathfrak{H}_{\mathrm{L}}^{\infty}$ is continuous in $\mathbb{D} \cup \{|z|=1\}$, the maximum modulus theorem implies

$$\|\check{P}\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}} := \sup_{|z|=1} \|\check{P}(z)\|_{\mathrm{L}}.$$

Just as we did in Section 2.3, we are interested in operators whose lifting-based transfer functions belong to $\mathfrak{H}_{\mathrm{L}}^{\infty}$. In fact, by noting the isometry between $\mathcal{L}^2$, $\ell^2_{\mathcal{L}^2[0,\tau)}$, and $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$, which was claimed in Proposition 2.13, we have the following result [85, p. 189] [9, Theorem 2 (ii)] [27, Proposition 2.10 (ii)] [97].

Proposition 2.16. *Suppose that a continuous-time operator P has its lifting-based transfer function in $\mathfrak{H}_{\mathrm{L}}^{\infty}$. Then, P has a bounded $\mathcal{L}^2$-induced norm and satisfies $\|P\| = \|\check{P}\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}$.*

Moreover, we can consider an analogue of $\mathcal{RH}^{\infty}$ and $\mathfrak{RH}^{\infty}$. Let $\mathfrak{RH}_{\mathrm{L}}^{\infty}$ be the subspace of $\mathfrak{H}_{\mathrm{L}}^{\infty}$ that consists of real rational functions only. This space is related to a sort of state-space representations. Suppose that we can express an operation of $P : \boldsymbol{a} \mapsto \boldsymbol{b}$ as

$$\boldsymbol{x}_{\mathrm{d}}[k+1] = \widetilde{A}\boldsymbol{x}_{\mathrm{d}}[k] + \widetilde{B}\widetilde{\boldsymbol{a}}[k], \quad \boldsymbol{x}_{\mathrm{d}}[0] = \boldsymbol{0}, \tag{2.4a}$$

$$\widetilde{\boldsymbol{b}}[k] = \widetilde{C}\boldsymbol{x}_{\mathrm{d}}[k] + \widetilde{D}\widetilde{\boldsymbol{a}}[k] \tag{2.4b}$$

using an n_x-dimensional-vector-valued sequence $\boldsymbol{x}_{\mathrm{d}}[k]$ and writing $\widetilde{\boldsymbol{a}} := W_\tau \boldsymbol{a}$ and $\widetilde{\boldsymbol{b}} := W_\tau \boldsymbol{b}$. Here, $\widetilde{A}$ is a real matrix and $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$ are real operators mapping $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathbb{C}^{n_x}$, $\mathbb{C}^{n_x}$ to $\mathcal{L}^2[0,\tau)^{n_b}$, and $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathcal{L}^2[0,\tau)^{n_b}$, respectively. Then, this is called a **lifting-based state-space representation** of P. Sometimes this representation is denoted by $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$. It is derived that $\check{P}(z) = \widetilde{D} + \widetilde{C}(zI - \widetilde{A})^{-1}\widetilde{B}$ in this case. Now, the next proposition easily follows.

Proposition 2.17. *A continuous-time operator P is bounded and has a lifting-based state-space representation, only if P has a lifting-based transfer function in $\mathfrak{RH}_{\mathrm{L}}^{\infty}$.*

Proof. If P has a lifting-based state-space representation, it is clear from the above expression of $\check{P}(z)$ that $\check{P}(z)$ is real and rational. Suppose $\check{P}(z)$ has a pole in $\mathbb{D}$ or on $|z| = 1$. Then, there exists $\check{\boldsymbol{a}} \in \mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$ such that $\check{P}(z)\check{\boldsymbol{a}}(z)$ does not belong to $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$. This means P is not bounded because of Proposition 2.13. Hence, $\check{P}(z)$ does not have a pole in $\mathbb{D}$ or on $|z| = 1$. From this, $\check{P} \in \mathfrak{RH}_{\mathrm{L}}^{\infty}$ follows. □

In contrast to the cases of $\mathcal{RH}^{\infty}$ and $\mathfrak{RH}^{\infty}$, a continuous-time operator having a transfer function in $\mathfrak{RH}_{\mathrm{L}}^{\infty}$ does not necessarily have a lifting-based state-space representation. This is because a control or an observer canonical form is not well-defined for $\mathfrak{RH}_{\mathrm{L}}^{\infty}$-functions. For more details, see [66].

When P has a *continuous-time* state-space representation, it also has a lifting-based state-space representation. Here, we present the explicit form of its lifting-based state-space representation for the later use [9].

Proposition 2.18. *Suppose that a continuous-time operator $P : \boldsymbol{a} \mapsto \boldsymbol{b}$ has a continuous-time state-space representation*

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{a}(t), \quad \boldsymbol{x}(0) = \boldsymbol{0},$$
$$\boldsymbol{b}(t) = C\boldsymbol{x}(t) + D\boldsymbol{a}(t),$$

where A, B, C, and D are finite-dimensional real matrices. Then, P has a lifting-based state-space representation as well and it is given by

$$\widetilde{A} := \mathrm{e}^{A\tau},$$
$$\widetilde{B}\widetilde{\boldsymbol{a}}[k] := \int_0^\tau \mathrm{e}^{A(\tau-t)} B\widetilde{\boldsymbol{a}}[k](t)\mathrm{d}t,$$
$$(\widetilde{C}\boldsymbol{x}(k\tau))(t) := C\mathrm{e}^{At}\boldsymbol{x}(k\tau),$$
$$(\widetilde{D}\widetilde{\boldsymbol{a}}[k])(t) := D\widetilde{\boldsymbol{a}}[k](t) + \int_0^t C\mathrm{e}^{A(t-r)} B\widetilde{\boldsymbol{a}}[k](r)\mathrm{d}r$$

in the notation of (2.4). Moreover, $\check{P}(z)$ has a pole at $z = z_0$ only if z_0 is expressed as $z_0 = \mathrm{e}^{s_0\tau}$ with s_0 being a pole of $\hat{P}(s)$.

2.4.3. Lifting of Other Types of Operators

Lifting of sampler-type and hold-type operators can be considered in a similar way.

Definition 2.19. A lifting of a sampler-type operator S is SW_τ^{-1}. A lifting of a hold-type operator H is $W_\tau H$. □

Suppose $\boldsymbol{b}_{\mathrm{d}} = S\boldsymbol{a}$, where $\boldsymbol{a}$ is a continuous-time signal and $\boldsymbol{b}_{\mathrm{d}}$ is a discrete-time signal. Writing $\widetilde{\boldsymbol{a}} := W_\tau \boldsymbol{a}$, we have $\boldsymbol{b}_{\mathrm{d}} = (SW_\tau^{-1})\widetilde{\boldsymbol{a}}$. In the sense that both $\widetilde{\boldsymbol{a}}$ and $\boldsymbol{b}_{\mathrm{d}}$ are sequences, SW_τ^{-1} resembles a discrete-time operator. Similarly, $W_\tau H$ can be regarded as a discrete-time operator because its input and output are sequences.

Suppose that a sampler-type operator S has the following representation, that is, there exists an operator sequence $\{S_k\}_{k=0}^{\infty}$ such that there holds

$$\boldsymbol{b}_{\mathrm{d}}[k] = \sum_{\ell=0}^{k} S_{k-\ell}\widetilde{\boldsymbol{a}}[\ell]$$

whenever $\boldsymbol{b}_{\mathrm{d}} = (SW_\tau^{-1})\widetilde{\boldsymbol{a}}$. Here, each S_k is an operator from $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathbb{C}^{n_b}$ with the dimensions of $\boldsymbol{a}$ and $\boldsymbol{b}_{\mathrm{d}}$ being n_a and n_b, respectively.

Definition 2.20. If a sampler-type operator S has the representation above, its **lifting-based transfer function** is the formal series

$$\check{S}(z) := \sum_{k=0}^{\infty} S_k z^{-k}.$$

□

Suppose that a hold-type operator H has an operator sequence $\{H_k\}_{k=0}^{\infty}$ such that $\breve{\boldsymbol{b}} = (W_\tau H)\boldsymbol{a}_{\rm d}$ implies

$$\breve{\boldsymbol{b}}[k] = \sum_{\ell=0}^{k} H_{k-\ell}\boldsymbol{a}_{\rm d}[\ell].$$

Here, each H_k is an operator from $\mathbb{C}^{n_a}$ to $\mathcal{L}^2[0,\tau)^{n_b}$ with the signal dimensions of $\boldsymbol{a}_{\rm d}$ and $\boldsymbol{b}$ being n_a and n_b, respectively.

Definition 2.21. If a hold-type operator H has a representation of the above form, its **lifting-based transfer function** is the formal series

$$\breve{H}(z) := \sum_{k=0}^{\infty} H_k z^{-k}.$$

□

In the previous subsection, we introduced the term "large operator," which means an operator from $\mathcal{L}^2[0,\tau)^n$ to $\mathcal{L}^2[0,\tau)^\ell$, where n and ℓ are positive integers. Correspondingly, let us call an operator from $\mathcal{L}^2[0,\tau)^n$ to $\mathbb{C}^\ell$ as a **flat operator** and an operator from $\mathbb{C}^n$ to $\mathcal{L}^2[0,\tau)^\ell$ as a **tall operator**. These names come from matrix representations of these operators. If we take bases in the spaces $\mathcal{L}^2[0,\tau)^n$ and $\mathcal{L}^2[0,\tau)^\ell$, respectively, we can represent a flat operator and a tall operator as infinite-dimensional matrices. A flat operator has a matrix representation with an infinite number of columns and a finite number of rows, that is, a "flat" representation. On the other hand, a matrix representation of a tall operator has a finite number of columns and an infinite number of rows, which means its shape is "tall." Using these terms, $\breve{S}(z)$ is a flat-operator-valued function while $\breve{H}(z)$ is a tall-operator-valued function.

Let us write the induced norm from $\mathcal{L}^2[0,\tau)^n$ to $\mathbb{C}^\ell$ as $\|\cdot\|_{\rm F}$, which is a norm of a flat operator. Likewise, the induced norm from $\mathbb{C}^n$ to $\mathcal{L}^2[0,\tau)^\ell$ is written as $\|\cdot\|_{\rm T}$, which works as a norm of a tall operator.

Furthermore, define the Hardy space $\mathfrak{H}_{\rm F}^\infty$ to be the set of flat-operator-valued functions analytic and uniformly bounded in $\mathbb{D}$. Here, a flat-operator-valued function $\breve{S}(z)$ is called **analytic** at $z = z_0$ if a scalar function $\boldsymbol{v}^*\breve{S}(z)\boldsymbol{f}$ is analytic at $z = z_0$ for any $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$ and any $\boldsymbol{v} \in \mathbb{C}^\ell$. In a similar way, a Hardy space $\mathfrak{H}_{\rm T}^\infty$ is defined. Namely, it is defined to be the set of tall-operator-valued functions analytic and uniformly bounded in $\mathbb{D}$. A tall-operator-valued function $\breve{H}(z)$ is called **analytic** at $z = z_0$, if a scalar function $(\boldsymbol{g}, \breve{H}(z)\boldsymbol{u})_{\mathcal{L}^2[0,\tau)}$ is analytic for any $\boldsymbol{u} \in \mathbb{C}^n$ and any $\boldsymbol{g} \in \mathcal{L}^2[0,\tau)^\ell$. For the spaces $\mathfrak{H}_{\rm F}^\infty$ and $\mathfrak{H}_{\rm T}^\infty$, their norms are defined as

$$\|\breve{S}\|_{\mathfrak{H}_{\rm F}^\infty} = \sup_{z\in\mathbb{D}} \|\breve{S}(z)\|_{\rm F},$$
$$\|\breve{H}\|_{\mathfrak{H}_{\rm T}^\infty} = \sup_{z\in\mathbb{D}} \|\breve{H}(z)\|_{\rm T}.$$

Note that all of these definitions are analogous to those of large-operator-valued functions discussed in the previous subsection.

Now the next proposition is derived from the isometry between $\mathcal{L}^2$, $\ell^2_{\mathcal{L}^2[0,\tau)}$, and $\mathfrak{H}^2_{\mathcal{L}^2[0,\tau)}$, which is established in Proposition 2.13.

Proposition 2.22. *Suppose that a sampler-type operator S has a lifting-based transfer function that belongs to $\mathfrak{H}_{\rm F}^\infty$. Then, S has a finite induced norm and satisfies $\|S\|_{\mathcal{L}^2\to\ell^2} = \|\breve{S}\|_{\mathfrak{H}_{\rm F}^\infty}$. Here, $\|\cdot\|_{\mathcal{L}^2\to\ell^2}$ stands for the induced norm from $\mathcal{L}^2$ to ℓ^2.*

Similarly, suppose that a hold-type operator H has a lifting-based transfer function belonging to $\mathfrak{H}_{\rm T}^\infty$. Then, H has a finite induced norm and there holds $\|H\|_{\ell^2\to\mathcal{L}^2} = \|\breve{H}\|_{\mathfrak{H}_{\rm T}^\infty}$. Here, $\|\cdot\|_{\ell^2\to\mathcal{L}^2}$ denotes the induced norm from ℓ^2 to $\mathcal{L}^2$.

Corresponding to $\mathfrak{RH}_{\rm L}^\infty$, the spaces $\mathfrak{RH}_{\rm F}^\infty$ and $\mathfrak{RH}_{\rm T}^\infty$ are defined. The space $\mathfrak{RH}_{\rm F}^\infty$ is defined as the subspace of $\mathfrak{H}_{\rm F}^\infty$ that consists of real rational functions only. The space $\mathfrak{RH}_{\rm T}^\infty$ is defined to be the subspace of $\mathfrak{H}_{\rm T}^\infty$ that is composed of real rational functions only. As is expected, the spaces $\mathfrak{RH}_{\rm F}^\infty$ and $\mathfrak{RH}_{\rm T}^\infty$ have relationships to state-space representations.

First, let us define lifting-based state-space representations of sampler-type and hold-type operators. Suppose that a sampler-type operator $S : \boldsymbol{a} \mapsto \boldsymbol{b}_{\rm d}$ has a representation:

$$\boldsymbol{x}_{\rm d}[k+1] = \widetilde{A}\boldsymbol{x}_{\rm d}[k] + \widetilde{B}\breve{\boldsymbol{a}}[k], \quad \boldsymbol{x}_{\rm d}[0] = \boldsymbol{0}, \tag{2.5a}$$
$$\boldsymbol{b}_{\rm d}[k] = \widetilde{C}\boldsymbol{x}_{\rm d}[k] + \widetilde{D}\breve{\boldsymbol{a}}[k], \tag{2.5b}$$

where $\widetilde{A}$ and $\widetilde{C}$ are real matrices and $\widetilde{B}$ and $\widetilde{D}$ are real flat operators. This is a **lifting-based state-space representation** of a sampler-type operator S. On the other hand, suppose that an operation of a hold-type operator $H : \boldsymbol{a}_{\rm d} \mapsto \boldsymbol{b}$ is represented as (2.5) with $\widetilde{A}$ and $\widetilde{B}$ being real matrices and $\widetilde{C}$ and $\widetilde{D}$ being real tall operators this time. Then, this is a **lifting-based state-space representation** of a hold-type operator H. Both representations are sometimes written as $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$.

Now, we have the next proposition. Note that the "if" part holds, too, this time unlike the case of $\mathfrak{RH}_{\rm L}^\infty$.

Proposition 2.23. *A sampler-type operator S is bounded and has a lifting-based state-space representation, if and only if it has a lifting-based transfer function belonging to $\mathfrak{RH}_{\rm F}^\infty$.*

Likewise, a hold-type operator H is bounded and has a lifting-based state-space representation, if and only if it has a lifting-based transfer function belonging to $\mathfrak{RH}_{\rm T}^\infty$.

Proof. The "only if" part can be shown similarly to Proposition 2.17. Let us consider the "if" part. Just as in [57, Section 6.1], we can construct the observer canonical form in the sampler-type operator case and the control canonical form in the hold-type operator case. Thus, S or H has a lifting-based state-space representation. Moreover, by Proposition 2.22, S or H is bounded. □

So far, we have considered lifting of continuous-time operators, sampler-type operators, and hold-type operators. There is no need to introduce lifting of discrete-time operators because their inputs and outputs are already discrete-time signals. However, it is also possible to consider that lifting of a discrete-time operator is this discrete-time operator itself. In this sense, by "a lifting-based transfer function of a discrete-time operator" we mean its usual discrete-time transfer function.

Now, all the four types of operators can be considered as discrete-time ones by a lifting technique and can be treated in the same framework. Especially, we do not have to be so nervous about distinguishing these four types of lifting-based transfer functions. This is a consequence of the next proposition.

Proposition 2.24. *Suppose that an operator P is either continuous-time, sampler-type, hold-type, or discrete-time and has a lifting-based transfer function. Suppose similarly regarding an operator Q and, moreover, assume that the operator composition PQ is well-defined. Then, a lifting-based transfer function of PQ exists and is equal to $\check{P}(z)\check{Q}(z)$.*

In order to prove this, just formally follow the proof of $(PQ)\check{}(z) = \check{P}(z)\check{Q}(z)$ for discrete-time operators P and Q.

In spite of the above result, we still continue to use symbols like $\mathfrak{H}_{\mathrm{L}}^\infty$, $\mathfrak{H}_{\mathrm{F}}^\infty$, $\mathfrak{H}_{\mathrm{T}}^\infty$, $\|\cdot\|_{\mathrm{L}}$, $\|\cdot\|_{\mathrm{F}}$, and $\|\cdot\|_{\mathrm{T}}$ making clear distinction between them. This is only to help readers' understanding when such detailed distinction is required.

The lifting-based notions introduced in this section are summarized in the table.

Table 2.3. Lifting-based notions concerning the four types of operators: (a) Typical representations of their lifting-based transfer functions; (b) Values of the lifting-based transfer functions; (c) The Hardy spaces that the lifting-based transfer functions belong to; (d) The real rational subspaces of the Hardy spaces.

Operator types	(a)	(b)	(c)	(d)
continuous-time operator (c.-t. sig. $\mapsto$ c.-t. sig.)	$\check{P}(z)$	large operator $(\mathcal{L}^2[0,\tau)^n \to \mathcal{L}^2[0,\tau)^\ell,\ \|\cdot\|_{\mathrm{L}})$	$\mathfrak{H}_{\mathrm{L}}^\infty$	$\mathfrak{RH}_{\mathrm{L}}^\infty$
sampler-type operator (c.-t. sig. $\mapsto$ d.-t. sig.)	$\check{S}(z)$	flat operator $(\mathcal{L}^2[0,\tau)^n \to \mathbb{C}^\ell,\ \|\cdot\|_{\mathrm{F}})$	$\mathfrak{H}_{\mathrm{F}}^\infty$	$\mathfrak{RH}_{\mathrm{F}}^\infty$
hold-type operator (d.-t. sig. $\mapsto$ c.-t. sig.)	$\check{H}(z)$	tall operator $(\mathbb{C}^n \to \mathcal{L}^2[0,\tau)^\ell,\ \|\cdot\|_{\mathrm{T}})$	$\mathfrak{H}_{\mathrm{T}}^\infty$	$\mathfrak{RH}_{\mathrm{T}}^\infty$
discrete-type operator (d.-t. sig. $\mapsto$ d.-t. sig.)	$\check{P}_{\mathrm{d}}(z)$	matrix $(\mathbb{C}^n \to \mathbb{C}^\ell,\ \overline{\sigma}\{\cdot\})$	$\mathfrak{H}^\infty$	$\mathfrak{RH}^\infty$

2.4.4. Matrix Representations of Operators

In this subsection, with the help of two families of operators, we consider matrix representations of large operators, flat operators, and tall operators. It will be seen in the succeeding chapters that this notion is useful to investigate these operators.

Definition 2.25. Let n be some positive integer. For each complex number $s \in \mathbb{C}$ and for each integer m, we define a tall operator $\grave{E}_m^s : \boldsymbol{v} \in \mathbb{C}^n \mapsto \boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$ by

$$\boldsymbol{f}(t) := \frac{1}{\sqrt{\tau}} \mathrm{e}^{(s+\mathrm{i}2\pi m/\tau)t}\boldsymbol{v} \qquad \text{for } 0 \le t < \tau.$$

On the other hand, a flat operator $\acute{E}_m^s$: $\boldsymbol{g} \in \mathcal{L}^2[0,\tau)^n \mapsto \boldsymbol{u} \in \mathbb{C}^n$ is defined as

$$\boldsymbol{u} := \int_0^\tau \frac{1}{\sqrt{\tau}} \mathrm{e}^{-(s+\mathrm{i}2\pi m/\tau)t}\boldsymbol{g}(t)\,\mathrm{d}t.$$

□

The accent marks of $\grave{E}_m^s$ and $\acute{E}_m^s$ symbolize their operations. Namely, $\grave{E}_m^s$ maps a finite-dimensional space to an infinite-dimensional one. Its associated accent mark expresses "from a small thing to a large thing." Conversely, the accent mark of $\acute{E}_m^s$ represents that this operator makes a large thing small. (The idea of this notation is adopted from [66].)

These operators have the following properties.

Proposition 2.26. *For any $s \in \mathbb{C}$ and any integers m and ℓ, we have $\acute{E}_m^s \grave{E}_\ell^s = \delta_{m,\ell} I$, where $\delta_{m,\ell}$ is Kronecker's delta. On the other hand, a series of large operators $\sum_{m=-\infty}^{\infty} \grave{E}_m^s \acute{E}_m^s$ strongly converges to the identity, that is, for any $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$,*

$$\lim_{M\to\infty} \left\| \sum_{m=-M}^{M} \grave{E}_m^s \acute{E}_m^s \boldsymbol{f} - \boldsymbol{f} \right\|_{\mathcal{L}^2[0,\tau)} = 0.$$

For any real number ω and any integer m, $(\grave{E}_m^{\mathrm{i}\omega})^* = \acute{E}_m^{\mathrm{i}\omega}$ and $\|\grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}} = \|\acute{E}_m^{\mathrm{i}\omega}\|_{\mathrm{F}} = 1$.

Proof. The first assertion is easily shown. The second assertion is proven by the fact that for any function in $\mathcal{L}^2[0,\tau)$ its Fourier series converges to the original function in the norm of $\mathcal{L}^2[0,\tau)$ [82, pp. 91–92]. The equality $(\grave{E}_m^{\mathrm{i}\omega})^* = \acute{E}_m^{\mathrm{i}\omega}$ is easy to prove. Finally, $\|\grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}} = 1$ because

$$\|\grave{E}_m^{\mathrm{i}\omega}\boldsymbol{v}\|_{\mathcal{L}^2[0,\tau)}^2 = (\grave{E}_m^{\mathrm{i}\omega}\boldsymbol{v}, \grave{E}_m^{\mathrm{i}\omega}\boldsymbol{v})_{\mathcal{L}^2[0,\tau)} = (\boldsymbol{v}, \acute{E}_m^{\mathrm{i}\omega}\grave{E}_m^{\mathrm{i}\omega}\boldsymbol{v})_{\mathcal{L}^2[0,\tau)} = \|\boldsymbol{v}\|_2^2$$

for any $\boldsymbol{v} \in \mathbb{C}^n$. Besides, $\|\grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}} = \|\acute{E}_m^{\mathrm{i}\omega}\|_{\mathrm{F}}$ follows from a property of adjoint operators. □

By the third assertion of Proposition 2.26, we can write $\boldsymbol{f} = \sum_{m=-\infty}^{\infty} \grave{E}_m^s \acute{E}_m^s \boldsymbol{f}$ for any $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$. This is equivalent to expanding a function $\boldsymbol{f} \in \mathcal{L}^2[0,\tau)^n$ with respect to a functional basis $\{(1/\sqrt{\tau})\mathrm{e}^{(s+\mathrm{i}2\pi m/\tau)t}\}_{m=-\infty}^{\infty}$, which is not orthogonal in general. Noting that each $\acute{E}_m^s \boldsymbol{f}$ is a finite-dimensional vector, let the expansion $\boldsymbol{f} = \sum_{m=-\infty}^{\infty} \grave{E}_m^s(\acute{E}_m^s \boldsymbol{f})$ correspond to an infinite-dimensional vector

$$\begin{bmatrix} \vdots \\ \acute{E}_{-1}^s \boldsymbol{f} \\ \acute{E}_0^s \boldsymbol{f} \\ \acute{E}_1^s \boldsymbol{f} \\ \vdots \end{bmatrix}.$$

Using this correspondence, we can represent a large operator L, a flat operator F, and a tall operator T as infinite-dimensional matrices:

$$L \sim \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \dots & \acute{E}_{-1}^s L \grave{E}_{-1}^s & \acute{E}_{-1}^s L \grave{E}_0^s & \acute{E}_{-1}^s L \grave{E}_1^s & \dots \\ \dots & \acute{E}_0^s L \grave{E}_{-1}^s & \acute{E}_0^s L \grave{E}_0^s & \acute{E}_0^s L \grave{E}_1^s & \dots \\ \dots & \acute{E}_1^s L \grave{E}_{-1}^s & \acute{E}_1^s L \grave{E}_0^s & \acute{E}_1^s L \grave{E}_1^s & \dots \\ & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2.6a}$$

$$F \sim \begin{bmatrix} \dots & F\grave{E}_{-1}^s & F\grave{E}_0^s & F\grave{E}_1^s & \dots \end{bmatrix}, \tag{2.6b}$$

$$T \sim \begin{bmatrix} \vdots \\ \acute{E}_{-1}^s T \\ \acute{E}_0^s T \\ \acute{E}_1^s T \\ \vdots \end{bmatrix}, \tag{2.6c}$$

respectively. The above matrices are referred to as **matrix representations** of the operators L, F, and T, respectively. Because of the shapes of these matrices, these operators are called "large," "flat," and "tall," respectively.

Let us consider the case that $s = \mathrm{i}\omega$ for a real ω. Note the basis $\{(1/\sqrt{\tau})\mathrm{e}^{\mathrm{i}(\omega+2\pi m/\tau)t}\}_{m=-\infty}^{\infty}$ is orthonormal in this case. This implies that correspondence between $\boldsymbol{f} = \sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \boldsymbol{f}$ and

$$\begin{bmatrix} \vdots \\ \acute{E}_{-1}^{\mathrm{i}\omega} \boldsymbol{f} \\ \acute{E}_0^{\mathrm{i}\omega} \boldsymbol{f} \\ \acute{E}_1^{\mathrm{i}\omega} \boldsymbol{f} \\ \vdots \end{bmatrix}$$

is isometric, where the norm of the above vector is defined as $(\sum_{m=-\infty}^{\infty} \|\acute{E}_m^{\mathrm{i}\omega} \boldsymbol{f}\|_2^2)^{1/2}$. Now, the next proposition is derived from this isometry.

Proposition 2.27. *For a large operator L, a flat operator F, and a tall operator T, consider their matrix representations (2.6) with $s = \mathrm{i}\omega$, where ω is a real number. Then, any finite-dimensional submatrices of these matrix representations have maximum singular values that are smaller than or equal to $\|L\|_{\mathrm{L}}$, $\|F\|_{\mathrm{F}}$, and $\|T\|_{\mathrm{T}}$, respectively. Especially for L, there hold $\|\acute{E}_m^{\mathrm{i}\omega} L\|_{\mathrm{F}} \le \|L\|_{\mathrm{L}}$ and $\|L\grave{E}_\ell^{\mathrm{i}\omega}\|_{\mathrm{T}} \le \|L\|_{\mathrm{L}}$.*

Furthermore, there hold

$$\|L\|_{\mathrm{L}}^2 \le \sum_{m=-\infty}^{\infty} \|\acute{E}_m^{\mathrm{i}\omega} L\|_{\mathrm{F}}^2 \le \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \overline{\sigma}(\acute{E}_m^{\mathrm{i}\omega} L \grave{E}_\ell^{\mathrm{i}\omega})^2,$$

$$\|L\|_{\mathrm{L}}^2 \le \sum_{\ell=-\infty}^{\infty} \|L\grave{E}_\ell^{\mathrm{i}\omega}\|_{\mathrm{T}}^2 \le \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \overline{\sigma}(\acute{E}_m^{\mathrm{i}\omega} L \grave{E}_\ell^{\mathrm{i}\omega})^2,$$

$$\|F\|_{\mathrm{F}}^2 \le \sum_{\ell=-\infty}^{\infty} \overline{\sigma}(F\grave{E}_\ell^{\mathrm{i}\omega})^2, \qquad \|T\|_{\mathrm{T}}^2 \le \sum_{m=-\infty}^{\infty} \overline{\sigma}(\acute{E}_m^{\mathrm{i}\omega} T)^2.$$

Since values of lifting-based transfer functions are either large operators, flat operators, or tall operators, matrix representations of lifting-based transfer functions can be considered. Here, we have the next proposition, which is important in Chapters 3 and 4. For its proof, see Appendix B.

Proposition 2.28. *Let P be a continuous-time operator having a continuous-time state-space representation:*

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{a}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{0},$$
$$\boldsymbol{b}(t) = C\boldsymbol{x}(t) + D\boldsymbol{a}(t),$$

with $\boldsymbol{a}$ and $\boldsymbol{b}$ being an input and an output of P, respectively. Moreover, let s be a complex number such that none of $s+\mathrm{i}2\pi m/\tau$, $m = 0, \pm 1, \dots$, is a pole of $\widehat{P}(s)$, which is the continuous-time transfer function of P. Then, for the lifting-based transfer function of P, i.e., $\check{P}(z)$, and any integers m and ℓ, there holds

$$\acute{E}_m^s \check{P}(\mathrm{e}^{s\tau}) \grave{E}_\ell^s = \widehat{P}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big)\, \delta_{m,\ell}.$$

Especially when s is equal to $\mathrm{i}\omega$ with a real number ω, there holds

$$\|\check{P}(\mathrm{e}^{\mathrm{i}\omega\tau})\|_{\mathrm{L}} = \sup_{m=0,\pm 1,\dots} \overline{\sigma}\Big\{\widehat{P}\Big(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\Big)\Big\}.$$

In a matrix form, the consequence of this proposition is expressed as

$$\check{P}(\mathrm{e}^{s\tau}) \sim \begin{bmatrix} \ddots & & & & 0 \\ & \widehat{P}\big(s - \frac{\mathrm{i}2\pi}{\tau}\big) & & & \\ & & \widehat{P}(s) & & \\ & & & \widehat{P}\big(s + \frac{\mathrm{i}2\pi}{\tau}\big) & \\ 0 & & & & \ddots \end{bmatrix}.$$

Note that only its diagonal blocks are nonzero. This structure originates from continuous-time time-invariance of P. If P is τ-periodic but not necessarily time-invariant, the off-diagonal blocks are nonzero in general.

As is seen so far, the form of $s + \mathrm{i}2\pi m/\tau$ often appears in relation with a matrix representation. Especially when s is represented as $\mathrm{i}\omega$ and ω is interpreted as a frequency, the frequencies $\omega + 2\pi m/\tau$, $m = \pm 1, \pm 2, \ldots$, are associated with it. They are called the **side-band frequencies** of ω.

Matrix representations of lifting-based transfer functions were heavily used by Araki and his co-workers under the name of "FR-operators" [4, 3, 42, 44, 5]. (Also see [96].) Our notion is more general than theirs in the point that we allow s to be a general complex number while they restricted s to be a pure imaginary number. This difference becomes essential in the proof of Theorem 3.34, which is important to derive the results of Chapter 4.

2.5. Continuous-Time Control Systems

Throughout this thesis, continuous-time control systems play an important role. In Chapter 3, a framework for sampled-data control systems are constructed so that it corresponds to a framework for continuous-time control systems. In Chapter 4, the best achievable performance of sampled-data control systems are investigated in comparison with that of continuous-time control systems. In order to prepare for the later use, we introduce several notions about continuous-time control systems and derive their relevant properties.

Figure 2.2 shows a **continuous-time control system** considered in this thesis. It consists of two continuous-time operators G and K, which exchange continuous-time signals shown by arrows. Any signal can be multi-dimensional. This configuration is quite standard in the $\mathcal{H}^\infty$-control literature [34, 26, 40, 99, 62].

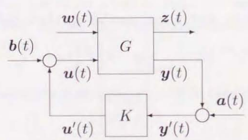


Figure 2.2. A continuous-time control system.

A continuous-time operator G is called a **generalized plant** and stands for an object

desired to be controlled. We assume that G has a continuous-time state-space representation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{0}, \tag{2.7a}$$

$$\boldsymbol{z}(t) = C_1\boldsymbol{x}(t) + D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t), \tag{2.7b}$$

$$\boldsymbol{y}(t) = C_2\boldsymbol{x}(t) + D_{21}\boldsymbol{w}(t) + D_{22}\boldsymbol{u}(t). \tag{2.7c}$$

Here, it is assumed that (A, B_2) is stabilizable and (C_2, A) is detectable, which means that there exist real matrices F and L such that $A + B_2F$ and $A + LC_2$ possess their eigenvalues only in $\operatorname{Re} s < 0$. The signal $\boldsymbol{w}(t)$ is an exogenous input consisting of command signals, disturbances, and sensor noises, for example; $\boldsymbol{u}(t)$ is a control input; $\boldsymbol{z}(t)$ is an output desired to be attenuated by control like tracking errors; $\boldsymbol{y}(t)$ is a measured output; $\boldsymbol{x}(t)$ is an internal state of G. The dimensions of $\boldsymbol{w}(t)$, $\boldsymbol{u}(t)$, $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$, and $\boldsymbol{x}(t)$ are denoted by n_w, n_u, n_z, n_y, and n_x, respectively. As usual, G is divided into four operators G_{11}, G_{12}, G_{21}, and G_{22} so that

$$\boldsymbol{z} = G_{11}\boldsymbol{w} + G_{12}\boldsymbol{u},$$

$$\boldsymbol{y} = G_{21}\boldsymbol{w} + G_{22}\boldsymbol{u}.$$

Note that no assumption is made on strict properness of G.

A continuous-time operator K is called a **continuous-time controller** and its purpose is to control G so that the system exhibits a desirable behavior. A continuous-time controller K is chosen from the set $\mathcal{K}$. Here, $\mathcal{K}$ consists of operators that have continuous-time state-space representations and have n_y-dimensional inputs and n_u-dimensional outputs.

With respect to a continuous-time control system in Figure 2.2, input-output stability is the most important property that is desired to be possessed. In order to define this notion, we need the fictitious inputs $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ together with the continuous-time transfer functions of G_{22} and K, i.e., $\widehat{G}_{22}(s)$ and $\widehat{K}(s)$.

Definition 2.29. A continuous-time control system in Figure 2.2 is called **input-output stable** or just **stable** if a function $\det\{I - \widehat{G}_{22}(s)\widehat{K}(s)\}$ takes a nonzero value at least at one $s \in \mathbb{C}_e$, and all the nine operators from $\boldsymbol{w}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{a}(t)$ to $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{u}'(t)$ have bounded $\mathcal{L}^2$-induced norms. □

In addition to input-output stability, a continuous-time system is desired to have a good performance. In the framework for the $\mathcal{H}^\infty$-control theory, a system performance is measured by the $\mathcal{L}^2$-induced norm of the operator from $\boldsymbol{w}(t)$ to $\boldsymbol{z}(t)$ and its small value is considered to show a good performance. Now, let us define the **lower fractional transform** by

$$\mathcal{F}(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

Then, the **best achievable performance of continuous-time control systems** (or the **best continuous-time control performance** in short) with respect to a provided G is expressed as

$$\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|. \tag{2.8}$$

Here, the symbol $\|\cdot\|$ stands for the $\mathcal{L}^2$-induced norm though we adopt the convention that the value of this norm is equal to infinity if the evaluated system is not input-output stable.

Example 2.30. Recall a continuous-time control system presented in Figure 1.7 (b). Here, P is a controlled plant and W is a weight to show how an amount of plant uncertainty depends on frequency. In Example 1.3, it is desired to reduce the $\mathcal{L}^2$-induced norm of the operator from $w(t)$ to $z(t)$ for the sake of robust stability. Now, let us represent this system in a standard configuration of a continuous-time control system. Assume that both P and PW are continuous-time operators having continuous-time state-space representations and put

$$G := \begin{bmatrix} O & I \\ PW & P \end{bmatrix}.$$

Then, the operator $\mathcal{F}(G, K)$ is equal to the operator from $w(t)$ to $z(t)$. This means that $\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$ implies the best achievable performance in the sense of robust stability. □

Remark 2.31. We have assumed stabilizability of (A, B_2) and detectability of (C_2, A) in the state-space representation of G, i.e., (2.7). As is seen below, this assumption is mild enough and also simplifies the treatment of our system.

Without loss of generality, we can assume stabilizability of $(A, [B_1 \;\; B_2])$ and detectability of $([C_1^{\mathrm{T}} \;\; C_2^{\mathrm{T}}]^{\mathrm{T}}, A)$ in the state-space representation of G, i.e., (2.7). Indeed, it suffices to consider a minimal state-space representation of G. Then, in this situation, it can be proven that (A, B_2) is stabilizable and (C_2, A) is detectable if and only if there exists $K \in \mathcal{K}$ with which the system consisting of G and K is input-output stable. Hence, as far as we are concerned with stabilization of G, the above assumption is mild enough. See Lemma A.4.2 of [40] for the proof of this result. (There, only the case of $D_{22} = O$ is considered. However, its generalization is straightforward.)

Next, assume stabilizability of (A, B_2) and detectability of (C_2, A). Then, it can be shown that a continuous-time control system is input-output stable if and only if the four operators from $\boldsymbol{b}(t)$, $\boldsymbol{a}(t)$ to $\boldsymbol{y}(t)$, $\boldsymbol{u}'(t)$ are bounded. This means that we do not have to care about the whole G but only G_{22} to inspect the stability of the system. The proof of this claim is found in Lemma A.4.3 of [40]. □

As is stated in this remark, under our standing assumption, there always exists a continuous-time controller $K \in \mathcal{K}$ that stabilizes the system. From this fact the next proposition follows.

Proposition 2.32. *For any generalized plant G, the best continuous-time control performance is finite, that is,*

$$\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\| < \infty.$$

Using a strong tool called the Youla parametrization [98, 24], we can parametrize all continuous-time controllers K that stabilize the system and analyze the infimum of (2.8). We introduce this parametrization following [24], [92, Chapters 4 and 5], and [34, Chapter 4].

First, we need the notions of unimodularity and coprimeness. Recall that $\mathcal{RH}^\infty$ is the set of all real rational functions analytic and bounded in $\operatorname{Re} s > 0$.

