Appendix D

Proof of Proposition 2.47

Here, Proposition 2.47, which is a flat-operator version of Nehari's theorem, is proven. Some preparation is needed for this.

Let F be a real flat operator that maps $\mathcal{L}^2[0,\tau)^n$ to $\mathbb{C}^\ell$. Since FF^* is a real symmetric matrix and is semi-positive definite, we can choose a real orthogonal matrix V and a real diagonal non-singular matrix Λ so that

$$FF^* = V^* \begin{bmatrix} \Lambda^2 & O \\ O & O \end{bmatrix} V.$$

Suppose that the matrix Λ has the size $f \times f$. Then, f is equal to $\operatorname{rank} FF^*$, and to $\operatorname{rank} F$ in turn. Define T by

$$T := F^* V^* \begin{bmatrix} \Lambda^{-1} \\ O \end{bmatrix},$$

which is a real tall operator mapping $\mathbb{C}^f$ into $\mathcal{L}^2[0,\tau)^n$. It is not difficult to show that $T^*T = I$ and TT^* is the orthogonal projection from $\mathcal{L}^2[0,\tau)^n$ to $\mathcal{N}(F)^\perp$. Here, let $\mathcal{N}(F)$ denote the null space of the operator F and let $\mathcal{N}(F)^\perp$ denote its orthogonal complement in $\mathcal{L}^2[0,\tau)^n$. In particular, there holds $F(I - TT^*) = O$. Furthermore, we can show that T is an isometric isomorphism from $\mathbb{C}^f$ to $\mathcal{N}(F)^\perp$. Here, we have the following lemmas.

Lemma D.1. *For any matrix M such that MF is well-defined, there holds*

$$\overline{\sigma}(MFT) = \|MF\|_{\mathrm{F}}.$$

Lemma D.2. *For any matrix-valued function $Q(z)$ having no pole on $|z| = 1$, we have*

$$\inf_{\Sigma' \in \mathfrak{RH}^\infty} \|QFT - \Sigma'\|_{\mathfrak{L}^\infty} = \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}.$$

These lemmas mean that problems on a flat-operator or a flat-operator-valued function can be simplified into corresponding problems on a matrix or a matrix-valued function, respectively, by concatenation of T.

Proof of Lemma D.1. Define an operator $\Xi : \mathbb{C}^f \oplus \mathcal{L}^2[0,\tau)^n \to \mathcal{L}^2[0,\tau)^n$ by

$$\Xi := [T \quad I - TT^*].$$

Here, $\oplus$ stands for the direct sum of linear spaces. Then, $\Xi\Xi^* = I$ holds. Just like the proof of Proposition 2.46, we can show that

$$\|MF\Xi\|_{\mathrm{ind}} = \|\Xi^* F^* M^*\|_{\mathrm{ind}} = \|F^* M^*\|_{\mathrm{T}} = \|MF\|_{\mathrm{F}}$$

and

$$\|MF\Xi\|_{\mathrm{ind}} = \|[MFT \quad O]\|_{\mathrm{ind}} = \overline{\sigma}(MFT).$$

Hence, we have shown the claim. $\square$

Proof of Lemma D.2. First, let us show

$$\inf_{\Sigma' \in \mathfrak{RH}^\infty} \|QFT - \Sigma'\|_{\mathfrak{L}^\infty} \le \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}. \tag{D.1}$$

Defining an operator Ξ as above, we have

$$\begin{aligned}\|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty} &= \sup_{|z|=1} \|Q(z)F - \Sigma(z)\|_{\mathrm{F}} = \sup_{|z|=1} \|\{Q(z)F - \Sigma(z)\}\Xi\|_{\mathrm{ind}} \\ &= \sup_{|z|=1} \left\| \left[Q(z)FT - \Sigma(z)T \quad -\Sigma(z)(I - TT^*) \right] \right\|_{\mathrm{ind}}.\end{aligned}$$

Hence, there holds

$$\begin{aligned}\inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}^2 &= \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \sup_{|z|=1} \left[\overline{\sigma}\{Q(z)FT - \Sigma(z)T\}^2 + \|\Sigma(z)(I - TT^*)\|_{\mathrm{F}}^2 \right] \\ &\ge \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \sup_{|z|=1} \overline{\sigma}\{Q(z)FT - \Sigma(z)T\}^2 \\ &= \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \|QFT - \Sigma T\|_{\mathfrak{L}^\infty}^2 \\ &\ge \inf_{\Sigma' \in \mathfrak{RH}^\infty} \|QFT - \Sigma'\|_{\mathfrak{L}^\infty}^2.\end{aligned}$$

Now, (D.1) is proven.

