Scattering Theory in the Energy Space for Nonlinear Klein-Gordon and Schrödinger Equations

(非線型クラインゴルドンおよびシュレディンガー方程式に対するエネルギー空間での散乱理論)

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Preface.

In this paper we study the scattering theory for nonlinear Klein-Gordon equations (NLKG):

$$\Box u + m^2 u + f(u) = 0,$$

and nonlinear Schrödinger equations (NLS):

$$i\dot{u} - \Delta u + f(u) = 0,$$

where $u = u(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$, $\dot{u} = \partial u/\partial t$, $\Box = \partial_t^2 - \Delta$, $n \in \mathbb{N}$ and $f : \mathbb{C} \to \mathbb{C}$. Our main objective is to prove that the wave operators and the scattering operators for NLKG and for NLS are well-defined and bijective in the whole energy space E (for NLKG, $E = H^1 \oplus L^2$ and for NLS, $E = H^1$). This means asymptotic completeness of the wave operators. The typical form of the nonlinearity is $f(u) = |u|^{p-1}u$. Then such results were known in the case where $n \geq 3$, $m > 0$ and $1 + 4/n < p < 1 + 4/(n-2)$, and for NLKG in the case where $n \geq 3$, $m = 0$ and $p = 1 + 4/(n-2)$ (in this case, $E = \dot{H}^1 \oplus L^2$). We extend such results in the following two cases.

In Part I, we will prove the asymptotic completeness for NLKG in the case where $n \geq 3$, $m \geq 0$ and $p = 1 + 4/(n-2)$, the Sobolev critical case. Although the mass term $m^2 u$ is not so important if we deal only with the local behavior of the solutions, it brings considerable difference to the asymptotic behavior of the solutions. In fact, the available proofs in the massless case ($m = 0$) depends in an essential way on the fact that in the massless case the distribution of the energy inside of light cones asymptotically gathers around the surface of the cones, which does not occur in the massive case. Thus our extention from $m = 0$ to $m \geq 0$ is far from trivial. Moreover, we can do better even for the local estimates. The essential difficulty in the Sobolev critical case is that because of the lack of local compactness of the Sobolev embedding, the standard energy estimates can not disprove the possibility of infinite concentration of the nonlinear part of the energy at the tip of light cones. So it is crucial to prove that such energy concentration can not occur to avoid singularities. In the preceding works, concentration phenomena were denied by contradictions, but no explicit estimate was known on the energy concentration. In this paper we will derive an estimate which explicitly bounds the energy concentration effect (Lemma 4.3 in Part I).

In Part II, we will prove the asymptotic completeness for NLKG and NLS in the case $n < 3$ and $p > 1 + 4/n$. The asymptotic completeness for $n < 3$ in the whole energy space has been one of the major open problems in this field, though there are several results on the lower dimesional scattering for NLS in a certain function space smaller than the energy space. The main difficulty for $n < 3$ is that we can not prove the Morawetz estimate, which has been essentially the only a priori estimate to start the proof of the asymptotic completeness in the energy space. In this paper we will derive some variants of the Morawetz estimate which hold in any spatial dimension (Lemmas 5.1 and 5.2). These estimates are weeker than the Morawetz estimate with respect to the weight function, but they contain some important informations on the asymptotic behavior of the energy which can not be observed by the Morawetz estimate.

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Part I.
Scattering Theory for Nonlinear Klein-Gordon Equation with Sobolev Critical Power

1. Introduction

In Part I, we study on the scattering theory in the energy space for nonlinear Klein-Gordon equations (NLKG) of the following form:

$$\Box u + m^2 u + f(u) = 0, \tag{1.1}$$

where $u = u(t,x)$, $(t,x) \in \mathbb{R}^{1+n}$ with $n \geq 3$, $m \geq 0$ and $\Box = \partial_t^2 - \Delta$. For simplicity we suppose that u is real valued, though we can deal with complex or vector valued functions as well. Then $f(\cdot)$ is also a real valued function, and we are interested particularly in the following nonlinearity:

$$\Box u + m^2 u + |u|^{p-2} u = 0, \tag{1.2}$$

with

$$p = 2^* := \frac{2n}{n-2}, \tag{1.3}$$

which is the Sobolev critical exponent. In the case where $m > 0$, $p < 2^*$ and not so small p, the scattering theory (namely, the existence of bijective wave operators) is well-known [5, 7]. In the case where $p = 2^*$ and $m = 0$ (NLW), the scattering theory has been obtained in a different way [2, 1, 12]. But neither method can be applied to the remaining case where $p = 2^*$ and $m > 0$, so we present in this paper another approach which can be applied to the case where $p = 2^*$ and $m \geq 0$. We should remark that in the radial case, one easily obtains the scattering result from the a priori estimate derived in [6]. Unique global existence of solutions of (1.2) with $p = 2^*$ and $m \geq 0$ is well-known (see, e.g., [10, 11, 14]).

Our approach is inspired by that of Bourgain's recent work [4] on the nonlinear Schrödinger equation with the Sobolev critical exponent (NLS). We obtain the scattering if we have global a priori estimates of certain space-time Lebesgue-Besov norms (which, for brevity, we call the ST-norms hereafter) by the energy size. Assuming that ST-norm is large enough, we have a point in space-time where the energy density is very highly concentrated. Since the wave component corresponding to the concentrated energy is also concentrated and decays very soon, we can isolate the concentrated wave and reduce the total energy size. Meanwhile, for sufficiently small energy data, we obtain the ST-norm estimate directly from the Strichartz estimate. Thus, by induction on the energy size, we obtain the desired estimate for ST-norm by the energy. This is just the strategy which Bourgain took in [4].

Now we should remark the essential differences between this paper and [4]. First, we do not assume the radial symmetry, so we can not predict where the concentration may occur in the space. Secondly, since NLKG does not have the homogeneous character such as in NLS and NLW, we have to deal with finite time intervals and infinite time intervals in different ways. In the cases of the critical NLW and NLS, the homogeneous character has always played an essential role (see [2, 1, 12, 4]).

Next we compare the massive case with the massless case. [6] is written for the massless case, but the arguments are also valid in the massive case (if the homogeneous spaces are replaced with the inhomogeneous counterparts), so that there is no difficulty in the radial case. But, in the nonsymmetric case (and the massless case) [2, 1, 12], the dilation identity for NLW has played an essential role, which does not yield any decay estimate for large time in the massive case.

We overcome these difficulties by the finite propagation property, Morawetz-type estimates which are rather stronger than that for NLS, and the decay property of the linear Klein-Gordon (LKG) equation for lower frequency which is faster than that for the linear wave equation (LW). In fact, for local ST-norm estimates, we do not need the induction process. In the massless case $m = 0$, the local estimate immediately becomes global by the homogeneity and so we obtain another proof of the scattering, with global a priori estimate for ST-norm, which was obtained in the special case where $n = 3$, $m = 0$ and $f(u) = u^5$ in [1], using the scattering operators. But in the massive case $m > 0$, for global ST-norm estimate, we need the induction together with the decay property of LKG in low frequency.

The rest of Part I is organized as follows. In Section 2, we introduce several notations used in Part I, and mention some basic estimates. In Section 3, we show that ST-norm concentration in time causes energy concentration in space-time. In Section 4, we derive a local ST-norm estimate. In Section 5, we derive a global ST-norm estimate. In Section 6, we present the main results of Part I: global a priori estimates, scattering and continuous dependence on the initial data both in the strong topology and in the weak topology. In Section 7, we prove several lemmas used in the previous sections.

2. Preliminaries

As usual, we denote by C auxiliary positive constants, and sometimes denote $C(a,b,\dots)$ to indicate that the constant depends only on $a,b,\dots$. We denote by $B^\sigma_{q,r}$ the usual inhomogeneous Besov spaces (see, e.g., [3]). We will use mainly the following particular space-time norms.

$$\begin{aligned} &\|u\|_{G(I)} := \|u\|_{L^{2^*-1}(I;L^{2(2^*-1)}(\mathbb{R}^n))}, \\ &\|u\|_{(j;I)} := \|u\|_{L^{p_j}(I;B^{\sigma_j}_{q_j,r_j})}, \quad \text{for } j=0,\dots,7, \end{aligned} \tag{2.1}$$

where $r_5=\infty$ and $r_j=2$ for $j\neq 5$. Sometimes we omit the interval I and write, e.g., $\|u\|_{(0)}$. Now we set the values of (p_j,q_j,σ_j). Denote $X_j := (1/p_j, 1/q_j, \sigma_j)$. Let

$$\begin{gathered} p_0 := \frac{2(n^2+2)}{(n+1)(n-2)}, \quad p_6 := \frac{2n}{n-1}, \\ X_0 := \left(\frac{1}{p_0}, \frac{1}{2^*}, \frac{1}{p_0}\right), \quad X_1 := \left(\frac{1}{p_0}, \frac{1}{q_0} - \frac{\sigma_0}{n}, 0\right), \\ X_2 := X_0 + (2^*-2)X_1, \\ X_3 := \left(\sigma_3, \frac{1}{2^*}, (2^*-2)\sigma_0\right), \quad X_4 := \left(\frac{1}{p_3}, \frac{1}{q_3} - \frac{\sigma_3}{n}, 0\right), \\ X_5 := X_3 + (2^*-2)X_1 = X_4 + (2^*-2)X_0, \\ X_6 := \left(\frac{1}{p_6}, \frac{1}{2^*}, \frac{1}{p_6}\right), \quad X_7 := \left(1 - \frac{1}{p_2}, 1 - \frac{1}{q_2}, -\sigma_2\right). \end{gathered}$$

There are two important numbers associated with X_j:

$$\begin{aligned} \mu_j &:= -\frac{1}{p_j} - n\left(\frac{1}{q_j} - \frac{1}{2}\right) + \sigma_j, \\ \nu_j &:= \frac{1}{p_j} + \frac{n-1}{2}\left(\frac{1}{q_j} - \frac{1}{2}\right). \end{aligned}$$

Then we have

$$\begin{gathered} \mu_0 = \mu_1 = \mu_3 = \mu_4 = \mu_6 = 1, \ \mu_7 = 0, \ \mu_2 = \mu_5 = -1, \\ \nu_0 < \nu_6 \le 0, \ \nu_2 \ge 1, \ \nu_5 > \nu_3 + 1, \ \nu_7 \le 0. \end{gathered} \tag{2.2}$$

For simplicity, we set $m=1$. Then the equations are

$$\Box u + u + f(u) = 0, \quad \text{(NLKG)} \tag{2.3}$$

$$\Box u + u = 0. \quad \text{(LKG)} \tag{2.4}$$

Let u be the solution of

$$\begin{cases} \Box u + u = g, \\ u(0) = \varphi, \quad \dot u(0) = \psi, \end{cases}$$

where $\dot{u} = \partial u/\partial t$. Then, by the Strichartz estimate (see, e.g., [5, 7]), we have

$$\begin{aligned}\|u\|_{(j;(0,T))} + \|\dot{u}\|_{(7;(0,T))} + \|u\|_{L^\infty(0,T;H^1)} + \|\dot{u}\|_{L^\infty(0,T;L^2)} \\ \leq C\|\varphi\|_{H^1} + C\|\psi\|_{L^2} + C\|g\|_{(2;(0,T))},\end{aligned} \tag{2.5}$$

for $j = 0, 1, 6$, and in the case $n \geq 6$, for $j = 3, 4$ also. By [7, Lemma 3.1], the Sobolev embedding and Hölder's inequality, we have

$$\|f(u)\|_{(2)} \leq C\|u\|_{(0)}\|u\|_{(1)}^{2^*-2} \leq C\|u\|_{(0)}^{2^*-1}, \tag{2.6}$$

under the assumption (2.10). We fix a radially symmetric cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Define $\chi_j(x) = \chi(2^{-j}x)$. Denote by $\mathcal{F}\varphi = \widetilde{\varphi}$ the Fourier transform of φ and define the Littlewood-Paley dyadic decomposition:

$$\begin{aligned}\psi_j &= \mathcal{F}^{-1}\chi_j,\\ \varphi_j &= \psi_j - \psi_{j-1}.\end{aligned} \tag{2.7}$$

Denote for any function φ,

$$\varphi(\omega) := \mathcal{F}^{-1}\varphi(\sqrt{1+|\xi|^2})\mathcal{F}. \tag{2.8}$$

We define the energy and related quantites.

$$\begin{aligned}F(u) &:= 2\int_0^u f(v)dv,\\ G(u) &:= uf(u) - F(u) = u^3\partial_u\left(\frac{F(u)}{2u^2}\right),\\ e_0(u;t) &:= |\dot{u}|^2 + |\nabla u|^2 + m^2|u|^2, \quad E_0(u;t) := \int_{\mathbb{R}^n} e_0(u)dx,\\ e(u;t) &:= e_0(u) + F(u), \quad E(u;t) := \int_{\mathbb{R}^n} e(u)dx,\end{aligned} \tag{2.9}$$

where $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. Now we give the hypotheses on the nonlinearity.

$$f(0) = 0 \quad \text{and} \quad |f(u) - f(v)| \leq C|u-v|(|u|+|v|)^{2^*-2}, \tag{2.10}$$

$$|f'(u) - f'(v)| \leq C|u-v|^{2^*-2} \qquad \text{if } n \geq 6, \tag{2.11}$$

$$G \geq 0. \tag{2.12}$$

These assumptions are the same as in [12] for the scattering of the critical NLW. The single critical power (1.2) with $p = 2^*$ satisfies these assumptions. For other examples of f, see [12]. Then, as is indicated in (2.9), we have in particular, $F \geq 0$. We define

$$K(t) := \omega^{-1}\sin\omega t. \tag{2.13}$$

Then the integral equation associated to NLKG is

$$u(t) = \dot{K}(t)u(0) + K(t)\dot{u}(0) - \int_0^t K(t-s)f(u(s))ds.$$

3. Energy concentration via space-time norm

In this section we show that ST-norm concentration in time causes energy concentration in space-time. This result corresponds to that in [4, Section 2]. Here we take a Besov-space approach instead of the Littlewood-Paley theorem (which was used in [4]). The arguments in this section work also in the massless case, but for simplicity we assume that $m = 1$. The objective of this section is the following lemma.

Lemma 3.1. *Assume* (2.10). *Let I be an interval and let u satisfy NLKG* (2.3) *on I with $E_0(u(t)) \le E < \infty$ for any $t \in I$ and*

$$0 < \eta/2 \le \|u\|_{(0;I)} \le 2\eta < \infty. \tag{3.1}$$

There exists a positive continuous function $\eta_0\colon [0,\infty) \to (0,\infty)$, such that if $\eta < \eta_0(E)$, we have a subinterval $J \subset I$, $c \in \mathbb{R}^n$ and $R > 0$ satisfying $R \le C_1|J|$ and

$$\begin{aligned}\int_{|x-c|<R} |\nabla u(t)|^2 + |u(t)|^2 dx > \eta^{2\alpha},\\ \int_{|x-c|<R} |u(t)|^{2^*} dx > \eta^{2^*\alpha},\end{aligned} \tag{3.2}$$

for any $t \in J$. Moreover, if $\|\psi_k * u\|_{(0;I)} \le \eta/4$ for some $k \ge 2$, we have $R \le C_2 2^{-k}$. Here $C_j = C_j(E,\eta) > 0$ and $\alpha = \alpha(n) > 0$ are certain constants.

Proof. η_0 may be given by

$$\eta_0(E) = (\gamma + E)^{-\gamma} \tag{3.3}$$

with some large $\gamma(n)$, which will be determined later. Let $I = (T, T')$, and let v be the solution of LKG (2.4) with the same initial data as u at $t = T$. By the Strichartz estimate, we have

$$\|u\|_{(6)} \le \|v\|_{(6)} + C\|u\|_{(0)}^{2^*-1} \le C(E) + C\eta^{2^*-1} \le C(E).$$

By the interpolation inequality and Hölder's inequality, we have

$$\begin{aligned}\eta/2 \le \|u\|_{(0)} &\le C\|u\|_{(6)}^{1-\theta}\|u\|_{L_t^\infty B^0_{2^*,\infty}}^{\theta}\\ &\le C(E)\|u\|_{L_t^\infty H^1}^{\theta(1-\lambda)}\|u\|_{L_t^\infty B^{1-n/2}_{\infty,\infty}}^{\theta\lambda} \le C(E)\|u\|_{L_t^\infty B^{1-n/2}_{\infty,\infty}}^{\theta\lambda},\end{aligned} \tag{3.4}$$

where $0 < \theta, \lambda < 1$ and

$$\frac{1}{p_0} = \frac{1-\theta}{p_6}, \quad 0 = 1 - \lambda + \lambda\left(1 - \frac{n}{2}\right).$$

Thus, denoting $\beta = 1/(\theta\lambda) < \infty$, we have some $(t_0, c) \in I \times \mathbb{R}^n$ such that either for some $N \in \mathbb{N}$

$$2^{(1-n/2)N}|\varphi_N * u(t_0, c)| \ge \eta^\beta/C(E) \tag{3.5}$$

or

$$|\psi_0 * u(t_0, c)| \ge \eta^\beta/C(E). \tag{3.6}$$

In the latter case, let $N := 1$. If $\|\psi_k * u\|_{(0)} < \eta/4$, we have similarly

$$\eta/4 \le \|u - \psi_k * u\|_{(0)} \le C(E)\|u - \psi_k * u\|^{\theta\lambda}_{L^\infty_t B^{1-n/2}_{\infty,\infty}},$$

so that we may assume $N \ge k$ in (3.5). By the Sobolev embedding, we have for $j \ge 1$, (remember (2.2))

$$\|\psi_j * u(t)\|_{B^{\sigma_0}_{q_0,2}} \le C\|\psi_j * u(t)\|_{H^{1+1/p_0}} \le C2^{j/p_0}\|\psi_j * u(t)\|_{H^1},$$

so that $\|\psi_j * u\|_{(0)} \le C(E)(2^j|I|)^{1/p_0}$. So we have $\|\psi_j * u\|_{(0)} \le \eta/4$ if $2^j|I| \le C(E,\eta)$. Thus we may assume $2^N|I| \ge C(E,\eta)$. Next we seek J where (3.5) or (3.6) remains valid. We treat only the case (3.5). The case (3.6) is similar. By the integral equation we have

$$\begin{aligned}\|\varphi_N * u(t) - \varphi_N * u(t_0)\|_{H^1} &\le \|(\dot K(t-t_0) - I)\varphi_N * u(t_0)\|_{H^1}\\ &+ \|K(t-t_0)\varphi_N * \dot u(t_0)\|_{H^1} + \left\|\int_{t_0}^t K(t-t_0)\varphi_N * f(u(s))ds\right\|_{H^1}.\end{aligned}$$

