博 士 学 位 論 文

Construction of heterotic string field theory including the Ramond sector

（ Ramond セクターを含む ヘテロティック弦の場の理論の構成 ）

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1 Introduction

One of the most significant problems in theoretical physics is quantization of gravity. In quantum field theory, if one tries to quantize gravity theory, there appear nonrenormalizable divergences. String theory is a candidate for a consistent theory containing the quantum gravity: it is known that the first quantization theory of the string is free from ultraviolet divergences since ultraviolet divergences can be interpreted as infrared divergences. In addition to gravity, string theory contains rich physics, such as dualities and physics of branes. While they play central roles in modern theoretical physics, they are non-perturbative aspects of string theory, and it is not easy to deal with them in the first quantization theory of the string. It is because the first quantization theory of the string is defined as a perturbation theory around a fixed background. To develop the understanding of the non-perturbative aspects of string theory, it would be necessary to provide a non-perturbative formulation of string theory.

String field theory is a field theoretic approach for a non-perturbative formulation of string theory. As in ordinary quantum field theory, a guiding principle in string field theory is a gauge symmetry. Gauge symmetry of string field theory is expected to be closely related to reproductions of the first quantized theory of the string, and the consistency of string theory. Therefore, it will be important to construct an action of string field theory and to understand structures of string field theory. The structures of bosonic string field theories are well-understood, while those of superstring field theories, especially of closed superstring field theories, are less understood. In this thesis, we focus on the heterotic string, a type of the closed string with supersymmetry, and discuss the construction of an action of heterotic string field theory.

Bosonic strings field theories

Structures of bosonic string field theories are well-understood. The structure of open bosonic string field theory [1, 2] and closed bosonic string field theory [3–8] can be understood in a framework of A_∞-algebras [9–13] and L_∞-algebras [14, 15], respectively. In these theories, interactions of strings are described by string products satisfying algebraic relations called A_∞-relations for the open string and L_∞-relations for the closed string, and the gauge invariances follow from them. It is known that the tree-level scattering amplitudes correctly reproduce those in string theory [8, 16–19], which is provided by an integration over the moduli space of Riemann surfaces.

Open superstring field theories

The understanding of the structure of open superstring field theory is developing remarkably. The open superstring in the Ramond-Neveu-Schwarz formalism consists of the Neveu-Schwarz (NS) sector and the Ramond sector. Recently, the first construction of an action including the both sectors is provided [20]. Let us introduce various approaches towards the construction of a complete action of open superstring field theory.

A superstring field theory which is a simple extension of bosonic open string field theory [1] was proposed by Witten [21]. The string field in the Witten theory carries natural picture, -1 for the NS sector and $-\frac{1}{2}$ for the Ramond sector, and belongs to the small Hilbert space which is a Hilbert space we usually use in the description of the superconformal ghost sector in terms of the $\beta\gamma$ ghosts.

While this theory has a nonlinear gauge invariance including the Ramond sector algebraically, it is not consistent since it suffers from singularities which come from collisions of certain local operators [22].

The modification of [21] was provided by [23, 24] where the string field carries picture number 0 for the NS sector and $-\frac{1}{2}$ for the Ramond sector. In this modified theory, it is necessary to use an operator which decreases the picture number by 2, and there is a difficulty with the nontrivial kernel of the operator. This modified theory also suffers from similar singularities to those in the Witten theory: it is pointed out in [25] that finite gauge transformation is singular.

A consistent theory for the NS sector of the open superstring was provided by Berkovits [26] based on the large Hilbert space, the Hilbert space naturally defined in the description of the superconformal ghost sector in terms of ξ, η, and ϕ [27]. The Berkovits theory has non-polynomial interactions. Unlike in the bosonic theory, the A_∞-structure is not manifest in the Berkovits theory, which makes the quantization of the Berkovits theory difficult [28–32]. The reproduction of the perturbative S-matrix in superstring theory was checked for four- and five-point amplitudes [33–35], but have not been considered to all orders. In [35], a consistent theory for the NS open string field which belongs to the small Hilbert space and carries natural picture was provided by partial gauge fixing of the Berkovits theory. This partially-gauge-fixed theory does not have manifest A_∞-structure, as in the Berkovits theory.

The equations of motions including the Ramond sector which are based on the large Hilbert space were constructed in [36].[1] These equations of motions can be obtained by imposing the constraint on the equations of motions derived from a "pseudo action" which is written using two Ramond string fields [37]. The Feynman rules which lead to the reproduction of the perturbative S-matrix in superstring theory with four external legs were proposed in [37]. The Feynman rules were modified in [38] so that the perturbative S-matrix in superstring theory with five external legs are reproduced.

A construction of actions for the NS open string with the A_∞-structure was given by Erler, Konopka, and Sachs in [39]. They provided a systematic construction of the string product satisfying the A_∞-relations and some other suitable properties. Utilizing the string products, an action was constructed in the same form as that in the bosonic theory. The same technique is used for the construction of an equation of motion including the Ramond sector [40]. It was shown based on the equation of motion with the A_∞-structure that this A_∞-theory including the Ramond sector correctly reproduces the perturbative S-matrix in superstring theory at tree level [41]. Also, it was shown in [42] that the A_∞-theory is related to the Berkovits theory by partial gauge fixing and field redefinition. The relations between the structures of the A_∞-theory and the Berkovits theory are well-investigated [43, 44].

Recently, the first construction of a complete action of open superstring field theory, including both the NS sector and the Ramond sector, was provided by Kunitomo and Okawa [20]. It is known that one can write an appropriate kinetic term for the Ramond string field which is restricted to certain subspace of the small Hilbert space [45–50]. In [20], starting with the Berkovits action, the complete action which contains the full interaction including the Ramond string field was constructed. In [51], the relation of the equation of motion in [20] and that in A_∞-formulation [40] is discussed.

[1] Also in [36], the action of open superstring field theory which can preserve d=8 Lorentz invariance or N=1 d=4 super Poincare invariance is provided.

Heterotic string field theories

The structure of heterotic string field theory is less understood than that in boson theories and open superstring field theory. The heterotic string in the Ramond-Neveu-Schwarz formalism consists of the NS sector and the Ramond sector, as in the open superstring. The difficulty stems from the complexity of the string products defining bosonic theories: while the string products in [1] consist of only 1- and 2-string products, those in [8] consist of an infinite number of string products. Let us introduce the previous works on heterotic string field theory.

In [52], the structure of the Berkovits action was represented in terms of functionals of the string field with a certain algebraic property, which we will call the WZW-like structure. This WZW-like structure was generalized to the NS sector of the heterotic string, and an action of NS heterotic string field theory was provided in [52], which correctly reproduces the partial construction [53]. We call the formulation based on the WZW-like structure the WZW-like formulation. As in the Berkovits theory, the quantization of the WZW-like theory for the heterotic string seems to be difficult, and the reproduction of the perturbative S-matrix in superstring theory was checked only for four-point amplitudes [54]. An action of NS-NS closed string field theory was also constructed in the WZW-like formulation [55, 56]. [2)]

The equations of motions including the Ramond sector were constructed in [57, 58] based on the large Hilbert space. As in the open superstring, these equations of motion can be obtained by imposing the constraint on the equations of motions derived from a "pseudo action," which is constructed perturbatively in [57]. The Feynman rules which reproduce the perturbative S-matrix in superstring theory were discussed in [59].

The construction of action for the NS heterotic string with the L_∞-structure was given by Erler, Konopka, and Sachs [60] in almost the same manner as that in the A_∞-theory. A systematic construction of string products satisfying the L_∞-relations and some other suitable properties is provided by a natural extension of that of the A_∞-theory. Utilizing the string products, actions are written in the same form as those in bosonic theories. The same technique is used for the construction of an equation of motion including the Ramond sector [40]. It was shown based on the equations of motion with the L_∞-structure that this L_∞-theory including the Ramond sector correctly reproduces the perturbative S-matrix in superstring theory at tree level [41]. Also, an action for the NS-NS closed string and an equation of motion for type II closed string field theory were provided in [60] and [40], respectively. While the on-shell equivalence of the L_∞-theory and the WZW-like theory is discussed in [61], the relation between the actions in both formulations remains to be understood.

The present thesis

An explicit construction of a complete action of heterotic string field theory is one of the remaining tasks. In this thesis, as a first step toward a complete action, we explicitly construct an action of heterotic string field theory up to quartic order in the Ramond string field, which is all-order in the NS string field at each order in the Ramond string field.

[2)] The type II closed superstring in the Ramond-Neveu-Schwarz formalism consists of four sectors: the NS-NS sector, the NS-R sector, the R-NS sector, and the R-R sector.

The present thesis is organized as follows. In part I, we provide reviews of bosonic string field theories and the A_∞- and L_∞-algebras. The A_∞- and L_∞-algebras are closely related to the gauge symmetry of bosonic string field theories, and play important roles also for the construction of superstring field theories, which is the subject of part II and part III. Open bosonic string field theory and its algebraic structure called A_∞-algebras are reviewed In section 2. Closed bosonic string field theory and its algebraic structure called L_∞-algebras are reviewed in section 3. We also introduce string products defining bosonic string field theories, which are the constituents of superstring products introduced in part II.

In part II, we provide the action for the NS sector which we will use as a starting point of construction of an action including the Ramond sector. We first review two successful and popular formulations: the A_∞/L_∞-formulation in section 4, and the WZW-like formulation in section 5. Then in section 6, based on structures introduced in sections 4 and 5, we define the dual WZW-like formulation: utilizing string products which are dual to those in the A_∞/L_∞-formulation, an gauge invariant action is provided by almost the same procedure as WZW-like formulation. Section 6 is based on the original work [62], in collaboration with H. Matsunaga.

Part III is a main part of the present thesis. We provide a construction of an action of heterotic string field theory including the Ramond sector. In section 7, we briefly review the construction of complete action of open superstring field theory. In section 8, starting with the dual WZW-like action for the NS sector, we naturally extend it to the heterotic string, and provide an action of heterotic string field theory up to quadratic order in the Ramond string field. While the complete action of open superstring field theory is quadratic order in the Ramond string field, it is not the case for the heterotic string: interaction terms of higher order in the Ramond string field will be necessary. In section 9, we construct an action of heterotic string field theory at quartic order in the Ramond string field. Note that our action is all-order in the NS string field at each order in the Ramond string field. The results in sections 8 and 9 are new results, based on the collaboration work with H,Kunitomo. Finally, section 10 is devoted to a conclusion and discussions. Several appendices are provided to supply details.

Concluding with the introduction, we should mention to another appealing construction of the actions for heterotic and type II strings provided by Sen [63–65]. The kinetic term of the action is written using an extra string field, which decouples from the interacting part of the theory. This characteristic kinetic term is used in a construction of a covariant action for type IIB supergravity [66]. It will be important to understand the relation of this action and our action.

Part I
Bosonic string field theories and A_∞/L_∞-algebras

In part I, we introduce preliminaries on A_∞- and L_∞-algebras. They are closely related to the gauge symmetry of bosonic string field theories, and play important roles also for the construction of superstring field theories, which is the subject of part II and part III. We also introduce string products $\mathbf{M}^{\mathrm{B}}$ for open string, and $\mathbf{L}^{\mathrm{B}}$ for closed string, which define bosonic string field theories, and which are the constituents of superstring products introduced in part II.

2 Open bosonic string field theory and A_∞-algebra

In this section we review open bosonic string field theory and A_∞-algebras. In section 2.1, we review open bosonic string field theory construct by Witten [1], whose interaction is cubic and is described by the *star product*. The algebraic relations of the BRST operator and the star product, which guarantee the gauge invariance of the theory, can be understood through a more general framework called A_∞-algebras [2, 9–13]. We define A_∞-algebras in terms of coalgebras in section 2.2, and write the Witten's open bosonic string field theory in terms of the A_∞-algebras in section 2.3. We also provide open bosonic string field theory whose interaction is described by general cyclic A_∞-products, which we call "open string with stubs", in section 2.4.

2.1 Witten's open bosonic string field theory

The open string is described by a holomorphic sector, which consists of the matter sector and the reparameterization ghost sector in terms of $b(z)$ and $c(z)$, The string field Ψ of bosonic open string field theory is a Grassmann-odd state with ghost number 1. For a pair of string fields, we use the BPZ-inner product

$$\langle A, B\rangle = \langle A|B\rangle \tag{2.1}$$

which satisfies

$$\langle A, B\rangle = (-)^{\epsilon(A)\epsilon(B)}\langle B, A\rangle, \tag{2.2}$$

where $\epsilon(A)$ denotes the Grassmann parity of A. Because of the anomaly in the bc reparameterization ghost sector, the inner product $\langle A, B\rangle$ vanishes unless the sum of the ghost number of A and B equals to 3.

The kinetic term for Ψ is given by

$$S = -\frac{1}{2}\langle \Psi, Q\Psi\rangle, \tag{2.3}$$

where Q is the BRST operator which is nilpotent and BPZ-odd:

$$Q^2 = 0, \tag{2.4}$$

$$\langle QA, B \rangle = (-)^{\epsilon(A)+1} \langle A, QB \rangle. \tag{2.5}$$

Note that Q is Grassmann odd and $\epsilon(QA) = \epsilon(A) + 1$. The equation of motion and the gauge transformation of this kinetic term are given by

$$\text{E.O.M.}: \quad Q\Psi = 0, \qquad \text{Gauge transformation}: \quad \delta\Psi = Q\Lambda, \tag{2.6}$$

where Λ is a gauge parameter which is Grassmann even and carries ghost number 0. The equation of motion and a gauge transformation correspond to the physical state condition and a change of the choice of the representative of BRST cohomology in the first-quantization of strings.

The action of bosonic string field theory constructed by Witten [1] consists of the kinetic term (2.3) and the cubic interaction which is described by the *star product* $A * B$ [3] which is associative and cyclic:

$$(A * B) * C = A * (B * C), \tag{2.7}$$

$$\langle A * B, C \rangle = \langle A, B * C \rangle, \tag{2.8}$$

and on which Q acts as a derivation

$$Q(A * B) = (QA) * B + (-)^{\epsilon(A)} A * QB. \tag{2.9}$$

Note that $\epsilon(A * B) = \epsilon(A) + \epsilon(B)$. The interacting action is given by

$$S = -\frac{1}{2}\langle \Psi, Q\Psi \rangle - \frac{1}{3}\langle \Psi, \Psi * \Psi \rangle. \tag{2.10}$$

The variation of the action is taken as

$$\delta S = -\langle \delta\Psi, Q\Psi + \Psi * \Psi \rangle, \tag{2.11}$$

Then the equation of motion reads

$$Q\Psi + \Psi * \Psi = 0. \tag{2.12}$$

The action is invariant under the gauge transformation

$$\delta\Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi = Q\Lambda + [\![\Psi, \Lambda]\!]^*, \tag{2.13}$$

where we defined the graded commutator [4] using the star product as

$$[\![A, B]\!]^* = A * B - (-)^{\epsilon(A)\epsilon(B)} B * A. \tag{2.15}$$

[3] The star product of two open string fields is defined as a glueing of the right half of the first string and the left half of the second string. It can be represented using conformal maps. See appendix A.

[4] In this thesis we denote the graded commutator by $[\![,]\!]$: for operators A and B,

$$[\![A, B]\!] = AB - (-)^{AB} BA. \tag{2.14}$$

In the present case, to denote that this graded commutator is with respect to the star product, we write $*$ in a superscript.

The gauge invariance follows form the properties of Q and the star product.

The gauge symmetry of this theory can be fixed using the Batalin-Vilkovisky formalism [69, 70], which was done in [67, 68] [5)]. It is shown [16] the tree-level scattering amplitudes of this theory correctly reproduce those in the world-sheet theory, which are given by integrations over the moduli spaces of open Riemann surfaces.

2.2 A_∞-algebras

The properties which guarantees the gauge invariance of the Witten theory, namely (2.4), (2.5), (2.7), (2.8), and (2.9) can be understood as an A_∞-algebra [2, 9–13]. In this subsection we provide the definition of cyclic A_∞-algebras in terms of coalgebras. See also [72, 73], where the general properties of classical string field theory are discussed based on the A_∞-algebras. We also define shifted structures of the A_∞-products and introduce their important properties.

2.2.1 Coalgebra and multilinear maps

Tensor algebras as coalgebras

Let $\mathcal{C}$ be a set. When a coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is defined on $\mathcal{C}$ and it is coassociative

$$(\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta, \tag{2.16}$$

then $(\mathcal{C}, \Delta)$ is called a *coalgebra*. For open string, $\mathcal{C}$ corresponds to the tensor algebra $\mathcal{T}(\mathcal{H})$ of the $\mathbb{Z}_2$-graded vector space $\mathcal{H}$. In the language of open string field theory, $\mathcal{H}$ is the state space for the string field, and the $\mathbb{Z}_2$-grading, called degree, equals to the Grassmann parity plus one:

$$\deg(A) \equiv \epsilon(A) + 1 \mod \mathbb{Z}_2. \tag{2.17}$$

We can construct a *tensor algebra* $\mathcal{T}(\mathcal{H})$ by

$$\mathcal{T}(\mathcal{H}) = \mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots . \tag{2.18}$$

We can define a coassociative coproduct $\Delta : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{H})$ and a set $(\mathcal{T}(\mathcal{H}), \Delta)$ gives coalgebra. The action of Δ on $\Psi_1 \otimes ... \otimes \Psi_n \in \mathcal{H}^{\otimes n}$ is given by

$$\Delta(\Psi_1 \otimes ... \otimes \Psi_n) = \sum_{k=0}^{n} (\Psi_1 \otimes ... \otimes \Psi_k) \otimes (\Psi_{k+1} \otimes ... \otimes \Psi_n). \tag{2.19}$$

We can naturally define a *projector* $\pi_n : \mathcal{T}(\mathcal{H}) \to \mathcal{H}^{\otimes n}$ whose action on $\Psi \in \mathcal{T}(\mathcal{H})$ is given by

$$\pi_n \widehat{\Psi} = \widehat{\Psi}_n, \qquad \text{where } \widehat{\Psi} = \sum_{k=1}^{\infty} \widehat{\Psi}_k \in \mathcal{T}(\mathcal{H}), \quad \widehat{\Psi}_k \in \mathcal{H}^{\otimes k}. \tag{2.20}$$

[5)] See also a helpful review paper of the Batalin-Vilkovisky formalism [71]. The gauge fixing of the Witten's open bosonic string field theory is contained as an example also in this paper.

Let $\mathcal{H}_0$ be the degree zero part of $\mathcal{H}$. The following geometrical series of $\Psi \in \mathcal{H}_0$ gives a special element of $\mathcal{T}(\mathcal{H})$ called a *group-like element*:

$$\frac{1}{1-\Psi} = \mathbf{1} + \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi \otimes \Psi + \cdots \in \mathcal{T}(\mathcal{H}), \tag{2.21}$$

which satisfies

$$\Delta \frac{1}{1-\Psi} = \frac{1}{1-\Psi} \otimes \frac{1}{1-\Psi}. \tag{2.22}$$

Multi-linear maps

From a multilinear map $b_n : \mathcal{H}^n \to \mathcal{H}$, a map $b_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ is naturally defined by

$$b_n(\Psi_1 \otimes \Psi_2 \otimes ... \otimes \Psi_n) = b_n(\Psi_1, \Psi_2, ..., \Psi_n). \tag{2.23}$$

The tensor product of two multilinear maps $A : \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes l}$ and $B : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$, $A \otimes B : \mathcal{H}^{\otimes k+m} \to \mathcal{H}^{\otimes l+n}$ can also be defined naturally by

$$A \otimes B(\Psi_1 \otimes ... \otimes \Psi_{k+m}) = (-)^{B(\Psi_1+...+\Psi_k)} A(\Psi_1 \otimes ... \otimes \Psi_k) \otimes B(\Psi_{k+1} \otimes ... \otimes \Psi_{k+m}). \tag{2.24}$$

The identity operator on $\mathcal{H}^{\otimes n}$ is defined by

$$\mathbb{I}_n = \mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}. \tag{2.25}$$

Multilinear maps with degree 1 and 0 naturally induce the maps from $\mathcal{T}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$. They are called a coderivation and a cohomomorphism respectively, and are the main focus of the rest of this subsection.

Multi-linear maps as a coderivation

A linear operator $\mathbf{b} : \mathcal{C} \to \mathcal{C}$ with degree one is called *coderivation* if it satisfies

$$\Delta \mathbf{b} = (\mathbf{b} \otimes \mathbb{1})\Delta + (\mathbb{1} \otimes \mathbf{b})\Delta. \tag{2.26}$$

From a map $b_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ which carries the degree one, the coderivation $\mathbf{b}_n : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is naturally defined by

$$\mathbf{b}_n \widehat{\Psi}_N = \sum_{k=0}^{N-n} (\mathbb{I}_k \otimes b_n \otimes \mathbb{I}_{N-n-k}) \widehat{\Psi}_N, \qquad \widehat{\Psi}_N \in \mathcal{H}^{\otimes N \geq n} \subset \mathcal{T}(\mathcal{H}), \tag{2.27}$$

and $\mathbf{b}_n$ vanishes when acting on $\mathcal{H}^{\otimes N<n}$. We will call $\mathbf{b}_n$ a *n-coderivation* . The n-coderivation $\mathbf{b}_n$ satisfies

$$\pi_1 \mathbf{b}_n \widehat{\Psi} = \pi_1 \mathbf{b}_n \widehat{\Psi}_n, \qquad \text{where } \widehat{\Psi} = \sum_{k=1}^{\infty} \widehat{\Psi}_k \in \mathcal{T}(\mathcal{H}), \quad \widehat{\Psi}_k \in \mathcal{H}^{\otimes k}. \tag{2.28}$$

The explicit actions of the one-coderivation $\mathbf{b}_1$ and the two-coderivation $\mathbf{b}_2$ is given by

$$\begin{array}{llcl} \mathbf{b}_1 : & 1 & \to & 0 \\ & \Psi_1 & \to & b_1(\Psi_1) \\ & \Psi_1 \otimes \Psi_2 & \to & b_1(\Psi_1) \otimes \Psi_2 + (-)^{\deg(\Psi_1)\deg(b_1)} \Psi_1 \otimes b_1(\Psi_2) \\ & \Psi_1 \otimes \Psi_2 \otimes \Psi_3 & \to & b_1(\Psi_1) \otimes \Psi_2 \otimes \Psi_3 + (-)^{\deg(\Psi_1)\deg(b_1)} \Psi_1 \otimes b_1(\Psi_2) \otimes \Psi_3 \\ & & & \quad +(-)^{(\deg(\Psi_1)+\deg(\Psi_2))\deg(b_1)} \Psi_1 \otimes \Psi_2 \otimes b_1(\Psi_3), \end{array} \tag{2.29}$$

$$\begin{array}{llll}\mathbf{b}_2: & 1 & \to & 0 \\ & \Psi_1 & \to & 0 \\ & \Psi_1\otimes\Psi_2 & \to & b_2(\Psi_1,\Psi_2) \\ & \Psi_1\otimes\Psi_2\otimes\Psi_3 & \to & b_2(\Psi_1,\Psi_2)\otimes\Psi_3+(-)^{\deg(\Psi_1)\deg(b_2)}\Psi_1\otimes b_2(\Psi_2,\Psi_3).\end{array} \tag{2.30}$$

Given two coderivations $\mathbf{b}_n$ and $\mathbf{c}_m$ which are derived from $b_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ and $c_m : \mathcal{H}^{\otimes m} \to \mathcal{H}$ respectively, the graded commutator $[\![\mathbf{b}_n, \mathbf{c}_m]\!]$ becomes the coderivation derived from the map $[\![b_n, c_m]\!] : \mathcal{H}^{\otimes n+m-1} \to \mathcal{H}$ which is defined by

$$[\![b_n, c_m]\!] = \sum_{k=0}^{n-1} b_n(\mathbb{I}_k \otimes c_m \otimes \mathbb{I}_{n-k-1}) - (-)^{\deg(b_n)\deg(c_m)} \sum_{k=0}^{m-1} c_m(\mathbb{I}_k \otimes b_n \otimes \mathbb{I}_{m-k-1}) \tag{2.31}$$

The action of the coderivation on the group-like element is given by

$$\mathbf{b}_n \frac{1}{1-\Psi} = \frac{1}{1-\Psi} \otimes \Big(\pi_1 \mathbf{b}_n \frac{1}{1-\Psi}\Big) \otimes \frac{1}{1-\Psi} = \frac{1}{1-\Psi} \otimes b_n(\Psi^{\otimes n}) \otimes \frac{1}{1-\Psi}. \tag{2.32}$$

Multilinear maps as a cohomomorphism

Given two coalgebras C, C', a *cohomomorphism* $\mathsf{f} : C \to C'$ is a map of degree zero satisfying

$$\Delta\mathsf{f} = (\mathsf{f}\otimes\mathsf{f})\Delta. \tag{2.33}$$

A set of degree zero multilinear maps $\{\mathsf{f}_n : \mathcal{H}^{\otimes n} \to \mathcal{H}'\}_{n=0}^{\infty}$ naturally induces a cohomomorphism $\mathsf{f} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}')$, which we denote as $\mathsf{f} = \{\mathsf{f}_n\}_{n=0}^{\infty}$. Its action on $\Psi_1 \otimes \cdots \otimes \Psi_n \in \mathcal{H}^{\otimes n} \subset \mathcal{T}(\mathcal{H})$ is defined by

$$\begin{aligned}\mathsf{f}(\Psi_1\otimes\cdots\otimes\Psi_n) = \sum_{\substack{i\le n \\ 1\le k_1<\cdots<k_i=n}} &\frac{1}{1-\mathsf{f}_0}\otimes \mathsf{f}_{k_1}(\Psi_1,\ldots,\Psi_{k_1})\otimes\frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_{k_2-k_1}(\Psi_{k_1+1},\ldots,\Psi_{k_2})\otimes\frac{1}{1-\mathsf{f}_0}\otimes \\ &\cdots\otimes\frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_{k_i-k_{i-1}}(\Psi_{k_{i-1}+1},\ldots,\Psi_n)\otimes\frac{1}{1-\mathsf{f}_0}.\end{aligned} \tag{2.34}$$

Its explicit actions are given as follows:

$$\begin{array}{llll}\mathsf{f}: & 1 & \to & \frac{1}{1-\mathsf{f}_0} \\ & \Psi & \to & \frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_1(\Psi)\otimes\frac{1}{1-\mathsf{f}_0} \\ & \Psi_1\otimes\Psi_2 & \to & \frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_1(\Psi_1)\otimes\frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_1(\Psi_2)\otimes\frac{1}{1-\mathsf{f}_0}+\frac{1}{1-\mathsf{f}_0}\otimes\mathsf{f}_2(\Psi_1,\Psi_2)\otimes\frac{1}{1-\mathsf{f}_0}.\end{array} \tag{2.35}$$

One of the important property of a cohomomorphism is its action on the group-like element:

$$\Delta\mathsf{f}\Big(\frac{1}{1-\Psi}\Big) = (\mathsf{f}\otimes\mathsf{f})\Delta\frac{1}{1-\Psi} = (\mathsf{f}\otimes\mathsf{f})\Big(\frac{1}{1-\Psi}\otimes\frac{1}{1-\Psi}\Big) = \Big(\mathsf{f}\frac{1}{1-\Psi}\Big)\otimes\Big(\mathsf{f}\frac{1}{1-\Psi}\Big). \tag{2.36}$$

We can see that the cohomomorphisms preserves the group-like element : $\mathsf{f}\frac{1}{1-\Psi} = \frac{1}{1-\Psi'}$. Then, $\Psi' = \pi_1\mathsf{f}\frac{1}{1-\Psi}$ and one can write

$$\mathsf{f}\frac{1}{1-\Psi} = \frac{1}{\pi_1\mathsf{f}\frac{1}{1-\Psi}}. \tag{2.37}$$

2.2.2 Cyclic A_∞-algebra

Cyclic A_∞-algebra $(\mathcal{H}, \mathbf{M}, \omega)$

Let $\mathcal{H}$ be a graded vector space and $\mathcal{T}(\mathcal{H})$ be its tensor algebra. A *weak A_∞-algebra* $(\mathcal{H}, \mathbf{M})$ is a coalgebra $\mathcal{T}(\mathcal{H})$ with a coderivation $\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 + \mathbf{M_2} + ...$ satisfying

$$(\mathbf{M})^2 = 0. \tag{2.38}$$

We denote the collection of the multilinear maps $\{M_k\}_{k\geq 0}$ also by $\mathbf{M}$. In particular, if $\mathbf{M}_0 = 0$, $(\mathcal{H}, \mathbf{M})$ is called an *A_∞-algebra.* In the case of an A_∞-algebra, the part of (2.38) that correspond to an n-fold multilinear map $\mathcal{H}^{\otimes n} \to \mathcal{H}$ is given by

$$\mathbf{M}_n \cdot \mathbf{M}_1 + \mathbf{M}_{n-1} \cdot \mathbf{M}_2 + \cdots + \mathbf{M}_2 \cdot \mathbf{M}_{n-1} + \mathbf{M}_1 \cdot \mathbf{M}_n = 0. \tag{2.39}$$

We can act it on $B_1 \otimes B_2 \otimes ... \otimes B_n \in \mathcal{H}^{\otimes n}$ to get the A_∞ relations for the multilinear maps $\{M_k\}$:

$$0 = \sum_{i+j=n+1} \sum_{k=0}^{n-i} M_j(B_1, ..., B_k, M_i(B_{k+1}, \dots, B_{k+i}), B_{k+i+1}, \dots, B_n). \tag{2.40}$$

We can define an inner product $\langle A, B\rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ satisfying the same property as (2.2). Given an operator $\mathcal{O}_n$, we can define its BPZ-conjugation $O_n^\dagger$ as follows [6]:

$$\langle B_1, \mathcal{O}_n(B_2, ..., B_{n+1})\rangle = (-)^{\deg(\mathcal{O}_n)(\deg(B_1)+1)+\deg(B_2)+...+\deg(B_n)} \langle \mathcal{O}_n^\dagger(B_1, ...B_n), B_{n+1}\rangle. \tag{2.43}$$

A set $(\mathcal{T}(\mathcal{H}), \mathbf{M}, \omega)$ is called *cyclic A_∞-algebra* if each M_n is BPZ-odd:

$$M_n^\dagger = -M_n. \tag{2.44}$$

The *Maurer-Cartan element* for an A_∞-algebra $(\mathcal{H}, \mathbf{M})$ is given by

$$\mathcal{F}(\Psi) := \pi_1 \mathbf{M} \frac{1}{1-\Psi} = M_1(\Psi) + M_2(\Psi, \Psi) + M_3(\Psi, \Psi, \Psi) + \cdots . \tag{2.45}$$

The *Maurer-Cartan equation* for an A_∞-algebra $(\mathcal{H}, \mathbf{M})$ is given by $\mathcal{F}(\Psi) = 0$, which correspond to the equation of motion in string field theory based on the A_∞-algebra$(\mathcal{H}, \mathbf{M})$.