Next, we show that for any $\Sigma' \in \mathfrak{RH}^\infty$ there exists $\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty$ that satisfies

$$\|QFT - \Sigma'\|_{\mathfrak{L}^\infty} = \|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}.$$

If it is shown, the proof is completed.

For a matrix-valued function $\Sigma' \in \mathfrak{RH}^\infty$, define a flat-operator-valued function Σ by $\Sigma := \Sigma' T^*$. Then there holds

$$\begin{aligned}\|QF - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}^2 &= \sup_{|z|=1} \|Q(z)F - \Sigma(z)\|_{\mathrm{F}}^2 = \sup_{|z|=1} \|\{Q(z)F - \Sigma(z)\}\Xi\|_{\mathrm{ind}}^2 \\ &= \sup_{|z|=1} \left\| \left[Q(z)FT - \Sigma(z)T \quad \{Q(z)F - \Sigma(z)\}(I - TT^*) \right] \right\|_{\mathrm{ind}}^2 \\ &= \sup_{|z|=1} \left\| \left[Q(z)FT - \Sigma'(z) \quad O \right] \right\|_{\mathrm{ind}}^2 \\ &= \|QFT - \Sigma'\|_{\mathfrak{L}^\infty}^2.\end{aligned}$$

$\square$

Now, Proposition 2.47 is proven.

Proof of Proposition 2.47. First, it is shown that we can express our $\Phi(z)$ as $\Psi(z)F$ using a real rational matrix-valued function $\Psi(z)$ and a flat operator F. Suppose for the time being that the provided $\Phi(z)$ has no pole at $z = \infty$. Then, by considering its observer canonical form just as we did to prove Proposition 2.23, we can represent $\Phi(z)$ as

$$\Phi(z) = D + C(zI - A)^{-1}B = \begin{bmatrix} I & C(zI - A)^{-1} \end{bmatrix} \begin{bmatrix} D \\ B \end{bmatrix}$$

using real matrices A and C and real flat operators B and D. Putting the first factor in the last expression as $\Psi(z)$ and the second factor as F, we see that our claim is proven in a special case. In the case that $\Phi(z)$ has a pole at $z = \infty$, transform $\Phi(z)$ by $z' = (-\alpha z + 1)/(z - \alpha)$ choosing $\alpha > 1$ so that $\Phi(z)$ has no pole at $z = \alpha$. Since in the z'-domain our $\Phi(z)$ does not have a pole at $z' = \infty$, we can represent our $\Phi(z)$ as above and transform it back to the z-domain. Then, we have a representation $\Phi(z) = \Psi(z)F$.

For a real flat operator F, define a tall operator T as before. Then, we have

$$\Phi(z)T = \Psi(z)FT$$

and $\Phi(z)T$ is a matrix-valued function. Note that Proposition 2.48 (Nehari's theorem for matrix-valued functions) implies

$$\inf_{\Sigma' \in \mathfrak{RH}^\infty} \|\Phi T - \Sigma'\|_{\mathfrak{L}^\infty} = \|\Phi T\|_{\mathrm{H}}.$$

In the following, we show

$$\inf_{\Sigma' \in \mathfrak{RH}^\infty} \|\Phi T - \Sigma'\|_{\mathfrak{L}^\infty} = \inf_{\Sigma \in \mathfrak{RH}_{\mathrm{F}}^\infty} \|\Phi - \Sigma\|_{\mathfrak{L}_{\mathrm{F}}^\infty}, \tag{D.2}$$

$$\|\Phi T\|_{\mathrm{H}} = \|\Phi\|_{\mathrm{H}}. \tag{D.3}$$

Then, the desired equality $\inf_{\Sigma\in\mathfrak{R}\mathfrak{H}^\infty_{\mathrm{F}}} \|\Phi - \Sigma\|_{\mathfrak{L}^\infty_{\mathrm{F}}} = \|\Phi\|_{\mathrm{H}}$ follows.