Definition 2.33. A function $A(s)$ is called **unimodular** in $\mathcal{RH}^\infty$ if both $A(s)$ and $A(s)^{-1}$ belong to $\mathcal{RH}^\infty$. □

Definition 2.34. Let $N(s)$ and $M(s)$ be elements of $\mathcal{RH}^\infty$. If there exist two functions $\widetilde{X}$, $\widetilde{Y} \in \mathcal{RH}^\infty$ such that $\widetilde{X}M - \widetilde{Y}N$ is unimodular in $\mathcal{RH}^\infty$, the pair (N, M) is called **right coprime** in $\mathcal{RH}^\infty$.

On the other hand, suppose that two functions $\widetilde{N}(s)$ and $\widetilde{M}(s)$ belong to $\mathcal{RH}^\infty$. If there exist two functions $X, Y \in \mathcal{RH}^\infty$ such that $\widetilde{M}X - \widetilde{N}Y$ is unimodular in $\mathcal{RH}^\infty$, the pair $(\widetilde{M}, \widetilde{N})$ is called **left coprime** in $\mathcal{RH}^\infty$. □

Definition 2.35. Suppose that a real rational function $A(s)$ is provided and it is expressed as $A = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$, where (N, M) is right coprime in $\mathcal{RH}^\infty$ and $(\widetilde{M}, \widetilde{N})$ is left coprime in $\mathcal{RH}^\infty$. Then, the pair (N, M) is called a **right-coprime factorization** of $A(s)$ in $\mathcal{RH}^\infty$, while the pair $(\widetilde{M}, \widetilde{N})$ is called a **left-coprime factorization** of $A(s)$ in $\mathcal{RH}^\infty$. □

Especially, the following type of coprime factorizations is important.

Definition 2.36. Suppose that, for a provided real rational function $A(s)$, there exist eight functions N, M, X, Y, $\widetilde{N}$, $\widetilde{M}$, $\widetilde{X}$, $\widetilde{Y} \in \mathcal{RH}^\infty$ satisfying

$$A(s) = N(s)M(s)^{-1} = \widetilde{M}(s)^{-1}\widetilde{N}(s),$$
$$\begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

Then, this octet of functions is called a **doubly-coprime factorization** of $A(s)$ in $\mathcal{RH}^\infty$. □

In the following, we do not explicitly describe the considered function class $\mathcal{RH}^\infty$ when we talk about unimodularity and coprimeness. Note that, with respect to a doubly-coprime factorization of $A(s)$, (N, M) is a right-coprime factorization of $A(s)$ and $(\widetilde{M}, \widetilde{N})$ is its left-coprime factorization. In particular, the next proposition shows that, for any real rational function $A(s)$, there exists its doubly-coprime factorization.

Proposition 2.37. *Suppose that $A(s)$ is a real rational function and let (N, M) and $(\widetilde{M}, \widetilde{N})$ be any right-coprime factorization and any left-coprime factorization of $A(s)$, respectively. Suppose that $\widetilde{X}, \widetilde{Y} \in \mathcal{RH}^\infty$ satisfy $\widetilde{X}M - \widetilde{Y}N = I$. Then, there exists $X, Y \in \mathcal{RH}^\infty$ such that*

$$\begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

Its proof is found in Theorem 4.1.60 of [92]. Reference [68] gave formulas to compute a doubly-coprime factorization of a provided $A(s)$ based on the state-space representations of $A(s)$.

Let us go back to the issue of stabilizing controllers. First, we have the following result.

Proposition 2.38. *Consider a continuous-time control system made of G and $K \in \mathcal{K}$. Let (N, M) and $(\widetilde{M}, \widetilde{N})$ be any right-coprime factorization and any left-coprime factorization of $\widehat{G}_{22}(s)$, respectively. Let (Y, X) and $(\widetilde{X}, \widetilde{Y})$ be any right-coprime factorization and any left-coprime factorization of $\widehat{K}(s)$, respectively. The the following are equivalent:*

(a) The considered system is input-output stable;

(b) The function $\widetilde{X}M - \widetilde{Y}N$ is unimodular;

(c) The function $\widetilde{M}X - \widetilde{N}Y$ is unimodular.

See Theorem 2 in [24] or Theorem 5.1.25 in [92] for its proof.

If we use the notion of doubly-coprime factorization, we can obtain a parametrization of all stabilizing controllers, which is called the **Youla parametrization**. For its proof, see [24, Theorem 3], [92, Theorem 5.2.1], or [34, Theorem 4.4.1], for example.

Proposition 2.39. *Suppose that a continuous-time control system is provided and a doubly-coprime factorization of $\widehat{G}_{22}(s)$ is given in the form of Definition 2.36. Then, a continuous-time controller $K \in \mathcal{K}$ makes this continuous-time control system input-output stable if and only if K is expressed as*

$$\widehat{K}(s) = \{\widetilde{X}(s) - Q(s)\widetilde{N}(s)\}^{-1}\{\widetilde{Y}(s) - Q(s)\widetilde{M}(s)\}$$

with some $Q \in \mathcal{RH}^\infty$ such that $\det\{\widetilde{X}(s) - Q(s)\widetilde{N}(s)\}$ is not constantly equal to zero.

From this parametrization, the next result follows. Its proof is found in [34, Theorem 4.5.1].

Proposition 2.40. *Define*

$$T_1(s) := \widehat{G}_{11}(s) + \widehat{G}_{12}M(s)\widetilde{Y}(s)\widehat{G}_{21}(s),$$
$$T_2(s) := \widehat{G}_{12}(s)M(s), \qquad T_3(s) := \widetilde{M}(s)\widehat{G}_{21}(s).$$

Then, all of T_1, T_2, and T_3 belong to $\mathcal{RH}^\infty$ and there holds

$$\widehat{\mathcal{F}}(G, K)(s) = T_1(s) - T_2(s)Q(s)T_3(s),$$

where $\widehat{\mathcal{F}}(G, K)(s)$ is the continuous-time transfer function of $\mathcal{F}(G, K)$.

By this proposition, another expression for (2.8) is obtained, that is,

$$\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\| = \inf_{\substack{Q \in \mathcal{RH}^\infty \\ \det(\widetilde{X} - Q\widetilde{N}) \neq 0}} \|T_1 - T_2QT_3\|_{\mathcal{H}^\infty}.$$

Note that the new expression is much easier to be treated because the free parameter Q is included in an affine manner there.

Next, we consider the best achievable performance of continuous-time control systems restricting K to be chosen from certain subsets of $\mathcal{K}$. In Chapter 4, it is revealed that these restricted performances are closely related to the best achievable performance of sampled-data control systems.

In the following discussion, the next results are useful. See [92, Lemma 5.2.25, Proposition 5.2.27] for their proofs.

Lemma 2.41. *Let $s_1, s_2, \ldots, s_\ell$ be distinct points in $\mathbb{C}_{+e}$ such that none is a pole of $\widehat{G}_{22}(s)$. Moreover, let $n_1, n_2, \ldots, n_\ell$ be positive integers. Then, there is a doubly-coprime factorization of $\widehat{G}_{22}(s)$ such that $Y(s)$ and $\widetilde{Y}(s)$ are equal to zero at each s_j, $j = 1, \ldots, \ell$, with multiplicity n_j or more, respectively.*

Proposition 2.42. *Under the same assumptions as the previous lemma, let N, M, X, Y, $\widetilde{N}$, $\widetilde{M}$, $\widetilde{X}$, $\widetilde{Y}$ be the doubly-coprime factorization given by the previous lemma. Then, a controller K stabilizes G and satisfies $\widehat{K}(s_j) = O$ at least with multiplicity n_j for each $j = 1, \ldots, \ell$ if and only if K is expressed as*

$$\widehat{K}(s) = \{\widetilde{X}(s) - Q(s)\widetilde{N}(s)\}^{-1}\{\widetilde{Y}(s) - Q(s)\widetilde{M}(s)\}$$

with $Q \in \mathcal{RH}^\infty$ such that $Q(s_j) = O$ at least with multiplicity n_j for each j.

By the definition of a doubly-coprime factorization, we have $\widetilde{X}M - \widetilde{Y}N = I$. Since $\widetilde{Y}(s_j) = O$, the matrix $\widetilde{X}(s_j)$ is invertible. Hence, $\det(\widetilde{X} - Q\widetilde{N}) \not\equiv 0$ is ensured by $Q(s_j) = O$, and thus it is not assumed explicitly this time.

We are particularly interested in the following subsets of $\mathcal{K}$:

$$\mathcal{K}_0 := \{K_0 \in \mathcal{K} : \widehat{K}_0(\infty) = O\},$$
$$\mathcal{K}_{00} := \{K_{00} \in \mathcal{K} : \widehat{K}_{00}(\infty) = O \text{ with multiplicity two or more}\}.$$

Let us consider $\mathcal{K}_0$ first. Under our standing assumption, that is, stabilizability of (A, B_2) and detectability of (C_2, A), actually we can find in $\mathcal{K}_0$ a controller that input-output stabilizes the system. (See the proof of Lemma A.4.2 in [40]. There, a stabilizing controller that belongs to $\mathcal{K}_0$ is actually constructed.) This gives the following result.

Proposition 2.43. *For any generalized plant G, there holds*

$$\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| < \infty.$$

Now, note $\mathcal{K}_0 \subsetneq \mathcal{K}$. This implies

$$\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| \geq \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$$

and the equality does not hold in general. However, for a special class of G, the equality holds.

Proposition 2.44. *If at least one of $\widehat{G}_{11}(s)$, $\widehat{G}_{12}(s)$, and $\widehat{G}_{21}(s)$ is strictly proper, we have*

$$\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|.$$

The proof is found in Appendix C.

Next, we consider $\mathcal{K}_{00}$. As we see below, the best achievable performance does not change even if the controller class is narrowed from $\mathcal{K}_0$ to $\mathcal{K}_{00}$. See Appendix C again for its proof.

Proposition 2.45. *For any generalized plant G, there holds*

$$\inf_{K_{00} \in \mathcal{K}_{00}} \|\mathcal{F}(G, K_{00})\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\|.$$

2.6. A Model-Matching Problem on Periodic Operators

In this section, we consider a model-matching problem on periodic operators in order to prepare for the investigation in Section 4.4, where a property of the best sampled-data control performance is discussed in connection with such a problem. The contents of this section is a new contribution. A model-matching problem on time-invariant continuous-time operators or time-invariant discrete-time operators has been examined well. Especially, it is known that such a problem can be solved with the help of inner-outer factorization and Nehari's theorem [25, Sections 2.3 and 2.4] [34, Chapters 7 and 8]. In this section, these techniques are translated so as to be applicable to our model-matching problem that includes *periodic* operators, particularly, a sampler-type operator and a hold-type operator. These two problems are considerably different at the point that the conventional problem can be considered using properties of *matrix*-valued functions, while our problem requires those of *operator*-valued functions. Therefore, an attention should be paid on the infinite-dimensional nature of operator-valued functions at the translation.

Our model-matching problem considered here is as follows.

Problem. Let P be a bounded continuous-time operator having a continuous-time state-space representation and let H be a hold-type operator whose lifting-based transfer function belongs

to $\mathfrak{RH}^\infty_{\mathrm{T}}$. Moreover, assume that, with respect to a matrix representation of $\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})$

$$\begin{bmatrix} \vdots \\ \acute{E}^{\mathrm{i}\omega}_{-1} \breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}^{\mathrm{i}\omega}_{0} \breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}^{\mathrm{i}\omega}_{1} \breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \vdots \end{bmatrix}, \tag{2.9}$$

all the columns are independent for any $\omega \in [-\pi/\tau, \pi/\tau)$. Under these conditions evaluate the value of

$$\inf_S \|P - HS\|, \tag{2.10}$$

where S varies over all sampler-type operators whose lifting-based transfer functions $\breve{S}(z)$ have the form of $z^{-1}\breve{S}'(z)$ with $\breve{S}' \in \mathfrak{RH}^\infty_{\mathrm{F}}$. □

Particularly in this section, we obtain an upper bound and a lower bound for the infimum (2.10).

Proposition 2.17 ensures that our P has its lifting-based transfer function $\breve{P}(z)$ in $\mathfrak{RH}^\infty_{\mathrm{L}}$; By assumption, our H has its lifting-based transfer function $\breve{H}(z)$ in $\mathfrak{RH}^\infty_{\mathrm{T}}$. With the help of Proposition 2.16, we can rewrite the above infimum in the frequency domain:

$$\inf_{\breve{S}' \in \mathfrak{RH}^\infty_{\mathrm{F}}} \|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}^\infty_{\mathrm{L}}}. \tag{2.11}$$

In the following, we mainly use this expression.

2.6.1. Inner-Outer Factorization

As our first step, we need the notion of inner-outer factorization for tall-operator-valued functions $\breve{H}(z)$.

In general, suppose that $A(z)$ is a function belonging to either $\mathfrak{RH}^\infty_{\mathrm{L}}$, $\mathfrak{RH}^\infty_{\mathrm{F}}$, $\mathfrak{RH}^\infty_{\mathrm{T}}$, or $\mathfrak{RH}^\infty$. Here, with respect to this $A(z)$, its **conjugate** $A^\sim(z)$ is defined as

$$A^\sim(z) := A\left(\frac{1}{\bar{z}}\right)^*.$$

A function $A^\sim(z)$ is well-defined for any z where $A(1/\bar{z})$ is well-defined. Note that whenever A is real and rational, so is $A^\sim(z)$. Note also that $A^\sim(z) = A(z)^*$ on $|z| = 1$. With this notation, a function $A(z)$ is called **inner** if $A(z)$ is an element of $\mathfrak{RH}^\infty_{\mathrm{T}}$ or $\mathfrak{RH}^\infty$ and satisfies

$$A^\sim(z)A(z) = I$$

for any z where both $A(z)$ and $A^{\sim}(z)$ are well-defined. Moreover, $A(z)$ is called **outer** if $A(z)$ is a square-matrix-valued function belonging to $\mathfrak{R}\mathfrak{H}^{\infty}$ and $\det A(z) \neq 0$ for any z in the set $\mathbb{D} \cup \{|z| = 1\}$. By definition, for an outer function $A(z)$, both $A(z)$ and $A(z)^{-1}$ belong to $\mathfrak{R}\mathfrak{H}^{\infty}$.

Now, for a tall-operator-valued function $\breve{H}(z)$, which is the lifting-based transfer function of the provided H in our problem, let us consider its factorization into a pair of an inner function and an outer function. This pair is called an **inner-outer factorization** of $\breve{H}(z)$. First, consider the conjugate of $\breve{H}(z)$. Since the function $\breve{H}(z)$ is real, rational, and tall-operator-valued, its conjugate $\breve{H}^{\sim}(z)$ is real, rational, and *flat*-operator-valued. From this it follows that a function product $\breve{H}^{\sim}(z)\breve{H}(z)$ is a real rational *matrix*-valued function. Since

$$\breve{H}^{\sim}(\mathrm{e}^{\mathrm{i}\omega\tau})\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) = \breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})^{*}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) = \breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})^{*}\left(\sum_{m=-\infty}^{\infty} \grave{E}_{m}^{\mathrm{i}\omega}\acute{E}_{m}^{\mathrm{i}\omega}\right)\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})$$

$$= \begin{bmatrix} \cdots & \left\{\acute{E}_{-1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})\right\}^{*} & \left\{\acute{E}_{0}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})\right\}^{*} & \left\{\acute{E}_{1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})\right\}^{*} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \acute{E}_{-1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}_{0}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}_{1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \vdots \end{bmatrix},$$

the assumption that the matrix in (2.9) has independent columns implies that the function $\breve{H}^{\sim}(z)\breve{H}(z)$ has a full rank at any points on $|z| = 1$. Therefore, using a spectral factorization technique for usual discrete-time transfer functions [51] [34, Section 7.3], we can obtain an outer function $\breve{H}^{\mathrm{out}}(z)$ such that

$$\breve{H}^{\mathrm{out}\sim}(z)\breve{H}^{\mathrm{out}}(z) = \breve{H}^{\sim}(z)\breve{H}(z)$$

for any z where $\breve{H}(z)$ and $\breve{H}^{\sim}(z)$ are well-defined. In fact, this $\breve{H}^{\mathrm{out}}(z)$ can be computed via matrix manipulations based on the discrete-time state-space representation of $\breve{H}^{\sim}(z)\breve{H}(z)$. Furthermore, define a tall-operator-valued function

$$\breve{H}^{\mathrm{in}}(z) := \breve{H}(z)\breve{H}^{\mathrm{out}}(z)^{-1}.$$

Then, there holds

$$\breve{H}^{\mathrm{in}\sim}(z)\breve{H}^{\mathrm{in}}(z) \equiv I.$$

Here, the function $\breve{H}^{\mathrm{in}}(z)$ belongs to $\mathfrak{R}\mathfrak{H}_{\mathrm{T}}^{\infty}$ since $\breve{H}(z)$ is an element of $\mathfrak{R}\mathfrak{H}_{\mathrm{T}}^{\infty}$ and $\breve{H}^{\mathrm{out}}(z)^{-1}$ belongs to $\mathfrak{R}\mathfrak{H}^{\infty}$. Hence, $\breve{H}^{\mathrm{in}}(z)$ is inner. Now, it can be seen that the pair $(\breve{H}^{\mathrm{in}}, \breve{H}^{\mathrm{out}})$ is an inner-outer factorization of $\breve{H}(z)$.

Using an inner-outer factorization of $\breve{H}(z)$, we can decompose the value of our concern (2.11) into two parts. For a large-operator-valued function $A(z)$, define

$$\|A\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}} := \operatorname*{ess\,sup}_{|z|=1} \|A(z)\|_{\mathrm{L}}.$$

Similarly for a flat-operator-valued function, a tall-operator-valued function, and a matrix-valued function, the norms $\|\cdot\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}$, $\|\cdot\|_{\mathfrak{L}_{\mathrm{T}}^{\infty}}$, and $\|\cdot\|_{\mathfrak{L}^{\infty}}$ are defined by replacing $\|\cdot\|_{\mathrm{L}}$ in the above definition by $\|\cdot\|_{\mathrm{F}}$, $\|\cdot\|_{\mathrm{T}}$, and $\overline{\sigma}\{\cdot\}$, respectively.

Then, we have the next proposition.

Proposition 2.46. *With the notation introduced so far, there holds*

$$\begin{aligned} \max\left\{\inf_{\breve{S}'\in\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}} \|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}^{2},\ \|(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^{2}\right\} & \\ \leq \inf_{\breve{S}'\in\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}} \|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}^{2} & \\ \leq \inf_{\breve{S}'\in\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}} \|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}^{2} + \|(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^{2}. & \end{aligned}$$

This proposition means that the value of (2.11) converges to zero if and only if the following two terms converge to zero, that is, $\inf_{\breve{S}'\in\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}} \|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}^{2}$ and $\|(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^{2}$. Although the latter term does not include $\breve{S}'$, the former term still includes it. In fact, by Nehari's theorem, we can show that the value of this former term is equal to the Hankel norm of $z\breve{H}^{\mathrm{in}\sim}\breve{P}$, which does not include $\breve{S}'$. This is what we consider in the next subsection.

Proof. Since our $\breve{P}(z)$, $\breve{H}(z)$, and $\breve{S}'(z)$ are rational functions, they are not only analytic in $\mathbb{D}$ but also continuous in $\mathbb{D} \cup \{|z| = 1\}$. The maximum modulus theorem implies

$$\|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}} = \|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}};$$

in other words, the $\mathfrak{H}_{\mathrm{L}}^{\infty}$-norm (the supremum in $\mathbb{D}$) is replaced by the $\mathfrak{L}_{\mathrm{L}}^{\infty}$-norm (the supremum on $|z| = 1$). Next, consider an operator-valued function

$$U(z) := \begin{bmatrix} z\breve{H}^{\mathrm{in}\sim}(z) \\ I - \breve{H}^{\mathrm{in}}(z)\breve{H}^{\mathrm{in}\sim}(z) \end{bmatrix}.$$

Then, $U^{\sim}(z)U(z) = I$ holds in the domain where $U(z)$ and $U^{\sim}(z)$ are well-defined. Note that both of the two are well-defined on $|z| = 1$. Furthermore, since the range of $U(z)$ is the direct sum of the finite-dimensional vector space and $\mathcal{L}^{2}[0,\tau)$, its norm and inner product are naturally introduced.

It is claimed that $\|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}} = \sup_{|z|=1}\|U(\breve{P} - z^{-1}\breve{H}\breve{S}')\|_{\mathrm{ind}}$ with $\|\cdot\|_{\mathrm{ind}}$ being an appropriate induced norm. To show this, note $U^{\sim}(z) = U(z)^{*}$ on $|z| = 1$. It can be derived on $|z| = 1$ that, for any $\boldsymbol{f} \in \mathcal{L}^{2}[0,\tau)$,

$$\begin{aligned} &\|U(z)\{\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\}\boldsymbol{f}\|^{2} \\ &\quad = \left(\{\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\}\boldsymbol{f},\ U^{\sim}(z)U(z)\{\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\}\boldsymbol{f}\right)_{\mathcal{L}^{2}[0,\tau)} \\ &\quad = \|\{\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\}\boldsymbol{f}\|_{\mathcal{L}^{2}[0,\tau)}^{2}, \end{aligned}$$

where the norm in the leftmost expression is the one appropriately defined in the range of $U(z)$. This equality means that $\|\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\|_{\mathrm{L}} = \|U(z)\{\breve{P}(z) - z^{-1}\breve{H}(z)\breve{S}'(z)\}\|_{\mathrm{ind}}$ on $|z| = 1$. Our claim is proven.

Now, there holds

$$\|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}} = \sup_{|z|=1} \|U(\breve{P} - z^{-1}\breve{H}\breve{S}')\|_{\mathrm{ind}} = \sup_{|z|=1} \left\| \begin{bmatrix} z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{H}^{\mathrm{out}}\breve{S}' \\ (I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P} \end{bmatrix} \right\|_{\mathrm{ind}}.$$

Here, we have used $\breve{H} = \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{out}}$ and $\breve{H}^{\mathrm{in}\sim}\breve{H}^{\mathrm{in}} = I$. Noting that $z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{H}^{\mathrm{out}}\breve{S}'$ is a flat-operator-valued function and $(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}$ is a large-operator-valued function, we conclude that

$$\begin{aligned} &\max\left\{\|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{H}^{\mathrm{out}}\breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}^2,\ \ \|(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2\right\} \\ &\qquad \le \|\breve{P} - z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}^2 \\ &\qquad \le \|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{H}^{\mathrm{out}}\breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}^2 + \|(I - \breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2. \end{aligned}$$

Take the infimum moving $\breve{S}'$ over $\mathfrak{RH}_{\mathrm{F}}^{\infty}$ in each expression. Then, since $\breve{H}^{\mathrm{out}}(z)$ has its inverse in $\mathfrak{RH}^{\infty}$, a function $\breve{H}^{\mathrm{out}}\breve{S}'$ moves all over $\mathfrak{RH}_{\mathrm{F}}^{\infty}$ as $\breve{S}'$ varies over $\mathfrak{RH}_{\mathrm{F}}^{\infty}$. Now, we have shown the claim. □

2.6.2. Hankel Norms and Nehari's Theorem

Let $\Phi(z)$ be a real rational operator-valued function. It is assumed that $\Phi(z)$ is analytic in $\rho_1 < |z| < \rho_2$, where $0 < \rho_1 < 1 < \rho_2$. Especially in this section, we are interested in the case that $\Phi(z)$ is a flat-operator-valued function or a matrix-valued function. In the following, we define the Hankel norm of such a function $\Phi(z)$ and see that this norm is equal to the infimum of $\|\Phi - \Sigma\|$ by Nehari's theorem, where Σ varies over $\mathfrak{RH}_{\mathrm{F}}^{\infty}$ or $\mathfrak{RH}^{\infty}$ depending on the function-type of $\Phi(z)$ and $\|\cdot\|$ stands for $\|\cdot\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}}$ or $\|\cdot\|_{\mathfrak{L}^{\infty}}$.

Let us consider the case that $\Phi(z)$ is a flat-operator-valued function first. That is, for each z, a function value $\Phi(z)$ is a flat operator mapping $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathbb{C}^{n_b}$ for some n_a and n_b. The notion of Laurent expansion is successfully extended to the case of operator-valued functions [49, p. 97]. Noting that $\Phi(z)$ is analytic in $\rho_1 < |z| < \rho_2$, write its Laurent expansion there as

$$\Phi(z) = \sum_{k=-\infty}^{\infty} L^k z^k, \qquad L^k := \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \Phi(z)\frac{1}{z^{k+1}}\,\mathrm{d}z.$$

Here, each L^k is a flat operator from $\mathcal{L}^2[0,\tau)^{n_a}$ to $\mathbb{C}^{n_b}$. Using this $\{L^k\}$, consider the operation

$$\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} L^1 & L^2 & L^3 & \dots \\ L^2 & L^3 & L^4 & \\ L^3 & L^4 & L^5 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \\ \vdots \end{bmatrix},$$

where each $\boldsymbol{f}_k$ belongs to $\mathcal{L}^2[0,\tau)^{n_a}$ and each $\boldsymbol{v}_k$ is an element of $\mathbb{C}^{n_b}$. The above operator composed of L^1, L^2, … is called the **Hankel operator** with the symbol Φ and is denoted by Γ_Φ. Define the norm in its domain as $(\sum_{k=1}^{\infty} \|\boldsymbol{f}_k\|_{\mathcal{L}^2[0,\tau)}^2)^{1/2}$ and define the norm in its range as $(\sum_{k=1}^{\infty} \|\boldsymbol{v}_k\|_2^2)^{1/2}$. Then, the induced norm of the Hankel operator Γ_Φ is well-defined. This norm is called the **Hankel norm** of Φ and is denoted by $\|\Phi\|_{\mathrm{H}}$. By definition, there holds

$$\|\Phi\|_{\mathrm{H}}^2 \le \sum_{k=1}^{\infty} \|L^k\|_{\mathrm{F}}^2. \tag{2.12}$$

In the case that $\Phi(z)$ is matrix-valued, its Hankel operator and Hankel norm are defined in the same way. Only the difference is that each L^k is a matrix and each $\boldsymbol{f}_k$ belongs to a finite-dimensional vector space.

A main result in this subsection is now presented.

Proposition 2.47. *Suppose that $\Phi(z)$ is a real rational flat-operator-valued function, which is analytic in $\rho_1 < |z| < \rho_2$, where $0 < \rho_1 < 1 < \rho_2$. Then, we have*

$$\inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^{\infty}} \|\Phi - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}} = \|\Phi\|_{\mathrm{H}}.$$

If the two functions Φ and Σ are not flat-operator-valued but matrix-valued, this result is well-known now in the control community as Nehari's theorem [25, Section 2.4] [34, Sections 6.2 and 8.1] [40, Theorem 10.4.6] [99, Section 8.8]. That is, there holds the next.

Proposition 2.48. *Suppose that $\Phi(z)$ is a real rational matrix-valued function, which is analytic in $0 < \rho_1 < |z| < \rho_2$, where $\rho_1 < 1 < \rho_2$. Then, we have*

$$\inf_{\Sigma \in \mathfrak{RH}^{\infty}} \|\Phi - \Sigma\|_{\mathfrak{L}^{\infty}} = \|\Phi\|_{\mathrm{H}}.$$

A flat-operator version of Nehari's theorem, Proposition 2.47, is proven by reducing it to its matrix counterpart, Proposition 2.48. See Appendix D for the proof. From this proof it can also be seen that computation of the Hankel norm of a flat-operator-valued function $\Phi(z)$ can be carried out by matrix manipulations and the same is true about computation of $\Sigma(z)$ that approximates $\Phi(z)$. Namely, these problems can be reduced to the corresponding problems on matrix-valued functions.

Let us see the implication of Proposition 2.47 on our model-matching problem. By this proposition, we have

$$\inf_{\breve{S}' \in \mathfrak{RH}_{\mathrm{F}}^{\infty}} \|z\breve{H}^{\mathrm{in}\sim}\breve{P} - \breve{S}'\|_{\mathfrak{L}_{\mathrm{F}}^{\infty}} = \|z\breve{H}^{\mathrm{in}\sim}\breve{P}\|_{\mathrm{H}},$$

which simplifies the formula obtained in the previous section. Our final result is summarized as follows. In Section 4.4, this proposition is utilized to simplify a condition for the convergence of the best sampled-data control performance.

Proposition 2.49. *Suppose that P is a continuous-time operator having a continuous-time state-space representation and H is a hold-type operator whose lifting-based transfer function belongs to $\mathfrak{RH}_{\mathrm{T}}^{\infty}$. Suppose also that, with respect to the matrix representation of $\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau})$, i.e.,*

$$\begin{bmatrix} \vdots \\ \acute{E}_{-1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}_{0}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \acute{E}_{1}^{\mathrm{i}\omega}\breve{H}(\mathrm{e}^{\mathrm{i}\omega\tau}) \\ \vdots \end{bmatrix},$$

all the columns are independent for any $\omega \in [-\pi/\tau, \pi/\tau)$. Then, there exists an inner-outer factorization of $\breve{H}(z)$. Moreover, when we write this factorization as $(\breve{H}^{\mathrm{in}}, \breve{H}^{\mathrm{out}})$, there holds

$$\begin{aligned} &\max\left\{\|z\breve{H}^{\mathrm{in}\sim}\breve{P}\|_{\mathrm{H}}^2,\ \|(I-\breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2\right\} \\ &\le \inf_{\breve{S}'\in\mathfrak{RH}_{\mathrm{F}}^{\infty}} \|\breve{P}-z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}^2 \\ &\le \|z\breve{H}^{\mathrm{in}\sim}\breve{P}\|_{\mathrm{H}}^2 + \|(I-\breve{H}^{\mathrm{in}}\breve{H}^{\mathrm{in}\sim})\breve{P}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2. \end{aligned}$$

2.6.3. A Dual Model-Matching Problem

So far, we have considered how to evaluate

$$\inf_{S}\|P-HS\|.$$

In Section 4.4 we need a corresponding result also on a problem that is dual to the above; namely, the value of

$$\inf_{H}\|P-HS\|$$

is desired to be obtained. Here, it is assumed that P is a continuous-time operator having a continuous-time state-space representation and that S is a sampler-type operator whose lifting-based transfer function $\breve{S}(z)$ can be expressed as $z^{-1}\breve{S}'(z)$ with $\breve{S}' \in \mathfrak{RH}_{\mathrm{F}}^{\infty}$. Moreover, in the above expression, H varies over all hold-type operators whose lifting-based transfer functions belong to $\mathfrak{RH}_{\mathrm{T}}^{\infty}$. Using lifting-based transfer functions, we can write our new model-matching problem as

$$\inf_{\breve{H}\in\mathfrak{RH}_{\mathrm{T}}^{\infty}} \|\breve{P}-z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}.$$

This expression resembles Equation (2.11) we have considered so far. Hence, by translating the discussion so far in an appropriate way, we can obtain a result on this problem, too. In this section, this result is presented briefly.

Some preparation is needed to state the result.

For a real rational function $A(z)$, $A(z)$ is said to be **co-inner** if $A^{\sim}(z)$ is inner; $A(z)$ is called **co-outer** if $A^{\sim}(z)$ is outer. For the above sampler-type operator S, consider its lifting-based transfer function $\breve{S}(z)$. Assume that, in its matrix representation

$$\begin{bmatrix}\cdots & \breve{S}(\mathrm{e}^{\mathrm{i}\omega\tau})\grave{E}_{-1}^{\mathrm{i}\omega} & \breve{S}(\mathrm{e}^{\mathrm{i}\omega\tau})\grave{E}_{0}^{\mathrm{i}\omega} & \breve{S}(\mathrm{e}^{\mathrm{i}\omega\tau})\grave{E}_{1}^{\mathrm{i}\omega} & \cdots\end{bmatrix}, \tag{2.13}$$

all the rows are independent for any $\omega \in [-\pi/\tau, \pi/\tau)$. Then, in fact, the function $\breve{S}(z)$ can be factored as

$$\breve{S}(z) = z^{-1}\breve{S}^{\mathrm{out}}(z)\breve{S}^{\mathrm{in}}(z)$$

so that $\breve{S}^{\mathrm{in}}(z)$ is co-inner and $\breve{S}^{\mathrm{out}}(z)$ is co-outer. If we write $\breve{S} = z^{-1}\breve{S}'$, this factorization becomes $\breve{S}'(z) = \breve{S}^{\mathrm{out}}(z)\breve{S}^{\mathrm{in}}(z)$, and the pair $(\breve{S}^{\mathrm{out}}, \breve{S}^{\mathrm{in}})$ is said to be a **co-inner-co-outer factorization** of $\breve{S}'$. In order to obtain this factorization, compute a spectral factorization of a matrix-valued function $\breve{S}'\breve{S}'^{\sim}$ $(= \breve{S}\breve{S}^{\sim})$. This factorization gives a co-outer function $\breve{S}^{\mathrm{out}}$ such that $\breve{S}'\breve{S}'^{\sim} = \breve{S}^{\mathrm{out}}\breve{S}^{\mathrm{out}\sim}$. Then, define $\breve{S}^{\mathrm{in}}(z) := \breve{S}^{\mathrm{out}}(z)^{-1}\breve{S}'(z)$ $(= z\breve{S}^{\mathrm{out}}(z)^{-1}\breve{S}(z))$. It is easy to see this function $\breve{S}^{\mathrm{in}}(z)$ is co-inner.

The Hankel operator and the Hankel norm are defined for a tall-operator-valued function $\Phi(z)$ by appropriate modification of the definitions in Subsection 2.6.2. Nehari's theorem holds in this setting, too. More precisely, Proposition 2.47 still holds even if Φ is a real rational tall-operator-valued operator and Σ moves in the set $\mathfrak{RH}_{\mathrm{T}}^{\infty}$.

We now present a result for our new model-matching problem, which is required in Section 4.4.

Proposition 2.50. *Suppose that P is a continuous-time operator having a continuous-time state-space representation and that S is a sampler-type operator whose lifting-based transfer function $\breve{S}(z)$ can be expressed as $z^{-1}S'(z)$ with $S' \in \mathfrak{RH}_{\mathrm{F}}^{\infty}$. Assume that, with respect to a matrix representation of $\breve{S}(\mathrm{e}^{\mathrm{i}\omega\tau})$ presented in (2.13), all the rows are independent for any $\omega \in [-\pi/\tau, \pi/\tau)$. Then, with respect to the function $\breve{S}'$, its co-inner-co-outer factorization $(\breve{S}^{\mathrm{out}}, \breve{S}^{\mathrm{in}})$ can be found. Moreover, there holds*

$$\begin{aligned} &\max\left\{\|z\breve{P}\breve{S}^{\mathrm{in}\sim}\|_{\mathrm{H}}^2,\ \|\breve{P}(I-\breve{S}^{\mathrm{in}\sim}\breve{S}^{\mathrm{in}})\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2\right\} \\ &\le \inf_{\breve{H}\in\mathfrak{RH}_{\mathrm{T}}^{\infty}} \|\breve{P}-z^{-1}\breve{H}\breve{S}'\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}}^2 \\ &\le \|z\breve{P}\breve{S}^{\mathrm{in}\sim}\|_{\mathrm{H}}^2 + \|\breve{P}(I-\breve{S}^{\mathrm{in}\sim}\breve{S}^{\mathrm{in}})\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}}^2. \end{aligned}$$

Chapter 3

A General Framework for Sampled-Data Control Systems

This chapter presents a framework for sampled-data control systems. The framework presented here is general in the sense that a large class of samplers and holds can be treated in it; moreover, this framework is clear in the sense that basic properties of sampled-data control systems are derived in a natural way. Although one purpose of this chapter is to give a solid theoretical basis for subsequent investigation, it is important in its own right. This is because this framework is believed to be useful in order to solve other advanced sampled-data control problems than the one considered here.

First, regular samplers and holds are defined. They are more general than the conventional notions of generalized samplers and holds. Namely, the kernel functions of our samplers and holds are defined on $[0, \infty)$, while those of conventional samplers and holds are only on $[0, \tau)$, where τ is the sampling period. Several properties of regular samplers and holds are stated on their transfer functions, state-space representations, and matrix representations. Next, a sampled-data control system is introduced and the notion of a sampling environment is given. Moreover, stability and the best achievable performance of sampled-data control systems are defined. Based on the constructed framework, three theorems are proven about properties of sampled-data control systems. Especially, the last theorem is important in the next chapter because it states a relationship between a sampled-data control system and a corresponding continuous-time control system.

3.1. Introduction

In a sampled-data control system, which was introduced in Section 1.1, a sampler was used for an analog-to-digital signal conversion and a hold was used for a digital-to-analog signal conversion. As was stated in Section 1.1, the most typical sampler is the ideal sampler and the most typical hold is the zero-order hold. However, if we choose a more generalized sampler and hold appropriately for a provided plant, it is possible to improve a control performance. Earlier studies on this topic are [55, 54]. After the lifting technique was introduced by [94, 95], lifting-based approaches have been tried on this topic [45, 86, 53, 56, 5, 66]. However, the frameworks used in these papers are not sufficient to analyze general configuration of sampled-data control systems. The reason is as follows.

In many of these papers, a generalized hold $H : \boldsymbol{q}_{\mathrm{d}} \mapsto \boldsymbol{q}$ is assumed to have the form

$$\boldsymbol{q}(k\tau + t) := \underline{H}(t)\boldsymbol{q}_{\mathrm{d}}[k] \quad \text{for } 0 \le t < \tau \text{ and } k = 0, 1, \ldots. \tag{3.1}$$

Here, τ is the sampling period, $\boldsymbol{q}_{\mathrm{d}}[k]$ is a discrete-time input to H, $\boldsymbol{q}(t)$ is the corresponding continuous-time output, and $\underline{H}(t)$ is a certain provided function. In fact, even the first-order hold, which is often quoted as an example of a generalized hold, cannot be modeled in this form. Indeed, in (3.1), the output $\boldsymbol{q}(t)$ during $k\tau \le t < (k+1)\tau$ depends only on $\boldsymbol{q}_{\mathrm{d}}[k]$. However, the output of the first-order hold during $k\tau \le t < (k+1)\tau$ depends not only on $\boldsymbol{q}_{\mathrm{d}}[k]$ but also on $\boldsymbol{q}_{\mathrm{d}}[k-1]$. Therefore, we have to use a more general form than (3.1).

The situation is similar as for generalized samplers. A generalized sampler $S : \boldsymbol{p} \mapsto \boldsymbol{p}_{\mathrm{d}}$ is typically assumed to have the form

$$\boldsymbol{p}_{\mathrm{d}}[k] := \int_{(k-1)\tau}^{k\tau} \underline{S}(k\tau - t)\boldsymbol{p}(t)\,\mathrm{d}t \quad \text{for } k = 0, 1, \ldots, \tag{3.2}$$

where $\boldsymbol{p}(t)$ is a continuous-time input to the sampler S, $\boldsymbol{p}_{\mathrm{d}}[k]$ is its discrete-time output, and $\underline{S}(t)$ is a provided function. In this form, the ideal sampler, which is the sampler most widely used in practice, is not easily treated. If one likes to model the ideal sampler, he has to set $\underline{S}(t)$ to be the delta function. This makes the subsequent mathematical treatment complicated.