A_∞-structure of the shifted product

We can define the *shifted structure* of A_∞-products $\mathbf{M}$, shifted by some field $\mathcal{G}$ with even degree, by

$$M_{n,\mathcal{G}}(B_1, ..., B_n) = \pi_1 \mathbf{M}\big(\frac{1}{1-\mathcal{G}} \otimes B_1 \otimes \frac{1}{1-\mathcal{G}} \otimes ... \otimes \frac{1}{1-\mathcal{G}} \otimes B_n \otimes \frac{1}{1-\mathcal{G}}\big). \tag{2.46}$$

[6] We can define an inner product $\langle A, B\rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ using the graded symplectic form $\langle \omega| : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ satisfying $\langle \omega|A \otimes B = (-)^{\deg(A)\deg(B)+1} \langle \omega|B \otimes A$, by:

$$\langle A, B\rangle = (-)^{\deg(A)} \langle \omega|A \otimes B. \tag{2.41}$$

Then, BPZ-conjugation $O_n^\dagger$ of the operator $\mathcal{O}_n$ is defined by

$$\langle \omega|\mathbb{I} \otimes \mathcal{O}_n = \langle \omega|\mathcal{O}_n^\dagger \otimes \mathbb{I}. \tag{2.42}$$

The BPZ properties of the shifted products follow from the original products $\mathbf{M}$: if $\mathbf{M}$ is BPZ-odd, $M_{n,\mathcal{G}}$ is also cyclic:

$$\langle B_1, M_{n,\mathcal{G}}(B_2, ..., B_{n+1})\rangle = (-)^{\deg(B_1)+\deg(B_2)+...+\deg(B_n)}\langle M_{n,\mathcal{G}}(B_1, B_2, ..., B_n), B_{n+1}\rangle. \tag{2.47}$$

One important property of the shifted structure is that the products $\mathbf{M}$ shifted by $\mathcal{G}$ (2.46) satisfy the A_∞-relations

$$0 = \sum_{i=1}^{n}\sum_{k=0}^{n-i} M_{n-i+1,\mathcal{G}}(B_1, ..., B_k, M_{i,\mathcal{G}}(B_{k+1}, ..., B_{k+i}), B_{k+i+1}, ..., B_n) \tag{2.48}$$

if $\mathcal{G}$ is a solution for the Maurer-Cartan equation for $\mathbf{M}$:

$$\mathcal{F}(\mathcal{G}) = \pi_1\mathbf{M}\frac{1}{1-\mathcal{G}} = 0. \tag{2.49}$$

To show (2.48), let us consider the action of coderivation $\mathbf{M}$ on $\frac{1}{1-\mathcal{G}}\otimes B_1\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes\frac{1}{1-\mathcal{G}}\otimes B_n\otimes\frac{1}{1-\mathcal{G}}$:

$$\begin{aligned}
&\mathbf{M}\big(\frac{1}{1-\mathcal{G}}\otimes B_1\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes\frac{1}{1-\mathcal{G}}\otimes B_n\otimes\frac{1}{1-\mathcal{G}}\big)\\
&=\sum_{k=0}^{n}\frac{1}{1-\mathcal{G}}\otimes B_1\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes B_k\otimes\frac{1}{1-\mathcal{G}}\otimes\big(\pi_1\mathbf{M}\frac{1}{1-\mathcal{G}}\big)\otimes\frac{1}{1-\mathcal{G}}\otimes B_{k+1}\otimes...\otimes B_n\otimes\frac{1}{1-\mathcal{G}}\\
&+\sum_{i=1}^{n}\sum_{k=0}^{n-i}\frac{1}{1-\mathcal{G}}\otimes B_1\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes\frac{1}{1-\mathcal{G}}\otimes\Big(\pi_1\mathbf{M}\big(\frac{1}{1-\mathcal{G}}\otimes B_{k+1}\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes B_{k+i}\otimes\frac{1}{1-\mathcal{G}}\big)\Big)\\
&\qquad\qquad\otimes\frac{1}{1-\mathcal{G}}\otimes...\otimes\frac{1}{1-\mathcal{G}}\otimes B_n\otimes\frac{1}{1-\mathcal{G}}.
\end{aligned}\tag{2.50}$$

Acting $\pi_1\mathbf{M}$ on both sides and using $\mathbf{M}^2 = 0$, one can obtain

$$\begin{aligned}
0 &= \sum_{k=0}^{n} M_{n+1,\mathcal{G}}(B_1, ..., B_k, \mathcal{F}(\mathcal{G}), B_{k+1}, ...B_n)\\
&\quad+\sum_{i=1}^{n}\sum_{k=0}^{n-i} M_{n-i+1,\mathcal{G}}(B_1, ..., B_k, M_{i,\mathcal{G}}(B_{k+1}, ..., B_{k+i}), B_{k+i+1}, ..., B_n).
\end{aligned}\tag{2.51}$$

Thus, it is shown from (2.51) that the shifted products (2.46) satisfy the A_∞-relations (2.48) if $\mathcal{G}$ is a solution for the Maurer-Cartan equation (2.49).

Another important property is that the $\mathcal{G}$-shifted 1-product $M_{1,\mathcal{G}}$ annihilates the Maurer-Cartan element $\mathcal{F}(\mathcal{G})$ for arbitrary $\mathcal{G}$ with even degree. It directly follows from $\mathbf{M}^2 = 0$:

$$M_{1,\mathcal{G}}(\mathcal{F}(\mathcal{G})) = \pi_1\mathbf{M}\Big(\frac{1}{1-\mathcal{G}}\otimes\big(\pi_1\mathbf{M}\frac{1}{1-\mathcal{G}}\big)\otimes\frac{1}{1-\mathcal{G}}\Big) = \pi_1\mathbf{M}\mathbf{M}\big(\frac{1}{1-\mathcal{G}}\big) = 0. \tag{2.52}$$

Generally, the gauge transformation is generated by the operator which annihilates the equation of motion. Since the equation of motion is given by the Maurer-Cartan element, $M_{1,\mathcal{G}}$ generate the gauge transformation.

2.3 Coalgebraic representation

In this subsection we see that the Witten theory can be written in terms of the A_∞-algebra. We define the degree of the string fields as Grassmann parity plus one:

$$\deg(A) \equiv \epsilon(A) + 1 \mod \mathbb{Z}_2. \tag{2.53}$$

For instance, Ψ is degree even. We define 2-string product m_2 by

$$m_2(A, B) = (-)^{\deg(A)} A * B. \tag{2.54}$$

Q and m_2 carries degree 1: since $\epsilon(QA) = \epsilon(A) + 1$ and $\epsilon(A * B) = \epsilon(A) + \epsilon(B)$,

$$\deg(QA) = \deg(A) + 1, \qquad \deg(m_2(A, B)) = \deg(A) + \deg(B) + 1. \tag{2.55}$$

In terms of the degree, the symmetric properties of the inner product and the cyclicity of Q and m_2 are written as

$$\langle A, B\rangle = (-)^{(\deg(A)+1)(\deg(B)+1)}\langle B, A\rangle, \tag{2.56}$$

$$\langle QA, B\rangle = (-)^{\deg(A)}\langle A, QB\rangle, \tag{2.57}$$

$$\langle A, m_2(B, C)\rangle = (-)^{\deg(A)+\deg(B)}\langle m_2(A, B), C\rangle. \tag{2.58}$$

The properties of Q and the star product (2.4), (2.9), and (2.7) can be written using m_2 as

$$0 = Q^2, \tag{2.59}$$

$$0 = Qm_2(A, B) + m_2(QA, B) + (-)^{\deg(A)} m_2(A, QB), \tag{2.60}$$

$$0 = m_2(m_2(A, B), C) + (-)^{\deg(A)} m_2(A, m_2(B, C)). \tag{2.61}$$

They are the A_∞-relations (2.40) with the products $M_1 = Q, M_2 = m_2, M_{n\geq 3} = 0$. Introducing a one-coderivation $\mathbf{Q}$ derived from Q, and a two-coderivation $\mathbf{m}_2$ derived from m_2, they can be written as

$$0 = \mathbf{Q}^2, \tag{2.62}$$

$$0 = \mathbf{Q}\mathbf{m}_2 + \mathbf{m}_2\mathbf{Q}, \tag{2.63}$$

$$0 = \mathbf{m}_2\mathbf{m}_2. \tag{2.64}$$

Finally we define

$$\mathbf{M}^{\mathrm{B}} = \mathbf{Q} + \mathbf{m}_2, \tag{2.65}$$

and they can be summarized into the nilpotency of $\mathbf{M}^{\mathrm{B}}$:

$$(\mathbf{M}^{\mathrm{B}})^2 = 0. \tag{2.66}$$

The action in terms of m_2 is written as

$$S = -\frac{1}{2}\langle\Psi, Q\Psi\rangle + \frac{1}{3}\langle\Psi, m_2(\Psi, \Psi)\rangle, \tag{2.67}$$

and its variation is taken as

$$\delta S = -\langle \delta\Psi, \mathcal{F}^{\mathrm{B}}(\Psi)\rangle, \tag{2.68}$$

where $\mathcal{F}^{\mathrm{B}}(\Psi)$ is the Maurer-Cartan element defined by

$$\mathcal{F}^{\mathrm{B}}(\Psi) = \pi_1 \mathbf{M}^{\mathrm{B}} \frac{1}{1-\Psi} = Q\Psi + m_2(\Psi,\Psi). \tag{2.69}$$

The equation of motion correspond with the Maurer-Cartan equation

$$\mathcal{F}^{\mathrm{B}}(\Psi) = 0, \tag{2.70}$$

We define the shifted BRST operator Q_Ψ by

$$Q_\Psi B = M^{\mathrm{B}}_{1,\Psi}(B) = Q\Lambda + m_2(\Psi,\Lambda) + m_2(\Lambda,\Psi) = Q\Lambda + [\![\Psi, B]\!]^*. \tag{2.71}$$

Note that

$$\begin{aligned} m_2(A,B) + (-)^{\deg(A)\deg(B)} m_2(B,A) &= (-)^{\deg(A)} A * B + (-)^{\deg(A)\deg(B)+\deg(B)} B * A \\ &= (-)^{\epsilon(A)+1}\big(A * B - (-)^{\epsilon(A)\epsilon(B)} B * A\big) \\ &= (-)^{\epsilon(A)+1} [\![A, B]\!]^*. \end{aligned} \tag{2.72}$$

Since $(\mathbf{M}^{\mathrm{B}})^2 = 0$, this shifted BRST operator Q_Ψ annihilate the Maurer-Cartan element $\mathcal{F}^{\mathrm{B}}(\Psi)$:

$$Q_\Psi \mathcal{F}^{\mathrm{B}}(\Psi) = 0, \tag{2.73}$$

and Q_Ψ generates the gauge transformation

$$\delta\Psi = Q_\Psi(\Lambda). \tag{2.74}$$

To write the action in terms of $\mathbf{M}^{\mathrm{B}}$, we introduce a parameter $t \in [0,1]$ and t-parameterized field $\Psi(t)$ satisfying $\Psi(0) = 0$ and $\Psi(1) = \Psi$, which is a path connecting 0 and the string field Ψ in the space of string fields. Using this $\Psi(t)$, the action can be written as

$$\begin{aligned} S &= -\langle \Psi, \frac{1}{2} Q\Psi + \frac{1}{3} m_2(\Psi,\Psi)\rangle \\ &= -\int_0^1 dt\, \partial_t \langle \Psi(t), \frac{1}{2} Q\Psi(t) + \frac{1}{3} m_2(\Psi(t),\Psi(t))\rangle \\ &= -\int_0^1 dt\, \langle \partial_t\Psi(t), Q\Psi(t) + m_2(\Psi(t),\Psi(t))\rangle \\ &= -\int_0^1 dt\, \langle \partial_t\Psi(t), \pi_1 \mathbf{M}^{\mathrm{B}} \frac{1}{1-\Psi(t)}\rangle. \end{aligned} \tag{2.75}$$

In the third line we act ∂_t and use the cyclicity of Q and m_2 to move $\partial_t\Psi(t)$ to the first slot of the inner product. Note that t-dependence is topological since the equation of motion does not depend on t.

2.4 Open bosonic string field theory with stubs

We can construct a gauge-invariant action of open bosonic string field theory in terms of more general cyclic A_∞-products [2]. Let us consider A_∞-products $\{M_k^{\rm B}\}_{k\geq 1}$ where $M_1^{\rm B}=Q$ and $M_n^{\rm B}$ carries ghost number $2-n$, and they are cyclic

$$\langle B_1, M_n(B_2,...,B_n,B_{n+1})\rangle = (-)^{\deg(B_1)+...+\deg(B_n)}\langle M_n(B_1,B_2,...,B_n),B_{n+1}\rangle. \tag{2.76}$$

We denote a coderivation derived from $M_k^{\rm B}$ by $\mathbf{M}_k^{\rm B}$, and define $\mathbf{M}^{\rm B}$ by

$$\mathbf{M}^{\rm B} = \sum_{n=1}^{\infty}\mathbf{M}_n^{\rm B} = \mathbf{M}_1^{\rm B}+\mathbf{M}_2^{\rm B}+\mathbf{M}_3^{\rm B}+\cdots. \tag{2.77}$$

The A_∞-relations can be written by

$$(\mathbf{M}^{\rm B})^2 = 0. \tag{2.78}$$

Some lowest order of the A_∞-relations read

$$0 = Q^2, \tag{2.79}$$
$$0 = QM_2^{\rm B}(A,B)+M_2^{\rm B}(QA,B)+(-)^{\deg(A)}M_2^{\rm B}(A,QB), \tag{2.80}$$
$$\begin{aligned}0 &= QM_3^{\rm B}(A,B,C)+M_2^{\rm B}(QA,B,C)+(-)^{\deg(A)}M_2^{\rm B}(A,QB,C)+(-)^{\deg(A)+\deg(B)}M_2^{\rm B}(A,B,QC)\\ &\quad + M_2^{\rm B}(M_2^{\rm B}(A,B),C)+(-)^{\deg(A)}M_2^{\rm B}(A,M_2^{\rm B}(B,C)).\end{aligned} \tag{2.81}$$

Note that $\mathbf{M}^{\rm B}=\mathbf{Q}+\mathbf{m}_2$ is the special case of this general class of products. A typical realization of $M_2^{\rm B}$ is the open string star product with "stubs" attached to each of the inputs and the output:

$$M_2^{\rm B}(\Psi_1,\Psi_2) = (-)^{\deg(A)}e^{-L_0}\Big((e^{-L_0}\Psi_1)*(e^{-L_0}\Psi_2)\Big). \tag{2.82}$$

The Feynman diagrams with this product and propagators do not cover the whole moduli space. A set of products $\{M_k^{\rm B}\}_{k\geq 3}$ which cover the missing regions can be defined by the decomposition of the moduli space of punctured Riemann surfaces, in almost the same manner as the closed string products. We refer to such general A_∞-products as *open string with stubs*. It is known that thus defined string products naturally satisfy the A_∞-relations [17, 18].

Utilizing $\{M_k^{\rm B}\}_{k\geq 1}$, the gauge-invariant action can be written as

$$\begin{aligned}S &= -\frac{1}{2}\langle\Psi,Q\Psi\rangle - \sum_{n=2}^{\infty}\frac{1}{n+1}\langle\Psi,M_n^{\rm B}(\overbrace{\Psi,\Psi,...,\Psi}^{n})\rangle\\ &= -\sum_{n=1}^{\infty}\frac{1}{n+1}\langle\Psi,M_n^{\rm B}(\overbrace{\Psi,\Psi,...,\Psi}^{n})\rangle\\ &= -\int_0^1 dt\,\langle\partial_t\Psi(t),\pi_1\mathbf{M}^{\rm B}\frac{1}{1-\Psi(t)}\rangle.\end{aligned} \tag{2.83}$$

Utilizing the cyclicity of M_n, its variation is taken as

$$\delta S = -\langle\delta\Psi,\pi_1\mathbf{M}^{\rm B}\frac{1}{1-\Psi}\rangle = -\langle\delta\Psi,\mathcal{F}^{\rm B}(\Psi)\rangle. \tag{2.84}$$

We define the shifted BRST operator $Q_\Psi B = M^{\rm B}_{1,\Psi}(B)$. Since Q_Ψ annihilate the Maurer-Cartan element $\mathcal{F}^{\rm B}(\Psi)$, this action is invariant under the gauge transformation

$$\begin{aligned}\delta\Psi &= Q_\Psi\Lambda \\ &= M^{\rm B}_{1,\Psi}(\Lambda) \\ &= Q\Lambda + M_2(\Psi,\Lambda) + M_2(\Lambda,\Psi) + M_2(\Psi,\Psi,\Lambda) + M_3(\Psi,\Lambda,\Psi) + M_3(\Lambda,\Psi,\Psi) + \cdots\,. \end{aligned} \tag{2.85}$$

The gauge invariance follows from the A_∞-relations of $\mathbf{M}^{\rm B}$.

In terms of $\mathbf{M}^{\rm B}$, one can find that the action, the equation of motion, and the gauge transformation are written using the same structure as those with the $\mathbf{M}^{\rm B} = \mathbf{Q} + \mathbf{m}_2$:

$$S = -\int_0^1 dt\,\langle \partial_t\Psi(t), \pi_1\mathbf{M}^{\rm B}\frac{1}{1-\Psi(t)}\rangle, \tag{2.86}$$

$$\delta S = -\langle\delta\Psi, \mathcal{F}^{\rm B}(\Psi)\rangle = -\langle\delta\Psi, \pi_1\mathbf{M}^{\rm B}\frac{1}{1-\Psi}\rangle, \tag{2.87}$$

$$\delta\Psi = Q_\Psi\Lambda = \pi_1\mathbf{M}^{\rm B}\big(\frac{1}{1-\Psi}\otimes\Lambda\otimes\frac{1}{1-\psi}\big). \tag{2.88}$$

These expressions are common with string field theories based on the cyclic A_∞-products with certain quantum number(s). We will see in section 4 that the NS open string field theory can be formulated on the basis of the cyclic A_∞-products, and the action, the equation of motion, and the gauge transformation are written in the same form.

3 Closed bosonic string field theory and L_∞-algebra

In this section, we provide brief reviews of bosonic closed string field theory, whose construction [3–7] was completed by Zwiebach in [8], and the L_∞-algebras [8, 14, 15].

3.1 Zwiebach's Closed bosonic string field theory

The string field Ψ of bosonic closed string field theory is a Grassmann even state with ghost number 2 which is annihilated by $b_0^- = b_0 - \bar{b}_0$ and $L_0^- = L_0 - \bar{L}_0$:

$$b_0^- \Psi = 0, \qquad L_0^- \Psi = 0, \tag{3.1}$$

where b_0, $\tilde{b}_0$, L_0, and $\tilde{L}_0$ are the zero modes of the b ghost in the holomorphic sector, the $\tilde{b}$ ghost in the antiholomorphic sector, the energy-momentum tensor $T(z)$ in the holomorphic sector, and the energy-momentum tensor $\tilde{T}(\bar{z})$ in the antiholomorphic sector, respectively. For the state satisfying (3.1) we use the inner product

$$\langle A, B \rangle = \langle A | c_0^- | B \rangle, \tag{3.2}$$

where $c_0^- = \frac{1}{2}(c_0 - \bar{c}_0)$ and $\langle A|B\rangle$ is the BPZ inner product. c_0 and $\tilde{c}_0$ are the zero modes the c ghost in the holomorphic sector and the $\tilde{c}$ ghost in the antiholomorphic sector, respectively. This inner product satisfies

$$\langle A, B \rangle = (-)^{(A+1)(B+1)} \langle B, A \rangle. \tag{3.3}$$

For the closed string, we define the degree to be equal to Grassmann parity, Here and in what follows a state in the exponent of $(-)$ represents its degree, or equivalently its Grassmann parity. Because of the anomaly in the conformal ghost sector, the inner product $\langle A, B \rangle$ vanishes unless the sum of the ghost number of A and B equals to 5.

The kinetic term for Ψ is given by

$$S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle, \tag{3.4}$$

where Q is the BRST operator which is nilpotent and BPZ-odd:

$$Q^2 = 0, \qquad (-)^A \langle QA, B \rangle = \langle A, QB \rangle. \tag{3.5}$$

The equation of motion and a gauge transformation correspond to the physical state condition and a change of the choice of the representative of BRST cohomology in the first-quantization of strings:

$$\text{E.O.M.}: \quad Q\Psi = 0, \qquad \text{Gauge transformation}: \quad \delta\Psi = Q\Lambda, \tag{3.6}$$

where Λ is a gauge parameter which is Grassmann odd, carries ghost number 1, and is annihilated by $b_0^- = b_0 - \bar{b}_0$ and $L_0^- = L_0 - \bar{L}_0$.

The cubic and higher interaction vertices are described by the string products $[\Psi, ..., \Psi]$ which are graded symmetric upon the interchange of the arguments and cyclic:

$$[B_{\sigma(1)}, \dots, B_{\sigma(k)}] = (-)^{\sigma(\{B\})} [B_1, \dots, B_k], \tag{3.7}$$

$$\langle B_1, [B_2, ..., B_{n+1}] \rangle = (-)^{B_1 + B_2 + ... + B_n} \langle [B_1, B_2, ..., B_n], B_{n+1} \rangle \tag{3.8}$$

where σ denote the permutation from $\{1,...,n\}$ to $\{\sigma(1),...,\sigma(n)\}$, $(-)^{\sigma(\{B\})}$ is the sign factor of the permutation from $\{B_1,...,B_n\}$ to $\{B_{\sigma(1)},...,B_{\sigma(n)}\}$, and we write $[B] = QB$. The string products satisfy the following relation called *the L_∞-relations*:

$$0 = \sum_\sigma \sum_{m=1}^{n} (-)^{\sigma(\{B\})} \frac{1}{m!(n-m)!} [\,[B_{\sigma(1)},\ldots,B_{\sigma(m)}], B_{\sigma(m+1)},\ldots,B_{\sigma(n)}], \tag{3.9}$$

A few orders of (3.9) read

$$0 = Q^2, \tag{3.10}$$

$$0 = Q[B_1,B_2] + [QB_1,B_2] + (-)^{B_1}[B_1,QB_2], \tag{3.11}$$

$$\begin{aligned} 0 = {} & Q[B_1,B_2,B_3] + [QB_1,B_2,B_3] + (-)^{B_1}[B_1,QB_2,B_3] + (-)^{B_1+B_2}[B_1,B_2,QB_3] \\ & + [[B_1,B_2],B_3] + (-)^{B_3B_2}[[B_1,B_3],B_2] + (-)^{B_1}[B_1,[B_2,B_3]]. \end{aligned} \tag{3.12}$$

In particular (3.12) means that the deviation of the associativity of the 2-product is compensated by the BRST variation of the 3-product. The interacting action is given by

$$\begin{aligned} S &= -\frac{1}{2}\langle\Psi,Q\Psi\rangle - \sum_{n=2}^{\infty}\frac{1}{(n+1)!}\langle\Psi,[\overbrace{\Psi,\Psi,...,\Psi}^{n}]\rangle \\ &= -\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\langle\Psi,[\overbrace{\Psi,\Psi,...,\Psi}^{n}]\rangle. \end{aligned} \tag{3.13}$$

Note that the n-string product carries ghost number $-2n+3$.

The L_∞-relations of the string products are closely related to the gauge invariance. Utilizing their commutativity and cyclicity, the variation of the action can be taken as

$$\delta S = -\sum_{n=1}^{\infty}\frac{1}{n!}\langle\delta\Psi,[\overbrace{\Psi,\Psi,...,\Psi}^{n}]\rangle, \tag{3.14}$$

and the equation of motion is given by

$$\sum_{n=1}^{\infty}\frac{1}{n!}[\overbrace{\Psi,...,\Psi}^{n}] = 0. \tag{3.15}$$

The action is invariant under the gauge transformation

$$\delta\Psi = \sum_{m=0}^{\infty}\frac{1}{m!}[\overbrace{\Psi,...,\Psi}^{m},\Lambda]. \tag{3.16}$$

The gauge invariance follows from the cyclicity and the L_∞-relations:

$$\delta S = -\sum_{n=1}^{\infty}\frac{1}{n!}\langle\delta\Psi,[\overbrace{\Psi,...,\Psi}^{n}]\rangle = -\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}\langle\Lambda,[[\overbrace{\Psi,...,\Psi}^{n}],\overbrace{\Psi,...,\Psi}^{m}]\rangle = 0. \tag{3.17}$$

In addition to the gauge invariance, the L_∞-relations would be important also for the reproduction of the scattering amplitudes in the first-quantization of strings, which are given by the integration over moduli spaces of punctured Riemann surfaces. The closed string products $\{L_k^{\mathrm{B}}\}_{k\geq 1}$ are defined by a decomposition of the moduli space of punctured sphere. See appendix A, or [8, 19] for more details. It is known that string products defined by the decomposition of the moduli space of punctured Riemann surfaces naturally satisfy the L_∞-relations [8, 19].

3.2 L_∞-algebras

In this subsection we define L_∞-algebras in terms of coalgebras, which is the key structure of the closed string field theory. Discussions are parallel to those for open string and A_∞-algebras. The main differences are the definition of the degree and the state space: for closed string the degree equals to the Grassmann-parity, and the L_∞-algebra is a structure on the symmetrized tensor algebra while the A_∞-algebra is a structure on the (unsymmetrized) tensor algebra. We also define the shifted structures of the L_∞-products and introduce their important properties.

3.2.1 Coalgebra and multilinear maps

Symmetrized tensor algebras as coalgebras

Let $\mathcal{C}$ be a set. When a coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is defined on $\mathcal{C}$ and it is coassociative

$$(\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta, \tag{3.18}$$

then $(\mathcal{C}, \Delta)$ is called a *coalgebra*. For closed string, $\mathcal{C}$ corresponds to the symmetrized tensor algebra $\mathcal{S}(\mathcal{H})$ of the $\mathbb{Z}_2$-graded vector space $\mathcal{H}$. In the language of closed string field theory, $\mathcal{H}$ is the state space for the string field, and the $\mathbb{Z}_2$-grading, called degree, equals to the Grassmann parity. The *symmetrized tensor product* $\wedge$ for elements of $\mathcal{H}$ is defined by

$$\Phi_1 \wedge \Phi_2 = \Phi_1 \otimes \Phi_2 + (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \otimes \Phi_1 \ , \ \ \Phi_i \in \mathcal{H}. \tag{3.19}$$

This product satisfies the following properties:

$$\Phi_1 \wedge \Phi_2 = (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \wedge \Phi_1, \tag{3.20}$$

$$(\Phi_1 \wedge \Phi_2) \wedge \Phi_3 = \Phi_1 \wedge (\Phi_2 \wedge \Phi_3), \tag{3.21}$$

$$\Phi_1 \wedge \Phi_2 \wedge ... \wedge \Phi_n = \sum_\sigma (-)^{\sigma\{\Phi\}} \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes ... \otimes \Phi_{\sigma(n)}. \tag{3.22}$$

We can construct a *symmetrized tensor algebra* $\mathcal{S}(\mathcal{H})$ by

$$\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots . \tag{3.23}$$

We can define a coassociative coproduct $\Delta : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \otimes \mathcal{S}(\mathcal{H})$ and a set $(\mathcal{S}(\mathcal{H}), \Delta)$ gives coalgebra. The action of coproduct Δ on $\Phi_1 \wedge ... \wedge \Phi_n \in \mathcal{H}^{\wedge n}$ is given by

$$\Delta(\Phi_1 \wedge ... \wedge \Phi_n) = \sum_\sigma \sum_{k=0}^{n} (-)^\sigma \frac{1}{k!(n-k)!} (\Phi_{\sigma(1)} \wedge ... \wedge \Phi_{\sigma(k)}) \otimes (\Phi_{\sigma(k+1)} \wedge ... \wedge \Phi_{\sigma(n)}). \tag{3.24}$$

We can naturally define a *projector* $\pi_n : \mathcal{S}(\mathcal{H}) \to \mathcal{H}^{\wedge n}$ whose action on $\Phi \in \mathcal{S}(\mathcal{H})$ is given by

$$\pi_n \widehat{\Phi} = \widehat{\Phi}_n, \qquad \text{where } \widehat{\Phi} = \sum_{k=1}^{\infty} \widehat{\Phi}_k \in \mathcal{S}(\mathcal{H}), \quad \widehat{\Phi}_k \in \mathcal{H}^{\wedge k}. \tag{3.25}$$

Let $\mathcal{H}_0$ be the degree zero part of $\mathcal{H}$. The following exponential map of $\Phi \in \mathcal{H}_0$ gives a special element of $\mathcal{S}(\mathcal{H})$ called a *group-like element*:

$$e^{\wedge \Phi} = \mathbf{1} + \Phi + \frac{1}{2} \Phi \wedge \Phi + \frac{1}{3!} \Phi \wedge \Phi \wedge \Phi + \cdots \in \mathcal{S}(\mathcal{H}), \tag{3.26}$$

which satisfies

$$\Delta e^{\wedge\Phi} = e^{\wedge\Phi} \otimes e^{\wedge\Phi}. \tag{3.27}$$

Multi-linear maps

From a multilinear map $b_n : \mathcal{H}^n \to \mathcal{H}$ which is graded symmetric upon the interchange of the arguments, a map $b_n : \mathcal{H}^{\wedge n} \to \mathcal{H}$ is naturally defined by

$$b_n(\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_n) = b_n(\Phi_1, \Phi_2, \ldots, \Phi_n). \tag{3.28}$$

The symmetric tensor product of two multilinear maps $A : \mathcal{H}^{\wedge k} \to \mathcal{H}^{\wedge l}$ and $B : \mathcal{H}^{\wedge m} \to \mathcal{H}^{\wedge n}$, $A \wedge B : \mathcal{H}^{\wedge k+m} \to \mathcal{H}^{\wedge l+n}$, can also be defined naturally by

$$\begin{aligned} &A \wedge B(\Phi_1 \wedge \ldots \wedge \Phi_{k+m}) \\ &= \sum_\sigma \frac{(-)^{\sigma(\{\Phi\})+B(\Phi_{\sigma(1)}+\ldots+\Phi_{\sigma(k)})}}{k!m!} A(\Phi_{\sigma(1)} \wedge \ldots \wedge \Phi_{\sigma(k)}) \wedge B(\Phi_{\sigma(k+1)} \wedge \ldots \wedge \Phi_{\sigma(k+m)}). \end{aligned} \tag{3.29}$$

The identity operator on $\mathcal{H}^{\wedge n}$ is defined by

$$\mathbb{I}_n = \frac{1}{n!} \mathbb{I} \wedge \mathbb{I} \wedge \ldots \wedge \mathbb{I} = \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}. \tag{3.30}$$

Note that we need the coefficient $\frac{1}{n!}$. Multilinear maps with degree 1 and 0 naturally induce the maps from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H})$. They are called a coderivation and a cohomomorphism respectively.