It is easy to see that the first and the second terms are bounded by $C2^N|t-t_0|E^{1/2}$. The third term is estimated by the Strichartz estimate as

$$\cdots \le C\|\varphi_N * f(u)\|_{(2;(t_0,t))} \le C|t-t_0|^{1/p_2}\|\varphi_N * f(u)\|_{L^\infty(B^{\sigma_2}_{q_2,2})}.$$

By the Sobolev embedding, we have

$$\begin{aligned}\|\varphi_N * f(u)\|_{B^{\sigma_2}_{q_2,2}} &\le C\|\varphi_N * f(u)\|_{B^{1/2+1/p_2}_{\frac{2n}{n+3},2}}\\ &\le C2^{N/p_2}\|f(u)\|_{B^{1/2}_{\frac{2n}{n+3},2}}\\ &\le C2^{N/p_2}\|u\|_{B^{1/2}_{\frac{2n}{n-1},2}}\|u\|^{2^*-2}_{L^{2^*}} \le C(E)2^{N/p_2},\end{aligned} \tag{3.7}$$

where we used [7, Lemma 3.1] in the third step. Summing up, we obtain

$$\|\varphi_N * u(t) - \varphi_N * u(t_0)\|_{H^1} \le C(E)\{2^N|t-t_0| + (2^N|t-t_0|)^{1/p_2}\}.$$

By the embedding $H^1 \hookrightarrow B^{1-n/2}_{\infty,\infty}$, this means that (3.5) remains valid for $|t-t_0| \le 2^{-N}C(E,\eta)$. Let J be the intersection of this interval and I, then we have $|J| \ge 2^{-N}C(E,\eta)$. Now we have only to find $R \le C(E,\eta)2^{-N}$ satisfying (3.2). There exists $\mathcal{S} \ni \varphi^{(1)}, \dots, \varphi^{(n)}$ satisfying

$$\varphi_0 = \sum_{k=1}^n \partial_k\varphi^{(k)}.$$

(See, e.g., [3]). Define $\varphi_j^{(k)}(x) = 2^{jn}\varphi^{(k)}(2^j x)$. Then we have for general φ,

$$\begin{aligned}
2^N|\varphi_N * \varphi(c)| &= \left|\sum_{k=1}^{n} \varphi_N^{(k)} * \partial_k\varphi(c)\right| \\
&\leq \sum_k \int |\varphi^{(k)}(y)\nabla\varphi(c - 2^{-N}y)|\,dy \\
&\leq \sum_k 2^{Nn/2}\left(\|\varphi^{(k)}\|_{L^2}\|\nabla\varphi\|_{L^2(|x-c|<2^{-N}R_0)} + \|\varphi^{(k)}\|_{L^2(|x|>R_0)}\|\nabla\varphi\|_{L^2}\right).
\end{aligned}$$

Since $\varphi^{(k)} \in \mathcal{S}$, if R_0 is sufficiently large, $\|\varphi^{(k)}\|_{L^2(|x|>R_0)}$ becomes small. Thus, in the case (3.5), we obtain

$$\eta^\beta/C(E) \leq \|\nabla u(t_0)\|_{L^2(|x-c|<2^{-N}R_0)}$$

for any $t_0 \in J$, if $R_0 \geq C(E,\eta)$. Similarly, we have

$$2^{N(1-n/2)}|\varphi_N * \varphi(c)| \leq \|\varphi_0\|_{L^q}\|\varphi\|_{L^{2^*}(|x-c|<2^{-N}R_0)} + \|\varphi_0\|_{L^q(|x|>R_0)}\|\varphi\|_{L^{2^*}},$$

with $q = 2n/(n+2)$, so that

$$\eta^\beta/C(E) \leq \|u(t_0)\|_{L^{2^*}(|x-c|<2^{-N}R_0)}$$

for any $t_0 \in J$, if $R_0 \geq C(E,\eta)$. In the case (3.6), we have similarly,

$$\begin{aligned}
\eta^\beta/C(E) &\leq \|u(t_0)\|_{L^2(|x-c|<R_0)}, \\
\eta^\beta/C(E) &\leq \|u(t_0)\|_{L^{2^*}(|x-c|<R_0)},
\end{aligned}$$

for any $t \in J$ and $R_0 \geq C(E,\eta)$. Taking $\alpha(n)$ and $\gamma(n)$ in (3.3) sufficiently large, we have

$$\eta^\beta/C(E) \geq \eta^\alpha, \tag{3.8}$$

thus we obtain the desired result. □

4. Local space-time estimate

In this section we derive an estimate of ST-norm by the energy on finite time interval. The key ingredient is a stronger version of the local Morawetz estimate (Lemma 4.3), which forbids time-like accumulation of concentrated energy. The arguments in this section work also in the massless case, but for simplicity we assume that $m = 1$.

The objective of this section is the following.

Proposition 4.1. *Assume* (2.10) *and* $G \geq 0$. *Let* u *be a solution of NLKG* (2.3) *satisfying* $E(u) \leq E < \infty$ *and* $\|u\|_{(0;(0,1))} < \infty$. *Then we have a bound* $B = B(E) < \infty$ *for the space-time norm:*

$$\|u\|_{(0;(0,1))} < B(E). \tag{4.1}$$

Proof. By [12], we have a unique global solution of NLKG with finite energy and locally finite ST-norm, whose energy is conserved. Let $\eta > 0$ be small and assume $\|u\|_{(0;(0,1))} \geq N\eta$ with $N \in \mathbb{N}$. We will show that if $\eta = \eta(E)$ is sufficiently small, N is bounded in terms of E and η. We have $0 = T_0 < T_1 < \cdots < T_N \leq 1$ such that $\|u\|_{(0;(T_j,T_{j+1}))} = \eta$. Denote $I_j = (T_j, T_{j+1})$. By Lemma 3.1, we obtain $J_j \subset I_j$, $c_j \in \mathbb{R}^n$ and $R_j > 0$. Choose $t_j \in J_j$ for each j. Now we want to extract a sequence dyadically converging to some point.

Lemma 4.2 (proved in Sect.7). *Let $\nu \in \mathbb{N}$, $\mathbb{R}^\nu \supset S$, $N \in \mathbb{N}$ and $\#S \geq \{4[\sqrt{\nu} + 1]\}^{\nu(N-1)}$. Then we have N mutually distinct points $x_1, \dots, x_N \in S$ satisfying*

$$|x_j - x_N| \leq \frac{1}{4}|x_{j-1} - x_N| \tag{4.2}$$

for $j = 1, \dots, N$.

By this lemma, if $M \in \mathbb{N}$ and $\{4[\sqrt{n+1} + 1]\}^{(n+1)(2M-1)} \leq N$, we obtain M mutually distinct points $y_1, \dots, y_M \in \{(t_1, c_1), \dots, (t_N, c_N)\}$ satisfying

$$|y_j - y_M| \leq \frac{1}{16}|y_{j-1} - y_M|. \tag{4.3}$$

We change the suffixes of $\{(I_j, J_j, t_j, c_j, R_j)\}$ such that for $j = 1, \dots, M$ we have $y_j = (t_j, c_j)$. Let $S := \{1, \dots, M\}$,

$$\begin{aligned} P &:= \{j \in S \mid |y_j - y_M| \leq 8R_j\}, \\ Q &:= \{j \in S \setminus P \mid |c_j - c_M| \leq 4|t_j - t_M|\}, \\ R &:= S \setminus (P \cup Q). \end{aligned}$$

Now let us bound $\#P$, $\#Q$ and $\#R$. By a variant of Morawetz estimate [12, Proposition 4.4], we have $C_4(E) < \infty$ such that

$$\int\int_{0 \leq t \leq 1} \frac{|u|^{2^*}}{|(t,x) - y_M|} dx dt \leq C_4(E). \tag{4.4}$$

The left hand side is bounded from below as follows.

$$\cdots \geq \sum_{j \in P} \int_{J_j} \frac{\eta^{2^*\alpha}}{9R_j + |J_j|} dt \geq \eta^{2^*\alpha} \sum_{j \in P} \frac{|J_j|}{9R_j + |J_j|} \geq \eta^{2^*\alpha} \#P \frac{1}{9C_1(E,\eta) + 1}. \tag{4.5}$$

Thus, we obtain

$$\#P \leq C_4 \eta^{-2^*\alpha}(9C_1 + 1).$$

We proceed to the bound of $\#Q$. For $j \in Q$ we have $|t_j - t_M| \leq |y_j - y_M| \leq \sqrt{17}|t_j - t_M|$ and $R_j \leq |t_j - t_M|\sqrt{17}/8$. Thus we have for $j, k \in Q$ with $j < k$,

$$|t_k - t_M| \leq |y_k - y_M| \leq \frac{1}{16}|y_j - y_M| \leq \frac{\sqrt{17}}{16}|t_j - t_M| \leq \frac{1}{2}|t_j - t_M|.$$

Let $B_j := \{(t_j, x) \mid |x - c_j| < R_j\}$ and $K = \{(t,x) | 0 \le t \le 1, |x - c_M| < 5|t - t_M|\}$. Then for $j \in Q$, we have $B_j \subset K$. Now we use a stronger version of the local Morawetz estimate on a fat cone.

Lemma 4.3 (proved in Sect.7). *Under the assumption of Proposition 4.1, let $c > 0$. Then we have $C_5(E,c) < \infty$ such that*

$$\sum_{j\in\mathbb{N}} \sup_{2^{-j}\le|t|\le 2^{-j+1}} \int_{|x|<c|t|} |u|^{2^*} dx \le C_5(E,c).$$

Applying this lemma on the cone K, we have

$$\begin{aligned} C_5(E,5) &\ge \sum_{j\in\mathbb{N}} \sup_{2^{-j}\le|t-t_M|\le 2^{-j+1}} \int_{|x-c_M|<5|t-t_M|} |u|^{2^*} dx \\ &\ge \#Q\eta^{2^*\alpha}. \end{aligned}$$

Thus, we obtain the bound for $\#Q$. Now we have only to bound $\#R$. For $j \in R$, we have $|c_j - c_M| \le |y_j - y_M| \le |c_j - c_M|\sqrt{17}/4$ and

$$R_j + |t_j - t_M| \le \left(\frac{1}{8}\frac{\sqrt{17}}{4} + \frac{1}{4}\right)|c_j - c_M| \le \frac{1}{2}|c_j - c_M|.$$

Denote $\widetilde{B}_j = \{(t_M, x) | \, |x - c_j| \le R_j + |t_j - t_M|\}$. Then, by the energy identity and positivity of the energy, we have

$$\int_{\widetilde{B}_j} e(u)dx \ge \int_{B_j} e(u)dx,$$

For $j, k \in R$ with $j < k$, we have

$$|c_k - c_M| \le |y_k - y_M| \le \frac{1}{16}|y_j - y_M| \le \frac{\sqrt{17}}{64}|c_j - c_M| < \frac{1}{4}|c_j - c_M|$$

and

$$\begin{aligned} \widetilde{B}_j &\subset \{(t_M, x) \mid |x - c_j| \le \frac{1}{2}|c_j - c_M|\} \\ &\subset \{(t_M, x) \mid \frac{1}{2}|c_j - c_M| \le |x - c_M| \le \frac{3}{2}|c_j - c_M|\}. \end{aligned}$$

So $\widetilde{B}_k \cap \widetilde{B}_j = \varnothing$. Thus we have

$$\begin{aligned} E = \int e(u(t_M))dx &\ge \sum_{j\in R} \int_{\widetilde{B}_j} e(u)dx \\ &\ge \sum_{j\in R} \int_{B_j} e(u)dx \ge \#R\eta^{2\alpha}, \end{aligned}$$

so that we obtain a bound for $\#R$ and the desired result. Restriction on the size of η comes only from Lemma 3.1. □

5. Global space-time estimate

In this section we derive an estimate of ST-norm on $\mathbb{R}$ by the energy. The objective of this section (and in essence the most important result in Part I) is the following proposition. Key ingredients are contained in Lemmas 5.2, 5.3 and 5.4, which are proved in Section 7. Lemma 5.3 works only in the massive case, and so we assume that $m = 1$.

Proposition 5.1. *Assume* (2.10), (2.11) *(if $n \geq 6$) and $G \geq 0$. Let u be a solution of NLKG* (2.3) *with finite energy $E(u) \leq E < \infty$ and locally finite ST-norm. Then we have a bound $B = B(E) < \infty$ such that*

$$\|u\|_{(0;\mathbb{R})} \leq B(E).$$

Proof. By [12], we have a unique global solution u with finite energy and locally finite ST-norm. By the local ST-norm estimate (Proposition 4.1), we have

$$\|u\|_{(0;(T,T+1))} \leq C(E)$$

for any $T \in \mathbb{R}$. Now we have to make this estimate global. For that purpose, we use an induction argument on the size of the energy E. If $E(u)$ is sufficiently small, we obtain the desired global estimate simply by applying the Strichartz estimate to the integral equation (see, e.g., [13, 6, 8] or the proof of Lemma 7.3). So what we have to prove is that:

For any $E > 0$, there exists $\varepsilon = \varepsilon(E) > 0$, continuous with respect to $E \in [0, \infty)$ (in fact, it suffices that $\inf_{0 \leq a < b} \varepsilon(a) > 0$ for any $b > 0$), such that if we have the global estimate

$$\|u\|_{(0;\mathbb{R})} \leq B(E, \varepsilon) < \infty, \tag{5.1}$$

for any solution of NLKG with locally finite ST-norm satisfying $E(u) \leq E - \varepsilon$, then we have also the estimate

$$\|u\|_{(0;\mathbb{R})} \leq \widetilde{B}(E, \varepsilon) < \infty, \tag{5.2}$$

for any solution of NLKG with locally finite ST-norm satisfying $E(u) \leq E$.

By the small data result mentioned above, we have $\varepsilon(E) > C > 0$ for sufficiently small E. Now suppose that $\|u\|_{(0;\mathbb{R})} > 3B'$. Then we have T_0 and T' satisfying $\|u\|_{(0;(-\infty,T_0))} > B'$, $\|u\|_{(0;(T_0,T'))} > B'$ and $\|u\|_{(0;(T',\infty))} > B'$. It suffices to bound B' in terms of E, ε, B for some $\varepsilon = \varepsilon(E) > 0$. In the following, we use two families of positive small parameters $\{\eta_j\}$ and $\{\kappa_j\}$. Those parameters should be determined in the order: $E, \eta_1, \eta_2, \varepsilon, B, \kappa_5, \ldots, \kappa_1$, such that all the conditions below are fulfilled (in other words, latter parameters may depend on former ones). Then, in terms of these parameters, a bound for B' will be given. First, we have the following lemma, which claims that we have either a very long interval with small ST-norm or very high concentration of energy.

Lemma 5.2 (proved in Sect.7). *Under the assumptions of Proposition 5.1, for any $\kappa > 0$, $L < \infty$ and $0 < \eta \leq \eta_0(E)$ (η_0 is given in Lemma 3.1), there exists $M = M(E, \eta, \kappa, L) > 0$ such that if $\|u\|_{(0;I_0)} \geq M$ for some interval I_0, then we have a subinterval $I \subset I_0$ such that $\|u\|_{(0;I)} \leq \eta$ and one of the following two conditions holds.*

(i) $|I| > L$.

(ii) $|I| \leq L$ *and*

$$\int_{|x-c|<\kappa|I|} |\nabla u(t)|^2 + |u(t)|^2 dx \geq \eta^{2\alpha},$$

for some $c \in \mathbb{R}^n$ and some $t \in I$, where $\alpha = \alpha(n)$ is the same as in Lemma 3.1.

By this lemma, if $\eta_1 \leq \eta_0(E)$ and $B' \geq B_1 := (M(E, \eta_1, \kappa_1, \kappa_2^{-1}) + \eta_1)N$ for some $N = N(E, \eta_1) \in \mathbb{N}$, then only the following two cases may occur.

(i) There exist $T_0 \leq T_1 < U_1 < V_1 \leq \cdots \leq T_N < U_N < V_N \leq T'$ satisfying $|U_j - T_j| \geq \kappa_2^{-1}$ and $\|u\|_{(0;(T_j,U_j))} \leq \eta_1 = \|u\|_{(0;(U_j,V_j))}$.

(ii) There exists $I \subset (T_0, T')$, satisfying $\|u\|_{(0;I)} \leq \eta_1$, $|I| \leq \kappa_2^{-1}$ and

$$\int_{|x-c|<\kappa_1|I|} |\nabla u(t)|^2 + |u(t)|^2 dx \geq \eta_1^{2\alpha} =: \eta_2,$$

for some $c \in \mathbb{R}$ and some $t \in I$.

Now we show that energy concentration occurs also in the case (i). Let v be the solution of LKG (2.4) with the same initial data as u at $t = T_0$. Then, by the Strichartz estimate $\|v\|_{(0;\mathbb{R})} \leq C(E)$, if $N = N(E, \eta_1)$ is sufficiently large, we have for some $j \leq N$, $\|v\|_{(0;(U_j,V_j))} \leq \eta_1^{2^*}$. Now we use the decay property of LKG in low frequency to obtain the following lemma, which claims that after a long interval with small ST-norm, the ST-norm may rally only from the high frequency.

Lemma 5.3 (proved in Sect.7). *Assume* (2.10). *Let $0 < T < U < V$ and let u be a solution of NLKG (2.3) on $[T, V]$ satisfying $E_0(u(t)) \leq E < \infty$ for any $t \in [T, V]$ and $\|u\|_{(0;(T,U))} \leq \eta = \|u\|_{(0;(U,V))}$. Let v be the solution of LKG (2.4) with the same initial data as u at $t = 0$, and assume $\|v\|_{(0;(U,V))} \leq \eta^{2^*}$. There exists $\delta_2 = \delta_2(E) > 0$, continuously depending on $E \in [0, \infty)$ with the following property: for $\eta \leq \delta_2(E)$ and any $k \in \mathbb{N}$, there exists $L = L(E, \eta, k) < \infty$ such that if $|T - U| > L$ then we have $\|\psi_k * u\|_{(0;(U,V))} \leq \eta/4$.*

If $\eta_1 \leq \delta_2(E)$, we can apply this lemma on (T_j, V_j), for $k = \kappa_3^{-1} \in \mathbb{N}$ provided $\kappa_2^{-1} > L(E, \eta_1, \kappa_3^{-1})$. Then, we apply Lemma 3.1 to (U_j, V_j), to obtain $c \in \mathbb{R}^n$,

$S \in (U_j, V_j)$ and $R < C_2(E, \eta_1)2^{-1/\kappa_3}$ satisfying

$$\int_{|x-c|<R} |\nabla u(S)|^2 + |u(S)|^2 dx \geq \eta_2. \tag{5.3}$$

Thus, in both cases, if we set

$$C_2(E, \eta_1)2^{-1/\kappa_3} < \kappa_4, \quad \kappa_1\kappa_2^{-1} < \kappa_4, \tag{5.4}$$

then we obtain some $S \in (T_0, T')$, $c \in \mathbb{R}^n$ and $R < \kappa_4$ satisfying (5.3). Then we cut off the concentrated energy by the following lemma.