Moreover, an anti-aliasing filter is modeled as a part of a plant in the existing frameworks. An anti-aliasing filter is different from a plant in the sense that it has some design flexibility although a plant is provided as fixed. Actually, we can improve control performance by choosing an anti-aliasing filter appropriately. However, if the filter is regarded as a part of a plant, this flexibility becomes implicit and gets difficult to be utilized.

Finally, many of the existing papers considered lifting of a plant only. They paid less attention on a sampler and a hold. In order to see how the control performance depends on a choice of a sampler and a hold, we have to treat these devices more seriously.

In this chapter, we construct a framework for sampled-data control systems so that these problems are resolved.

In particular, by assuming that the functions $\underline{S}(t)$ and $\underline{H}(t)$ are defined on $[0, \infty)$ (as opposed to $[0, \tau)$), we extend the class of samplers and holds. Our class includes the classes of [55, 54, 45, 86, 53, 56, 5, 66] as its subclasses and can model the ideal sampler and the first-order hold in a natural way. Furthermore, this extension enables us to treat an anti-aliasing filter as a part of a sampler (as opposed to a plant), which means that analysis and synthesis of the filter can be formulated as those of a sampler.

Next, not only lifting of a controlled plant but also that of a sampler and a hold is considered. Although this was already tried by Mirkin and Rotstein [66], our class of a sampler and a hold is more general than theirs. We obtain explicit formulas that express lifting-based transfer functions of a sampler and a hold in terms of $\underline{S}(t)$ and $\underline{H}(t)$. These formulas are useful in the subsequent chapters.

Finally, some important properties of sampled-data control systems are derived based on the constructed framework. Here, the notions of a lifting-based transfer function and its matrix representation play an important role. Especially, a property about a relationship between a continuous-time control system and a sampled-data control system is shown. This is a new result and works as a key when we compare the best achievable performance of these two types of systems in the next chapter.

3.2. Regular Samplers and Holds

Before considering sampled-data control systems, we prepare our class of samplers and holds. Our class is large enough to cover many practically important samplers and holds. Especially, it includes the sampler and hold classes formerly proposed by [55, 54, 45, 86, 53, 56, 5, 66]. Next, we investigate properties of samplers and holds that belong to the presented class. The obtained properties are utilized to derive useful formulas on sampled-data control systems in Section 3.4.

We need the following functional space to define our class of samplers and holds.

Definition 3.1. Suppose that $a(t)$ is a real function such that $\mathrm{e}^{\epsilon t}a(t)$ belongs to $\mathcal{L}^2$ for some $\epsilon > 0$. Let $\mathcal{D}$ be the space of all such functions. Here, ϵ may vary depending on $a \in \mathcal{D}$. □

With the help of this space, a considered class of samplers and holds is defined as follows. Let the sampling period be τ. This is a positive number, which associates a discrete-time k with a continuous-time t by $t = k\tau$.

Definition 3.2. A sampler-type operator $S : \boldsymbol{p} \mapsto \boldsymbol{p}_{\mathrm{d}}$ is called a **regular sampler** or simply a **sampler** if its operation is represented as

$$\boldsymbol{p}_{\mathrm{d}}[k] = \int_0^{k\tau} \underline{S}(k\tau - t)\boldsymbol{p}(t)\,\mathrm{d}t \quad \text{for } k = 0, 1, \ldots$$

using a matrix-valued function $\underline{S}(t)$ whose elements belong to $\mathcal{D}$. Note that $\boldsymbol{p}(t)$ is a continuous-time signal and $\boldsymbol{p}_{\mathrm{d}}[k]$ is a discrete-time signal.

On the other hand, a hold-type operator $H : \boldsymbol{q}_\mathrm{d} \mapsto \boldsymbol{q}$ is said to be a **regular hold** or simply a **hold** if it is represented as

$$\boldsymbol{q}(k\tau + t) = \sum_{\ell=0}^{k} \underline{H}(k\tau + t - \ell\tau)\boldsymbol{q}_\mathrm{d}[\ell] \quad \text{for } k = 0, 1, \dots \text{ and } t \in [0, \tau)$$

with $\underline{H}(t)$ being a function every element of which belongs to $\mathcal{D}$.

The functions $\underline{S}(t)$ and $\underline{H}(t)$ are called the **kernel functions** of S and H, respectively. □

We express kernel functions by putting underlines on the corresponding symbols of a sampler and a hold.

The definitions above are different from (3.1) and (3.2), which are typical definitions of a generalized sampler and hold in the literature. That is, in our definitions kernel functions are defined on $[0, \infty)$, while in the conventional definitions only on $[0, \tau)$. This means that our class of samplers and holds includes the conventional classes as its subclasses.

Example 3.3. Define a sampler-type operator S_τ^id so that it maps a continuous-time signal $\boldsymbol{p}(t)$ to a discrete-time signal $\boldsymbol{p}_\mathrm{d}$ in accordance with

$$\boldsymbol{p}_\mathrm{d}[k] := \boldsymbol{p}(k\tau) \quad \text{for } k = 0, 1, \dots .$$

Figure 3.1 shows how this works in the one-dimensional case. (This is essentially the same figure as Figure 1.3.) This S_τ^id is called the **ideal sampler** with the sampling period τ. It is not a regular sampler because its kernel function turns out to be the delta function, which does not belong to $\mathcal{D}$.

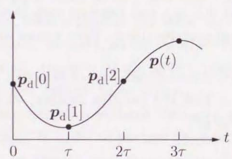


Figure 3.1. The operation of the ideal sampler S_τ^id.

In order to avoid this difficulty, consider a bounded continuous-time operator P that can be described by a strictly proper state-space representation (A, B, C, O). Without loss of generality it can be assumed that the matrix A has all of its eigenvalues in $\mathrm{Re}\, s < 0$. Here, an operator $S_\tau^\mathrm{id} P$ is a regular sampler. Indeed, its kernel function is $Ce^{At}B$. □

Example 3.4. Let us consider a sampler-type operator $S_\tau^\mathrm{mr} : \boldsymbol{p} \mapsto \boldsymbol{p}_\mathrm{d}$ that works as

$$\boldsymbol{p}_\mathrm{d}[k] = \begin{bmatrix} \boldsymbol{p}(k\tau) \\ \boldsymbol{p}((k - 1/2)\tau) \end{bmatrix} \quad \text{for } k = 0, 1, \dots .$$

Here, suppose $\boldsymbol{p}(-1/2)$ is equal to zero. This sampler picks up the input signal $\boldsymbol{p}$ not only at $t = k\tau$ but also at $t = (k - 1/2)\tau$. Let us call this operator the **multirate sampler**. By using this sampler, we can deal with a certain type of multirate sampled-data control systems as will be seen in Examples 3.26 and 3.27. Although the multirate sampler S_τ^mr is not a regular sampler, the operator $S_\tau^\mathrm{mr} P$ is a regular sampler by a proper choice of a continuous-time operator P just as the previous example. In this case, the kernel function of $S_\tau^\mathrm{mr} P$ is

$$\begin{bmatrix} Ce^{At}B \\ 1(t - \tau/2)Ce^{A(t-\tau/2)}B \end{bmatrix},$$

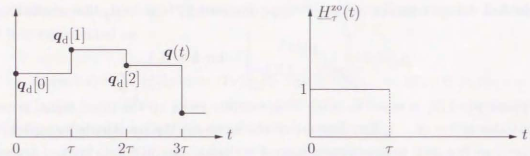
where $1(t) := 1$ for $t \geq 0$ and $1(t) := 0$ for $t < 0$. It is seen from this example that a kernel function may not be square. □

Remark 3.5. In our framework, the ideal sampler S_τ^id is always treated as in Example 3.3. Practically, we do as follows.

In the recent sampled-data control studies, it is standard to assume that a continuous-time operator F that has a strictly proper state-space representation precedes the ideal sampler. One way to use the technique of Example 3.3 is to prepare a bounded continuous-time operator P having a strictly proper state-space representation, and decompose $S_\tau^\mathrm{id} F$ into a successive operation of $S_\tau^\mathrm{id} P$ and $P^{-1}F$. Since $S_\tau^\mathrm{id} P$ is a regular sampler and $P^{-1}F$ is a continuous-time operator with a state-space representation, their treatment is now easy.

There is another way when the operator F is bounded itself. This is often the case when F is an anti-aliasing filter. In this case, F has a state-space representation whose "A"-matrix has all of its eigenvalues in $\mathrm{Re}\, s < 0$. Then, it is possible to regard $S_\tau^\mathrm{id} F$ as a regular sampler.

This shows another possibility of our framework. That is, an anti-aliasing filter F can be treated as a part of a sampler in our framework. In the lifting-based studies so far, an anti-aliasing filter was often modeled as a part of a plant. However, our formulation is considered to be more natural than the conventional ones because of the following reasons. First, since both anti-aliasing filter and sampler work to convert a continuous-time signal into a discrete-time signal, it is appropriate to treat them in a combined way. Next, an anti-aliasing filter has some design flexibility and can be designed in accordance with engineer's preferences. At this point an anti-aliasing filter is different from a plant because a plant is given to an engineer as fixed. Once we regard an anti-aliasing filter as a part of a sampler, we can formulate a design problem of this filter as that of a sampler. In addition, note that the kernel function of $S_\tau^\mathrm{id} F$ has the form of $Ce^{At}B$ and, thus, it is nonzero all over $[0, \infty)$ in general. Therefore, this operator cannot be



(a) (b)

Figure 3.2. (a) The operation and (b) the kernel function of the zero-order hold H_τ^{zo}.

treated as a sampler in the conventional frameworks, where kernel functions are defined only on $[0, \tau)$. □

Example 3.6. Let a hold-type operator H_τ^{zo} maps a discrete-time signal $q_d[k]$ to a continuous-time signal $q(t)$ as

$$q(k\tau + t) = q_d[k] \quad \text{for } 0 \le t < \tau \text{ and } k = 0, 1, \dots .$$

Figure 3.2 (a) illustrates its operation in the case that both $q_d[k]$ and $q(t)$ are one-dimensional. (This is almost the same figure as Figure 1.3 (b).) This operator H_τ^{zo} is called the **zero-order hold** with the sampling period τ. Its kernel function $\underline{H}_\tau^{zo}(t)$ turns out to be

$$\underline{H}_\tau^{zo}(t) = \begin{cases} I & \text{for } 0 \le t < \tau, \\ O & \text{for } \tau \le t. \end{cases}$$

This function is presented in Figure 3.2 (b). It is seen that the zero-order hold is a regular hold. □

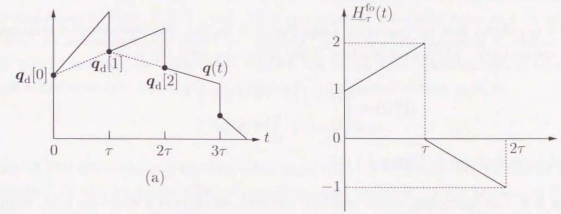
Example 3.7. Consider a hold-type operator $H_\tau^{fo} : q_d \mapsto q$ such that

$$q(k\tau + t) = -\frac{t}{\tau} q_d[k-1] + \Big(1 + \frac{t}{\tau}\Big) q_d[k].$$

Figure 3.3 (a) shows how this operator works in the one-dimensional case. This operator H_τ^{fo} is called the **first-order hold**. Its kernel function is obtained as

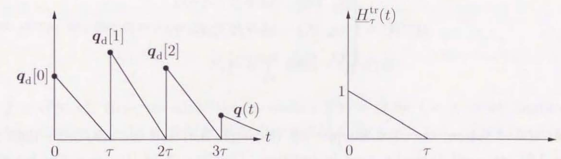
$$\underline{H}_\tau^{fo}(t) = \begin{cases} (1 + \frac{t}{\tau})I & \text{for } 0 \le t < \tau, \\ (1 - \frac{t}{\tau})I & \text{for } \tau \le t < 2\tau, \\ O & \text{for } 2\tau \le t. \end{cases}$$

See Figure 3.3 (b) for the shape of this function. Because this kernel function takes a nonzero value in $\tau \le t$, the first-order hold cannot be modeled in the conventional frameworks. □



(a) (b)

Figure 3.3. (a) The operation and (b) the kernel function of the first-order hold H_τ^{fo}.



(a) (b)

Figure 3.4. (a) The operation and (b) the kernel function of the triangular hold H_τ^{tr}.

Example 3.8. For comparison, we define a rather artificial hold. Consider a hold-type operator $H_\tau^{\rm tr}: \boldsymbol{q}_{\rm d} \mapsto \boldsymbol{q}$ such that

$$\boldsymbol{q}(k\tau + t) = (1 - \frac{t}{\tau})\boldsymbol{q}_{\rm d}[k] \quad \text{for } 0 \le t < \tau \text{ and } k = 0, 1, \dots .$$

Figure 3.4 (a) shows how its operation looks like. Let us call this operator the **triangular hold** for convenience. The kernel function of the triangular hold $H_\tau^{\rm tr}$ is as

$$\underline{H}_\tau^{\rm tr}(t) = \begin{cases} (1 - \frac{t}{\tau})I & \text{for } 0 \le t < \tau, \\ O & \text{for } \tau \le t. \end{cases}$$

This function is depicted in Figure 3.4 (b). □

Example 3.9. Corresponding to the multirate sampler, we think of a multirate hold. Suppose that $H_\tau^{\rm mr}: \boldsymbol{q}_{\rm d} \mapsto \boldsymbol{q}$ works as

$$\boldsymbol{q}(k\tau + t) = \begin{cases} [I_n \;\; O_n]\boldsymbol{q}_{\rm d}[k] & \text{for } 0 \le t < \tau/2, \\ [O_n \;\; I_n]\boldsymbol{q}_{\rm d}[k] & \text{for } \tau/2 \le t < \tau. \end{cases}$$

Here, n is the dimension of the output signal $\boldsymbol{q}(t)$, I_n denotes the $n \times n$-identity matrix, and O_n stands for the $n \times n$-zero matrix. This hold gives the upper half of $\boldsymbol{q}_{\rm d}[k]$ during $k\tau \le t < (k + 1/2)\tau$ and then gives the lower half during $(k + 1/2)\tau \le t < (k + 1)\tau$. In this sense, the output of this hold is switched not only at $t = k\tau$ but also at $t = (k+1/2)\tau$. Therefore, it is appropriate to call it the **multirate hold**. Its kernel function $\underline{H}_\tau^{\rm mr}(t)$ is expressed as

$$\underline{H}_\tau^{\rm mr}(t) = \begin{cases} [I_n \;\; O_n] & \text{for } 0 \le t < \tau/2, \\ [O_n \;\; I_n] & \text{for } \tau/2 \le t < \tau, \\ [O_n \;\; O_n] & \text{for } \tau \le t. \end{cases}$$

□

As was stated before, the ideal sampler $S_\tau^{\rm id}$ (Example 3.3) and the zero-order hold $H_\tau^{\rm zo}$ (Example 3.6) are most typically used in practice. Corollary 2 of [17] gave the following interesting property of them. This result is utilized in the next chapter.

Proposition 3.10. *Let P be a bounded continuous-time operator having a strictly proper continuous-time state-space representation. Then, for the ideal sampler $S_\tau^{\rm id}$ and the zero-order hold $H_\tau^{\rm zo}$ that have consistent dimensions, there holds*

$$\lim_{\tau \to 0} \|P - H_\tau^{\rm zo} S_\tau^{\rm id} P\| = 0.$$

Remark 3.11. We assumed that kernel functions of a regular sampler S and a regular hold H, i.e., $\underline{S}(t)$ and $\underline{H}(t)$, belong to $\mathcal{D}$. Roughly speaking, this assumption requires these functions to decrease exponentially or faster as t goes to infinity. Although their *exponential* decrease is assumed for technical reasons, it is reasonable to assume that these functions decrease by the following reasons. First, if $\underline{S}(t)$ (resp. $\underline{H}(t)$) converges to a nonzero value as $t \to \infty$, S (resp. H) must be unbounded, and thus it is irrelevant. Let us see this regarding S. Define $\boldsymbol{p}(t) := \mathbf{1}$ for $0 \le t < \tau$ and $\boldsymbol{p}(t) := \mathbf{0}$ otherwise, where $\mathbf{1}$ is a vector whose elements are all equal to one. If this continuous-time signal $\boldsymbol{p}(t)$ is given to S, the obtained output $\boldsymbol{p}_{\rm d}[k]$ is

$$\boldsymbol{p}_{\rm d}[k] = \int_0^\tau \underline{S}(k\tau - t)\mathbf{1}\,{\rm d}t.$$

Hence, if $\underline{S}(t)$ converges to a nonzero value, $\boldsymbol{p}_{\rm d}[k]$ does not approach zero no matter how large k becomes. In this sense, the effect of a nonzero input during $0 \le t < \tau$ remains in the output forever. Therefore S is unbounded. The proof is similar regarding H. Moreover, it is natural to assume decrease of kernel functions considering realization of S and H as practical devices. Indeed, decrease of a kernel function means that the effect of an input at some particular time gradually decreases in the output as the time passes by. □

In Section 2.4, lifting-based transfer functions were considered for sampler-type and hold-type operators. Let us pay special attention to regular samplers and holds and investigate their lifting-based transfer functions.

By the definition of a regular sampler $S: \boldsymbol{p} \mapsto \boldsymbol{p}_{\rm d}$, there holds

$$\boldsymbol{p}_{\rm d}[k] = \int_0^{k\tau} \underline{S}(k\tau - t)\boldsymbol{p}(t)\,{\rm d}t = \sum_{\ell=0}^{k-1} \int_0^\tau \underline{S}((k-\ell)\tau - t)\boldsymbol{p}(\ell\tau + t)\,{\rm d}t.$$

Here, define the flat operators S_k, $k = 0, 1, \dots$, by

$$S_k \boldsymbol{f} := \int_0^\tau \underline{S}(k\tau - t)\boldsymbol{f}(t)\,{\rm d}t \tag{3.3}$$

for $\boldsymbol{f} \in \mathcal{L}^2[0, \tau)$. Here, we adopt the convention $\underline{S}(t) = O$ for $t < 0$, which implies $S_0 = O$. Moreover, write $\tilde{\boldsymbol{p}} := W_\tau \boldsymbol{p}$, that is, $\tilde{\boldsymbol{p}}[\ell](t) := \boldsymbol{p}(\ell t + t)$ for $\ell = 0, 1, \dots$. Then, we have

$$\boldsymbol{p}_{\rm d}[k] = \sum_{\ell=0}^{k-1} S_{k-\ell}\tilde{\boldsymbol{p}}[\ell]. \tag{3.4}$$

Therefore, the lifting-based transfer function of S is given by

$$\breve{S}(z) := \sum_{k=0}^{\infty} S_k z^{-k}$$

with S_k defined in (3.3) (see Definition 2.20).

A similar discussion is possible with respect to a regular hold H. Namely, define tall operators H_k, $k = 0, 1, \ldots$, by

$$(H_k \boldsymbol{v})(t) := \underline{H}(k\tau + t)\boldsymbol{v}, \tag{3.5}$$

where $\boldsymbol{v}$ is a vector and t runs over $[0, \tau)$. Then, the operation of a regular hold $H : \boldsymbol{q}_{\mathrm{d}} \mapsto \boldsymbol{q}$ is expressed as

$$\tilde{\boldsymbol{q}}[k] = \sum_{\ell=0}^{k} H_{k-\ell}\, \boldsymbol{q}_{\mathrm{d}}[\ell],$$

where $\tilde{\boldsymbol{q}} := W_\tau \boldsymbol{q}$. Hence, with H_k in (3.5), the lifting-based transfer function of H is defined as

$$\breve{H}(z) := \sum_{k=0}^{\infty} H_k z^{-k}$$

in accordance with Definition 2.21.

Remark 3.12. There holds $S_0 = O$ while $H_0 = O$ does not hold in general. We need the equality $S_0 = O$ in order to guarantee causality of a sampler S. Indeed, if $S_0 \neq O$, $\boldsymbol{p}_{\mathrm{d}}[k]$ depends on $\tilde{\boldsymbol{p}}[k]$ by (3.4); in a word, the output of S at the discrete time k depends on the future input given in $k\tau \leq t < (k+1)\tau$. □

From the explicit forms of S_k and H_k, the next proposition follows, which is about lifting-based transfer functions of regular samplers and holds.

Proposition 3.13. *For a regular sampler S, there exists $0 < \rho_0 < 1$ such that its lifting-based transfer function $\breve{S}(z)$ is analytic in $z \in \mathbb{D}_{\rho_0}$.*

Similarly for a regular hold H, we can choose $0 < \rho_0 < 1$ so that its lifting-based transfer function $\breve{H}(z)$ is analytic in $z \in \mathbb{D}_{\rho_0}$.

Particularly, $\breve{S}$ belongs to $\mathfrak{H}_{\mathrm{F}}^{\infty}$ and $\breve{H}$ belongs to $\mathfrak{H}_{\mathrm{T}}^{\infty}$. Moreover, $\breve{S}(\infty) = O$.

Proof. By the definition of S_k (3.3) and the Schwarz inequality, there holds

$$\|S_k\|_{\mathrm{F}}^2 \leq \int_0^{\tau} \bar{\sigma}\{\underline{S}(k\tau - t)\}^2\, \mathrm{d}t = \int_{(k-1)\tau}^{k\tau} \bar{\sigma}\{\underline{S}(t)\}^2\, \mathrm{d}t. \tag{3.6}$$

Since each element of $\underline{S}(t)$ belongs to $\mathcal{D}$, there exist $\epsilon > 0$ and $V > 0$ such that $\int_0^\infty \mathrm{e}^{2\epsilon t}\bar{\sigma}\{\underline{S}(t)\}^2\, \mathrm{d}t < V^2$. Then, it is derived that

$$\mathrm{e}^{2\epsilon(k-1)\tau} \int_{(k-1)\tau}^{k\tau} \bar{\sigma}\{\underline{S}(t)\}^2\, \mathrm{d}t \leq \int_{(k-1)\tau}^{k\tau} \mathrm{e}^{2\epsilon t}\bar{\sigma}\{\underline{S}(t)\}^2\, \mathrm{d}t < V^2.$$

Combining this with (3.6), we obtain the bound $\|S_k\|_{\mathrm{F}}^2 < \mathrm{e}^{-2\epsilon(k-1)\tau}V^2$. Hence, $\breve{S}(z) = \sum_{k=0}^{\infty} S_k z^{-k}$ absolutely converges for $|z^{-1}| < \mathrm{e}^{\epsilon\tau}$. This means that there exists $0 < \rho_0 < 1$

such that $\breve{S}(z)$ is analytic in $z \in \mathbb{D}_{\rho_0}$. In particular, $\breve{S}(z)$ is analytic and uniformly bounded in $\mathbb{D}$; hence $\breve{S} \in \mathfrak{H}_{\mathrm{F}}^{\infty}$. It is easy to see $\breve{S}(\infty) = S_0 = O$.

In a similar way, the results on H can be derived. □

For notational convenience, let us introduce the following notation. Recall that $\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$ is a subspace of $\mathfrak{H}_{\mathrm{F}}^{\infty}$ that consists of real rational functions only (Section 2.4.3).

Definition 3.14. Let $z^{-1}\mathfrak{H}_{\mathrm{F}}^{\infty}$ denote the set of functions having the form of $z^{-1}A(z)$ with $A \in \mathfrak{H}_{\mathrm{F}}^{\infty}$. Similarly, let $z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$ denote the set of functions with the form of $z^{-1}A(z)$ for some $A \in \mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$. □

With this notation, the claim of Proposition 3.13 can be restated as $\breve{S} \in z^{-1}\mathfrak{H}_{\mathrm{F}}^{\infty}$.

Combining the above proposition 3.13 with Proposition 2.22, we can immediately obtain the next result.

Proposition 3.15. *A regular sampler S has a finite induced norm and satisfies $\|S\|_{\mathcal{L}^2 \to \ell^2} = \|\breve{S}\|_{\mathfrak{H}_{\mathrm{F}}^{\infty}}$. Likewise, a regular hold H has a finite induced norm and satisfies $\|H\|_{\ell^2 \to \mathcal{L}^2} = \|\breve{H}\|_{\mathfrak{H}_{\mathrm{T}}^{\infty}}$.*

From Proposition 3.15 together with Proposition 2.23, it follows that, if a regular sampler S (resp. a regular hold H) has a lifting-based state-space representation, its lifting-based transfer function $\breve{S}(z)$ (resp. $\breve{H}(z)$) belongs to $z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$ (resp. $\mathfrak{R}\mathfrak{H}_{\mathrm{T}}^{\infty}$). The converse is true in a slightly stronger form. The result is summarized as follows.

Proposition 3.16. *A sampler-type operator S is a regular sampler and has a lifting-based state-space representation if and only if S has a lifting-based transfer function belonging to $z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$. A hold-type operator H is a regular hold and has a lifting-based state-space representation if and only if H has a lifting-based transfer function that belongs to $\mathfrak{R}\mathfrak{H}_{\mathrm{T}}^{\infty}$.*

Proof. Let us prove the "only if" part for a sampler-type operator S. Suppose that S is a regular sampler and has a lifting-based state-space representation. Then, by Proposition 3.15, S is bounded. Applying Proposition 2.23, we see that S has a lifting-based transfer function belonging to $z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$.

Next, the "if" part is proven. If S has a lifting-based transfer function in $z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty} \subset \mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$, S is bounded and has a lifting-based state-space representation by virtue of Proposition 2.23. Let us write this state-space representation as $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, where $\tilde{A}$ and $\tilde{C}$ are real matrices and $\tilde{B}$ and $\tilde{D}$ are real flat operators. Here, $\breve{S}(z) = \tilde{D} + \tilde{C}(zI - \tilde{A})^{-1}\tilde{B}$. Since $\breve{S} \in z^{-1}\mathfrak{R}\mathfrak{H}_{\mathrm{F}}^{\infty}$, $\tilde{D} = O$. Moreover, it can be assumed that $\tilde{A}$ has its eigenvalues only in $|z| < 1$. (If it is not the case, we can get rid of all the eigenvalues in $|z| \geq 1$ by obtaining a minimal state-space representation of S.) Since $\tilde{B}$ is a linear operator mapping $\mathcal{L}^2[0, \tau)$ to a finite-dimensional vector space, the Riesz representation theorem implies that there exists a real function $\underline{B} \in \mathcal{L}^2[0, \tau)$ such that

$$\tilde{B}\boldsymbol{f} = \int_0^{\tau} \underline{B}(\tau - t)\boldsymbol{f}(t)\, \mathrm{d}t \quad \text{for any } \boldsymbol{f} \in \mathcal{L}^2[0, \tau).$$

Define a function $\underline{S}(t)$ by $\underline{S}(k\tau+t) := \widetilde{C}\widetilde{A}^k\underline{B}(t)$ for $k=0,1,\dots$ and $0 \le t < \tau$. Then, this $\underline{S}(t)$ turns out to be the kernel function of S. Since all the eigenvalues of $\widetilde{A}$ are located in $|z|<1$, each element of $\underline{S}(t)$ belongs to $\mathcal{D}$. Hence, S is a regular sampler.

As for a hold-type operator H the claim is proven similarly. □

Here, we present an easily testable sufficient condition in order that a regular sampler and a regular hold have their lifting-based state-space representations.

Proposition 3.17. *Suppose that S is a regular sampler. If its kernel function $\underline{S}(t)$ has a bounded support or if the Laplace transform of $\underline{S}(t)$ is rational, then S has a lifting-based state-space representation.*

Suppose that H is a regular hold. If its kernel function $\underline{H}(t)$ has a bounded support or if the Laplace transform of $\underline{H}(t)$ is rational, then H has a lifting-based state-space representation.

Proof. Suppose that $\underline{S}(t)$ has a bounded support. Then, there exists $k_0>0$ such that $S_k=O$ for any $k>k_0$. This means that $\check{S}(z)$ is equal to $\sum_{k=0}^{k_0} S_k z^{-k}$, which is rational. Next, suppose that the Laplace transform of the kernel function $\underline{S}(t)$ is a rational function. Then, this Laplace transform $\widehat{\underline{S}}(s)$ is expressed as $D+C(sI-A)^{-1}B$ using real matrices A, B, C, and D. In this case, we have $\underline{S}(t) = D + Ce^{At}B$. Since $\underline{S}$ belongs to $\mathcal{D}$ by assumption, D must be a zero matrix and we can assume A has its eigenvalues only in $\mathrm{Re}\, s<0$ without loss of generality. Define a flat operator $\widetilde{B}$ so that

$$\widetilde{B}\boldsymbol{f} := \int_0^\tau e^{A(\tau-t)}B\boldsymbol{f}(t)\,dt \tag{3.7}$$

for any $\boldsymbol{f}\in\mathcal{L}^2[0,\tau)$. Then, there hold $S_k = Ce^{A(k-1)\tau}\widetilde{B}$ for $k=1,2,\dots$ and $S_0=O$. This implies that $\check{S}(z) = C(zI-e^{A\tau})^{-1}\widetilde{B} = z^{-1}C(I-z^{-1}e^{A\tau})^{-1}\widetilde{B}$; hence $\check{S}\in z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$.

The proof is similar for a hold. Especially, once we obtain the form $\underline{H}(t)=Ce^{At}B$, define tall operators $\widetilde{C}$ and $\widetilde{D}$ by

$$(\widetilde{C}\boldsymbol{v})(t) := Ce^{A(\tau+t)}\boldsymbol{v} \qquad \text{and} \qquad (\widetilde{D}\boldsymbol{u})(t) := Ce^{At}B\boldsymbol{u}. \tag{3.8}$$

Then, there holds $\check{H}(z) = \widetilde{D}+\widetilde{C}(zI-e^{A\tau})^{-1}B$. Hence, $\check{H}\in\mathfrak{RH}_{\mathrm{T}}^\infty$. □

Note that the above proof gives explicit forms for the lifting-based state-space representations of S and H, when their kernel functions are represented as $Ce^{At}B$. Namely, if $\underline{S}(t)=Ce^{At}B$, the lifting-based state-space representation of S is $(e^{A\tau},\widetilde{B},C,O)$, where $\widetilde{B}$ is as in (3.7); if $\underline{H}(t)=Ce^{A\tau}B$, the lifting-based state-space representation of H is $(e^{A\tau},B,\widetilde{C},\widetilde{D})$ with $\widetilde{C}$ and $\widetilde{D}$ being as in (3.8).

From Proposition 3.17, we can see that all the samplers and holds in Examples 3.3–3.9 satisfy $\check{S}\in z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\check{H}\in\mathfrak{RH}_{\mathrm{T}}^\infty$. This suggests that many of practically important samplers and holds can be covered by the classes $z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\mathfrak{RH}_{\mathrm{T}}^\infty$, respectively.

In the case that a continuous-time operator P has a continuous-time state-space representation, an explicit formula can be obtained for a matrix representation of the lifting-based transfer function of P (Proposition 2.18). A similar thing is possible with respect to a regular sampler and a regular hold. Here, we need a tall operator $\grave{E}_m^s$ and a flat operator $\acute{E}_m^s$, which were defined in Definition 2.25.

Proposition 3.18. *Let S be a regular sampler and let $\check{S}(z)$ be analytic in $\mathbb{D}_{\rho_0}$ for $0<\rho_0<1$. Then, for any complex number s such that $e^{s\tau}\in\mathbb{D}_{\rho_0}$, there holds*

$$\check{S}(e^{s\tau})\grave{E}_m^s = \frac{1}{\sqrt{\tau}}\widehat{\underline{S}}\Big(s+\frac{i2\pi m}{\tau}\Big)$$

for $m=0,\pm1,\pm2,\dots$, where $\widehat{\underline{S}}(s)$ is the Laplace transform of the kernel function $\underline{S}(t)$.

Suppose that H is a regular hold and $\check{H}(z)$ is analytic in $\mathbb{D}_{\rho_0}$ for $0<\rho_0<1$. Then, for any complex number satisfying $e^{s\tau}\in\mathbb{D}_{\rho_0}$, there holds

$$\acute{E}_m^s\check{H}(e^{s\tau}) = \frac{1}{\sqrt{\tau}}\widehat{\underline{H}}\Big(s+\frac{i2\pi m}{\tau}\Big)$$

for $m=0,\pm1,\pm2,\dots$, where $\widehat{\underline{H}}(s)$ is the Laplace transform of the kernel function $\underline{H}(t)$.

Writing this proposition in a matrix form, we have

$$\check{S}(e^{s\tau}) \sim \frac{1}{\sqrt{\tau}}\begin{bmatrix}\cdots & \widehat{\underline{S}}\big(s-\frac{i2\pi}{\tau}\big) & \widehat{\underline{S}}(s) & \widehat{\underline{S}}\big(s+\frac{i2\pi}{\tau}\big) & \cdots\end{bmatrix},$$

$$\check{H}(e^{s\tau}) \sim \frac{1}{\sqrt{\tau}}\begin{bmatrix}\vdots \\ \widehat{\underline{H}}\big(s-\frac{i2\pi}{\tau}\big) \\ \widehat{\underline{H}}(s) \\ \widehat{\underline{H}}\big(s+\frac{i2\pi}{\tau}\big) \\ \vdots\end{bmatrix}.$$

These formulas are quite useful to investigate sampled-data control systems and play an important role in the sequel. Similar formulas were presented in [4, 3, 5, 42] under the name of "FR-operators." However, their class of samplers and holds is smaller than ours and they considered only the case that s is a pure imaginary number.

3.3. A Structure of Sampled-Data Control Systems

Figure 3.5 shows a **sampled-data control system** considered in this thesis. It is made of four operators G, K_{d}, S, and H and signals connecting them. The signals shown by solid arrows are continuous-time signals, while those shown by broken arrows are discrete-time signals. All the signals may be multi-dimensional. As is seen from the figure, G is a continuous-time operator, K_{d} is a discrete-time operator, S is a sampler-type operator, and H is a hold-type operator. This system configuration is quite standard in the recent sampled-data control literature (see [9, 86, 20, 27] for example).

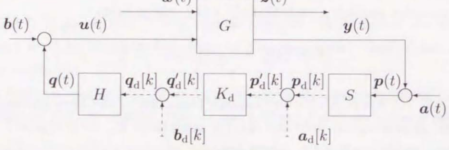


Figure 3.5. A sampled-data control system.

A continuous-time operator G is a generalized plant and is defined exactly in the same way as in Section 2.5. Especially, the dimension of the signals $\boldsymbol{w}(t)$, $\boldsymbol{u}(t)$, $\boldsymbol{z}(t)$, and $\boldsymbol{y}(t)$ are written as n_w, n_u, n_z, and n_y. Besides, G is divided into four operators so that

$$\boldsymbol{z} = G_{11}\boldsymbol{w} + G_{12}\boldsymbol{u},$$
$$\boldsymbol{y} = G_{21}\boldsymbol{w} + G_{22}\boldsymbol{u}.$$

A positive number τ is associated with the sampled-data control system. An operator $S : \boldsymbol{p} \mapsto \boldsymbol{p}_{\mathrm{d}}$ is a regular sampler with this τ being its sampling period. The dimension of its input $\boldsymbol{p}(t)$ is denoted by n_p and that of its output $\boldsymbol{p}_{\mathrm{d}}[k]$ is denoted by n_p^{d}. On the other hand, an operator $H : \boldsymbol{q}_{\mathrm{d}} \mapsto \boldsymbol{q}$ is a regular hold with τ being its sampling period. The dimension of its input $\boldsymbol{q}_{\mathrm{d}}[k]$ is written as n_q^{d} and that of its output $\boldsymbol{q}(t)$ is written as n_q.

Here, some terminology is introduced.

Definition 3.19. Suppose that τ is a positive number, S is a regular sampler with the sampling period τ, and H is a regular hold with the sampling period τ. Then, the triplet (τ, S, H) is called a **sampling environment**. □

Definition 3.20. Suppose that a generalized plant G and a sampling environment (τ, S, H) are provided. If there hold $n_u = n_q$ and $n_y = n_p$, G and (τ, S, H) are said to be **consistent** with each other. □

If a sampling environment is consistent with a provided G, then a sampler S and a hold H have input- and output-signal dimensions that match those of G, so that S and H can be connected

to G. Once a sampling environment (τ, S, H), which is consistent with G, is fixed, what is left for us is only to choose K_{d}. In this sense, (τ, S, H) prepares an environment to make a sampled-data control system. This is the reason why (τ, S, H) is called a sampling environment.

A discrete-time operator K_{d} is called a **discrete-time controller**. It is assumed to be chosen from a set $\mathcal{K}_{\mathrm{d}}$. The set $\mathcal{K}_{\mathrm{d}}$ is defined to be composed of all discrete-time operators that have discrete-time state-space representations, n_p^{d}-dimensional input, and n_q^{d}-dimensional output. Moreover, the operator composition $HK_{\mathrm{d}}S$ is called a **sampled-data controller** as a whole.

Remark 3.21. In view of design flexibility left to an engineer, a sampling environment (τ, S, H) is located between a plant G and a discrete-time controller K_{d}. A plant G is provided to an engineer as fixed; a discrete-time controller K_{d} can be chosen almost freely by an engineer, though practically some constraints are posed on the choice of K_{d} because of a cost, technical difficulty, and so on. On the other hand, a sampling environment (τ, S, H) has some design flexibility but it is not so flexible as a discrete-time controller K_{d}. For example, a sampling period τ cannot be made too small because devices with a small sampling period cost much. Moreover, we cannot assign a too complicated function to a sampler S and a hold H. This is because τ is chosen small usually and the operations of S and H are to integrate and interpolate signals, respectively, in the time range shorter than τ. □

Next, stability of this sampled-data control system is considered. The signals $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{a}_{\mathrm{d}}[k]$, and $\boldsymbol{b}_{\mathrm{d}}[k]$ are fictitious inputs introduced to define stability of this system. Here, we also need lifting-based transfer functions of G_{22}, S, H, and K_{d}, which are written as $\breve{G}_{22}(z)$, $\breve{S}(z)$, $\breve{H}(z)$, and $\breve{K}_{\mathrm{d}}(z)$, respectively. Note that $\breve{S}\breve{G}_{22}\breve{H}\breve{K}_{\mathrm{d}}$ gives a matrix-valued function.