Multi-linear maps as a coderivation

A linear operator $\mathbf{b} : \mathcal{C} \to \mathcal{C}$ which raise the degree one is called *coderivation* if it satisfies

$$\Delta \mathbf{b} = (\mathbf{b} \otimes \mathbb{1})\Delta + (\mathbb{1} \otimes \mathbf{b})\Delta. \tag{3.31}$$

From a map $b_n : \mathcal{H}^{\wedge n} \to \mathcal{H}$ which carries the degree one, the coderivation $\mathbf{b}_n : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is naturally defined by

$$\mathbf{b}_n \widehat{\Phi}_N = (b_n \wedge \mathbb{I}_{N-n})\widehat{\Phi}_N, \qquad \widehat{\Phi}_N \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H}), \tag{3.32}$$

and $\mathbf{b}_n$ vanishes when acting on $\mathcal{H}^{\wedge N<n}$. We will call $\mathbf{b}_n$ a *n-coderivation* . The n-coderivation $\mathbf{b}_n$ satisfies

$$\pi_1 \mathbf{b}_n \widehat{\Phi} = \pi_1 \mathbf{b}_n \widehat{\Phi}_n, \qquad \text{where } \widehat{\Phi} = \sum_{k=1}^{\infty} \widehat{\Phi}_k \in \mathcal{S}(\mathcal{H}), \quad \widehat{\Phi}_k \in \mathcal{H}^{\wedge k}. \tag{3.33}$$

The explicit actions of the one-coderivation $\mathbf{b}_1$ and the two-coderivation $\mathbf{b}_2$ is given by

$$\begin{array}{rrcl} \mathbf{b}_1 : & 1 & \to & 0 \\ & \Phi_1 & \to & b_1(\Phi_1) \\ & \Phi_1 \wedge \Phi_2 & \to & b_1(\Phi_1) \wedge \Phi_2 + (-)^{\deg(\Phi_1)\deg(b_1)} \Phi_1 \wedge b_1(\Phi_2) \\ & \Phi_1 \wedge \Phi_2 \wedge \Phi_3 & \to & b_1(\Phi_1) \wedge \Phi_2 \wedge \Phi_3 + (-)^{\deg(\Phi_1)\deg(b_1)} \Phi_1 \wedge b_1(\Phi_2) \wedge \Phi_3 \\ & & & \quad +(-)^{(\deg(\Phi_1)+\deg(\Phi_2))\deg(b_1)} \Phi_1 \wedge \Phi_2 \wedge b_1(\Phi_3), \end{array} \tag{3.34}$$

$$
\begin{array}{llll}
\mathbf{b}_2 : & 1 & \to & 0 \\
& \Phi_1 & \to & 0 \\
& \Phi_1 \wedge \Phi_2 & \to & b_2(\Phi_1, \Phi_2) \\
& \Phi_1 \wedge \Phi_2 \wedge \Phi_3 & \to & b_2(\Phi_1, \Phi_2) \wedge \Phi_3 + (-)^{\deg(\Phi_1)\deg(b_2)} \Phi_1 \wedge b_2(\Phi_2, \Phi_3) \\
& & & \qquad + (-)^{\deg(\Phi_2)\deg(\Phi_3)} b_2(\Phi_1, \Phi_3) \wedge \Phi_2.
\end{array}
\tag{3.35}
$$

Given two coderivations $\mathbf{b}_n$ and $\mathbf{c}_m$ which are derived from $b_n : \mathcal{H}^{\wedge n} \to \mathcal{H}$ and $c_m : \mathcal{H}^{\wedge m} \to \mathcal{H}$ respectively, the graded commutator $[\![\mathbf{b}_n, \mathbf{c}_m]\!]$ becomes the coderivation derived from the map $[\![b_n, c_m]\!] : \mathcal{H}^{\wedge n+m-1} \to \mathcal{H}$ which is defined by

$$
[\![b_n, c_m]\!] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1}). \tag{3.36}
$$

The action of a coderivation on a group-like element is given by

$$
\mathbf{b}_n(e^{\wedge \Phi}) = \Big(\pi_1 \mathbf{b}_n(e^{\wedge \Phi})\Big) \wedge e^{\wedge \Phi} = \frac{1}{n!} b_n(\Phi^{\wedge n}) \wedge e^{\wedge \Phi}. \tag{3.37}
$$

Where we promise $0! = 1$.

Multilinear maps as a cohomomorphism

Given two coalgebras C, C', a *cohomomorphism* $\mathsf{f} : C \to C'$ is a map of degree zero satisfying

$$
\Delta \mathsf{f} = (\mathsf{f} \otimes \mathsf{f}) \Delta. \tag{3.38}
$$

A set of degree zero multilinear maps $\{\mathsf{f}_n : \mathcal{H}^{\wedge n} \to \mathcal{H}'\}_{n=0}^{\infty}$ naturally induces a cohomomorphism $\mathsf{f} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}')$, which we denote as $\mathsf{f} = \{\mathsf{f}_n\}_{n=0}^{\infty}$. Its action on $\Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$ is defined by

$$
\begin{aligned}
\mathsf{f}(\Phi_1 \wedge \cdots \wedge \Phi_n) = \sum_{i \le n} \sum_{k_1 < \cdots < k_i} & e^{\wedge \mathsf{f}_0} \wedge \mathsf{f}_{k_1}(\Phi_1, \dots, \Phi_{k_1}) \wedge \mathsf{f}_{k_2 - k_1}(\Phi_{k_1+1}, \dots, \Phi_{k_2}) \wedge \\
& \cdots \wedge \mathsf{f}_{k_i - k_{i-1}}(\Phi_{k_{i-1}+1}, \dots, \Phi_n).
\end{aligned}
\tag{3.39}
$$

Its explicit actions are given as follows:

$$
\begin{array}{llll}
\mathsf{f} : & 1 & \to & e^{\wedge \mathsf{f}_0} \\
& \Phi & \to & e^{\wedge \mathsf{f}_0} \wedge \mathsf{f}_1(\Phi) \\
& \Phi_1 \wedge \Phi_2 & \to & e^{\wedge \mathsf{f}_0} \wedge \mathsf{f}_1(\Phi_1) \wedge \mathsf{f}_1(\Phi_2) + e^{\wedge \mathsf{f}_0} \wedge \mathsf{f}_2(\Phi_1 \wedge \Phi_2).
\end{array}
\tag{3.40}
$$

One of the important property of a cohomomorphism is its action on the group-like element:

$$
\Delta \mathsf{f}(e^{\wedge \Phi}) = (\mathsf{f} \otimes \mathsf{f}) \Delta e^{\wedge \Phi} = (\mathsf{f} \otimes \mathsf{f})(e^{\wedge \Phi} \otimes e^{\wedge \Phi}) = \mathsf{f}(e^{\wedge \Phi}) \otimes \mathsf{f}(e^{\wedge \Phi}). \tag{3.41}
$$

We can see that the cohomomorphisms preserves the group-like element : $\mathsf{f}(e^{\wedge \Phi}) = e^{\wedge \Phi'}$. Then, $\Phi' = \pi_1 \mathsf{f}(e^{\wedge \Phi})$ and one can write

$$
\mathsf{f}(e^{\wedge \Phi}) = e^{\wedge \pi_1 \mathsf{f}(e^{\wedge \Phi})}. \tag{3.42}
$$

3.2.2 Cyclic L_∞-algebra

Cyclic L_∞-algebra $(\mathcal{H}, \mathbf{L}, \omega)$

Let $\mathcal{H}$ be a graded vector space and $\mathcal{S}(\mathcal{H})$ be its symmetrized tensor algebra. A *weak L_∞-algebra* $(\mathcal{H}, \mathbf{L})$ is a coalgebra $\mathcal{S}(\mathcal{H})$ with a coderivation $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_1 + \mathbf{L_2} + ...$ satisfying

$$(\mathbf{L})^2 = 0. \tag{3.43}$$

We denote the collection of the multilinear maps $\{L_k\}_{k\geq 0}$ also by $\mathbf{L}$. In particular, if $\mathbf{L}_0 = 0$, $(\mathcal{H}, \mathbf{L})$ is called an *L_∞-algebra*. In the case of an L_∞-algebra, the part of (3.43) that correspond to an n-fold multilinear map $\mathcal{H}^{\wedge n} \to \mathcal{H}$ is given by

$$\mathbf{L}_n \cdot \mathbf{L}_1 + \mathbf{L}_{n-1} \cdot \mathbf{L}_2 + \cdots + \mathbf{L}_2 \cdot \mathbf{L}_{n-1} + \mathbf{L}_1 \cdot \mathbf{L}_n = 0. \tag{3.44}$$

We can act it on $B_1 \wedge B_2 \wedge ... \wedge B_n \in \mathcal{H}^{\wedge n}$ to get the L_∞ relations for the multilinear maps $\{L_k\}$:

$$0 = \sum_\sigma \sum_{m=1}^{n} \frac{(-)^{\sigma(\{B\})}}{m!(n-m)!} L_{n-m+1}(L_m(B_{\sigma(1)}, \ldots, B_{\sigma(m)}), B_{\sigma(m+1)}, \ldots, B_{\sigma(n)}), \tag{3.45}$$

where $\pi_1 \mathbf{L}_n(B_1 \wedge ... \wedge B_n) = L_n(B_1, ..., B_n)$.

We can define an inner product $\langle A, B\rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ satisfying the same property as (3.3). Given the operator $\mathcal{O}_n$, we can define its BPZ-conjugation $O_n^\dagger$ as follows [7)]:

$$\langle B_1, \mathcal{O}_n(B_2, ..., B_{n+1})\rangle = (-)^{\mathcal{O}_n(B_1+1)+B_2+...+B_n} \langle \mathcal{O}_n^\dagger(B_1, ...B_n), B_{n+1}\rangle. \tag{3.48}$$

A set $(\mathcal{S}(\mathcal{H}), \mathbf{L}, \omega)$ is called *cyclic L_∞-algebra* if each L_n is BPZ-odd:

$$L_n^\dagger = -L_n. \tag{3.49}$$

The *Maurer-Cartan element* for an L_∞-algebra $(\mathcal{H}, \mathbf{L})$ is given by

$$\mathcal{F}(\Phi) := \pi_1 \mathbf{L}(e^{\wedge\Phi}) = L_1(\Phi) + \frac{1}{2} L_2(\Phi, \Phi) + \frac{1}{3!} L_3(\Phi, \Phi, \Phi) + \cdots . \tag{3.50}$$

The *Maurer-Cartan equation* for an L_∞-algebra $(\mathcal{H}, \mathbf{L})$ is given by $\mathcal{F}(\Phi) = 0$, which correspond to the equation of motion in string field theory based on the L_∞-algebra $(\mathcal{H}, \mathbf{L})$.

L_∞-structure of the shifted product

We can define the *shifted structure* of L_∞-products $\mathbf{L}$, shifted by some field $\mathcal{G}$ with even degree, by

$$L_{n,\mathcal{G}}(B_1, ..., B_n) = \sum_{m=0}^{\infty} \frac{1}{m!} L_{n+m}(\overbrace{\mathcal{G}, ..., \mathcal{G}}^{m}, B_1, ..., B_n). \tag{3.51}$$

7) We can define an inner product $\langle A, B\rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ using the graded symplectic form $\langle \omega| : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ satisfying $\langle \omega | A \otimes B = (-)^{AB+1} \langle \omega | B \otimes A$, by:

$$\langle A, B\rangle = (-)^A \langle \omega | A \otimes B. \tag{3.46}$$

Then, the BPZ-conjugation $O_n^\dagger$ of the operator $\mathcal{O}_n$ is defined by

$$\langle \omega | \mathbb{I} \otimes \mathcal{O}_n = \langle \omega | \mathcal{O}_n^\dagger \otimes \mathbb{I}. \tag{3.47}$$

The BPZ properties of the shifted products follows from the original products $\mathbf{L}$: if $\mathbf{L}$ is BPZ-odd, $L_{n,\mathcal{G}}$ is also cyclic:

$$\langle B_1, L_{n,\mathcal{G}}(B_2, ..., B_{n+1})\rangle = (-)^{B_1+B_2+...+B_n}\langle L_{n,\mathcal{G}}(B_1, B_2, ..., B_n), B_{n+1}\rangle. \tag{3.52}$$

One important property of the shifted structure is that the products $\mathbf{L}$ shifted by $\mathcal{G}$ (3.51) satisfy the L_∞-relations [74]

$$0 = \sum_{\sigma}\sum_{k=1}^{n}\frac{(-)^{\sigma(\{B\})}}{k!(n-k)!}L_{n-k+1,\mathcal{G}}\Big(L_{k,\mathcal{G}}(B_{\sigma(1)}, ..., B_{\sigma(k)}), B_{\sigma(k+1)}, ..., B_{\sigma(n)}\Big), \tag{3.53}$$

if $\mathcal{G}$ is a solution for the Maurer-Cartan equation for $\mathbf{L}$:

$$\mathcal{F}(\mathcal{G}) = \pi_1\mathbf{L}(e^{\wedge\mathcal{G}}) = 0. \tag{3.54}$$

To show (3.53), it is convenient to represent them in the coalgebraic notation: we write

$$L_{n,\mathcal{G}}(B_1, ..., B_n) = \pi_1\mathbf{L}(B_1\wedge ...\wedge B_n\wedge e^{\wedge\mathcal{G}}). \tag{3.55}$$

Recall that $\mathbf{L}$ is a coderivation and it acts on $B_1\wedge ...\wedge B_n\wedge e^{\wedge\mathcal{G}}$ as

$$\begin{aligned}\mathbf{L}(B_1\wedge ...\wedge B_n\wedge e^{\wedge\mathcal{G}}) &= \pi_1\mathbf{L}(e^{\wedge\mathcal{G}})\wedge B_1\wedge ...\wedge B_n\wedge e^{\wedge\mathcal{G}}\\ &\quad+\sum_{\sigma}\sum_{k=1}^{n}\frac{(-)^{\sigma(\{B\})}}{k!(n-k)!}\pi_1\mathbf{L}(B_{\sigma(1)}\wedge ...\wedge B_{\sigma(k)}\wedge e^{\wedge\mathcal{G}})\wedge B_{\sigma(k+1)}\wedge ...\wedge B_{\sigma(n)}\wedge e^{\wedge\mathcal{G}}\\ &= \mathcal{F}(\mathcal{G})\wedge B_1\wedge ...\wedge B_n\wedge e^{\wedge\mathcal{G}}\\ &\quad+\sum_{\sigma}\sum_{k=1}^{n}\frac{(-)^{\sigma(\{B\})}}{k!(n-k)!}L_{k,\mathcal{G}}(B_{\sigma(1)}, ..., B_{\sigma(k)})\wedge B_{\sigma(k+1)}\wedge ...\wedge B_{\sigma(n)}\wedge e^{\wedge\mathcal{G}}.\end{aligned} \tag{3.56}$$

Acting $\pi_1\mathbf{L}$ on both sides and using $\mathbf{L}^2 = 0$, one can obtain

$$\sum_{\sigma}\sum_{k=1}^{n}\frac{(-)^{\sigma}}{k!(n-k)!}L_{n-k+1,\mathcal{G}}(L_{k,\mathcal{G}}(B_{\sigma(1)}, ..., B_{\sigma(k)}), B_{\sigma(k+1)}, ..., B_{\sigma(n)}) = -L_{n+1,\mathcal{G}}(\mathcal{F}(\mathcal{G}), B_1, ..., B_n). \tag{3.57}$$

Thus, it is shown from (3.57) that the shifted products (3.51) satisfy the L_∞-relations (3.53) if $\mathcal{G}$ is a solution for the Maurer-Cartan equation (3.54).

Another important property is that the $\mathcal{G}$-shifted 1-product $L_{1,\mathcal{G}}$ annihilated the Maurer-Cartan element $\mathcal{F}(\mathcal{G})$ for arbitrary $\mathcal{G}$ with even degree. It directly follows from $\mathbf{L}^2 = 0$:

$$L_{1,\mathcal{G}}(\mathcal{F}(\mathcal{G})) = \pi_1\mathbf{L}(\pi_1\mathbf{L}(e^{\wedge\mathcal{G}})\wedge e^{\wedge\mathcal{G}}) = \pi_1\mathbf{L}\mathbf{L}(e^{\wedge\mathcal{G}}) = 0. \tag{3.58}$$

Generally, the gauge transformation is generated by the operator which annihilates the equation of motion. Since the equation of motion is given by the Maurer-Cartan element, $L_{1,\mathcal{G}}$ generates the gauge transformation, as we will see in the next subsection.

3.3 Coalgebraic representation

To conclude this section, let us describe closed bosonic string field theory in the coalgebraic representation. String products in closed bosonic string field theory can be represented by a set of multilinear maps $L_n^{\rm B} : {\mathcal{H}_{\rm B}}^{\wedge n} \to \mathcal{H}_{\rm B}$:

$$[\Psi_1, \Psi_2, ..., \Psi_n] = L_n^{\rm B}(\Psi_1, \Psi_2, ..., \Psi_n) = L_n^{\rm B}(\Psi_1 \wedge \Psi_2 \wedge ... \wedge \Psi_n), \tag{3.59}$$

and the set of $\{L_n^{\rm B}\}$ naturally defines a set of coderivations $\{\mathbf{L}_n^{\rm B}\}$. Because of the L_∞ relations (3.9) for the original products, $\mathbf{L}^{\rm B} = \sum_{n=1}^{\infty} \mathbf{L}_n^{\rm B}$ is nilpotent

$$(\mathbf{L}^{\rm B})^2 = 0. \tag{3.60}$$

The cyclicity of original string products (3.8) corresponds to $L_n^{\rm B\dagger} = -L_n^{\rm B}$. Therefore the algebraic properties of the string products is encoded to the fact that $(\mathcal{H}_{\rm B}, \mathbf{L}^{\rm B}, \omega_{\rm B})$ defines a cyclic-L_∞ algebra.

We can transform the action in terms of the string products $\mathbf{L}$ and the group-like element $e^{\wedge\Phi}$. Let t be a real parameter $t \in [0, 1]$. We introduce a t-parametrized string field $\Psi(t)$ satisfying $\Psi(0) = 0$ and $\Psi(1) = \Psi$, which is a path connecting 0 and the string field Ψ in the space of string fields. Using this $\Psi(t)$,

$$\begin{aligned} S &= -\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \langle \Psi, L_n^{\rm B}(\overbrace{\Psi, \Psi, ..., \Psi}^{n}) \rangle \\ &= -\int_0^1 dt \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_t \langle \Psi(t), L_n^{\rm B}(\overbrace{\Psi(t), \Psi(t), ..., \Psi(t)}^{n}) \rangle \\ &= -\int_0^1 dt \sum_{n=1}^{\infty} \frac{1}{n!} \langle \partial_t \Psi(t), L_n^{\rm B}(\overbrace{\Psi(t), \Psi(t), ..., \Psi(t)}^{n}) \rangle. \end{aligned} \tag{3.61}$$

In the last line we act ∂_t and use the cyclicity of the string products to move $\partial_t \Psi(t)$ to the first slot of the inner product. The action can be represented using the coderivations and the group-like element as follows:

$$\begin{aligned} S &= -\int_0^1 dt \langle \partial_t \Psi(t), \pi_1 \mathbf{L}^{\rm B}(e^{\wedge\Psi(t)}) \rangle \\ &= -\int_0^1 dt \langle \pi_1 \boldsymbol{\partial}_t (e^{\wedge\Psi(t)}), \pi_1 \mathbf{L}^{\rm B}(e^{\wedge\Psi(t)}) \rangle, \end{aligned} \tag{3.62}$$

where we denote the one-coderivation derived from ∂_t as $\boldsymbol{\partial}_t$. Note that we conventionally use the group-like element in the first slot of the inner product of (3.62). This expression is useful in a case of the superstring.

Utilizing the cyclicity, the variation of the action can be taken as

$$\delta S = -\langle \delta\Psi, \mathcal{F}^{\rm B}(\Psi) \rangle, \tag{3.63}$$

where $\mathcal{F}^{\rm B}(\Psi)$ is the Maurer-Cartan element of the L_∞-algebra $(\mathcal{H}_{\rm B}, \mathbf{L}^{\rm B}, \omega_{\rm B})$ which is given by

$$\mathcal{F}^{\rm B}(\Psi) = \pi_1 \mathbf{L}^{\rm B}(e^{\wedge\Psi}) = \sum_{n=1}^{\infty} \frac{1}{n!} L_n^{\rm B}(\overbrace{\Psi, \Psi, ..., \Psi}^{n}). \tag{3.64}$$

The equation of motion is given by the Maurer-Cartan equation

$$\mathcal{F}^{\mathrm{B}}(\Psi) = 0. \tag{3.65}$$

Note that t-dependence is topological: the equation of motion does not depend of t. We define the shifted BRST operator $Q_\Psi(B) = L^{\mathrm{B}}_{1,\Psi}(B) = \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi} \wedge B)$. Since $(\mathbf{L}^{\mathrm{B}})^2 = 0$, this shifted BRST operator Q_Ψ annihilate the Maurer-Cartan element $\mathcal{F}^{\mathrm{B}}(\Psi)$:

$$Q_\Psi \mathcal{F}^{\mathrm{B}}(\Psi) = \pi_1 \mathbf{L}^{\mathrm{B}}\big(e^{\wedge\Psi} \wedge \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi})\big) = \pi_1 \mathbf{L}^{\mathrm{B}} \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi}) = 0, \tag{3.66}$$

which means that Q_Ψ generates the gauge transformation

$$\delta\Psi = Q_\Psi \Lambda = \sum_{m=0}^{\infty} \frac{1}{m!} [\overbrace{\Psi, ..., \Psi}^{m}, \Lambda]. \tag{3.67}$$

Note that the shifted BRST operator Q_Ψ is BPZ-odd:

$$\langle Q_\Psi A, B \rangle = (-)^A \langle A, Q_\Psi B \rangle. \tag{3.68}$$

These expressions of the action, the equation of motion, and the gauge transformation

$$S = -\int_0^1 dt\, \langle \partial_t \Psi(t), \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi(t)}) \rangle, \tag{3.69}$$

$$\delta S = -\langle \delta\Psi, \mathcal{F}^{\mathrm{B}}(\Psi) \rangle = -\langle \delta\Psi, \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi}) \rangle, \tag{3.70}$$

$$\delta\Psi = Q_\Psi \Lambda = \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge\Psi} \wedge \Lambda) \tag{3.71}$$

are common with string field theories based on the cyclic L_∞-products. We will see in section 4 that the NS heterotic string field theory can be formulated on the basis of the cyclic L_∞-products, and the action, the equation of motion, and the gauge transformation are written in the same form.

Part II

Formulations of superstring field theories for the Neveu-Schwarz sector

The goal of part II is to define *dual WZW-like action* for the NS sector which we will use as a starting point of construction of an action including the Ramond sector. We first review two successful and popular formulations: the A_∞- and L_∞-formulation in section 4, and the WZW-like formulation in section 5. Then in section 6, based on structures introduced in sections 4 and 5, we define the dual WZW-like formulation: utilizing string products which are dual to those in the A_∞/L_∞-formulation, an gauge invariant action is provided by almost the same procedure as WZW-like formulation.

4 A_∞- and L_∞-formulation for the NS sector

In this section, we provide a brief review of the L_∞-formulation [60]: we construct the NS superstring products $\mathbf{L}^{\mathrm{NS}}$ which satisfy the L_∞-relations and a consistency condition for a state space, and are cyclic. A gauge-invariant action for heterotic string field theory can be constructed in terms of them. To begin with, let us first introduce the superconformal ghost sector which we use to describe superstrings.

Superconformal ghost sector

In the heterotic string, one of the holomorphic and antiholomorphic sectors is supersymmetric and the other is bosonic. We choose the holomorphic sector to be supersymmetric, which consists of the matter sector, the reparameterization ghost sector in terms of $b(z)$ and $c(z)$, and the superconformal ghost sector. The antiholomorphic sector is bosonic, and it consists of the matter sector and the reparameterization ghost sector in terms of $\tilde{b}(\bar{z})$ and $\tilde{c}(\bar{z})$. We describe the superconformal ghost sector in terms of $\xi(z)$, $\eta(z)$, and $\phi(z)$ [27], where $\xi(z)$ and $\eta(z)$ are fermionic and $\phi(z)$ is bosonic. They are related to the description by $\beta(z)$ and $\gamma(z)$ as follows:

$$\beta(z) = \partial\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = e^{\phi(z)}\eta(z). \tag{4.1}$$

We can consider the two Hilbert spaces for the superconformal ghost sector. One is called the *large Hilbert space*, the Hilbert space for the system of $\xi(z)$, $\eta(z)$, and $\phi(z)$. For a pair of states of heterotic string V_1 and V_2 which belong to the large Hilbert space and are annihilated by b_0^- and L_0^-, the inner product is defined by

$$\langle V_1, V_2\rangle = \langle V_1|c_0^-|V_2\rangle. \tag{4.2}$$

The inner product in the large Hilbert space is nonvanishing when the sums of the ghost number and the picture number of the two input states are $(4, -1)$.

The other is called the *small Hilbert space*, the Hilbert space we usually use in the description of the superconformal ghost sector in terms of $\beta(z)$ and $\gamma(z)$. A state Φ in the small Hilbert space is annihilated by the zero mode of $\eta(z)$:

$$\eta\Phi = 0, \tag{4.3}$$

where we denote the zero mode of $\eta(z)$ by η:

$$\eta = \oint \frac{dz}{2\pi i}\eta(z). \tag{4.4}$$

Since η is nilpotent and there exists an operator ξ satisfying

$$\eta^2 = 0, \qquad [\![\xi, \eta]\!] = 1, \qquad \xi^2 = 0, \qquad \langle \xi A, B\rangle = (-)^{A+1}\langle A, \xi B\rangle, \tag{4.5}$$

the cohomology of η is trivial in the large Hilbert space and $\eta\xi$ works as a projector on the small Hilbert space. For the heterotic string, the consistency with the L_0^- constraint requires ξ to be the zero mode of $\xi(z)$:

$$\xi = \oint \frac{dz}{2\pi i}\,\frac{1}{z}\,\xi(z). \tag{4.6}$$

For a pair of states of heterotic string Φ_1 and Φ_2 which belong to the small Hilbert space and are annihilated by b_0^- and L_0^-, the inner product is defined by

$$\langle\!\langle \Phi_1, \Phi_2\rangle\!\rangle = \langle \xi\Phi_1, \Phi_2\rangle. \tag{4.7}$$

The inner product in the small Hilbert space is nonvanishing when the sums of the ghost number and the picture number of the two input states are $(5, -2)$.

For the open string, which consists of the supersymmetric holomorphic sector, we can define the small and large Hilbert spaces in the same manner. The inner product for a pair of states of open NS string ϕ_1 and ϕ_2 which belong to the large Hilbert space is defined by

$$\langle \phi_1, \phi_2\rangle = \langle \phi_1|\phi_2\rangle. \tag{4.8}$$

The inner product in the small Hilbert space is related to that in the large Hilbert space in the same manner as the case of heterotic string:

$$\langle\!\langle \Psi_1, \Psi_2\rangle\!\rangle = \langle \xi\Psi_1, \Psi_2\rangle. \tag{4.9}$$

Note that we do not need to impose the subsidiary condition with b_0^- and L_0^- on states, and ξ can be chosen arbitrary line integral of $\xi(z)$ as long as the conditions (4.5) are satisfied [35]. The inner product in the large/small Hilbert space is nonvanishing when the sums of the ghost number and the picture number of the two input states are $(2, -1)/(3, -2)$ for open string.

4.1 Regularization of the Witten theory

Covariant open superstring field theory proposed by Witten [21] was the natural extension of the bosonic theory [1]. The action for the NS sector is given by [8)]

$$S_{\text{Witten}} = -\frac{1}{2}\langle\!\langle \Psi, Q\Psi\rangle\!\rangle - \frac{g}{3}\langle\!\langle \Psi, \mathcal{X}(i)(\Psi * \Psi)\rangle\!\rangle, \tag{4.10}$$

[8)] For superstring, we use the star product as a glueing of two string fields, see appendix A. Note that, in oscillator representation, the Neumann coefficients differ depending on the theories.

where the string field Ψ is a state in the small Hilbert space carrying ghost number 1 and picture number -1, the natural picture for the NS sector. The cubic interaction contains the local insertion of the picture changing operator at the midpoint of the string

$$\mathcal{X}(i) = [\![Q, \xi(i)]\!]. \tag{4.11}$$

On the one hand, the midpoint insertion of local operators does not break the associativity of cubic interaction, which enables the action to be (formally) gauge-invariant without higher interactions. On the other hand, it can make the theory singular: the singularity can arise from the collisions of local operators. Actually, in the Witten theory, collisions of the picture changing operators make the scattering amplitude and the gauge symmetry singular [22]. Since then, the construction of the theory for the string field belonging to the small Hilbert space in the natural picture with the consistent gauge invariance has been a problem to be solved, even for the NS sector.