Lemma 5.4 (proved in Sect.7). *Assume* (2.10) *and* $G \geq 0$. *Let* u *be a solution of NLKG* (2.3) *with locally finite ST-norm and* $E(u) \leq E < \infty$. *Let* $(S, c) \in \mathbb{R}^{1+n}$, $R, \eta > 0$. *Suppose*

$$\int_{|x-c|<R} e(u; S)dx \geq \eta.$$

For any $\kappa > 0$, *there exists* $\delta_1 = \delta_1(E, \eta, \kappa) > 0$ *such that if* $R < \delta_1$, *then we have* $T \in (S, S+1)$ *and a solution* v *of LKG* (2.4) *satisfying*

$$\begin{aligned} E_0(v) &\leq E + \kappa, \\ \|v\|_{L_t^\infty(T,\infty;L^{2^*})} &< \kappa, \\ E(u - v; T) &\leq E - \eta/2. \end{aligned} \tag{5.5}$$

By this lemma, if we set

$$\kappa_4 < \delta_1(E, \eta_2, \kappa_5),$$

we obtain some $T \in (S, S+1)$ and a solution v of LKG, satisfying (5.5) with $\kappa = \kappa_5$ and $\eta = \eta_2$. Now we can determine the size of the induction step as $\varepsilon(E) = \eta_2/2 = \eta_1^{2\alpha}/4$, where we set $\eta_1 := \min(\eta_0(E), \delta_2(E))$ (η_0 is defined in Lemma 3.1 and δ_2 is defined in Lemma 5.3). Then we may apply the induction hypothesis to the solution W of NLKG with the same initial data as $u - v$ at T, so that $\|W\|_{(0;(T,\infty))} \leq B = B(E) < \infty$. Then, the desired estimate for u comes from the following estimate. (2.11) is required only in this lemma.

Lemma 5.5 (proved in Sect.7). *Assume* (2.10) *and* (2.11) *(if* $n \geq 6$*). Let* u, W *be two solutions of NLKG* (2.3), v *be a solution of LKG* (2.4) *satisfying* $(u(0), \dot{u}(0)) = (v(0), \dot{v}(0)) + (W(0), \dot{W}(0))$, $E_0(u(t)), E_0(W(t)) \leq E$ *for any* $t \geq 0$, $\|W\|_{(0;(0,\infty))} \leq M$, $\|v\|_{L_t^\infty(0,\infty;L_x^{2^*})} < \varepsilon$ *and* $\|u\|_{(0;(0,T))} < \infty$ *for any* $T > 0$. *Then, there exists* $\varepsilon_3 = \varepsilon_3(E, M) > 0$ *and* $B_2(E, M) < \infty$ *such that if* $\varepsilon < \varepsilon_3$ *we have* $\|u\|_{(0;(0,\infty))} \leq B_2(E, M)$.

By this lemma, if $\kappa_5 < \varepsilon_3(E, B(E, \varepsilon))$, then we have $\|u\|_{(0;(T,\infty))} \leq B_2(E, B)$. Since $S < T'$ and $|T - S| < 1$, by the local estimate, we have $B' < \|u\|_{(0;(S,\infty))} \leq B_2 + C(E)$, so that we obtain the desired global estimate. □

6. Main results

The scaling $u(t,x) \mapsto u' = \lambda^{n/2-1}u(\lambda t, \lambda x)$ with $\lambda > 0$ transforms a solution of NLKG on $[0,T]$ into a solution on $[0,T/\lambda]$ of another NLKG with the mass $m' = \lambda m$ and the nonlinearity $f' = \lambda^{n/2+1}f(\lambda^{1-n/2}\cdot)$. u' has the same energy as u, and f' satisfies (2.10) and (2.11) with the same constants as f. So, in the case $m = 0$, we obtain the global estimate from the local estimate by the scaling. We have proved the local estimate in the case $m = 1$, but the argument is also valid in the case $m = 0$, if we replace the inhomogeneous spaces by the homogeneous ones. In fact, the proof of the local estimate is more suited to NLW, rather than NLKG, for we used an estimate related to the conformal invariance of LW (in the proof of Lemma 4.3). That estimate holds globally for NLW, but not globally for NLKG. Moreover, once we have obtained the global estimate in the case $m = 1$, we obtain the global estimate for any $m > 0$ by the scaling, and the estimate for the homogeneous ST-norm (e.g., $\|u\|_{(1;\mathbb{R})}$) is independent of m. (Remark that $\|u\|_{(1;\mathbb{R})} \leq C\|u\|_{(0;\mathbb{R})}$ by the Sobolev embedding.) Thus we have obtained the global estimate independent of m for the homogeneous ST-norm :

Theorem 6.1. *Let $n \geq 3$ and $m \geq 0$. Assume* (2.10) *and $G \geq 0$. In the case where $m > 0$ and $n \geq 6$, assume* (2.11) *in addition. Then, for any finite energy solution u of NLKG* (1.1) *with locally finite ST-norm, we have a global bound for ST-norm:*

$$\|u\|_{(1;\mathbb{R})} \leq B < \infty,$$

where B depends only on n, $E(u)$ and the constants in (2.10) *and* (2.11).

It is well-known that one can derive a priori estimates for any (appropriate) ST-norm from an estimate for a particular ST-norm (see, e.g., [6, Proposition 2.6]).

From this estimate, we obtain the continuous dependence on the initial data. Define the energy space

$$X := \{(\varphi,\psi) \mid \|(\varphi,\psi)\|_X^2 := \|\nabla\varphi\|_{L^2}^2 + m^2\|\varphi\|_{L^2}^2 + \|\psi\|_{L^2}^2 < \infty\}. \tag{6.1}$$

Corollary 6.2. *Under the same assumption of Theorem 6.1, the finite energy solution of NLKG* (1.1) *with locally finite ST-norm depends on the initial data continuously both in the strong topology of X and in the weak topology of X.*

Proof. Suppose that the initial data converges weakly in X. Then the corresponding solution converges weakly in X at $t = 0$, and, by the boundedness, converges weakly in $(1;\mathbb{R})$, if we extract some subsequence. Then the limit function is also a finite energy solution of NLKG with finite ST-norm, so by the uniqueness, the weak continuity follows. Then, the strong continuity follows from the weak continuity and the energy conservation. □

We have also the scattering result.

Corollary 6.3. *Let $n \geq 3$, $m \geq 0$ and $G \geq 0$. Assume (2.10) and (2.11) (if $n \geq 6$). Then any finite energy solution u of NLKG (1.1) with locally finite ST-norm approaches to some solutions $v_\pm$ of LKG*

$$\Box v_\pm + m^2 v_\pm = 0,$$

as $t \to \pm\infty$ in X. Moreover, the correspondences $M_\pm : (u(0), \dot{u}(0)) \mapsto (v_\pm(0), \dot{v}_\pm(0))$ define homeomorphisms in X and we have

$$E(u) = E_0(v_\pm).$$

$M_\pm$ and $M_\pm^{-1}$ are continuous also in the weak topology of X.

Proof. It suffices to consider the case $t \to \infty$. For simplicity, we consider the case $m = 1$. The arguments in the other cases are similar. Since $\|u\|_{(0;\mathbb{R})} < \infty$, we have $\|u\|_{(0;(T,\infty))} \to 0$ as $T \to \infty$. By the Strichartz estimate, we have

$$\left\| \int_S^t \left(-\frac{\sin \omega s}{\omega}, \cos \omega s \right) f(u(s)) ds \right\|_X \leq C \|f(u)\|_{(2;(S,t))} \leq C \|u\|_{(0;(S,\infty))}^{2^*-1} \to 0,$$

as $t > S \to \infty$. So there exists the limit in X:

$$(\Phi, \Psi) := \int_0^\infty \left(-\frac{\sin \omega s}{\omega}, \cos \omega s \right) f(u(s)) ds.$$

We may define

$$v_+(t) := \dot{K}(T)(u(0) + \Phi) + K(t)(\dot{u}(0) + \Psi).$$

Then we have

$$\begin{aligned} \|(u, \dot{u})(t) - (v_+, \dot{v}_+)(t)\|_X &= \left\| \int_t^\infty (K(t-s), \dot{K}(t-s)) f(u(s)) ds \right\|_X \\ &\leq C \|u\|_{(0;(t,\infty))}^{2^*-1} \to 0, \end{aligned}$$

as $t \to \infty$. This property uniquely determines v_+, so M_+ is a map in X. Since $\|v_+(t)\|_{L^{2^*}} \to 0$ as $t \to \infty$, by the Sobolev embedding we have $\|u(t)\|_{L^{2^*}} \to 0$, so that

$$E(u) = \lim_{t \to \infty} E_0(u; t) = E_0(v_+). \tag{6.2}$$

By the same argument as in [8], we obtain the wave operator $W_+ = M_+^{-1}$. We consider the weak continuity. Suppose $X \ni (\varphi_+^\nu, \psi_+^\nu)$ converges to (φ_+, ψ_+) weakly in X. Let $(\varphi^\nu, \psi^\nu) := W_+(\varphi_+^\nu, \psi_+^\nu)$ and $(\varphi, \psi) := W_+(\varphi_+, \psi_+)$. By the definition, $(\varphi^\nu, \psi^\nu) = (u^\nu(0), \dot{u}^\nu(0))$ and $(\varphi, \psi) = (u(0), \dot{u}(0))$ where u^ν and u are the solutions of

$$u^\nu(t) = \dot{K}(t)\varphi_+^\nu + K(t)\psi_+^\nu - \int_\infty^t K(t-s) f(u^\nu(s)) ds, \tag{6.3}$$

$$u(t) = \dot{K}(t)\varphi_+ + K(t)\psi_+ - \int_\infty^t K(t-s) f(u(s)) ds. \tag{6.4}$$

By Theorem 6.1, $\{u^\nu\}$ is bounded in $(0;\mathbb{R})$, so is $\{f(u^\nu)\}$ in $(2;\mathbb{R})$. So, after extracting some subsequence, we may assume that u^ν converges to some u^∞ weakly in $(0;\mathbb{R})$ and $f(u^\nu) \to f(u^\infty)$ weakly in $(2;\mathbb{R})$. Then, by the Strichartz estimate, letting $\nu \to \infty$ in (6.3), we have

$$u^\infty(t) = \dot{K}(t)\varphi_+ + K(t)\psi_+ - \int_\infty^t K(t-s)f(u^\infty(s))ds.$$

Thus we obtain $u^\infty = u$ and W_+ is weakly continuous. Then, the strong continuity follows from the weak continuity and (6.2). The continuity of M_+ can be proved in a similar way. □

7. Proofs of lemmas

In this section, we prove those lemmas in the previous sections whose proofs have been postponed. As in Sections 4 and 5, we assume that $m = 1$.

Proof of Lemma 4.2. Let $L := [\sqrt{\nu}+1]$. We may assume $\#S = (4L)^{\nu(N-1)}$. Then, there exists a cube $C_1 \supset S$. Now we take a finite number of cubes $\{C_j\}$ and of points $\{x_j\}$ by the following procedure. First, let $j = 1$. Then, repeat the following routine until $\#(C_j \cap S) = 1$.

- Divide C_j into $(4L)^\nu$ disjoint subcubes, from which choose a subcube C that contains the most points of S among the subcubes.
- Let $\widetilde{C}$ be the cube composed of 3^ν subcubes including C and all the neighboring subcubes.
- If $(C_j \setminus \widetilde{C}) \cap S = \varnothing$, replace C_j by $\widetilde{C}$, and repeat the routine. Otherwise, choose a point $x_j \in (C_j \setminus \widetilde{C}) \cap S$ and let $C_{j+1} := C$, increase j by 1 and repeat the routine.

It is obvious that this procedure ends in finite times, and we obtain a sequence of cubes $C_1 \supset \cdots \supset C_J$ and of points $x_1, \ldots, x_{J-1} \in S$. Let $C_J \cap S = \{x_J\}$. Denote by ℓ_j the length of the edges of C_j, and let $N_j = \#(C_j \cap S)$. By the construction, we have

$$N_{j+1} \geq \frac{1}{(4L)^\nu} N_j, \quad \ell_{j+1} \leq \frac{1}{4L}\ell_j,$$
$$\ell_{j+1} \leq |x_J - x_j| \leq \sqrt{\nu}\ell_j.$$

Thus we obtain $1 = N_J \geq (4L)^{-\nu(J-1)}N_1 = (4L)^{-\nu(J-1)}\#S = (4L)^{\nu(N-J)}$, so that $J \geq N$, and

$$|x_J - x_{j+1}| \leq \sqrt{\nu}\ell_{j+1} \leq \frac{\sqrt{\nu}}{4L}\ell_j \leq \ell_j/4 \leq |x_J - x_j|/4.$$

Thus, $x_{J-N+1}, \ldots, x_J$ is the desired sequence. □

To prove Lemma 4.3, it is convenient to introduce the following norm.

$$\|f(t)\|_{\ell^1 L^\infty} := \sum_{j\in\mathbb{N}} \sup_{2^{-j}\le t\le 2^{-j+1}} |f(t)|. \tag{7.1}$$

Then, it is clear that

$$\int_0^1 |f(t)|\frac{dt}{t} \le \|f\|_{\ell^1 L^\infty}. \tag{7.2}$$

To derive estimates for $\ell^1 L^\infty$-norms, we will use the following lemma.

Lemma 7.1. *Let $s > 0$, $0 \le f(t) \in L^\infty(0,1)$ and $0 \le g(t) \in L^1(0,1)$. Suppose that for any $0 < S < T < 1$ it holds that*

$$[t^s f(t)]_S^T \le \int_S^T t^s g(t)dt.$$

Then we have

$$\|f\|_{\ell^1 L^\infty} \le C\|g\|_{L^1},$$

where $C > 0$ depends only on s.

Proof. Let

$$q_j := f(2^{-j}), \quad r_j := \int_{2^{-j}}^{2^{-j+1}} g(t)dt.$$

Then we have

$$\begin{aligned} q_{j-1} &\le 2^{-s}q_j + r_j \le 2^{-2s}q_{j+1} + 2^{-s}r_{j+1} + r_j \le \cdots \\ &\le \sum_{k\ge j} 2^{(j-k)s} r_k, \end{aligned}$$

Thus we obtain

$$\sum_{j\ge 1} q_j \le \sum_{k\ge j\ge 1} 2^{(j-k)s} r_k \le C_s \sum_{k\ge 1} r_k \le C_s \|g\|_{L^1(0,1)}.$$

Since

$$\sup_{2^{-j}\le t\le 2^{-j+1}} f(t) \le q_j + r_j,$$

we obtain the desired result. □

For the proof of Lemma 4.3, we introduce several notations.

Definition 7.2.

$$r := |x|, \quad \theta := \frac{x}{r}, \quad u_r := \theta\cdot\nabla u, \quad u_\theta := \nabla u - \theta u_r,$$

$$H(u) := \frac{n-1}{2}G(u) - F(u),$$

$$t^2 Q_0(u;t) := (t\dot{u} + ru_r + (n-1)u)^2 + (r\dot{u} + tu_r)^2 + (t^2 + r^2)(|u_\theta|^2 + u^2),$$

$$t^2 Q_1(u;t) := t^2 Q_0 + (t^2 + r^2)\frac{u^2}{r^2}.$$

Now we prove Lemma 4.3. We do not use the nonlinear term to estimate $|u|^{2^*}$, but use the quantity $|u_\theta|^2 + u^2/r^2$, as in [12]. Such an approach would be crucial in the proof of Lemma 5.4, where we must apply such estimates to a solution of LKG.