Equation (D.2) follows from Lemma D.2 easily. Let us consider Equation (D.3). When we expand $\Phi(z) = \Psi(z)F$ into the Laurent expansion $\sum_{k=-\infty}^{\infty} L^k z^k$, each L^k can be written as $M^k F$ using a matrix M^k. Then, just like the proof of Lemma D.1, it can be shown that

$$\left\| \begin{bmatrix} L^1 & L^2 & \cdots \\ L^2 & L^3 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \Xi & & 0 \\ & \Xi & \\ 0 & & \ddots \end{bmatrix} \right\|_{\mathrm{ind}} = \left\| \begin{bmatrix} L^1 & L^2 & \cdots \\ L^2 & L^3 & \\ \vdots & & \ddots \end{bmatrix} \right\|_{\mathrm{ind}} = \|\Phi\|_{\mathrm{H}},$$

and

$$\begin{aligned} \left\| \begin{bmatrix} L^1 & L^2 & \cdots \\ L^2 & L^3 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \Xi & & 0 \\ & \Xi & \\ 0 & & \ddots \end{bmatrix} \right\|_{\mathrm{ind}} &= \left\| \begin{bmatrix} L^1T & O & L^2T & O & \cdots \\ L^2T & O & L^3T & O & \\ \vdots & & & & \ddots \end{bmatrix} \right\|_{\mathrm{ind}} \\ &= \left\| \begin{bmatrix} L^1T & L^2T & \cdots \\ L^2T & L^3T & \\ \vdots & & \ddots \end{bmatrix} \right\|_{\mathrm{ind}} = \|\Phi T\|_{\mathrm{H}}. \end{aligned}$$

Hence, Equation (D.3) is established. □

Note that the above proof shows how we can compute the Hankel norm of a flat-operator-valued function $\Phi(z)$ and how we can obtain $\Sigma(z)$ that approximates $\Phi(z)$. That is, using the formula (D.3), computation of $\|\Phi\|_{\mathrm{H}}$ is reduced to that of $\|\Phi T\|_{\mathrm{H}}$, which is carried out via matrix calculations. Besides, if a matrix-valued function $\Sigma'(z)$ approximates $\Phi(z)T$ well, then $\Sigma(z) := \Sigma'(z)T^*$ approximates $\Phi(z)$ equally well.

Appendix E

Proof of Proposition 3.24

We prove Proposition 3.24, which states properties of the 25 operators from $\boldsymbol{w}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{a}(t)$, $\boldsymbol{a}_{\mathrm{d}}[k]$, $\boldsymbol{b}_{\mathrm{d}}[k]$ to $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{p}_{\mathrm{d}}[k]$, $\boldsymbol{q}'_{\mathrm{d}}[k]$, $\boldsymbol{q}(t)$ in Figure E.1. (This figure is the same as Figure 3.5.) Namely, it is proven that, if the system in Figure E.1 is input-output stable, these 25 operators have lifting-based transfer functions, and there exists $0 < \rho < 1$ such that $\breve{S}(z)$, $\breve{H}(z)$, and these 25 transfer functions are analytic in $\mathbb{D}_\rho$.

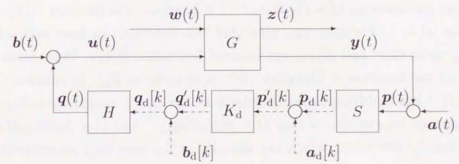

Figure E.1. A sampled-data control system.

First, let us consider the operator $(I - SG_{22}HK_{\mathrm{d}})^{-1}$, which is the discrete-time operator mapping $\boldsymbol{a}_{\mathrm{d}}$ to $\boldsymbol{p}'_{\mathrm{d}}$. Write this operator as V_{d} temporarily. Because the operator from $\boldsymbol{a}_{\mathrm{d}}$ to $\boldsymbol{p}_{\mathrm{d}}$ is bounded by assumption, this operator V_{d} is bounded, too. Note that both $\breve{G}_{22}(z)$ and $\breve{K}_{\mathrm{d}}(z)$ are rational functions. Moreover, $\breve{S}(z)$ and $\breve{H}(z)$ are analytic in $\mathbb{D}_{\rho_0}$ for some $0 < \rho_0 < 1$ by Proposition 3.13. Hence, $v(z) := \det(I - \breve{S}\breve{G}_{22}\breve{H}\breve{K}_{\mathrm{d}})(z)$ is meromorphic in $\mathbb{D}_{\rho_0}$.