Recently, the breakthrough for this problem was found and such a consistent theory was constructed. It is leaded by the work by Iimori, Noumi, Okawa, and Torii [35]. In this work, the consistent theory for open string field belonging to the small Hilbert space in the natural picture is provided by the partial gauge fixing of the large Hilbert space formulation constructed by Berkovits [26]. The problematic local insertions of the picture changing operator are replaced with its line integrals, and the theory does not suffer from any divergences. Since the Berkovits theory has nonpolynomial interaction, so does the partially gauge fixed theory, but the theory obtained in [35] does not have a manifest A_∞-structure.

The use of the line integral typically breaks the associativity of the 2-string product describing the cubic interaction, and then for the gauge invariance the higher vertices become necessary, even for open string. The novel construction of the higher vertices which satisfy A_∞/L_∞-relation was given by Erler, Konopka, and Sachs, for open NS string [39] and for NS heterotic and NS-NS closed string [60], which provides the actions in terms of the vertices, with the same structure as bosonic theories. The same technique is used for the construction of the equations of motion including the Ramond sector [40]. Also, it is shown based on the A_∞ and L_∞ structure that these theories correctly reproduce the scattering amplitude in first quantized string [41].

In this section we review the L_∞-formulation provided in [60]. In the rest of this subsection, we see explicitly how the Witten theory is regularized. In subsection 4.2 and 4.3, we focus on heterotic string field theory. We provide the all-order construction of the NS L_∞-products and the gauge-invariant actions in terms of them. For open string with and without stubs, the construction of the NS A_∞-products is parallel to the NS L_∞-products. In subsection 4.4, we mention it and provide the actions for open string with and without stubs.

Regularization of the Witten theory

Let us see how one can regularize the Witten theory. The string field Ψ is Grassmann odd, belongs to the small Hilbert space, and carries ghost number 1 and picture number -1: The kinetic term for open string field Ψ is given by

$$S_2 = -\frac{1}{2}\langle\!\langle \Psi, Q\Psi \rangle\!\rangle. \tag{4.12}$$

Since $Q^2 = 0$, S_2 is invariant under the following linearized gauge transformation

$$\delta_0 \Psi = Q\Lambda^{\mathrm{NS}} \tag{4.13}$$

where Λ^{NS} is a gauge parameter which is Grassmann even, belongs to the small Hilbert space, and carries ghost number 0 and picture number -1. We denoted the order of the string field by subscripts.

As a regularization of (4.10), we give the cubic interaction in the following form:

$$S_3 = -\frac{1}{3}\langle\!\langle \Psi, M_2^{\mathrm{NS}}(\Psi, \Psi)\rangle\!\rangle, \tag{4.14}$$

$$M_2^{\mathrm{NS}}(\Psi_1, \Psi_2) = \frac{1}{3}\big(\mathcal{X} M_2^{\mathrm{B}}(\Psi_1, \Psi_2) + M_2^{\mathrm{B}}(\mathcal{X}\Psi_1, \Psi_2) + M_2^{\mathrm{B}}(\Psi_1, \mathcal{X}\Psi_2)\big), \tag{4.15}$$

where $\mathcal{X} = [\![Q, \xi]\!]$ is a picture changing operator and M_2^{B} is the 2-string product of the string products $\{M_k^{\mathrm{B}}\}_{k\geq 1}$ which define bosonic open string field theory.[9] M_2^{B} commutes with Q. One can choose $M_2^{\mathrm{B}}(A, B) = m_2(A, B) = (-)^{\deg(A)} A * B$. Note that η acts as a derivation with respect to M_2^{NS} and then $M_2^{\mathrm{NS}}(\Psi, \Psi)$ belongs to the small Hilbert space. The gauge transformation at next order $\delta_1\Psi$ is given by

$$\delta_1 \Psi = M_2^{\mathrm{NS}}(\Psi, \Lambda^{\mathrm{NS}}) + M_2^{\mathrm{NS}}(\Lambda^{\mathrm{NS}}, \Psi), \tag{4.16}$$

then $S_2 + S_3$ is invariant under $\delta_0\Psi + \delta_1\Psi$ at cubic order in fields:

$$\delta_0 S_3 + \delta_1 S_2 = \langle\!\langle \Lambda^{\mathrm{NS}}, Q M_2^{\mathrm{NS}}(\Psi, \Psi) + M_2^{\mathrm{NS}}(Q\Psi, \Psi) + M_2^{\mathrm{NS}}(\Psi, Q\Psi)\rangle\!\rangle = 0, \tag{4.17}$$

which follows from the derivation property of Q with respect to M_2^{NS}:

$$Q M_2^{\mathrm{NS}}(\Psi_1, \Psi_2) + M_2^{\mathrm{NS}}(Q\Psi_1, \Psi_2) + (-)^{\deg(\Psi_1)} M_2^{\mathrm{NS}}(\Psi, Q\Psi) = 0. \tag{4.18}$$

Recall that the degree of the string field is defined by Grassmann parity plus one.

The NS 2-product M_2^{NS} is no longer associative, even if M_2^{B} is the associative star product $M_2^{\mathrm{B}}(A, B) = m_2(A, B) = (-)^{\deg(A)} A * B$. Then $\delta_1 S_3 \neq 0$, and we need higher corrections to compensate it. We write the quartic interaction in terms of a product M_3^{NS} as

$$S_4 = -\frac{1}{4}\langle\!\langle \Psi, M_3^{\mathrm{NS}}(\Psi, \Psi, \Psi)\rangle\!\rangle. \tag{4.19}$$

We assume that M_3^{NS} is degree odd, cyclic, and of suitable ghost and picture numbers, and that η acts as a derivation with respect to M_3^{NS} and then $M_3^{\mathrm{NS}}(\Psi, \Psi, \Psi)$ belongs to the small Hilbert space. We also assume the gauge transformation is given by the same product:

$$\delta_2 \Psi = M_3^{\mathrm{NS}}(\Lambda^{\mathrm{NS}}, \Psi, \Psi) + M_3^{\mathrm{NS}}(\Psi, \Lambda^{\mathrm{NS}}, \Psi) + M_3^{\mathrm{NS}}(\Psi, \Psi, \Lambda^{\mathrm{NS}}), \tag{4.20}$$

then the variation of the action at this order reads

$$\begin{aligned}\delta_2 S_2 + \delta_1 S_3 + \delta_0 S_4 = \Big\langle\!\!\Big\langle \Lambda^{\mathrm{NS}}, & Q M_3^{\mathrm{NS}}(\Psi, \Psi, \Psi) + M_3^{\mathrm{NS}}(Q\Psi, \Psi, \Psi) + M_2^{\mathrm{NS}}(\Psi, Q\Psi, \Psi) + M_2^{\mathrm{NS}}(\Psi, \Psi, Q\Psi) \\ & + M_2^{\mathrm{NS}}(M_2^{\mathrm{NS}}(\Psi, \Psi), \Psi) + M_2^{\mathrm{NS}}(\Psi, M_2^{\mathrm{NS}}(\Psi, \Psi))\Big\rangle\!\!\Big\rangle.\end{aligned} \tag{4.21}$$

[9] As in the star product, for superstring we use M_k^{B} as a glueing of string fields, see appendix A. Note that, in oscillator representation, the Neumann coefficients differ depending on the theories.

This gauge invariance holds if the product $M_3^{\rm NS}$ satisfy the A_∞-relation:

$$\begin{aligned}0 =&QM_3^{\rm NS}(\Psi_1,\Psi_2,\Psi_3)\\ &+M_3^{\rm NS}(Q\Psi_1,\Psi_2,\Psi_3)+(-)^{\deg(\Psi_1)}M_2^{\rm NS}(\Psi_1,Q\Psi_2,\Psi_3)+(-)^{\deg(\Psi_1)+\deg(\Psi_2)}M_2^{\rm NS}(\Psi_1,\Psi_2,Q\Psi_3)\\ &\quad+M_2^{\rm NS}(M_2^{\rm NS}(\Psi_1,\Psi_2),\Psi_3)+(-)^{\deg(\Psi_1)}M_2^{\rm NS}(\Psi_1,M_2^{\rm NS}(\Psi_2,\Psi_3)).\end{aligned} \tag{4.22}$$

As in the case of bosonic theories, if the interactions and the gauge transformation are given by the A_∞-products, the gauge invariance follows from the A_∞-relations. Introducing coderivations $\mathbf{Q}$ and $\mathbf{M}_k^{\rm NS}$ which are derived from Q and $M_k^{\rm NS}$, the nilpotency of Q, (4.18), and (4.22) can be written as

$$\mathbf{Q}\mathbf{Q}=0,\qquad \mathbf{Q}\mathbf{M}_2^{\rm NS}+\mathbf{M}_2^{\rm NS}\mathbf{Q}=0,\qquad \mathbf{Q}\mathbf{M}_3^{\rm NS}+\mathbf{M}_2^{\rm NS}\mathbf{M}_2^{\rm NS}+\mathbf{M}_3^{\rm NS}\mathbf{Q}=0,\quad\cdots, \tag{4.23}$$

and they can be summarized by introducing the generating function $\mathbf{M}^{\rm NS}=\sum_{n=0}\mathbf{M}_{n+1}^{\rm NS}$ as

$$[\![\mathbf{M}^{\rm NS},\mathbf{M}^{\rm NS}]\!]=0. \tag{4.24}$$

Here we write $\mathbf{M}_1^{\rm NS}=\mathbf{Q}$. We also require $M_n^{\rm NS}(\Psi,...,\Psi)$ belongs to the small Hilbert space for any n. It is satisfied if η acts as a derivation with respect to $M_n^{\rm NS}$, which can be written as

$$[\![\boldsymbol{\eta},\mathbf{M}^{\rm NS}]\!]=0, \tag{4.25}$$

where $\boldsymbol{\eta}$ is a one-coderivation derived from η. We call (4.25) *the η-derivation property* of $\mathbf{M}^{\rm NS}$. A gauge-invariant action for open string can be constructed in terms of the string products satisfying (4.24) and (4.25).

Gauge-invariant actions for the open string with stubs, heterotic string, and NS-NS closed string can be constructed in the same manner: in terms of the A_∞- or L_∞-products with η-derivation properties. In the following we focus on the heterotic string, and review the construction of the L_∞-products $\{L_k^{\rm NS}\}_{k\geq 1}$ from the string products defining closed bosonic string field theory $\{L_k^{\rm B}\}_{k\geq 1}$[10], and operators η and ξ. Hereafter we write $\mathbf{L}_k^{\rm B}$ for a coderivations derived from $L_k^{\rm B}$, and define $\mathbf{L}^{\rm B}=\sum_{n=0}^{\infty}\mathbf{L}_{n+1}^{\rm B}$. We may symbolically write $\mathbf{L}^{\rm B}$ to denote a set of string products $\{L_k^{\rm B}\}_{k\geq 1}$.

4.2 The NS L_∞ products

To construct a gauge-invariant action of heterotic string field theory, in this subsection we provide the construction of the NS products $\{L_k^{\rm NS}\}_{k\geq 1}$ which describe fundamental interactions of string fields. We introduce coderivations $\mathbf{L}_k^{\rm NS}$ which are derived from $L_k^{\rm NS}$, and define $\mathbf{L}^{\rm NS}(\tau)$ by

$$\mathbf{L}^{\rm NS}(\tau)=\sum_{n=0}^{\infty}\tau^n\mathbf{L}_{n+1}^{\rm NS}. \tag{4.26}$$

Hereafter we consider the construction of this coderivation $\mathbf{L}^{\rm NS}$ from $\mathbf{L}^{\rm B}$, η, ξ, and $\mathcal{X}=[\![Q,\xi]\!]$. We require the N-th product $\mathbf{L}_N^{\rm NS}$ to carry the same ghost number as $\mathbf{L}_N^{\rm B}$, $3-2N$, and picture number $N-1$. We also require $\mathbf{L}^{\rm NS}$ to satisfy the L_∞-relation and the η-derivation property,

$$0=[\![\mathbf{L}^{\rm NS}(\tau),\mathbf{L}^{\rm NS}(\tau)]\!], \tag{4.27}$$

$$0=[\![\boldsymbol{\eta},\mathbf{L}^{\rm NS}(\tau)]\!], \tag{4.28}$$

[10] For superstring we use $L_k^{\rm B}$ as a glueing of string fields, see appendix A. Note that, in oscillator representation, the Neumann coefficients differ depending on the theories.

and to be cyclic, $(\mathbf{L}^{\text{NS}})^\dagger = -\mathbf{L}^{\text{NS}}$. The construction in this subsection also works for the NS open string by replacing $\mathbf{L}^{\text{B}}$ to $\mathbf{M}^{\text{B}}$.

In the construction, it is convenient to label the picture number of the product by its deficit relative to what is needed for the NS products. We write NS products as

$$\mathbf{L}^{\text{NS}} = \mathbf{L}^{[0]}. \tag{4.29}$$

Since N-th NS-product carries picture number $N-1$, the picture number of the product of deficit picture d is $N-1-d$. The same notation will be used for the gauge products $\boldsymbol{\lambda}$ which are defined shortly.

4.2.1 Defining differential equations

In this subsubsection, the differential equations which lead to the suitable properties of products are derived perturbatively in the picture deficit.

L_∞-relation of $\mathbf{L}^{[0]}(\tau)$

Let us consider the condition for the L_∞-relations for $\mathbf{L}^{[0]}(\tau)$. We first check the initial condition: at $\tau = 0$, we identify

$$\mathbf{L}^{[0]}(0) = \mathbf{Q}, \tag{4.30}$$

and the L_∞-relations (4.27) at $\tau = 0$ is the nilpotency of $\mathbf{Q}$: $[\![\mathbf{L}^{[0]}(0), \mathbf{L}^{[0]}(0)]\!] = [\![\mathbf{Q}, \mathbf{Q}]\!] = 0$. Differentiating (4.27) by τ, the right hand side of (4.27) becomes

$$\partial_\tau [\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] = 2[\![\mathbf{L}^{[0]}(\tau), \partial_\tau \mathbf{L}^{[0]}(\tau)]\!]. \tag{4.31}$$

Then, the following differential equation ensures $\mathbf{L}^{[0]}(\tau)$ to satisfy the L_∞-relations:

$$\partial_\tau \mathbf{L}^{[0]}(\tau) = [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!], \tag{4.32}$$

where $\boldsymbol{\lambda}^{[0]}(\tau)$ is defined by the sum of coderivations $\{\boldsymbol{\lambda}^{[0]}_{n+2}\}_{n\geq 0}$ which are derived from a set of products $\{\lambda^{[0]}_{n+2}\}_{n\geq 0}$ called *gauge products*:

$$\boldsymbol{\lambda}^{[0]}(\tau) = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\lambda}^{[0]}_{n+2}. \tag{4.33}$$

N-th gauge product $\lambda^{[0]}_N$ carries ghost number $2-2N$ and picture number $N-1$. Utilizing this, (4.31) becomes homogeneous in $[\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!]$:

$$\partial_\tau [\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] = 2[\![\mathbf{L}^{[0]}(\tau), [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!]]\!] = [\![[\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!]. \tag{4.34}$$

Since the L_∞-relation is satisfied at $\tau = 0$, $[\![\mathbf{L}^{[0]}(0), \mathbf{L}^{[0]}(0)]\!] = 0$, this homogeneous differential equation ensures the L_∞-relation at arbitrary τ, $[\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] = 0$.

η-derivation property of $\mathbf{L}^{[0]}(\tau)$

The logic to derive the condition for η-derivation property of $\mathbf{L}^{[0]}$ is almost the same as that for the L_∞-relations. At $\tau = 0$, it holds: $[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(0)]\!] = [\![\boldsymbol{\eta}, \mathbf{Q}]\!] = 0$. Differentiating by τ, the right-hand side of (4.28) becomes

$$\begin{aligned}\partial_\tau [\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!] &= [\![\boldsymbol{\eta}, \partial_\tau \mathbf{L}^{[0]}(\tau)]\!] \\ &= [\![\boldsymbol{\eta}, [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!]]\!] \\ &= [\![[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] - [\![[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!]. \end{aligned} \tag{4.35}$$

If we take the choice of $\boldsymbol{\lambda}^{[0]}$ so that the second term vanishes, the differential equation becomes homogeneous in $[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!]$. Then $[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(0)]\!] = 0$ leads to the solution $[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!] = 0$, and therefore the product $\mathbf{L}^{[0]}$ has η-derivation property at arbitrary τ.

Such a product $\boldsymbol{\lambda}^{[0]}$ can be constructed in terms of the product with deficit picture 1

$$\mathbf{L}^{[1]}(\tau) = \sum_{n=0}^{\infty} \tau^n \mathbf{L}^{[1]}_{n+2} \tag{4.36}$$

which commutes with $\mathbf{L}^{[0]}(\tau)$ and $\boldsymbol{\eta}$

$$[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] = 0, \tag{4.37}$$

$$[\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!] = 0. \tag{4.38}$$

The following identification provides the suitable definition of $\boldsymbol{\lambda}^{[0]}$:

$$\mathbf{L}^{[1]}(\tau) = [\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau)]\!]. \tag{4.39}$$

In the reverse direction, $\boldsymbol{\lambda}^{[0]}$ is given by $\mathbf{L}^{[1]}$ as

$$\boldsymbol{\lambda}^{[0]}(\tau) = \xi \circ \mathbf{L}^{[1]}(\tau), \tag{4.40}$$

where $\xi\circ$ is an operation satisfying

$$[\![\boldsymbol{\eta}, \xi \circ \mathbf{b}]\!] + \xi \circ [\![\boldsymbol{\eta}, \mathbf{b}]\!] = \mathbf{b}. \tag{4.41}$$

For the cyclicity of $\mathbf{L}^{[0]}$, one can define $\xi\circ$ by any cyclic assignment of the ξ. One conventional choice is as follows: for n-coderivation $\mathbf{b}_n$, $\xi \circ \mathbf{b}_n$ is a coderivation derived from a map

$$\xi \circ b_n \equiv \frac{1}{n+1}\Big(\xi b_n - b_n(\xi \wedge \mathbb{I}_{n-1})\Big). \tag{4.42}$$

Next, let us consider how such products $\mathbf{L}^{[1]}$ satisfying (4.37) and (4.38) can be constructed.

Commutativity of $\mathbf{L}^{[1]}$ and $\mathbf{L}^{[0]}$

The condition for products $\mathbf{L}^{[1]}(\tau)$ to commute with $\mathbf{L}^{[0]}(\tau)$ can be derived again in the parallel way. We identify $\mathbf{L}^{[1]}_2 = \mathbf{L}^{\mathrm{B}}_2$. Then,

$$\mathbf{L}^{[1]}(0) = \mathbf{L}^{\mathrm{B}}_2. \tag{4.43}$$

Utilizing (4.43), we find (4.37) holds at $\tau = 0$, because of the L_∞-relation of $\mathbf{L}^{\mathrm{B}}$: $[\![\mathbf{L}^{[1]}(0), \mathbf{L}^{[0]}(0)]\!] = [\![\mathbf{L}_2^{\mathrm{B}}, \mathbf{Q}]\!] = 0$. Differentiating (4.37) in τ, the left-hand side of (4.37) becomes

$$\begin{aligned}
\partial_\tau [\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] &= [\![\mathbf{L}^{[1]}(\tau), [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!]]\!] + [\![\partial_\tau \mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] \\
&= [\![[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] - [\![[\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!] + [\![\partial_\tau \mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] \\
&= [\![[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![\partial_\tau \mathbf{L}^{[1]}(\tau) - [\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!]. \qquad (4.44)
\end{aligned}$$

It is sufficient to require $\mathbf{L}^{[1]}(\tau)$ to satisfy the following differential equation:

$$\partial_\tau \mathbf{L}^{[1]}(\tau) = [\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[1]}(\tau)]\!], \qquad (4.45)$$

where $\boldsymbol{\lambda}^{[1]}$ is the new gauge product

$$\boldsymbol{\lambda}^{[1]}(\tau) = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\lambda}_{n+3}^{[1]}. \qquad (4.46)$$

Then, (4.45) becomes homogeneous in $[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!]$ utilizing the L_∞-relation of $\mathbf{L}^{[0]}$:

$$\begin{aligned}
\partial_\tau [\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] &= [\![[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![[\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[1]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!] \\
&= [\![[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] - \frac{1}{2}[\![\boldsymbol{\lambda}^{[1]}(\tau), [\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!]]\!] \\
&= [\![[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!], \qquad (4.47)
\end{aligned}$$

which ensures $[\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[0]}(\tau)]\!] = 0$ at arbitrary τ, since it holds at $\tau = 0$.

η-derivation property of $\mathbf{L}^{[1]}$

Since we want to identify $\mathbf{L}^{[1]}(\tau) = [\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau)]\!]$, we require $\mathbf{L}^{[1]}$ to satisfy η-derivation property

$$[\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!] = 0. \qquad (4.48)$$

Differentiating in τ, the left-hand side of (4.48) becomes

$$\begin{aligned}
&\partial_\tau [\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!] \\
&= [\![\boldsymbol{\eta}, [\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[1]}(\tau)]\!]]\!] \\
&= [\![[\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] - [\![[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau)]\!], \mathbf{L}^{[1]}(\tau)]\!] + [\![[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[1]}(\tau)]\!] - [\![[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[1]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!] \\
&= [\![[\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!], \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![[\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!], \boldsymbol{\lambda}^{[1]}(\tau)]\!] - [\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[1]}(\tau)]\!] - [\![[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[1]}(\tau)]\!], \mathbf{L}^{[0]}(\tau)]\!]. \qquad (4.49)
\end{aligned}$$

If there exist the new products

$$\mathbf{L}^{[2]}(\tau) = \sum_{n=0}^{\infty} \tau^n \mathbf{L}_{n+3}^{[2]} \qquad (4.50)$$

which satisfy

$$0 = [\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[1]}(\tau)]\!] + 2[\![\mathbf{L}^{[2]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \qquad (4.51)$$

$$0 = [\![\boldsymbol{\eta}, \mathbf{L}^{[2]}]\!], \qquad (4.52)$$

the identification[11)]

$$[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[1]}(\tau)]\!] = 2\mathbf{L}^{[2]}(\tau), \tag{4.53}$$

or equivalently $\boldsymbol{\lambda}^{[1]}(\tau) = 2\xi \circ \mathbf{L}^{[2]}(\tau)$ provides the suitable choice of $\boldsymbol{\lambda}^{[1]}$: utilizing (4.51) and the η-derivation property of $\mathbf{L}^{[0]}$, the differential equation (4.49) becomes homogeneous in $[\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!]$, which imply the η-derivation property of $\mathbf{L}^{[1]}$ (4.48) at arbitrary τ since (4.48) holds at $\tau = 0$.

Summary so far

In this subsubsection, we have demonstrated that the NS string products $\mathbf{L}^{\mathrm{NS}} = \mathbf{L}^{[0]}$ satisfying

$$0 = [\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[0]}(\tau)]\!], \qquad 0 = [\![\boldsymbol{\eta}, \mathbf{L}^{[0]}(\tau)]\!], \tag{4.54}$$

can be constructed by introducing the products $\mathbf{L}^{[d]}$ satisfying

$$0 = [\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[1]}(\tau)]\!], \qquad 0 = 2[\![\mathbf{L}^{[0]}(\tau), \mathbf{L}^{[2]}(\tau)]\!] + [\![\mathbf{L}^{[1]}(\tau), \mathbf{L}^{[1]}(\tau)]\!], \tag{4.55}$$

$$0 = [\![\boldsymbol{\eta}, \mathbf{L}^{[1]}(\tau)]\!], \qquad 0 = [\![\boldsymbol{\eta}, \mathbf{L}^{[2]}(\tau)]\!]. \tag{4.56}$$

We can define such products $\mathbf{L}^{[d]}$ by differential equations

$$\partial_\tau \mathbf{L}^{[0]}(\tau) = [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!], \qquad \partial_\tau \mathbf{L}^{[1]}(\tau) = [\![\mathbf{L}^{[1]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!] + [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[1]}(\tau)]\!], \tag{4.57}$$

where $\boldsymbol{\lambda}^{[d]}$ are the gauge products defined by

$$\boldsymbol{\lambda}^{[0]}(\tau) = \xi \circ \mathbf{L}^{[1]}(\tau), \qquad \boldsymbol{\lambda}^{[1]}(\tau) = 2\xi \circ \mathbf{L}^{[2]}(\tau). \tag{4.58}$$

For the cyclicity of $\mathbf{L}^{[d]}$, one can define $\xi\circ$ by (4.42). In that way, we can define a product with suitable properties by introducing a new products with a certain properties.

4.2.2 Generating functions

To complete the all-order construction of the products $\mathbf{L}^{[0]}$ from $\mathbf{L}^{\mathrm{B}}$, let us consider the generating functions of $\mathbf{L}^{[d]}$:

$$\mathbf{L}(s,\tau) = \sum_{m=0}^{\infty} s^m \mathbf{L}^{[m]}(\tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s^m \tau^n \mathbf{L}^{[m]}_{m+n+1}, \tag{4.59}$$

and require $\mathbf{L}(s,\tau)$ to satisfy

$$0 = [\![\mathbf{L}(s,\tau), \mathbf{L}(s,\tau)]\!], \tag{4.60}$$

$$0 = [\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!]. \tag{4.61}$$

We illustrate the products contained in the generating functions in figure 1. Note that the lowest three orders in s of (4.60) and (4.61) give (4.54), (4.55), and (4.56). At $\tau = 0$ and $s = 0$, $\mathbf{L}(s,\tau)$ is identified with the string products $\mathbf{L}^{\mathrm{B}}$ and $\mathbf{L}^{\mathrm{NS}}$, respectively:

$$\mathbf{L}(s,0) = \mathbf{L}^{\mathrm{B}}(s) = \sum_{n=0}^{\infty} s^n \mathbf{L}^{\mathrm{B}}_{n+1}, \qquad \mathbf{L}(0,\tau) = \mathbf{L}^{[0]}(\tau) = \mathbf{L}^{\mathrm{NS}}(\tau). \tag{4.62}$$

[11)] Note that the coefficient is just a convention for the natural L_∞-relation.

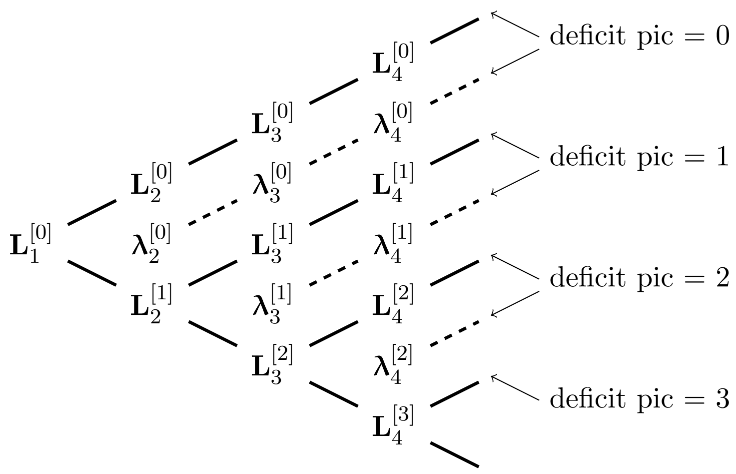

Figure 1: The products contained in the generating functions $\mathbf{L}(s,\tau)$ and $\boldsymbol{\lambda}(s,\tau)$.

Such products $\mathbf{L}(s,\tau)$ can be constructed by the following differential equations

$$\partial_\tau \mathbf{L}(s,\tau) = [\![\mathbf{L}(s,\tau), \boldsymbol{\lambda}(s,\tau)]\!], \tag{4.63}$$

$$\xi \circ \partial_s \mathbf{L}(s,\tau) = \boldsymbol{\lambda}(s,\tau), \tag{4.64}$$

where $\boldsymbol{\lambda}(s,\tau)$ is the generating function for the gauge products with deficit picture $\boldsymbol{\lambda}^{[d]}$, defined by

$$\boldsymbol{\lambda}(s,\tau) = \sum_{m=0}^{\infty} s^m \boldsymbol{\lambda}^{[m]}(\tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s^m \tau^n \boldsymbol{\lambda}^{[m]}_{m+n+2}. \tag{4.65}$$

Note that the lowest two orders in s of (4.63) and (4.64) give (4.57) and (4.58).

The L_∞-relations and the η-derivation properties

Differentiating $[\![\mathbf{L}(s,\tau), \mathbf{L}(s,\tau)]\!]$ in τ, and utilizing (4.63), we obtain the following differential equation

$$\partial_\tau [\![\mathbf{L}(s,\tau), \mathbf{L}(s,\tau)]\!] = [\![[\![\mathbf{L}(s,\tau), \mathbf{L}(s,\tau)]\!], \boldsymbol{\lambda}(s,\tau)]\!]. \tag{4.66}$$

This homogeneous differential equation leads to the L_∞-relations at arbitrary τ since the initial condition at $\tau = 0$ corresponds to the L_∞-relations of $\mathbf{L}^{\mathrm{B}}$: $[\![\mathbf{L}(s,0), \mathbf{L}(s,0)]\!] = [\![\mathbf{L}^{\mathrm{B}}(s), \mathbf{L}^{\mathrm{B}}(s)]\!] = 0$.

Differentiating $[\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!]$ in τ, we obtain the following differential equation which is homogeneous in $[\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!]$:

$$\partial_\tau [\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!] = [\![[\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!], \boldsymbol{\lambda}(s,\tau)]\!] - \frac{1}{2}\partial_s [\![\mathbf{L}(s,\tau), \mathbf{L}(s,\tau)]\!] + [\![\mathbf{L}(s,\tau), \xi \circ \big(\partial_s [\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!]\big)]\!], \tag{4.67}$$

where we used the L_∞-relation of $\mathbf{L}(s,\tau)$ and the definition of $\xi\circ$ (4.41). This homogeneous differential equation ensures the η-derivation property at arbitrary τ, since $[\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!] = 0$ holds at $\tau = 0$: $[\![\boldsymbol{\eta}, \mathbf{L}^{\mathrm{B}}(s)]\!] = 0$.