Proof of Lemma 4.3. It suffices to consider the time interval $t \in [0, 1]$. (For the estimate on $[-1, 0]$, just reverse the time direction.) We may assume $c > 1$. We work only with smooth solutions. The estimate for general finite energy solutions can be obtained by approximation arguments (see, e.g., [6, 14, 1]). We use the inversional identity (see, e.g., [15, (2.20)], [9, (2.2a)]):

$$\begin{aligned}(\Box u + u + f(u))m(u)\\ &= \partial_t(t^2 Q'(u)) + \nabla \cdot \{-m(u)\nabla u + 2tx(e(u) - 2\dot{u}^2)\} + 4t(H(u) - u^2), \quad (7.3)\end{aligned}$$

where

$$\begin{aligned}m(u) &:= 2(t^2 + r^2)\dot{u} + 4tru_r + 2(n-1)tu,\\ t^2 Q'(u) &:= (t^2 + r^2)e(u) + 2\dot{u}(2tru_r + (n-1)tu) - (n-1)u^2\\ &= t^2 Q_0(u) + (t^2 + r^2)F(u) - (n-1)\nabla \cdot (xu^2).\end{aligned}$$

Integrating (7.3) over the truncated fat cone $K := \{(t, x) | S < t < T, r < ct\}$ for $0 < S < T < 1$, we obtain

$$\begin{aligned}\left[\int_{r<ct} t^2 Q_0(u) + (t^2 + r^2)F(u)dx\right]_{t=S}^{t=T} &= \int_{cS<r<cT} r^2 P_c(u)(r/c, x)dx\\ &\quad + \int_K 4t(u^2 - H(u))dxdt, \quad (7.4)\end{aligned}$$

where $P_c(u)$ is a certain quantity satisfying

$$|P_c(u)(r/c, x)| \le C_c \left(e(u) + \frac{u^2}{r^2}\right)(r/c, x).$$

By the energy identity and Hardy's inequality, we have for $c > 1$,

$$\int_{0<r/c<1} \left(e(u) + \frac{u^2}{r^2}\right)(r/c, x)dx \le C(E, c). \quad (7.5)$$

Using Hardy's inequality and a variant of Morawetz estimate [12, Proposition 4.4], we have

$$\int_{\substack{r<ct\\ 0<t<1}} \frac{|u^2 - H(u)|}{t} dxdt \le \int_{\substack{r<ct\\ 0<t<1}} t\frac{u^2}{r^2} dxdt + C\int_{\substack{r<ct\\ 0<t<1}} \frac{|u|^{2^*}}{t + r} dxdt \le C(E, c).$$

Thus we may apply Lemma 7.1 to (7.4) and obtain

$$\left\| \int_{r<ct} Q_0(u)dx \right\|_{\ell^1 L^\infty} \le C(E, c). \quad (7.6)$$

Next, integrate the inequality

$$\begin{aligned}\partial_t(u^2) &= 2u\left(\dot{u} + \frac{t}{r}u_r\right) + \nabla\cdot\left(\frac{t}{r}u^2\theta\right) - (n-2)\frac{tu^2}{r^2}\\ &\le \frac{(r\dot{u}+tu_r)^2}{t} + \nabla\cdot\left(\frac{t}{r}u^2\theta\right),\end{aligned} \tag{7.7}$$

over K. Then we have

$$\left[\int_{r<ct} u^2 dx\right]_S^T \le \int_{cS<r<cT} (1+c^{-2})u^2(r/c,x)dx + \int_K \frac{(r\dot{u}+tu_r)^2}{t}dxdt. \tag{7.8}$$

By (7.5), we have

$$\int_{0<r/c<1} u^2(r/c,x)\frac{dx}{r^2} \le C(E,c).$$

By (7.6), we have

$$\int_{\substack{r<ct\\0<t<1}} \frac{(r\dot{u}+tu_r)^2}{t^3}dxdt \le C(E,c).$$

So, we may apply Lemma 7.1 to (7.8) and obtain

$$\left\|\int_{r<ct}\frac{u^2}{t^2}dx\right\|_{\ell^1L^\infty} \le C(E,c). \tag{7.9}$$

By a Hardy-type inequality, we have (see, e.g., [12, Proposition 3.6])

$$\int_{r<ct}\frac{u^2}{r^2}dx \le C(c)\int_{r<ct}\frac{u^2}{t^2} + \left(\frac{t-r}{t}\right)^2 u_r^2 dx.$$

From this and the inequality

$$\frac{u^2}{t^2} + \left(\frac{t-r}{t}\right)^2 u_r^2 \le C\left\{\frac{u^2}{t^2} + \frac{(t\dot{u}+ru_r+(n-1)u)^2 + (r\dot{u}+tu_r)^2}{t^2}\right\},$$

we obtain

$$\int_{r<ct} Q_1(u;t)dx \le C(c)\int_{r<ct} Q_0(u;t) + \frac{u^2}{t^2}dx, \tag{7.10}$$

so that

$$\left\|\int_{r<ct} Q_1(u;t)dx\right\|_{\ell^1L^\infty} \le C(E,c). \tag{7.11}$$

Now the desired result follows from the following Hardy-type inequality (see [12, proof of Proposition 4.5]):

$$\int_{r<R}|\varphi|^{2^*}dx \le C\|\nabla\varphi\|_{L^2}^{2^*-2}\int_{r<R}|\varphi_\theta|^2 + \frac{\varphi^2}{r^2}dx. \tag{7.12}$$

□

Proof of Lemma 5.2. Suppose that $N \in \mathbb{R}$, $T_1 < \cdots < T_{N+1}$ satisfies $(T_1, T_{N+1}) \subset I_0$ and $\|u\|_{(0;I_j)} = \eta$ for $I_j = (T_j, T_{j+1})$. By Lemma 3.1, we obtain $J_j \subset I_j$, $c_j \in \mathbb{R}^n$ and $R_j > 0$. Choose $t_j \in J_j$ for each j. Suppose that we have for any j,

$$|I_j| \le L \quad\text{and}\quad R_j \ge \kappa|I_j|. \tag{7.13}$$

Then, by the local ST-estimate (Proposition 4.1), it suffices to get a bound for $|T_{N+1} - T_1|$ in terms of E, η, κ and L under the assumption (7.13). Denote $S = \{1, \dots, N\}$. Let $P \subset S$ satisfy that for any distinct $j, k \in P$ it holds

$$|c_k - c_j| > |t_k - t_j| + R_k + R_j. \tag{7.14}$$

Now we will bound $\#P$. Denote $B_j = \{(t_j, x) \,|\, |x - c_j| < R_j\}$, $K_j = \{(t, x) \,|\, |x - c_j| < R_j + |t_j - t|, t \geq t_j\}$ and $K = \bigcup_{j \in P} K_j$. Then, by the energy identity and the positivity of the energy, $E_K(t) := \int_K e(u; t) dx$ is nondecreasing. From (7.14), if $j \in P$, we have $B_j \cap K_k = \varnothing$ for any other $k \in P$. So, $E_K(t)$ increases at least by $\int_{B_j} e(u) dx \geq \eta^{2\alpha}$ at $t = t_j$. Thus we obtain

$$E = \int e(u(T_{N+1})) dx \geq E_K(T_{N+1}) \geq \#P \eta^{2\alpha},$$

so that $\#P \leq M := [E\eta^{-2\alpha}]$. Now we divide S into mutually disjoint sets $P, A_1, \dots, A_M$ by the following procedure. First, set $P = \{1\}$, $A_1 = \dots = A_M = \varnothing$, $q(1) := 1$ and $j = 2$. Now repeat the following routine for $j = 2, \dots, N$:

- If (7.14) is satisfied with any $k \in P$, add j to P and then let $q(j) := \#P$.
- Otherwise, choose some $k \in P$ such that (7.14) does not hold and add j to $A_{q(k)}$.

Then we obtain $P, A_1, \dots, A_M$ satisfying $S = P \sqcup A_1 \sqcup \dots \sqcup A_M$ (disjoint union) and we have (7.14) for any mutually distinct $j, k \in P$. Now by a variant of Morawetz estimate [12, Proposition 4.4], we have

$$\begin{aligned} MC(E) &\geq \sum_{k \in P} \int \int_{T_1 < t < T_{N+1}} \frac{|u|^{2^*}}{|(t, x) - (t_k, c_k)|} dx dt \\ &\geq \sum_{j \in S} \sum_{k \in P} \int_{J_j} \frac{\eta^{2^* \alpha}}{|(t_j, c_j) - (t_k, c_k)| + R_j + |I_j|} dt. \end{aligned}$$

In the case $j \in P$, when $j = k \in P$ we have

$$|(t_j, c_j) - (t_k, c_k)| + R_j + |I_j| = R_j + |I_j| \leq (C_1(E, \eta) + 1)L,$$

where C_1 is given in Lemma 3.1, and we used (7.13). In the case $j \in A_{q(k)}$, we have

$$\begin{aligned} |(t_j, c_j) - (t_k, c_k)| + R_j + |I_j| &\leq 2|t_j - t_k| + R_k + 2R_j + |I_j| \\ &\leq 2|t_j - T_1| + (3C_1(E, \eta) + 1)L, \end{aligned}$$

where we used (7.13) and $t_k < t_j$. Thus we obtain

$$\begin{aligned} MC(E) &\geq \sum_{j\in S} \eta^{2^*\alpha} \frac{|J_j|}{2|t_j - T_1| + (3C_1+1)L} \\ &\geq \eta^{2^*\alpha} \inf_j \frac{|J_j|}{|I_j|} \sum_{j\in S} \int_{I_j} \frac{dt}{2|t-T_1| + (3C_1+3)L} \\ &\geq \eta^{2^*\alpha} \kappa C_1^{-1} \log \frac{|T_{N+1} - T_1|}{(3C_1+3)L}. \end{aligned}$$

Thus we obtain the desired result. □

Proof of Lemma 5.3. Let $u_k = \psi_k * u$ and $v_k = \psi_k * v$. For $t \in (U, V)$, we have the integral equation:

$$u_k - v_k = -\int_0^t K(t-s)\psi_k * f(u(s))ds.$$

We split the integral into those on $[0, t-L]$ and $[t-L, t]$, and denote by $\mathcal{I}_1$, $\mathcal{I}_2$ the corresponding integrals. For $\mathcal{I}_1$, we use the following decay estimate of LKG (see, e.g., [7, Lemma 2.1]):

$$\begin{aligned} \|K(t)\psi_k * f(\varphi)\|_{B^{-1/2}_{\frac{2n}{n-3},2}} &\leq C|t|^{-3/2} \|\psi_k * f(\varphi)\|_{B^{3/n}_{\frac{2n}{n+3},2}} \\ &\leq C(k)|t|^{-3/2} \|\varphi\|_{H^1}^{2^*-1}, \end{aligned}$$

where in the second step, we used similar estimates with (3.7). So we obtain

$$\begin{aligned} \|\mathcal{I}_1\|_{L^\infty(U,V;B^{-1/2}_{\frac{2n}{n-3},2})} &\leq C(k,E) \int_0^{t-L} |t-s|^{-3/2} ds \\ &\leq C(k,E)L^{-1/2}. \end{aligned}$$

By Lemma 7.3, if we set $2\eta \leq \delta_0(E)$, we have $\|u\|_{(6;(T,V))} \leq C(E)$. So we have also $\|u_k - v_k\|_{(6;(T,V))} \leq C(E)$ and, by the Strichartz estimate,

$$\|\mathcal{I}_2\|_{(6;(U,V))} \leq C\|u\|_{(0;(T,V))}^{2^*-1} \leq C(E)\eta^{2^*-1},$$

so that $\|\mathcal{I}_1\|_{(6;(U,V))} \leq C(E)$. By the complex interpolation and the Sobolev embedding, we have as in (3.4),

$$\|\mathcal{I}_1\|_{(0;(U,V))} \leq C(E) \|\mathcal{I}_1\|^{\theta\lambda}_{L^\infty(U,V;B^{-1/2}_{\frac{2n}{n-3},2})} \|\mathcal{I}_1\|^{1-\theta}_{(6;(U,V))} \leq C(E,k)L^{-\theta\lambda/2}$$

where $\theta, \lambda > 0$ are the same as in (3.4). So, setting L sufficiently large relative to $C(E,k)$, we have $\|\mathcal{I}_1\|_{(0;(U,V))} \leq \eta^{2^*}$. By the Strichartz estimate again, we have

$$\|\mathcal{I}_2\|_{(0;(U,V))} \leq C(E)\eta^{2^*-1}.$$

Since $2^* - 1 > 1$, if we set $\delta_2(E)$ sufficiently small (for example, let $\delta_2(E) = (\gamma+E)^{-\gamma}$ with large $\gamma(n)$), we have

$$\|u_k\|_{(0;(U,V))} \leq \|v_k\|_{(0;(U,V))} + \|\mathcal{I}_1\|_{(0;(U,V))} + \|\mathcal{I}_2\|_{(0;(U,V))} < \eta/4.$$

□

Lemma 7.3. *Assume* (2.10). *Let* $T > 0$, *let* u *be the solution of NLKG* (2.3) *with* $E_0(u(t)) \le E < \infty$ *for any* $t \in [0, T]$, *and let* v *be the solution of LKG* (2.4) *with the same initial data at* $t = 0$. *There exists* $\delta_0 = \delta_0(E)$ *such that if* $\|u\|_{(0;(0,T))} \le \eta \le \delta_0$, *we have*

$$\|u\|_{(6;(0,T))} \le C_0(E), \quad \|\dot{u}\|_{(7;(0,T))} \le C_0(E),$$
$$\|v\|_{(0;(0,T))} \le C_0(E)\eta, \quad \|\omega^{-1}\dot{u}\|_{(0;(0,T))} \le C_0(E).$$

Proof. By the Strichartz estimate, we have for $j = 6$ or 0,

$$\begin{aligned}\|u\|_{(j;(0,T))} + \|\dot{u}\|_{(7;(0,T))} + \|\omega^{-1}\dot{u}\|_{(0;(0,T))} &\le CE_0(v)^{1/2} + C\|f(u)\|_{(2;(0,T))} \\ &\le CE^{1/2} + C\|u\|_{(0;(0,T))}^{2^*-1} \le CE^{1/2} + C\eta^{2^*-1},\end{aligned}$$

and

$$\|v\|_{(0;(0,T))} \le \|u\|_{(0;(0,T))} + C\|f(u)\|_{(2;(0,T))} \le \eta + C\eta^{2^*-1}.$$

So, setting $\delta_0 < 1$ and $\delta_0 < E$, we obtain the desired result. □

Proof of Lemma 5.4. By translation, we may assume $c = 0$ and $S = R$. Let $J \in \mathbb{N}$ (large) and suppose $S = R < 2^{-2J}$. Let $\varepsilon := 2^{-2J} + 1/J < 1$. From now on, we denote by C any positive constant dependent only on E. By (7.11), there exists some $j \in \{J, \dots, 2J\}$ such that for $T := 2^{-j}$ we have

$$\int_{r<4T} Q_1(u;T)dx \le C/J \le C\varepsilon.$$

By the energy conservation, we have

$$\int_{r<T} e(u;T)dx \ge \int_{r<S} e(u;S)dx \ge \eta.$$

Let $\zeta = \chi(x/2T)$ and $v_0 = \zeta u$. Let v be the solution of LKG with the same initial data as v_0 at $t = T$ and $w = u - v$. We have for $t = T$,

$$|\nabla v|^2 \le |\zeta \nabla u|^2 + 2|\nabla u||u\nabla\zeta| + |u\nabla\zeta|^2,$$

and

$$\int |u\nabla\zeta|^2 dx \le \int_{r<4T} |r\nabla\zeta|^2 \frac{u^2}{r^2}dx \le C\int_{r<4T} \frac{u^2}{r^2}dx \le C\varepsilon,$$

so that, by Schwarz' inequality and the monotonicity of F (which follows from $G \ge 0$),

$$E_0(v;T) \le E_0(u;T) + C\sqrt{\varepsilon} \le E + C\sqrt{\varepsilon}.$$

Similarly, we have

$$E(w;T) \le \int_{r>2T} e(u;T)dx + C\sqrt{\varepsilon} \le E - \eta + C\sqrt{\varepsilon}.$$

Moreover, we have at $t = T$,

$$Q_1(v) \le 2Q_1(u) + 2\left(\frac{r^2}{t^2} + 1\right)(\zeta_r u)^2,$$

and

$$\left(\frac{r^2}{t^2} + 1\right)(\zeta_r u)^2 \le C\frac{r^2}{t^2}(\zeta_r u)^2 \le C(r\zeta_r)^2\frac{u^2}{t^2} \le CQ_1(u).$$

Thus we obtain

$$\int_{r<4T} Q_1(v;T)dx \le C\int_{r<4T} Q_1(u;T)dx \le C\varepsilon.$$

Since $v = 0$ if $t \ge T$ and $r \ge 4T$, we have from (7.4) (with $f \equiv 0$),

$$\left[\int_{r<4t} t^2 Q_0(v;t)dx\right]_T^U \le \int_{\substack{r<4t \\ T<t<U}} 4tv^2 dxdt, \tag{7.15}$$

for any $U > T$. Similarly, from (7.8), we have

$$\left[\int_{r<4t} v^2(t)dx\right]_T^U \le \int_{\substack{r<4t \\ T<t<U}} \frac{(r\dot{v} + tv_r)^2}{t} dxdt. \tag{7.16}$$

By the energy conservation for $\omega^{-1}v$, we have

$$\begin{aligned}\|v(t)\|_{L^2}^2 &\le \|v(T)\|_{L^2}^2 + \|\dot{v}(T)\|_{H^{-1}}^2 \\ &\le C\|v(T)\|_{L^{2^*}}^2 T^2 + \|\dot{v}(T)\|_{L^2}^2 T^2 \le C\varepsilon,\end{aligned}$$

since $\operatorname{supp}(v(T), \dot{v}(T)) \subset \{x | r < 4T\}$. From (7.15), we have for any $U > T$,

$$\begin{aligned}\int_{r<4U} Q_0(v;U)dx &\le \frac{T^2}{U^2}\int_{r<4T} Q_0(v;T)dx + \frac{4}{U}\int_{\substack{r<4t \\ T<t<U}} v^2 dxdt \\ &\le C\varepsilon.\end{aligned}$$

From this and (7.16), in the same way we obtain

$$\int_{r<4U} \frac{v^2(U)}{U^2} dx \le C\varepsilon.$$

From these estimates, (7.10) and (7.12), we have for any $t > T$,

$$\|v(t)\|_{L^{2^*}}^{2^*} = \int_{r<4t} v^{2^*}(t)dx \le C\varepsilon.$$

Thus we obtain the desired result. □

We will prove Lemma 5.5 for $n \le 5$ and $n \ge 6$ separately. First, we consider the easier case $n \le 5$.