Definition 3.22. With respect to a sampled-data control system in Figure 3.5, suppose that the function $\det(I - \breve{S}\breve{G}_{22}\breve{H}\breve{K}_{\mathrm{d}})$ takes a nonzero value at least at one point $z \in \mathbb{D}$, and all the 25 operators mapping $\boldsymbol{w}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{a}(t)$, $\boldsymbol{a}_{\mathrm{d}}[k]$, $\boldsymbol{b}_{\mathrm{d}}[k]$ to $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{p}_{\mathrm{d}}[k]$, $\boldsymbol{q}'_{\mathrm{d}}[k]$, $\boldsymbol{q}(t)$ are bounded in respect of their appropriate induced norms. Then, this sampled-data control system is called **input-output stable** or just **stable**. □

Remark 3.23. Just like the case of a continuous-time control system, it is conjectured that, under some condition, we can decrease the number of operators whose stability should be checked. A clue to consider this problem can be found in the work of Francis and Georgiou [35]. That is, using the notion of a *non-pathological sampling period*, they gave a condition in order that the stability of some special sampled-data control systems can be checked by their observation at sampling instants only. It is expected that by generalizing this notion we can obtain a condition to decrease the number of operators. The research is now proceeding in this direction. (See [65] for another attempt to generalize this notion.) □

Usually, we want the system not only to be stable but also to possess a good performance. In a standard formulation of a sampled-data $\mathfrak{H}^\infty$-control problem [11, 9, 86, 89, 56, 83, 48, 20], the system performance is measured by the $\mathcal{L}^2$-induced norm of the operator that maps $\boldsymbol{w}(t)$ to $\boldsymbol{z}(t)$. Using the symbol of the lower fractional transform $\mathcal{F}(\cdot,\cdot)$, which was introduced in Section 2.5, this norm is expressed as $\|\mathcal{F}(G, HK_\mathrm{d}S)\|$. The symbol $\|\cdot\|$ denotes the $\mathcal{L}^2$-induced norm. Just as in Section 2.5, we define that this norm is equal to infinity when the considered sampled-data system is not input-output stable. In a standard formulation, the smaller the value of $\|\mathcal{F}(G, HK_\mathrm{d}S)\|$ is, the better the system performance is. Hence, the **best achievable performance of sampled-data control systems** (or **the best sampled-data control performance** in short) with respect to a provided G and (τ, S, H) is expressed as

$$\inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\|.$$

Unlike the case of a continuous-time control system, it is possible that this value is infinite. In other words, there is a case that there exists no discrete-time controller $K_\mathrm{d} \in \mathcal{K}_\mathrm{d}$ that input-output stabilizes the system.

For an input-output stable sampled-data control system, a lifting-based transfer function is well-defined for each of the 25 operators above.

Proposition 3.24. *For an input-output stable sampled-data control system, each of the 25 operators that were used to define input-output stability has a lifting-based transfer function. Moreover, there exists $0 < \rho < 1$ such that these 25 lifting-based transfer functions together with $\check{S}(z)$, $\check{H}(z)$ are analytic in $\mathbb{D}_\rho$.*

For its proof, see Appendix E. By this proposition, each of these 25 lifting-based transfer functions belongs to either $\mathfrak{H}^\infty_\mathrm{L}$, $\mathfrak{H}^\infty_\mathrm{F}$, $\mathfrak{H}^\infty_\mathrm{T}$, or $\mathfrak{H}^\infty$ depending on its function type.

Let us formulate a sampled-data control system considered in Section 1.1 into the standard form.

Example 3.25. In Example 1.3, robust stabilization by means of sampled-data control was considered. The diagram of the considered system is redrawn in Figure 3.6. (This is essentially the same figure as Figure 1.8 (b).) Here, P is a continuous-time operator having a continuous-time state-space representation; W is an operator such that PW is a continuous-time operator having a continuous-time state-space representation; F is a bounded continuous-time operator having a strictly proper continuous-time state-space representation. This P stands for a plant to be controlled; W is a weight to express how the amount of uncertainty included in the plant model depends on the frequency; F stands for an anti-aliasing filter. Suppose that a typical sampler and hold are chosen here, that is, the ideal sampler S^id_τ and the zero-order hold H^zo_τ both having the sampling period τ. (See Examples 3.3 and 3.6 for definitions of these devices.)

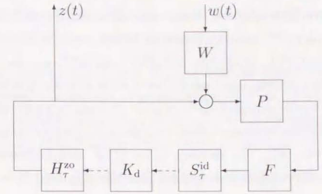


Figure 3.6. The system considered in Example 1.3. The $\mathcal{L}^2$-induced norm of the operator from $w(t)$ to $z(t)$ should be reduced for robust stabilization of this system.

It is desired to minimize the $\mathcal{L}^2$-induced norm of the operator from $w(t)$ to $z(t)$ for robust stabilization.

Now, let us formulate this robust stability problem into the standard configuration introduced above. This is achieved by putting a generalized plant G as

$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} O & I \\ PW & P \end{bmatrix}$$

and defining a sampler S to be $S^\mathrm{id}_\tau F$ and a hold H to be H^zo_τ. Then, since $S = S^\mathrm{id}_\tau F$ is a regular sampler and $H = H^\mathrm{zo}_\tau$ is a regular hold by Examples 3.3 and 3.6, the triplet (τ, S, H) is a sampling environment. The operator from $w(t)$ to $z(t)$ is expressed by $\mathcal{F}(G, HK_\mathrm{d}S)$. Therefore, the best achievable performance in the sense of robust stability is exactly equal to $\inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\|$, that is, the best achievable performance in the standard configuration. □

In general, a system is called a **multirate sampled-data control system** if each of a sampler, a hold, and a discrete-time controller works with its own time period and there is not necessarily one unified sampling period. By adopting the least common multiple of the periods of the sampler, the hold, and the discrete-time controller as the unified sampling period, we can regard a multirate system as a usual single-rate sampled-data control system. In this case, the sampler and the hold are interpreted to have general functions in the sense that they produce and receive multiple discrete-time signals in one sampling period. See [63, 58, 6, 41, 67, 43, 33] for earlier results on multirate sampled-data control systems, and see [93, 21, 81] for their lifting-based treatment.

Some types of multirate sampled-data control systems can be treated in our framework.

Example 3.26. In the system in Figure 3.6, replace the ideal sampler S_τ^{id} by the multirate sampler S_τ^{mr} considered in Example 3.4. While the ideal sampler S_τ^{id} samples the input signal only at $t = k\tau$, this sampler S_τ^{mr} samples it also at $t = (k-1/2)\tau$. In other words, this sampler S_τ^{mr} works with the period $\tau/2$. Since the included hold H_τ^{zo} and the discrete-time controller K_{d} have the period τ, this system has two different time periods inside. Accordingly, this is a special example of a multirate sampled-data control system. In particular, this system belongs to a special class of multirate systems called *two-delay control systems* and is known to have some interesting properties [67]. By putting $S := S_\tau^{\mathrm{mr}} F$ and following the procedure of the previous example, we can formulate this system into the standard form again. □

Example 3.27. In the system in Figure 3.6, replace not only S_τ^{id} by S_τ^{mr} but also the zero-order hold H_τ^{zo} by the multirate hold H_τ^{mr}, which was considered in Example 3.9. The resulting system is another example of a multirate sampled-data control system because a sampler and a hold work with the period $\tau/2$ while a discrete-time controller has the period τ. Just as the preceding two examples, this system can be formulated into the standard form by putting $S := S_\tau^{\mathrm{mr}} F$ and $H := H_\tau^{\mathrm{mr}}$. □

Remark 3.28. Let us consider the sampled-data controller $HK_{\mathrm{d}}S = H_\tau^{\mathrm{mr}} K_{\mathrm{d}} S_\tau^{\mathrm{mr}} F$, which was obtained in Example 3.27. By the definition of H_τ^{mr}, its output during the time $k\tau \le t < (k+1)\tau$ is produced from the output of K_{d} at the time $t = k\tau$. Moreover, the input signal that arrives at S after $t = k\tau$ is not sent to K_{d} until $t = (k+1)\tau$ because the kernel function of S, i.e., $\underline{S}(t)$, is equal to zero in $t < 0$. In summary, the input signal given to $HK_{\mathrm{d}}S$ after $t = k\tau$ does not affect the output of $HK_{\mathrm{d}}S$ until $t = (k+1)\tau$. However, in a general formulation of multirate sampled-data control systems, it is allowed that the input to $HK_{\mathrm{d}}S$ during $k\tau \le t < (k+1)\tau$ is reflected in its output before $t = (k+1)\tau$ [6, 41, 43, 93, 21, 81]. In this sense, our framework for sampled-data control systems do not cover general multirate systems. It is considered that this problem is will be resolved if the kernel function $\underline{S}(t)$ is allowed to have a nonzero value in $t < 0$. However, it is not clear how we can consistently extend our framework in this direction. This is an interesting topic and left as a future research theme. □

At the end of this section, a relationship to the result of Mirkin and Rotstein [66] is discussed. In their paper, they assumed that a sampler has the form of (3.2) and a hold is represented as in (3.1). In our terminology, they allowed kernel functions to have nonzero values only in $[0, \tau)$, which means their samplers and holds are *individually* quite special compared with ours. Nevertheless, the class of *sampled-data controllers* constructed from their samplers and holds is the same as our corresponding class. This is a consequence of their main result Theorem 1 and is formally stated as follows.

Proposition 3.29. *Suppose that a regular sampler S and a regular hold H have their lifting-based state-space representations and a discrete-time controller K_{d} belongs to the set $\mathcal{K}_{\mathrm{d}}$. Then, the sampled-data controller $HK_{\mathrm{d}}S$ can be expressed as $H'K'_{\mathrm{d}}S'$ so that K'_{d} is a discrete-time controller belonging to $\mathcal{K}_{\mathrm{d}}$ and S' and H' are a regular sampler and a regular hold, respectively, whose kernel functions have nonzero values only in $[0, \tau)$.*

Proof. By assumption, each of K_{d}, S, and H has its lifting-based state-space representation. Moreover, the lifting-based state-space representation of S is strictly proper. This implies that the sampled-data controller $HK_{\mathrm{d}}S$ has a lifting-based state-space representation and it is strictly proper. Specifically, $HK_{\mathrm{d}}S$ has a lifting-based transfer function of the form

$$\tilde{C}(zI - \tilde{A})^{-1}\tilde{B} = \tilde{C}\{I + (zI - \tilde{A})^{-1}\tilde{A}\}z^{-1}\tilde{B},$$

where $\tilde{A}$ is a matrix, $\tilde{B}$ is a flat operator, and $\tilde{C}$ is a tall operator. Now, define a discrete-time operator K'_{d}, a sampler-type operator S', and a hold-type operator H' by giving their lifting-based transfer functions as

$$\breve{K}'_{\mathrm{d}}(z) := I + (zI - \tilde{A})^{-1}\tilde{A}, \qquad \breve{S}'(z) := z^{-1}\tilde{B}, \qquad \breve{H}'(z) := \tilde{C}.$$

Then, it is easy to see that K'_{d}, S', and H' are the desired operators. □

Significance to define $\underline{S}(t)$ and $\underline{H}(t)$ on $[0, \infty)$ rather than $[0, \tau)$ is not lost because of this result. Proposition 3.29 only claims that combination of a sampler, a discrete-time controller, and a hold can be expressed by a combination of special devices whose kernel functions are defined only on $[0, \tau)$. However, in order to analyze a sampler and a hold themselves, it is desirable that they can be expressed by a single operators, respectively, not by a combination of multiple operators. Furthermore, it is possible to interpret an anti-aliasing filter as a part of a sampler only when the kernel function $\underline{S}(t)$ is defined on $[0, \infty)$ as we saw in Example 3.3.

3.4. Basic Properties of Sampled-Data Control Systems

In this section, some basic properties of sampled-data control systems are derived. These properties are shown to be quite useful in the next chapter. By applying technical tools such as lifting-based transfer functions and their matrix representations, we can obtain these properties.

First, we present properties of functions belonging to the set $\mathcal{D}$. Because the kernel functions of a regular sampler and a regular hold belong to this set, these properties are important for the subsequent analysis.

Proposition 3.30. *Suppose that a function $a(t)$ is an element of $\mathcal{D}$ and $e^{\epsilon t}a(t)$ belongs to $\mathcal{L}^2$ with $\epsilon > 0$. Write the Laplace transform of $a(t)$ as $\hat{a}(s)$. Then, the following properties hold.*

(a) The function $\hat{a}(s - \epsilon)$ belongs to the Hardy space $\mathcal{H}^2$.

(b) *In the half plane* $\operatorname{Re} s \geq 0$, *the function* $\widehat{a}(s)$ *converges to zero uniformly as* $|s|$ *approaches infinity.*

(c) *Let* B *be any bounded closed set that is contained in the open half plane* $\operatorname{Re} s > -\epsilon$. *Then, the infinite series* $\sum_{m=-\infty}^{\infty} |\widehat{a}(s + \mathrm{i}2\pi m/\tau)|^2$ *converges uniformly for all* $s \in B$.

Proof. Note that the Laplace transform of $\mathrm{e}^{\epsilon t}a(t)$ is $\widehat{a}(s-\epsilon)$ by a property of the Laplace transform. Then, Property (a) immediately follows from the equivalence between $\mathcal{L}^2$ and $\mathcal{H}^2$ stated in Proposition 2.1. Moreover, applying Proposition 2.3 to the $\mathcal{H}^2$-function $\widehat{a}(s-\epsilon)$, we obtain Properties (b) and (c). □

In a sampled-data control system, an operator composition $SG_{22}H$ is a discrete-time operator since its input and output are discrete-time signals. This operator has the next important property.

Theorem 3.31. *Consider a sampled-data control system in Figure 3.5 and let* $0 < \rho_0 < 1$ *be a number such that* $\breve{S}(z)$ *and* $\breve{H}(z)$ *are analytic in* $\mathbb{D}_{\rho_0}$. *Moreover, let* s *be any complex number such that* $\mathrm{e}^{s\tau} \in \mathbb{D}_{\rho_0}$ *and none of* $s + \mathrm{i}2\pi m/\tau$, $m = 0, \pm 1, \ldots$, *is a pole of* $\widehat{G}_{22}(s)$. *Then, there holds*

$$(SG_{22}H)^{\smile}(\mathrm{e}^{s\tau}) = \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \underline{\widehat{S}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \widehat{G}_{22}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \underline{\widehat{H}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big). \tag{3.9}$$

Note that such a number ρ_0 always exists by Proposition 3.13. The formula (3.9) is known as a consequence of an impulse modulation formula and is used as a basis of sampled-data systems analysis in [4, 3, 5], for example. In [12], this formula is called a key sampling formula and is proven rigorously. However, in [4, 3, 5], the complex number s is restricted on the imaginary axis. Besides, the proof in [12] requires the assumption that $\operatorname{Re} s$ is larger than the real part of any pole of $\widehat{G}_{22}(s)$. Theorem 3.31 holds without such assumptions and this fact is essential when this theorem is applied in the proof of Theorem 3.34, which is the main result of this chapter. (Another proof, which is more general than that of [12], is found in [71] though it treated only the case that the ideal sampler and the zero-order hold are equipped.)

Proof. Note that Proposition 2.24 implies $(SG_{22}H)^{\smile}(\mathrm{e}^{s\tau}) = \breve{S}(\mathrm{e}^{s\tau})\breve{G}_{22}(\mathrm{e}^{s\tau})\breve{H}(\mathrm{e}^{s\tau})$. By the assumptions on s, Propositions 2.28 and 3.18 can be applied and the following formulas are obtained:

$$\acute{E}^s_m \breve{G}_{22}(\mathrm{e}^{s\tau}) \grave{E}^s_\ell = \widehat{G}_{22}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big)\delta_{m,\ell}, \tag{3.10a}$$

$$\breve{S}(\mathrm{e}^{s\tau})\grave{E}^s_m = \frac{1}{\sqrt{\tau}}\underline{\widehat{S}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big), \tag{3.10b}$$

$$\acute{E}^s_\ell \breve{H}(\mathrm{e}^{s\tau}) = \frac{1}{\sqrt{\tau}}\underline{\widehat{H}}\Big(s + \frac{\mathrm{i}2\pi \ell}{\tau}\Big). \tag{3.10c}$$

Here, $\acute{E}^s_m$ and $\grave{E}^s_\ell$ are as defined in Definition 2.25.

In the following, dependence on $\mathrm{e}^{s\tau}$ is not described explicitly. Combining the above formulas and applying Propositions 2.26, we obtain that

$$\begin{aligned}(SG_{22}H)^{\smile}(\mathrm{e}^{s\tau}) &= \breve{S}\left(\sum_{m=-\infty}^{\infty} \grave{E}^s_m \acute{E}^s_m\right)\breve{G}_{22}\left(\sum_{\ell=-\infty}^{\infty} \grave{E}^s_\ell \acute{E}^s_\ell\right)\breve{H} \\ &= \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} (\breve{S}\grave{E}^s_m)(\acute{E}^s_m \breve{G}_{22}\grave{E}^s_\ell)(\acute{E}^s_\ell \breve{H}) \\ &= \frac{1}{\tau}\sum_{m=-\infty}^{\infty} \underline{\widehat{S}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \widehat{G}_{22}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \underline{\widehat{H}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big).\end{aligned}$$

Each infinite series above converges strongly. In particular, the last series converges also *absolutely* as a matrix series because $\widehat{G}_{22}(s + \mathrm{i}2\pi m/\tau)$ is uniformly bounded for any integer m and each of $\underline{\widehat{S}}(s + \mathrm{i}2\pi m/\tau)$ and $\underline{\widehat{H}}(s + \mathrm{i}2\pi m/\tau)$ is square summable as a sequence indexed by m by Proposition 3.30 (c). □

Remark 3.32. Using matrix representations discussed in Subsection 2.4.4, the procedure of the above proof can be expressed as follows:

$$\begin{aligned}&(SG_{22}H)^{\smile}(\mathrm{e}^{s\tau}) = \breve{S}(\mathrm{e}^{s\tau})\breve{G}_{22}(\mathrm{e}^{s\tau})\breve{H}(\mathrm{e}^{s\tau}) \\ &= \frac{1}{\tau}\begin{bmatrix} \cdots & \underline{\widehat{S}}\big(s - \frac{\mathrm{i}2\pi}{\tau}\big) & \underline{\widehat{S}}(s) & \underline{\widehat{S}}\big(s + \frac{\mathrm{i}2\pi}{\tau}\big) & \cdots \end{bmatrix} \begin{bmatrix} \ddots & & & & 0 \\ & \widehat{G}_{22}\big(s - \frac{\mathrm{i}2\pi}{\tau}\big) & & & \\ & & \widehat{G}_{22}(s) & & \\ & & & \widehat{G}_{22}\big(s + \frac{\mathrm{i}2\pi}{\tau}\big) & \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \underline{\widehat{H}}\big(s - \frac{\mathrm{i}2\pi}{\tau}\big) \\ \underline{\widehat{H}}(s) \\ \underline{\widehat{H}}\big(s + \frac{\mathrm{i}2\pi}{\tau}\big) \\ \vdots \end{bmatrix} \\ &= \frac{1}{\tau}\sum_{m=-\infty}^{\infty} \underline{\widehat{S}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \widehat{G}_{22}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big) \underline{\widehat{H}}\Big(s + \frac{\mathrm{i}2\pi m}{\tau}\Big).\end{aligned}$$

This helps us to understand the essence of the proof. However, it cannot be a rigorous proof of Theorem 3.31 because convergence of the series is treated in a naive way. □

The next result of ours is about the lifting-based transfer function of $\mathcal{F}(G, HK_\mathrm{d}S)$, which is the closed-loop operator mapping $\boldsymbol{w}(t)$ to $\boldsymbol{z}(t)$ in Figure 3.5.

Theorem 3.33. *For a provided sampled-data control system, choose* $0 < \rho_0 < 1$ *so that* $\breve{S}(z)$ *and* $\breve{H}(z)$ *are analytic in* $\mathbb{D}_{\rho_0}$. *Consider a discrete-time operator* $K_\mathrm{d}(I - SG_{22}HK_\mathrm{d})^{-1}$ *and write it as* L_d. *Finally, let* s *be a complex number such that (i) there holds* $\mathrm{e}^{s\tau} \in \mathbb{D}_{\rho_0}$, *(ii) the discrete-time transfer function of* L_d, *i.e.,* $\breve{L}_\mathrm{d}(z)$, *is analytic at* $z = \mathrm{e}^{s\tau}$, *and (iii) none of* $s + \mathrm{i}2\pi m/\tau$, $m = 0, \pm 1, \ldots$, *is a pole of* $\widehat{G}(s)$. *Then, we have*

$$\acute{E}^s_m \breve{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{s\tau})\grave{E}^s_\ell = \widehat{G}_{11}(s_m)\delta_{m,\ell} + \frac{1}{\tau}\widehat{G}_{12}(s_m)\underline{\widehat{H}}(s_m)\breve{L}_\mathrm{d}(\mathrm{e}^{s\tau})\underline{\widehat{S}}(s_\ell)\widehat{G}_{21}(s_\ell), \tag{3.11}$$

where $s_m := s + \mathrm{i}2\pi m/\tau$.

If the large operator $\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})$ is represented in the matrix form of (2.6a), the above quantity $\acute{E}^s_m\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})\grave{E}^s_\ell$ corresponds to the (m, ℓ)-block of this matrix. In the case that s is a pure imaginary number, the formula (3.11) is equivalent to the one obtained in the papers of Araki *et al.* [4, 3, 42, 44, 5], where sampled-data control systems were analyzed based on their matrix representations (or FR-operators in their words).

Proof. The proof is carried out similarly to that of Theorem 3.31.

Note that

$$\mathcal{F}(G, HK_{\mathrm{d}}S) = G_{11} + G_{12}HK_{\mathrm{d}}(I - SG_{22}HK_{\mathrm{d}})^{-1}SG_{21} = G_{11} + G_{12}HL_{\mathrm{d}}SG_{21}.$$

Obtain the lifting-based transfer functions of the both sides. Then, by the assumptions on s, the values of the functions $\breve{G}_{11}(z)$, $\breve{G}_{12}(z)$, $\breve{H}(z)$, $\breve{L}_{\mathrm{d}}(z)$, $\breve{S}(z)$, and $\breve{G}_{21}(z)$ are bounded operators at $z = \mathrm{e}^{s\tau}$. Therefore, it follows that

$$\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau}) = \breve{G}_{11}(\mathrm{e}^{s\tau}) + \breve{G}_{12}(\mathrm{e}^{s\tau})\breve{H}(\mathrm{e}^{s\tau})\breve{L}_{\mathrm{d}}(\mathrm{e}^{s\tau})\breve{S}(\mathrm{e}^{s\tau})\breve{G}_{21}(\mathrm{e}^{s\tau}).$$

In the following, we omit the dependence on $\mathrm{e}^{s\tau}$. Apply $\acute{E}^s_m$ and $\grave{E}^s_\ell$ on both sides of the above equality. Moreover, use Proposition 2.26 and substitute (3.10). Then, it is obtained that

$$\begin{aligned}\acute{E}^s_m\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})\grave{E}^s_\ell &= \acute{E}^s_m\breve{G}_{11}\grave{E}^s_\ell + \acute{E}^s_m\breve{G}_{12}\left(\sum_{j=-\infty}^{\infty}\grave{E}^s_j\acute{E}^s_j\right)\breve{H}\breve{L}_{\mathrm{d}}\breve{S}\left(\sum_{k=-\infty}^{\infty}\grave{E}^s_k\acute{E}^s_k\right)\breve{G}_{21}\grave{E}^s_\ell \\ &= \widehat{G}_{11}(s_m)\delta_{m,\ell} + \frac{1}{\tau}\widehat{G}_{12}(s_m)\underline{\widehat{H}}(s_m)\breve{L}_{\mathrm{d}}(\mathrm{e}^{s\tau})\underline{\widehat{S}}(s_\ell)\widehat{G}_{21}(s_\ell).\end{aligned}$$

Now the proof is completed. □

Our final result in this section is about a relationship between a sampled-data control system and a continuous-time control system, which was considered in Section 2.5. This theorem was first obtained by Oishi [72] (whose contents were published as [77]) with restricted to a special case. It is a powerful tool to compare the best sampled-data control performance with the best continuous-time control performance and works as the basis of the analysis in the next chapter.

Let us briefly review a continuous-time control system. In Figure 3.7, a continuous-time control system is depicted compared with a sampled-data control system. (Figure 3.7 (a) is the same as Figure 3.5 and Figure 3.7 (b) is the same as Figure 2.2.) A continuous-time control system is composed of two continuous-time operators G and K as is shown in Figure 3.7 (b). The operator G stands for a generalized plant and is assumed in the same way as in a sampled-data control system. The operator K is a continuous-time controller and is assumed to be an element of the set $\mathcal{K}$, that is, the set of continuous-time operators having continuous-time state-space representations and consistent dimensions with G. A continuous-time control system is called input-output stable if the nine operators mapping $\boldsymbol{w}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{a}(t)$ to $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{u}'(t)$ are all bounded. Sometimes, a continuous-time controller K is required to be chosen from a subset of $\mathcal{K}$. The set $\mathcal{K}_0$ is one of such subsets and consists of continuous-time operators having *strictly proper* continuous-time state-space representations and consistent dimensions with G.

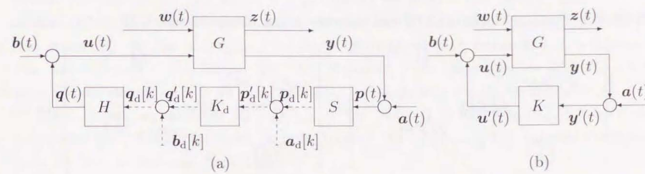


Figure 3.7. Standard configurations of (a) a sampled-data control system and (b) a continuous-time control system.

Now the result is stated.

Theorem 3.34. *Suppose that an input-output stable sampled-data control system depicted in Figure 3.7 (a) is provided. Let us write a discrete-time operator* $K_{\mathrm{d}}(I - SG_{22}HK_{\mathrm{d}})^{-1}$ *as* L_{d}. *Then, there exists a sequence of continuous-time controllers* $\{K_j\}_{j=1}^{\infty}$, $K_j \in \mathcal{K}_0$, *that satisfies the following conditions:*

(a) The continuous-time control system constructed by G *and* K_j *in the configuration of Figure 3.7 (b) is input-output stable for every* j;

(b) The closed-loop transfer function of the continuous-time control system above, i.e., $\widehat{\mathcal{F}}(G, K_j)(s)$, *converges as* $j \to \infty$ *to the function*

$$\widehat{G}_{11}(s) + \frac{1}{\tau}\widehat{G}_{12}(s)\underline{\widehat{H}}(s)\breve{L}_{\mathrm{d}}(\mathrm{e}^{s\tau})\underline{\widehat{S}}(s)\widehat{G}_{21}(s) \tag{3.12}$$

uniformly for any $\mathrm{Re}\, s \geq 0$.

An interpretation of Theorem 3.34 is as follows. Regarding the closed-loop operator of the sampled-data control system, i.e., $\mathcal{F}(G, HK_{\mathrm{d}}S)$, consider its lifting-based transfer function $\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(z)$. We can write its matrix representation at $z = \mathrm{e}^{s\tau}$ as

$$\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau}) \sim \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \acute{E}^s_{-1}\breve{\mathcal{F}}\grave{E}^s_{-1} & \acute{E}^s_{-1}\breve{\mathcal{F}}\grave{E}^s_0 & \acute{E}^s_{-1}\breve{\mathcal{F}}\grave{E}^s_1 & \cdots \\ \cdots & \acute{E}^s_0\breve{\mathcal{F}}\grave{E}^s_{-1} & \acute{E}^s_0\breve{\mathcal{F}}\grave{E}^s_0 & \acute{E}^s_0\breve{\mathcal{F}}\grave{E}^s_1 & \cdots \\ \cdots & \acute{E}^s_1\breve{\mathcal{F}}\grave{E}^s_{-1} & \acute{E}^s_1\breve{\mathcal{F}}\grave{E}^s_0 & \acute{E}^s_1\breve{\mathcal{F}}\grave{E}^s_1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \tag{3.13}$$

following the form of (2.6a). Here, $\breve{\mathcal{F}}$ is a shorthand for $\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})$. Now, let us write the function in (3.12) as $\Theta(s)$. Then, from (3.11), it follows that

$$\acute{E}_m^s \breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau}) \grave{E}_m^s = \Theta\left(s + \frac{\mathrm{i}2\pi m}{\tau}\right).$$

Extract the diagonal blocks of (3.13) and construct a new infinite matrix. This matrix satisfies

$$\begin{bmatrix} \ddots & & & 0 \\ & \acute{E}_{-1}^s \breve{\mathcal{F}} \grave{E}_{-1}^s & & \\ & & \acute{E}_0^s \breve{\mathcal{F}} \grave{E}_0^s & \\ & & & \acute{E}_1^s \breve{\mathcal{F}} \grave{E}_1^s \\ 0 & & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & 0 \\ & \Theta\left(s - \frac{\mathrm{i}2\pi}{\tau}\right) & & \\ & & \Theta(s) & \\ & & & \Theta\left(s + \frac{\mathrm{i}2\pi}{\tau}\right) \\ 0 & & & & \ddots \end{bmatrix}. \tag{3.14}$$

That is, $\Theta(s)$ interpolates the diagonal blocks of the matrix representation of $\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})$. Now, consider the matrix representation of $\breve{\mathcal{F}}(G, K_j)(\mathrm{e}^{s\tau})$. Since $\mathcal{F}(G, K_j)$ is a continuous-time operator having a continuous-time state-space representation, Proposition 2.28 implies

$$\breve{\mathcal{F}}(G, K_j)(\mathrm{e}^{s\tau}) \sim \begin{bmatrix} \ddots & & & 0 \\ & \hat{\mathcal{F}}(G, K_j)\left(s - \frac{\mathrm{i}2\pi}{\tau}\right) & & \\ & & \hat{\mathcal{F}}(G, K_j)(s) & \\ & & & \hat{\mathcal{F}}(G, K_j)\left(s + \frac{\mathrm{i}2\pi}{\tau}\right) \\ 0 & & & & \ddots \end{bmatrix}. \tag{3.15}$$

Since Theorem 3.34 claims that, as $j \to \infty$, the function $\hat{\mathcal{F}}(G, K_j)(s)$ approximates $\Theta(s)$ uniformly in $\operatorname{Re} s \geq 0$, the matrix (3.15) converges to (3.14) uniformly. Let us summarize this interpretation. Theorem 3.34 claims that there exists a continuous-time controller sequence $\{K_j\}$ such that $\breve{\mathcal{F}}(G, K_j)(\mathrm{e}^{s\tau})$ approximates $\breve{\mathcal{F}}(G, HK_{\mathrm{d}}S)(\mathrm{e}^{s\tau})$ in its diagonal blocks.

It is conjectured that $\Theta(s)$ is a continuous-time transfer function of some continuous-time time-invariant operator that optimally approximates $\mathcal{F}(G, HK_{\mathrm{d}}S)$ with respect to some norm. Interpreting this theorem in this direction is an interesting subject and research is proceeding in this direction. See [22] for this kind of approximation.

The proof of Theorem 3.34 is given in the next section.

3.5. Proof of Theorem 3.34

Theorem 3.34, which plays a key role in Chapter 4, is proven here.

First, we present one lemma. Recall that $\mathcal{A}_{\mathrm{R}}$ is the set of all real functions analytic in $\operatorname{Re} s > 0$ and continuous in $\mathbb{C}_{+\mathrm{e}} = \{s \in \mathbb{C}_{\mathrm{e}} : \operatorname{Re} s \geq 0 \text{ or } s = \infty\}$. (It was defined before Proposition 2.7.)

Lemma 3.35. *A function $Q(s)$ belongs to the set $\mathcal{A}_{\mathrm{R}}$ and $Q(\infty) = O$, if and only if there exists a functional sequence $\{Q_j\}_{j=1}^{\infty}$, $Q_j \in \mathcal{RH}^{\infty}$, that satisfies $Q_j(\infty) = O$ and $\|Q - Q_j\|_{\mathcal{H}^{\infty}} \to 0$ as $j \to \infty$.*

Proof. Proving the "if" part is easy. Let us consider the proof of the "only if" part.

Since $Q \in \mathcal{A}_{\mathrm{R}}$, Proposition 2.7 implies the existence of a sequence $\{Q_j'\}_{j=1}^{\infty}$, $Q_j' \in \mathcal{RH}^{\infty}$, that attains $\|Q - Q_j'\|_{\mathcal{H}^{\infty}} < 1/2j$ for $j = 1, 2, \ldots$. Since $Q(\infty) = O$, we can find $\Omega_j > 0$ for each j so that $\overline{\sigma}\{Q(\mathrm{i}\omega)\} < 1/4j$ for any $|\omega| \geq \Omega_j$. Moreover, if we define $Q_j(s) := Q_j'(s)/(\alpha_j s + 1)$ using a small enough $\alpha_j > 0$, we can attain $\overline{\sigma}\{Q_j'(\mathrm{i}\omega) - Q_j(\mathrm{i}\omega)\} < 1/2j$ for any $|\omega| < \Omega_j$.

Then, in fact, we can show $\|Q - Q_j\|_{\mathcal{H}^{\infty}} < 1/j$. Since $Q_j \in \mathcal{RH}^{\infty}$ and $Q_j(\infty) = O$, this means that the proof is completed. In order to show $\|Q - Q_j\|_{\mathcal{H}^{\infty}} < 1/j$, suppose the case of $|w| < \Omega_j$ first. In this case, there holds

$$\overline{\sigma}\{Q(\mathrm{i}\omega) - Q_j(\mathrm{i}\omega)\} \leq \overline{\sigma}\{Q(\mathrm{i}\omega) - Q_j'(\mathrm{i}\omega)\} + \overline{\sigma}\{Q_j'(\mathrm{i}\omega) - Q_j(\mathrm{i}\omega)\} < \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j}. \tag{3.16}$$

Next, suppose $|\omega| \geq \Omega_j$. Then, we have $|\overline{\sigma}\{Q(\mathrm{i}\omega)\} - \overline{\sigma}\{Q_j'(\mathrm{i}\omega)\}| < 1/2j$. Note that $\overline{\sigma}\{Q_j(\mathrm{i}\omega)\} < \overline{\sigma}\{Q_j'(\mathrm{i}\omega)\}$ by definition. Therefore,

$$\overline{\sigma}\{Q_j(\mathrm{i}\omega)\} < \overline{\sigma}\{Q_j'(\mathrm{i}\omega)\} \leq |\overline{\sigma}\{Q_j'(\mathrm{i}\omega)\} - \overline{\sigma}\{Q(\mathrm{i}\omega)\}| + \overline{\sigma}\{Q(\mathrm{i}\omega)\} < \frac{1}{2j} + \frac{1}{4j}.$$

It is derived that

$$\overline{\sigma}\{Q(\mathrm{i}\omega) - Q_j(\mathrm{i}\omega)\} \leq \overline{\sigma}\{Q(\mathrm{i}\omega)\} + \overline{\sigma}\{Q_j(\mathrm{i}\omega)\} < \frac{1}{4j} + \left(\frac{1}{2j} + \frac{1}{4j}\right) = \frac{1}{j}. \tag{3.17}$$

Combining (3.16) and (3.17), we can see $\|Q - Q_j\|_{\mathcal{H}^{\infty}} < 1/j$. The proof is now completed. □

In order to prove Theorem 3.34, we need a doubly-coprime factorization of $\widehat{G}_{22}(s)$, which was introduced in Definition 2.36. That is, obtain eight functions N, M, X, Y, $\widetilde{N}$, $\widetilde{M}$, $\widetilde{X}$, $\widetilde{Y} \in \mathcal{RH}^{\infty}$ such that

$$\widehat{G}_{22}(s) = N(s)M(s)^{-1} = \widetilde{M}(s)^{-1}\widetilde{N}(s),$$

$$\begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

Note that $\widehat{G}_{22}(s)$ cannot have a pole at $s = \infty$ because $\widehat{G}_{22}(\infty) = D_{22}$ with the state-space representation of G. Hence, by Lemma 2.41, it is possible to assume $Y(\infty) = O$ and $\widetilde{Y}(\infty) = O$ without loss of generality. Then, from Proposition 2.42, it follows that a continuous-time operator $K_0 \in \mathcal{K}_0$ stabilizes a provided continuous-time control system if and only if K_0 is represented as

$$\widehat{K}_0(s) = \{\widetilde{X}(s) - Q_0(s)\widetilde{N}(s)\}^{-1}\{\widetilde{Y}(s) - Q_0(s)\widetilde{M}(s)\} \tag{3.18}$$

using some $Q_0 \in \mathcal{RH}^\infty$ such that $Q_0(\infty) = O$. Moreover, define

$$T_1(s) := \widehat{G}_{11}(s) + \widehat{G}_{12}M(s)\widetilde{Y}(s)\widehat{G}_{21}(s),$$
$$T_2(s) := \widehat{G}_{12}(s)M(s), \qquad T_3(s) := \widetilde{M}(s)\widehat{G}_{21}(s).$$

Then, Proposition 2.40 shows that these three functions belong to $\mathcal{RH}^\infty$ and satisfy

$$\widehat{\mathcal{F}}(G, K_0)(s) = T_1(s) - T_2(s)Q_0(s)T_3(s)$$

with (3.18).

The key step of the proof is to show that the function

$$Q_{\rm sd}(s) := \left\{\widetilde{Y}(s) - \frac{1}{\tau}M(s)^{-1}\widehat{\underline{H}}(s)\check{L}_{\rm d}(e^{s\tau})\widehat{\underline{S}}(s)\right\}\widetilde{M}(s)^{-1}$$

belongs to $\mathcal{A}_{\rm R}$ and satisfies $Q_{\rm sd}(\infty) = O$. Suppose that it is proven. Then, Lemma 3.35 shows existence of a functional sequence $\{Q_j\}_{j=1}^\infty$, $Q_j \in \mathcal{RH}^\infty$, such that $Q_j(\infty) = O$ and $\|Q_{\rm sd} - Q_j\|_{\mathcal{H}^\infty} \to 0$ as $j \to \infty$. Note that $\|Q_{\rm sd} - Q_j\|_{\mathcal{H}^\infty} \to 0$ means that $Q_j(s)$ converges to $Q_{\rm sd}(s)$ uniformly in $\mathrm{Re}\, s > 0$. Recall that the functions $T_1(s)$, $T_2(s)$, $T_3(s)$ are bounded in $\mathrm{Re}\, s > 0$. Therefore, $T_1 - T_2Q_jT_3$ converges to $T_1 - T_2Q_{\rm sd}T_3$ uniformly in $\mathrm{Re}\, s > 0$. Here, it is easy to see

$$T_1(s) - T_2(s)Q_{\rm sd}(s)T_3(s) = \widehat{G}_{11}(s) + \frac{1}{\tau}\widehat{G}_{12}(s)\widehat{\underline{H}}(s)\check{L}_{\rm d}(e^{s\tau})\widehat{\underline{S}}(s)\widehat{G}_{21}(s).$$

This is the function that appeared in the theorem statement as (3.12). Moreover, define a continuous-time controller K_j by substituting each Q_j, $j = 1, 2, \ldots$, into Q_0 in (3.18). Then, K_j belongs to $\mathcal{K}_0$, stabilizes the continuous-time control system, and satisfies

$$T_1(s) - T_2(s)Q_j(s)T_3(s) = \widehat{\mathcal{F}}(G, K_j)(s).$$

Now the theorem is proven.