The logic of the recursive construction

To conclude this subsubsection, we explain the logic of the recursive construction of $\mathbf{L}^{\mathrm{NS}}$ from $\mathbf{L}^{\mathrm{B}}$. For convenience, let us label the products with their picture (not deficit picture), $\mathbf{L}_N^{[d]} = \mathbf{L}_N^{(N-1-d)}$ and $\boldsymbol{\lambda}_N^{[d]} = \boldsymbol{\lambda}_N^{(N-1-d)}$, and write

$$\mathbf{L}^{(n)}(s) = \sum_{m=0}^{\infty} s^m \mathbf{L}^{(n)}_{m+n+1} \quad , \quad \boldsymbol{\lambda}^{(n+1)}(s) = \sum_{m=0}^{\infty} s^m \boldsymbol{\lambda}^{(n+1)}_{m+n+2}. \tag{4.68}$$

The generating functions $\mathbf{L}(s,\tau)$ and $\boldsymbol{\lambda}(s,\tau)$ are expanded in powers of τ as

$$\mathbf{L}(s,\tau) = \sum_{n=0}^{\infty} \tau^n \mathbf{L}^{(n)}(s) \quad , \quad \boldsymbol{\lambda}(s,\tau) = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\lambda}^{(n+1)}(s). \tag{4.69}$$

Picking up the τ^{n-1} part of the differential equation $\partial_\tau \mathbf{L}(s,\tau) = [\![\mathbf{L}(s,\tau), \boldsymbol{\lambda}(s,\tau)]\!]$, we obtain

$$n\mathbf{L}^{(n)}(s) = \sum_{n_1+n_2=n-1} [\![\mathbf{L}^{(n_1)}(s), \boldsymbol{\lambda}^{(n_2+1)}(s)]\!]. \tag{4.70}$$

We call (4.70) as $\mathscr{A}(n)$ to denote the explicit n dependence. $\mathscr{A}(n)$ determines $\mathbf{L}^{(n)}(s)$ from $\mathbf{L}^{(n_1)}(s)$ for $n_1 \leq n-1$ and $\boldsymbol{\lambda}^{(n_2+1)}(s)$ for $n_2+1 \leq n$.

The τ^n part of differential equation $\boldsymbol{\lambda}(s,\tau) = \xi \circ \partial_s \mathbf{L}(s,\tau)$ determines $\boldsymbol{\lambda}^{(n+1)}(s)$ from $\mathbf{L}^{(n)}(s)$:

$$\boldsymbol{\lambda}^{(n+1)}(s) = \xi \circ \partial_s \mathbf{L}^{(n)}(s). \tag{4.71}$$

We call (4.71) as $\mathscr{B}(n+1)$ to denote the explicit n dependence. More explicitly, its s^m part reads

$$\boldsymbol{\lambda}^{(n+1)}_{m+n+2} = (m+1)\xi \circ \mathbf{L}^{(n)}_{m+n+2}. \tag{4.72}$$

The construction starts with the initial condition: $\mathbf{L}(s,0) = \mathbf{L}^{(0)}(s) = \mathbf{L}^{\mathrm{B}}(s)$. One can determine $\boldsymbol{\lambda}^{(1)}(s)$ by $\mathscr{B}(1)$, then $\mathbf{L}^{(1)}(s)$ by $\mathscr{A}(1)$, then $\boldsymbol{\lambda}^{(2)}(s)$ by $\mathscr{B}(2)$, and then $\mathbf{L}^{(2)}(s)$ by $\mathscr{A}(2)$. Thus the iterated use of $\mathscr{B}(n)$ and $\mathscr{A}(n)$ determines $\mathbf{L}^{(n)}(s)$ recursively in n, then one can obtain $\mathbf{L}(s,\tau)$, and $\mathbf{L}^{\mathrm{NS}}(\tau) = \mathbf{L}(0,\tau)$. The procedure is illustrated in figure 2.

4.2.3 Solutions by path-ordered exponentials

Utilizing the above constructed gauge products, the NS products $\mathbf{L}^{\mathrm{NS}}$ can be written as the similarity transformation of $\mathbf{Q}$:

$$\mathbf{L}^{\mathrm{NS}}(\tau) = \mathbf{G}^{-1}(\tau)\mathbf{Q}\mathbf{G}(\tau), \tag{4.73}$$

where $\mathbf{G}(\tau)$ and its inverse $\mathbf{G}^{-1}(\tau)$ are defined by the following path-ordered exponential maps:

$$\begin{aligned}\mathbf{G}(\tau) &= \overleftarrow{\mathcal{P}}\exp\left(\int_0^\tau d\tau' \boldsymbol{\lambda}^{[0]}(\tau')\right)\\ &= \mathbb{1} + \left(\int_0^\tau d\tau_1 \boldsymbol{\lambda}^{[0]}(\tau_1)\right) + \sum_{n=2}^{\infty}\left(\int_0^{\tau_{n-1}} d\tau_n \boldsymbol{\lambda}^{[0]}(\tau_n)\right)\cdots\left(\int_0^{\tau_1} d\tau_2 \boldsymbol{\lambda}^{[0]}(\tau_2)\right)\left(\int_0^{\tau} d\tau_1 \boldsymbol{\lambda}^{[0]}(\tau_1)\right)\\ &= \mathbb{1} \ + \tau\boldsymbol{\lambda}^{[0]}_2 \ + \frac{\tau^2}{2}\left(\boldsymbol{\lambda}^{[0]}_2\boldsymbol{\lambda}^{[0]}_2 + \boldsymbol{\lambda}^{[0]}_3\right) \ + \frac{\tau^3}{3!}\left(\boldsymbol{\lambda}^{[0]}_2\boldsymbol{\lambda}^{[0]}_2\boldsymbol{\lambda}^{[0]}_2 + 2\boldsymbol{\lambda}^{[0]}_2\boldsymbol{\lambda}^{[0]}_3 + \boldsymbol{\lambda}^{[0]}_3\boldsymbol{\lambda}^{[0]}_2 + 2\boldsymbol{\lambda}^{[0]}_4\right) + \cdots ,\end{aligned} \tag{4.74}$$

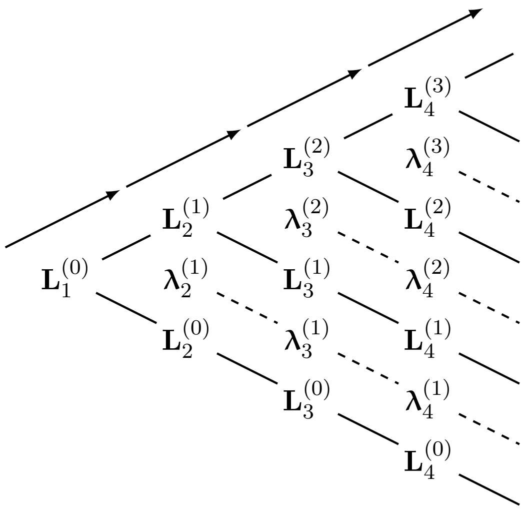

Figure 2: The logic of the recursive construction of NS products $\mathbf{L}^{\mathrm{NS}} = \sum_{n=1}^{\infty} \mathbf{L}_n^{(n-1)}$. By recursive application of $\mathscr{B}(n)$ and $\mathscr{A}(n)$, one can define products carrying higher picture number from $\mathbf{L}^{\mathrm{B}} = \sum_{n=1}^{\infty} \mathbf{L}_n^{(0)}$.

$$
\begin{aligned}
\mathbf{G}(\tau)^{-1} &= \overrightarrow{\mathcal{P}} \exp\left(-\int_0^{\tau} d\tau' \boldsymbol{\lambda}^{[0]}(\tau')\right) \\
&= \mathbb{1} + \left(-\int_0^{\tau} d\tau_1 \boldsymbol{\lambda}^{[0]}(\tau_1)\right) + \sum_{n=2}^{\infty} \left(-\int_0^{\tau} d\tau_1 \boldsymbol{\lambda}^{[0]}(\tau_1)\right)\left(-\int_0^{\tau_1} d\tau_2 \boldsymbol{\lambda}^{[0]}(\tau_2)\right) \cdots \left(-\int_0^{\tau_{n-1}} d\tau_n \boldsymbol{\lambda}^{[0]}(\tau_n)\right) \\
&= \mathbb{1} - \tau \boldsymbol{\lambda}_2^{[0]} + \frac{\tau^2}{2}\left(\boldsymbol{\lambda}_2^{[0]}\boldsymbol{\lambda}_2^{[0]} - \boldsymbol{\lambda}_3^{[0]}\right) - \frac{\tau^3}{3!}\left(\boldsymbol{\lambda}_2^{[0]}\boldsymbol{\lambda}_2^{[0]}\boldsymbol{\lambda}_2^{[0]} - \boldsymbol{\lambda}_2^{[0]}\boldsymbol{\lambda}_3^{[0]} - 2\boldsymbol{\lambda}_3^{[0]}\boldsymbol{\lambda}_2^{[0]} + 2\boldsymbol{\lambda}_4^{[0]}\right) + \cdots . \qquad (4.75)
\end{aligned}
$$

The $\leftarrow$ ($\rightarrow$) over $\mathcal{P}$ denote the ordering of the operations in which the "late-time" operator will act from the right (left). For more detail of the path-ordered maps, see appendix B. The important property of $\mathbf{G}$ and $\mathbf{G}^{-1}$ is that they satisfy the following differential equations

$$\partial_\tau \mathbf{G}(\tau) = \mathbf{G}(\tau)\boldsymbol{\lambda}^{[0]}(\tau), \qquad (4.76)$$

$$\partial_\tau \mathbf{G}^{-1}(\tau) = -\boldsymbol{\lambda}^{[0]}(\tau)\mathbf{G}^{-1}(\tau), \qquad (4.77)$$

and the initial conditions $\mathbf{G}(0) = \mathbb{1}$ and $\mathbf{G}^{-1}(0) = \mathbb{1}$. It follows from these properties that (4.73) provides the solution for the differential equation $\partial_\tau \mathbf{L}^{[0]}(\tau) = [\![\mathbf{L}^{[0]}(\tau), \boldsymbol{\lambda}^{[0]}(\tau)]\!]$, with the initial condition $\mathbf{L}^{[0]}(0) = \mathbf{Q}$, and therefore $\mathbf{L}^{\mathrm{NS}} = \mathbf{L}^{[0]}$ satisfies the L_∞-relation.

In fact, L_∞-relations follows only from the fact that the NS products $\mathbf{L}^{\mathrm{NS}}$ are written as the similarity transformations of the nilpotent operator $\mathbf{Q}$. That is, any choice of the gauge products leads to the L_∞-relations of $\mathbf{L}^{\mathrm{NS}}$. One can derive the condition of the gauge products so that the NS products satisfy the η-derivation property and are cyclic, based on the representation using the path-ordered exponential maps, see appendix C.

4.3 Action for heterotic string field theory in the L_∞-formulation

Utilizing the above constructed product $\mathbf{L}^{\rm NS} = \{L_k^{\rm NS}\}_{k\geq 1}$, the gauge-invariant action of heterotic string field theory in the L_∞-formulation is constructed by [60].

The string field Φ in the L_∞ formulation carries ghost number 2 and picture number -1 and belongs to the small Hilbert space $\mathcal{H}_{\rm small}$: $\eta\Phi = 0$. We use the inner product in the small Hilbert space introduced in (4.7), which satisfies

$$\langle\!\langle A, B\rangle\!\rangle = (-)^{(A+1)(B+1)}\langle\!\langle B, A\rangle\!\rangle. \tag{4.78}$$

The NS products $\mathbf{L}^{\rm NS}$ which are constructed in the previous subsection satisfy the L_∞-relations, the η-derivation properties, and the cyclicity: for the inner product in the large Hilbert space,

$$\langle B_1, L_n^{\rm NS}(B_2, ..., B_{n+1})\rangle = (-)^{B_1+B_2+...+B_n}\langle L_n^{\rm NS}(B_1, B_2, ..., B_n), B_{n+1}\rangle, \tag{4.79}$$

and for the inner product in the small Hilbert space,

$$\langle\!\langle B_1, L_n^{\rm NS}(B_2, ..., B_{n+1})\rangle\!\rangle = (-)^{B_1+B_2+...+B_n}\langle\!\langle L_n^{\rm NS}(B_1, B_2, ..., B_n), B_{n+1}\rangle\!\rangle, \tag{4.80}$$

which follows from (4.79), the η-derivation property of $L_n^{\rm NS}$, and that B_i belongs to the small Hilbert space: $\eta B_i = 0$. Note that if all B_i belong to the small Hilbert space, $L_n^{\rm NS}(B_2, ..., B_{n+1})$ also belongs to the small Hilbert space, which follows from the η-derivation property of $L_n^{\rm NS}$.

Utilizing these cyclic L_∞-products, the gauge-invariant action is given in the same form as closed bosonic string theory by

$$\begin{aligned}
S_{\rm EKS} &= \sum_{n=0}^{\infty}\frac{\kappa^n}{(n+2)!}\langle\!\langle \Phi, L_{n+1}^{\rm NS}(\overbrace{\Phi, \Phi, ..., \Phi}^{n+1})\rangle\!\rangle, \\
&= \int_0^1 dt\,\frac{\partial}{\partial t}\Big(\sum_{n=0}^{\infty}\frac{\kappa^n}{(n+2)!}\langle\!\langle \Phi(t), L_{n+1}^{\rm NS}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1})\rangle\!\rangle\Big) \\
&= \int_0^1 dt\sum_{n=0}^{\infty}\frac{\kappa^n}{(n+1)!}\langle\!\langle \partial_t\Phi(t), L_{n+1}^{\rm NS}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1})\rangle\!\rangle \\
&= \int_0^1 dt\langle\!\langle \partial_t\Phi(t), \pi_1\mathbf{L}^{\rm NS}(e^{\wedge\Phi(t)})\rangle\!\rangle,
\end{aligned} \tag{4.81}$$

where we introduced the t-parametrized string field $\Phi(t)$ with $t \in [0,1]$ satisfying $\Phi(0) = 0$ and $\Phi(1) = \Phi$. Note that $\pi_1\mathbf{L}_n^{\rm NS}(B_1\wedge ...\wedge B_n) = L_n^{\rm NS}(B_1, ..., B_n)$ and $L_1^{\rm NS} = Q$. We call this action the L_∞-action. Utilizing the cyclicity of $\mathbf{L}^{\rm NS}$, the variation of the action is taken as

$$\delta S_{\rm EKS} = \sum_{n=0}^{\infty}\frac{\kappa^n}{(n+1)!}\langle\!\langle \delta\Phi, L_{n+1}^{\rm NS}(\overbrace{\Phi, \Phi, ..., \Phi}^{n+1})\rangle\!\rangle = \big\langle\!\big\langle \delta\Phi, \pi_1\mathbf{L}^{\rm NS}(e^{\wedge\Phi})\big\rangle\!\big\rangle, \tag{4.82}$$

and the equation of motion is given by

$$0 = \sum_{n=0}^{\infty}\frac{\kappa^n}{(n+1)!}L_{n+1}^{\rm NS}(\overbrace{\Phi, \Phi, ..., \Phi}^{n+1}) = \pi_1\mathbf{L}^{\rm NS}(e^{\wedge\Phi}). \tag{4.83}$$

Note that this t-dependence is topological, since it does not appear in the variation of the action. Since the equation of motion is annihilated by $\pi_1 \mathbf{L}^{\mathrm{NS}}(\,\cdot\, \wedge e^{\wedge \Phi})$:

$$\pi_1 \mathbf{L}^{\mathrm{NS}}(\pi_1 \mathbf{L}^{\mathrm{NS}}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}) = \pi_1 \mathbf{L}^{\mathrm{NS}} \mathbf{L}^{\mathrm{NS}}(e^{\wedge\Phi}) = 0, \tag{4.84}$$

the action is invariant under the following gauge transformation

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} L^{\mathrm{NS}}_{n+1}(\Lambda^{\mathrm{NS}}, \overbrace{\Phi, \Phi, ..., \Phi}^{n}) = \pi_1 \mathbf{L}^{\mathrm{NS}}(\Lambda^{\mathrm{NS}} \wedge e^{\wedge\Phi}), \tag{4.85}$$

where Λ^{NS} is the gauge parameter in the small Hilbert space carrying ghost number 1 and picture number -1. Note that $\pi_1 \mathbf{L}^{\mathrm{NS}}(\,\cdot\, \wedge e^{\wedge\Phi})$ is BPZ-odd. One can find that the action (4.81), the variation of the action (4.82), and the gauge transformation (4.85) are given in the same form as those in bosonic theory (3.69), (3.70), and (3.71).

For later use, we provide other expressions of the action. Utilizing $\langle\!\langle A, B\rangle\!\rangle = \langle \xi A, B\rangle$,

$$S_{\mathrm{EKS}} = \int_0^1 dt \sum_{n=0}^{\infty} \frac{\kappa^n}{(n+1)!} \langle \partial_t \xi \Phi(t), L^{\mathrm{NS}}_{n+1}(\overbrace{\Phi(t), \ldots, \Phi(t)}^{n+1})\rangle. \tag{4.86}$$

Then, using the language of coalgebra, we can represent the action as follows:

$$S_{\mathrm{EKS}} = \int_0^1 dt\, \langle \pi_1(\boldsymbol{\xi}_t\, e^{\wedge\Phi(t)}), \pi_1\big(\mathbf{L}^{\mathrm{NS}}(e^{\wedge\Phi(t)})\big)\rangle, \tag{4.87}$$

where $\boldsymbol{\xi}_t$ is a one-coderivation derived from $\partial_t \xi$ [12]. Utilizing $\mathbf{L}^{\mathrm{NS}} = \mathbf{G}^{-1}\mathbf{Q}\mathbf{G}$, the action can be transformed as

$$\begin{aligned} S_{\mathrm{EKS}} &= \int_0^1 dt\, \langle \pi_1(\boldsymbol{\xi}_t\, e^{\wedge\Phi(t)}),\, \pi_1\big(\mathbf{G}^{-1}\, \mathbf{Q}\, \mathbf{G}(e^{\wedge\Phi(t)})\big)\rangle \\ &= \int_0^1 dt\, \langle \pi_1\big(\mathbf{G}(\boldsymbol{\xi}_t e^{\wedge\Phi(t)})\big),\, \pi_1\, \mathbf{Q}\big(\mathbf{G}(e^{\wedge\Phi(t)})\big)\rangle \\ &= \int_0^1 dt\, \langle \pi_1\big(\mathbf{G}(\boldsymbol{\xi}_t e^{\wedge\Phi(t)})\big), Q\, \pi_1\, \big(\mathbf{G}(e^{\wedge\Phi(t)})\big)\rangle. \end{aligned} \tag{4.88}$$

In the last line we used $\pi_1 \mathbf{Q} = Q\pi_1$. This expression is originally obtained for the NS open string in [42]. For the heterotic string, see also [61]. This expression is important to clarify the WZW-like structure which the L_∞-formulation naturally possesses, which is discussed in section 6.

4.4 Action for open NS string field theory in the A_∞-formulation

The NS A_∞-products $\mathbf{M}^{\mathrm{NS}}(\tau)$ for open superstring can be constructed in the same way as $\mathbf{L}^{\mathrm{NS}}(\tau)$, by replacing the starting string products $\mathbf{L}^{\mathrm{B}}$ with $\mathbf{M}^{\mathrm{B}}$. We denote the gauge product for open string by $\boldsymbol{\mu}$ instead of $\boldsymbol{\lambda}$.

The string field Ψ in the A_∞-formulation carries ghost number 1 and picture number -1 and belongs to the small Hilbert space $\mathcal{H}_{\mathrm{small}}$: $\eta\Psi = 0$. As heterotic string field theory, the gauge-invariant

[12] We will define $\xi_{\mathbb{X}}$ for the more general class of $\mathbb{X}$ later. We write $\xi_t = \xi_{\partial_t}$. For $\mathbb{X} = \partial_t$, definitions are equivalent.

action is given in the same form as bosonic theory:

$$S_{\text{EKS}} = -\sum_{n=0}^{\infty} \frac{g^n}{n+2} \langle\!\langle \Psi, M^{\text{NS}}_{n+1}(\overbrace{\Psi, \Psi, ..., \Psi}^{n+1}) \rangle\!\rangle,$$

$$= -\int_0^1 dt\, \langle\!\langle \partial_t \Psi(t), \pi_1 \big(\mathbf{M}^{\text{NS}} \frac{1}{1-\Psi(t)}\big) \rangle\!\rangle, \tag{4.89}$$

where $\Psi(t)$ with $t \in [0,1]$ is the t-parametrized string field satisfying $\Psi(0) = 0$ and $\Psi(1) = \Psi$. The variation of the action and the gauge transformation are also in the same form as bosonic theory:

$$\delta S_{\text{EKS}} = -\langle\!\langle \delta\Psi, \pi_1 \big(\mathbf{M}^{\text{NS}} (\frac{1}{1-\Psi})\big) \rangle\!\rangle, \tag{4.90}$$

$$\delta\Psi = \pi_1 \mathbf{M}^{\text{NS}} (\frac{1}{1-\Psi} \otimes \Lambda^{\text{NS}} \otimes \frac{1}{1-\Psi}), \tag{4.91}$$

where Λ^{NS} is the gauge parameter for open NS string carries ghost number 0 and picture number -1. The gauge invariance follows from the A_∞-relations and the cyclic properties of the NS products $\mathbf{M}^{\text{NS}}$. The action can be written in the following form [42]:

$$\begin{aligned} S_{\text{EKS}} &= -\int_0^1 dt\, \big\langle \partial_t \xi \Psi(t), \pi_1 \big(\mathbf{M}^{\text{NS}} (\frac{1}{1-\Psi(t)})\big) \big\rangle \\ &= -\int_0^1 dt\, \big\langle \pi_1 (\boldsymbol{\xi}_t \frac{1}{1-\Psi(t)}), \pi_1 \big(\mathbf{M}^{\text{NS}} (\frac{1}{1-\Psi(t)})\big) \big\rangle \\ &= -\int_0^1 dt\, \big\langle \pi_1 \mathbf{G} (\boldsymbol{\xi}_t \frac{1}{1-\Psi(t)}), Q \pi_1 \mathbf{G} (\frac{1}{1-\Psi(t)}) \big\rangle. \end{aligned} \tag{4.92}$$

Note on the case of the star product

If $\mathbf{M}^{\text{B}}_2$ is associative, $\mathbf{M}^{\text{B}}_2 \mathbf{M}^{\text{B}}_2 = 0$, higher products are not necessary for A_∞-relations, and can be set to zero: $\mathbf{M}^{\text{B}}_{N>3} = 0$. Then, $\mathbf{M}^{\text{B}}$ consists of $\mathbf{M}^{\text{B}}_1$ and $\mathbf{M}^{\text{B}}_2$:

$$\mathbf{M}^{\text{B}}(s) = \mathbf{M}^{\text{B}}_1 + s\mathbf{M}^{\text{B}}_2. \tag{4.93}$$

In this case, one can set $\mathbf{M}^{[d\geq 2]} = 0$ and $\boldsymbol{\mu}^{[d\geq 1]} = 0$. The nonvanishing products are the NS products with deficit picture 0 and 1, $\mathbf{M}^{[0]}$ and $\mathbf{M}^{[1]}$, and the gauge products with deficit picture 0, $\boldsymbol{\mu}^{[0]}$. The generating functions are truncated:

$$\mathbf{M}(s,\tau) = \mathbf{M}^{[0]}(\tau) + s\mathbf{M}^{[1]}(\tau), \qquad \boldsymbol{\mu}(s,\tau) = \boldsymbol{\mu}^{[0]}(\tau). \tag{4.94}$$

Since $\boldsymbol{\mu}(s,\tau) = \boldsymbol{\mu}^{[0]}(\tau)$, $\mathbf{G}(s,\tau)$ equals to $\mathbf{G}(\tau)$:

$$\mathbf{G}(s,\tau) = \overleftarrow{\mathcal{P}} \exp\Big(\int_0^\tau d\tau' \boldsymbol{\mu}^{[0]}(\tau') \Big) = \mathbf{G}(\tau), \tag{4.95}$$

and therefore

$$\mathbf{M}(s,\tau) = \mathbf{G}^{-1}(\tau) \mathbf{M}^{\text{B}}(s) \mathbf{G}(\tau). \tag{4.96}$$

Then one can find that, similarly to $\mathbf{M}^{[0]} = \mathbf{G}^{-1}(\tau)\mathbf{Q}\mathbf{G}(\tau)$, $\mathbf{M}^{[1]}$ also can be written as the similarity transformation of $\mathbf{M}^{\text{B}}_2$:

$$\mathbf{M}^{[1]} = \mathbf{G}^{-1}(\tau, 0) \mathbf{M}^{\text{B}}_2 \mathbf{G}(0,\tau). \tag{4.97}$$

It follows from (4.97) that $\mathbf{M}^{[1]}$ is also nilpotent:

$$[\![\mathbf{M}^{[1]}, \mathbf{M}^{[1]}]\!] = 0. \tag{4.98}$$

Note that (4.98) can be derived from the A_∞-relation of $\mathbf{M}^{\rm NS}(s,\tau)$. For more details, see appendix C.

5 WZW-like formulation

In this section, a successful approach to the formulation for superstring field theory for Neveu-Schwarz sector, which is called WZW-like formulation, is reviewed.

Originally, an action for open superstring field theory using the string field ϕ in the large Hilbert space is constructed in [26]:

$$S = -\frac{1}{2g^2}\int_0^1 dt\Big(\partial_t\langle e^{-g\phi(t)}\eta e^{g\phi(t)}, e^{-g\phi(t)}Qe^{g\phi(t)}\rangle + \langle e^{-g\phi(t)}\partial_t e^{g\phi(t)}, [\![e^{-g\phi(t)}Qe^{g\phi(t)}, e^{-g\phi(t)}\eta e^{g\phi(t)}]\!]^*\rangle\Big), \tag{5.1}$$

where $\phi(t)$ is a string field parameterized by t satisfying $\phi(0) = 0$ and $\phi(1) = \phi$, g is a coupling constant, and $[\![A, B]\!]^* \equiv A * B - (-)^{\epsilon(A)\epsilon(B)}B * A$ is the graded commutator using the star product. In [52], the algebraic structure for the action (5.1) to be gauge-invariant is elucidated: the gauge-invariant action for NS open string can be formulated in the following Wess-Zumino-Witten-like form:

$$S = \int_0^1 dt\langle\Psi_t, \eta\Psi_Q\rangle, \tag{5.2}$$

where the functionals Ψ_Q and Ψ_t are given by $\Psi_Q = e^{-\phi}(Qe^{\phi})$ and $\Psi_t = e^{-\phi}(\partial_t e^{\phi})$. The important properties of Ψ_Q and $\Psi_{\mathbb{X}}$ for operators $\mathbb{X} = \eta, \partial_t, \delta$ [13)] which acts as a derivation with respect to the star product are the following relations:

$$0 = Q\Psi_Q + g\Psi_Q * \Psi_Q, \tag{5.3}$$

$$(-)^{\mathbb{X}}\mathbb{X}\Psi_Q = Q\Psi_{\mathbb{X}} + g[\![\Psi_Q, \Psi_{\mathbb{X}}]\!]^*. \tag{5.4}$$

The gauge invariance of the Berkovits action follows from these relations. The formulation based on the functionals satisfying the algebraic relations, which we call the WZW-like formulation, is generalized to the heterotic string and the all-order action for heterotic string field theory was constructed in this way [52], which correctly reproduces the partial construction [53]. Later, the action for type II closed string field theory is also constructed using this WZW-like formulation [55, 56].

In the WZW-like formulation, the actions are written in terms of the functional of string field Ψ_Q and $\Psi_{\mathbb{X}}$ with a certain algebraic property, and the A_∞/L_∞ structures are not manifest. It makes the quantization of WZW-like theory difficult. If the action is written in terms of the cyclic L_∞- or A_∞-products, the classical Batalin-Vilkovisky (BV) master action is obtained by replacing the fundamental string field with the string field carrying unconstrained ghost number. While the classical BV master action for the Berkovits theory has been constructed partially [28–32], its complete form remains unknown. The reproduction of scattering amplitudes in the first quantization of string is checked for some lower point amplitudes [33–35], but there has been no all-order consideration.

In the first two subsections of this section, we summarize the the algebraic properties for Ψ_Q which we call a pure-gauge-like field and $\Psi_{\mathbb{X}}$ which we call an associated field, and see how the gauge-invariant action for heterotic string field theory can be constructed using them. After that, we provide the explicit construction of Ψ_Q and $\Psi_{\mathbb{X}}$ using the string field in the large Hilbert space. In the

13) We write Ψ_t for Ψ_{∂_t}.

last subsection, we explain the construction for open string with and without stubs, and the specific property for open string without stubs, which we call $\mathbb{Z}_2$-reversing. The relation of the Berkovits theory and the A_∞-theory is well-understood using this $\mathbb{Z}_2$-reversing [42–44].

5.1 Pure-gauge-like field and Associated fields

In this subsection we summarize the the algebraic properties for Ψ_Q which we call a pure-gauge-like field and $\Psi_{\mathbb{X}}$ which we call an associated field. In this subsection and the following subsections, we do not specify the parameterization of Ψ_Q and $\Psi_{\mathbb{X}}$. We assume that the parameter t is carried only by the t-parameterized fundamental string field, then ∂_t acts only on it. We also assume the t-parameterized fundamental string field vanishes at $t=0$. It will be shown that the t-dependence is topological: the equation of motion is independent of t.