Proof of Lemma 5.5 for $n \le 5$. By [6, Proposition 2.6] we have the following estimate. (That proposition is written for the massless case, but the arguments are valid also in the massive case. We get the estimate on unbounded intervals by a simple limiting argument because the estimate is independent of the length of the interval.)

$$\|W\|_{G(0,\infty)} \le {}^{\exists}M_1(E, M) < \infty.$$

By the Strichartz estimate and Hölder's inequality, we have

$$\|v\|_{G(0,\infty)} \le \|v\|_{L_t^\infty L_x^{2^*}}^{1-\theta} \|v\|_{L_t^p L_x^q}^{\theta} \le C(E)\varepsilon^{1-\theta},$$

where

$$\frac{1}{p} = \frac{1}{2} - \delta, \quad \frac{1}{q} = \frac{1}{2} - \frac{3}{2n} + \frac{\delta}{n},$$
$$\frac{n-2}{n+2} = \frac{\theta}{p}, \quad (0 < \theta < 1),$$

with sufficiently small δ, say $1/20$. So we may assume

$$\|v\|_{G(0,\infty)} \le \varepsilon_1(E, \varepsilon),$$

where $\varepsilon_1 \searrow 0$ as $\varepsilon \searrow 0$. For $\eta > 0$, there exist $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\|W\|_{G(T_j,T_{j+1})} \le \eta \quad \text{and} \quad N^{\frac{n-2}{n+2}}\eta \le M_1.$$

Let $\Gamma = u - v - W$. Then we have the integral equation

$$\Gamma(t) = \Gamma_j(t) + \int_{T_j}^t K(t-s)(f(W) - f(\Gamma + v + W))(s)ds, \tag{7.17}$$

where Γ_j is the solution of LKG with the same initial data as Γ at $t = T_j$. By the Strichartz estimate, we have

$$\|\Gamma\|_{G(T_j,T)} + E_0(\Gamma;T)^{1/2} \le CE_0(\Gamma;T_j)^{1/2} + C\|f(W) - f(\Gamma + v + W)\|_{L_t^1(T_j,T;L_x^2)}.$$

By (2.10) and Hölder's inequality, we have

$$\|f(W) - f(\Gamma + v + W)\|_{L_t^1 L_x^2} \le C(\|\Gamma + v\|_G + \|W\|_G)^{2^*-2}\|v + \Gamma\|_G.$$

Denote $q_j(T) := \|\Gamma\|_{G(T_j,T)} + E_0(\Gamma;T)^{1/2}$ and $\widetilde{q}_j := q_j(T_{j+1})$. Let $\widetilde{q}_{-1} := 0$. Then $q_j(T)$ is continuous with respect to T and

$$q_j(T_j) = E_0(\Gamma;T_j)^{1/2} \le \widetilde{q}_{j-1},$$
$$q_j(T) \le C_1\widetilde{q}_{j-1} + C_2(q_j(T) + \varepsilon_1 + \eta)^{2^*-2}(q_j(T) + \varepsilon_1),$$

for some $C_1, C_2 \ge 1$. We set $C_2(3\eta)^{2^*-2} < 1/4$ and $(2C_1)^2\varepsilon_1 < \eta$. If $\varepsilon_1 \le q_j(T) \le \eta$, we have

$$q_j(T) \le C_1\widetilde{q}_{j-1} + q_j(T)/2,$$

so that $q_j(T) \le 2C_1\widetilde{q}_{j-1}$. Thus, if $2C_1\widetilde{q}_{j-1} < \eta$, by the continuity, we have $q_j(T) \le 2C_1\widetilde{q}_{j-1} < \eta$ for any $T \le T_{j+1}$, so that $\widetilde{q}_j \le 2C_1\widetilde{q}_{j-1}$. If $\widetilde{q}_j \le \varepsilon_1$, we have either $\widetilde{q}_j \le \varepsilon_1$ or $\widetilde{q}_j \le 2C_1\widetilde{q}_{j-1} \le 2C_1\varepsilon_1$. So, if we set

$$(2C_1)^{N+1}\varepsilon_1 < \eta,$$

then we obtain $\widetilde{q}_j \le (2C_1)^j\varepsilon_1 < \eta$ for any j and the desired result follows. □

The proof in the case $n \geq 6$ is more complicated, because we do not have a simple estimate for the difference of two solutions, as in the case $n \leq 5$. In particular, we cannot estimate the energy norm of the difference, so we introduce the following substitute.

$$\|(\varphi, \psi)\|_W := \left\|\dot{K}(t)\varphi + K(t)\psi\right\|_{(3;(0,\infty))}. \tag{7.18}$$

We will use the following variant of the Strichartz estimate.

Lemma 7.4. *Let*

$$1 < p < p' < \infty, \tag{7.19}$$

$$\frac{2(n-1)}{n+1} < q < 2 < q' < \frac{2(n-1)}{n-3}, \tag{7.20}$$

$$\frac{1}{p} + \frac{n}{q} - \sigma - 2 = \frac{1}{p'} + \frac{n}{q'} - \sigma', \tag{7.21}$$

$$\frac{1}{p} + (n-1)\left(\frac{1}{q} - \frac{1}{2}\right) - 1 > 0 > \frac{1}{p'} + (n-1)\left(\frac{1}{q'} - \frac{1}{2}\right), \tag{7.22}$$

$$\frac{1}{p} + \frac{n-1}{2}\frac{1}{q} - 1 > \frac{1}{p'} + \frac{n-1}{2}\frac{1}{q'}. \tag{7.23}$$

Then we have

$$\left\|\int_0^t \frac{e^{\pm i\omega(t-s)}}{\omega} f(s)ds\right\|_{L^{p'}(0,T;B^{\sigma'}_{q',1})} \leq C\|f\|_{L^p(0,T;B^{\sigma}_{q,\infty})}, \tag{7.24}$$

where $C > 0$ depends only on the exponents $(p, q, \sigma, p', q', \sigma')$.

Proof. First we consider a weaker estimate where the third exponents of Besov spaces are replaced with 2:

$$\left\|\int_0^t \frac{e^{\pm i\omega(t-s)}}{\omega} f(s)ds\right\|_{L^{p'}(0,T;B^{\sigma'}_{q',2})} \leq C\|f\|_{L^p(0,T;B^{\sigma}_{q,2})}. \tag{7.25}$$

In the case (in addition to (7.19)–(7.23))

$$\frac{1}{p} + \frac{n-1}{2}\left(\frac{1}{q} - \frac{1}{2}\right) - 1 \geq 0 \geq \frac{1}{p'} + \frac{n-1}{2}\left(\frac{1}{q'} - \frac{1}{2}\right), \tag{7.26}$$

(7.25) is the standard (generalized) Strichartz estimate (see, e.g., [5, 7, 13]). For sufficiently small $\varepsilon > 0$, let

$$\begin{aligned}\left(\frac{1}{p}, \frac{1}{q}, \sigma\right) &= \left(\varepsilon, \frac{1}{2} + \frac{1}{n-1} - \frac{\varepsilon}{n}, \sigma\right),\\ \left(\frac{1}{p'}, \frac{1}{q'}, \sigma'\right) &= \left(\frac{\varepsilon}{n}, \frac{1}{2} - \frac{1}{n-1} + \frac{\varepsilon}{n}, \sigma - \frac{2}{n-1} + \frac{n+1}{n}\varepsilon\right).\end{aligned} \tag{7.27}$$

Then, (7.19)–(7.23) are satisfied and, by [7, Lemma 2.1], we have

$$\left\|\frac{e^{\pm i\omega t}}{\omega} f(s)\right\|_{B^{\sigma'}_{q',2}} \leq C|t-s|^{1-\frac{n-1}{n}\varepsilon}\|f(s)\|_{B^{\sigma}_{q,2}}.$$

From this and the Hardy-Littlewood-Sobolev inequality, we obtain (7.25) for this exponent. By the complex interpolation between the case (7.26) and the case (7.27), and by the Sobolev embedding, we obtain (7.25) in the case

$$\frac{1}{p} + \frac{n-1}{2}\left(\frac{1}{q} - \frac{1}{2}\right) - 1 \leq 0.$$

Then, by the duality argument and the interpolation, we obtain (7.25) in the remaining case. If $(p, q, \sigma, p', q', \sigma')$ satisfies (7.19)–(7.23), then $(p_\pm, q, \sigma_\pm, p'_\pm, q', \sigma'_\pm)$ with

$$\frac{1}{p_\pm} = \frac{1}{p} \pm \varepsilon, \quad \sigma_\pm = \sigma \pm \varepsilon, \quad \frac{1}{p'_\pm} = \frac{1}{p'} \pm \varepsilon, \quad \sigma'_\pm = \sigma' \pm \varepsilon,$$

satisfy (7.19)–(7.23) for sufficiently small $\varepsilon > 0$. By the interpolation inequality and Hölder's inequality, we have

$$L^{p'_+}(B^{\sigma'_+}_{q',2}) \cap L^{p'_-}(B^{\sigma'_-}_{q',2}) \hookrightarrow L^{p'}(B^{\sigma'}_{q',1}),$$

and by the duality argument we have

$$L^{p_+}(B^{\sigma_+}_{q,2}) + L^{p_-}(B^{\sigma_-}_{q,2}) \hookleftarrow L^p(B^{\sigma}_{q,\infty}).$$

So, (7.24) follows from (7.25) with the exponents $(p_\pm, q, \sigma_\pm, p'_\pm, q', \sigma'_\pm)$. □

Then we have the following estimate for W-norm.

Lemma 7.5. *Let $n \geq 6$. Let u be the solution of*

$$\begin{cases} \Box u + u = g, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi. \end{cases}$$

Then we have for any $T \geq 0$,

$$\|(u(T), \dot{u}(T))\|_W \leq \|(\varphi, \psi)\|_W + C\|g\|_{(5;(0,T))}.$$

Proof. Let v be the solution of LKG with the same initial data as u at $t = T$. Let w be the solution of

$$\begin{cases} \Box w + w = g\chi_{(0,T)}, \\ w(0) = \varphi, \quad \dot{w}(0) = \psi, \end{cases}$$

where $\chi_{(0,T)}$ denotes the characteristic function of the interval $(0, T)$. Then we have $v = w$ for $t > T$, so that by Lemma 7.4 we have

$$\|v\|_{(3;(T,\infty))} \leq \|w\|_{(3;(0,\infty))} \leq \|(\varphi, \psi)\|_W + \|g\|_{(5;(0,T))},$$

which is the desired result. □

Proof of Lemma 5.5 for $n \geq 6$. We have some $M_2 = M_2(E, M) < \infty$ such that

$$\|W\|_{(j;(0,\infty))} \leq M_2(E, M)/3 < \infty,$$

for $j = 3, 0$ or 6, by [6, Proposition 2.6]. (That proposition is written for the massless case, but the arguments are valid also in the massive case. We get the estimate on unbounded intervals by simple limiting argument because the estimate is independent of the length of the interval.) By the Strichartz estimate and the complex interpolation, we have for $j = 3$ or 0,

$$\|v\|_{(j;(0,\infty))} \le C\|v\|^{\theta}_{L^\infty_t L^{2^*}_x} \|v\|^{1-\theta}_{(6;(0,\infty))} \le C(E)\varepsilon^\theta \le \varepsilon_2(E,\varepsilon),$$

where $1/p_j = (1-\theta)/p_6$ and $\varepsilon_2 \searrow 0$ as $\varepsilon \searrow 0$. For $\eta > 0$, there exist $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\|W\|_{(3;(T_j,T_{j+1}))} + \|W\|_{(0;(T_j,T_{j+1}))} + \|W\|_{(6;(T_j,T_{j+1}))} \le \eta,$$

and

$$N^{1/p_3}\eta \le M_2.$$

Let $\Gamma = u - v - W$. Applying Lemmas 7.4 and 7.5 to (7.17), we have

$$\begin{aligned}\|\Gamma\|_{(3;(T_j,T))} &+ \|(\Gamma(T),\dot\Gamma(T))\|_W \\ &\le C\|(\Gamma(T_j),\dot\Gamma(T_j))\|_W + C\|f(W) - f(\Gamma+v+W)\|_{(5;(T_j,T))}.\end{aligned}$$

By [8, Lemma 2.3(3)] and the Sobolev embedding, we have

$$\begin{aligned}\|f(W) - f(\Gamma+v+W)\|_{(5)} &\le C(\|\Gamma+v\|_{(1)} + \|W\|_{(1)})^{2^*-2}\|\Gamma+v\|_{(3)} \\ &\quad + C(\|\Gamma+v\|_{(0)} + \|W\|_{(0)})^{2^*-2}\|\Gamma+v\|_{(4)} \\ &\le C(\|\Gamma+v\|_{(0)} + \|W\|_{(0)})^{2^*-2}\|\Gamma+v\|_{(3)}.\end{aligned}$$

Denote $Q_j(T) := \|\Gamma\|_{(3;(T_j,T))} + \|(\Gamma(T),\dot\Gamma(T))\|_W$ and $\widetilde{Q}_j := Q_j(T_{j+1})$. Let $\widetilde{Q}_{-1} := 0$. $Q_j(T)$ is continuous with respect to T and

$$\begin{aligned}&Q_j(T_j) \le \widetilde{Q}_{j-1}, \\ &Q_j(T) \le C_1\widetilde{Q}_{j-1} + C(\|\Gamma\|_{(0)} + \varepsilon_2 + \eta)^{2^*-2}(Q_j(T) + \varepsilon_2).\end{aligned}$$

While $Q_j(T) \le \eta$, we have $\|u\|_{(3;(T_j,T))} \le 2\eta + \varepsilon_2 < E$. So, by [6, Proposition 2.6], we have $\|u\|_{(6;(T_j,T))} \le C(E)$. Since $\|W\|_{(6;(T_j,T))} \le \eta < E$ and $\|v\|_{(6;(T_j,T))} \le C(E)$, we have

$$\|\Gamma\|_{(0;(T_j,T))} \le C\|\Gamma\|^{\beta}_{(3;(T_j,T))}\|\Gamma\|^{1-\beta}_{(6;(T_j,T))} \le C(E)Q_j(T)^\beta,$$

where $0 < \beta < 1$ and

$$\frac{1}{p_0} = \frac{\beta}{p_3} + \frac{1-\beta}{p_6}.$$

Thus we obtain

$$Q_j(T) \le C_1\widetilde{Q}_{j-1} + C_2(Q_j(T)^\beta + \varepsilon_2 + \eta)^{2^*-2}(Q_j(T) + \varepsilon_2),$$

as long as $Q_j(T) \le \eta$. Then, by a similar argument with the case $n \le 5$, if we set $C_2(3\eta)^{2^*-2} < 1/4$ and $(2C_1)^{N+1}\varepsilon_2 < \eta^{1/\beta}$, then we obtain $\widetilde{Q}_j \le (2C_1)^j\varepsilon_2 < \eta^{1/\beta}$ for any j, from which the desired result follows. □

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Part II.
Energy Scattering for Nonlinear Klein-Gordon and Schrödinger Equations in Spatial Dimensions 1 and 2

1. Introduction

In Part II, we study the scattering theory in the energy space for nonlinear Klein-Gordon equations (NLKG):

$$\Box u + m^2 u + |u|^{p-1} u = 0, \tag{1.1}$$

and for nonlinear Schrödinger equations (NLS):

$$i\dot{u} - \Delta u + |u|^{p-1} u = 0, \tag{1.2}$$

where $u = u(t,x)$, $(t,x) \in \mathbb{R}^{1+n}$, $\dot{u} = \partial u/\partial t$, $\Box = \partial_t^2 - \Delta$, $m > 0$, $n \leq 2$ and $p > 1 + 4/n$. We will prove that the wave operators and the scattering operators for (1.1) and for (1.2) are well-defined and bijective in the whole energy space E (for NLKG, $E = H^1 \oplus L^2$ and for NLS, $E = H^1$). It is well known that there exist injective wave operators defined everywhere in E. So, the main problem is the surjectivity, which means the asymptotic completeness of the wave operators. Such results are known in the case where $n \geq 3$ and $p > 1 + 4/n$, in the case of small energy data and, in the NLS case, in certain function spaces smaller than the energy space (see, e.g., [4, 5, 6, 7, 8, 9, 10, 14, 15, 16]). But, as far as the author knows, no result is known for the scattering in the whole energy space when $n \leq 2$. In particular, this is the first result on the large data scattering of NLKG for $n \leq 2$, which was left as one of the major open problems in [13, pp. 247]. The difficulty for $n \leq 2$ consists mainly in the two points: the breakdown of the Morawetz estimate and the unintegrability of the time decay order of the free equations. We overcome the first difficulty by certain variants of the Morawetz estimate with space-time weights. Such estimates seem to have first appeared in [11] for $n \geq 3$. Moreover, we do not need the integrability of the time decay order if we use the argument of 'separation of localized energy', which was invented by Bourgain [3] and was used also in Part I.

2. Notations and conventions

In this section, we introduce several notations and conventions. In order to state the results and the proofs in a unified way for both NLKG and NLS, we use several notations whose meanings differ depending on whether we are considering NLKG or NLS. As usual, we denote by C auxiliary *positive* constants, and sometimes write as $C(a, b, \dots)$ to indicate that the constant depends only on $a, b, \dots$ and that the dependence is continuous (we will use this convention for constants which are not denoted by 'C'). We fix n and p, and so we ignore the dependence of the constants on n and p. We denote by $B^{\sigma}_{q,r}$ the usual inhomogeneous Besov spaces (see, e.g., [1]). We will use the following space-time norms. We will sometimes abbreviate them as 'ST-norms'.

$$\begin{aligned} &(B; I) := L^\infty(I; B^{-\sigma}_{\infty,\infty}(\mathbb{R}^n)), && (X; I) := L^q(I \times \mathbb{R}^n), \\ &(K; I) := L^\rho(I; B^{\sigma_K}_{\rho,2}(\mathbb{R}^n)), && (\bar{K}; I) := L^{\bar{\rho}}(I; B^{\sigma_K}_{\bar{\rho},2}(\mathbb{R}^n)), \\ &(Y; I) := L^q(I; B^{\sigma_Y}_{\rho,2}(\mathbb{R}^n)), && (\widetilde{Y}; I) := L^{q/p}(I; B^{\sigma_Y}_{\bar{\rho},2}(\mathbb{R}^n)), \end{aligned} \tag{2.1}$$

where $\rho = (2n+4)/n$, $1/\rho + 1/\bar{\rho} = 1$, $(p-1)/q + 1/\rho = 1/\bar{\rho}$, $\sigma = \rho/(2q)$, $\sigma_Y = n/\rho - n/q$ and

$$\sigma_K = \begin{cases} 1/2, & \text{in the NLKG case,} \\ 1, & \text{in the NLS case.} \end{cases} \tag{2.2}$$

The condition that $p > 1 + 4/n$ is equivalent to that $q > \rho$. We will sometimes omit the interval I in (2.1). For simplicity, we set $m = 1$ for NLKG. Then the equation is

$$\Box u + u + |u|^{p-1} u = 0 \quad \text{(NLKG)}. \tag{2.3}$$

We fix a smooth cut-off function h satisfying

$$h \in C^\infty(\mathbb{R}), \quad 0 \le h \le 1, \quad h(t) = \begin{cases} 1, & t \ge 1, \\ 0, & t \le 0. \end{cases} \tag{2.4}$$

Denote by $\mathcal{F}\varphi = \widetilde{\varphi}$ the Fourier transform of φ and define the Littlewood-Paley dyadic decomposition:

$$\begin{aligned} &\psi_j := \mathcal{F}^{-1} h(2 - 2^{-j}|\xi|) \in \mathcal{S}(\mathbb{R}^n), \\ &\varphi_j := \psi_j - \psi_{j-1} \in \mathcal{S}(\mathbb{R}^n) \text{ for } j \in \mathbb{N}, \\ &\varphi_0 := \psi_0. \end{aligned} \tag{2.5}$$

We define the energy and several related quantites.