In the rest of this section, it is shown that $Q_{\rm sd} \in \mathcal{A}_{\rm R}$ and $Q_{\rm sd}(\infty) = O$.

First, let us prove $Q_{\rm sd}(\infty) = O$. The proof proceeds by three steps: that is, we show that first $\widehat{\underline{H}}(s)\check{L}_{\rm d}(e^{s\tau})\widehat{\underline{S}}(s) \to O$ as $s \to \infty$, then $\widetilde{Y}(\infty) = O$, and finally $M(s)^{-1}$ and $\widetilde{M}(s)^{-1}$ are bounded at $s = \infty$. Since a provided sampled-data control system is input-output stable, Proposition 3.24 ensures that we can choose $0 < \rho < 1$ so that the lifting-based transfer functions of the 25 operators in Definition 3.22 are analytic in $\mathbb{D}_\rho$. In particular, $\check{L}_{\rm d}(z)$ is analytic in $\mathbb{D}_\rho$ because $L_{\rm d} = K_{\rm d}(I - SG_{22}HK_{\rm d})^{-1}$ is an operator mapping $\boldsymbol{a}_{\rm d}[k]$ to $\boldsymbol{q}'_{\rm d}[k]$ in Figure 3.7 (a). This means that $\check{L}_{\rm d}(e^{s\tau})$ is analytic and bounded in $\mathrm{Re}\, s \geq 0$ as a function of s. However, we should note that this function has an essential singularity at $s = \infty$. On the other hand, since the kernel functions $\underline{S}(t)$ and $\underline{H}(t)$ belong to the set $\mathcal{D}$, Proposition 3.30 implies that their Laplace transforms $\widehat{\underline{S}}(s)$ and $\widehat{\underline{H}}(s)$ are analytic and bounded in $\mathrm{Re}\, s \geq 0$ and satisfy

$$\widehat{\underline{S}}(s) \to O \quad \text{and} \quad \widehat{\underline{H}}(s) \to O$$

as s approaches infinity in $\mathrm{Re}\, s \geq 0$. Therefore, a function $\widehat{\underline{H}}(s)\check{L}_{\rm d}(e^{s\tau})\widehat{\underline{S}}(s)$ is analytic in $\mathrm{Re}\, s \geq 0$ and approaches zero as s goes to infinity. On the other hand, recall that $Y(\infty) = O$ and $\widetilde{Y}(\infty) = O$. Since $\widetilde{X}M - \widetilde{Y}N \equiv I$, there holds $\widetilde{X}(s)M(s) = I$ at $s = \infty$, which means that $M(s)^{-1}$ is bounded at $s = \infty$. Similarly, from the fact that $\widetilde{M}X - \widetilde{N}Y \equiv I$, boundedness of $\widetilde{M}(s)^{-1}$ at $s = \infty$ is derived, too. Now, it is easy to see $Q_{\rm sd} = \{\widetilde{Y} - (1/\tau)M^{-1}\widehat{\underline{H}}\check{L}_{\rm d}\widehat{\underline{S}}\}\widetilde{M}^{-1}$ vanishes at $s = \infty$.

Next, it is shown that $Q_{\rm sd} \in \mathcal{A}_{\rm R}$. Since the all functions that appear in the definition of $Q_{\rm sd}(s)$ are real functions, $Q_{\rm sd}(s)$ is real, too. As we saw above, the function $\widehat{\underline{H}}(s)\check{L}_{\rm d}(e^{s\tau})\widehat{\underline{S}}(s)$ is analytic in $\mathrm{Re}\, s \geq 0$. Therefore, at any s satisfying $\mathrm{Re}\, s \geq 0$, the function $Q_{\rm sd}(s)$ is analytic or possibly has a pole when $M(s)^{-1}$ or $\widetilde{M}(s)^{-1}$ has a pole there. Moreover, $Q_{\rm sd}(s)$ is continuous at $s = \infty$ as is shown above; indeed, $Q_{\rm sd}(s)$ approaches the zero matrix as s goes to infinity. Hence, if we can show $Q_{\rm sd}(s)$ has no pole in $\mathrm{Re}\, s \geq 0$, we have $Q_{\rm sd} \in \mathcal{A}_{\rm R}$ by definition. Let us show this next.

With slight abuse of notation, temporarily let the symbols N, M, X, Y, $\widetilde{N}$, $\widetilde{M}$, $\widetilde{X}$, $\widetilde{Y}$ denote the continuous-time operators whose continuous-time transfer functions are N, M, $\ldots$, $\widetilde{Y}$, respectively. These eight continuous-time operators are bounded because their continuous-time transfer functions belong to $\mathcal{RH}^\infty$. Now, consider the operator composition

$$[\widetilde{X} \ \ -\widetilde{Y}]\begin{bmatrix} HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1}G_{22} & -HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1} \\ (I - G_{22}HK_{\rm d}S)^{-1}G_{22} & -(I - G_{22}HK_{\rm d}S)^{-1}\end{bmatrix}\begin{bmatrix} Y \\ X\end{bmatrix}. \tag{3.19}$$

In fact, each of the four blocks in the second operator matrix is bounded. Indeed, the operator of the $(1,1)$-block, i.e., $HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1}G_{22}$, is equal to the operator mapping $\boldsymbol{b}(t)$ to $\boldsymbol{q}(t)$ in Figure 3.7 (a), which is bounded by the definition of input-output stability. Boundedness of the other three operators is shown in a similar fashion. Since operators X, Y, $\widetilde{X}$, $\widetilde{Y}$ are bounded, too, the operator composition (3.19) is bounded.

The expression of the operator composition (3.19) can be simplified. For this purpose, note that there hold the following equalities among operators: $\widetilde{X}M - \widetilde{Y}N = I$, $\widetilde{M}X - \widetilde{N}Y = I$, $G_{22} = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$, and $L_{\rm d} = K_{\rm d}(I - G_{22}HK_{\rm d}S)^{-1}$. We can show the first three by the fact that the corresponding equalities hold among transfer functions. The last one is the definition of $L_{\rm d}$. Moreover, there hold the identities $A(I - BA)^{-1} = (I - AB)^{-1}A$ and $(I - AB)^{-1} = I + AB(I - AB)^{-1}$ for general operators A and B. Using these equalities, we can derive the following:

$$\begin{aligned}
&[\widetilde{X} \ \ -\widetilde{Y}]\begin{bmatrix} HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1}G_{22} & -HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1} \\ (I - G_{22}HK_{\rm d}S)^{-1}G_{22} & -(I - G_{22}HK_{\rm d}S)^{-1}\end{bmatrix}\begin{bmatrix} Y \\ X\end{bmatrix} \\
&= \widetilde{X}\left\{HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1}G_{22}Y - HK_{\rm d}S(I - G_{22}HK_{\rm d}S)^{-1}X\right\} \\
&\quad - \widetilde{Y}\left\{(I - G_{22}HK_{\rm d}S)^{-1}G_{22}Y - (I - G_{22}HK_{\rm d}S)^{-1}X\right\}
\end{aligned}$$

$$
\begin{aligned}
&= (\widetilde{X}HK_{\mathrm{d}}S - \widetilde{Y})(I - G_{22}HK_{\mathrm{d}}S)^{-1}(G_{22}Y - X) \\
&= (\widetilde{X}HK_{\mathrm{d}}S - \widetilde{Y})(I - G_{22}HK_{\mathrm{d}}S)^{-1}\widetilde{M}^{-1}(\widetilde{N}Y - \widetilde{M}X) \\
&= -(\widetilde{X}HK_{\mathrm{d}}S - \widetilde{Y})(I - G_{22}HK_{\mathrm{d}}S)^{-1}\widetilde{M}^{-1} \\
&= -\left\{\widetilde{X}HK_{\mathrm{d}}S(I - G_{22}HK_{\mathrm{d}}S)^{-1} - \widetilde{Y}(I - G_{22}HK_{\mathrm{d}}S)^{-1}\right\}\widetilde{M}^{-1} \\
&= -\left\{\widetilde{X}HL_{\mathrm{d}}S - \widetilde{Y}(I - G_{22}HK_{\mathrm{d}}S)^{-1}\right\}\widetilde{M}^{-1} \\
&= -\left[\widetilde{X}HL_{\mathrm{d}}S - \widetilde{Y}\left\{I + G_{22}HK_{\mathrm{d}}S(I - G_{22}HK_{\mathrm{d}}S)^{-1}\right\}\right]\widetilde{M}^{-1} \\
&= -(\widetilde{X}HL_{\mathrm{d}}S - \widetilde{Y} - \widetilde{Y}G_{22}HL_{\mathrm{d}}S)\widetilde{M}^{-1} \\
&= \left\{\widetilde{Y} - (\widetilde{X}M - \widetilde{Y}N)M^{-1}HL_{\mathrm{d}}S\right\}\widetilde{M}^{-1} \\
&= (\widetilde{Y} - M^{-1}HL_{\mathrm{d}}S)\widetilde{M}^{-1}.
\end{aligned}
$$

Since the leftmost operator composition is bounded, so is the rightmost composition $(\widetilde{Y} - M^{-1}HL_{\mathrm{d}}S)\widetilde{M}^{-1}$.

From the above result, it follows that, for any $\operatorname{Re} s \geq 0$,

$$\acute{E}_0^s(\widetilde{Y} - M^{-1}HL_{\mathrm{d}}S)\widetilde{M}^{-1}\grave{E}_0^s$$

is bounded. The definitions of $\acute{E}_0^s$ and $\grave{E}_0^s$ were given in Definition 2.25. Actually, just as the proof of Theorem 3.33, we can prove that

$$\acute{E}_0^s(\widetilde{Y} - M^{-1}HL_{\mathrm{d}}S)\widetilde{M}^{-1}\grave{E}_0^s = \left\{\widetilde{Y}(s) - \frac{1}{\tau}M(s)^{-1}\underline{\widehat{H}}(s)\check{L}_{\mathrm{d}}(\mathrm{e}^{s\tau})\underline{\widehat{S}}(s)\right\}\widetilde{M}(s)^{-1} = Q_{\mathrm{sd}}(s),$$

where the symbols M, $\widetilde{M}$, $\widetilde{Y}$ in the central expression stand for functions rather than operators. Therefore, $Q_{\mathrm{sd}}(s)$ cannot have a pole in $\operatorname{Re} s \geq 0$.

The proof of the theorem is now completed.

3.6. Conclusion

In this chapter, a framework for sampled-data control systems was presented. In Section 3.2, regular samplers and holds were defined in terms of kernel functions. They are more general than the conventional generalized samplers and holds because the kernel functions of our samplers and holds are allowed to have nonzero values over $[0, \infty)$ while the kernel functions of the conventional ones only on $[0, \tau)$. Section 3.3 gave a standard structure of sampled-data control systems and presented the notion of sampling environments. Moreover, properties of a sampled-data control system were derived in Section 3.4 as three theorems. The first two theorems can be regarded as a generalization of already known results. From the fact that these generalized results are obtained in a natural way, we can see usefulness of our framework. The last theorem claimed that any stable sampled-data control system can be approximated in the diagonal

blocks of its matrix representation by a sequence of corresponding continuous-time control systems. This result is important in the next chapter. Section 3.5 gave the proof of this last theorem.

Our framework is general enough to cover systems with a large class of samplers and holds. Especially, an anti-aliasing filter can be regarded as a part of a sampler. Since we have some flexibility on the choice of an anti-aliasing filter, it is possible to improve control performance by choosing an appropriate filter. This problem is formulated as a design of a sampler in our framework. Furthermore, it is noteworthy that a sampler and a hold are treated in a symmetric way here. Indeed, it is the case in the definitions of a sampler and a hold, the Hardy spaces of their lifting-based transfer functions, and their matrix representations, for example. It is considered that this fact shows our framework is mathematically natural. This kind of symmetry is found also in the frameworks of [86, 53, 66].

Based on our framework, established methodologies for analysis and synthesis of sampled-data control systems can be extended to more generally configured systems. Moreover, this framework can be a basis to consider more advanced problems on sampled-data systems such as analysis and design of a sampling environment aiming at further improvement of control performance. Especially, lifting-based transfer functions and their matrix representations are considered to be strong tools for these problems.

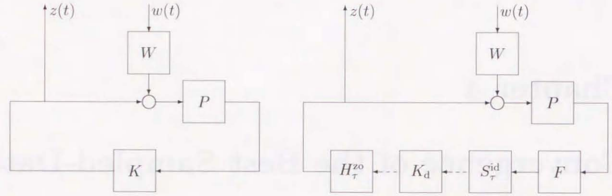
Chapter 4

Convergence of the Best Sampled-Data Control Performance

This is the main chapter of this thesis and is devoted to investigation on the best achievable performance of sampled-data control systems. Here, the $\mathcal{L}^2$-induced norm is adopted as a performance measure. Since the best sampled-data control performance varies depending on the sampling environment, it is possible to improve it by a proper choice of the environment. What we consider first in this chapter is to relate a theoretical bound of this improvement with the best continuous-time control performance. Next, we obtain a necessary and sufficient condition in order that the best sampled-data control performance converges to this theoretical bound. The condition for the convergence is obtained not only in a general case but also in special cases, which are of practical importance. The condition in the special cases has a simpler form and is easier to be tested. This study is motivated by an experimental result presented in Example 1.3, where the best sampled-data control performance did not converge to the best continuous-time control performance even though the sampling period approaches zero.

4.1. Introduction

Intuitively, it seems obvious that the best achievable performance of sampled-data control systems approaches that of continuous-time control systems as the sampling period goes to zero. Furthermore, this conjecture forms a basis to use a sampled-data controller in place of a continuous-time controller. Indeed, Osburn and Bernstein [80] and Trentelman and Stoorvogel [91] proved correctness of this conjecture in the case that the control performance is measured by the $\mathcal{H}^2$-norm. Moreover, Hara *et al.* [45] did the same thing in the case that the performance is measured by the $\mathcal{L}^2$-induced norm. (Tadmor [86] obtained a closely related result, too.) However, these results are valid only in special cases as is seen from Example 1.3. Here, let us recall this example and restate it using notions introduced in the preceding chapters.

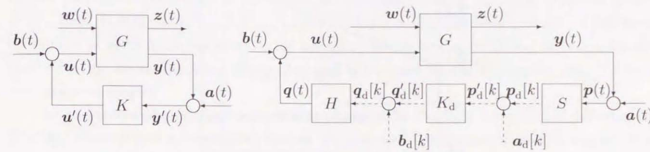


(a) (b)

Figure 4.1. The systems examined in Example 1.3 for robust stabilization: (a) a continuous-time control case; (b) a sampled-data control case.

Example 4.1. Recall the robust stabilization problem considered in Example 1.3. Here, a plant has a continuous-time transfer function $1/(s-1)$ and its robust stabilization is tried both with a continuous-time controller and with a sampled-data controller. This problem is formulated into reduction of the $\mathcal{L}^2$-induced norm of the operator from $w(t)$ to $z(t)$ in each diagram of Figure 4.1. (These figures are essentially the same as Figures 1.7 (b) and 1.8 (b), respectively.) Here, W is a weight assumed to have a continuous-time transfer function $s+1$. Accordingly, what we have to do with the continuous-time control system of Figure 4.1 (a) is to minimize the norm above by choosing an appropriate continuous-time controller K from the set $\mathcal{K}$. Here, $\mathcal{K}$ is the set of continuous-time operators having state-space representations and consistent input- and output-signal dimensions with P. Regarding the sampled-data control system of Figure 4.1 (b), minimization is carried out by an appropriate choice of a discrete-time controller K_d from the set $\mathcal{K}_d$. This set $\mathcal{K}_d$ consists of discrete-time operators having discrete-time state-space representations and consistent input- and output-signal dimensions with the sampler S_τ^{id} and the hold H_τ^{zo}. Here, the symbols S_τ^{id} and H_τ^{zo} denote the ideal sampler and the zero-order hold, respectively, both having the sampling period $\tau > 0$. Their definitions are found in Examples 3.3 and 3.6, respectively. A continuous-time operator F stands for an anti-aliasing filter, and two cases are considered for this F. In the first case, F is taken to be R_τ, which is a continuous-time operator having a transfer function $\widehat{R}_\tau(s) = 1/(\tau s+1)$. In the second case, F is chosen as R, where R is defined to be a continuous-time operator whose transfer function is $\widehat{R}(s) = 1/(s+1)$. In a word, the bandwidth of F is taken to be proportional to the Nyquist frequency π/τ in the first case; it is fixed irrespective of τ in the second case. Apparently, the first choice of F seems reasonable because undesirable aliases that should be

cut off by F appear mostly in the frequency range higher than the Nyquist frequency.



(a) (b)

Figure 4.2. Standard configurations of (a) a continuous-time control system and (b) a sampled-data control system.

In Example 2.30, we have seen that our continuous-time control system in Figure 4.1 (a) can be modified into the standard configuration depicted in Figure 4.2 (a). (This is the same figure as Figure 2.2.) Indeed, if we put

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} := \begin{bmatrix} O & I \\ PW & P \end{bmatrix},$$

the operator from $w(t)$ to $z(t)$ in the original figure is expressed as $\mathcal{F}(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. The best continuous-time control performance is written as $\inf_{K\in\mathcal{K}} \|\mathcal{F}(G,K)\|$, which is the same as in the standard configuration.

Likewise, our sampled-data control system in Figure 4.1 (b) can be modified into the standard structure shown in Figure 4.2 (b) as was done in Example 3.25. (Figure 4.2 (b) is the same as Figure 3.5.) Set G be as before. Choose a sampler S to be $S_\tau^{id}R_\tau$ in the first case, and to be $S_\tau^{id}R$ in the second case. Finally, set a hold H as H_τ^{zo}. Then, the $\mathcal{L}^2$-induced norm $\|\mathcal{F}(G, HK_dS)\|$ in Figure 4.2 (b) is exactly equal to the $\mathcal{L}^2$-induced norm of the operator from $w(t)$ to $z(t)$ in Figure 4.1 (b). The best sampled-data control performance is expressed as $\inf_{K_d\in\mathcal{K}_d} \|\mathcal{F}(G, HK_dS)\|$. This is exactly what we considered in the standard configuration.

Figure 4.3 shows the best continuous-time control performance and the best sampled-data control performance. (This is the same figure as Figure 1.9.) We can analytically obtain the best continuous-time control performance just as in Example 6.1.2 of [34], that is, by transforming this problem into a model-matching problem. Computation of the best sampled-data control performance is carried out by using the algorithm of [9]. The solid line stands for the best sampled-data control performance in the first case $S = S_\tau^{id}R_\tau$; The broken line shows the best performance in the second case $S = S_\tau^{id}R$. It is observed that in the first case the best sampled-data control performance *does not* converge to the best continuous-time control performance

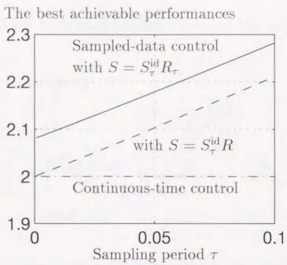


Figure 4.3. The best sampled-data control performance does not always converge to the best continuous-time control performance.

expressed by the dot-dash line, even if the sampling period approaches zero. On the other hand, this convergence is accomplished in the second case. □

This example shows that our conjecture about the best sampled-data control performance is not always correct. Since this conjecture is fundamental in the use of a sampled-data controller, we have to clarify when it is correct and when it is not. The same example shows that convergence to the best continuous-time control performance depends on the choice of an anti-aliasing filter. Actually, from another simulation result, it is seen that the choice of a hold also affects the convergence. However, these system components have never been investigated from this viewpoint.

In this chapter, we first consider a theoretical bound that shows how much we can improve the best sampled-data control performance $\inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\|$ by choosing a sampling environment (τ, S, H), where τ denotes a sampling period, S a sampler, and H a hold. Then, this theoretical bound of the best sampled-data control performance is compared with the best continuous-time control performance. It is shown that there exists a gap between these two in general, which means that sometimes we cannot make the best sampled-data control performance approach the best continuous-time control performance no matter how we choose a sampling period, a sampler, and a hold. Next, it is considered when the theoretical bound of the best sampled-data control performance is attained. Namely, supposing that a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ is given, we obtain a necessary and sufficient condition in order that the best sampled-data control performance for each environment converges to its theoretical bound for all plants. This condition is split into a condition on a hold and a condition on a sampler and these two are symmetric to each other. If we notice a class of plants

with which there is no gap between the theoretical bound and the best continuous-time control performance, the above condition is necessary and sufficient in order that the best sampled-data control performance converges to the best continuous-time control performance for all plants in this class. Furthermore, by applying techniques for a model-matching problem, which were introduced in Section 2.6, we simplify the necessary and sufficient condition. It is also shown that when the kernel functions of samplers and holds have special structures, the condition becomes even simpler.

The non-converging phenomenon such as observed in Example 4.1 was first reported in [73, 74]. These papers theoretically clarified the reason of this phenomenon with respect to a particular example. It is also proven that the pair of $S_\tau^{\mathrm{id}} R$ and H_τ^{zo}, which are used in the second example of Example 4.1, guarantees convergence of the best sampled-data control performance to the best continuous-time control performance for some large class of plants. Later, Reference [79] presented a class of systems that have a non-converging property and gave a necessary and sufficient condition for the convergence in a general case. The contents of this chapter are based on the results of [75, 76, 78], which were obtained by further investigation on this topic. Recently, Hara *et al.* reported several interesting simulation results on sampled-data control systems with small sampling periods, which include a non-converging phenomenon [46].

4.2. The Theoretical Bound of the Best Sampled-Data Control Performance

In this section, we consider a theoretical bound of the best sampled-data control performance and compare it with the best continuous-time control performance. Let us review the systems in Figure 4.2. Figure 4.2 (a) shows a standard continuous-time control system introduced in Section 2.5 and its best achievable performance is $\inf_{K\in\mathcal{K}} \|\mathcal{F}(G, K)\|$. Here, $\|\cdot\|$ is the $\mathcal{L}^2$-induced norm though its value is defined to be equal to infinity if the evaluated system is not input-output stable. The set $\mathcal{K}$ is composed of all continuous-time operators that have state-space representations and appropriate signal dimensions. On the other hand, Figure 4.2 (b) shows a standard sampled-data control system considered in the previous chapter and its best achievable performance is $\inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\|$. The set $\mathcal{K}_{\mathrm{d}}$ consists of all discrete-time operators having state-space representations and appropriate input- and output-signal dimensions.

Note that the best sampled-data control performance $\inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\|$ depends not only on a generalized plant G but also on a sampling environment (τ, S, H), where τ is a sampling period, S is a sampler, and H is a hold. By using an appropriate (τ, S, H), it is possible to improve the control performance, that is, to make the value of $\inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\|$ smaller. From now on, we mean the bound of this improvement by the **theoretical bound of the best sampled-data control performance** or simply the **theoretical bound**. This can be written as

$$\inf_{(\tau,S,H)} \inf_{K_d \in \mathcal{K}_d} \|\mathcal{F}(G, HK_dS)\|,$$

where the left infimum is taken over all sampling environments consistent with the provided G. (See Definition 3.19 for the definition of a sampling environment.) Then, what property does this bound have? Is it equal to the best continuous-time control performance $\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$? Our task in this section is to answer these questions. Here, we need the set $\mathcal{K}_0$, which was defined in Section 2.5 as

$$\mathcal{K}_0 := \{K_0 \in \mathcal{K} : \widehat{K}_0(\infty) = O\};$$

in a word, $\mathcal{K}_0$ is a subset of $\mathcal{K}$ and consists of continuous-time operators whose continuous-time transfer functions are equal to zero at $s = \infty$. Recall that a continuous-time transfer function having a zero at $s = \infty$ is called strictly proper. (See Section 2.3.1.)

Theorem 4.2. *For any generalized plant G, there holds*

$$\inf_{(\tau,S,H)} \inf_{K_d \in \mathcal{K}_d} \|\mathcal{F}(G, HK_dS)\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\|. \tag{4.1}$$

Here, the leftmost infimum is taken over all sampling environments (τ, S, H) consistent with the given G.

Here, it is noteworthy that the theoretical bound of the best sampled-data control performance is *not* equal to the best continuous-time control performance in general. Indeed, since $\mathcal{K}_0 \subsetneq \mathcal{K}$, Theorem 4.2 implies

$$\inf_{(\tau,S,H)} \inf_{K_d \in \mathcal{K}_d} \|\mathcal{F}(G, HK_dS)\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| \geq \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$$

and the equality does not hold in general in the last inequality. When the rightmost quantity is strictly smaller than the quantity in the middle, there is a gap between the best sampled-data control performance and the best continuous-time control performance. In other words, the best continuous-time control performance cannot be recovered by a sampled-data controller no matter how we choose a sampling environment and a discrete-time controller. This recovery is possible by a sampled-data controller if and only if there is no gap, that is,

$$\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|. \tag{4.2}$$

These results are understood as follows. By definition, a continuous-time state-space representation of $K_0 \in \mathcal{K}_0$ is strictly proper. That is, if we represent a continuous-time operator $K_0 \in \mathcal{K}_0 : \boldsymbol{y}' \mapsto \boldsymbol{u}'$ as

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{y}'(t), \qquad \boldsymbol{x}(0) = \boldsymbol{0},$$

$$\boldsymbol{u}'(t) = C\boldsymbol{x}(t) + D\boldsymbol{y}'(t),$$

then we have $D = O$. In a word, a controller $K_0 \in \mathcal{K}_0$ has no direct feedthrough term or the input of K_0 cannot instantaneously affect its output.

On the other hand, a sampled-data controller HK_dS has a similar property. We assumed that our regular sampler $S : \boldsymbol{p} \mapsto \boldsymbol{p}_d$ is described in an integral form:

$$\boldsymbol{p}_d[k] = \int_0^{k\tau} \underline{S}(k\tau - t)\boldsymbol{p}(t)\,dt.$$

Since $\underline{S}(t)$ cannot be the delta function, the input of S cannot instantaneously affect its output. Therefore, a whole of a sampled-data controller HK_dS does not instantaneously pass its input to its output. This similarity between a continuous-time controller in $\mathcal{K}_0$ and a sampled-data controller is considered to be a reason why Equation (4.1) holds.

Remark 4.3. One may think that, if we allow $\underline{S}(t)$ to be the delta function, then a sampler is allowed to respond instantaneously to its input and eventually the left-hand side of (4.1) would become equal to the best continuous-time control performance $\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$. However, this idea has some difficulties. First, practically, such a sampler is sensitive to a high-frequency noise just like the ideal sampler. (Recall that the ideal sampler has the delta function as its kernel function.) It is problematic to feed a noisy sensor output directly to such a sampler. If we use an anti-aliasing filter before the sampler, then the instantaneous response is impossible. Second, theoretically, such a sampler has an infinite induced norm and a sampled-data control system with it never can be input-output stable in the sense of Definition 3.22. □

A sufficient condition for Equation (4.2) to be satisfied was given in Proposition 2.44. From this, we can see that most of plants that appear in practical problems satisfy (4.2), though not all of them. The result is summarized as follows.

Corollary 4.4. *If at least one of $\widehat{G}_{11}(s)$, $\widehat{G}_{12}(s)$, and $\widehat{G}_{21}(s)$ is strictly proper, then the theoretical bound of the best sampled-data control performance is equal to the best continuous-time control performance, that is,*

$$\inf_{(\tau,S,H)} \inf_{K_d \in \mathcal{K}_d} \|\mathcal{F}(G, HK_dS)\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|. \tag{4.3}$$

Example 4.5. Let us recall the systems considered in Example 4.1. Here, a generalized plant G is provided as $G_{11} = O$, $G_{12} = I$, and $G_{21} = PW$, and $G_{22} = P$. Since G_{11} is strictly proper, Corollary 4.4 ensures that Equation (4.3) holds in this case; in other words, we can make the best sampled-data control performance converge to the best continuous-time control performance by choosing an appropriate sequence of sampling environments.

In the first case of Example 4.1, the sampling environment is set as $(\tau, S, H) := (\tau, S_\tau^{\mathrm{id}} R_\tau, H_\tau^{\mathrm{zo}})$, while in the second case it is chosen as $(\tau, S, H) := (\tau, S_\tau^{\mathrm{id}} R, H_\tau^{\mathrm{zo}})$. Here, R_τ and R are continuous-time operators whose transfer functions are $1/(\tau s+1)$ and $1/(s+1)$, respectively. According to the experimental result presented in Example 4.1, the convergence

$$\inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\| \to \inf_{K\in\mathcal{K}} \|\mathcal{F}(G,K)\| \qquad (\tau\to 0)$$

is *not* accomplished in the first case while it *is* accomplished in the second case. This means that the first sampling environment $(\tau, S, H) = (\tau, S_\tau^{\mathrm{id}}R_\tau, H_\tau^{\mathrm{zo}})$ *does not* attain the theoretical bound

$$\inf_{(\tau,S,H)} \inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\|$$

as $\tau\to 0$, whereas the second environment $(\tau, S, H) = (\tau, S_\tau^{\mathrm{id}}R, H_\tau^{\mathrm{zo}})$ *does* attain it. □

In the next section, we seek for a condition in order to ensure that a provided sequence of sampling environments attains the theoretical bound. We close this section by giving a proof to Theorem 4.2.

Proof of Theorem 4.2. First, let us prove

$$\inf_{(\tau,S,H)} \inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\| \geq \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\|.$$

For this purpose, it suffices to show $\|\mathcal{F}(G, HK_\mathrm{d}S)\| \geq \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\|$ for any (τ, S, H) and $K_\mathrm{d}\in\mathcal{K}_\mathrm{d}$. If a sampled-data control system is not input-output stable, the value of $\|\mathcal{F}(G, HK_\mathrm{d}S)\|$ is infinite by definition. In this case, the desired inequality is trivial. Thus, let us consider an input-output stable sampled-data control system. Applying Theorem 3.34 on this system, we can obtain a continuous-time controller sequence $\{K_j\}_{j=1}^{\infty}$, $K_j\in\mathcal{K}_0$, such that $\widehat{\mathcal{F}}(G,K_j)(s)$ converges to the function

$$\widehat{G}_{11}(s) + \frac{1}{\tau}\widehat{G}_{12}(s)\underline{\widehat{H}}(s)\check{L}_\mathrm{d}(\mathrm{e}^{s\tau})\underline{\widehat{S}}(s)\widehat{G}_{21}(s)$$

uniformly in $\mathrm{Re}\, s\geq 0$. Here, L_d is a discrete-time operator defined as $K_\mathrm{d}(I-SG_{22}HK_\mathrm{d})^{-1}$. Let us write the function displayed above as $\Theta(s)$. Then, we have

$$\begin{aligned}\inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\| &\leq \lim_{j\to\infty} \|\mathcal{F}(G,K_j)\| = \lim_{j\to\infty} \|\widehat{\mathcal{F}}(G,K_j)\|_{\mathcal{H}^\infty} = \|\Theta\|_{\mathcal{H}^\infty} \\ &= \sup_{\omega\in\mathbb{R}} \overline{\sigma}\{\Theta(\mathrm{i}\omega)\} = \sup_{\omega\in[-\pi/\tau,\pi/\tau)} \sup_{m=0,\pm1,\ldots} \overline{\sigma}\Big\{\Theta\Big(\mathrm{i}\omega+\frac{\mathrm{i}2\pi m}{\tau}\Big)\Big\}. \end{aligned} \tag{4.4}$$

Here, the first inequality follows from $K_j\in\mathcal{K}_0$, and the second equality from the relationship between the $\mathcal{L}^2$-induced norm and the $\mathcal{H}^\infty$-norm (Proposition 2.4). The third equality is obtained from uniform convergence of $\widehat{\mathcal{F}}(G,K_j)(s)$ to $\Theta(s)$. Now, as we have noticed just after Theorem 3.34, this function $\Theta(s)$ is related to the matrix representation of $\check{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{s\tau})$, that is, there holds

$$\Theta\Big(\mathrm{i}\omega+\frac{\mathrm{i}2\pi m}{\tau}\Big) = \acute{E}_m^{\mathrm{i}\omega}\check{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{\mathrm{i}\omega\tau})\grave{E}_m^{\mathrm{i}\omega}.$$

Here, the right-hand side of the above equation stands for the (m,m)-block of the matrix representation of $\check{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{\mathrm{i}\omega\tau})$. Hence, the former half of Proposition 2.27 ensures that

$$\overline{\sigma}\Big\{\Theta\Big(\mathrm{i}\omega+\frac{\mathrm{i}2\pi m}{\tau}\Big)\Big\} \leq \|\check{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{\mathrm{i}\omega\tau})\|_\mathrm{L}. \tag{4.5}$$

Substitute (4.5) into (4.4). Then, since the right-hand side of (4.5) does not depend on m, there holds

$$\inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\| \leq \sup_{\omega\in[-\pi/\tau,\pi/\tau)} \|\check{\mathcal{F}}(G, HK_\mathrm{d}S)(\mathrm{e}^{\mathrm{i}\omega\tau})\|_\mathrm{L} = \|\check{\mathcal{F}}(G, HK_\mathrm{d}S)\|_{\mathfrak{H}_\mathrm{L}^\infty}.$$

The rightmost quantity is equal to $\|\mathcal{F}(G, HK_\mathrm{d}S)\|$ by Proposition 2.16.

Next, it is proven that

$$\inf_{(\tau,S,H)} \inf_{K_\mathrm{d}\in\mathcal{K}_\mathrm{d}} \|\mathcal{F}(G, HK_\mathrm{d}S)\| \leq \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\|.$$

By Proposition 2.43, the right-hand side is finite. Moreover, according to Proposition 2.45,

$$\inf_{K_{00}\in\mathcal{K}_{00}} \|\mathcal{F}(G,K_{00})\| = \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G,K_0)\|,$$

where $\mathcal{K}_{00}$ is a subset of $\mathcal{K}$ defined as

$$\mathcal{K}_{00} := \{K_{00}\in\mathcal{K} : \widehat{K}_{00}(\infty) = O \text{ with multiplicity two or more}\}$$

in Section 2.5. It is possible to choose K_{00} from $\mathcal{K}_{00}$ so that $\|\mathcal{F}(G,K_{00})\|$ is arbitrarily close to the above infimum. Therefore, the desired inequality is proven if we can find a sampling environment family $\{(\tau, S_\tau, H_\tau)\}_{\tau>0}$ and a controller family $\{K_{\mathrm{d},\tau}\}_{\tau>0}$, $K_{\mathrm{d},\tau}\in\mathcal{K}_\mathrm{d}$, such that

$$\|\mathcal{F}(G, H_\tau K_{\mathrm{d},\tau}S_\tau)\| \to \|\mathcal{F}(G,K_{00})\| \qquad (\tau\to 0).$$

In the following we do this.

The basic idea of the proof is borrowed from the proof of Theorem 4 in [18]. Let K_{00} be any operator in $\mathcal{K}_{00}$. We can assume that the continuous-time closed-loop system made of G and K_{00} is input-output stable because we are interested in this case only. Define R^n to be a continuous-time operator whose transfer function is $\{1/(s+1)\}I_n$, where I_n is the $n\times n$-identity matrix. Then, since $K_{00}\in\mathcal{K}_{00}$, K_{00} can be expressed as $R^{n_u}K_1R^{n_y}$ with some $K_1\in\mathcal{K}$. The resulting system is shown in Figure 4.4 (a). Define an operator G_ref so that $[\boldsymbol{z}^\mathrm{T} \;\; \boldsymbol{y}^\mathrm{T} \;\; \boldsymbol{b}_0^\mathrm{T}]^\mathrm{T} = G_\mathrm{ref}[\boldsymbol{w}^\mathrm{T} \;\; \boldsymbol{y}_0^\mathrm{T} \;\; \boldsymbol{b}^\mathrm{T}]^\mathrm{T}$ in this figure. Since $K_1\in\mathcal{K}$, K_1 has a state-space representation. Hence, at $s=\infty$, $\widehat{K}_1(s)$ is equal to the "D"-matrix of this state-space representation; in particular $\widehat{K}_1(s)$ is bounded there. From this, we can derive that G_ref is a bounded operator as follows.

Let us show G_ref is bounded. The operator G_ref is composed of nine operators. Because boundedness of these nine operators can be proven similarly, we give an explicit proof only on

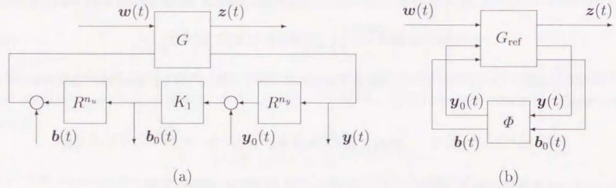


Figure 4.4. (a) Definition of G_{ref}; (b) The feedback connection between G_{ref} and Φ.

the operator from $\boldsymbol{w}(t)$ to $\boldsymbol{b}_0$ for instance. Let us write $(I - G_{22}R^{n_u}K_1R^{n_y})^{-1}G_{21} =: A$ in short. In the sequel, we call a pole in $\{s : \mathrm{Re}\, s \geq 0 \text{ or } s = \infty\}$ an *unstable pole*. Then, the considered operator from $\boldsymbol{w}(t)$ to $\boldsymbol{b}_0(t)$ is expressed as $K_1R^{n_y}A$. By the assumption that $K_{00} = R^{n_u}K_1R^{n_y}$ stabilizes G, the operators $R^{n_u}K_1R^{n_y}A$ and A are bounded. (Recall the definition of input-output stability as for a continuous-time control system, which was given in Definition 2.29.) Hence, the continuous-time transfer function of A, i.e., $\widehat{A}(s)$, does not have an unstable pole. This means that $(K_1R^{n_y}A)\widehat{\ }(s)$ can have an unstable pole only at the point where $\widehat{K_1}(s)$ has its unstable pole. Since $\widehat{K_1}(\infty)$ is equal to the "D"-matrix of its state-space representation, $\widehat{K_1}(s)$ does not have a pole at $s = \infty$. However, if $(K_1R^{n_y}A)\widehat{\ }(s)$ has an unstable pole at some point other than $s = \infty$, so does $(R^{n_u}K_1R^{n_y}A)\widehat{\ }(s)$ since $\widehat{R}^{n_u}(s)$ is invertible at $s \neq \infty$. This contradicts with boundedness of $R^{n_u}K_1R^{n_y}A$. Therefore, $(K_1R^{n_y}A)\widehat{\ }(s)$ has no unstable pole, which means boundedness of $K_1R^{n_y}A$.