Since $v(z)$ takes a nonzero value at least at one z in $\mathbb{D}$ by assumption, a function $(I - \breve{S}\breve{G}_{22}\breve{H}\breve{K}_{\mathrm{d}})^{-1}$ is well-defined and is meromorphic in $\mathbb{D}_{\rho_0}$. This is exactly the lifting-based transfer function of V_{d}, i.e., $\breve{V}_{\mathrm{d}}(z)$. Suppose that $v(z) = 0$ for some z in $\mathbb{D}$ or on $|z| = 1$. Then, $\breve{V}_{\mathrm{d}}(z)$ has a pole there. However, if $\breve{V}_{\mathrm{d}}(z)$ has a pole in $\mathbb{D}$ or on $|z| = 1$, there exists $\breve{\boldsymbol{a}}_{\mathrm{d}} \in \mathfrak{H}^2$ such that $\breve{V}_d\breve{a}_d$ does not belong to $\mathfrak{H}^2$. This means that the operator V_d is not bounded, which contradicts the assumption. Hence, $v(z) \neq 0$ in $\mathbb{D}$ or on $|z| = 1$. Next, we show that we can choose ρ_1 so that $0 < \rho_0 < \rho_1 < 1$ and $v(z)$ is not equal to zero in $\mathbb{D}_{\rho_1}$. In order to show it, suppose that $v(z)$ has a zero in $\mathbb{D}_{\rho_1}$ for *any* $\rho_0 < \rho_1 < 1$. Define $v^-(w) := v(1/z)$. Then, $v^-(w)$ has a zero in $|w| < 1/\rho_1$ for any $\rho_0 < \rho_1 < 1$. Because the disc $|w| \leq 1$ is compact, there is a sequence $\{w_j\}_{j=1}^{\infty}$ such that $v^-(w_j)$ is equal to zero and w_j converges to some point in $|w| \leq 1$. This contradicts with the previous result. Now, we have shown that $v(z) \neq 0$ in $\mathbb{D}_{\rho_1}$ for some $\rho_0 < \rho_1 < 1$. This implies that the operator has the transfer function $\breve{V}_d(z)$, which is analytic in $\mathbb{D}_{\rho_1}$.

Next, we consider the operator $(I - G_{22}HK_dS)^{-1}$. Since this is the operator from $\boldsymbol{a}$ to $\boldsymbol{p}$, it is bounded by assumption. Note that

$$I = (I - G_{22}HK_dS)\{I + G_{22}HK_d(I - SG_{22}HK_d)^{-1}S\} = (I - G_{22}HK_dS)(I + G_{22}HK_dV_dS).$$

We have shown that $\breve{V}_d(z)$ is analytic in $\mathbb{D}_{\rho_1}$. In the same region, $\breve{G}(z)$ and $\breve{K}_d(z)$ are meromorphic and $\breve{S}(z)$ and $\breve{H}(z)$ are analytic. This means that the function

$$I + \breve{G}_{22}\breve{H}\breve{K}_d\breve{V}_d\breve{S} \tag{E.1}$$

is meromorphic in $\mathbb{D}_{\rho_1}$ and it is exactly the transfer function of $(I - G_{22}HK_dS)^{-1}$. From the assumption that the operator $(I - G_{22}HK_dS)^{-1}$ is bounded, the function (E.1) cannot have a pole in $\mathbb{D}$ or on $|z| = 1$. Furthermore, note that this function can have only a finite number of poles in $\mathbb{D}_{\rho_1}$ since $\breve{G}(z)$ and $\breve{K}_d(z)$ are rational functions. Hence, there exists ρ_2 such that $\rho_1 < \rho_2 < 1$ and the function of Equation (E.1) is analytic in $\mathbb{D}_{\rho_2}$. In summary, the operator $(I - G_{22}HK_dS)^{-1}$ has its lifting-based transfer function and it is analytic in $\mathbb{D}_{\rho_2}$.

In a similar way, we can derive that $(I - HK_dSG_{22})^{-1}$ and $(I - K_dSG_{22}H)^{-1}$ have their lifting-based transfer functions, which are analytic in $\mathbb{D}_{\rho_3}$ and $\mathbb{D}_{\rho_4}$, respectively, where $\rho_1 < \rho_3 < 1$ and $\rho_1 < \rho_4 < 1$.