Shifted structure

To begin with, let us recall the shifted structure. The shifted products of $\mathbf{L}^{\mathrm{B}}$ are defined by

$$[B_1, B_2, \cdots, B_n]_A = \sum_{m=0}^{\infty} \frac{1}{m!} [\underbrace{A, A, \cdots, A}_{m}, B_1, B_2, \cdots, B_n]. \tag{5.5}$$

Let $\mathcal{G}$ be a solution for the Maurer-Cartan equation for $\mathbf{L}^{\mathrm{B}}$:

$$0 = \pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge \mathcal{G}}) = Q\mathcal{G} + \frac{1}{2}[\mathcal{G}, \mathcal{G}] + \frac{1}{3!}[\mathcal{G}, \mathcal{G}, \mathcal{G}] + \cdots . \tag{5.6}$$

The $\mathcal{G}$-shifted products are defined by

$$[B_1, ..., B_n]_{\mathcal{G}} = \sum_{m=0}^{\infty} \frac{1}{m!} [\underbrace{\mathcal{G}, \mathcal{G}, \cdots, \mathcal{G}}_{m}, B_1, ..., B_n], \tag{5.7}$$

and they satisfy the L_∞-relations

$$\sum_{\sigma} \sum_{k=1}^{n} \frac{1}{k!(n-k)!} (-)^{|\sigma|} \big[[B_{i_{\sigma(1)}}, \ldots, B_{i_{\sigma(k)}}]_{\mathcal{G}}, B_{i_{\sigma(k+1)}}, \ldots, B_{i_{\sigma(n)}} \big]_{\mathcal{G}} = 0. \tag{5.8}$$

In particular we write $Q_{\mathcal{G}}$ for the $\mathcal{G}$-shifted 1-product: $Q_{\mathcal{G}} B = [B]_{\mathcal{G}}$. From the L_∞-relation of the $\mathcal{G}$-shifted $\mathbf{L}^{\mathrm{B}}$, $Q_{\mathcal{G}}$ is nilpotent:

$$(Q_{\mathcal{G}})^2 B = -[\pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge \mathcal{G}}), B]_{\mathcal{G}} = 0, \tag{5.9}$$

and acts on the $\mathcal{G}$-shifted 2-product $[B_1, B_2]_{\mathcal{G}}$ as a derivation:

$$Q_{\mathcal{G}}[B_1, B_2]_{\mathcal{G}} + [Q_{\mathcal{G}} B_1, B_2]_{\mathcal{G}} + (-)^{B_1}[B_1, Q_{\mathcal{G}} B_2]_{\mathcal{G}} = -[\pi_1 \mathbf{L}^{\mathrm{B}}(e^{\wedge \mathcal{G}}), B_1, B_2]_{\mathcal{G}} = 0. \tag{5.10}$$

Because of the shift, the operators $\mathbb{X} = \{\eta, \partial_t, \delta\}$ are not derivations on the shifted product:

$$\mathbb{X}\big[B_1, \ldots, B_n\big]^{\eta}_{\mathcal{G}} = \sum_{i=1}^{n-1} (-)^{\mathbb{X}(B_1 + \cdots + B_{k-1} + 1)} \big[B_1, \ldots, \mathbb{X}B_k, \ldots, B_n\big]_{\mathcal{G}} + (-)^{\mathbb{X}}[\mathbb{X}\mathcal{G}, B_1, \ldots, B_n]_{\mathcal{G}}, \tag{5.11}$$

where $\mathbb{X}$ in the exponent of $(-)$ represents its Grassmann parity. In particular,

$$[\![\mathbb{X}, Q_{\mathcal{G}}]\!]B = (-)^{\mathbb{X}}[\mathbb{X}\mathcal{G}, B]_{\mathcal{G}}. \tag{5.12}$$

The shifted products are cyclic, which follows from that of string products $\mathbf{L}^{\mathrm{B}}$:

$$\langle B_1, [B_2, \cdots, B_{n+1}]_{\mathcal{G}}\rangle = (-)^{B_1+B_2+\cdots+B_n}\langle [B_1, \cdots, B_n]_{\mathcal{G}}, B_{n+1}\rangle. \tag{5.13}$$

Pure-gauge-like field

The key ingredient in WZW-like theories is the pure-gauge-like field $\Psi_Q(t)$ which is a solution for the Maurer-Cartan equation for $\mathbf{L}^{\mathrm{B}}$:

$$0 = \mathcal{F}^{\mathrm{B}}(\Psi_Q(t)) = \pi_1 \mathbf{L}^{\mathrm{B}} e^{\Psi_Q(t)} = Q\Psi_Q(t) + \frac{1}{2}[\Psi_Q(t), \Psi_Q(t)] + \frac{1}{3!}[\Psi_Q(t), \Psi_Q(t), \Psi_Q(t)] + \cdots. \tag{5.14}$$

In other words, it is a solution for the equation of motion of *bosonic* string field theory. $\Psi_Q(t)$ is Grassmann-even and carries ghost number 2 and picture number 0.

Associated fields

For the derivation operators $\mathbb{X} = \eta, \partial_t, \delta$ on the string products $\mathbf{L}^{\mathrm{B}}$, $[\![\mathbb{X}, \mathbf{L}^{\mathrm{B}}]\!] = 0$, $\mathbb{X}\Psi_Q(t)$ is annihilated by the $\Psi_Q(t)$-shifted BRST operator $Q_{\Psi_Q(t)}$. Acting with $\mathbb{X}$ on (5.14), we can directly check it:

$$0 = \mathbb{X}\pi_1 \mathbf{L}^{\mathrm{B}}(e^{\Psi_Q(t)}) = (-)^{\mathbb{X}}\pi_1 \mathbf{L}^{\mathrm{B}}\big(\mathbb{X}\Psi_Q(t) \wedge e^{\Psi_Q(t)}\big) = (-)^{\mathbb{X}} Q_{\Psi_Q(t)}(\mathbb{X}\Psi_Q(t)). \tag{5.15}$$

Here we used $Q_{\Psi_Q(t)}B = \pi_1 \mathbf{L}^{\mathrm{B}}(B \wedge e^{\Psi_Q(t)})$. Since $Q_{\Psi_Q(t)}$ is nilpotent and cohomology of Q is trivial in the large Hilbert space, $\mathbb{X}\Psi_Q(t)$ is $Q_{\Psi_Q(t)}$-exact. The associated fields $\Psi_{\mathbb{X}}(t)$ can be defined to satisfy

$$(-)^{\mathbb{X}}\mathbb{X}\Psi_Q(t) = Q_{\Psi_Q(t)}\Psi_{\mathbb{X}}(t). \tag{5.16}$$

We write $\Psi_t(t)$ for $\Psi_{\partial_t}(t)$. $\Psi_\eta(t)$ is Grassmann-even and carries ghost number 2 and picture number -1, and Ψ_t and $\Psi_\delta(t)$ are Grassmann-odd and carry ghost number 1 and picture number 0.

For $\mathbb{X}, \mathbb{Y} = \eta, \partial_t, \delta$, the following relation can be derived from (5.16):

$$Q_{\Psi_Q(t)}\big(\mathbb{X}\Psi_{\mathbb{Y}}(t) - (-)^{\mathbb{X}\mathbb{Y}}\mathbb{Y}\Psi_{\mathbb{X}}(t) - (-)^{\mathbb{X}}[\Psi_{\mathbb{X}}(t), \Psi_{\mathbb{Y}}(t)]_{\Psi_Q(t)}\big) = 0. \tag{5.17}$$

We may write

$$F_{\mathbb{X}\mathbb{Y}}(t) = \mathbb{X}\Psi_{\mathbb{Y}}(t) - (-)^{\mathbb{X}\mathbb{Y}}\mathbb{Y}\Psi_{\mathbb{X}}(t) - (-)^{\mathbb{X}}[\Psi_{\mathbb{X}}(t), \Psi_{\mathbb{Y}}(t)]_{\Psi_Q(t)}, \tag{5.18}$$

then (5.17) reads $Q_{\Psi_Q(t)}F_{\mathbb{X}\mathbb{Y}}(t) = 0$.

5.2 Gauge-invariant action in the WZW-like form

The gauge-invariant action can be written in the following form which we call *the WZW-like form*:

$$S_{\mathrm{WZW}} = -\int_0^1 dt \langle \Psi_t(t), \eta\Psi_Q(t)\rangle. \tag{5.19}$$

Here we explicitly write the t-dependence of the functionals. In the following we see that this action is gauge invariant, which follows from the WZW-like relations (5.14) and (5.16).

Variation of the action

Let us take the variation of the action. First, consider the variation of the integrand:

$$\delta\big\langle \Psi_t(t), \eta\Psi_Q(t)\big\rangle = \big\langle \delta\Psi_t(t), \eta\Psi_Q(t)\big\rangle + \big\langle \Psi_t(t), \eta\delta\Psi_Q(t)\big\rangle. \tag{5.20}$$

Utilizing $\big\langle F_{\delta t}, \eta\Psi_Q(t)\big\rangle = 0$, the first term on the right-hand side of (5.20) becomes

$$\big\langle \delta\Psi_t(t), \eta\Psi_Q(t)\big\rangle = \big\langle \partial_t\Psi_\delta(t) + [\Psi_\delta(t), \Psi_t(t)]_{\Psi_Q(t)}, \eta\Psi_Q(t)\big\rangle. \tag{5.21}$$

The second term on the right-hand side of (5.20) can be transformed as follows:

$$\begin{aligned}\big\langle \Psi_t(t), \eta\delta\Psi_Q(t)\big\rangle &= \big\langle \Psi_t(t), \eta Q_{\Psi_Q(t)}\Psi_\delta(t)\big\rangle \\ &= \big\langle Q_{\Psi_Q(t)}\eta\Psi_t(t), \Psi_\delta(t)\big\rangle \\ &= \big\langle -\eta Q_{\Psi_Q(t)}\Psi_t(t) - [\eta\Psi_Q(t), \Psi_t(t)]_{\Psi_Q(t)}, \Psi_\delta(t)\big\rangle. \end{aligned}\tag{5.22}$$

The second term on the right-hand side of (5.21) and the second term on the right-hand side of (5.22) are canceled because of the cyclicity of $L^{\mathrm{B}}_{\Psi_Q(t)}$. Then, the variation becomes a total derivative in t:

$$\begin{aligned}\delta\big\langle \Psi_t(t), \eta\Psi_Q(t)\big\rangle &= \big\langle \partial_t\Psi_\delta(t), \eta\Psi_Q(t)\big\rangle - \big\langle \eta Q_{\Psi_Q(t)}\Psi_t(t), \Psi_\delta(t)\big\rangle \\ &= \big\langle \partial_t\Psi_\delta(t), \eta\Psi_Q(t)\big\rangle + \big\langle \Psi_\delta(t), \eta\partial_t\Psi_Q(t)\big\rangle \\ &= \partial_t\big\langle \Psi_\delta(t), \eta\Psi_Q(t)\big\rangle. \end{aligned}\tag{5.23}$$

Integrating over t, the variation of the action is given by

$$\delta S_{\mathrm{WZW}} = -\int_0^1 dt\,\delta\langle\Psi_t(t), \eta\Psi_Q(t)\rangle = -\int_0^1 dt\,\partial_t\langle\Psi_\delta(t), \eta\Psi_Q(t)\rangle = -\Big[\langle\Psi_\delta(t), \eta\Psi_Q(t)\rangle\Big]_0^1 \tag{5.24}$$

Since the pure-gauge-like field and associated field vanish at $t = 0$, the variation of the action becomes

$$\delta S_{\mathrm{WZW}} = -\langle\Psi_\delta, \eta\Psi_Q\rangle. \tag{5.25}$$

Here we omit the argument $t = 1$ of the fields. The variation of the action does not depend on t, and therefore t-dependence is topological. The equation of motion is given by

$$\eta\Psi_Q = 0. \tag{5.26}$$

Note that, to identify $\eta\Psi_Q = 0$ as an equation of motion, we actually need to impose the regularity condition on Ψ_δ: an arbitrary variation of fundamental string field provides arbitrary Ψ_δ. See also [44].

Gauge invariance

Since the equation of motion belongs to the kernels of Q_{Ψ_Q} and η, they generate the gauge transformations in WZW-like action:

$$\Psi_\delta = Q_{\Psi_Q}\widetilde{\Lambda} + \eta\widetilde{\Omega}. \tag{5.27}$$

The gauge parameters $\widetilde{\Lambda}$ and $\widetilde{\Omega}$ carry ghost number 0 and picture number 0 and 1, respectively. The gauge invariance follows from the nilpotency of Q_{Ψ_Q} and η.

5.3 Realization of the pure-gauge-like field and associated fields

In this subsection we introduce an explicit realization of the pure-gauge-like and associated fields as functionals of the string field. We use a string field $\widetilde{V}$ of heterotic string field theory which is a Grassmann-odd state in the large Hilbert space of ghost number 1 and picture number 0. It satisfies the constraints

$$b_0^- \widetilde{V} = 0\,, \qquad L_0^- \widetilde{V} = 0\,. \tag{5.28}$$

We introduce a t-parametrized string field $\widetilde{V}(t)$ satisfying $\widetilde{V}(0) = 0$ and $\widetilde{V}(1) = \widetilde{V}$.

Pure-gauge-like field

The pure-gauge-like field Ψ_Q can be defined in the same manner as the pure gauge construction in the bosonic theory: it is obtained by the successive infinitesimal gauge transformation of the bosonic theory from 0 along the gauge orbit parameterized by τ. The pure-gauge-like string field Ψ_Q is obtained by replacing the gauge parameter in bosonic theory with the string field $\widetilde{V}$ in WZW-like theory. Note that $\widetilde{V}$ is Grassmann odd and carries ghost number 1 and picture number 0, the same as the gauge parameter in bosonic theory. Then, Ψ_Q is defined by the following differential equation:

$$\partial_\tau \Psi_{Q[\tau]} = Q_{\Psi_{Q[\tau]}} \widetilde{V}. \tag{5.29}$$

The pure-gauge-like string field $\Psi_{Q[\tau]}$ satisfies

$$\Psi_{Q[0]} = 0, \qquad \Psi_{Q[\tau]} = \int_0^\tau d\tau' Q_{\Psi_{Q[\tau']}} \widetilde{V}. \tag{5.30}$$

Their explicit forms are given by [14)]

$$\begin{aligned}\Psi_{Q[\tau]} &= \tau Q\widetilde{V} + \frac{\kappa\tau^2}{2}[\widetilde{V}, Q\widetilde{V}] + \frac{\kappa^2\tau^3}{3!}\big([\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}] + [\widetilde{V}, [\widetilde{V}, Q\widetilde{V}]]\big)\\ &+ \frac{\kappa^3\tau^4}{4!}\Big([\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}] + [\widetilde{V}, [\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}]] + [\widetilde{V}, [\widetilde{V}, [\widetilde{V}, Q\widetilde{V}]]] + 3[\widetilde{V}, [\widetilde{V}, Q\widetilde{V}], Q\widetilde{V}]\Big) + \cdots\,,\end{aligned} \tag{5.31}$$

$$\begin{aligned}Q_{\Psi_{Q[\tau]}} B &= QB + \kappa\tau[Q\widetilde{V}, B] + \frac{\kappa^2\tau^2}{2}\big([[\widetilde{V}, Q\widetilde{V}], B] + [Q\widetilde{V}, Q\widetilde{V}, B]\big)\\ &+ \frac{\kappa^3\tau^3}{6}\Big([[\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}], B] + [[\widetilde{V}, [\widetilde{V}, Q\widetilde{V}]], B] + 3[Q\widetilde{V}, [\widetilde{V}, Q\widetilde{V}], B] + [Q\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}, B]\Big) + \cdots\,.\end{aligned} \tag{5.32}$$

We can check that it satisfies the Maurer-Cartan equation by differentiating (5.14) in τ:

$$\partial_\tau \mathcal{F}^{\rm B}(\Psi_Q) = \pi\mathbf{L}^{\rm B}\big(\partial_\tau\Psi_Q \wedge e^{\wedge\Psi_Q}\big) = Q_{\Psi_Q}(\partial_\tau\Psi_Q) = Q_{\Psi_Q}Q_{\Psi_Q}\widetilde{V} = -[\mathcal{F}^{\rm B}(\Psi_Q), \widetilde{V}]_{\Psi_Q}. \tag{5.33}$$

Since $\Psi_Q(\tau = 0) = 0$, $\mathcal{F}^{\rm B}(\Psi_Q) = 0$ holds at $\tau = 0$, and then this homogeneous differential equation ensures that $\mathcal{F}^{\rm B}(\Psi_Q) = 0$ holds at arbitrary τ.

14) Ψ_Q agree with the pure gauge in closed bosonic string field theory [75].

Associated fields

The associated fields are required to satisfy $(-)^{\mathbb{X}}\mathbb{X}\Psi_Q = Q_{\Psi_Q}\Psi_{\mathbb{X}}$. Differentiating it by τ, one can derive the differential equation which defines $\Psi_{\mathbb{X}}$. Let us define $\mathcal{I}(\tau) = Q_{\Psi_Q}\Psi_{\mathbb{X}} - (-)^{\mathbb{X}}\mathbb{X}\Psi_Q$ and consider its differentiation in τ:

$$\partial_\tau \mathcal{I}(\tau) = [\widetilde{V}, \mathcal{I}(\tau)]_{\Psi_Q} + Q_{\Psi_Q}\Big(\partial_\tau \Psi_{\mathbb{X}} - \mathbb{X}\widetilde{V} - [\widetilde{V}, \Psi_{\mathbb{X}}]_{\Psi_Q}\Big). \tag{5.34}$$

We define the associated fields $\Psi_{\mathbb{X}}[\widetilde{V}]$ by the following differential equation:

$$\frac{\partial}{\partial \tau}\Psi_{\mathbb{X}}[\tau] = \mathbb{X}\widetilde{V} + \kappa\big[\widetilde{V}, \Psi_{\mathbb{X}}[\tau]\big]_{\Psi_Q[\tau]} \tag{5.35}$$

with the initial condition $\Psi_{\mathbb{X}}[\tau = 0] = 0$. Then, the equation (5.34) becomes $\partial_\tau \mathcal{I}(\tau) = [\widetilde{V}, \mathcal{I}(\tau)]_{\Psi_Q}$, and leads to the vanishing of $\mathcal{I}(\tau)$ at arbitrary τ since $\mathcal{I}(\tau = 0) = 0$, which means $(-)^{\mathbb{X}}\mathbb{X}\Psi_Q = Q_{\Psi_Q}\Psi_{\mathbb{X}}$. The explicit forms of $\Psi_{\mathbb{X}}$ in lower order is given by

$$\begin{aligned}\Psi_{\mathbb{X}[\tau]} =& \tau\mathbb{X}\widetilde{V} + \frac{\tau^2}{2}[\widetilde{V}, \mathbb{X}\widetilde{V}] + \frac{\tau^3}{6}\big(2[\widetilde{V}, Q\widetilde{V}, \mathbb{X}\widetilde{V}] + [\widetilde{V}, [\widetilde{V}, \mathbb{X}\widetilde{V}]]\big) + \frac{\tau^4}{24}\Big(3[\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}, \mathbb{X}\widetilde{V}] \\ &+ 2[\widetilde{V}, [\widetilde{V}, Q\widetilde{V}, \mathbb{X}\widetilde{V}]] + [\widetilde{V}, [\widetilde{V}, [\widetilde{V}, \mathbb{X}\widetilde{V}]]] + 3[\widetilde{V}, [\widetilde{V}, Q\widetilde{V}], \mathbb{X}\widetilde{V}] + 3[\widetilde{V}, Q\widetilde{V}, [\widetilde{V}, \mathbb{X}\widetilde{V}]]\Big) + \cdots .\end{aligned} \tag{5.36}$$

Note that $\Psi_{\mathbb{X}}$ is invertible as a function of $\mathbb{X}\widetilde{V}$. In particular, Ψ_δ is invertible as a function of $\delta\widetilde{V}$, which guarantees the regularity condition, and $\eta\Psi_Q$ provides correct equation of motion.

Gauge transformation in terms of $\widetilde{V}$

We can solve $\Psi_\delta = Q_{\Psi_Q}\widetilde{\Lambda} + \eta\widetilde{\Omega}$ for $\delta\widetilde{V}$ perturbatively. For the computation, see [52] [15]:

$$\begin{aligned}\delta_Q\widetilde{V} =& Q\widetilde{\Lambda} + \frac{1}{2}\big(2[Q\widetilde{V}, \widetilde{\Lambda}] - [\widetilde{V}, Q\widetilde{\Lambda}]\big) + \frac{1}{12}\Big(6[[\widetilde{V}, Q\widetilde{V}], \widetilde{\Lambda}] + 6[Q\widetilde{V}, Q\widetilde{V}, \widetilde{\Lambda}] - 6[\widetilde{V}, [Q\widetilde{V}, \widetilde{\Lambda}]] \\ &- 4[\widetilde{V}, Q\widetilde{V}, Q\widetilde{\Lambda}] + [\widetilde{V}, [\widetilde{V}, Q\widetilde{\Lambda}]]\Big) + \cdots ,\end{aligned} \tag{5.37}$$

$$\delta_\eta\widetilde{V} = \eta\widetilde{\Omega} - \frac{1}{2}[\widetilde{V}, \eta\widetilde{\Omega}] + \frac{1}{6}\Big(- 2[\widetilde{V}, Q\widetilde{V}, \eta\widetilde{\Omega}] + \frac{1}{2}[\widetilde{V}, [\widetilde{V}, \eta\widetilde{\Omega}]]\Big) + \cdots . \tag{5.38}$$

5.4 $\mathbb{Z}_2$-reversing property in open superstring based on the star product

5.4.1 Characterizations and parameterizations for open string

The above construction works also for the open string, basically by replacing the starting string products $\mathbf{L}^{\mathrm{B}}$ with $\mathbf{M}^{\mathrm{B}}$ and the inner product $\langle A, B\rangle = \langle A|c_0^-|B\rangle$ with $\langle A, B\rangle = \langle A|B\rangle$. The degree for open string field is defined to be Grassmann parity plus one.

[15] We can redefine the gauge parameter $\widetilde{\Lambda}$ to $\widetilde{\Lambda}'$ so that the gauge parameter only appears as $Q\widetilde{\Lambda}'$ [52], then the gauge transformation can be written as

$$\delta_Q\widetilde{V} = Q\widetilde{\Lambda} + \frac{1}{2}[\widetilde{V}, Q\widetilde{\Lambda}] + \frac{1}{6}[\widetilde{V}, Q\widetilde{V}, Q\widetilde{\Lambda}'] + \frac{1}{12}[\widetilde{V}, [\widetilde{V}, Q\widetilde{\Lambda}']] + \cdots .$$

Characterizations

The pure-gauge-like string field is defined as a solution for the Maurer-Cartan equation for $\mathbf{M}^{\mathrm{B}}$:

$$0 = \pi_1 \mathbf{M}^{\mathrm{B}} \frac{1}{1-\Psi_Q} = Q\Psi_Q + M_2^{\mathrm{B}}(\Psi_Q, \Psi_Q) + M_3^{\mathrm{B}}(\Psi_Q, \Psi_Q, \Psi_Q) + \cdots . \tag{5.39}$$

Ψ_Q is degree even. Since $\mathbf{M}_n^{\mathrm{B}}$ carries ghost number $1-n$ and picture number 0, Ψ_Q carries ghost number 1 and picture number 0. The associated string fields are defined by $(-)^{\mathbb{X}}\mathbb{X}\Psi_Q = Q_{\Psi_Q}\Psi_{\mathbb{X}}$, the same equation as (5.16). Note that the shifted structure for the open superstring is defined by

$$M^{\mathrm{B}}_{n,\Psi_Q}(B_1, ..., B_n) = \pi_1 \mathbf{M}^{\mathrm{B}}\big(\frac{1}{1-\Psi_Q} \otimes B_1 \otimes \frac{1}{1-\Psi_Q} \otimes ... \otimes \frac{1}{1-\Psi_Q} \otimes B_n \otimes \frac{1}{1-\Psi_Q}\big). \tag{5.40}$$

Ψ_η is degree even and carries ghost number 1 and picture number -1, and Ψ_t and Ψ_δ are degree odd and carry ghost number 0 and picture number 0.

Recall that the shifted structure for heterotic string is defined by

$$[B_1, ..., B_n]_{\Psi_Q} = \pi_1 \mathbf{L}^{\mathrm{B}}(B_1 \wedge ... \wedge B_n \wedge e^{\Psi_Q}). \tag{5.41}$$

The relations in subsections 5.2 and 5.3 hold also for the open string with the reinterpretation

$$[B_1, ..., B_n]_{\Psi_Q} \longleftrightarrow \pi_1 \mathbf{M}^{\mathrm{B}}(B_1 \wedge ... \wedge B_n \wedge e^{\Psi_Q}) = \sum_\sigma (-)^{\sigma(\{B\})} M^{\mathrm{B}}_{n,\Psi_Q}(B_{\sigma(1)}, ..., B_{\sigma(n)}). \tag{5.42}$$

For example, the commutator of $\mathbb{X} = \{\eta, \partial_t, \delta\}$ and Q_{Ψ_Q}, and $F_{\mathbb{X}\mathbb{Y}}$ read

$$[\![\mathbb{X}, Q_{\Psi_Q}]\!] B = (-)^{\mathbb{X}}\big(M^{\mathrm{B}}_{2,\Psi_Q}(\mathbb{X}\Psi_Q, B) + (-)^{\deg(B)\mathbb{X}} M^{\mathrm{B}}_{2,\Psi_Q}(B, \mathbb{X}\Psi_Q)\big), \tag{5.43}$$

$$F_{\mathbb{X}\mathbb{Y}} = \mathbb{X}\Psi_{\mathbb{Y}} - (-)^{\mathbb{X}\mathbb{Y}}\mathbb{Y}\Psi_{\mathbb{X}} - (-)^{\mathbb{X}} M^{\mathrm{B}}_{2,\Psi_Q}(\Psi_{\mathbb{X}}, \Psi_{\mathbb{Y}}) - (-)^{\mathbb{X}+(\mathbb{X}+1)(\mathbb{Y}+1)} M^{\mathrm{B}}_{2,\Psi_Q}(\Psi_{\mathbb{Y}}, \Psi_{\mathbb{X}}). \tag{5.44}$$

Using the BPZ inner product $\langle A, B\rangle = \langle A|B\rangle$ for the open string, the WZW-like action is given by

$$S_{\mathrm{WZW}} = \int_0^1 dt \big\langle \Psi_t(t), \eta\Psi_Q(t)\big\rangle. \tag{5.45}$$

The equation of motion $\eta\Psi_Q = 0$ and the gauge transformations $\Psi_\delta = Q_{\Psi_Q}\widetilde{\Lambda} + \eta\widetilde{\Omega}$ can be derived and checked in the same manner as the heterotic string. The gauge parameters $\widetilde{\Lambda}$ and $\widetilde{\Omega}$ carry ghost number -1 and picture number 0 and 1, respectively.

Parameterizations

The string field ϕ of open superstring field theory is a Grassmann-even state in the large Hilbert space of ghost number 0 and picture number 0. We do not need b_0^- and L_0^- conditions. We introduce a t-parametrized string field $\phi(t)$ satisfying $\phi(0) = 0$ and $\phi(1) = \phi$. The construction in subsection 5.3 works, with the following reinterpretation of the shifted structure:

$$\partial_\tau \Psi_{Q[\tau]} = Q_{\Psi_{Q[\tau]}}\phi, \tag{5.46}$$

$$\partial_\tau \Psi_{\mathbb{X}}[\tau] = \mathbb{X}\phi + M^{\mathrm{B}}_{2,\Psi_{Q[\tau]}}(\phi, \Psi_{\mathbb{X}}[\tau]) + (-)^{\mathbb{X}+1} M^{\mathrm{B}}_{2,\Psi_{Q[\tau]}}(\Psi_{\mathbb{X}}[\tau], \phi). \tag{5.47}$$

Without stubs

If we use the star product as M_2^{B}:

$$M_2^{\mathrm{B}}(A,B)=m_2(A,B)=(-)^{\deg(A)}A*B, \tag{5.48}$$

$\mathbf{M}_2^{\mathrm{B}}$ is associative: $[\![\mathbf{M}_2^{\mathrm{B}},\mathbf{M}_2^{\mathrm{B}}]\!]=0$, and we can set $\mathbf{M}_{n\geq 3}^{\mathrm{B}}=0$. Then the shifted structures become

$$Q_{\Psi_Q}B=QB+M_2^{\mathrm{B}}(B,\Psi_Q)+M_2^{\mathrm{B}}(\Psi_Q,B), \tag{5.49}$$

$$M_{2,\Psi_Q}^{\mathrm{B}}(A,B)=M_2^{\mathrm{B}}(A,B) \tag{5.50}$$

and the differential equations (5.46) and (5.47) become

$$\partial_\tau\Psi_{\mathbb{X}}[\tau]=\mathbb{X}\phi-[\![\phi,\Psi_{\mathbb{X}}[\tau]]\!]^*, \tag{5.51}$$

the common form to $\mathbb{X}=Q,\eta,\partial_t,\delta$. Note that ϕ is degree odd. The solutions are given by

$$\Psi_{\mathbb{X}}[\tau]=e^{-\tau\phi}(\mathbb{X}e^{\tau\phi}). \tag{5.52}$$

For these $\Psi_{\mathbb{X}}$, $F_{\mathbb{X}\mathbb{Y}}=0$ holds, and the action (5.45) can be transformed into the standard WZW-like form:

$$S_{\mathrm{WZW}}=-\frac{1}{2}\int_0^1 dt\Big(\partial_t\langle e^{-\phi}\eta e^{\phi},e^{-\phi}Qe^{\phi}\rangle+\langle e^{-\phi}\partial_t e^{\phi},[\![e^{-\phi}Qe^{\phi},e^{-\phi}\eta e^{\phi}]\!]^*\rangle\Big). \tag{5.53}$$

Note that the action can be transformed into the standard WZW-like form if $F_{\eta t}=0$, see [52].

5.4.2 $\mathbb{Z}_2$-reversing property in open string and equivalence to the A_∞-action

In the action (5.53), Q and η appears symmetrically and the action preserves its form under the exchange of Q and η, which we call $\mathbb{Z}_2$-reversing property. In the following, we explain this $\mathbb{Z}_2$-reversing property of the WZW-like action (5.53), and show the equivalence of the action (5.53) and the A_∞-action using it.