Define $S_\tau^n := S_\tau^{\mathrm{id}}R^n$, where S_τ^{id} is the ideal sampler whose sampling period is τ. As we saw in Example 3.3, this S_τ^n is a regular sampler. Moreover, let H_τ^n be the zero-order hold such that its sampling period is τ and its input signal and output signal have the dimension n, respectively. This H_τ^n is a regular hold as is seen in Example 3.6. By Proposition 3.10, there holds $\|H_\tau^nS_\tau^n - R^n\| \to 0$ as $\tau \to 0$. Now, consider feedback connection between G_{ref} and

$$\begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} H_\tau^{n_y}S_\tau^{n_y} - R^{n_y} & O \\ O & H_\tau^{n_u}S_\tau^{n_u} - R^{n_u} \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{b}_0 \end{bmatrix},$$

and construct a closed-loop system. We write the above displayed operator as Φ. Figure 4.4 (b) shows the constructed closed-loop system. Note that G_{ref} is a bounded operator and Φ satisfies $\|\Phi\| \to 0$ as $\tau \to 0$. Therefore, the closed-loop system consisting of G_{ref} and Φ is input-output stable for sufficiently small τ. Moreover, it satisfies

$$\|\mathcal{F}(G_{\mathrm{ref}}, \Phi)\| \to \|\mathcal{F}(G_{\mathrm{ref}}, O)\| = \|\mathcal{F}(G, K_{00})\| \qquad (\tau \to 0).$$

Besides, it can be shown that

$$\mathcal{F}(G_{\mathrm{ref}}, \Phi) = \mathcal{F}(G, H_\tau^{n_u}S_\tau^{n_u}K_1H_\tau^{n_y}S_\tau^{n_y})$$

by definition. Write $K_{\mathrm{d},\tau} := S_\tau^{n_u}K_1H_\tau^{n_y}$. Then, we have $K_{\mathrm{d},\tau} \in \mathcal{K}_{\mathrm{d}}$ and $\|\mathcal{F}(G, H_\tau^{n_u}K_{\mathrm{d},\tau}S_\tau^{n_y})\| \to \|\mathcal{F}(G, K_{00})\|$ as $\tau \to 0$. □

Remark 4.6. In Theorem 4.2, Equation (4.1) remains to hold even if we restrict the class of samplers and holds by assuming $\breve{S} \in z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\breve{H} \in \mathfrak{RH}_{\mathrm{T}}^\infty$. Here, $\breve{S} \in z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\breve{H} \in \mathfrak{RH}_{\mathrm{T}}^\infty$ mean that a sampler S and a hold H have their lifting-based state-space representations, respectively (Proposition 3.16). In order to see this, note that the former half of the above proof works as it is even if $\breve{S} \in z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\breve{H} \in \mathfrak{RH}_{\mathrm{T}}^\infty$ are assumed. In the latter half of the proof, recall that S_τ^n and H_τ^n have transfer functions in $z^{-1}\mathfrak{RH}_{\mathrm{F}}^\infty$ and $\mathfrak{RH}_{\mathrm{T}}^\infty$, respectively, as were commented before Proposition 3.18.

Furthermore, Proposition 3.29 stated that the class of sampled-data controllers does not change even if the kernel functions $\underline{S}(t)$ and $\underline{H}(t)$ are allowed to take a nonzero value only in $[0, \tau)$. Therefore, under this restriction, Equation (4.1) still holds.

Here, we like to comment on a result of Tadmor. In [86], he considered an optimal design of a sampler and a hold as well as a discrete-time controller assuming that $\widehat{G}_{11}(s)$ is strictly proper. As a corollary to one of his main results, he gave a result about the best achievable performance (Corollary 3.1 in [86]). In our terms, his result can be stated as

$$\lim_{\tau \to 0} \inf_{(S,H)} \inf_{K_{\mathrm{d}} \in \mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\| = \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|,$$

where the leftmost infimum is taken over all samplers and holds under the condition that their sampling periods are equal to τ and their kernel functions are nonzero only in $[0, \tau)$. From the above discussion, Equation 4.1 continues to be valid even if the kernel functions of a sampler and a hold are allowed to be nonzero only in $[0, \tau)$. Since strict properness of $\widehat{G}_{11}(s)$ is assumed here, Corollary 4.4 implies that the theoretical bound of the best sampled-data control performance is equal to the best continuous-time control performance. Therefore, his result on the best achievable performance can be understood as a special case of ours. □

4.3. A Necessary and Sufficient Condition for the Convergence

In the previous section, we observe that there holds

$$\inf_{(\tau,S,H)} \inf_{K_{\mathrm{d}} \in \mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, HK_{\mathrm{d}}S)\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\|;$$

in other words, the theoretical bound of the best sampled-data control performance is equal to the best continuous-time control performance when a controller class is limited. By choosing an appropriate sampling environment, we can make the best sampled-data control performance as close to this bound as we wish. Then, how should we choose the environment for this? To state this problem precisely, we suppose that a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ is provided. Because there must exist a plant G that is consistent with all sampling environments (τ_j, S_j, H_j), the dimensions n_p and n_q are constant for all environments. (Recall that n_p and n_q stand for the input-signal dimension of S_j and the output-signal dimension of H_j, respectively.) Then, our purpose in this section is to obtain a condition in order to guarantee the performance convergence:

$$\lim_{j\to\infty} \inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| = \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\| \tag{4.6}$$

for all plants G consistent with this n_p and n_q. Note that, especially for plants that satisfy (4.2), Equation (4.6) ensures convergence to the best continuous-time control performance.

The reasons to consider this problem are as follow. The first and main reason is that non-converging examples such as Example 4.1 inspire our theoretical interests. Intuitively, it is obvious that the best sampled-data control performance converges to the best continuous-time control performance. It is considered that this conjecture helped the sampled-data control scheme to be accepted widely in practice. However, a non-converging example such as Example 4.1 tells us that this conjecture is not always correct. Then, we have to clarify why such a non-converging phenomenon occurs and how we can avoid it in order to keep the sampled-data control scheme being acceptable. Another reason is that such investigation on the performance convergence gives one way to appraise existing samplers and holds from an asymptotic viewpoint. For instance, Example 4.1 suggests that bandwidth of an anti-aliasing filter should not be taken proportionally to the Nyquist frequency though some textbooks say the opposite. Regarding an anti-aliasing filter as a part of a sampler, we can also say that such choice of a sampler is not appropriate for a good performance. If we can obtain a condition for the performance convergence, we should be able to find inappropriateness of this sampler without doing a simulation. Finally, through a convergence analysis, we can see what is important in samplers and holds to improve the best achievable performance. Such knowledge is believed to be useful to design an efficient sampler and hold for a given plant.

Our first result on this problem is a condition that the sampling period has to satisfy for the performance convergence. We have assumed that the signal dimensions n_p and n_q (i.e., the input-signal dimension of S_j and the output-signal dimension of H_j) are constant for all environments (τ_j, S_j, H_j). In order to obtain the following result, we need to further assume that the dimensions n_p^{d} and n_q^{d} (i.e., the output-signal dimension of S_j and the input-signal dimension of H_j) are bounded uniformly for all (τ_j, S_j, H_j).

Proposition 4.7. *Suppose we are provided a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$. Assume that the dimensions n_p and n_q are constant for all (τ_j, S_j, H_j) and the dimensions n_p^{d} and n_q^{d} are bounded uniformly for all (τ_j, S_j, H_j). Then, the performance convergence:*

$$\lim_{j\to\infty} \inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| = \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\|$$

is accomplished for all generalized plants G consistent with n_p and n_q, only if τ_j converges to zero as j increases.

Proof. First, let us consider the case of $n_p = n_q = n_p^{\mathrm{d}} = n_q^{\mathrm{d}} = 1$ for simplicity. This means that the kernel function of S_j amd H_j are scalar-valued. Suppose that G is a plant such that $\widehat{G}_{11} = 1/(s+1)$, $\widehat{G}_{12} = \widehat{G}_{21} = 1$, and $\widehat{G}_{22} = 0$. If we put K_0 to have $\widehat{K_0}(s) = -1/(s+1)$, there holds $\|\mathcal{F}(G, K_0)\| = 0$. Hence, $\inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = 0$.

Now, let us consider sampled-data control of this G under an environment (τ_j, S_j, H_j) and show that its best performance does not approach zero unless $\tau_j \to 0$. When a sampled-data controller $H_j K_{\mathrm{d}} S_j$ stabilizes G, Proposition 3.24 implies that the closed-loop operator $\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)$ has its transfer function and this transfer function is analytic in $\mathbb{D}_\rho$ for some $0 < \rho < 1$. Hence, by Propositions 2.16 and the comments preceding it, we have

$$\|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| = \sup_{\omega\in[-\pi/\tau_j, \pi/\tau_j)} \|\breve{\mathcal{F}}(G, H_j K_{\mathrm{d}} S_j)(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}.$$

Using $\acute{E}_m^{\mathrm{i}\omega}$ and $\grave{E}_\ell^{\mathrm{i}\omega}$, which were introduced in Section 2.4.4, we can represent the operator $\breve{\mathcal{F}}(G, H_j K_{\mathrm{d}} S_j)(\mathrm{e}^{\mathrm{i}\omega\tau_j})$ in the matrix form

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \acute{E}_{-1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{-1}^{\mathrm{i}\omega} & \acute{E}_{-1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{0}^{\mathrm{i}\omega} & \acute{E}_{-1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{1}^{\mathrm{i}\omega} & \cdots \\ \cdots & \acute{E}_{0}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{-1}^{\mathrm{i}\omega} & \acute{E}_{0}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{0}^{\mathrm{i}\omega} & \acute{E}_{0}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{1}^{\mathrm{i}\omega} & \cdots \\ \cdots & \acute{E}_{1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{-1}^{\mathrm{i}\omega} & \acute{E}_{1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{0}^{\mathrm{i}\omega} & \acute{E}_{1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{1}^{\mathrm{i}\omega} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix},$$

where $\breve{\mathcal{F}}$ is a shorthand for $\breve{\mathcal{F}}(G, H_j K_{\mathrm{d}} S_j)(\mathrm{e}^{\mathrm{i}\omega\tau_j})$. Now, Proposition 2.27 claims that the value of $\|\breve{\mathcal{F}}(G, H_j K_{\mathrm{d}} S_j)(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}$ is larger than or equal to the maximum singular value of any submatrix in the above matrix representation. Therefore, we have

$$\begin{aligned} \|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| &\geq \|\breve{\mathcal{F}}(G, H_j K_{\mathrm{d}} S_j)(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}\Big|_{\omega=-\pi/\tau_j} \\ &\geq \overline{\sigma}\left\{\begin{bmatrix} \acute{E}_{0}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{0}^{\mathrm{i}\omega} & \acute{E}_{0}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{1}^{\mathrm{i}\omega} \\ \acute{E}_{1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{0}^{\mathrm{i}\omega} & \acute{E}_{1}^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_{1}^{\mathrm{i}\omega} \end{bmatrix}\right\}\Bigg|_{\omega=-\pi/\tau_j}. \end{aligned} \tag{4.7}$$

Theorem 3.33 gives the explicit formula for $\acute{E}_m^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_\ell^{\mathrm{i}\omega}$, that is,

$$\acute{E}_m^{\mathrm{i}\omega}\breve{\mathcal{F}}\grave{E}_\ell^{\mathrm{i}\omega} = \frac{1}{\mathrm{i}\omega + \mathrm{i}2\pi m/\tau_j + 1}\delta_{m,\ell} + \frac{1}{\tau_j}\underline{\widehat{H}}_j\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\breve{L}_{\mathrm{d}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\underline{\widehat{S}}_j\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi\ell}{\tau_j}\right).$$

Hence,

$$\|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| \geq \overline{\sigma}\left\{\begin{bmatrix} \frac{1}{-\mathrm{i}\pi/\tau_j+1} & 0 \\ 0 & \frac{1}{\mathrm{i}\pi/\tau_j+1} \end{bmatrix} + \frac{1}{\tau_j}\begin{bmatrix} \underline{\widehat{H}}_j\left(-\frac{\mathrm{i}\pi}{\tau_j}\right) \\ \underline{\widehat{H}}_j\left(\frac{\mathrm{i}\pi}{\tau_j}\right) \end{bmatrix} \check{L}_{\mathrm{d}}(-1)\begin{bmatrix} \underline{\widehat{S}}_j\left(-\frac{\mathrm{i}\pi}{\tau_j}\right) & \underline{\widehat{S}}_j\left(\frac{\mathrm{i}\pi}{\tau_j}\right) \end{bmatrix}\right\}. \tag{4.8}$$

Note that the functions $\underline{\widehat{S}}_j(s)$ and $\underline{\widehat{H}}_j(s)$ are scalar-valued. Since the second term of the right-hand side of (4.8) has a rank one or less, this maximum singular value must be greater than or equal to

$$\left|\frac{1}{-\mathrm{i}\pi/\tau_j+1}\right| = \left|\frac{1}{\mathrm{i}\pi/\tau_j+1}\right| = \frac{\tau_j}{\sqrt{\pi^2+\tau_j^2}}.$$

It is clear now that $\|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\|$ does not converge to zero unless $\tau_j \to 0$ as $j \to \infty$.

Next, we consider the case that $n_p = n_q = 1$ but not necessarily $n_p^{\mathrm{d}} = n_q^{\mathrm{d}} = 1$. Since n_p^{d} and n_q^{d} are bounded uniformly for all (τ_j, S_j, H_j), it is possible to find their upper bound, say n^{d}. The proposition is proven similarly to the previous case except that we choose in (4.7) a submatrix having at least $n^{\mathrm{d}}+1$ rows and columns. Indeed, the input and output of the discrete-time operator $L_{\mathrm{d}} = K_{\mathrm{d}}(I - S_j G_{22} H_j K_{\mathrm{d}})^{-1}$ have dimensions n^{d} or less this time. Therefore, the second term of the right-hand side of (4.8) does not have a full rank again, which enables us to use a similar reasoning.

Finally in the case that $n_p \neq 1$ or $n_q \neq 1$, consider a plant G such that the $(1,1)$-elements of $\widehat{G}_{11}(s)$, $\widehat{G}_{12}(s)$, $\widehat{G}_{21}(s)$, and $\widehat{G}_{22}(s)$ are as above and other elements are all equal to zero. This time, each $\acute{E}_m^{\mathrm{i}\omega}\check{\mathcal{F}}\grave{E}_\ell^{\mathrm{i}\omega}$ is not a scalar but a matrix. However, applying the same procedure blockwise, we can show the claim. □

In the sequel, we do not especially assume that n_p^{d} and n_q^{d} are bounded uniformly for all environments (τ_j, S_j, H_j). However, as is seen from Example 4.5, they are considered to be bounded in many situations which are of practical importance. Hence, in the following, we assume $\tau_j \to 0$ as $j \to \infty$. Proposition 4.7 guarantees that not so much generality is lost because of this assumption.

If we restrict ourselves to the case that a sampling period τ_j approaches zero and a sampler S_j and a hold H_j have their lifting-based state-space representations, a necessary and sufficient condition for convergence can be obtained. Since many of practically important samplers and holds require $\tau_j \to 0$ (Proposition 4.7), and also many of them have state-space representations (Proposition 3.17), this result is significant. Recall that a regular sampler S_j has a lifting-based state-space representation if and only if its lifting-based transfer function $\check{S}_j(z)$ belongs to $z^{-1}\mathfrak{RH}_{\mathrm{F}}^{\infty}$. Similarly, a regular hold H_j has a lifting-based state-space representation if and only if its lifting-based transfer function $\check{H}_j(z)$ belongs to $\mathfrak{RH}_{\mathrm{T}}^{\infty}$ (Proposition 3.16).

Theorem 4.8. *Let $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ be a sequence of sampling environments whose dimensions n_p and n_q are constant independently of j. Assume that $\tau_j \to 0$ as $j \to \infty$ and S_j and H_j have*

their lifting-based state-space representations for each j. Then, the performance convergence:

$$\lim_{j\to\infty} \inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\| = \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\|$$

is accomplished for all plants G consistent with n_p and n_q if and only if there exist a regular sampler sequence $\{S_j^0\}_{j=1}^{\infty}$ and a regular hold sequence $\{H_j^0\}_{j=1}^{\infty}$ such that each S_j^0 and H_j^0 have their state-space representations and there hold

$$\|R^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty), \tag{4.9}$$

$$\|R^{n_p} - H_j^0 S_j\| \to 0 \qquad (j \to \infty). \tag{4.10}$$

Here, R^n is a continuous-time operator whose continuous-time transfer function is $\{1/(s+1)\}I_n$.

If a provided plant satisfies Equation (4.2), that is,

$$\inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{K\in\mathcal{K}} \|\mathcal{F}(G, K)\|,$$

the above condition ensures convergence to the best continuous-time control performance. This fact can be strengthened as is stated below.

Corollary 4.9. *Suppose that an environment sequence as in Theorem 4.8 is provided. Consider the set of all plants G that satisfy (4.2) and are consistent with n_p and n_q. Then, this environment sequence guarantees, for any plant in this set, that the best sampled-data control performance $\inf_{K_{\mathrm{d}}\in\mathcal{K}_{\mathrm{d}}} \|\mathcal{F}(G, H_j K_{\mathrm{d}} S_j)\|$ converges to the best continuous-time control performance $\inf_{K\in\mathcal{K}} \|\mathcal{F}(G, K)\|$, if and only if there exist a sampler sequence $\{S_j^0\}_{j=1}^{\infty}$ and a hold sequence $\{H_j^0\}_{j=1}^{\infty}$ such that each S_j^0 and H_j^0 have their state-space representation and there hold (4.9) and (4.10).*

Note that the condition given in Theorem 4.8 is split into a condition on holds (4.9) and a condition on samplers (4.10). It is interesting that these two conditions are symmetric to each other.

Proof of Theorem 4.8. [if] Again, we use the set $\mathcal{K}_{00}$, which was defined as

$$\mathcal{K}_{00} := \{K_{00} \in \mathcal{K} : \widehat{K}_{00}(\infty) = O \text{ with multiplicity two or more}\}.$$

Recall that

$$\inf_{K_{00}\in\mathcal{K}_{00}} \|\mathcal{F}(G, K_{00})\| = \inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\|$$

by Proposition 2.45.

Now, choose a $K_{00} \in \mathcal{K}_{00}$ so that $\|\mathcal{F}(G, K_{00})\|$ is close enough to $\inf_{K_{00}\in\mathcal{K}_{00}} \|\mathcal{F}(G, K_{00})\|$. Since $K_{00} \in \mathcal{K}_{00}$, it can be decomposed as $R^{n_q} K_1 R^{n_p}$ using some $K_1 \in \mathcal{K}$. Define $K_{\mathrm{d},j} :=$

$S_j^0 K_1 H_j^0$. Then $K_{d,j} \in \mathcal{K}_d$. Moreover, in a similar way to the latter half of the proof of Theorem 4.2, we can show that $\|\mathcal{F}(G, H_j K_{d,j} S_j)\|$ converges to $\|\mathcal{F}(G, K_{00})\|$. Now the "if" part is proven.

[only if] We prove the existence of $\{S_j^0\}$ only. The existence of $\{H_j^0\}$ is similarly proven.

Let us consider the case $n_q \le n_p$ first. Define G by putting $G_{11} = [R^{n_q} \ \ O]$, $G_{12} = I_{n_q}$, $G_{21} = I_{n_p}$, and $G_{22} = O$. If we put $K_0 := [-R^{n_q} \ \ O]$, then $\|\mathcal{F}(G, K_0)\| = 0$ and $K_0 \in \mathcal{K}_0$. Hence, $\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\| = \inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = 0$ in this case. Therefore, by assumption, we can choose $\{K_{d,j}\}_{j=1}^{\infty}$, $K_{d,j} \in \mathcal{K}_d$, so that $\|\mathcal{F}(G, H_j K_{d,j} S_j)\| \to 0$ as $j \to \infty$. Let us consider a sampler-type operator $K_{d,j}(I - S_j G_{22} H_j K_{d,j})^{-1} S_j$ and write it as S_j^1. This S_j^1 is an operator from $\boldsymbol{a}(t)$ to $\boldsymbol{q}_d'[k]$ in Figure 4.2 (b). Hence, it is bounded. Proposition 3.24 implies that its lifting-based transfer function $\breve{S}_j^1(z)$ has no pole in $\mathbb{D}_\rho$ for some $0 < \rho < 1$. Furthermore, since $K_{d,j}$, S_j, G_{22}, and H_j have rational lifting-based transfer functions, so does S_j^1. Finally, since $\breve{S}_j(\infty) = O$, there holds $\breve{S}_j^1(\infty) = O$. In summary, $\breve{S}_j^1$ belongs to $z^{-1}\mathfrak{R}\mathfrak{H}_F^{\infty}$. On the other hand, it can be seen that $\mathcal{F}(G, H_j K_{d,j} S_j) = [R^{n_q} \ \ O] - H_j S_j^1$, which means $\|[R^{n_q} \ \ O] - H_j S_j^1\| \to 0$ as $j \to \infty$. Here, collect the first n_q columns of S_j^1 and write them as S_j^0. It is now easy to see that $\|R^{n_q} - H_j S_j^0\| \to 0$ as $j \to \infty$ and its transfer function $\breve{S}_j^0$ belongs to $z^{-1}\mathfrak{R}\mathfrak{H}_F^{\infty}$. This means that $\{S_j^0\}$ is exactly what we want.

Next, suppose $n_p < n_q \le 2n_p$. Define G by

$$G_{11} := \begin{bmatrix} R^{n_p} \\ O \end{bmatrix}, \quad G_{12} := I_{n_q}, \quad G_{21} := I_{n_p}, \quad G_{22} := O$$

and go through the previous procedure. Then, writing the operator $K_{d,j}(I - S_j G_{22} H_j K_{d,j})^{-1} S_j$ as S_j^1, we have

$$\left\| \begin{bmatrix} R^{n_p} \\ O \end{bmatrix} - H_j S_j^1 \right\| \to 0$$

as $j \to \infty$. Furthermore, replace G_{11} by

$$G_{11} := \begin{bmatrix} O & O \\ R^{n_q - n_p} & O \end{bmatrix}$$

and repeat the same procedure. Pick the first $n_q - n_p$ columns out of the obtained sampler, and write them as S_j^2. Then, there holds

$$\left\| \begin{bmatrix} O \\ R^{n_q - n_p} \end{bmatrix} - H_j S_j^2 \right\| \to 0$$

as $j \to \infty$. If we put $S_j^0 := [S_j^1 \ \ S_j^2]$, this S_j^0 satisfies all the requirements.

The proof is similar in the case of $2n_p < n_q$. □

Proof of Corollary 4.9. The "if" part directly follows from Theorem 4.8.

In the "only if" part of the proof of Theorem 4.8, all the considered G's have strictly proper G_{11}'s. This means that they satisfy (4.2) according to Corollary 4.4. Therefore, even if we concentrate only on the plants that attain (4.2), still we can derive the existence of $\{S_j^0\}$ and $\{H_j^0\}$ that satisfy the requirements. □

Remark 4.10. The conditions (4.9) and (4.10) include the continuous-time operators R^{n_q} and R^{n_p} whose continuous-time transfer functions are $\{1/(s+1)\}I_{n_q}$ and $\{1/(s+1)\}I_{n_p}$, respectively. These operators can be replaced by other operators to some degree.

First, let U^{n_q} be any continuous-time operator whose continuous-time transfer function $\widehat{U}^{n_q}(s)$ is $n_q \times n_q$ and is unimodular in $\mathcal{RH}^{\infty}$. Here, we say $\widehat{U}^{n_q}(s)$ is unimodular in $\mathcal{RH}^{\infty}$ if both $\widehat{U}^{n_q}(s)$ and $\widehat{U}^{n_q}(s)^{-1}$ belong to $\mathcal{RH}^{\infty}$ (Definition 2.33). The continuous-time operator having $\widehat{U}^{n_q}(s)^{-1}$ as its transfer function is the operator inverse of U^{n_q} and is denoted by $(U^{n_q})^{-1}$. It is clear that both U^{n_q} and $(U^{n_q})^{-1}$ have bounded $\mathcal{L}^2$-induced norms. Now, note that

$$\|R^{n_q} - H_j S_j^0\| \, \|U^{n_q}\| \ge \|R^{n_q} U^{n_q} - H_j S_j^0 U^{n_q}\|$$

and if S_j^0 is a regular sampler having a lifting-based state-space representation, so is $S_j^0 U^{n_q}$. Moreover, there holds

$$\|R^{n_q} U^{n_q} - H_j S_j^0\| \, \|(U^{n_q})^{-1}\| \ge \|R^{n_q} - H_j S_j^0 (U^{n_q})^{-1}\|$$

and $S_j^0 (U^{n_q})^{-1}$ is a regular sampler with a lifting-based state-space representation. From these facts, it is seen that there exists $\{S_j^0\}$ satisfying $\|R^{n_q} - H_j S_j^0\| \to 0$ if and only if there exists $\{S_j^0\}$ that satisfies $\|R^{n_q} U^{n_q} - H_j S_j^0\| \to 0$. This means that Theorem 4.8 and Corollary 4.9 remain to hold even with Equation (4.9) being replaced by

$$\|R^{n_q} U^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty).$$

Similar replacement is possible about Equation (4.10).

In fact, it is possible to further replace (4.9) and (4.10) by

$$\|(R^{n_q})^m U^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty), \tag{4.11}$$

$$\|U^{n_p} (R^{n_p})^{m'} - H_j^0 S_j\| \to 0 \qquad (j \to \infty), \tag{4.12}$$

respectively. Here, m and m' are positive integers and $(R^{n_q})^m$ stands for an operator whose continuous-time transfer function is $\{1/(s+1)^m\}I_{n_q}$. In order to show this, consider a subclass of $\mathcal{K}$ that consists of a continuous-time operator having a zero at $s = \infty$ with multiplicity $m + m'$ or more. Then, actually we can show that the infimum of $\|\mathcal{F}(G, K)\|$ when K varies in this class is equal to $\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\|$. (It is proven in a similar way to Proposition 2.45.)

Therefore, we can mimic the proofs of Theorem 4.8 and Corollary 4.9 replacing R^{n_q} and R^{n_p} by $(R^{n_q})^m$ and $(R^{n_p})^{m'}$ and finally obtain the equations

$$\|(R^{n_q})^m - H_j S_j^0\| \to 0 \qquad (j \to \infty),$$
$$\|(R^{n_p})^{m'} - H_j^0 S_j\| \to 0 \qquad (j \to \infty)$$

instead of (4.9) and (4.10). From the previous discussion, it is clear that these equations can further be replaced by (4.11) and (4.12). □

Example 4.11. Let us examine Example 4.1 using the results of Theorem 4.8 and Corollary 4.9.

As we saw in Example 4.5, there holds

$$\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|$$

in this case. Therefore, according to Corollary 4.9, if a provided sampling period approaches zero and there exist $\{S_j^0\}$ and $\{H_j^0\}$ satisfying (4.9) and (4.10), respectively, then the best sampled-data control performance converges to the best continuous-time control performance.

Let us consider the second case where the sampling environment was chosen as $(\tau, S_\tau^{\mathrm{id}} R, H_\tau^{\mathrm{zo}})$. Here, R is the continuous-time operator whose continuous-time transfer function is $1/(s+1)$. Let $\{\tau_j\}_{j=1}^\infty$ be any sequence of sampling periods that approaches zero. Then, $\{(\tau_j, S_{\tau_j}^{\mathrm{id}} R, H_{\tau_j}^{\mathrm{zo}})\}_{j=1}^\infty$ defines a sequence of sampling environments. Proposition 3.10 claims

$$\|R - H_{\tau_j}^{\mathrm{zo}} S_{\tau_j}^{\mathrm{id}} R\| \to 0 \qquad (j \to \infty). \tag{4.13}$$

This implies that there exist $\{S_j^0\}$ and $\{H_j^0\}$ that satisfy (4.9) and (4.10) in this case. Indeed, putting $S_j^0 := S_{\tau_j}^{\mathrm{id}} R$ and $H_j^0 := H_{\tau_j}^{\mathrm{zo}}$, we obtain (4.9) and (4.10). Thus, the best sampled-data control performance converges to the best continuous-time control performance in this case. This is consistent with the experimental result.

Next, let us examine the first case. There, the sampling environment was $(\tau, S_\tau^{\mathrm{id}} R_\tau, H_\tau^{\mathrm{zo}})$. Choose any sequence of sampling periods $\{\tau_j\}_{j=1}^\infty$ and consider the sequence $\{(\tau_j, S_{\tau_j}^{\mathrm{id}} R_{\tau_j}, H_{\tau_j}^{\mathrm{zo}})\}$. Equation (4.13) shows that there exists $\{S_j^0\}$ such that (4.9) is satisfied. On the other hand, since the best sampled-data control performance does not converge to the best continuous-time control performance, there should no $\{H_j^0\}$ satisfying (4.10). However, the inexistence of such $\{H_j^0\}$ is not clear itself. □

One may notice in the above example that the condition for convergence that Theorem 4.8 and Corollary 4.9 give is not so easy to be tested because the existence of $\{S_j^0\}$ and $\{H_j^0\}$ is not always obvious. In the next section, we obtain conditions easier to be tested.

4.4. Simpler Conditions for Convergence

In Section 4.3, we provided a necessary and sufficient condition in order that the best sampled-data control performance converges to its theoretical bound. Namely, the provided condition is the existence of a sampler sequence $\{S_j^0\}$ satisfying Equation (4.9)

$$\|R^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty)$$

and the existence of a hold sequence $\{H_j^0\}$ satisfying Equation (4.10)

$$\|R^{n_p} - H_j^0 S_j\| \to 0 \qquad (j \to \infty).$$

Here, R^n is a continuous-time operator whose continuous-time transfer function is $\{1/(s+1)\}I_n$. The former half of the condition is concerned with the provided holds $\{H_j\}$ and the latter half is concerned with the provided $\{S_j\}$.

A problem here is that, in order to check this condition, we have to find a sampler sequence $\{S_j^0\}$ and a hold sequence $\{H_j^0\}$ having particular characteristics. This is not an easy problem.

In this section, we try to simplify this condition. The basic idea is to note that Equation (4.9) resembles the model-matching problem considered in Section 2.6 and to apply techniques introduced there. Then, it is derived that the existence of a sampler sequence $\{S_j^0\}$ satisfying (4.9) is equivalent to two conditions: one condition implies that the Hankel norm of some function converges to zero as j increases; the other condition means that the side-band-frequency components of H_j disappear as $j \to \infty$ in some sense. Corresponding results can be obtained also on Equation (4.10).

Suppose that a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^\infty$ is provided. In this section, we put the following assumptions on this sequence. The first one is about the sampling periods τ_j:

(A1) the sampling period τ_j approaches zero as j goes to infinity.

As for the regular holds H_j we put the following assumptions:

(A2H) the output signal of each hold H_j has a constant dimension n_q irrespective of j;

(A3H) each hold H_j has a lifting-based state-space representation;

(A4H) each hold H_j has a lifting-based transfer function $\breve{H}_j(z)$ such that its matrix representation at $z = e^{i\omega\tau_j}$, i.e.,

$$\frac{1}{\sqrt{\tau_j}} \begin{bmatrix} \vdots \\ \widehat{H}_j\left(i\omega - \frac{i2\pi}{\tau_j}\right) \\ \widehat{H}_j(i\omega) \\ \widehat{H}_j\left(i\omega + \frac{i2\pi}{\tau_j}\right) \\ \vdots \end{bmatrix}$$

is column full rank (i.e., all the columns are independent) for any $\omega \in [-\pi/\tau_j, \pi/\tau_j)$.

Correspondingly, we require the regular samplers S_j to satisfy the following assumptions:

(A2S) the input signal of each sampler S_j has a constant dimension n_p independently of j;

(A3S) each sampler S_j has a lifting-based state-space representation;

(A4S) each sampler S_j has a lifting-based transfer function $\breve{S}_j(z)$ such that its matrix representation at $z = \mathrm{e}^{\mathrm{i}\omega\tau_j}$, i.e.,

$$\frac{1}{\sqrt{\tau_j}}\left[\cdots \ \underline{\widehat{S}}_j\left(\mathrm{i}\omega - \tfrac{\mathrm{i}2\pi}{\tau_j}\right) \ \underline{\widehat{S}}_j(\mathrm{i}\omega) \ \underline{\widehat{S}}_j\left(\mathrm{i}\omega + \tfrac{\mathrm{i}2\pi}{\tau_j}\right) \ \cdots\right],$$

is row full rank (i.e., all the rows are independent) for any $\omega \in [-\pi/\tau_j, \pi/\tau_j)$.

By Proposition 4.7, Assumption (A1) has to be satisfied in order that the best sampled-data performance converges to its theoretical bound when we consider practically important sampling environment sequences. Therefore, generality is not lost so much even if we assume it. Assumptions (A2H) and (A2S) are natural assumptions to ensure that all the sampling environments are consistent with a certain generalized plant. The remaining assumptions (A3H), (A4H), (A3S), and (A4S) are important here because they enable us to apply techniques introduced in Section 2.6. Nevertheless they are mild enough. Indeed, many practical samplers and holds satisfy them.

Now, let us consider the condition on holds, that is, the existence of a sampler sequence $\{S_j^0\}$ satisfying Equation (4.9). As for (4.9), there holds

$$\|R^{n_q} - H_j S_j^0\| = \|\breve{R}^{n_q} - \breve{H}_j \breve{S}_j^0\|_{\mathfrak{H}_{\mathrm{L}}^\infty}.$$

Since S_j^0 is required to have a lifting-based state-space representation in Theorem 4.8, its lifting-based transfer function has the form $\breve{S}_j^0 = z^{-1}\breve{S}_j'$ for some $\breve{S}_j' \in \mathfrak{R}\mathfrak{H}_{\mathrm{F}}^\infty$ (Proposition 3.16). Therefore, existence of such $\{S_j^0\}$ is equivalent to

$$\inf_{\breve{S}_j' \in \mathfrak{R}\mathfrak{H}_{\mathrm{F}}^\infty} \|\breve{R}^{n_q} - z^{-1}\breve{H}_j \breve{S}_j'\|_{\mathfrak{H}_{\mathrm{L}}^\infty} \to 0 \qquad (j \to \infty). \tag{4.14}$$

In order to see this, suppose (4.14) holds. Then, if we choose each S_j' so that $\|\breve{R}^{n_q} - z^{-1}\breve{H}_j\breve{S}_j'\|_{\mathfrak{H}_{\mathrm{L}}^\infty}$ is close enough to its infimum and put $\breve{S}_j^0(z) := z^{-1}\breve{S}_j'(z)$, this $\{S_j^0\}$ accomplishes (4.9). The converse is also easy, too. Now, note that Equation (4.14) resembles Equation (2.11), which was investigated in Section 2.6. Therefore, by application of the techniques there, we can simplify the provided condition.

By Assumption (A4H), the matrix representation of a tall operator $\breve{H}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j})$ has independent columns for any $\omega \in [-\pi/\tau_j, \pi/\tau_j)$. Therefore, following the procedure in Subsection 2.6.1, we can choose an inner function $\breve{H}_j^{\mathrm{in}}(z)$ and an outer function $\breve{H}_j^{\mathrm{out}}(z)$ so that $\breve{H}_j(z) = \breve{H}_j^{\mathrm{in}}(z)\breve{H}_j^{\mathrm{out}}(z)$ for each j. This is an inner-outer factorization of $\breve{H}_j(z)$. Recall that

a real rational function $A(z)$ is called inner if $A^\sim(z)A(z) \equiv I$. Recall also that a real rational function $A(z)$ is said to be outer if $A(z)$ is square-matrix-valued and $\det A(z) \neq 0$ for any z in the set $\{z : z \in \mathbb{D} \text{ or } |z| = 1\}$. Here, $A^\sim(z)$ is a function defined as $A^\sim(z) := A(1/\bar{z})^*$. In our case, it is important that this inner-outer factorization can be obtained via matrix computations though $\breve{H}_j(z)$ and $\breve{H}_j^{\mathrm{in}}(z)$ are tall-operator-valued functions. Next, write the Laurent expansion of the flat-operator-valued function $z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}^{n_q}(z)$ into the form

$$z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}^{n_q}(z) = \sum_{k=-\infty}^{\infty} L^k z^k, \qquad L^k := \frac{1}{2\pi\mathrm{i}}\oint_{|z|=1} \frac{1}{z^{k+1}} z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}^{n_q}(z)\,\mathrm{d}z.$$

Here, each L^k is a flat operator. Then, the Hankel operator with the symbol $z\breve{H}_j^{\mathrm{in}\sim}\breve{R}^{n_q}$ is

$$\begin{bmatrix} L^1 & L^2 & L^3 & \cdots \\ L^2 & L^3 & L^4 & \\ L^3 & L^4 & L^5 & \\ \vdots & & & \ddots \end{bmatrix}$$

and its induced norm is the Hankel norm of the function $z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}^{n_q}(z)$, which is denoted by $\|z\breve{H}_j^{\mathrm{in}\sim}\breve{R}^{n_q}\|_{\mathrm{H}}$.

Now, we have the following result.

Theorem 4.12. *Suppose that a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^\infty$ satisfies Assumptions (A1), (A2H)–(A4H). Then, there exists a sampler sequence $\{S_j^0\}_{j=1}^\infty$ such that each sampler S_j^0 has a lifting-based state-space representation and there holds Equation (4.9), i.e.,*

$$\|R^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty),$$

if and only if the following two conditions are satisfied:

(a) $\|z\breve{H}_j^{\mathrm{in}\sim}\breve{R}^{n_q}\|_{\mathrm{H}} \to 0 \qquad (j \to \infty)$;

(b) For any $\Omega > 0$,

$$\bar{\sigma}\left[\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^{-1/2}\left\{\sum_{m\neq 0} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^{-1/2}\right] \tag{4.15}$$

converges to zero uniformly for any $|\omega| < \Omega$ as $j \to \infty$. Here, $\omega_m := \omega + 2\pi m/\tau_j$.

Proof. As is seen above, existence of $\{S_j^0\}$ satisfying (4.9) is equivalently restated as Equation (4.14). By Proposition 2.49, Equation (4.14) is further restated as

(a) $\|z\breve{H}_j^{\mathrm{in}\sim}\breve{R}^{n_q}\|_{\mathrm{H}} \to 0 \qquad (j \to \infty)$, (b′) $\|(I - \breve{H}_j^{\mathrm{in}}\breve{H}_j^{\mathrm{in}\sim})\breve{R}^{n_q}\|_{\mathfrak{L}_{\mathrm{L}}^\infty} \to 0 \qquad (j \to \infty).$

It is left for us to show (b′) is equivalent to (b). Since its proof is long, it is given in Appendix F. □

Interpretation of Condition (a) is not easy. In a rough sense, this condition is considered to mean that the effect of unstable zeros of $\underline{\widehat{H}}_j(s)$, i.e., zeros in $\mathrm{Re}\, s \geq 0$, decreases as $j \to \infty$. This interpretation comes from an analogy with a usual model-matching problem on matrix-valued functions. Another reason why we interpret so is that an equivalent expression is obtained for Condition (a) in some special case and it is related to unstable zeros. This is discussed after Theorem 4.18.