Now, we consider the 25 operators in question. The claim is proven almost in the same procedure for each operator. For example, let us consider the operator from $\boldsymbol{w}$ to $\boldsymbol{z}$. This operator is expressed as

$$G_{11} + G_{12}HK_d(I - SG_{22}HK_d)^{-1}SG_{21}. \tag{E.2}$$

Since each of G_{11}, G_{12}, H, K_d, $(I - SG_{22}HK_d)^{-1}$, S, and G_{21} has its lifting-based transfer function, so does the above operator. Furthermore, $\breve{S}$, $\breve{H}$, and the transfer function of $(I - SG_{22}HK_d)^{-1}$ is analytic in $\mathbb{D}_{\rho_1}$; $\breve{G}(z)$ and $\breve{K}_d(z)$ are meromorphic and have only finite number of poles. Therefore, the lifting-based transfer function of (E.2) is meromorphic in $\mathbb{D}_{\rho_1}$ and has finite number of poles there. From the assumption that the considered operator is bounded, it follows that this transfer function cannot have a pole in $\mathbb{D}$ or on $|z| = 1$. Since this function can have finite number of poles in $\mathbb{D}_{\rho_1}$, there exists $\rho_1 < \rho < 1$ such that the considered transfer function is analytic in $\mathbb{D}_\rho$.

It is possible to choose a common $0 < \rho < 1$ so that the transfer functions of all the 25 operators together with $\breve{S}(z)$ and $\breve{H}(z)$ are analytic in $\mathbb{D}_\rho$. Now the proof is completed.

Appendix F

Supplement to the Proof of Theorem 4.12

The aim of this appendix is to prove the following two statements are equivalent. This fact was left unproven in Section 4.4.

(b) For any $\Omega > 0$,

$$\overline{\sigma}\left[\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^{-1/2} \left\{\sum_{m\neq 0} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\} \left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^{-1/2}\right]$$

converges to zero uniformly for any $|\omega| < \Omega$ as $j \to \infty$. Here, $\omega_m := \omega + 2\pi m/\tau_j$.

(b′) $\|(I - \breve{H}_j^{\mathrm{in}} \breve{H}_j^{\mathrm{in}\sim})\breve{R}^{n_q}\|_{\mathfrak{L}_{\mathrm{L}}^\infty} \to 0 \qquad (j \to \infty)$.

The proof proceeds by two steps. At the first step, (b′) is shown to be equivalent to

(b″) $\forall\Omega > 0$, $\forall\epsilon > 0$, $\exists J > 0$ s.t. $\|\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\}\grave{E}_0^{\mathrm{i}\omega}\|_{\mathrm{T}} < \epsilon$ for any $j > J$ and $|\omega| < \Omega$.

Then, at the second step, equivalence between (b″) and (b) is proven. Hereafter, let us write $V_j^{\omega} := I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})$ and write R^{n_q} just as R.

[(b′)⇒(b″)] Assume (b″) does not hold. That is, we assume that there exists $\Omega > 0$ and $\epsilon > 0$ such that we can attain

$$\left\|\left\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega_j\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega_j\tau_j})\right\}\grave{E}_0^{\mathrm{i}\omega_j}\right\|_{\mathrm{T}} = \|V_j^{\omega_j}\grave{E}_0^{\mathrm{i}\omega_j}\|_{\mathrm{T}} \geq \epsilon$$

for infinitely many j's by choosing each ω_j appropriately with $|\omega_j| < \Omega$. By redefining the sequence $\{(\tau_j, S_j, H_j)\}_{j=1}^{\infty}$, we can assume $\|V_j^{\omega_j}\grave{E}_0^{\mathrm{i}\omega_j}\|_{\mathrm{T}} \geq \epsilon$ for any j without loss of generality. Since $\{\omega_j\}$ is a bounded sequence, there exists $r > 0$ such that $\underline{\sigma}\{\widehat{R}(\mathrm{i}\omega_j)\} \geq r$ for each j. In the sequel, we will show that $\|(I - \breve{H}_j^{\mathrm{in}}\breve{H}_j^{\mathrm{in}\sim})\breve{R}\|_{\mathfrak{L}_{\mathrm{L}}^\infty} \geq r\epsilon$. Because this contradicts with (b′), (b′)⇒(b″) is confirmed.