$\mathbb{Z}_2$-dual description

Let us first introduce the $\mathbb{Z}_2$-reversing property. We denote the action (5.53) with the explicit dependence on its ingredients as

$$S(g,\eta,Q)=-\frac{1}{2g^2}\int_0^1 dt\Big(\partial_t\langle e^{-g\phi}\eta e^{g\phi},e^{-g\phi}Qe^{g\phi}\rangle+\langle e^{-g\phi}\partial_t e^{g\phi},[\![e^{-g\phi}Qe^{g\phi},e^{-g\phi}\eta e^{g\phi}]\!]^*\rangle\Big). \tag{5.54}$$

For a derivation operator $\mathbb{X}=\{Q,\eta,\partial,\delta\}$ which acts on the star product, it follows from $0=\mathbb{X}(1)=\mathbb{X}(e^{-g\phi}*e^{g\phi})$ that

$$e^{-g\phi}*(\mathbb{X}e^{g\phi})=-(\mathbb{X}e^{-g\phi})*e^{g\phi}. \tag{5.55}$$

Utilizing this and the properties of the inner product and the star product

$$\langle A,B\rangle=(-)^{\epsilon(A)\epsilon(B)}\langle B,A\rangle,\quad \langle A,B*C\rangle=\langle A*B,C\rangle,\quad \langle A,B*C\rangle=(-)^{\epsilon(C)(\epsilon(A)+\epsilon(B))}\langle C*A,B\rangle, \tag{5.56}$$

the action (5.54) can be transformed as follows:

$$\begin{aligned}
S(g,\eta,Q) &= -\frac{1}{2g^2}\int_0^1 dt\Big(\partial_t\langle e^{-g\phi}\eta e^{g\phi}, e^{-g\phi}Qe^{g\phi}\rangle + \langle e^{-g\phi}\partial_t e^{g\phi}, [\![e^{-g\phi}Qe^{g\phi}, e^{-g\phi}\eta e^{g\phi}]\!]^*\rangle\Big)\\
&= -\frac{1}{2g^2}\int_0^1 dt\Big(-\partial_t\langle e^{-g\phi}Qe^{g\phi}, e^{-g\phi}\eta e^{g\phi}\rangle + \langle e^{-g\phi}\partial_t e^{g\phi}, [\![e^{-g\phi}\eta e^{g\phi}, e^{-g\phi}Qe^{g\phi}]\!]^*\rangle\Big)\\
&= -\frac{1}{2g^2}\int_0^1 dt\Big(-\partial_t\langle (Qe^{-g\phi})e^{g\phi}, \eta(e^{-g\phi})e^{g\phi}\rangle - \langle (\partial_t e^{-g\phi})e^{g\phi}, [\![(\eta e^{-g\phi})e^{g\phi}, (Qe^{-g\phi})e^{g\phi}]\!]^*\rangle\Big)\\
&= \frac{1}{2g^2}\int_0^1 dt\Big(\partial_t\langle e^{g\phi}(Qe^{-g\phi}), e^{g\phi}\eta(e^{-g\phi})\rangle + \langle e^{g\phi}(\partial_t e^{-g\phi}), [\![e^{g\phi}(\eta e^{-g\phi}), e^{g\phi}(Qe^{-g\phi})]\!]^*\rangle\Big).
\end{aligned} \tag{5.57}$$

The last line is in the same form as the first line, with replacing $\eta \leftrightarrow Q$ and reversing the signs of coupling constant $g \to -g$ and the action $S \to -S$, and provides the $\mathbb{Z}_2$-dual description of the action:

$$S(g,\eta,Q) = -S(-g,Q,\eta). \tag{5.58}$$

We call this property the $\mathbb{Z}_2$-reversing property.

Utilizing this $\mathbb{Z}_2$-reversing property, the action and its variation can be written in terms of the functionals $A_{\mathbb{X}} \equiv \frac{1}{-g}e^{g\phi}(\mathbb{X}e^{-g\phi})$ which are $\mathbb{Z}_2$-dual to the functionals $\Psi_{\mathbb{X}}$, as

$$S_{\mathrm{WZW}} = -\int_0^t dt\langle A_t(t), QA_\eta(t)\rangle, \tag{5.59}$$

$$\delta S_{\mathrm{WZW}} = -\langle A_\delta, QA_\eta\rangle. \tag{5.60}$$

In particular, A_η is $\mathbb{Z}_2$-dual to pure-gauge-like field Ψ_Q, and satisfies

$$\eta A_\eta - gA_\eta * A_\eta = 0. \tag{5.61}$$

The equivalence of the Berkovits theory and the A_∞-theory is shown based on this $\mathbb{Z}_2$-reversed expressions [42]. Let us see how this $\mathbb{Z}_2$-reversing plays important roles.

Equivalence to the A_∞-action

For the open string without stubs, it is shown that the WZW-like action (5.45), or equivalently (5.59) is equivalent to the A_∞-action. In the A_∞-formulation, the constraint and the equation of motion for the fundamental string field Ψ are given by η and Q-based A_∞ products $\mathbf{M}^{\mathrm{NS}}$ respectively:

$$\mathit{Constraint}: \quad \eta\Psi = 0, \qquad \mathit{E.O.M.}: \quad \mathbf{M}^{\mathrm{NS}}\frac{1}{1-\Psi} = 0. \tag{5.62}$$

In conventional WZW-like formulation, the constraint and the equation of motion for the pure-gauge-like string field $\Psi_Q = \frac{1}{g}e^{-g\phi}(Qe^{g\phi})$ are given by Q-based A_∞ products $Q + gm_2$ and η respectively:

$$\mathit{Constraint}: \quad Q\Psi_Q + g\Psi_Q * \Psi_Q = 0, \qquad \mathit{E.O.M.}: \quad \eta\Psi_Q = 0. \tag{5.63}$$

We find that the roles of Q and η are reversed. While it is not problem on shell, this skew makes it difficult to discuss the off-shell relation of these theories. However, if we start with the action in the $\mathbb{Z}_2$-dual description (5.59), the constraint and the equation of motion in terms of the functional $A_\eta = \frac{1}{-g} e^{g\phi}(\eta e^{-g\phi})$ are given by η-based A_∞ products $\eta - gm_2$ and Q, respectively:

$$Constraint: \quad \eta A_\eta - g A_\eta * A_\eta = 0, \qquad E.O.M.: \quad Q A_\eta = 0. \tag{5.64}$$

Q and η play the same role as those in the A_∞-formulation. In fact, (5.62) can be written in terms of the redefined string field $\Psi' = \pi_1 \mathbf{G}^{-1} \frac{1}{1-\Psi}$ as

$$Constraint: \quad \eta \Psi' - g \Psi' * \Psi' = 0, \qquad E.O.M.: \quad Q\Psi' = 0. \tag{5.65}$$

The identification $\Psi' = A_\eta$ provides not only the equivalence of the equation of motion, but also the equivalence of the A_∞-action and the WZW-like action for open string without stubs, which is shown in [42]. Also, the relation between the A_∞ structures on both sides is provided based on the small Hilbert space in [43], and the relation between the gauge symmetries of the theories are provided based on the large Hilbert space in [44].

For NS open string with stubs, NS heterotic string, and NS-NS closed string, the $\mathbb{Z}_2$-reversing property of the WZW-like actions is not known, and the relation of the WZW-like actions and the A_∞- and L_∞-actions remains to be understood.[16)] However, alternative WZW-like actions can be constructed based on the $\mathbb{Z}_2$-dual characterization of the functionals [62], the generalization of the *constraint* in (5.64), which is the content in the next section. They are shown to be off-shell equivalent with L_∞/A_∞-actions, while the all-order relations with the conventional WZW-like actions have not yet understood.

[16)] Their on-shell equivalence is discussed in [61].

6 Dual WZW-like formulation

As mentioned in section 5.4, for NS open string with stubs, NS heterotic string, and NS-NS closed string, it is not understood whether and how the WZW-like actions have the $\mathbb{Z}_2$-reversing property. Then, in the WZW-like formulation, the constraint and the equation of motion require the pure-gauge-like string field Ψ_Q to be annihilated by the Q-based L_∞ products $\mathbf{L}^{\mathrm{B}}$ and η, respectively:

$$\text{Constraint:} \qquad 0 = Q\Psi_Q + \frac{1}{2}[\Psi_Q, \Psi_Q] + \frac{1}{3!}[\Psi_Q, \Psi_Q, \Psi_Q] + \cdots\,, \tag{6.1}$$

$$\text{E.O.M.:} \qquad 0 = \eta\Psi_Q. \tag{6.2}$$

While, in the L_∞-type formulation, the small Hilbert space constraint and the equation of motion require $e^{\wedge\Phi}$ to be annihilated by $\mathbf{L}^{\mathrm{B}}$ and η, respectively:

$$\text{Constraint:} \qquad 0 = \eta\Phi, \tag{6.3}$$

$$\text{E.O.M.:} \qquad 0 = Q\Phi + \frac{1}{2}L_2^{\mathrm{NS}}(\Phi,\Phi) + \frac{1}{3!}L_3^{\mathrm{NS}}(\Phi,\Phi,\Phi) + \cdots\,. \tag{6.4}$$

There is a skew between the roles of Q and η of both sides, which is one of the obstacle for the simple relation between the A_∞/L_∞-type formulation and the WZW-like formulation.

In this section, starting with the generalization of the $\mathbb{Z}_2$-dual characterization of the functionals (5.64), we introduce an alternative WZW-like formulation which is naturally related to the L_∞-formulation. We define the dual L_∞-products

$$\eta, \quad [\,\cdot\,,\,\cdot\,]^\eta, \quad [\,\cdot\,,\,\cdot\,,\,\cdot\,]^\eta, \quad \cdots\,, \tag{6.5}$$

and introduce the pure-gauge-like field A_η and the associated fields $A_{\mathbb{X}}$ which satisfy

$$0 = \eta\, A_\eta + \sum_{k=2}^{\infty}\frac{1}{k!}\,[\,\overbrace{A_\eta,\ldots,A_\eta}^{k}\,]^\eta, \tag{6.6}$$

$$(-)^{\mathbb{X}}\mathbb{X}\,A_\eta = \eta\,A_{\mathbb{X}} + \sum_{k=1}^{\infty}\frac{1}{k!}\,[\,\overbrace{A_\eta,\ldots,A_\eta}^{k},A_{\mathbb{X}}\,]^\eta, \tag{6.7}$$

which are $\mathbb{Z}_2$-reversed versions of WZW-like relations in [52], namely (5.14) and (5.16). Then we show that once A_η and $A_{\mathbb{X}}$ are given as functionals of some dynamical string field, the gauge-invariant action can be constructed in terms of them, as

$$S_\eta = \int_0^1 dt\langle A_t(t), QA_\eta(t)\rangle. \tag{6.8}$$

The constraint and the equation of motions, which will be derived in section 6.3, are summarized as

$$\text{Constraint:} \qquad 0 = \eta A_\eta + \frac{1}{2}[A_\eta, A_\eta]^\eta + \frac{1}{3!}[A_\eta, A_\eta, A_\eta]^\eta + \cdots\,, \tag{6.9}$$

$$\text{E.O.M.:} \qquad 0 = QA_\eta. \tag{6.10}$$

There is not a skew of the roles of Q and η, and then they can be naturally related to those in the L_∞-type formulation. Besides, it is helpful in the construction of the action including the Ramond sector. The NS sector part of the complete action of open superstring field theory [20] is based on the $\mathbb{Z}_2$-dual characterization (5.64), see also section 7. In the same way, the $\mathbb{Z}_2$-dual plays a crucial role in the construction of the action of heterotic string field theory including the Ramond sector, which is performed in sections 8 and 9.

In this section, we first introduce the dual products and show that the gauge invariance of the action (6.8) follows from the WZW-like relations (6.6) and (6.7). Then we show that the L_∞-action is a WZW-like action which is parameterized by Φ, $S_{\text{EKS}}[\Phi] = S_\eta[\Phi]$. The functionals appearing in the L_∞-action satisfy the relations (6.6) and (6.7), which provides the equivalence of (6.3) and (6.9), and of (6.4) and (6.10). We also give another parameterization, a parameterization by the string field in the large Hilbert space V: $S_\eta[V]$. We show that the actions in different parameterizations, namely $S_\eta[V]$ and $S_{\text{EKS}}[\Phi]$, are equivalent under the identification of the pure-gauge-like fields A_η of both sides, by almost the same procedure performed in [42], We also derive the relation between two dynamical string fields Φ and V, and discuss the equivalence of S_η and the conventional WZW-like action S_{WZW}.

Note that we use A_η, $A_{\mathbb{X}}$ for the functionals in the dual WZW-like formulation, while those in the conventional WZW-like formulation are written as Ψ_Q, $\Psi_{\mathbb{X}}$. This section is based on [62] in collaboration work H. Matsunaga.

6.1 η-based L_∞ products

Let us introduce the dual products $\mathbf{L}^\eta$ which play a role as a starting point of the dual WZW-like formulation. We first summarize the required property of the dual products, and then we provide its construction.

Properties of dual products

We require the dual products $\mathbf{L}^\eta$

$$\mathbf{L}^\eta = \sum_{n=1}^{\infty} \mathbf{L}^\eta_n, \tag{6.11}$$

to be degree odd, to satisfy the L_∞-relations, and to be cyclic:

$$[\![\mathbf{L}^\eta, \mathbf{L}^\eta]\!] = 0, \qquad (\mathbf{L}^\eta)^\dagger = -\,\mathbf{L}^\eta. \tag{6.12}$$

We also require that Q acts as a derivation with respect to $\mathbf{L}^\eta$:

$$[\![\mathbf{Q}, \mathbf{L}^\eta]\!] = 0, \tag{6.13}$$

which we call the *Q-derivation property.* The n-th product L^η_n carries ghost number $3-2n$ and picture number $n-2$. We write

$$\pi_1 \mathbf{L}^\eta_n (B_1 \wedge ... \wedge B_n) = [B_1, ..., B_n]^\eta, \tag{6.14}$$

and $[B]^\eta = \eta B$. The L_∞-relations, the derivation properties with the operators $\mathbb{X} = \{Q, \partial_t, \delta\}$, and the cyclic properties are written as

$$\sum_{\sigma}\sum_{k=1}^{n}\frac{1}{k!(n-k)!}(-)^{\sigma(\{B\})}\big[[B_{i_{\sigma(1)}},\dots,B_{i_{\sigma(k)}}]^{\eta},B_{i_{\sigma(k+1)}},\dots,B_{i_{\sigma(n)}}\big]^{\eta}=0, \tag{6.15}$$

$$\mathbb{X}\big[B_1,\dots,B_n\big]^{\eta}=\sum_{i=1}^{n}(-)^{\mathbb{X}(B_1+\cdots+B_{k-1}+1)}\big[B_1,\dots,\mathbb{X}B_k,\dots,B_n\big]^{\eta}, \tag{6.16}$$

$$\langle B_1,[B_2,\cdots,B_{n+1}]^{\eta}\rangle=(-)^{B_1+B_2+\cdots+B_n}\langle[B_1,\cdots,B_n]^{\eta},B_{n+1}\rangle, \tag{6.17}$$

where $(-)^{\sigma}$ is the sign factor of the permutation from $\{B_1,...,B_n\}$ to $\{B_{\sigma(1)},...,B_{\sigma(n)}\}$.

Construction of the dual L_∞-products $\mathbf{L}^{\eta}$

Such products can be constructed using the cohomomorphism $\mathbf{G}$ which provides the NS heterotic string products $\mathbf{L}^{\rm NS}=\mathbf{G}^{-1}\,\mathbf{Q}\,\mathbf{G}$. The product $\mathbf{L}^{\eta}$ is defined as the similarity transformation of η:

$$\mathbf{L}^{\boldsymbol{\eta}}=\mathbf{G}\,\boldsymbol{\eta}\,\mathbf{G}^{-1}. \tag{6.18}$$

The $\mathbf{L}^{\eta}$ are degree odd, and the n-th product $\mathbf{L}^{\eta}_n$ carries ghost number $3-2n$ and picture number $n-2$. $\mathbf{L}^{\eta}$ satisfies the L_∞-relations, which follow from its definition:

$$[\![\mathbf{L}^{\eta},\mathbf{L}^{\eta}]\!]=2\mathbf{L}^{\eta}\mathbf{L}^{\eta}=2\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1}\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1}=2\mathbf{G}\boldsymbol{\eta}\boldsymbol{\eta}\mathbf{G}^{-1}=0. \tag{6.19}$$

The Q-derivation properties of $\mathbf{L}^{\eta}$ follow from $[\![\boldsymbol{\eta},\mathbf{L}^{\rm NS}]\!]=0$:

$$[\![\mathbf{Q},\mathbf{L}^{\eta}]\!]=[\![\mathbf{Q},\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1}]\!]=\mathbf{G}[\![\mathbf{G}^{-1}\mathbf{Q}\mathbf{G},\boldsymbol{\eta}]\!]\mathbf{G}^{-1}=\mathbf{G}[\![\mathbf{L}^{\rm NS},\boldsymbol{\eta}]\!]\mathbf{G}^{-1}=0. \tag{6.20}$$

The cyclicity of L^{η} follows from that of the gauge products, which leads to $\mathbf{G}^{-1}=\mathbf{G}^{\dagger}$ and then

$$(\mathbf{L}^{\boldsymbol{\eta}})^{\dagger}=(\mathbf{G}\,\boldsymbol{\eta}\,\mathbf{G}^{-1})^{\dagger}=(\mathbf{G}^{-1})^{\dagger}\,\boldsymbol{\eta}^{\dagger}\,\mathbf{G}^{\dagger}=-\mathbf{G}\,\boldsymbol{\eta}\,\mathbf{G}^{-1}. \tag{6.21}$$

Expanding the path-ordered exponential $\mathbf{G}$ in τ, the explicit forms of $\mathbf{L}^{\eta}$ in lower orders read

$$\begin{aligned}\mathbf{L}^{\eta}(\tau)&=\boldsymbol{\eta}+\tau[\![\boldsymbol{\lambda}^{[0]}_2,\boldsymbol{\eta}]\!]+\frac{\tau^2}{2}\Big([\![\boldsymbol{\lambda}^{[0]}_3,\boldsymbol{\eta}]\!]+[\![\boldsymbol{\lambda}^{[0]}_2,[\![\boldsymbol{\lambda}^{[0]}_2,\boldsymbol{\eta}]\!]]\!]\Big)+\\&\quad+\frac{\tau^3}{3!}\Big(2[\![\boldsymbol{\lambda}^{[0]}_4,\boldsymbol{\eta}]\!]+2[\![\boldsymbol{\lambda}^{[0]}_2,[\![\boldsymbol{\lambda}^{[0]}_3,\boldsymbol{\eta}]\!]]\!]+[\![\boldsymbol{\lambda}^{[0]}_3,[\![\boldsymbol{\lambda}^{[0]}_2,\boldsymbol{\eta}]\!]]\!]+[\![\boldsymbol{\lambda}^{[0]}_2,[\![\boldsymbol{\lambda}^{[0]}_2,[\![\boldsymbol{\lambda}^{[0]}_2,\boldsymbol{\eta}]\!]]\!]]\!]\Big)+\cdots\\&=\boldsymbol{\eta}-\tau\mathbf{L}^{[1]}_2+\frac{\tau^2}{2}\Big(-\mathbf{L}^{[1]}_3-[\![\boldsymbol{\lambda}^{[0]}_2,\mathbf{L}^{[1]}_2]\!]\Big)+\\&\quad+\frac{\tau^3}{3!}\Big(-2\mathbf{L}^{[1]}_4-2[\![\boldsymbol{\lambda}^{[0]}_2,\mathbf{L}^{[1]}_3]\!]-[\![\boldsymbol{\lambda}^{[0]}_3,\mathbf{L}^{[1]}_2]\!]-[\![\boldsymbol{\lambda}^{[0]}_2,[\![\boldsymbol{\lambda}^{[0]}_2,\mathbf{L}^{[1]}_2]\!]]\!]\Big)+\cdots.\end{aligned} \tag{6.22}$$

Note that $\mathbf{L}^{\eta}_2=-\mathbf{L}^{[1]}_2=-\mathbf{L}^{\rm B}_2$.

Shifted dual products

The shifted products of $\mathbf{L}^{\eta}$ are defined by

$$[B_1,B_2,\cdots,B_n\,]^{\eta}_A=\sum_{m=0}^{\infty}\frac{1}{m!}[\,\underbrace{A,A,\cdots,A}_{m},B_1,B_2,\cdots,B_n\,]^{\eta}. \tag{6.23}$$

Let A_η be a solution for the Maurer-Cartan equation for $\mathbf{L}^\eta$:

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta}) = \eta A_\eta + \frac{1}{2}[A_\eta, A_\eta]^\eta + \frac{1}{3!}[A_\eta, A_\eta, A_\eta]^\eta + \cdots . \tag{6.24}$$

The A_η-shifted dual products are defined by

$$[B_1, ..., B_n]^\eta_{A_\eta} = \sum_{m=0}^{\infty} \frac{1}{m!}[\underbrace{A_\eta, A_\eta, \cdots, A_\eta}_{m}, B_1, ..., B_n]^\eta, \tag{6.25}$$

and they satisfy the L_∞-relations

$$\sum_\sigma \sum_{k=1}^{n} \frac{1}{k!(n-k)!}(-)^{\sigma(\{B\})}\big[[B_{i_{\sigma(1)}}, \dots, B_{i_{\sigma(k)}}]^\eta_{A_\eta}, B_{i_{\sigma(k+1)}}, \dots, B_{i_{\sigma(n)}}\big]^\eta_{A_\eta} = 0. \tag{6.26}$$

In particular we write D_η for the A_η-shifted 1-product: $D_\eta B = [B]^\eta_{A_\eta}$. From the L_∞-relation of the A_η-shifted $\mathbf{L}^\eta$, D_η is nilpotent:

$$(D_\eta)^2 B = -[\pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta}), B]^\eta_{A_\eta} = 0, \tag{6.27}$$

and acts on the A_η-shifted 2-product $[B_1, B_2]^\eta_{A_\eta}$ as a derivation:

$$D_\eta[B_1, B_2]^\eta_{A_\eta} + [D_\eta B_1, B_2]^\eta_{A_\eta} + (-)^{B_1}[B_1, D_\eta B_2]^\eta_{A_\eta} = -[\pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta}), B_1, B_2]^\eta_{A_\eta} = 0. \tag{6.28}$$

Because of the shift, the operators $\mathbb{X} = \{Q, \partial_t, \delta\}$ are not derivations on the shifted product:

$$\mathbb{X}\big[B_1, \dots, B_n\big]^\eta_{A_\eta} = \sum_{i=1}^{n-1}(-)^{\mathbb{X}(B_1+\cdots+B_{k-1}+1)}\big[B_1, \dots, \mathbb{X}B_k, \dots, B_n\big]^\eta_{A_\eta} + (-)^{\mathbb{X}}[\mathbb{X}A_\eta, B_1, \dots, B_n]^\eta_{A_\eta}. \tag{6.29}$$

In particular,

$$[\![\mathbb{X}, D_\eta]\!]B = (-)^{\mathbb{X}}[\mathbb{X}A_\eta, B]^\eta_{A_\eta}. \tag{6.30}$$

The shifted dual products are cyclic, which follows from the cyclicity of the dual products:

$$\langle B_1, [B_2, \cdots, B_{n+1}]^\eta_{A_\eta}\rangle = (-)^{B_1+B_2+\cdots+B_n}\langle [B_1, \cdots, B_n]^\eta_{A_\eta}, B_{n+1}\rangle. \tag{6.31}$$

6.2 Dual WZW-like action

As mentioned at the beginning of the section, the gauge-invariant action can be constructed using the functionals of the string field satisfying certain relations which we call WZW-like relations. In this subsection we do not specify the explicit parameterizations of the functionals but assume that they are parameterized by t and vanish at $t = 0$. The gauge invariance follows from the WZW-like relations, and does not depend on the specific choice of the parameterization of the functionals.

Pure-gauge-like field

The key ingredient of the dual WZW-like formulation is the pure-gauge-like field A_η which is defined to satisfy (6.6), that is, A_η is defined to be a solution to the Maurer-Cartan equation for $\mathbf{L}^\eta$:

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta}) = \eta A_\eta + \frac{1}{2}[A_\eta, A_\eta]^\eta + \frac{1}{3!}[A_\eta, A_\eta, A_\eta]^\eta + \cdots . \tag{6.32}$$

A_η is Grassmann even and carries ghost number 2 and picture number -1.

Associated fields

Since $\mathbf{L}^\eta$ commute with $\mathbb{X} = \{Q, \partial_t, \delta\}$, acting with $\mathbb{X}$ on $0 = \pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta})$, one can obtain

$$0 = \mathbb{X}\pi_1 \mathbf{L}^\eta(e^{\wedge A_\eta}) = (-)^{\mathbb{X}} \pi_1 \mathbf{L}^\eta(\mathbb{X}A_\eta \wedge e^{\wedge A_\eta}) = (-)^{\mathbb{X}} D_\eta(\mathbb{X}A_\eta). \tag{6.33}$$

We find that $\mathbb{X}A_\eta$ is annihilated by D_η and then $\mathbb{X}A_\eta$ should be D_η-exact. We can define the associated field $A_{\mathbb{X}}$ to satisfy (6.7) which can be written as

$$(-)^{\mathbb{X}} \mathbb{X} A_\eta = D_\eta A_{\mathbb{X}}. \tag{6.34}$$

We write A_t fo A_{∂_t}. A_Q is Grassmann even and carries ghost number 2 and picture number 0, and $A_{\mathbb{X}=\{\delta,\partial_t\}}$ is Grassmann odd and carries ghost number 1 and picture number 0.

From (6.34), we can derive the follow relation: for the pair of derivations $\mathbb{X}, \mathbb{Y} = \{Q, \partial_t, \delta\}$,

$$D_\eta \big(\mathbb{X}A_{\mathbb{Y}} - (-)^{\mathbb{X}\mathbb{Y}} \mathbb{Y}A_{\mathbb{X}} - (-)^{\mathbb{X}} [A_{\mathbb{X}}, A_{\mathbb{Y}}]^\eta_{A_\eta} \big) = 0. \tag{6.35}$$

To derive it, consider the action of the commutator of $\mathbb{X}$ and $\mathbb{Y}$ on A_η:

$$\begin{aligned}
[\![\mathbb{X}, \mathbb{Y}]\!] A_\eta &= \mathbb{X}\mathbb{Y}A_\eta - (-)^{\mathbb{X}\mathbb{Y}} \mathbb{Y}\mathbb{X}A_\eta \\
&= (-)^{\mathbb{Y}} \mathbb{X} D_\eta A_{\mathbb{Y}} - (-)^{\mathbb{X}\mathbb{Y}} (-)^{\mathbb{X}} \mathbb{Y} D_\eta A_{\mathbb{X}} \\
&= (-)^{\mathbb{Y}} \big([D_\eta A_{\mathbb{X}}, A_{\mathbb{Y}}]^\eta_{A_\eta} + (-)^{\mathbb{X}} D_\eta \mathbb{X} A_{\mathbb{Y}} \big) - (-)^{\mathbb{X}\mathbb{Y}+\mathbb{X}} \big([D_\eta A_{\mathbb{Y}}, A_{\mathbb{X}}]^\eta_{A_\eta} + (-)^{\mathbb{Y}} D_\eta \mathbb{Y} A_{\mathbb{X}} \big) \\
&= (-)^{\mathbb{Y}+\mathbb{X}} D_\eta \big(\mathbb{X}A_{\mathbb{Y}} - (-)^{\mathbb{X}\mathbb{Y}} \mathbb{Y}A_{\mathbb{X}} - (-)^{\mathbb{X}} [A_{\mathbb{X}}, A_{\mathbb{Y}}]^\eta_{A_\eta} \big).
\end{aligned} \tag{6.36}$$

We used $[\![\mathbb{X}, D_\eta]\!] B = [D_\eta A_{\mathbb{X}}, B]^\eta_{A_\eta}$ which follows from (6.30) and (6.34). Since the commutator equals zero: $[\![\mathbb{X}, \mathbb{Y}]\!] = 0$, we obtain (6.35). We may write

$$F^\eta_{\mathbb{X}\mathbb{Y}} = \mathbb{X}A_{\mathbb{Y}} - (-)^{\mathbb{X}\mathbb{Y}} \mathbb{Y}A_{\mathbb{X}} - (-)^{\mathbb{X}} [A_{\mathbb{X}}, A_{\mathbb{Y}}]^\eta_{A_\eta}, \tag{6.37}$$

then (6.35) reads $D_\eta F^\eta_{\mathbb{X}\mathbb{Y}} = 0$.

Dual WZW-like action

We define the η-based WZW-like action by

$$S_\eta = \int_0^1 dt \langle A_t(t), Q A_\eta(t) \rangle. \tag{6.38}$$

In the following we see that this action is gauge invariant.

Variation of the action

Let us take the variation of the action (6.38). Note that the computations are based on the WZW-like relations (6.32) and (6.34), and the specific parameterization of the functionals A_η and $A_{\mathbb{X}}$ are not used.