The meaning of Condition (b) becomes clearer if we consider the case that the input- and output-signal dimensions of H_j are equal to one. In this case, the quantity in this condition can be written as

$$\frac{\displaystyle\sum_{m\neq 0}\left|\underline{\widehat{H}}_j\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\right|^2}{\displaystyle\sum_{m=-\infty}^{\infty}\left|\underline{\widehat{H}}_j\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\right|^2}. \tag{4.16}$$

The function $\underline{\widehat{H}}_j(s)$ is the Laplace transform of the kernel function $\underline{H}_j(t)$. The values of $\underline{\widehat{H}}_j(\mathrm{i}\omega_m) = \underline{\widehat{H}}_j(\mathrm{i}\omega + \mathrm{i}2\pi m/\tau_j)$, $m \neq 0$, are the frequency components of $\underline{H}_j(t)$ at the side-band frequencies of ω. Therefore, the above value (4.16) and, in turn, the quantity in Condition (b) express a relative amount of the frequency components of $\underline{H}_j(t)$ at the side-band frequencies ω_m compared with the one at the original frequency ω. Or one may say that it stands for the amount of aliasing effects in the hold H_j. Condition (b) claims that this amount should converge to zero uniformly in $|\omega| < \Omega$.

Example 4.13. Recall a triangular hold H_τ^{tr}, which was introduced in Example 3.8. Its kernel function was defined as

$$\underline{H}_\tau^{\mathrm{tr}}(t) = \begin{cases} (1 - \frac{t}{\tau})I & \text{for } 0 \leq t < \tau, \\ O & \text{for } \tau \leq t. \end{cases}$$

Let $\{\tau_j\}$ be any sequence of sampling periods such that $\tau_j \to 0$ as $j \to \infty$. Moreover, let $\{S_j\}$ be any sequence of regular samplers such that the sampling period of S_j is τ_j for each j. Then, the triplet $(\tau_j, S_j, H_{\tau_j}^{\mathrm{tr}})$ forms a sampling environment and the sequence $\{(\tau_j, S_j, H_{\tau_j}^{\mathrm{tr}})\}$ satisfies (A1), (A2H)–(A4H).

Indeed, it is obvious that both (A1) and (A2H) are satisfied. From Proposition 3.17, (A3H) is correct. In order to see that (A4H) is fulfilled, note

$$\underline{\widehat{H}}_\tau^{\mathrm{tr}}(s) = \frac{\tau s - 1 + \mathrm{e}^{-\tau s}}{\tau s^2} I. \tag{4.17}$$

Hence, $\det \underline{\widehat{H}}_\tau^{\mathrm{tr}}(\mathrm{i}\omega) = 0$ if and only if $\omega = 0$. This means that it is not possible that all of

$$\det \underline{\widehat{H}}_\tau^{\mathrm{tr}}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right), \qquad m = 0, \pm 1, \pm 2, \dots,$$

are equal to zero simultaneously.

Now, let us apply Theorem 4.12 to this environment sequence $\{(\tau_j, S_j, H_{\tau_j}^{\mathrm{tr}})\}$. Equation (4.17) implies

$$\frac{1}{\tau_j}\underline{\widehat{H}}_{\tau_j}^{\mathrm{tr}}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right) = \frac{\mathrm{i}\omega\tau_j + \mathrm{i}2\pi m - 1 + \mathrm{e}^{-\mathrm{i}\omega\tau_j}}{(\mathrm{i}\omega\tau_j + \mathrm{i}2\pi m)^2} I \to \begin{cases} \frac{1}{2}I & \text{for } m = 0, \\ \frac{1}{\mathrm{i}2\pi m}I & \text{for } m \neq 0 \end{cases}$$

as $j \to \infty$. Therefore, the quantity (4.15) converges to a nonzero value. Here, the order of summation and limitation is converted. This is allowed because the quantity (4.15) is equal to

$$\bar{\sigma}\left[I - \left\{\sum_{m=-\infty}^{\infty}\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)\right\}^{-1/2}\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega)^*\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega)\left\{\sum_{m=-\infty}^{\infty}\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)\right\}^{-1/2}\right]$$

and $\sum_{m=-M}^{M}\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)$ converges to $\sum_{m=-\infty}^{\infty}\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)^*\underline{\widehat{H}}_j^{\mathrm{tr}}(\mathrm{i}\omega_m)$ as $M \to \infty$ uniformly to j.

The above result means that, for our environment sequence, there exists no $\{S_j^0\}$ satisfying (4.9). In other words, if a triangular hold H_τ^{tr} is used in a sampled-data control system, no matter what a sampler would be, there exists a plant G such that the best sampled-data control performance does not converge to its theoretical bound. □

As is seen in the above example, testing Condition (b) is easily done based on the kernel function of a hold H_j. Since (b) itself is a necessary condition for the existence of $\{S_j^0\}$, if (b) is not satisfied then there always exists G such that the best sampled-data control performance does not converge to its theoretical bound whatever the used sampler would be. Condition (a) is more complicated than (b). However, since the Hankel norm can be computed through matrix calculations, its test is not difficult.

A similar discussion is possible about the condition on samplers, that is, the existence of a hold sequence $\{H_j^0\}$ satisfying (4.10). Just like the case of the condition on holds, one can rewrite the above condition in the frequency domain as

$$\inf_{\breve{H}_j^0 \in \mathfrak{RH}_{\mathrm{T}}^\infty}\|\breve{R}^{n_p} - z^{-1}\breve{H}_j^0\breve{S}_j'\|_{\mathfrak{H}_{\mathrm{L}}^\infty} \to 0 \qquad (j \to \infty).$$

Here, $\breve{S}_j'(z)$ is a function in $\mathfrak{RH}_{\mathrm{F}}^\infty$ such that $\breve{S}_j(z) = z^{-1}\breve{S}_j'(z)$. Then, by virtue of Assumptions (A3S) and (A4S) we can obtain a co-inner-co-outer factorization of $\breve{S}_j'(z)$ as $\breve{S}_j'(z) = \breve{S}_j^{\mathrm{out}}(z)\breve{S}_j^{\mathrm{in}}(z)$. Here, $\breve{S}_j^{\mathrm{out}}(z)$ is a co-outer function and $\breve{S}_j^{\mathrm{in}}(z)$ is a co-inner function. See Section 2.6 for the definitions of these terms. Finally, let $\|z\breve{R}^{n_p}\breve{S}_j^{\mathrm{in}\sim}\|_{\mathrm{H}}$ be the Hankel norm of a tall-operator-valued function $z\breve{R}^{n_p}(z)\breve{S}_j^{\mathrm{in}\sim}(z)$. Then, using Proposition 2.50 we can obtain the next result.

Theorem 4.14. *Suppose that a sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ satisfies Assumptions (A1), (A2S)–(A4S). Then, there exists a hold sequence $\{H_j^0\}_{j=1}^{\infty}$ such that each hold H_j^0 has a lifting-based state-space representation and there holds Equation (4.10), that is,*

$$\|R^{n_p} - H_j^0 S_j\| \to 0 \qquad (j \to \infty),$$

if and only if the two conditions below are satisfied:

(a) $\|z\breve{R}^{n_p}\breve{S}_j^{\mathrm{in}\sim}\|_{\mathrm{H}} \to 0 \qquad (j \to \infty)$;

(b) For any $\Omega > 0$,

$$\bar{\sigma}\left[\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{S}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_m)^*\right\}^{-1/2}\left\{\sum_{m\neq 0} \underline{\widehat{S}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_m)^*\right\}\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{S}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_m)^*\right\}^{-1/2}\right]$$

converges to zero uniformly for any $|\omega| < \Omega$ as $j \to \infty$. Here, $\omega_m := \omega + 2\pi m/\tau_j$.

Remark 4.15. As we saw in Remark 4.10, the existence of an appropriate $\{S_j^0\}$ and $\{H_j^0\}$ satisfying

$$\|(R^{n_q})^m U^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty),$$
$$\|U^{n_p}(R^{n_p})^{m'} - H_j^0 S_j\| \to 0 \qquad (j \to \infty)$$

also guarantees the performance convergence to the theoretical bound. Here, U^n is a continuous-time operator whose continuous-time transfer function $\widehat{U}^n(s)$ is $n \times n$ and is unimodular in $\mathcal{RH}^{\infty}$. Starting from this expression, we can obtain different forms of conditions for the convergence in place of those given in Theorems 4.12 and 4.14. Namely, Condition (a) in Theorem 4.12 may be replaced by

$$\|z\breve{H}_j^{\mathrm{in}\sim}(\breve{R}^{n_q})^m\breve{U}^{n_q}\|_{\mathrm{H}} \to 0 \qquad (j \to \infty),$$

and Condition (a) in Theorem 4.14 may be replaced by

$$\|z\breve{U}^{n_p}(\breve{R}^{n_p})^{m'}\breve{S}_j^{\mathrm{in}\sim}\|_{\mathrm{H}} \to 0 \qquad (j \to \infty).$$

□

Let us go back to the hold case. When a hold sequence $\{H_j\}_{j=1}^{\infty}$ has some special structure, there is an even simpler condition, which is necessary and sufficient for the existence of an appropriate $\{S_j^0\}$. Suppose that the kernel function of each hold H_j, i.e., $\underline{H}_j(t)$, can be written as $\Xi(t/\tau_j)\Upsilon_j$, where $\Xi(t)$ is a fixed function belonging to $\mathcal{D}$ elementwise and Υ_j is an invertible matrix. Here, the set $\mathcal{D}$ consists of all real functions $a(t)$ such that $e^{\epsilon t}a(t)$ belongs to $\mathcal{L}^2$ for some $\epsilon > 0$ (Definition 3.1). In this case, we call a hold sequence $\{H_j\}$ a **proportional-type** hold sequence. This name comes from the fact that the shape of the graph of $\underline{H}_j(t)$ shrinks

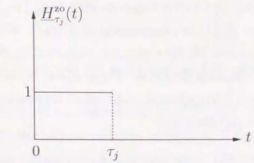


Figure 4.5. The kernel function of the zero-order hold H_{τ}^{zo}.

proportionally to τ_j as j increases. On the other hand, suppose that $\underline{H}_j(t)$ can be expressed as $\Xi(t)\Upsilon_j$ with $\Xi(t)$ and Υ_j being as before. In this case, we call a hold sequence $\{H_j\}$ a **fixed-type** hold sequence. This is because the shape of the kernel function $\underline{H}_j(t)$ is fixed irrespective of j.

Example 4.16. Recall a zero-order hold H_{τ}^{zo}, a first-order hold H_{τ}^{fo}, and a triangular hold H_{τ}^{tr}, which were introduced in Examples 3.6, 3.7, and 3.8, respectively. If we choose a sequence of sampling periods $\{\tau_j\}$ so that it converges to zero, each of the hold sequences $\{H_{\tau_j}^{\mathrm{zo}}\}$, $\{H_{\tau_j}^{\mathrm{fo}}\}$, and $\{H_{\tau_j}^{\mathrm{tr}}\}$ is proportional-type. Indeed, with respect to the zero-order holds for example, if we define

$$\Xi(t) = \begin{cases} I & \text{for } 0 \le t < \tau, \\ O & \text{for } \tau \le t \end{cases}$$

and $\Upsilon_j = I$, then $\underline{H}_{\tau_j}^{\mathrm{zo}}(t) = \Xi(t/\tau_j)\Upsilon_j$. Besides, noticing Figure 4.5, we can see that the graph of its kernel function $\underline{H}_{\tau_j}^{\mathrm{zo}}(t)$ shrinks proportionally to τ_j. (This is almost the same figure as Figure 3.2 (b).) □

Now, we have the following theorems. Their proofs will be given at the end of this section.

Theorem 4.17. *Suppose that a provided sequence of sampling environments $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ satisfies Assumptions (A1), (A2H)–(A4H). Moreover, assume $\{H_j\}_{j=1}^{\infty}$ is a proportional-type hold sequence. Then, there exists a sampler sequence $\{S_j^0\}_{j=1}^{\infty}$ such that each sampler S_j^0 has a lifting-based state-space representation and there holds (4.9), that is,*

$$\|R^{n_q} - H_j S_j^0\| \to 0 \qquad (j \to \infty),$$

if and only if Condition (b) in Theorem 4.12 is satisfied.

Theorem 4.18. *Suppose that a provided sequence of sampling environments* $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ *satisfies Assumptions (A1), (A2H)–(A4H). Suppose also that* $\{H_j\}_{j=1}^{\infty}$ *is a fixed-type hold sequence and the kernel function* $\underline{H}_j(t)$ *is represented as* $\Xi(t)\Upsilon_j$*. Furthermore, assume that the input signal and the output signal of* H_j *have the same dimension, that is,* $n_q = n_q^{\mathrm{d}}$*, and that the Laplace transform of* $\Xi(t)$ *is a rational function. Then, there exists a sampler sequence* $\{S_j^0\}_{j=1}^{\infty}$ *such that each sampler* S_j^0 *has a lifting-based state-space representation and there holds (4.9), if and only if* $\det \widehat{\Xi}(s) \neq 0$ *for any* $\mathrm{Re}\, s \geq 0$.

According to Theorem 4.8, the best sampled-data control performance converges to its theoretical bound if and only if there exist a sampler sequence $\{S_j^0\}$ satisfying (4.9) and a hold sequence $\{H_j^0\}$ satisfying (4.10). Theorem 4.12 claims that in order that such $\{S_j^0\}$ exists it is necessary and sufficient that both Conditions (a) and (b) in the theorem are satisfied. Now, the claim of Theorem 4.17 is that, when a provided hold sequence $\{H_j\}$ is a proportional type, only Condition (b) is necessary and sufficient for the existence of such $\{S_j^0\}$. (In other words, Condition (a) is always satisfied in this case.) This result simplifies a lot checking the condition for the performance convergence. On the other hand, when a provided hold sequence $\{H_j\}$ is a fixed type, a desired $\{S_j^0\}$ exists if and only if the Laplace transform of the kernel function of each H_j has no unstable zero. This is the claim of Theorem 4.18. In this case, if we further assume $\widehat{\Xi}(\mathrm{i}\omega) \neq 0$ for any finite real number ω, Condition (b) in Theorem 4.12 is fulfilled actually. This means that Condition (a) is equivalent to the existence of an appropriate $\{S_j^0\}$ and then to the no-unstable-zero condition. This is one reason why Condition (a) was interpreted in connection with unstable zeros just after Theorem 4.12.

Example 4.19. Let us consider a zero-order hold H_τ^{zo}, a first-order hold H_τ^{fo}, and a triangular hold H_τ^{tr} again. We have seen in Example 4.16 that each of $\{H_{\tau_j}^{\mathrm{zo}}\}$, $\{H_{\tau_j}^{\mathrm{fo}}\}$, and $\{H_{\tau_j}^{\mathrm{tr}}\}$ forms a proportional-type hold sequence when $\tau_j \to 0$ $(j \to \infty)$. Theorem 4.17 claims that with respect to these sequences Condition (b) is necessary and sufficient for the existence of an appropriate $\{S_j^0\}$. We have already seen that, in the case of a triangular hold H_τ^{tr}, Condition (b) is not satisfied. Hence, no appropriate $\{S_j^0\}$ exists for this hold. As for a zero-order hold and a first-order hold, Condition (b) is satisfied actually. Consequently, there exists an appropriate $\{S_j^0\}$ for each of these two holds.

Let us show that Condition (b) holds in the case of a zero-order hold H_τ^{zo}. Recall that

$$\underline{H}_\tau^{\mathrm{zo}}(t) = \begin{cases} I & \text{for } 0 \leq t < \tau, \\ O & \text{for } \tau \leq t. \end{cases}$$

Its Laplace transform is computed as

$$\widehat{\underline{H}}_\tau^{\mathrm{zo}}(s) = \frac{1 - \mathrm{e}^{-\tau s}}{s} I.$$

Hence,

$$\frac{1}{\tau}\widehat{\underline{H}}_{\tau_j}^{\mathrm{zo}}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right) = \frac{1 - \mathrm{e}^{-\mathrm{i}\omega\tau}}{\mathrm{i}\omega\tau + \mathrm{i}2\pi m} I \to \begin{cases} I & \text{for } m = 0, \\ O & \text{for } m \neq 0 \end{cases}$$

as $j \to \infty$. This means that Condition (b) is fulfilled in this case. □

With respect to Theorem 4.14, corresponding results hold. A sequence of regular samplers $\{S_j\}_{j=1}^{\infty}$ is called a **proportional-type** sampler sequence, if the kernel function $\underline{S}_j(t)$ has the form $\Upsilon_j \Xi(t/\tau_j)$, where $\Xi(t)$ is a fixed function belonging to $\mathcal{D}$ elementwise and Υ_j is an invertible matrix. Besides, $\{S_j\}_{j=1}^{\infty}$ is called a **fixed-type** sampler sequence, if the kernel function $\underline{S}_j(t)$ can be written in the form of $\Upsilon_j \Xi(t)$, where $\Xi(t)$ is a fixed function whose elements belong to $\mathcal{D}$ and Υ_j are invertible matrices. With this terminology, the following theorems hold.

Theorem 4.20. *Suppose that a provided sequence of sampling environments* $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ *satisfies Assumptions (A1), (A2S)–(A4S). Moreover, assume* $\{S_j\}_{j=1}^{\infty}$ *is a proportional-type sampler sequence. Then, there exists a hold sequence* $\{H_j^0\}_{j=1}^{\infty}$ *such that each hold* H_j^0 *has a lifting-based state-space representation and there holds (4.10), that is,*

$$\|R^{n_p} - H_j^0 S_j\| \to 0 \qquad (j \to \infty),$$

if and only if Condition (b) in Theorem 4.14 is satisfied.

Theorem 4.21. *Suppose that a provided sequence of sampling environments* $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$ *satisfies Assumptions (A1), (A2S)–(A4S). Furthermore, suppose that* $\{S_j\}_{j=1}^{\infty}$ *is a fixed-type sampler sequence, that is, the kernel function* $\underline{S}_j(t)$ *has the form of* $\Upsilon_j \Xi(t)$*. Finally, assume that the input signal and the output signal of* S_j *have the same dimension, that is,* $n_p = n_p^{\mathrm{d}}$*, and that the Laplace transform of* $\Xi(t)$ *is a rational function. Then, we can find a hold sequence* $\{H_j^0\}_{j=1}^{\infty}$ *such that each hold* H_j^0 *has a lifting-based state-space representation and there holds (4.10), if and only if* $\det \widehat{\Xi}(s) \neq 0$ *for any* $\mathrm{Re}\, s \geq 0$.

Example 4.22. Let us consider regular samplers $S_\tau^{\mathrm{id}} R_\tau$ and $S_\tau^{\mathrm{id}} R$, which appeared in Example 4.1. Their kernel functions are $(1/\tau)\mathrm{e}^{-t/\tau} I$ and $\mathrm{e}^{-t} I$, respectively. If $\{\tau_j\}$ is a sequence of sampling periods that converges to zero, $\{S_{\tau_j}^{\mathrm{id}} R_{\tau_j}\}$ is a proportional-type sampler sequence with $\Upsilon_j = (1/\tau_j) I$ and $\Xi(t) = \mathrm{e}^{-t} I$; $\{S_{\tau_j}^{\mathrm{id}} R\}$ is a fixed-type sampler sequence with $\Upsilon_j = I$ and $\Xi(t) = \mathrm{e}^{-t} I$. Application of Theorem 4.20 to the sampler sequence $\{S_{\tau_j}^{\mathrm{id}} R_{\tau_j}\}$ shows that there exists no hold sequence $\{H_j^0\}$ satisfying the requirement (4.9). Let us see this. Putting $S_j := S_{\tau_j}^{\mathrm{id}} R_{\tau_j}$, we have $\underline{S}_j(t) = (1/\tau_j)\mathrm{e}^{-t/\tau_j} I$ and thus

$$\widehat{\underline{S}}_j(s) = \frac{1}{\tau_j s + 1} I.$$

This implies

$$\underline{\widehat{S}}_j\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right) = \frac{1}{\mathrm{i}\omega\tau_j + \mathrm{i}2\pi m + 1} I \to \frac{1}{\mathrm{i}2\pi m + 1} I \qquad (j \to \infty).$$

Note that the quantity in Condition (b) in Theorem 4.14 is equal to

$$\overline{\sigma}\left[I - \left\{ \sum_{m=-\infty}^{\infty} \underline{\widehat{S}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_m)^* \right\}^{-1/2} \underline{\widehat{S}}_j(\mathrm{i}\omega)\underline{\widehat{S}}_j(\mathrm{i}\omega)^* \left\{ \sum_{m=-\infty}^{\infty} \underline{\widehat{S}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_m)^* \right\}^{-1/2} \right].$$

Since the convergence of the above infinite sum is uniform with respect to j, we can compute the limit of $j \to \infty$ before the infinite sum. Then, it is seen that the above quantity converges to a nonzero value as $j \to \infty$.

On the other hand, if we apply Theorem 4.21 to the sampler sequence $\{S^{\mathrm{id}}_{\tau_j} R\}$, we can see that there exists an appropriate hold sequence $\{H^0_j\}$ in this case. Indeed, since $\Xi(t) = \mathrm{e}^{-t} I$, there holds $\widehat{\Xi}(s) = \{1/(s+1)\}I$, which has no zero in $\operatorname{Re} s \geq 0$. □

Example 4.23. Combining the results of Examples 4.19 and 4.22, we can explain the simulation results of Example 4.1 (or Example 1.3), which motivated this research.

In the first case of Example 4.1, a sampling environment was chosen as $(\tau, S^{\mathrm{id}}_\tau R_\tau, H^{\mathrm{zo}}_\tau)$. Letting $\{\tau_j\}_{j=1}^\infty$ be any sequence of sampling periods that converges to zero, we consider the sampling environment sequence $\{(\tau_j, S^{\mathrm{id}}_{\tau_j} R_{\tau_j}, H^{\mathrm{zo}}_{\tau_j})\}_{j=1}^\infty$. As was shown in Example 4.19, for the considered hold sequence $\{H^{\mathrm{zo}}_{\tau_j}\}$, there exists an appropriate sampler sequence $\{S^0_j\}$ that satisfies (4.9). However, according to the result of Example 4.22, there is no hold sequence $\{H^0_j\}$ that satisfies (4.10) with the sampler sequence $\{S^{\mathrm{id}}_{\tau_j} R_{\tau_j}\}$. Hence, the consequence of Corollary 4.9 is that there exists a plant G with which the best sampled-data control performance for each environment does not converge to the best continuous-time control performance as $j \to \infty$. Our G examined in Example 4.1 was one of such plants.

The second case in Example 4.1 dealt with a sampling environment $(\tau, S^{\mathrm{id}}_\tau R, H^{\mathrm{zo}}_\tau)$. Choosing $\{\tau_j\}$ as before, we consider a sampling environment sequence $\{(\tau_j, S^{\mathrm{id}}_{\tau_j} R, H^{\mathrm{zo}}_{\tau_j})\}$. We have already seen that the hold sequence $\{H^{\mathrm{zo}}_{\tau_j}\}$ has an appropriate $\{S^0_j\}$. Moreover, Example 4.22 says that, for the considered sampler sequence $\{S^{\mathrm{id}}_{\tau_j} R\}$, there is a hold sequence $\{H^0_j\}$ satisfying (4.10). Therefore, Corollary 4.9 concludes that the best sampled-data control performance converges to the best continuous-time control performance as $j \to \infty$ for any plant G in a certain set. Since our plant G belongs to this set (Example 4.5), the simulation result of Example 4.1 is explained. □

We conclude this section by giving proofs for Theorems 4.17 and 4.18. Theorems 4.20 and 4.21 are proven similarly.

Proof of Theorem 4.17. Because we can let the factor Υ_j be absorbed in $\breve{H}^{\mathrm{out}}_j(z)$, we can consider that Υ_j does not affect $\breve{H}^{\mathrm{in}}_j(z)$. Since only $\breve{H}^{\mathrm{in}}_j(z)$ appears in Condition (a), it can be

assumed that $\Upsilon_j = I$ without loss of generality. First, let us show the existence of $0 < \rho_0 < 1$ such that $\|\breve{H}^{\mathrm{in}}_j(z)\|_{\mathrm{T}}$ is bounded uniformly for any $z \in \mathbb{D}_{\rho_0}$ and any j.

Since $\{H_j\}$ is a proportional-type hold sequence, a function $\breve{H}^{\mathrm{out}}_j(z)$ has a similar shape irrespective of j. Let us see this. Consider the lifting-based transfer function of the hold H_j, that is, $\breve{H}_j(z)$. Just as we did before Remark 3.12, represent this $\breve{H}_j(z)$ in the form of $\breve{H}_j(z) = \sum_{k=0}^\infty H^k_j z^{-k}$ for each j. Here, H^k_j is a tall operator mapping $\boldsymbol{v} \in \mathbb{C}^{n^{\mathrm{d}}_q}$ to a function

$$(H^k_j \boldsymbol{v})(t) = \underline{H}_j(k\tau_j + t)\boldsymbol{v} = \Xi(k + t/\tau_j)\boldsymbol{v},$$

which belongs to $\mathcal{L}^2[0,\tau_j)^{n_q}$. Define a regular hold $H_{\tau=1}$ by setting its kernel function to be $\underline{H}_{\tau=1}(t) := \Xi(t)$. Expand its lifting-based transfer function $\breve{H}_{\tau=1}(z)$ into a series $\sum_{k=0}^\infty H^k_{\tau=1} z^{-k}$ by defining the operator $H^k_{\tau=1}$ as

$$(H^k_{\tau=1}\boldsymbol{v})(t) := \Xi(k+t)\boldsymbol{v}.$$

Now, observe that

$$\begin{aligned}
\breve{H}^{\mathrm{out}\sim}_j(z)\breve{H}^{\mathrm{out}}_j(z) = \breve{H}^{\sim}_j(z)\breve{H}_j(z) &= \left\{\sum_{k=0}^\infty (H^k_j)^* z^k\right\}\left\{\sum_{k=0}^\infty H^k_j z^{-k}\right\} \\
&= \sum_{\ell=0}^\infty (H^\ell_j)^* H^\ell_j + \sum_{k=1}^\infty \left\{ z^k \sum_{\ell=0}^\infty (H^{k+\ell}_j)^* H^\ell_j + z^{-k} \sum_{\ell=0}^\infty (H^\ell_j)^* H^{k+\ell}_j \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
(H^{k+\ell}_j)^* H^\ell_j &= \int_0^{\tau_j} \underline{H}_j((k+\ell)\tau_j + t)^* \underline{H}_j(\ell\tau_j + t)\,\mathrm{d}t \\
&= \tau_j \int_0^1 \Xi(k+\ell+r)^* \Xi(\ell+r)\,\mathrm{d}r = \tau_j (H^{k+\ell}_{\tau=1})^* H^\ell_{\tau=1},
\end{aligned}$$

we can write $\breve{H}^{\mathrm{out}}_j(z) = \sqrt{\tau_j}\breve{H}^{\mathrm{out}}_{\tau=1}(z)$, where $\breve{H}^{\mathrm{out}}_{\tau=1}(z)$ is an outer factor of $\breve{H}_{\tau=1}(z)$. Hence, each of the functions $\breve{H}^{\mathrm{out}}_j(z)$, $j = 1, 2, \ldots$, has a similar shape. By definition, $\breve{H}^{\mathrm{out}}_{\tau=1}(z)^{-1}$ is a rational function and is bounded in $\mathbb{D}$. This implies the existence of $0 < \rho_1 < 1$ and $M_1 > 0$ such that $\overline{\sigma}\{\breve{H}^{\mathrm{out}}_{\tau=1}(z)^{-1}\} < M_1$ for $z \in \mathbb{D}_{\rho_1}$.

By assumption, there is $\epsilon > 0$ for which $\mathrm{e}^{\epsilon t}\Xi(t)$ belongs to $\mathcal{L}^2$. Using this ϵ, we obtain

$$(H^k_j)^* H^k_j = \tau_j \int_0^1 \Xi(k+r)^* \Xi(k+r)\mathrm{d}r \leq \tau_j \mathrm{e}^{-2\epsilon k} \int_0^1 \mathrm{e}^{\epsilon(k+r)} \Xi(k+r)^* \Xi(k+r) \mathrm{e}^{\epsilon(k+r)}\mathrm{d}r,$$

where the inequality $A \leq B$ means that the matrix $B - A$ is positive semi-definite. Since $\mathrm{e}^{\epsilon t}\Xi(t)$ belongs to $\mathcal{L}^2$, the integral in the last expression has an upper bound independent of k. Thus, $\|H^k_j\|_{\mathrm{T}}$ is bounded by $\sqrt{\tau_j}\mathrm{e}^{-\epsilon k} V$, where V is independent of k. Since $\|\breve{H}_j(z)\|_{\mathrm{T}} \leq \sum_{k=0}^\infty \|H^k_j\|_{\mathrm{T}} |z|^{-k} \leq \sum_{k=0}^\infty \sqrt{\tau_j} V (\mathrm{e}^\epsilon |z|)^{-k}$, we can find $0 < \rho_2 < 1$ and $M_2 > 0$ so that $\|\breve{H}_j(z)\|_{\mathrm{T}} < \sqrt{\tau_j} M_2$ for any $z \in \mathbb{D}_{\rho_2}$. Now, put $\rho_0 := \max\{\rho_1, \rho_2\}$. Then, for any $z \in \mathbb{D}_{\rho_0}$, there holds $\|\breve{H}_j^{\mathrm{in}}(z)\|_{\mathrm{T}} \leq \|\breve{H}_j(z)\{\sqrt{\tau_j}\breve{H}_{\tau=1}^{\mathrm{out}}(z)\}^{-1}\|_{\mathrm{T}} < M_2 M_1$. Defining M to be $M_2 M_1$, we have proven the claim.

Next, consider the flat-operator-valued function $z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}^{n_q}(z)$. In the following, $\breve{R}^{n_q}(z)$ is written as $\breve{R}(z)$ for simplicity. Note that $\breve{H}_j^{\mathrm{in}\sim}(z)$ is analytic in $0 \leq |z| < 1/\rho_0$ since $\breve{H}_j^{\mathrm{in}}(z)$ is analytic in $\mathbb{D}_{\rho_0}$. Furthermore, $\breve{R}(z)$ is analytic in $1 \leq |z|$. Therefore, if we choose $1 < \rho_3 < 1/\rho_0$, the function $z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}(z)$ is analytic in $1 \leq |z| \leq \rho_3$. Expand it into the Laurent series $\sum_{k=-\infty}^{\infty} L^k z^k$ with putting $L^k := (1/2\pi\mathrm{i}) \oint_{|z|=\rho_3} z\breve{H}_j^{\mathrm{in}\sim}(z)\breve{R}(z)z^{-k-1}\mathrm{d}z$. Since $\|\breve{H}_j^{\mathrm{in}}(z)\|_{\mathrm{T}} < M$ in $\mathbb{D}_{\rho_0}$, $\|\breve{H}_j^{\mathrm{in}\sim}(z)\|_{\mathrm{F}} < M$ on $|z| = \rho_3$. Then, there holds

$$\|L^k\|_{\mathrm{F}} \leq \frac{1}{2\pi}\oint_{|z|=\rho_3} \|\breve{H}_j^{\mathrm{in}\sim}(z)\|_{\mathrm{F}}\|\breve{R}(z)\|_{\mathrm{L}}\rho_3^{-k}|\mathrm{d}z| < \frac{M}{\rho_3^{k-1}} \sup_{|z|=\rho_3} \|\breve{R}(z)\|_{\mathrm{L}}.$$

By (2.12), the Hankel norm $\|z\breve{H}_j^{\mathrm{in}\sim}\breve{R}\|_{\mathrm{H}}$ can be bounded as

$$\|z\breve{H}_j^{\mathrm{in}\sim}\breve{R}\|_{\mathrm{H}}^2 \leq \sum_{k=1}^{\infty} k\|L^k\|_{\mathrm{F}}^2 \leq \frac{M^2\rho_3^4}{(\rho_3^2-1)^2}\left(\sup_{|z|=\rho_3}\|\breve{R}(z)\|_{\mathrm{L}}\right)^2.$$

In the last expression, $\sup_{|z|=\rho_3}\|\breve{R}(z)\|_{\mathrm{L}}$ tends to zero as $j \to \infty$ in fact, which implies that Condition (a) is satisfied. In order to see this, note that a continuous-time state-space representation of R is $(-I, I, I, O)$. By application of Proposition 2.18, the lifting-based state-space representation of R is obtained as $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$, where

$$\begin{aligned}
\widetilde{A} &:= \mathrm{e}^{-\tau_j} I, \\
\widetilde{B}\boldsymbol{f} &:= \int_0^{\tau_j} \mathrm{e}^{-(\tau_j - t)}\boldsymbol{f}(t)\,\mathrm{d}t, \\
(\widetilde{C}\boldsymbol{v})(t) &:= \mathrm{e}^{-t}\boldsymbol{v} \qquad \text{for } 0 \leq t < \tau_j, \\
(\widetilde{D}\boldsymbol{f})(t) &:= \int_0^t \mathrm{e}^{-(t-r)}\boldsymbol{f}(r)\,\mathrm{d}r \qquad \text{for } 0 \leq t < \tau_j.
\end{aligned}$$

Here, $\boldsymbol{f}$ is an arbitrary function in $\mathcal{L}^2[0,\tau_j)^{n_q}$ and $\boldsymbol{v}$ is an arbitrary vector in $\mathbb{C}^{n_q}$. By definition, $\|\widetilde{D}\|_{\mathrm{L}}$, $\|\widetilde{C}\|_{\mathrm{T}}$, and $\|\widetilde{B}\|_{\mathrm{F}}$ approach zero as $j \to \infty$ while $\widetilde{A}$ converges to the identity. Since $\breve{R}(z) = \widetilde{D} + \widetilde{C}(zI - \widetilde{A})^{-1}\widetilde{B}$, the value of $\sup_{|z|=\rho_3}\|\breve{R}(z)\|_{\mathrm{L}}$ goes to zero as j increases. Now, the claim is proven. □

Proof of Theorem 4.18. [if] Let us simply write R in place of R^{n_q}. It suffices to show that for any $\epsilon > 0$ there exists J such that $\inf_S \|R - H_j S\| < \epsilon$ holds for any $j > J$, where S varies over all regular samplers having state-space representations. Note that $\inf_S \|R - H_j S\| = \inf_S \|R - H_j \Upsilon_j^{-1}\Upsilon_j S\| = \inf_S \|R - H_j\Upsilon_j^{-1}S\|$. Hence, it is enough to prove above in the case of $\Upsilon_j = I$.

Let ϵ be any positive number. Since $\widehat{\Xi}(s)$ is rational and has a zero at $s = \infty$, there exists a positive integer r such that $\{1/(s+1)^r\}\widehat{\Xi}(s)^{-1}$ is bounded at $s = \infty$. Write as Q_α a continuous-time operator whose transfer function is $\widehat{Q}_\alpha(s) = \{1/(\alpha s + 1)^r\}I$. By using a small enough $\alpha > 0$, we can guarantee $\|R - Q_\alpha R\| = \|\widehat{R} - \widehat{Q}_\alpha\widehat{R}\|_{\mathcal{H}^\infty} < \epsilon/2$. Here, $\widehat{R}(s)$ and $\widehat{Q}(s)$ are continuous-time transfer functions of R and Q, respectively, and $\|\cdot\|_{\mathcal{H}^\infty}$ denotes the usual $\mathcal{H}^\infty$-norm (not in the lifted domain). The proof is completed if we can find a sequence of regular samplers $\{S_j\}_{j=1}^\infty$ such that each S_j has its lifting-based state-space representation and it is possible to find J so that

$$\|Q_\alpha R - H_j S_j\| < \epsilon/2$$

for any $j > J$. Let us construct such $\{S_j\}$.

Consider a sequence of functions $\widehat{\underline{S}}_j(s) := \tau_j\widehat{\Xi}(s)^{-1}\widehat{Q}_\alpha(s)\widehat{R}(s)$, $j = 1, 2, \ldots$. Then, actually, it is possible to define a regular sampler S_j so that it has $\widehat{\underline{S}}_j(s)$ as its kernel function. Indeed, $\widehat{\underline{S}}_j(s)$ is a rational function, has a zero at $s = \infty$, and has no pole in $\mathrm{Re}\, s \geq 0$. Therefore, there exists $\epsilon > 0$ such that $\widehat{\underline{S}}_j(s - \epsilon)$ belongs to $\mathcal{H}^2$ as a function of s. Noting that the equivalence between $\mathcal{H}^2$ and $\mathcal{L}^2$, which was stated in Proposition 2.4, it is seen that $\mathrm{e}^{\epsilon t}\underline{S}_j(t)$ belongs to $\mathcal{L}^2$ as a function of t. Hence, $\underline{S}_j$ belongs to $\mathcal{D}$ and, consequently, S_j can be defined as a regular sampler. Moreover, since $\widehat{\underline{S}}_j(s)$ is rational, Proposition 3.17 implies that S_j has a lifting-based state-space representation. Now, it is left for us to show the existence of J such that $\|Q_\alpha R - H_j S_j\| < \epsilon/2$ for any $j > J$.

The considered norm is bounded as

$$\begin{aligned}
\|Q_\alpha R - H_j S_j\|^2 &= \sup_{\omega\in[-\pi/\tau_j,\pi/\tau_j)} \|(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}^2 \\
&\leq \sup_{\omega\in[-\pi/\tau_j,\pi/\tau_j)} \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \overline{\sigma}\left\{\acute{E}_m^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega}\right\}^2.
\end{aligned}$$

This inequality follows from Proposition 2.27. The summation in the last expression is evaluated by being classified into the following four groups:

(i) $m = 0$ and $\ell = 0$;
(ii) $m \neq 0$ and $\ell = 0$;
(iii) $m = 0$ and $\ell \neq 0$;
(iv) $m \neq 0$ and $\ell \neq 0$.