Note that

$$\|(I - \breve{H}_j^{\mathrm{in}} \breve{H}_j^{\mathrm{in}\sim})\breve{R}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}} \geq \|V_j^{\omega_j} \breve{R}(\mathrm{e}^{\mathrm{i}\omega_j \tau_j})\|_{\mathrm{L}} \geq \|V_j^{\omega_j} \breve{R}(\mathrm{e}^{\mathrm{i}\omega_j \tau_j}) \grave{E}_0^{\mathrm{i}\omega_j}\|_{\mathrm{T}}.$$

Here, the first inequality is derived from the definition of the $\mathfrak{L}_{\mathrm{L}}^{\infty}$-norm (Section 2.6.1) and the second from Proposition 2.27. By Propositions 2.26 and 2.28, there holds

$$\breve{R}(\mathrm{e}^{\mathrm{i}\omega_j \tau_j}) \grave{E}_0^{\mathrm{i}\omega_j} = \sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega_j} \acute{E}_m^{\mathrm{i}\omega_j} \breve{R}(\mathrm{e}^{\mathrm{i}\omega_j \tau_j}) \grave{E}_0^{\mathrm{i}\omega_j} = \grave{E}_0^{\mathrm{i}\omega_j} \widehat{R}(\mathrm{i}\omega_j). \tag{F.1}$$

Using this, we obtain

$$\|(I - \breve{H}_j^{\mathrm{in}} \breve{H}_j^{\mathrm{in}\sim})\breve{R}\|_{\mathfrak{L}_{\mathrm{L}}^{\infty}} \geq \|V_j^{\omega_j} \grave{E}_0^{\mathrm{i}\omega_j} \widehat{R}(\mathrm{i}\omega_j)\|_{\mathrm{T}} \geq \|V_j^{\omega_j} \grave{E}_0^{\mathrm{i}\omega_j}\|_{\mathrm{T}}\, \underline{\sigma}\{\widehat{R}(\mathrm{i}\omega_j)\} \geq \epsilon r.$$

Now, the proof of this part is completed.

[(b′)⇐(b″)] Let ϵ be any positive number. Choose Ω so that $\overline{\sigma}\{\widehat{R}(\mathrm{i}\omega)\} < \epsilon$ for any $|\omega| \geq \Omega$. Moreover, let us consider large enough j's such that $\Omega < \pi/\tau_j$. We will show that $\|\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\}\breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}} = \|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}$ converges to zero as $j \to \infty$ uniformly for any $|\omega| \leq \pi/\tau_j$. This implies (b′).

First, we consider the range of $\Omega \leq |\omega| \leq \pi/\tau_j$. In this range, there holds

$$\|\breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}} = \sup_{m=0,\pm 1,\ldots} \overline{\sigma}\left\{\widehat{R}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\right\} < \epsilon.$$

Here, the first equality is based on Proposition 2.28. Since the $U(z)$ in Section 2.6.1 satisfies $U^{\sim}(z)U(z) \equiv I$, there holds $\|V_j^{\omega}\|_{\mathrm{L}} \leq 1$. This gives

$$\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}} \leq \|V_j^{\omega}\|_{\mathrm{L}} \|\breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}} < \epsilon.$$

Hence, the uniform convergence we are interested in has been proven in $\Omega \leq |\omega| \leq \pi/\tau_j$.

Next, consider the range of $|\omega| < \Omega$. Here, we use the relationship

$$\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}^2 \leq \sum_{m=-\infty}^{\infty} \|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}}^2, \tag{F.2}$$

which is obtained from Proposition 2.27. Note that there holds

$$\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_0^{\mathrm{i}\omega}\|_{\mathrm{T}} = \|V_j^{\omega} \grave{E}_0^{\mathrm{i}\omega} \widehat{R}(\mathrm{i}\omega)\|_{\mathrm{T}} \leq \|V_j^{\omega} \grave{E}_0^{\mathrm{i}\omega}\|_{\mathrm{T}}\, \overline{\sigma}\{\widehat{R}(\mathrm{i}\omega)\},$$

where the first equality follows from (F.1). By assumption, the above quantity $\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_0^{\mathrm{i}\omega}\|_{\mathrm{T}}$ converges to zero uniformly in $|\omega| < \Omega$. On the other hand, for any nonzero integer m,

$$\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}} = \left\|V_j^{\omega} \grave{E}_m^{\mathrm{i}\omega} \widehat{R}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\right\|_{\mathrm{T}} \leq \|V_j^{\omega}\|_{\mathrm{L}}\, \|\grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}}\, \overline{\sigma}\left\{\widehat{R}\left(\mathrm{i}\omega + \frac{\mathrm{i}2\pi m}{\tau_j}\right)\right\}.$$