$$
\begin{gathered}
f(u) := |u|^{p-1}u, \quad F(u) := \frac{2}{p+1}|u|^{p+1}, \\
G(u) := \Re(\bar{u}f(u)) - F(u) = \left(1 - \frac{2}{p+1}\right)|u|^{p+1} \geq \frac{|u|^{p+1}}{2}, \\
e_L(u;t) := \begin{cases} |\dot{u}|^2 + |\nabla u|^2 + |u|^2, & \text{in the NLKG case,} \\ |\nabla u|^2 + |u|^2, & \text{in the NLS case,} \end{cases} \\
e_N(u) := e_L(u) + F(u), \\
E_L(u;t) := \int_{\mathbb{R}^n} e_L(u;t)dx, \quad E_N(u;t) := \int_{\mathbb{R}^n} e_N(u;t)dx,
\end{gathered}
\tag{2.6}
$$

where $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. E_N is a conserved quantity for NLKG and NLS, and E_L is a conserved quantity for the free equations. Denote $\langle a \rangle := \sqrt{1+|a|^2}$. Denote for any function φ,

$$
\begin{aligned}
\varphi(\omega) &:= \mathcal{F}^{-1}\varphi(\langle\xi\rangle)\mathcal{F}, \\
\varphi(\Delta) &:= \mathcal{F}^{-1}\varphi(-|\xi|^2)\mathcal{F}.
\end{aligned}
\tag{2.7}
$$

Using these notations, we define

$$
U(t) := \begin{cases} \omega^{-1}\sin\omega t, & \text{in the NLKG case,} \\ -ie^{-i\Delta t}, & \text{in the NLS case.} \end{cases}
\tag{2.8}
$$

Then the integral equations associated to NLKG and NLS are respectively

$$
u(t) = \dot{U}(t)u(0) + U(t)\dot{u}(0) - \int_0^t U(t-s)f(u(s))ds, \tag{2.9}
$$

$$
u(t) = iU(t)u(0) - \int_0^t U(t-s)f(u(s))ds. \tag{2.10}
$$

3. Basic estimates on ST-norms

In this section we collect basic and well-known estimates on the space-time norms introduced in the previous section. By the Sobolev embedding, we have for any $j \in \mathbb{N}$,

$$
\begin{gathered}
\|u\|_{(B)} \leq C\|u\|_{L_t^\infty(H^1)}, \quad \|u\|_{(X)} \leq C\|u\|_{(Y)}, \\
\|\varphi_j * u\|_{(B)} \leq C2^{-\sigma j}\|u\|_{L_t^\infty(H^1)}.
\end{gathered}
\tag{3.1}
$$

By the Sobolev embedding and the well-known nonlinear estimates for the Besov norms (see, e.g., [9, Lemma 3.1]), we have

$$
\|f(u)\|_{(\bar{K})} \leq C\|u\|_{(K)}\|u\|_{(X)}^{p-1}, \tag{3.2}
$$

$$
\|f(u) - f(v)\|_{(\bar{Y})} \leq C\|u-v\|_{(Y)}(\|u\|_{(Y)} + \|v\|_{(Y)})^{p-1}. \tag{3.3}
$$

By the complex interpolation and the Sobolev embedding, we have

$$\|u\|_{(X)} \leq C\|u\|_{(K)}^{\rho/q}\|u\|_{(B)}^{1-\rho/q}, \tag{3.4}$$

$$\|u\|_{(Y)} \leq C\|u\|_{(X)}^{1-2/\rho}\|u\|_{(K)}^{2/q}\|u\|_{L_t^\infty(H^1)}^{2/\rho-2/q}. \tag{3.5}$$

We have the following decay estimate for $U(t)$ (see, e.g., [9])

$$\|U(t)\varphi\|_{B^0_{\rho,2}} \leq C|t|^{-\mu}\|\varphi\|_{B^0_{\bar\rho,2}}, \tag{3.6}$$

where $\mu = n(1/2 - 1/\rho) = 1 + 1/q - p/q$. So, by Young's inequality, we have for any $T > 0$,

$$\begin{aligned}\left\|\int_0^t U(t-s)v(s)ds\right\|_{(Y;(0,T))} &\leq C\left\|\int_0^t |t-s|^{-\mu}\|v(s)\|_{B^{\sigma_Y}_{\bar\rho,2}} ds\right\|_{L^q(0,T)} \\ &\leq C\|v\|_{(\tilde{Y};(0,T))}.\end{aligned} \tag{3.7}$$

Denote

$$eq_L(u) := \begin{cases} \Box u + u, & \text{in the NLKG case,} \\ i\dot{u} - \Delta u, & \text{in the NLS case.}\end{cases} \tag{3.8}$$

Then, by the Strichartz estimate (see, e.g., [4, 9]), we have for any $t > 0$,

$$E_L(u;t)^{1/2} + \|u\|_{(K;(0,t))} + \|u\|_{(X;(0,t))} \leq CE_L(u;0)^{1/2} + C\|eq_L(u)\|_{(\bar{K};(0,t))}. \tag{3.9}$$

Using the above estimates, we have the following lemma.

Lemma 3.1. *Let u be a solution of (2.3) or (1.2) on an interval $I = (S, T)$ with $E_L(u;S) \leq E < \infty$ and $\|u\|_{(X;I)} = \eta$. Let v be the solution of the free equation with the same initial data as u at $t = S$. There exists a constant $\eta_0(E) \in (0,1)$ such that if $\eta \leq \eta_0(E)$ we have*

$$\begin{gathered}\|u - v\|_{(K;I)} + \|u - v\|_{(X;I)} < \eta, \\ \|v\|_{(X;I)} < 2\eta, \quad \|u\|_{(K;I)} \leq C(E).\end{gathered} \tag{3.10}$$

Proof. By (3.9) and (3.2), we have

$$\begin{aligned}\|u - v\|_{(K;I)} + \|u - v\|_{(X;I)} &\leq C\|f(u)\|_{(\bar{K};I)} \\ &\leq C\|u\|_{(K;I)}\|u\|_{(X;I)}^{p-1} \\ &\leq C\eta^{p-1}\|u\|_{(K;I)}.\end{aligned} \tag{3.11}$$

Now we set η_0 so small that $C\eta_0^{p-1} < 1/2$. Then we have

$$\|u\|_{(K;I)} \leq 2\|v\|_{(K;I)} \leq C(E), \tag{3.12}$$

where the last inequality follows from the Strichartz estimate. Thus, from (3.11), we have

$$\|u - v\|_{(K;I)} + \|u - v\|_{(X;I)} \leq C(E)\eta^{p-1}. \tag{3.13}$$

Setting η_0 so small that $C(E)\eta_0^{p-2} < 1$, we obtain the desired estimate. □

4. Distribution of ST-norms

In this section we prove the following lemma, which relates the time-distribution of the ST-norm with the space-time distribution. The lemma is merely a reproduction of Lemma 3.1 of Part I in the present context, and the idea is essentially due to [3]. Since (2.3) and (1.2) are H^1-subcritical, the situation is much simpler than those in [3] and Part I.

Lemma 4.1. *Let u satisfy (2.3) or (1.2) on an interval I with $E_N(u) \le E < \infty$. Suppose that $\|u\|_{(X;I)} = \eta \in (0, \eta_0(E)]$ (η_0 is given by Lemma 3.1). Let $s \ge 1$. Then, there exist a subinterval $J \subset I$, $R > 0$ and $c \in \mathbb{R}^n$ satisfying $|J| \ge C(E,\eta)$, $R \le C(E,\eta)$ and*

$$\int_{|x-c|<R} \min(|u(t)|, |u(t)|^s)\,dx \ge C(E,\eta,s), \tag{4.1}$$

for any $t \in J$.

Proof. By Lemma 3.1 and (3.4), we have

$$\eta = \|u\|_{(X)} \le C\|u\|_{(K)}^{\rho/q}\|u\|_{(B)}^{1-\rho/q} \le C(E)\|u\|_{(B)}^{1-\rho/q}, \tag{4.2}$$

so that we have some $T \in I$, $c \in \mathbb{R}^n$ and $j \in \mathbb{N} \cup \{0\}$ such that

$$|2^{-\sigma j}\varphi_j * u(T,c)| \ge C(E,\eta). \tag{4.3}$$

On the other hand, by (3.1), we have

$$|2^{-\sigma j}\varphi_j * u(T,c)| \le 2^{-\sigma j}C(E), \tag{4.4}$$

so that we have $j \le C(E,\eta)$. By the Sobolev embedding and Hölder's inequality, we have

$$\eta = \|u\|_{(X;I)} \le C(E)|I|^{1/q}, \tag{4.5}$$

so that we have $|I| \ge C(E,\eta)$. From the equation and the Sobolev embedding, we have

$$\|\varphi_j * (u(t) - u(T))\|_{L^\infty} \le C(j)\|u(t) - u(T)\|_{H^{-1}} \le C(E,\eta)|t-T|. \tag{4.6}$$

Thus, we have some interval $J \subset I$ such that $|J| \ge C(E,\eta)$ and we have (4.3) for any $T \in J$ (of course, the constant C should be changed). Denote

$$\Phi := \begin{cases} \psi_0 - \psi_{-1} & \text{if } j \ge 1, \\ \psi_0 & \text{if } j = 0. \end{cases} \tag{4.7}$$

Then we have for any $t \in J$,

$$\begin{aligned} C(E,\eta) \le |\varphi_j * u(t,c)| &= \left| \int 2^{jn}\Phi(2^j y)u(t, c-y)dy \right| \\ &\le 2^{jn}\|\Phi\|_{L^\infty}\|u(t)\|_{L^1(|x-c|<R)} + 2^{jn/2}\|\Phi\|_{L^2(|x|>2^j R)}\|u(t)\|_{L^2} \\ &\le C(E,\eta)\left\{\|u(t)\|_{L^1(|x-c|<R)} + \|\Phi\|_{L^2(|x|>2^j R)}\right\}. \end{aligned} \tag{4.8}$$

Since $\Phi \in \mathcal{S}$, we can make $\|\Phi\|_{L^2(|x|>2^j R)}$ arbitrarily small if we take R sufficiently large. Thus we obtain some $R \le C(E,\eta)$ such that for any $t \in J$, we have

$$\|u(t)\|_{L^1(|x-c|<R)} \ge C(E,\eta). \tag{4.9}$$

Denote

$$A := \int_{\substack{|u|\le 1\\ |x-c|<R}} |u|^s dx, \quad B := \int_{\substack{|u|>1\\ |x-c|<R}} |u| dx. \tag{4.10}$$

Then, from (4.9) we have for $t \in J$,

$$\begin{aligned} C(E,\eta) &\le B + \int_{\substack{|u|\le 1\\ |x-c|<R}} |u| dx \\ &\le B + C(R)A^{1/s} \le C(E,\eta)\{A + B + (A+B)^{1/s}\}, \end{aligned} \tag{4.11}$$

so that we obtain the desired estimate:

$$A + B \ge C(E,\eta,s). \tag{4.12}$$

□

5. Morawetz-type estimates

In this section, we derive certain variants of the Morawetz estimates with space-time weights. Such an estimate for NLKG was derived for $n \ge 3$ in [11, Proposition 4.4]. Here we are concerned only with the asymptotic behaviour of the solutions for large time. The estimate (5.3) for NLS is a new estimate, which might be useful also for $n \ge 3$.

Lemma 5.1. *Let u be a global solution of NLKG (2.3) with $E_N(u) = E < \infty$. In the case $n = 2$, we have*

$$\iint_{\mathbb{R}^{1+2}} \frac{|u_\omega|^2}{\langle t\rangle + |x|} + \frac{\langle t\rangle^2 G(u)}{\langle t\rangle^3 + |x|^3} dxdt \le C(E), \tag{5.1}$$

where u_ω is the projection of $(\dot u, \nabla u)$ to the tangent space of the hyperboloid $t^2 - |x|^2 =$ constant. In the case $n = 1$, we have

$$\iint_{\mathbb{R}^{1+1}} \frac{\min(|u|^2, G(u))}{\langle t\rangle \log(|t|+2)\log(\max(r-t,2))} dxdt \le C(E). \tag{5.2}$$

Lemma 5.2. *Let u be a global solution of NLS (1.2) with $E_N(u) = E < \infty$. Then we have*

$$\iint_{\mathbb{R}^{1+n}} \frac{|2t\nabla u + ixu|^2}{\langle t\rangle^3 + |x|^3} + \frac{\langle t\rangle^2 G(u)}{\langle t\rangle^3 + |x|^3} dxdt \le C(E). \tag{5.3}$$

For the proof of the above estimates, we introduce several notations.

$$\begin{aligned} r = |x|, \quad \theta = \frac{x}{r}, \quad \lambda = \sqrt{t^2+r^2}, \quad \Theta = \frac{(-t,x)}{\lambda}, \\ u_r = \theta \cdot \nabla u, \quad u_\theta = \nabla u - \theta u_r, \\ \ell_K(u) = \frac{1}{2}\left\{-|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + F(u)\right\}, \\ \ell_S(u) = \frac{1}{2}\left\{\Re(i\dot{u}\bar{u}) + |\nabla u|^2 + F(u)\right\}. \end{aligned} \tag{5.4}$$

Proof of Lemma 5.1. It suffices to prove the estimate for C^2 solutions and on the interval $(2,\infty)$. We have the following identity (see [11, Proof of Lemma 4.2 (4.4)]):

$$\begin{aligned} \Re\{(\Box u + u + f(u))\bar{m}_h\} = \sum_{\alpha=0}^{n} \partial_\alpha \Re\left(-\bar{m}_h \partial^\alpha u + \ell_K(u)\Theta_\alpha + \frac{|u|^2}{2}\partial^\alpha g\right) \\ + \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}\Box g + G(u)g, \end{aligned} \tag{5.5}$$

where $(\partial_0, \partial_1, \dots, \partial_n) = (-\partial^0, \partial^1, \dots, \partial^n) = (\partial_t, \nabla)$ and

$$\begin{aligned} g = \frac{n-1}{2\lambda} + \frac{t^2-r^2}{2\lambda^3}, \quad m_h = \Theta\cdot(\dot{u}, \nabla u) + ug, \\ \Box g = \frac{(n-3)(n+3)}{2\lambda^3} + 3(n-1)\frac{t^2-r^2}{\lambda^5} + 15\frac{(t^2-r^2)^2}{2\lambda^7}. \end{aligned} \tag{5.6}$$

Since g is smooth for $t > 0$, we can integrate (5.5) over $(2,T)\times\mathbb{R}^n$ for $T > 2$, and by the divergence theorem we obtain

$$\begin{aligned} \left[\int_{\mathbb{R}^n} -\Re\,\dot{u}\bar{m}_h + \ell_K(u)\frac{t}{\lambda} + \frac{|u|^2}{2}\dot{g}\,dx\right]_{t=2}^{t=T} \\ = \int_2^T \int_{\mathbb{R}^n} \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}\Box g + G(u)g\,dxdt. \end{aligned} \tag{5.7}$$

The left hand side is bounded by the energy, and

$$\left|\int_2^T \int_{\mathbb{R}^n} \frac{|u|^2}{2}\Box g dxdt\right| \le C\int_2^T \int_{\mathbb{R}^n} \frac{|u|^2}{t^3} dxdt \tag{5.8}$$

is also bounded by the energy. In the case $n \ge 2$, the remaining terms in the right hand side of (5.7) are nonnegative. So we obtain the desired result in the case $n = 2$. In the case $n = 1$, we have $g \ge 0$ only if $r \le |t|$. So we integrate (5.5) over the region $\{(t,x) \mid 2 < t < T, r < t\}$. Then we have by the divergence theorem,

$$\begin{aligned} \left[\int_{r<t} -\Re\,\dot{u}\bar{m}_h + \ell_K(u)\frac{t}{\lambda} + \frac{|u|^2}{2}\dot{g}\,dx\right]_{t=2}^{t=T} \\ = \int_2^T \int_{r<t} \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}\Box g + G(u)g\,dxdt \\ + \Re\int_{2<r=t<T} -\bar{m}_h(\dot{u}+u_r) + \sqrt{2}\ell_K(u)dx. \end{aligned} \tag{5.9}$$

By the energy estimate on the surface of the light cone, the last term of (5.9) is estimated as

$$\begin{aligned} \cdots &= \frac{\sqrt{2}}{2}\int_{2<r=t<T} |u_\theta|^2 + |u|^2 + F(u)dx \\ &\leq \frac{\sqrt{2}}{2}\int_{2<r=t<T} |\theta\dot{u} + \nabla u|^2 + |u|^2 + F(u)dx \leq C(E). \end{aligned} \tag{5.10}$$

So, as in the case $n = 2$, we obtain

$$\int_2^\infty \int_{r<t} \frac{|u_\omega|^2}{\lambda} + G(u)g\, dxdt \leq C(E). \tag{5.11}$$

By the energy estimate on the surface of the light cones, we have

$$\int_{\mathbb{R}} |u(r+t,x)|^2 dx \leq C(E), \tag{5.12}$$

for any $t \in \mathbb{R}$. Now we integrate (5.12) multiplied with

$$w(t) := \frac{1}{|t|^{1/2}\langle t\rangle^{1/2}(\log(|t|+2))^{3/2}} \tag{5.13}$$

over $\mathbb{R}$. Since $w(t)$ is integrable, we obtain

$$\iint_{\mathbb{R}^{1+1}} |u|^2 w(t-r) dxdt \leq C(E). \tag{5.14}$$

From (5.11) and (5.14), we obtain

$$\iint_{t>2} \min(|u|^2, G(u)) \max(\widetilde{g}, w(t-r)) dxdt \leq C(E), \tag{5.15}$$

where we denote

$$\widetilde{g}(t,x) := \begin{cases} g(t,x), & \text{if } r < t, \\ 0, & \text{if } r \geq t. \end{cases} \tag{5.16}$$

Since for $r < t$ we have

$$\begin{aligned} \max(\widetilde{g}, w(t-r)) &\geq |g|^{1/3}\{w(t-r)\}^{2/3} \\ &= \frac{(t+r)^{1/3}}{2^{1/3}\lambda\langle t-r\rangle^{1/3}(\log(|t-r|+2))} \\ &\geq \frac{C}{t\log(t+2)}\left(\frac{t}{\langle t\rangle}\right)^{1/3}, \end{aligned} \tag{5.17}$$

we obtain the desired result from (5.15). □

Proof of Lemma 5.2. We will use the following new multiplier:

$$m_p := 2\frac{r}{\lambda}u_r + \left(\frac{n-1-it}{\lambda} + \frac{t^2}{\lambda^3}\right)u. \tag{5.18}$$

We have the following identity for a general multiplier $m = a \cdot \nabla u + gu$ with $a : \mathbb{R}^{1+n} \to \mathbb{R}^n$ and $g : \mathbb{R}^{1+n} \to \mathbb{C}$ (cf. [13, Theorem 2.2]).