Propositions 2.28 and 3.18 give

$$\begin{aligned}
\acute{E}_m^{\mathrm{i}\omega}\breve{Q}_\alpha(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega} &= \widehat{Q}_\alpha(\mathrm{i}\omega_m)\delta_{m,\ell}, \\
\acute{E}_m^{\mathrm{i}\omega}\breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega} &= \widehat{R}(\mathrm{i}\omega_m)\delta_{m,\ell}, \\
\acute{E}_m^{\mathrm{i}\omega}\breve{H}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j}) &= \frac{1}{\sqrt{\tau_j}}\widehat{\underline{H}}_j(\mathrm{i}\omega_m) = \frac{1}{\sqrt{\tau_j}}\widehat{\Xi}(\mathrm{i}\omega_m), \\
\breve{S}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega} &= \frac{1}{\sqrt{\tau_j}}\widehat{\underline{S}}_j(\mathrm{i}\omega_\ell) = \sqrt{\tau_j}\widehat{\Xi}(\mathrm{i}\omega_\ell)^{-1}\widehat{Q}_\alpha(\mathrm{i}\omega_\ell)\widehat{R}(\mathrm{i}\omega_\ell),
\end{aligned}$$

where $\omega_m := \omega + 2\pi m/\tau_j$. Combining them, we have

$$\acute{E}_m^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega} = \widehat{Q}_\alpha(\mathrm{i}\omega_m)\widehat{R}(\mathrm{i}\omega_m)\delta_{m,\ell} - \frac{1}{\tau_j}\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\underline{\widehat{S}}_j(\mathrm{i}\omega_\ell).$$

Therefore, for the group (i),

$$\overline{\sigma}\left\{\acute{E}_0^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_0^{\mathrm{i}\omega}\right\}^2 = 0;$$

for the group (ii),

$$\sum_{m\neq 0}\overline{\sigma}\left\{\acute{E}_m^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_0^{\mathrm{i}\omega}\right\}^2 \leq \left[\sum_{m\neq 0}\overline{\sigma}\left\{\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^2\right]\overline{\sigma}\left\{\frac{1}{\tau_j}\underline{\widehat{S}}_j(\mathrm{i}\omega)\right\}^2; \tag{4.18}$$

for the group (iii),

$$\sum_{\ell\neq 0}\overline{\sigma}\left\{\acute{E}_0^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega}\right\}^2 \leq \overline{\sigma}\left\{\underline{\widehat{H}}_j(\mathrm{i}\omega)\right\}^2\left[\sum_{\ell\neq 0}\overline{\sigma}\left\{\frac{1}{\tau_j}\underline{\widehat{S}}_j(\mathrm{i}\omega_\ell)\right\}^2\right]; \tag{4.19}$$

for the group (iv),

$$\sum_{m\neq 0}\sum_{\ell\neq 0}\overline{\sigma}\left\{\acute{E}_m^{\mathrm{i}\omega}(Q_\alpha R - H_j S_j)^{\vee}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_\ell^{\mathrm{i}\omega}\right\}^2 \leq \left[\sum_{m\neq 0}\overline{\sigma}\left\{\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^2\right]\left[\sum_{\ell\neq 0}\overline{\sigma}\left\{\frac{1}{\tau_j}\underline{\widehat{S}}_j(\mathrm{i}\omega_\ell)\right\}^2\right]. \tag{4.20}$$

Here, note that $\underline{\widehat{H}}_j(s) = \widehat{\Xi}(s)$ and $(1/\tau_j)\underline{\widehat{S}}_j(s) = \widehat{\Xi}(s)^{-1}\widehat{Q}_\alpha(s)\widehat{R}(s)$ do not depend on j actually. Therefore, as $j \to \infty$, the values of

$$\left[\sum_{m\neq 0}\overline{\sigma}\left\{\underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^2\right] \quad\text{and}\quad \left[\sum_{\ell\neq 0}\overline{\sigma}\left\{\frac{1}{\tau_j}\underline{\widehat{S}}_j(\mathrm{i}\omega_\ell)\right\}^2\right]$$

converge to zero because $\omega_m = \omega + 2\pi m/\tau_j \to \infty$ when $m \neq 0$. Hence, the right-hand sides of (4.18)–(4.20) converge to zero, which implies the existence of J such that $\|Q_\alpha R - H_j S_j\| < \epsilon/2$ for each $j > J$.

[only if] Suppose that there exists a sequence $\{S_j^0\}_{j=1}^{\infty}$ such that each S_j^0 is a regular sampler having a state-space representation and there holds $\|R - H_j S_j^0\| \to 0$ as $j \to \infty$. In the following, let us assume $\det\widehat{\Xi}(s) = 0$ for some $\operatorname{Re} s \geq 0$ and derive contradiction.

First, note that

$$\begin{aligned}\|\acute{E}_0^s\|_{\mathrm{F}}\|(R - H_j S_j^0)^{\vee}(\mathrm{e}^{s\tau_j})\|_{\mathrm{L}}\|\grave{E}_0^s\|_{\mathrm{T}} &\geq \overline{\sigma}\left\{\acute{E}_0^s(R - H_j S_j^0)^{\vee}(\mathrm{e}^{s\tau_j})\grave{E}_0^s\right\}\\ &= \overline{\sigma}\left\{\widehat{R}(s) - \frac{1}{\tau_j}\underline{\widehat{H}}_j(s)\underline{\widehat{S}}_j^0(s)\right\}.\end{aligned} \tag{4.21}$$

Here, since $\|\grave{E}_0^s\|_{\mathrm{T}}^2 \leq (1/\tau_j)\int_0^{\tau_j}|\mathrm{e}^{st}|^2\,\mathrm{d}t$ by definition, it is seen that $\|\grave{E}_0^s\|_{\mathrm{T}} \to 1$ as $j \to \infty$. Similarly, $\|\acute{E}_0^s\|_{\mathrm{F}} \to 1$. Moreover, because

$$\|R - H_j S_j^0\| = \|(R - H_j S_j^0)^{\vee}\|_{\mathfrak{H}_{\mathrm{L}}^{\infty}} = \sup_{z\in\mathbb{D}}\|(R - H_j S_j^0)^{\vee}(z)\|_{\mathrm{L}} = \sup_{\operatorname{Re} s\geq 0}\|(R - H_j S_j^0)^{\vee}(\mathrm{e}^{s\tau_j})\|_{\mathrm{L}}$$

and $\|R - H_j S_j^0\|$ approaches zero as j goes to infinity, there holds $\|(R - H_j S_j^0)^{\vee}(\mathrm{e}^{s\tau_j})\|_{\mathrm{L}} \to 0$ as $j \to \infty$ for each point in $\operatorname{Re} s \geq 0$. Using these results in (4.21), we can conclude that

$$\overline{\sigma}\left\{\widehat{R}(s) - \frac{1}{\tau_j}\underline{\widehat{H}}_j(s)\underline{\widehat{S}}_j^0(s)\right\} \to 0$$

as j goes to infinity for each point in $\operatorname{Re} s \geq 0$. However, at s such that $\det\widehat{\Xi}(s) = 0$, we have $\det\underline{\widehat{H}}_j(s) = 0$ for each j but $\det\widehat{R}(s) \neq 0$. This is a contradiction. □

4.5. Conclusion

This chapter was devoted to an analysis of the best sampled-data control performance especially about its convergence to the best continuous-time control performance. Section 4.2 noticed that the best sampled-data control performance can be improved by an appropriate choice of a sampling environment (the triplet of a sampling period, a sampler, and a hold) and related the theoretical bound of this improvement with the best continuous-time control performance. Then, Section 4.3 presented a necessary and sufficient condition in order that a provided sampling environment sequence ensures convergence to this theoretical bound for all plants. If we concentrate only on the plants with which recovery of the best continuous-time control performance is potentially possible, the above condition is necessary and sufficient for convergence to the best continuous-time performance. In Section 4.4, this condition was made easier to be tested by use of techniques for a model-matching problem in the $\mathcal{H}^{\infty}$-control theory. For special types of samplers and holds, this condition was further simplified.

A control theory has been developed mostly about continuous-time controllers and it enables us to compute the best achievable performance of continuous-time control systems. However, since it is difficult in practice to make a continuous-time controller that realizes a complicated function with a high precision, a sampled-data controller is usually used instead. In Theorem 4.2, we have seen that it may not be possible to recover the best achievable performance of continuous-time controllers by means of sampled-data controllers no matter how a sampling environment is chosen. Fortunately, this recovery is possible when at least one of G_{11}, G_{12}, and G_{21} is strictly proper. Hence, the above fact does not cause a problem in many of practical systems. However, this result suggests that serious care is necessary to handle direct feedthrough terms of G, which are often treated lightly.

This chapter gave a condition on a sequence of sampling environments in order that the best sampled-data control performance for each environment converges to the theoretical bound of the best sampled-data control performance. That is, the sampling period should converge to zero, the effect of unstable zeros should decrease in a sampler and a hold, and side-band-frequency components should diminish also in a sampler and a hold. These results give us some insight about how we should choose a sampling environment. Then one might ask whether these results give us a quantitative index to appraise a provided sampling environment; whether it is possible to optimize a sampling environment with respect to that index. Recall Conditions (a) and (b) in Theorem 4.12. Since the quantity in Condition (b) depends only on the provided sampling environment, this may work as a performance index of the provided environment. However, the norm in Condition (a) depends on an operator R, and this R can be replaced by many other continuous-time operators (Remark 4.15). Therefore, it is not expected that the specific value of this norm expresses goodness of the provided environment. Therefore, the mentioned questions cannot be positively answered right now. Nevertheless, the results of this chapter are expected to be a starting point to consider those questions.

It can be seen from the results of this chapter that naive inference based on intuition is dangerous with respect to sampled-data control systems because it sometimes leads to erroneous consequences. For example, although it seems natural to choose the bandwidth of an anti-aliasing filter proportional to the Nyquist frequency, this choice does not guarantee even convergence to the theoretical performance bound as the sampling period tends to zero. In spite that the zero-order hold H_τ^{zo} and the triangular hold H_τ^{tr} look similar in the sense that their kernel functions change their shapes proportionally to the sampling period, the hold H_τ^{zo} does satisfy the condition for convergence while H_τ^{tr} does not. Besides, it is seen from this research that a lifting-based approach is powerful for a careful treatment of sampled-data control systems.

Chapter 5

Topics for Further Research

At the end of this thesis, topics that are considered to be interesting for further research are listed up.

Chapter 3

Chapter 3 provided a framework for sampled-data control systems. This framework is general enough to cover many of practically important samplers and holds. Based on this framework, useful properties of sampled-data control systems were derived.

The following problems need to be investigated further in relation to this framework.

- Generalizing the framework so that general multirate sampled-data control systems can be treated there.

In order to treat a multirate system in our framework, we choose for the sampling period τ the least common multiple of the all periods included in the system. Then, a continuous-time signal sampled at multiple time points in $[k\tau, (k+1)\tau)$ are regarded as a discrete-time signal at the time k. In this setting, actually we can allow devices such as a sampler to work in an apparently non-causal way. However, in this thesis, each of a sampler, a discrete-time controller, and a hold are required to be causal. This is the reason why a general multirate system cannot be treated in our framework. (See Remark 3.28.) It is considered that we can resolve this problem by allowing a sampler to work in a non-causal way to some degree. For this purpose, the framework itself should be modified so as to be consistent with this extension.

- Extension and application of the approximation theorem.

The main result in Chapter 3 is a sort of approximation theorem (Theorem 3.34). This reveals a relationship between a sampled-data control system and a continuous-time control system for the same plant. It is conjectured that this theorem can be generalized more so as to hold between a general periodic control system and a continuous-time time-invariant control system. This is considered to be useful for analysis of the best performance of a periodic control scheme, which has interesting properties [60, 35, 47, 67]. On the other hand, this theorem was applied to the performance analysis of control systems in this thesis. It must be interesting to consider another application of it. For example, in identification of a continuous-time system it is usual that the system is identified in a discrete-time sense first, and then the obtained discrete-time system is approximated by a continuous-time system. It is considered to be interesting if this theorem is applicable to the analysis of this procedure.

Chapter 4

In Chapter 4, the best achievable performance of sampled-data control system was investigated. Especially, we obtained a necessary and sufficient condition in order that the best sampled-data control performance converges to its theoretical bound. In many cases, this theoretical bound is equal to the best continuous-time control performance, though not always.

In relation to this chapter, the following topics are considered to be interesting.

- Obtaining a change rate of the best sampled-data control performance as the sampling period approaches zero.

In this thesis, we considered *whether* the best sampled-data control performance converges to its theoretical bound. However, it would be good if we can also see *how* it converges. If we use a small sampling period, the best achievable performance is improved usually, but at the same time, more expensive devices are needed to realize a controller. In order to see this tradeoff, the rate of convergence is desired to be computed. To consider this problem, again the approximation theorem is expected to be a strong tool.

- Optimization of a sampler or a hold in a limited class.

Tadmor [86] considered optimization of a sampler or a hold for a provided plant and sampling period. He assumed that any sampler and hold can be realized. However, it is impossible practically. Especially when the sampling period is small, only a limited class of samplers and holds having rather simple functions can be realized. Therefore, a practically important problem is optimization of a sampler or a hold in a limited class. One approach toward this problem is to choose some simple samplers (or holds) as a basis and express a realizable sampler as a linear combination of the basis samplers. It is not difficult to describe this problem using bilinear matrix inequalities. It is known that the global optimum can be computed for a problem expressed by bilinear matrix inequalities [39, 38]. Therefore, our problem is solved at least in principle. However, this algorithm is based on the branch-and-bound method and is

not efficient unfortunately. Furthermore, a problem expressed by bilinear matrix inequalities is NP-hard in general [90]. It should be clarified whether our problem, that is, optimization of a sampler or a hold in a limited class, is NP-hard itself. If NP-hard, it is unlikely that there exists an efficient algorithm to optimize a sampler or a hold; hence, we have to consider to obtain a good sampler or a hold based on a different scheme. One possibility for this is the use of the change rate of the best achievable performance.

- A performance index of a sampling environment.

Theorems 4.12 and 4.14 gave conditions on a sequence of sampling environments in order that the best sampled-data control performance for each environment converges to its theoretical bound. As we have seen in Section 4.5, this result does not directly give a performance index of a sampling environment. Nevertheless, it would be interesting if we can derive some kind of index to measure goodness of a sampling environment from the quantities that appeared in the mentioned conditions. This is because these quantities are independent of a provided plant and, thus, express properties of a sampling environment, which do not depend on a particular plant. We have only limited freedom in the choice of samplers and holds due to a restriction on their physical realization. Therefore, it seems to be more practical to pursue a sampler and hold universally good to all plants rather than to try fine tuning of them for each of a provided plant. Moreover, such an approach may be effective in order to choose a basis of a sampler and a hold for their optimization, which was considered in the previous topic.

- Information-based approach in the control theory.

Theorem 4.2 showed that the theoretical bound of the best sampled-data control performance is equal to the best performance achievable by strictly proper continuous-time controllers. What does this theorem imply theoretically? It means that, no matter how fast the sampling period is, a sampled-data controller cannot compensate the plant dynamics at $s = \infty$. In other words, some information about the plant is inevitably lost in the sampling process. Then, what kind of information is lost? In view of the best achievable control performance, how a notion of information should be defined? Recently, in the field of control-oriented identification, the set of models unfalsified by provided input-output data is explicitly obtained, and then by measuring the diameter of this set a value of a prior information is evaluated [101, 15, 100, 52]. Also from this example, we can see how important a notion of information is in the control theory. It is a challenging problem to reconsider control and identification from an information-based viewpoint. This problem is expected to have relationships to other areas like statistics, a learning theory, and an information theory.

Appendix A

Proof of Property (b) of Proposition 2.3

Here, a property of a scalar-valued function $\widehat{a}(s)$ belonging to $\mathcal{H}^2$ is proven. This property is important because a regular sampler and hold, which are defined in Section 3.2, are closely related to functions in $\mathcal{H}^2$. Although the property itself is simple, its proof has to be rather long.

Two lemmas are prepared first. By Proposition 2.1, it is possible to find a function $a(t)$ in $\mathcal{L}^2$ so that the Laplace transform of $a(t)$ is our $\widehat{a}(s)$. Let ℓ_{b}^2 denote the set of two-sided square-summable scalar-valued sequences. For any sequence $\alpha = \{\alpha[k]\}_{k=-\infty}^{\infty}$ in ℓ_{b}^2, its ℓ_{b}^2-norm is defined as

$$\|\alpha\|_{\ell_{\mathrm{b}}^2} := \left(\sum_{k=-\infty}^{\infty} |\alpha[k]|^2 \right)^{1/2}.$$

Lemma A.1. *For any s such that $\operatorname{Re} s > 0$, the sequence $\{\widehat{a}(s + \mathrm{i}2\pi m/\tau)\}_{m=-\infty}^{\infty}$ belongs to ℓ_{b}^2 and satisfies*

$$\sum_{m=-\infty}^{\infty} \left| \widehat{a}\left(s + \frac{\mathrm{i}2\pi m}{\tau}\right) \right|^2 = \sum_{m=-\infty}^{\infty} \left| \int_0^{\infty} a(t) \mathrm{e}^{-(s+\mathrm{i}2\pi m/\tau)t} \mathrm{d}t \right|^2 \leq \frac{\tau J^2}{\{1 - \mathrm{e}^{-(\operatorname{Re} s)\tau}\}^2}, \tag{A.1}$$

where $J^2 := \int_0^{\infty} |a(t)|^2 \mathrm{d}t < \infty$.

Proof. Choose any s so that $\operatorname{Re} s > 0$. Define

$$a_m^k := \int_{k\tau}^{(k+1)\tau} a(t) \mathrm{e}^{-(s+\mathrm{i}2\pi m/\tau)t} \mathrm{d}t = \int_0^{\tau} a(k\tau + t) \mathrm{e}^{-s(k\tau+t)} \mathrm{e}^{-(\mathrm{i}2\pi m/\tau)t} \mathrm{d}t.$$

Here, $a(k\tau + t)\mathrm{e}^{-s(k\tau+t)}$ belongs to $\mathcal{L}^2[0, \tau)$ as a function of $t \in [0, \tau)$ and its Fourier coefficients are $\{a_m^k\}_{m=-\infty}^{\infty}$. Therefore, from Parseval's identity, there hold $\{a_m^k\}_{m=-\infty}^{\infty} \in \ell_{\mathrm{b}}^2$ and

$$\sum_{m=-\infty}^{\infty} |a_m^k|^2 = \tau \int_0^{\tau} |a(k\tau + t)\mathrm{e}^{-s(k\tau+t)}|^2 \mathrm{d}t = \tau \int_{k\tau}^{(k+1)\tau} |a(t)\mathrm{e}^{-st}|^2 \mathrm{d}t$$

$$= \tau \int_{k\tau}^{(k+1)\tau} |a(t)|^2 |\mathrm{e}^{-(\mathrm{Re}\, s)t}|^2 \mathrm{d}t \le \tau \mathrm{e}^{-2(\mathrm{Re}\, s)k\tau} J^2.$$

If we describe the sequence $\{a_m^k\}_{m=-\infty}^{\infty}$ as a^k, the above formula shows that the ℓ_{b}^2-norm of a^k is less than or equal to $\sqrt{\tau}\mathrm{e}^{-(\mathrm{Re}\, s)k\tau} J$. Therefore, we can see that the infinite series of the sequences, $\sum_{k=0}^{\infty} a^k$, absolutely converges in ℓ_{b}^2 for $\mathrm{Re}\, s > 0$. Because the space ℓ_{b}^2 is complete, $\sum_{k=0}^{\infty} a^k$ belongs to ℓ_{b}^2 and it is identical to the sequence in the claim. In order to derive (A.1), note that its left-hand side equals $\| \sum_{k=0}^{\infty} a^k \|_{\ell_{\mathrm{b}}^2}^2$ and is bounded from above by $(\sum_{k=0}^{\infty} \|a^k\|_{\ell_{\mathrm{b}}^2})^2$. Now use $\|a^k\|_{\ell_{\mathrm{b}}^2} \le \sqrt{\tau}\mathrm{e}^{-(\mathrm{Re}\, s)k\tau} J$. □

Lemma A.2. *The function of* s,

$$\beta_\infty(s) := \sum_{m=-\infty}^{\infty} \left| \hat{a}\left(s + \frac{\mathrm{i}2\pi m}{\tau}\right) \right|^2,$$

is continuous in $\mathrm{Re}\, s > 0$.

Proof. The function $\beta_\infty(s)$ is well-defined from Lemma A.1. Let s_0 be any complex number with $\mathrm{Re}\, s_0 > 0$. In order to prove the continuity of $\beta_\infty(s)$, it suffices to show that we can make $\left|\sqrt{\beta_\infty(s)} - \sqrt{\beta_\infty(s_0)}\right|$ arbitrarily small by letting s be close enough to s_0.

Since $\sqrt{\beta_\infty(s)}$ is the ℓ_{b}^2-norm of the sequence $\{\hat{a}(s + \mathrm{i}2\pi m/\tau)\}_{m=-\infty}^{\infty}$, the triangle inequality induces

$$\begin{aligned} \left|\sqrt{\beta_\infty(s)} - \sqrt{\beta_\infty(s_0)}\right|^2 &\le \sum_{m=-\infty}^{\infty} \left| \hat{a}\left(s + \frac{\mathrm{i}2\pi m}{\tau}\right) - \hat{a}\left(s_0 + \frac{\mathrm{i}2\pi m}{\tau}\right) \right|^2 \\ &= \sum_{m=-\infty}^{\infty} \left| \int_0^\infty a(t)(\mathrm{e}^{-st} - \mathrm{e}^{-s_0 t})\mathrm{e}^{-(\mathrm{i}2\pi m/\tau)t} \mathrm{d}t \right|^2. \qquad \text{(A.2)} \end{aligned}$$

If we choose a positive number δ small enough, it is possible to find a neighborhood of s_0, say U, so that U is contained in the open half plane $\mathrm{Re}\, s > \delta$. Then, for any $s \in U$, the function $a(t)(\mathrm{e}^{-st} - \mathrm{e}^{-s_0 t})\mathrm{e}^{\delta t}$ belongs to $\mathcal{L}^2$. Apply Lemma A.1, with substituting δ into s and $a(t)(\mathrm{e}^{-st} - \mathrm{e}^{-s_0 t})\mathrm{e}^{\delta t}$ into $a(t)$. Then we have

$$\sum_{m=-\infty}^{\infty} \left| \int_0^\infty a(t)(\mathrm{e}^{-st} - \mathrm{e}^{-s_0 t})\mathrm{e}^{\delta t} \cdot \mathrm{e}^{-(\delta + \mathrm{i}2\pi m/\tau)t} \mathrm{d}t \right|^2 \le \frac{\tau J^2}{(1 - \mathrm{e}^{-\delta\tau})^2},$$

where $J^2 = \int_0^\infty |a(t)(\mathrm{e}^{-st} - \mathrm{e}^{-s_0 t})\mathrm{e}^{\delta t}|^2 \mathrm{d}t$. Combine this inequality with (A.2) to have

$$\left|\sqrt{\beta_\infty(s)} - \sqrt{\beta_\infty(s_0)}\right|^2 \le \frac{\tau J^2}{(1 - \mathrm{e}^{-\delta\tau})^2}.$$

It is easy to show that we can make this J arbitrarily small by letting s approach s_0. □

Proof of Property (b). Define

$$\beta_M(s) := \sum_{m=-M}^{M} \left| \hat{a}\left(s + \frac{\mathrm{i}2\pi m}{\tau}\right) \right|^2$$

for $s \in B$ and for $M = 0, 1, \ldots$. Note that, for each M, $\beta_M(s)$ is continuous in $s \in B$. Moreover, for each $s \in B$, the sequence $\beta_0(s), \beta_1(s), \ldots$ increases monotonically and converges to $\beta_\infty(s)$, which is continuous due to Lemma A.2. Now, applying Dini's theorem [13, Theorem 4.5.5], which is presented below, we can show the claim. □

Proposition A.3 (Dini's Theorem). *Let* X *be a compact metric space, and* $\{f_n\}$ *be an increasing sequence of continuous real-valued functions that converges to a continuous real-valued function* f *at each* $x \in X$. *Then,* $\{f_n\}$ *converges to* f *uniformly.*

Appendix B

Proof of Proposition 2.28

Proposition 2.28, which gives a matrix representation of a continuous-time operator having a state-space representation, is proven here.

Without loss of generality, we can assume the continuous-time state-space representation of P, i.e., (A, B, C, D), has the property that the matrix A has an eigenvalue p only if $\widehat{P}(s)$ has a pole at $s = p$. (See Section 6.1 and Exercise 6.5.8 of [57].) Let us write $s_m := s + \mathrm{i}2\pi m/\tau$ for an integer m. If we choose s so that none of s_m is a pole of $\widehat{P}(s)$, each of $s_m I - A$ is invertible and, then, so is $\mathrm{e}^{s\tau} I - \mathrm{e}^{A\tau}$.

By Proposition 2.18, the lifting-based state-space representation of P is given by $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$, where

$$\begin{aligned}
\widetilde{A} &:= \mathrm{e}^{A\tau},\\
\widetilde{B}\tilde{\boldsymbol{a}}[k] &:= \int_0^\tau \mathrm{e}^{A(\tau-t)} B\tilde{\boldsymbol{a}}[k](t)\mathrm{d}t,\\
(\widetilde{C}\boldsymbol{x}(k\tau))(t) &:= C\mathrm{e}^{At}\boldsymbol{x}(k\tau),\\
(\widetilde{D}\tilde{\boldsymbol{a}}[k])(t) &:= D\tilde{\boldsymbol{a}}[k](t) + \int_0^t C\mathrm{e}^{A(t-r)} B\tilde{\boldsymbol{a}}[k](r)\mathrm{d}r.
\end{aligned}$$

With this representation, $\breve{P}(z) = \widetilde{D} + \widetilde{C}(zI - \widetilde{A})^{-1}\widetilde{B}$. Hence, $\acute{E}_m^s \breve{P}(\mathrm{e}^{s\tau})\grave{E}_\ell^s = \acute{E}_m^s \widetilde{D}\grave{E}_\ell^s + \acute{E}_m^s \widetilde{C}(\mathrm{e}^{s\tau} I - \widetilde{A})^{-1}\widetilde{B}\grave{E}_\ell^s$. Straightforward calculation gives

$$\begin{aligned}
\acute{E}_m^s \widetilde{C} &= \frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{e}^{-s_m t} C\mathrm{e}^{At}\,\mathrm{d}t = \frac{1}{\sqrt{\tau}} C(s_m I - A)^{-1}\{I - \mathrm{e}^{-(s_m I - A)\tau}\},\\
\widetilde{B}\grave{E}_\ell^s &= \frac{1}{\sqrt{\tau}}\int_0^\tau \mathrm{e}^{A(\tau-t)} B\mathrm{e}^{s_\ell t}\,\mathrm{d}t = \frac{1}{\sqrt{\tau}}(\mathrm{e}^{s_\ell \tau} - \mathrm{e}^{A\tau})(s_\ell I - A)^{-1} B,\\
\acute{E}_m^s \widetilde{D}\grave{E}_\ell^s &= \acute{E}_m^s \left\{\frac{1}{\sqrt{\tau}} D\mathrm{e}^{s_\ell t} + \frac{1}{\sqrt{\tau}} C\int_0^t \mathrm{e}^{A(t-r)} B\mathrm{e}^{s_\ell r}\,\mathrm{d}r\right\}\\
&= \acute{E}_m^s \left\{\frac{1}{\sqrt{\tau}} D\mathrm{e}^{s_\ell t} + \frac{1}{\sqrt{\tau}} C(\mathrm{e}^{s_\ell t} I - \mathrm{e}^{At})(s_\ell I - A)^{-1} B\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tau}\left\{\int_0^\tau \mathrm{e}^{-s_m t} D \mathrm{e}^{s_\ell t}\,\mathrm{d}t + \int_0^\tau \mathrm{e}^{-s_m t} C \mathrm{e}^{s_\ell t}(s_\ell I - A)^{-1} B\,\mathrm{d}t\right. \\
&\qquad \left. - \int_0^\tau \mathrm{e}^{-s_m t} C \mathrm{e}^{At}(s_\ell I - A)^{-1} B\,\mathrm{d}t\right\} \\
&= \widehat{P}(s_m)\delta_{m,\ell} - \frac{1}{\tau} C(s_m I - A)^{-1}\{I - \mathrm{e}^{-(s_m I - A)\tau}\}(s_\ell I - A)^{-1} B.
\end{aligned}
$$

These equations establish the relationship

$$
\acute{E}_m^s \breve{P}(\mathrm{e}^{s\tau}) \grave{E}_\ell^s = \widehat{P}(s_m)\delta_{m,\ell}.
$$

Next, we consider the second equation in the proposition. Since Proposition 2.27 implies $\|\breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau})\|_{\mathrm{L}} \geq \overline{\sigma}\{\acute{E}_m^{\mathrm{i}\omega} \breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau}) \grave{E}_m^{\mathrm{i}\omega}\}$ for each m, the left-hand side of the considered equation is larger than or equal to its right-hand side. To show the reversed inequality, suppose $\boldsymbol{f}$ is an arbitrary function in $\mathcal{L}^2[0,\tau)$. Then, by Proposition 2.26,

$$
\begin{aligned}
\breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau})\boldsymbol{f} &= \left(\sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega}\right) \breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau}) \left(\sum_{\ell=-\infty}^{\infty} \grave{E}_\ell^{\mathrm{i}\omega} \acute{E}_\ell^{\mathrm{i}\omega}\right) \boldsymbol{f} \\
&= \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau}) \grave{E}_\ell^{\mathrm{i}\omega} \acute{E}_\ell^{\mathrm{i}\omega} \boldsymbol{f} \\
&= \sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \widehat{P}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right)(\acute{E}_m^{\mathrm{i}\omega}\boldsymbol{f}).
\end{aligned}
$$

Noting that $\widehat{P}(\mathrm{i}\omega + \mathrm{i}2\pi m/\tau)(\acute{E}_m^{\mathrm{i}\omega}\boldsymbol{f})$ is just a finite-dimensional vector, we obtain

$$
\begin{aligned}
\|\breve{P}(\mathrm{e}^{\mathrm{i}\omega\tau})\boldsymbol{f}\|_{\mathcal{L}^2[0,\tau)}^2 &\leq \sum_{m=-\infty}^{\infty} \left\|\widehat{P}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right)(\acute{E}_m^{\mathrm{i}\omega}\boldsymbol{f})\right\|_2^2 \\
&\leq \sup_{m=0,\pm1,\ldots} \overline{\sigma}\left\{\widehat{P}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right)\right\}^2 \cdot \sum_{m=-\infty}^{\infty} \|\acute{E}_m^{\mathrm{i}\omega}\boldsymbol{f}\|_2^2 \\
&= \sup_{m=0,\pm1,\ldots} \overline{\sigma}\left\{\widehat{P}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau}\right)\right\}^2 \cdot \|\boldsymbol{f}\|_{\mathcal{L}^2[0,\tau)}^2.
\end{aligned}
$$

This confirms that the desired inequality holds.

Appendix C

Proofs of Propositions 2.44 and 2.45

Here, two propositions 2.44 and 2.45 are proven, which are concerned with the best continuous-time control performance when the controller class is restricted.

Proof of Proposition 2.44. It is obvious that

$$
\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| \geq \inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\|.
$$

Hence, we show the reversed inequality in the following.

Note that $\widehat{G}_{22}(s)$ cannot have a pole at $s = \infty$ because $\widehat{G}_{22}(\infty) = D_{22}$. Applying Lemma 2.41, we obtain a doubly coprime factorization of $\widehat{G}_{22}(s)$ such that $Y(\infty) = O$ and $\widetilde{Y}(\infty) = O$. Define

$$
\mathcal{RH}_0^\infty := \{Q \in \mathcal{RH}^\infty : Q(\infty) = O\}. \tag{C.1}
$$

Then, if we define $T_1, T_2, T_3 \in \mathcal{RH}^\infty$ as in the statement of Proposition 2.40, it is derived from Propositions 2.40 and 2.42 that

$$
\inf_{K \in \mathcal{K}} \|\mathcal{F}(G, K)\| = \inf_{\substack{Q \in \mathcal{RH}^\infty \\ \det(\widetilde{X} - Q\widetilde{N}) \not\equiv 0}} \|T_1 - T_2 Q T_3\|_{\mathcal{H}^\infty}
$$

and

$$
\inf_{K_0 \in \mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{Q_0 \in \mathcal{RH}_0^\infty} \|T_1 - T_2 Q_0 T_3\|_{\mathcal{H}^\infty}.
$$

Let ϵ be any positive number and choose one $Q \in \mathcal{RH}^\infty$ so that $\det(\widetilde{X} - Q\widetilde{N}) \not\equiv 0$ and

$$
\|T_1 - T_2 Q T_3\|_{\mathcal{H}^\infty} < \inf_{\substack{Q \in \mathcal{RH}^\infty \\ \det(\widetilde{X} - Q\widetilde{N}) \not\equiv 0}} \|T_1 - T_2 Q T_3\|_{\mathcal{H}^\infty} + \frac{\epsilon}{2}.
$$

Define $Q_\alpha(s) := \{1/(\alpha s + 1)\} Q(s)$ for $\alpha > 0$. From now on, we show that

$$
\|T_1 - T_2 Q_\alpha T_3\|_{\mathcal{H}^\infty} < \|T_1 - T_2 Q T_3\|_{\mathcal{H}^\infty} + \frac{\epsilon}{2} \tag{C.2}
$$

for small enough α. If it is shown, since $Q_\alpha \in \mathcal{RH}_0^\infty$, the desired inequality follows.

Now note that at least one of T_1, T_2, and T_3 is strictly proper, since at least one of $\widehat{G}_{11}$, $\widehat{G}_{12}$, and $\widehat{G}_{21}$ is strictly proper. Suppose T_1 is strictly proper. Continuity of $T_1(s)$, $T_2(s)$, and $T_3(s)$ in $\mathbb{C}_{+e}$ implies

$$\|T_1 - T_2 Q_\alpha T_3\|_{\mathcal{H}^\infty} = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\{T_1(\mathrm{i}\omega) - T_2(\mathrm{i}\omega)Q_\alpha(\mathrm{i}\omega)T_2(\mathrm{i}\omega)\},$$

as is noted before Proposition 2.7. Here, we have

$$\begin{aligned}
&\overline{\sigma}\{T_1(\mathrm{i}\omega) - T_2(\mathrm{i}\omega)Q_\alpha(\mathrm{i}\omega)T_2(\mathrm{i}\omega)\}\\
&\quad = \overline{\sigma}\Big\{(T_1 - T_2QT_3)\frac{1}{\alpha\mathrm{i}\omega+1} + T_1\Big(1-\frac{1}{\alpha\mathrm{i}\omega+1}\Big)\Big\}\\
&\quad \le \overline{\sigma}(T_1 - T_2QT_3) + \overline{\sigma}(T_1)\left|1-\frac{1}{\alpha\mathrm{i}\omega+1}\right|.
\end{aligned}$$

Dependence on $\mathrm{i}\omega$ is not described above for notational convenience. This inequality means that, if we take a large enough $\omega_0 > 0$, we can ensure

$$\overline{\sigma}\{T_1(\mathrm{i}\omega) - T_2(\mathrm{i}\omega)Q_\alpha(\mathrm{i}\omega)T_3(\mathrm{i}\omega)\} < \overline{\sigma}\{T_1(\mathrm{i}\omega) - T_2(\mathrm{i}\omega)Q(\mathrm{i}\omega)T_3(\mathrm{i}\omega)\} + \frac{\epsilon}{2} \tag{C.3}$$

for any $|\omega| \ge \omega_0$. On the other hand, there also holds

$$\begin{aligned}
&\overline{\sigma}\{T_1(\mathrm{i}\omega) - T_2(\mathrm{i}\omega)Q_\alpha(\mathrm{i}\omega)T_2(\mathrm{i}\omega)\}\\
&\quad = \overline{\sigma}\Big\{T_1 - T_2QT_3 + T_2QT_3\Big(1-\frac{1}{\alpha\mathrm{i}\omega+1}\Big)\Big\}\\
&\quad \le \overline{\sigma}(T_1 - T_2QT_3) + \overline{\sigma}(T_2QT_3)\left|1-\frac{1}{\alpha\mathrm{i}\omega+1}\right|.
\end{aligned}$$

Hence, also for $|\omega| < \omega_0$, by taking small enough $\alpha > 0$, we can guarantee (C.3). Now, the inequality (C.2) is confirmed.

Also in the case that T_2 or T_3 is strictly proper, (C.2) can be shown to hold for small enough $\alpha > 0$ by a similar technique. The proof is completed. □

Proof of Proposition 2.45. Define $\mathcal{RH}_0^\infty$ as in (C.1). Define $\mathcal{RH}_{00}^\infty$ as

$$\mathcal{RH}_{00}^\infty := \{Q \in \mathcal{RH}^\infty : Q(\infty) = O \text{ with multiplicity two or more}\}.$$

By Lemma 2.41, we can find a coprime factorization of $\widehat{G}_{22}(s)$ such that $Y(s)$ and $\tilde{Y}(s)$ equal to zero at $s = \infty$ with multiplicity two or more. Using this factorization, define functions T_1, T_2, T_3 as in Proposition 2.40. Then, Proposition 2.42 implies that

$$\inf_{K_0\in\mathcal{K}_0} \|\mathcal{F}(G, K_0)\| = \inf_{Q_0\in\mathcal{RH}_0^\infty} \|T_1 - T_2Q_0T_3\|_{\mathcal{H}^\infty},$$

$$\inf_{K_{00}\in\mathcal{K}_{00}} \|\mathcal{F}(G, K_{00})\| = \inf_{Q_{00}\in\mathcal{RH}_{00}^\infty} \|T_1 - T_2Q_{00}T_3\|_{\mathcal{H}^\infty}.$$

Let ϵ be some positive number. Choose $Q_0 \in \mathcal{RH}_0^\infty$ so that

$$\|T_1 - T_2Q_0T_3\|_{\mathcal{H}^\infty} < \inf_{Q_0\in\mathcal{RH}_0^\infty} \|T_1 - T_2Q_0T_3\|_{\mathcal{H}^\infty} + \frac{\epsilon}{2}.$$

Define $Q_\alpha(s) := \{1/(\alpha s+1)\}Q_0(s)$ for $\alpha > 0$. Then, $Q_\alpha \in \mathcal{RH}_{00}^\infty$ and we can show

$$\|T_1 - T_2Q_\alpha T_3\|_{\mathcal{H}^\infty} < \|T_1 - T_2Q_0T_3\|_{\mathcal{H}^\infty} + \frac{\epsilon}{2}$$

for small enough α using a technique similar to the one used in the previous proof. Therefore, it is shown that $\inf_{Q_{00}\in\mathcal{RH}_{00}^\infty} \|T_1 - T_2Q_{00}T_3\|_{\mathcal{H}^\infty} \le \inf_{Q_0\in\mathcal{RH}_0^\infty} \|T_1 - T_2Q_0T_3\|_{\mathcal{H}^\infty}$. Since the reversed inequality is obvious, the desired equality is obtained. □