As we saw above, $\|V_j^{\omega}\|_{\mathrm{L}} \leq 1$. Besides, $\|\grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}} = 1$ by Proposition 2.26. Noting that $\overline{\sigma}\{\widehat{R}(\mathrm{i}\omega + \mathrm{i}2\pi m/\tau_j)\}$ converges to zero uniformly for any $|\omega| < \Omega$ as $j \to \infty$, we can see that $\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_m^{\mathrm{i}\omega}\|_{\mathrm{T}}$ converges to zero uniformly for any $|\omega| < \Omega$ and any nonzero integer m. By these facts together with (F.2), the uniform convergence of $\|V_j^{\omega} \breve{R}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\|_{\mathrm{L}}$ is confirmed in $|\omega| < \Omega$, too.

[(b″)⇒(b)] Define a finite-dimensional matrix Z_m by

$$Z_m := \acute{E}_m^{\mathrm{i}\omega} \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j}).$$

Although Z_m depends on j and ω, this dependence is suppressed for notational simplicity. Then, there holds

$$\left\|\left\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\right\} \grave{E}_0^{\mathrm{i}\omega}\right\|_{\mathrm{T}} \geq \overline{\sigma}\left[\acute{E}_0^{\mathrm{i}\omega}\left\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\right\} \grave{E}_0^{\mathrm{i}\omega}\right] = \overline{\sigma}(I - Z_0 Z_0^*).$$

We used Proposition 2.27 to obtain the first inequality and used $\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) = \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})^*$ and $\grave{E}_0^{\mathrm{i}\omega} = (\acute{E}_0^{\mathrm{i}\omega})^*$ to have the last equality. Note that for a square matrix M, there holds $\overline{\sigma}(I - MM^*) = \overline{\sigma}(I - M^*M)$ in general. Furthermore, from Proposition 2.26, it follows that

$$I = \breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) = \sum_{m=-\infty}^{\infty} \breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) = \sum_{m=-\infty}^{\infty} Z_m^* Z_m.$$

Hence, it is derived that

$$\left\|\left\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\right\} \grave{E}_0^{\mathrm{i}\omega}\right\|_{\mathrm{T}} \geq \overline{\sigma}(I - Z_0 Z_0^*) = \overline{\sigma}(I - Z_0^* Z_0) = \overline{\sigma}\left\{\sum_{m\neq 0} Z_m^* Z_m\right\}. \tag{F.3}$$

On the other hand, by the definition of $\breve{H}_j^{\mathrm{out}}(z)$,

$$\begin{aligned} \breve{H}_j^{\mathrm{out}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{out}}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) = \breve{H}_j^{\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j}) &= \sum_{m=-\infty}^{\infty} \breve{H}_j^{\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \breve{H}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \\ &= \sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m). \end{aligned}$$

Hence, $\breve{H}_j^{\mathrm{out}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})$ can be represented as $W\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\}^{1/2}$ with an appropriate unitary matrix W. This further gives a representation

$$Z_m = \acute{E}_m^{\mathrm{i}\omega} \breve{H}_j(\mathrm{e}^{\mathrm{i}\omega\tau_j}) \breve{H}_j^{\mathrm{out}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})^{-1} = \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\left\{\sum_{m=-\infty}^{\infty} \underline{\widehat{H}}_j(\mathrm{i}\omega_m)^* \underline{\widehat{H}}_j(\mathrm{i}\omega_m)\right\}^{-1/2} W^*.$$

Substituting this into (F.3), we can see that (b″) implies (b).