$$\begin{aligned}\Re\{(i\dot{u} - \Delta u + f(u))\bar{m}\} &= \frac{\partial_t}{2}\Im\left\{a \cdot \bar{u}\nabla u + |u|^2 g\right\} \\ &+ \nabla \cdot \Re\left\{-\nabla u\bar{m} + a\ell_S(u) + \frac{|u|^2}{2}\nabla g\right\} + \sum_{i,j=1}^{n} \partial_i u\, \partial_i a_j\, \bar{\partial_j} u + G(u)\Re g \\ &+ \frac{|u|^2}{2}\Re(i\dot{g} - \Delta g) + (2\nabla\Im g - \dot{a}) \cdot \frac{1}{2}\Im(\bar{u}\nabla u) + (2\Re g - \nabla \cdot a)\ell_S(u).\end{aligned} \tag{5.19}$$

Now let $m = m_p$. Then the last term in (5.19) vanishes, and we have

$$\begin{aligned}&\sum_{i,j=1}^{n} \partial_i u\, \partial_i a_j\, \bar{\partial_j} u + \frac{|u|^2}{2}\Re(i\dot{g}) + (2\nabla\Im g - \dot{a}) \cdot \frac{1}{2}\Im(\nabla u\bar{u}) \\ &= \frac{2t^2}{\lambda^3}|\nabla u|^2 + \frac{2r^2}{\lambda^3}|u_\theta|^2 + \frac{r^2}{2\lambda^3}|u|^2 + 2\frac{xt}{\lambda^3}\Im(\nabla u\bar{u}) \\ &= \frac{|2t\nabla u + ixu|^2}{2\lambda^3} + \frac{2r^2}{\lambda^3}|u_\theta|^2,\end{aligned} \tag{5.20}$$

$$-\Re\Delta g = \frac{(n-1)(n-3)}{\lambda^3} + \frac{6(n-3)t^2}{\lambda^5} + \frac{15t^4}{\lambda^7}, \tag{5.21}$$

so that $|\Re\Delta g| < C/\lambda^3$. Thus we obtain

$$\begin{aligned}\Re\{(i\dot{u} - \Delta u + f(u))\bar{m}_p\} &\geq \partial_t\left\{\frac{r}{\lambda}\Im(\bar{u}u_r) - \frac{|u|^2 t}{2\lambda}\right\} \\ &+ \nabla \cdot \Re\left\{-\nabla u\bar{m}_p + \frac{2x}{\lambda}\ell_S(u) - |u|^2\left(\frac{(n-1)x}{2\lambda^3} + \frac{3xt^2}{2\lambda^5}\right)\right\} \\ &+ \frac{|2t\nabla u + ixu|^2}{2\lambda^3} + G(u)\left(\frac{n-1}{\lambda} + \frac{t^2}{\lambda^3}\right) \\ &+ \frac{2r^2}{\lambda^3}|u_\theta|^2 - C\frac{|u|^2}{\lambda^3}.\end{aligned} \tag{5.22}$$

Integrating this inequality over $(1, \infty) \times \mathbb{R}^n$, we obtain the desired result as in the proof of Lemma 5.1. □

As was shown in [11] and Part I, $\|u/r\|_{L^2}$ is an important quantity to control the energy when $n \geq 3$. Although we can not have $u/r \in L^2(\mathbb{R}^n)$ for $n \leq 2$, we still have the following decay estimate for $\|u/\langle x\rangle\|_{L^2}$.

Lemma 5.3. *Let u be a global solution of (2.3) or (1.2) with $E_N(u) = E < \infty$. Then we have*

$$\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^n}\frac{|u|^2}{\langle x\rangle^2}dx\right\}^{(p+1)/2}\frac{dt}{\langle t\rangle} \leq C(E). \tag{5.23}$$

Proof. By Hölder's inequality, we have

$$\int \frac{|u|^2}{\langle x\rangle^2}dx \leq \left(\int |u|^{p+1}dx\right)^{2/(p+1)} \left(\int \langle x\rangle^{-2(p+1)/(p-1)}dx\right)^{1-2/(p+1)}. \tag{5.24}$$

Since $n \leq 2$, the second integral term in the right hand side is finite. From (5.1), (5.11) or (5.3), we have

$$\begin{aligned}\int_{\mathbb{R}}\left\{\int_{r<|t|/2}\frac{|u|^2}{\langle x\rangle^2}dx\right\}^{(p+1)/2}\frac{dt}{\langle t\rangle} &\leq C\iint_{r<|t|/2}\frac{|u|^{p+1}}{\langle t\rangle}dxdt \\ &\leq C\iint_{r<|t|/2}\frac{G(u)}{\langle t\rangle}dxdt \\ &\leq C(E),\end{aligned} \tag{5.25}$$

and

$$\int_{\mathbb{R}}\left\{\iint_{r>|t|/2}\frac{|u|^2}{\langle x\rangle^2}dx\right\}^{(p+1)/2}\frac{dt}{\langle t\rangle} \leq \int_{\mathbb{R}}\frac{C(E)}{\langle t\rangle^{2+p}}dt \leq C(E). \tag{5.26}$$

Thus we obtain the desired result. □

6. Weighted global estimate for ST-norms

In this section we will prove the following lemma, which is a reproduction of Lemma 5.2 of Part I for the present subject, but the situation is simpler because of the subcriticality. There is a similar argument in [2]. Here we use the generalized Morawetz estimates and the finite propagation property. We do not have the finite propagation property for NLS in the strict sense as in NLKG, but we have certain approximate finiteness of propagation (Lemma 6.2 below).

Lemma 6.1. *Let u be a global solution of (2.3) or (1.2) with $E_N(u) = E < \infty$. Let $0 = T_0 < T_1 < \cdots$, $I_j = (T_{j-1}, T_j)$, $0 < \eta \leq \eta_0(E)$ (η_0 is as in Lemma 3.1) and $\eta/2 \leq \|u\|_{(X;I_j)} \leq \eta$ for any j. Let S be the totality of the indices j, which may be finite or infinite. Then, there exist $t_j \in I_j$ for each $j \in S$ such that*

$$\sum_{j\in S}\frac{1}{(t_j+1)\log(t_j+2)} \leq C(E,\eta). \tag{6.1}$$

Lemma 6.2. *Let u be a global solution of NLS (1.2) with $E_N(u) = E < \infty$. Let B be a compact subset of $\mathbb{R}^n$. Then, for any $R > 0$ and $T > 0$, we have*

$$\int_{B(R)}|u(T,x)|^2dx \geq \int_B |u(0,x)|^2dx - C(E)T/R, \tag{6.2}$$

where $B(R) := \{\, x \in \mathbb{R}^n \mid {}^{\exists}y \in B \text{ s.t. } |x-y| \leq R \,\}$.

Proof. Define

$$d(x) := \inf_{y\in B}|x-y|. \tag{6.3}$$

Then, $x \in B(R)$ if and only if $d(x) \le R$, and we have $|\nabla d(x)| \le 1$. We define

$$\chi(x) = h(1 - d(x)/R). \tag{6.4}$$

Then we have

$$\chi(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in \mathbb{R}^n \setminus B(R), \end{cases} \tag{6.5}$$

and $|\nabla \chi| \le C/R$. By the equation (1.2), we have

$$\begin{aligned} \partial_t \|\chi u\|_{L_x^2}^2 &= 2\Re(\chi u, \chi(-i\Delta u + if(u))) \\ &= 4\Im(\chi(\nabla\chi)u, \nabla u) \ge -\frac{C}{R}E, \end{aligned} \tag{6.6}$$

where $(\cdot,\cdot)$ denotes the inner-product in $L^2(\mathbb{R}^n)$. From this we obtain the desired result. □

Proof of Lemma 6.1. By Lemma 4.1, for each $j \in S$, there exist $J_j \subset I_j$, $c_j \in \mathbb{R}^n$ and $R > 0$ such that $|J_j| \ge C(E,\eta)$, $R \le C(E,\eta)$ and

$$\int_{|x-c_j|<R} \min(|u|^2, G(u(t)))dx \ge C(E,\eta), \tag{6.7}$$

for any $t \in J_j$. Let $t_j = \inf J_j$. Now, in order to use the finite propagation property, we consider the following proposition for $j, k \in S$:

$$|c_j - c_k| \le M|t_j - t_k| + 2R, \tag{6.8}$$

$$|c_j - c_k| > M|t_j - t_k| + 2R, \tag{6.9}$$

where $M = 1$ in the NLKG case, while in the NLS case $M = M(E,\eta)$ should be taken so large that $C_1/2 \ge C_2/M$, where $C_2 = C_2(E)$ is the constant in (6.2) and $C_1 = C_1(E,\eta)$ is the constant in (6.7). Let $p_1 := 1$, and define p_{a+1} for $a = 1, 2, \dots$, inductively as the minimal $k \in S$ satisfying (6.9) for $j = p_1, \dots, p_a$. Denote $P = \{p_1, p_2, \dots\}$. For $j \in P$, denote $A_j := \{\, k \in S \mid k \ge j \text{ and (6.8) holds} \,\}$. By the definition of P, we have $S = \bigcup_{j \in P} A_j$. Using the generalized Morawetz estimates (Lemmas 5.1 and 5.2) and (6.7), we have

$$\begin{aligned} &\#PC(E,\eta) \\ &\ge \sum_{j\in P} \iint_{|x-c_j| \le M|t-t_j|+3R} \frac{\min(|u|^2, G(u))}{C(M,R)(|t-t_j|+1)\log(|t-t_j|+2)} dxdt \\ &\ge \sum_{j\in P}\sum_{k\in A_j} \int_{J_k} \frac{C(E,\eta)}{(t+1)\log(t+2)} dt \\ &\ge \sum_{k\in S} \frac{C(E,\eta)}{(t_k+1)\log(t_k+2)}. \end{aligned} \tag{6.10}$$

So, the desired result follows if we can estimate as $\#P \le C(E,\eta)$. Let $k \in P$ and $P_k = \{j \in P \mid j \le k\}$. In the NLKG case, by the finite propagation property and the definition of P, we have

$$\begin{aligned} E &\ge \int_{\bigcup_{j\in P} B(c_j, R+|t_j-t_k|)} e_N(u;t_k)dx \\ &\ge \sum_{j\in P_k} \int_{B(c_j,R)} e_N(u;t_j)dx \ge \#P_k C(E,\eta), \end{aligned} \tag{6.11}$$

where $B(c,r) = \{\, x \in \mathbb{R}^n \mid |x-c| < r\,\}$. So $\#P$ is bounded. In the NLS case, using Lemma 6.2 step by step, we obtain

$$\begin{aligned} E &\ge \int_{\bigcup_{j\in P} B(c_j, R+M|t_j-t_k|)} |u(t_k)|^2 dx \\ &\ge \sum_{j\in P_k} \int_{B(c_j,R)} |u(t_j)|^2 dx - \#P_k C_2/M \ge \#P_k C_1/2, \end{aligned} \tag{6.12}$$

so that $\#P$ is bounded. □

7. Space-time localized energy

In this section, we show that if the ST-norm is sufficiently large, there exists a very long interval with small ST-norm, in which somewhere a certain amount of energy is localized. The length of the interval is much larger than the spatial extent of the localized energy, and the quantity of the ST-norm is smaller than that of the localized energy.

Lemma 7.1. *Let u be a global solution of (2.3) or (1.2) with $E_N(u) = E < \infty$. Let $\nu, \varepsilon > 0$ and $M < \infty$. There exists $\nu_1 = \nu_1(E) > 0$, $N = N(E,\nu,M,\varepsilon) < \infty$ with the following properties. If $\nu \le \nu_1$ and $\|u\|_{(X;I)} > N$ on some interval I, then there exist $(S,T) \subset I$, $c \in \mathbb{R}^n$ and $R \in (1,\infty)$ such that $|T-S| > MR$ and that for $t = S$ or $t = T$ we have*

$$\begin{aligned} \|u\|^2_{(X;(S,T))} + \|u\|^2_{(K;(S,T))} \le \nu^2 \le \int_{|x-c|<R} e_N(u;t)dx, \\ \left\| \frac{u(t)}{\langle x - c\rangle} \right\|_{L^2} < \varepsilon. \end{aligned} \tag{7.1}$$

Proof. We divide I into subintervals $I_j = (T_{j-1}, T_j)$ such that $\eta/2 \le \|u\|_{(X;I_j)} \le \eta := \eta_0(E)/2$ for any j. By Lemma 4.1, for any j, there exist $t_j \in I_j$, $R' < C(E)$ and $c_j \in \mathbb{R}^n$ such that

$$\int_{|x-c_j|<R'} |u(t_j)|^2\, dx > C_1(E). \tag{7.2}$$

We may assume $R' \geq 1$. Now we set $\nu_1(E) := \sqrt{C_1(E)}/2$. By the finite propagation property for NLKG and by Lemma 6.2 for NLS, there exists $1 \leq \lambda < C(E)$ such that for any t we have

$$\int_{|x-c_j|<R'+\lambda|t-t_j|} e_N(u;t)dx > \nu^2. \tag{7.3}$$

Now for each j, we divide I_j into subintervals J_k^j with $k \in P_j \subset \mathbb{Z}$, such that $J_k^j = (S_{k-1}^j, S_k^j)$, $S_0^j = t_j$, $\|u\|_{(X;J_k^j)} + \|u\|_{(K;J_k^j)} \leq \nu$ and $\#P_j < C(E,\nu)$. By Lemma 5.3, we have some $L < C(E,\varepsilon)$ such that we have some $T_k^j \in (S_{k-1}^j, S_{k-1}^j + L\langle S_{k-1}^j - t_j\rangle)$ for $k > 0$ and $T_k^j \in (S_k^j - L\langle S_k^j - t_j\rangle, S_k^j)$ for $k \leq 0$ satisfying

$$\left\| \frac{u(T_k^j)}{\langle x - c_j\rangle} \right\|_{L^2} < \varepsilon. \tag{7.4}$$

Now let $M' = M'(E, M, \varepsilon)$ be a large constant satisfying

$$\begin{aligned} M'\lambda - L &> M\lambda(L+1), \\ M'R' - L &> M(R' + \lambda(L+1)). \end{aligned} \tag{7.5}$$

Suppose that for any $k \in P_j$ we have

$$\begin{cases} |S_{k-1}^j - S_k^j| < M'(R' + \lambda|S_{k-1}^j - t_j|), & \text{if } k > 0, \\ |S_{k-1}^j - S_k^j| < M'(R' + \lambda|S_k^j - t_j|), & \text{if } k \leq 0. \end{cases} \tag{7.6}$$

Then we have for any $k \in P_j$,

$$|S_k^j - t_j| + R' \leq (2M'\lambda)^{|k|}R'. \tag{7.7}$$

Then, we have

$$|I_j| \leq (2M'\lambda)^{C(E,\nu)}R' < C(E,\nu,M,\varepsilon). \tag{7.8}$$

By Lemma 6.1, there exists $N = N(E,\nu,M,\varepsilon)$ such that if $\|u\|_{(X;I)} > N$ then for some j (7.8) does not hold. Thus, for this j, there exists some $k \in P_j$ such that (7.6) does not hold. Assume that $k > 0$. Then, by (7.5), we have

$$\begin{aligned} S_k^j - T_k^j &> S_k^j - S_{k-1}^j - L\langle S_{k-1}^j - t_j\rangle \\ &\geq M'(R' + \lambda|S_{k-1}^j - t_j|) - L\langle S_{k-1}^j - t_j\rangle \\ &\geq M(R' + \lambda(L+1)\langle S_{k-1}^j - t_j\rangle) \\ &\geq M(R' + \lambda|T_k^j - t_j|). \end{aligned} \tag{7.9}$$

Thus we obtain the desired result with $t = S := T_k^j$, $T := S_k^j$, $c := c_j$ and $R := R' + \lambda|T_k^j - t_j|$. In the case $k \leq 0$, the argument is similar. □

8. Separation of the localized energy

In this section, we show that we can separate the localized energy obtained in the previous section, if its spatial extent is sufficiently small relative to the length of the interval with the small ST-norm where the localization occurs. Remark that the absolute size of the spatial extent of the localized energy might be in fact very large. For brevity, we consider only the case $t = S$ in (7.1).

Lemma 8.1. *Let u be a global solution of (2.3) or (1.2) with $E_N(u) = E < \infty$. Assume that for some $\nu > 0$, $\varepsilon > 0$, $c \in \mathbb{R}^n$, $R > 1$, and $T > S > 0$, we have*

$$\|u\|^2_{(X;(S,T))} + \|u\|^2_{(K;(S,T))} \le \nu^2 \le \int_{|x-c|<R} e_N(u;S)dx, \tag{8.1}$$

and

$$\left\| \frac{u(S)}{\langle x-c \rangle} \right\|_{L^2} < \varepsilon. \tag{8.2}$$

We have some positive $\nu_2 = \nu_2(E)$ and $\varepsilon_0 = \varepsilon_0(E,\nu)$ such that if $\nu \le \nu_2$ and $\varepsilon \le \varepsilon_0(E,\nu)$, then there exists a solution v of the free equation satisfying

$$E_N(u-v;T) < E - \frac{\nu^2}{4}, \tag{8.3}$$

$$E_L(v;T) < 2\nu^2, \tag{8.4}$$

$$\|v\|_{(X;(T,\infty))} < C(E,\nu)\left(\frac{R}{|T-S|}\right)^{\alpha}, \tag{8.5}$$

where α is a positive constant dependent only on n and p.