[(b″)⇐(b)] Define a matrix Z_m as before. By assumption, for any $\Omega > 0$ and $\epsilon > 0$, there exists $J > 0$ such that

$$\overline{\sigma}\left(\sum_{m\neq 0} Z_m^* Z_m\right) < \min\left\{\frac{\epsilon}{\sqrt{2}},\ \frac{\epsilon^2}{2(1+\epsilon)}\right\} \tag{F.4}$$

holds for any $j > J$ and $|\omega| < \Omega$. Recall $I = \sum_{m=-\infty}^{\infty} Z_m^* Z_m$. Hence, there holds

$$\overline{\sigma}(I - Z_0^* Z_0) = \overline{\sigma}\left(\sum_{m\neq 0} Z_m^* Z_m\right) < \frac{\epsilon}{\sqrt{2}}.$$

This implies that

$$\|\grave{E}_0^{\mathrm{i}\omega}(I - Z_0 Z_0^*)\|_{\mathrm{T}}^2 = \overline{\sigma}(I - Z_0 Z_0^*)^2 < \frac{\epsilon^2}{2} \tag{F.5}$$

On the other hand, again from (F.4), it is derived that

$$|1 - \overline{\sigma}(Z_0)^2| = |1 - \overline{\sigma}(Z_0^* Z_0)| \leq \overline{\sigma}(I - Z_0^* Z_0) = \overline{\sigma}\left(\sum_{m\neq 0} Z_m^* Z_m\right) \leq \frac{\epsilon}{\sqrt{2}}.$$

This gives

$$\overline{\sigma}(Z_0)^2 \leq 1 + \frac{\epsilon}{\sqrt{2}} < 1 + \epsilon. \tag{F.6}$$

We note

$$\overline{\sigma}\left(\sum_{m\neq 0} Z_m^* Z_m\right) < \frac{\epsilon^2}{2(1+\epsilon)}$$

by (F.4) and multiply Z_0 and Z_0^* to the sum in the left-hand side. Then, from (F.6) it follows that

$$\overline{\sigma}\left\{ Z_0 \left(\sum_{m\neq 0} Z_m^* Z_m\right) Z_0^* \right\} < \frac{\epsilon^2}{2}.$$

This relationship implies

$$\left\|\sum_{m\neq 0} \grave{E}_m^{\mathrm{i}\omega} Z_m Z_0^*\right\|_{\mathrm{T}}^2 = \overline{\sigma}\left\{ \left(\sum_{m\neq 0} Z_0 Z_m^* \acute{E}_m^{\mathrm{i}\omega}\right) \left(\sum_{\ell\neq 0} \grave{E}_\ell^{\mathrm{i}\omega} Z_\ell Z_0^*\right) \right\} = \overline{\sigma}\left\{ \sum_{m\neq 0} Z_0 Z_m^* Z_m Z_0^* \right\} < \frac{\epsilon^2}{2}. \tag{F.7}$$

Now, using Propositions 2.26 and 2.27, we have

$$\begin{aligned}
\left\|\left\{I - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\right\} \grave{E}_0^{\mathrm{i}\omega}\right\|_{\mathrm{T}}^2 &= \left\|\sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \left\{\grave{E}_0^{\mathrm{i}\omega} - \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\grave{E}_0^{\mathrm{i}\omega}\right\}\right\|_{\mathrm{T}}^2 \\
&= \left\|\grave{E}_0^{\mathrm{i}\omega} - \sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} \acute{E}_m^{\mathrm{i}\omega} \breve{H}_j^{\mathrm{in}}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\breve{H}_j^{\mathrm{in}\sim}(\mathrm{e}^{\mathrm{i}\omega\tau_j})\acute{E}_0^{\mathrm{i}\omega}\right\|_{\mathrm{T}}^2 \\
&= \left\|\grave{E}_0^{\mathrm{i}\omega} - \sum_{m=-\infty}^{\infty} \grave{E}_m^{\mathrm{i}\omega} Z_m Z_0^*\right\|_{\mathrm{T}}^2 \\
&= \|\grave{E}_0^{\mathrm{i}\omega} - \grave{E}_0^{\mathrm{i}\omega} Z_0 Z_0^*\|_{\mathrm{T}}^2 + \left\|\sum_{m\neq 0} \grave{E}_m^{\mathrm{i}\omega} Z_m Z_0^*\right\|_{\mathrm{T}}^2.
\end{aligned}$$

By Equations (F.5) and (F.7), this expression is less than ϵ^2.

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