Proof. By the finiteness of the energy, there exists some $c' \in \mathbb{R}^n$ with $d := |c-c'| < C(E,\nu)$ such that

$$\int_{|x-c'|<2} e_N(u;S)dx \le \nu^2/2. \tag{8.6}$$

Let v be the solution of the free equation satisfying

$$\begin{aligned} &v(S) = \chi_\Gamma u(S), \\ &\dot v(S) = \chi_\Gamma \dot u(S), \quad \text{in the NLKG case,} \end{aligned} \tag{8.7}$$

where $\chi_\Gamma = h(2 - |x-c'|/\Gamma)$ is a cut-off function (h is given in (2.4)), and $\Gamma \in (1, R+d)$ should be taken such that

$$\int \chi_\Gamma^2 e_N(u;S)dx = \nu^2. \tag{8.8}$$

Such a choice of Γ is possible, since for $\Gamma = 1$ we have

$$\int \chi_1^2 e_N(u;s)dx \le \int_{|x-c'|<2} e_N(u;S)dx \le \nu^2/2 \tag{8.9}$$

by (8.6), and for $\Gamma = R + d$ we have

$$\int \chi_{R+d}^2 e_N(u; s)dx \geq \int_{|x-c'|<R+d} e_N(u; S)dx \geq \nu^2, \tag{8.10}$$

by (8.1). Then we have

$$\|\nabla v(S)\|_{L^2}^2 \leq \|\chi_\Gamma \nabla u(S)\|_{L^2}^2 + C(E)(a + a^2), \tag{8.11}$$

where

$$\begin{aligned} a := \|u(S)\nabla\chi_\Gamma\|_{L^2} &\leq C \left\| \frac{u(S)}{\langle x - c' \rangle} \right\|_{L^2} \\ &\leq C(d) \left\| \frac{u(S)}{\langle x - c \rangle} \right\|_{L^2} < C(E, \nu)\varepsilon. \end{aligned} \tag{8.12}$$

So, we have

$$\begin{aligned} E_L(v; S) \leq E_N(v; S) &\leq \int \chi_\Gamma^2 e_N(u; S)dx + C(E, \nu)\varepsilon \\ &\leq \nu^2 + C(E, \nu)\varepsilon. \end{aligned} \tag{8.13}$$

So, taking $\varepsilon_0(E, \nu)$ sufficiently small, we have $E_L(v; S) < 2\nu^2$. Let $w := u - v$. In a similar way as above, we have

$$\|\nabla w(S)\|_{L^2}^2 \leq \|(1 - \chi_\Gamma)\nabla u(S)\|_{L^2}^2 + C(E)(a + a^2), \tag{8.14}$$

where a is the same as in (8.12). So, taking $\varepsilon_0 = \varepsilon_0(E, \nu)$ small again if necessary, we have

$$\begin{aligned} E_N(w; S) &\leq \int (1 - \chi_\Gamma)^2 e_N(u; S)dx + C(E, \nu)\varepsilon \\ &\leq \int (1 - \chi_\Gamma^2) e_N(u; S)dx + C(E, \nu)\varepsilon_0 \\ &\leq E - \nu^2 + C(E, \nu)\varepsilon_0 \\ &\leq E - \nu^2/2. \end{aligned} \tag{8.15}$$

In the NLKG case, by the decay property of the linear Klein-Gordon (see, e.g., [9]) and the support property of v, we have,

$$\begin{aligned} \|v(t)\|_{B^{-3}_{\infty,2}} &\leq C|t - S|^{-n/2} \left(\|v(S)\|_{B^{-1}_{1,2}} + \|\dot{v}(S)\|_{B^{-2}_{1,2}} \right) \\ &\leq C|t - S|^{-n/2} E_L(v)^{1/2} \Gamma^{n/2} \leq C \left(\frac{\Gamma}{|t - S|} \right)^{n/2} \nu, \end{aligned} \tag{8.16}$$

for any $t \in \mathbb{R}$. We obtain the same estimate for NLS in a similar way. By the interpolation inequalities and the Strichartz estimate, we obtain

$$\begin{aligned}
\|v\|_{(X;(T,\infty))} &\le C\|v\|^{1-\beta}_{L^q(T,\infty;B^{s_0}_{q,2})}\|v\|^{\beta}_{L^q(T,\infty;B^{s_1}_{q,2})} \\
&\le C\nu^{1-\beta}\|v\|^{\beta\gamma}_{(K;(T,\infty))}\|v\|^{\beta(1-\gamma)}_{L^\infty(T,\infty;B^{-3}_{\infty,2})} \\
&\le C\nu^{1-\beta+\beta\gamma}\left(\frac{\Gamma}{|T-S|}\right)^{n\beta(1-\gamma)/2}\nu^{\beta(1-\gamma)} \qquad (8.17)\\
&\le C(E,\nu)\left(\frac{R+1}{|T-S|}\right)^{n\beta(1-\gamma)/2}.
\end{aligned}$$

where $s_0 = \sigma_K\rho/q > 0$, $\gamma = \rho/q$, $s_1 = \sigma_K\gamma - 3(1-\gamma)$ and $\beta \in (0,1)$ should be chosen such that $(1-\beta)s_0 + \beta s_1 > 0$. In the NLKG case, we obtain from the Strichartz estimate in the same way as in the proof of Lemma 3.1,

$$\|\omega^{-1}\dot{w}\|_{(K;(S,T))} \le C(E). \qquad (8.18)$$

Then, by the energy identity, (3.2) and the duality between $\omega(K)$ and $(\bar{K})$, we have

$$\begin{aligned}
E_N(w;T) &= E_N(w;S) + \int_S^T 2\Re(\Box w + w + f(w), \dot{w})dt \\
&= E_N(w;S) + \int_S^T 2\Re(f(w) - f(u), \dot{w})dt \\
&\le E_N(w;S) + C\left(\|w\|^{p-1}_{(X;I)}\|w\|_{(K;I)} + \|u\|^{p-1}_{(X;I)}\|u\|_{(K;I)}\right) \qquad (8.19)\\
&\qquad \times \|\omega^{-1}\dot{w}\|_{(K;I)} \\
&\le E_N(w;S) + C(E)\nu^p,
\end{aligned}$$

where $I = (S,T)$ and $(\cdot,\cdot)$ denotes the inner-product in $L^2(\mathbb{R}^n)$. Since $p > 2$, if we set $\nu_2 = \nu_2(E)$ sufficiently small, then we have $C(E)\nu^p < \nu^2/4$ in the last member of (8.19) and we obtain the desired result. In the NLS case, similarly we have

$$\begin{aligned}
E_N(w;T) &= E_N(w;S) + \int_S^T 2\Re(i\dot{w} - \Delta w + f(w), \dot{w} + iw)dt \\
&= E_N(w;S) + \int_S^T 2\Re(f(w) - f(u), -i\Delta w + if(u) + iw)dt \\
&\le E_N(w;S) + C\left(\|w\|^{p-1}_{(X;I)}\|w\|_{(K;I)} + \|u\|^{p-1}_{(X;I)}\|u\|_{(K;I)}\right) \qquad (8.20)\\
&\qquad \times \left(\|w\|_{(K;I)} + \|f(u)\|_{L^\rho(I\times\mathbb{R}^n)}\right),
\end{aligned}$$

where $I = (S,T)$. By Hölder's inequality, the complex interpolation and the Sobolev embedding, we have

$$\begin{aligned}
\|f(u)\|_{L^\rho(I\times\mathbb{R}^n)} &\le C\|u\|^p_{L^{p\rho}(I\times\mathbb{R}^n)} \\
&\le C\|u\|_{(K;I)}\|u\|^{p-1}_{L^\infty(I;B^0_{\infty,2})} \le C(E). \qquad (8.21)
\end{aligned}$$

So we obtain $E_N(w;T) \leq E_N(w;S) + C(E)\nu^p$ and the desired result as in the NLKG case. □

9. Perturbation argument

In this section, we show that if we can separate the wave component corresponding to the localized energy as in the previous section, we can estimate the ST-norm of the solution by the ST-norm of the remaining component, provided the separated wave component has decayed sufficiently in the sense of ST-norms. The idea and the proof of the lemma below are essentially due to [3].

Lemma 9.1. *Let u be a global solution of (2.3) or (1.2) with $E_N(u) = E < \infty$. Let v be a global solution of the free equation with $E_L(v) \leq 2E$, and let w be the global solution of the same equation as u and with the same initial data as $u - v$ at $t = 0$. For any $L < \infty$, there exists $\kappa = \kappa(E, L) > 0$ such that if $\|w\|_{(X;(0,\infty))} < L$ and $\|v\|_{(X;(0,\infty))} < \kappa$, we have $\|u\|_{(X;(0,\infty))} < C(E, L)$.*

Proof. By (3.5), we have

$$\|v\|_{(Y;(0,\infty))} \leq C\kappa^{1-2/\rho}E^{2/\rho} =: \kappa'. \tag{9.1}$$

Let $\eta \in (0, \eta_0(E))$. There exist $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\|w\|_{(X;(T_j,T_{j+1}))} \leq \eta \quad \text{and} \quad N^{1/q}\eta \leq L. \tag{9.2}$$

Then, by Lemma 3.1 and (3.5), we have

$$\|w\|_{(Y;(T_j,T_{j+1}))} \leq C(E)\eta^{1-2/\rho} =: \eta'. \tag{9.3}$$

Let $\Gamma = u - v - w$. Then we have the integral equation

$$\Gamma(t) = \Gamma_j(t) + \int_{T_j}^{t} U(t-s)(f(w) - f(\Gamma + v + w))(s)ds, \tag{9.4}$$

where Γ_j is the solution of the free equation with the same initial data as Γ at $t = T_j$. By (3.7) and (3.3), we have for $I = (T_j, T)$ with $T > T_j$,

$$\begin{aligned}\|\Gamma\|_{(Y;I)} &\leq \|\Gamma_j\|_{(Y;I)} + C\|f(w) - f(\Gamma + v + w)\|_{(\tilde{Y};I)} \\ &\leq \|\Gamma_j\|_{(Y;I)} + C(\|\Gamma + v\|_{(Y;I)} + \|w\|_{(Y;I)})^{p-1}\|v + \Gamma\|_{(Y;I)}.\end{aligned} \tag{9.5}$$

Moreover, since

$$\Gamma_{j+1}(t) = \Gamma_j(t) + \int_{T_j}^{T_{j+1}} U(t-s)(f(w) - f(\Gamma + v + w))(s)ds, \tag{9.6}$$

we have

$$\begin{aligned}\|\Gamma_{j+1}\|_{(Y;(T_{j+1},\infty))} &\leq \|\Gamma_j\|_{(Y;(T_j,\infty))} \\ &\quad + C(\|\Gamma + v\|_{(Y;(T_j,T_{j+1}))} + \|w\|_{(Y;(T_j,T_{j+1}))})^{p-1}\|v + \Gamma\|_{(Y;(T_j,T_{j+1}))}.\end{aligned} \tag{9.7}$$

Denote $p_j := \|\Gamma_j\|_{(Y;(T_j,\infty))}$ and $q_j(T) := \|\Gamma\|_{(Y;(T_j,T))}$. Then $q_j(T)$ is continuous with respect to T and we have

$$p_0 = 0, \quad q_j(T_j) = 0, \tag{9.8}$$

$$q_j(T) \leq p_j + C_1(q_j(T) + \kappa' + \eta')^{p-1}(q_j(T) + \kappa'), \tag{9.9}$$

$$p_{j+1} \leq p_j + C_1(q_j(T_{j+1}) + \kappa' + \eta')^{p-1}(q_j(T_{j+1}) + \kappa'), \tag{9.10}$$

where C_1 is the constant in (9.5) and (9.7). Now we fix η so small that $C_1(3\eta')^{p-1} < 1/4$, and we set κ so small that we have $2^{N+1}\kappa' < \eta'$. If $\kappa' \leq q_j(T) \leq \eta'$, we have from (9.9),

$$q_j(T) \leq p_j + C_1(3\eta')^{p-1}(2q_j(T)) \leq p_j + \frac{q_j(T)}{2}, \tag{9.11}$$

so that $q_j(T) \leq 2p_j$. Now suppose $2p_j < \eta'$. Then, by the continuity of $q_j(T)$, we have $q_j(T) \leq \max(\kappa', 2p_j) < \eta'$ for any $T \leq T_{j+1}$. Then, from (9.10), we have

$$p_{j+1} \leq p_j + C_1(3\eta')^{p-1}(\max(\kappa', 2p_j) + \kappa') \leq \max(\kappa', 2p_j). \tag{9.12}$$

Thus we obtain $p_j \leq 2^j\kappa' < \eta'/2$ and $q_j(T_{j+1}) < \eta'$ for any $j \leq N$. Then, by the Sobolev embedding, we have

$$\|u\|_{(X;(0,\infty))} \leq C\|u\|_{(Y;(0,\infty))} \leq CN\eta' \leq C(E, L). \tag{9.13}$$

□

10. Global space-time integrability

To obtain the scattering result, it suffices to show that any finite energy solution has a finite global space-time norm. So, the following proposition is essentially the main result of Part II. The strategy for the proposition is inspired by [3].

Proposition 10.1. *Let u be a global solution of* (2.3) *or* (1.2) *with finite energy $E_N(u) = E < \infty$. Then we have*

$$\|u\|_{(X;\mathbb{R})} < C(E). \tag{10.1}$$

Proof. Here we use the induction argument on the size of $E_N(u)$ as in [3] and Part I. For small energy data, the desired estimate can be easily obtained directly by the Strichartz estimate as in Lemma 3.1. So, the proof will be finished if for any $E > 0$ we can derive (10.1) for any solution u with $E_N(u) \leq E$ from the hypothesis that we have (10.1) for any solution u with $E_N(u) \leq E - \delta$, where $\delta = \delta(E) > 0$ satisfies that

$$\inf_{0 \leq E \leq E'} \delta(E) > 0, \tag{10.2}$$

for any $E' > 0$. For (10.2), it suffices that δ is a positive continuous function of E. Now, assume the induction hypothesis with $\delta = \nu^2/4$, where $\nu = \nu(E) :=$

$\min(\nu_1(E), \nu_2(E), \sqrt{E}/2)$ is given by Lemmas 7.1 and 8.1. Suppose that u is a solution satisfying $E_N(u) \leq E$ and $\|u\|_{(X;\mathbb{R})} \geq 3B$. We will show that there exists some bound $B_0 < C(E)$ for B (B_0 depends on the induction hypothesis). There exist $T_0 < T_1$ such that $\|u\|_{(X;(-\infty,T_0))} > B$, $\|u\|_{(X;(T_0,T_1))} > B$ and $\|u\|_{(X;(T_1,\infty))} > B$. By the induction hypothesis, we have $\|u\|_{(X;\mathbb{R})} < C_1(E)$ for any solution u with $E_N(u) \leq E - \delta$. Let $\kappa := \kappa(E, C_1(E))$ be given by Lemma 9.1. Then, there exists $M = M(E)$ such that the right hand side of (8.5) becomes smaller than κ if $|T - S| > MR$. Let $\varepsilon := \varepsilon_0(E, \nu(E))$ be given by Lemma 8.1. Now we can use Lemma 7.1 on the interval (T_0, T_1) if we assume $B > N(E, \nu(E), M(E), \varepsilon(E))$. Assume $t = S$ in (7.1). Then, by Lemma 8.1, we have a solution v of the free equation satisfying

$$E_L(v) < 2\nu^2 < E, \quad E_N(u - v; T) < E - \delta, \tag{10.3}$$

$$\|v\|_{(X;(T,\infty))} < \kappa. \tag{10.4}$$

Now we can use the induction hypothesis on the solution w of NLKG (or NLS) with the same initial data as $u - v$ at $t = T$. Then, by Lemma 9.1, we obtain $\|u\|_{(X;(T,\infty))} < C_2(E)$. Since $T \leq T_1$, we obtain $B \leq C_2(E)$. In the case $t = T$ in (7.1), we obtain similarly that $\|u\|_{(X;(-\infty,T_0))} < C_2(E)$, provided $B > N$. Thus, we have $\|u\|_{(X;\mathbb{R})} \leq B_0(E) := 3\max(N, C_2)$ for any solution u of (2.3) with $E_N(u) \leq E$, under the induction hypothesis. Thus we obtain the desired result.

□

11. Scattering

After we obtained the global space-time integrability (Proposition 10.1), it is easy to derive the scattering result (see, e.g., [4, 8]). So, we merely state the results.

Theorem 11.1. *Let $m > 0$, $n \leq 2$ and $p > 1 + 4/n$. Then, there exist homeomorphisms $W_\pm$ on $H^1 \oplus L^2$ with the following property. For any $(\varphi, \psi) \in H^1 \oplus L^2$, let v be the solution to*

$$\begin{cases} \Box v + m^2 v = 0, \\ (v(0), \dot{v}(0)) = (\varphi, \psi), \end{cases} \tag{11.1}$$

and let $u_\pm$ be the global solution to

$$\begin{cases} \Box u_\pm + m^2 u_\pm + |u_\pm|^{p-1} u_\pm = 0, \\ (u_\pm(0), \dot{u}_\pm(0)) = W_\pm(\varphi, \psi). \end{cases} \tag{11.2}$$

Then we have

$$\lim_{t \to \pm\infty} \|(v(t), \dot{v}(t)) - (u_\pm(t), \dot{u_\pm}(t))\|_{H^1 \oplus L^2} = 0. \tag{11.3}$$

Moreover, this property uniquely determines $W_\pm$. Thus the scattering operator $S = W_+^{-1} W_-$ is also a homeomorphism on $H^1 \oplus L^2$.

Theorem 11.2. *Let $n \le 2$ and $p > 1 + 4/n$. Then, there exist homeomorphisms $W_\pm$ on H^1 with the following property. For any $\varphi \in H^1$, let v be the solution to*

$$\begin{cases} i\dot{v} - \Delta v = 0, \\ v(0) = \varphi, \end{cases} \tag{11.4}$$

and let $u_\pm$ be the global solution to

$$\begin{cases} i\dot{u_\pm} - \Delta u_\pm + |u_\pm|^{p-1} u_\pm = 0, \\ u_\pm(0) = W_\pm \varphi. \end{cases} \tag{11.5}$$

Then we have

$$\lim_{t \to \pm\infty} \|v(t) - u_\pm(t)\|_{H^1} = 0. \tag{11.6}$$

Moreover, this property uniquely determines $W_\pm$. Thus the scattering operator $S = W_+^{-1} W_-$ is also a homeomorphism on H^1.

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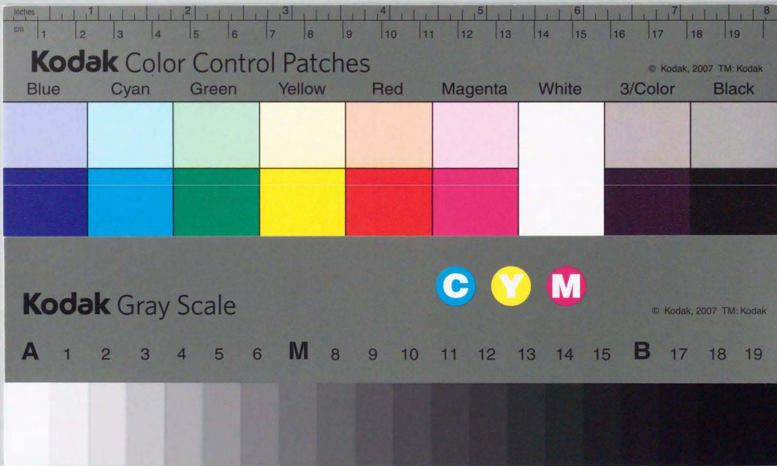

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