First, consider the variation of the integrand of (6.38):

$$\delta \big\langle A_t(t), Q A_\eta(t) \big\rangle = \big\langle \delta A_t(t), Q A_\eta(t) \big\rangle + \big\langle A_t(t), Q \delta A_\eta(t) \big\rangle. \tag{6.39}$$

Utilizing (6.35), the first term of the right-hand side of (6.39) can be transformed into

$$\big\langle \delta A_t(t), QA_\eta(t)\big\rangle = \big\langle \partial_t A_\delta(t) + [A_\delta(t), A_t(t)]^\eta_{A_\eta(t)}, QA_\eta(t)\big\rangle. \tag{6.40}$$

Utilizing (6.34) and (6.30), the second term of the right-hand side of (6.39) can be transformed into

$$\begin{aligned}\big\langle A_t(t), Q\delta A_\eta(t)\big\rangle &= \big\langle A_t(t), QD_\eta A_\delta(t)\big\rangle \\ &= \big\langle D_\eta QA_t(t), A_\delta(t)\big\rangle \\ &= \big\langle -QD_\eta A_t(t) - [QA_\eta(t), A_t(t)]^\eta_{A_\eta(t)}, A_\delta(t)\big\rangle.\end{aligned} \tag{6.41}$$

The second terms of the right-hand side of (6.40) and the right-hand side of (6.41) are canceled because of the cyclicity of A_η-shifted $\mathbf{L}^\eta$. Then we find that the variation of the integrand of (6.38) becomes a total derivative in t:

$$\begin{aligned}\delta\big\langle A_t(t), QA_\eta(t)\big\rangle &= \big\langle \partial_t A_\delta(t), QA_\eta(t)\big\rangle - \big\langle QD_\eta A_t(t), A_\delta(t)\big\rangle \\ &= \big\langle \partial_t A_\delta(t), QA_\eta(t)\big\rangle + \big\langle A_\delta(t), Q\partial_t A_\eta(t)\big\rangle \\ &= \partial_t\big\langle A_\delta(t), QA_\eta(t)\big\rangle.\end{aligned} \tag{6.42}$$

Integrating it over t, the variation of the action is given by

$$\int_0^1 dt\, \delta\langle A_t(t), QA_\eta(t)\rangle = \int_0^1 dt\, \partial_t\langle A_\delta(t), QA_\eta(t)\rangle = \langle A_\delta(1), QA_\eta(1)\rangle, \tag{6.43}$$

where we used that the pure-gauge-like field A_η and the associated field $A_{\mathbb{X}}$ vanish at $t = 0$. Then the variation of the action becomes

$$\delta S_\eta = \langle A_\delta(1), QA_\eta(1)\rangle. \tag{6.44}$$

We find that the variation of the action does not depend on t, and therefore the t-dependence of the dynamical string field is topological. Then, the equation of motion reads

$$QA_\eta(1) = 0. \tag{6.45}$$

Note that, as in the conventional WZW-like formulation, to identify $QA_\eta = 0$ as an equation of motion, we actually need to impose the regularity condition on A_δ: an arbitrary variation of fundamental string field provides arbitrary A_δ.

Gauge invariances

It follows from the nilpotency of Q and D_η that the WZW-like action S_η is invariant under the following form of gauge transformations:

$$A_\delta(1) = D_\eta\Omega + Q\Lambda, \tag{6.46}$$

where Ω and Λ are gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0, and picture numbers 1 and 0, respectively.

6.3 Parameterization by string field in the small Hilbert space

In this subsection, we see that the L_∞-action is one parameterization of the η-based WZW-like action:

$$S_{\text{EKS}}[\Phi] = S_\eta[\Phi]. \tag{6.47}$$

Recall that the action in the L_∞-formulation is given by

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt\, \langle \pi_1\big(\mathbf{G}(\boldsymbol{\xi}_t e^{\wedge\Phi(t)})\big), Q\,\pi_1\,\big(\mathbf{G}(e^{\wedge\Phi(t)})\big)\rangle. \tag{6.48}$$

One can show that the functionals of the dynamical string field Φ appearing in the action

$$A_\eta[\Phi(t)] = \pi_1\mathbf{G}\big(e^{\wedge\Phi(t)}\big), \tag{6.49}$$
$$A_{\mathbb{X}}[\Phi(t)] = \pi_1\mathbf{G}\big(\boldsymbol{\xi}_{\mathbb{X}} e^{\wedge\Phi(t)}\big), \tag{6.50}$$

satisfy the WZW-like relations:

$$0 = \pi_1\mathbf{L}^{\boldsymbol{\eta}}\big(e^{\wedge A_\eta}\big), \tag{6.51}$$
$$(-)^{\mathbb{X}}\mathbb{X}\,A_\eta = D_\eta\,A_{\mathbb{X}}. \tag{6.52}$$

Let us confirm the fields $A_\eta[\Phi(t)]$ and $A_{\mathbb{X}}[\Phi(t)]$ satisfy the WZW-like relations. The first relation (6.51) directly follows from the constraint that Φ belongs to the small Hilbert space:

$$\mathbf{L}^{\boldsymbol{\eta}}\big(e^{\wedge A_\eta[\Phi(t)]}\big) = \mathbf{L}^{\boldsymbol{\eta}}\big(e^{\wedge\pi_1\mathbf{G}(e^{\wedge\Phi(t)})}\big) = (\mathbf{G}\,\boldsymbol{\eta}\,\mathbf{G}^{-1})\,\mathbf{G}\big(e^{\wedge\Phi(t)}\big) = \mathbf{G}\,\boldsymbol{\eta}\big(e^{\wedge\Phi(t)}\big) = \mathbf{G}\big(\eta\Phi(t)\wedge e^{\wedge\Phi(t)}\big) = 0. \tag{6.53}$$

The second relation (6.52) can be confirmed similarly. For the operator $\mathbb{X}$ which commutes with $\mathbf{L}^\eta$, $[\![\mathbb{X}, \mathbf{L}^\eta]\!] = 0$, since $[\![\mathbf{G}^{-1}\mathbb{X}\mathbf{G}, \boldsymbol{\eta}]\!] = 0$ holds, one can define the coderivation $\boldsymbol{\xi}_{\mathbb{X}}$ such that

$$\mathbf{G}^{-1}\mathbb{X}\mathbf{G} = (-)^{\mathbb{X}}[\![\boldsymbol{\eta}, \boldsymbol{\xi}_{\mathbb{X}}]\!]. \tag{6.54}$$

Note that for the operator $\mathbb{X}$ which commutes also with $\mathbf{G}$, such as ∂_t and δ, $\boldsymbol{\xi}_{\mathbb{X}}$ is a coderivation derived from $\mathbb{X}\xi$. Then, utilizing this $\boldsymbol{\xi}_{\mathbb{X}}$, the following relation holds:

$$\begin{aligned}
(-)^{\mathbb{X}}\mathbb{X}\mathbf{G}\,\big(e^{\wedge\Phi(t)}\big) &= (-)^{\mathbb{X}}\mathbf{G}\,(\mathbf{G}^{-1}\mathbb{X}\mathbf{G})\,\big(e^{\wedge\Phi(t)}\big) \\
&= \mathbf{G}\,\boldsymbol{\eta}\,\boldsymbol{\xi}_{\mathbb{X}}\big(e^{\wedge\Phi(t)}\big) \\
&= \mathbf{L}^{\boldsymbol{\eta}}\,\mathbf{G}\,\boldsymbol{\xi}_{\mathbb{X}}\,\Big(e^{\wedge\Phi(t)}\Big) \\
&= \mathbf{L}^{\boldsymbol{\eta}}\,\Big(\pi_1\mathbf{G}\,\boldsymbol{\xi}_{\mathbb{X}}\big(e^{\wedge\Phi(t)}\big)\wedge e^{\wedge\pi_1\mathbf{G}(e^{\wedge\Phi(t)})}\Big).
\end{aligned} \tag{6.55}$$

Since $D_\eta = \pi_1\mathbf{L}^{\boldsymbol{\eta}}\big(\mathbb{I}\wedge e^{\wedge A_\eta}\big)$, we can see $A_\eta[\Phi(t)]$ and $A_{\mathbb{X}}[\Phi(t)]$ satisfy the WZW-like relation (6.52).

Thus the functionals A_η and $A_{\mathbb{X}}$ are the pure-gauge-like field and the associated field, and the action in the L_∞-formulation is one realization of the WZW-like action:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt\,\langle A_t[\Phi(t)], QA_\eta[\Phi(t)]\rangle = S_\eta[\Phi]. \tag{6.56}$$

The variation of the action, the equation of motion, and the gauge transformation can be written in the WZW-like form (6.44), (6.45), and (6.46) [17]. For competition, let us consider the regularity of A_δ in this case. Utilizing $QA_\eta = -D_\eta A_Q$, $D_\eta = \pi_1 \mathbf{L}^\eta(\,\cdot\, \wedge e^{A_\eta})$, $\mathbf{L}^\eta = \mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1}$, and $e^{\wedge A_\eta} = \mathbf{G}e^{\wedge \Phi}$, we can transform QA_η as follows:

$$QA_\eta = -\pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(A_Q \wedge e^{\wedge A_\eta}) = -\pi_1 \mathbf{G}\big(\pi_1 \boldsymbol{\eta}\mathbf{G}^{-1}(A_Q \wedge e^{\wedge A_\eta}) \wedge e^{\wedge \Phi}\big). \tag{6.57}$$

Then, utilizing $A_\delta = \pi_1 \mathbf{G}(\xi\delta\Phi \wedge e^{\wedge\Phi})$, and the cyclicity of the cohomomorphism $\mathbf{G}$

$$\langle \pi_1 \mathbf{G}(A \wedge e^{\wedge C}), \pi_1 \mathbf{G}(B \wedge e^{\wedge C})\rangle = \langle A, B\rangle, \tag{6.58}$$

the variation of the action can be written in the following form:

$$\delta S_\eta = \big\langle A_\delta, QA_\eta \big\rangle = \big\langle \xi\delta\Phi, -\pi_1\boldsymbol{\eta}\mathbf{G}^{-1}(A_Q \wedge e^{\wedge A_\eta})\big\rangle = \big\langle\!\big\langle \delta\Phi, -\pi_1\boldsymbol{\eta}\mathbf{G}^{-1}(A_Q \wedge e^{\wedge A_\eta})\big\rangle\!\big\rangle. \tag{6.59}$$

This parameterization is regular in this sense. Utilizing $A_Q = \pi_1 \mathbf{G}\boldsymbol{\xi}_Q(e^{\wedge\Phi})$, one can check that the dual WZW-like action gives an equation of motions of L_∞-formulation (4.83) as follows:

$$-\pi_1\boldsymbol{\eta}\mathbf{G}^{-1}(A_Q \wedge e^{\wedge A_\eta}) = -\pi_1\boldsymbol{\eta}\mathbf{G}^{-1}(\pi_1\mathbf{G}\boldsymbol{\xi}_Q(e^{\wedge\Phi}) \wedge e^{\wedge A_\eta}) = -\pi_1\boldsymbol{\eta}\boldsymbol{\xi}_Q e^{\wedge\Phi} = \pi_1 \mathbf{L}^{\mathrm{NS}} e^{\wedge\Phi}, \tag{6.60}$$

where we used $-[\![\boldsymbol{\eta}, \boldsymbol{\xi}_Q]\!] = \mathbf{L}^{\mathrm{NS}}$ which follows from (6.54). One may check the equivalence of (6.45) and (4.83) more easily by

$$\pi_1 \mathbf{L}(e^{\wedge\Phi}) = \pi_1 \mathbf{G}^{-1}\mathbf{Q}\,\mathbf{G}\big(e^{\wedge\Phi}\big) = \pi_1\mathbf{G}^{-1}\mathbf{Q}\left(e^{\wedge A_\eta[\Phi]}\right) = \pi_1\mathbf{G}^{-1}\big((QA_\eta[\Phi]) \wedge e^{\wedge A_\eta[\Phi]}\big). \tag{6.61}$$

Since $\pi\mathbf{G}^{-1}(\,\cdot\, \wedge e^{\wedge A_\eta})$ is invertible, (6.45) and (4.83) are equivalent.

6.4 Parameterization by string field in the large Hilbert space

Let V be a dynamical string field which belongs to the large Hilbert space and carries ghost number 1 and picture number 0. In this subsection, we provide a parameterization of the pure-gauge-like field $A_\eta = A_\eta[V]$ and the associated fields $A_{\mathbb{X}} = A_{\mathbb{X}}[V]$ by a set of differential equations which are the $\mathbb{Z}_2$-reversed version of those in [52].

Pure-gauge-like field $A_\eta[V]$

A pure-gauge-like field of A_η is a solution of the the Maurer-Cartan equation for $\mathbf{L}^\eta$:

$$\mathcal{F}^\eta(A_\eta) = \pi_1 \mathbf{L}^\eta e^{\wedge A_\eta} = 0. \tag{6.62}$$

$A_\eta[V]$, the pure-gauge-like field parameterized by V, is obtained by the same procedure in the pure gauge construction of [52]. Let us introduce a real parameter $\tau \in [0,1]$ and a functional parameterized by τ, $A_\eta[\tau; V]$. We define $A_\eta[\tau; V]$ by the differential equation

$$\begin{aligned} \partial_\tau A_\eta[\tau; V] &= D_\eta(\tau) V \\ &= \eta\, V + \sum_{k=1}^{\infty} \frac{\kappa^{m-1}}{m!} \big[\underbrace{A_\eta[\tau; V], \ldots, A_\eta[\tau; V]}_{m}, V\big]^\eta, \end{aligned} \tag{6.63}$$

[17] The relations of the gauge transformations in the WZW-form (6.46) and the original form (4.85) are discussed in [62].

with the initial condition $A_\eta[\tau=0;V]=0$. $A_\eta[V]$ is given by setting $\tau=1$ of $A_\eta[\tau;V]$:

$$A_\eta[V] \equiv A_\eta[\tau=1;V]. \tag{6.64}$$

One can check that the differential equation (6.63) actually provides a solution for the Maurer-Cartan equation for $\mathbf{L}^\eta$

$$\mathcal{F}^\eta(A_\eta[\tau;V]) = \pi_1 \mathbf{L}^\eta e^{\wedge A_\eta[\tau;V]} = 0. \tag{6.65}$$

Differentiating $\mathcal{F}^\eta(A_\eta[\tau;V])$ in τ, we obtain

$$\partial_\tau \mathcal{F}^\eta(A_\eta) = \pi \mathbf{L}^\eta\big(\partial_\tau A_\eta \wedge e^{\wedge A_\eta}\big) = D_\eta(\partial_\tau A_\eta) = D_\eta D_\eta V = -[\mathcal{F}^\eta(A_\eta), V]^\eta_{A_\eta}. \tag{6.66}$$

Note that we do not use the nilpotency of D_η since it follows from (6.65), what we are going to show. At $\tau=0$, it satisfies the initial condition $\mathcal{F}^\eta(A_\eta[\tau=0;V])=0$. Then this homogeneous differential equation (6.66) ensures (6.65) for arbitrary τ, namely by setting $\tau=1$, (6.62).

Associated fields $A_{\mathbb{X}}[V]$

The associated fields $A_{\mathbb{X}}$ are the functionals satisfying

$$(-)^{\mathbb{X}} \mathbb{X} A_\eta = D_\eta A_{\mathbb{X}}. \tag{6.67}$$

Again, we introduce a real parameter $\tau \in [0,1]$ and a τ-parameterized functional $A_{\mathbb{X}}[\tau;V]$. Let us introduce a functional $\mathcal{I}(\tau)$ by

$$\mathcal{I}(\tau) \equiv D_\eta(\tau) A_{\mathbb{X}}[\tau;V] - (-)^{\mathbb{X}} \mathbb{X} A_\eta[\tau;V], \tag{6.68}$$

and consider its differentiation in τ:

$$\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]^\eta_{A_\eta} + D_\eta\big(\partial_\tau A_{\mathbb{X}} - \mathbb{X}V - \kappa[V, A_{\mathbb{X}}]^\eta_{A_\eta}\big). \tag{6.69}$$

We define the associated fields $A_{\mathbb{X}}[\tau;V]$ by the differential equation

$$\partial_\tau A_{\mathbb{X}}[\tau;V] = \mathbb{X}V + \kappa\big[V, A_{\mathbb{X}}[\tau;V]\big]^\eta_{A_\eta[\tau;V]}, \tag{6.70}$$

with the initial condition $A_{\mathbb{X}}[\tau=0;V]=0$. Then, the equation (6.69) becomes homogeneous, $\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]^\eta_{A_\eta}$, and leads to $\mathcal{I}(\tau)=0$ at arbitrary τ since $\mathcal{I}(\tau=0)=0$. We define the associated field by setting $\tau=1$:

$$A_{\mathbb{X}}[V] \equiv A_{\mathbb{X}}[\tau=1;V]. \tag{6.71}$$

$A_{\mathbb{X}}[V]$ satisfies the WZW-like relation (6.67) since $\mathcal{I}(1)=0$. Note that $A_{\mathbb{X}}$ is invertible as a function of $\mathbb{X}V$. In particular, A_δ is invertible as a function of δV, which guarantees the regularity condition, and QA_η provides correct equation of motion.

Action $S_\eta[V]$

Utilizing the pure-gauge-like string field $A_\eta[V]$ and associated fields $A_{\mathbb{X}}[V]$ which are defined by (6.63) and (6.70) to satisfy the WZW-like relations (6.62) and (6.67), one can construct a gauge-invariant action for the dynamical string field V by

$$S_\eta[V] = \int_0^1 dt \, \langle A_t[V(t)], \, Q \, A_\eta[V(t)] \rangle. \tag{6.72}$$

In the same manner as section 6.2, the variation of the action cam be taken and the gauge transformations can be derived as follows:

$$\delta S_\eta[V] = \langle A_\delta[V], \, Q \, A_\eta[V] \rangle, \tag{6.73}$$

$$A_\delta[V] = D_\eta \Omega + Q\Lambda, \tag{6.74}$$

where Ω and Λ are the gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0, and picture numbers 1 and 0, respectively.

6.5 Equivalence to the L_∞-type formulation

The η-based WZW-like actions in different parameterizations, namely $S_\eta[V]$ and $S_{\text{EKS}}[\Phi] = S_\eta[\Phi]$, are equivalent if the pure-gauge-like string fields in both parameterizations are identified [18)]

$$A_\eta[\Phi(t)] \equiv A_\eta[V(t)]. \tag{6.75}$$

Note that it provides the equivalence of the equations of motions $QA_\eta[V] = QA_\eta[\Phi]$.

Under this identification $A_\eta[\Phi] \equiv A_\eta[V]$, the associated fields in both parameterizations are equivalent up to D_η-exact terms $A_{\mathbb{X}}[\Phi] = A_{\mathbb{X}}[V] + (D_\eta\text{-exact terms})$, which is guaranteed by the WZW-like relation $(-)^{\mathbb{X}}\mathbb{X}A_\eta = D_\eta A_{\mathbb{X}}$:

$$D_\eta\big(A_{\mathbb{X}}[\Phi(t)] - A_{\mathbb{X}}[V(t)]\big) = (-)^{\mathbb{X}}\mathbb{X}\big(A_\eta[\Phi(t)] - A_\eta[V(t)]\big) = 0. \tag{6.76}$$

Recall that there exists arbitrariness to add D_η-exact terms in the associated fields, since they do not affect to the WZW-like relation $(-)^{\mathbb{X}}\mathbb{X}A_\eta = D_\eta A_{\mathbb{X}}$. Besides, since QA_η is D_η-exact, the difference between $A_t[\Phi]$ and $A_t[V]$ does not contribute to the action.

Then, the two actions are shown to be equivalent:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \, \langle A_t[\Phi(t)], QA_\eta[\Phi(t)] \rangle = \int_0^1 dt \, \langle A_t[V(t)], QA_\eta[V(t)] \rangle = S_\eta[V]. \tag{6.77}$$

Note that we only use the WZW-like relations here, and therefore this identification provides the equivalence of the WZW-like actions in the arbitrary parameterizations as long as the WZW-like relations hold.

[18)] The equivalence is shown in almost the same procedure in [42]: the equivalence of A_∞-action and the Berkovits action are shown by identifying A_η and $\pi\mathbf{G}\frac{1}{1-\Psi}$.

Relation of the fields Φ and V, and Partial-gauge-fixing

We also derive the relation of two dynamical string fields Φ and V from the identification of the pure-gauge-like fields.

The identification $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ can be solved by Φ. Exponentiating both hand side, $e^{\Psi_\eta[\Phi]} = e^{\Psi_\eta[V]}$, and using the property of the cohomomorphism and group-like element, the condition becomes

$$e^{\wedge \Psi_\eta[\Phi]} = e^{\wedge \pi_1 \mathbf{G}(e^{\wedge \Phi})} = \mathbf{G}(e^{\wedge \Phi}) = e^{\wedge \Psi_\eta[V]}. \tag{6.78}$$

Since $\mathbf{G}$ is invertible, by acting $\mathbf{G}^{-1}$ and projecting by π_1, the condition which provides $S_\eta[V]$ from the L_∞-action $S_{\text{EKS}}[\Phi]$ is obtained:

$$\Phi[V] = \pi_1 \mathbf{G}^{-1}(e^{\wedge \Psi_\eta[V]}). \tag{6.79}$$

Expanding it in powers of V, it reads

$$\begin{aligned}\Phi[V] = \eta V - \frac{\kappa}{2}\eta \lambda_2^{[0]}(V, \eta V) + \frac{\kappa^2}{12}\eta\Big(&- \lambda_3^{[0]}(V, \eta V, \eta V) + 2\lambda_2^{[0]}(\lambda_2^{[0]}(V, \eta V), \eta V) \\ &- \lambda_2^{[0]}(V, \lambda_2^{[0]}(\eta V, \eta V)) + 2\lambda_2^{[0]}(V, \eta\lambda_2^{[0]}(V, \eta V))\Big) + O(\kappa^3).\end{aligned} \tag{6.80}$$

The identification $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ can be solved also by V when the η-symmetry is fixed. By expanding $V = V[\Phi] = V_1(\Phi) + V_2(\Phi, \Phi) + V_3(\Phi, \Phi, \Phi) + \cdots$ in powers of Φ and acting with ξ on both hand sides of $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$, one can determine V_n perturbatively. A simple choice of the partial-gauge-fixing condition is $\xi V = 0$, which provides $\xi\eta V = V$. Then the explicit form of the partially-gauge-fixed string field $V(\Phi)$ which provides the L_∞-action $S_{\text{EKS}}[\Phi]$ from $S_\eta[V]$ is obtained as follows:

$$\begin{aligned}V[\Phi] = \xi\Phi + \frac{\kappa}{2}\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi) + \frac{\kappa^2}{12}\xi\eta\Big(&\lambda_3^{[0]}(\xi\Phi, \Phi, \Phi) - 2\lambda_2^{[0]}(\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi) + \lambda_2^{[0]}(\xi\Phi, \lambda_2^{[0]}(\Phi, \Phi)) \\ &+ \lambda_2^{[0]}(\xi\Phi, \eta\lambda_2^{[0]}(\xi\Phi, \Phi)) + 3\lambda_2^{[0]}(\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi)\Big) + O(\kappa^3).\end{aligned} \tag{6.81}$$

6.6 Towards the equivalence to the conventional WZW-like formulation

6.6.1 Open string with/without stubs

This formulation also works for the open string by replacing the product $\mathbf{L}^\eta$ with $\mathbf{M}^\eta$, as in section 5.4.1. We first explain the characterization and the parameterization for the open string with stubs. Then, we focus on the special case, open string without stubs, and see the equivalence to the conventional WZW-like formulation.

Characterization for the open string with stubs

The pure-gauge-like string field A_η is defined as a solution for the Maurer-Cartan equation for $\mathbf{M}^\eta$:

$$0 = \pi_1 \mathbf{M}^\eta \frac{1}{1 - A_\eta} = \eta A_\eta + M_2^\eta(A_\eta, A_\eta) + M_3^\eta(A_\eta, A_\eta, A_\eta) + \cdots . \tag{6.82}$$

A_η is degree even. $\mathbf{M}_n^\eta$ carries ghost number $2 - n$ and picture number $n - 2$, and A_η carries ghost number 1 and picture number -1. The associated string fields are defined by $(-)^{\mathbb{X}}\mathbb{X}A_\eta = D_\eta A_{\mathbb{X}}$. A_Q

is degree even and carries ghost number 1 and picture number 0, and A_t and A_δ are degree odd and carry ghost number 0 and picture number 0. Then, the dual WZW-like action is given by

$$S_\eta = -\int_0^1 dt \langle A_t(t), QA_\eta(t)\rangle. \tag{6.83}$$

The equation of motion $QA_\eta = 0$ and the gauge transformations $A_\delta = D_\eta\Omega + Q\Lambda$ can be derived in the same manner as in the heterotic string. The gauge parameters Λ and Ω carry ghost number -1 and picture number 0 and 1, respectively.

Note that the shifted structure for the open string is defined by

$$M^\eta_{n,A_\eta}(B_1,...,B_n) = \pi_1\mathbf{M}^\eta\big(\frac{1}{1-A_\eta}\otimes B_1\otimes\frac{1}{1-A_\eta}\otimes...\otimes\frac{1}{1-A_\eta}\otimes B_n\otimes\frac{1}{1-A_\eta}\big). \tag{6.84}$$

Recall that the shifted structure for the heterotic string is defined by

$$[B_1,...,B_n]^\eta_{A_\eta} = \pi_1\mathbf{L}^\eta(B_1\wedge...\wedge B_n\wedge e^{A_\eta}). \tag{6.85}$$

The relations in from the section 6.1 to the section 6.5 hold also for the open string under the reinterpretation

$$[B_1,...,B_n]^\eta_{A_\eta} \longleftrightarrow \pi_1\mathbf{M}^\eta(B_1\wedge...\wedge B_n\wedge e^{A_\eta}) = \sum_\sigma(-)^{\sigma(\{B\})}M^\eta_{n,A_\eta}(B_{\sigma(1)},...,B_{\sigma(n)}). \tag{6.86}$$

Note that the sign factor $(-)^{\sigma(\{B\}}$ is based on the degree. For example, the commutator of $\mathbb{X} = \{Q,\partial_t,\delta\}$ and D_η, and $F^\eta_{\mathbb{XY}}$ given by (6.37) read

$$[\![\mathbb{X}, D_\eta]\!]B = (-)^{\mathbb{X}}\big(M^\eta_{n,A_\eta}(\mathbb{X}A_\eta, B) + (-)^{\deg(B)\mathbb{X}}M^\eta_{n,A_\eta}(B,\mathbb{X}A_\eta)\big), \tag{6.87}$$

$$F_{\mathbb{XY}} = \mathbb{X}A_{\mathbb{Y}} - (-)^{\mathbb{XY}}\mathbb{Y}A_{\mathbb{X}} - (-)^{\mathbb{X}}M^\eta_{n,A_\eta}(A_{\mathbb{X}},A_{\mathbb{Y}}) - (-)^{\mathbb{X}+(\mathbb{X}+1)(\mathbb{Y}+1)}M^\eta_{n,A_\eta}(A_{\mathbb{Y}},A_{\mathbb{X}}). \tag{6.88}$$

Parameterizations for the open string with stubs

Let us consider the parameterization by the string field ϕ of open NS string, which is a Grassmann-even state in the large Hilbert space of ghost number 0 and picture number 0. We do not need b_0^- and L_0^- conditions. We introduce a string field $\phi(t)$ satisfying $\phi(0) = 0$ and $\phi(1) = \phi$. The construction in section 6.4 works under the reinterpretation of the shifted structure (6.86):

$$\partial_\tau A_{\eta[\tau]} = D_\eta\phi, \tag{6.89}$$

$$\partial_\tau A_{\mathbb{X}}[\tau] = \mathbb{X}\phi + M^\eta_{n,A_\eta}(\phi, A_{\mathbb{X}}[\tau]) + (-)^{\mathbb{X}+1}M^\eta_{n,A_\eta}(A_{\mathbb{X}}[\tau],\phi). \tag{6.90}$$

The open string without stubs

If we start with the star product, we obtain $\mathbf{M}_2^{\rm B} = \mathbf{m}_2$ and $\mathbf{M}^{\rm B}_{n\geq 3} = 0$, and one can find the dual products are also truncated:

$$M_1^\eta(B) = \eta B, \qquad M_2^\eta(B_1,B_2) = -m_2(B_1,B_2) = -(-)^{\deg(A)}A*B, \qquad M^\eta_{n\geq 3} = 0, \tag{6.91}$$

and so do the dual products shifted by some field A with even degree:

$$D_\eta B = \eta B + M_2^\eta(A,B) + M_2^\eta(B,A) = \eta B - [\![A,B]\!]^*, \tag{6.92}$$

$$M_{2,A}^\eta(B_1,B_2) = M_2^\eta(B_1,B_2) = -(-)^{\deg(B_1)} B_1 * B_2, \tag{6.93}$$

$$M_{n\geq 3,A}^\eta = 0. \tag{6.94}$$

In this case the interpretation (6.86) becomes

$$D_\eta B_1 \longleftrightarrow D_\eta B_1 = \eta B - [\![A,B]\!]^*, \tag{6.95}$$

$$[B_1,B_2]_{A_\eta}^\eta \longleftrightarrow M_2^\eta(B_1,B_2) + (-)^{\deg(B_1)\deg(B_2)} M_2^\eta(B_2,B_1) = -(-)^{\deg(B_1)}[\![B_1,B_2]\!]^*, \tag{6.96}$$

$$[B_1,...,B_{n\geq 3}]_{A_\eta}^\eta \longleftrightarrow 0. \tag{6.97}$$

Then (6.89) and (6.90) become the same form for $\mathbb{X} = \eta, Q, \partial_t, \delta$,

$$\partial_\tau A_{\mathbb{X}}[\tau] = \mathbb{X}\phi + [\![\phi, A_{\mathbb{X}}[\tau]]\!]^*, \tag{6.98}$$

and the solutions are given by

$$A_{\mathbb{X}}[\tau] = -e^{\tau\phi}(\mathbb{X}e^{-\tau\phi}). \tag{6.99}$$

The dual WZW-like action (6.83) is written in terms of $A_{\mathbb{X}}$ as

$$S_\eta = -\int_0^1 \langle A_t(t), QA_\eta(t)\rangle, \tag{6.100}$$

which are equivalent to the Berkovits action in $\mathbb{Z}_2$-reversed description (5.59). Then we obtain the equivalence of the conventional and dual WZW-like actions for the open string without stubs, and under the identification (6.75) they are also equivalent to the A_∞-action

$$S_{\text{WZW}} = S_\eta = S_{\text{EKS}}. \tag{6.101}$$

For completion, one can verify that the equation of motion and the constraint in terms of A_η are given by

$$\textit{Constraint}: \quad \eta A_\eta - gA_\eta * A_\eta = 0, \qquad \textit{E.O.M.}: \quad QA_\eta = 0, \tag{6.102}$$

which are equivalent to (5.64), and to (5.65) under the identification $\Psi' = A_\eta$.

6.6.2 Heterotic string

For the heterotic string, however, it is not known whether and how the WZW-like action has this $\mathbb{Z}_2$-reversing property. Then, to show the equivalence between S_η and the conventional WZW-like action S_{WZW} remains to be understood.

At least perturbatively, one can discuss the equivalence of $S_\eta[V]$ and $S_{\text{WZW}}[\widetilde{V}]$, where we denoted the dynamical string field for the conventional WZW-like action by $\widetilde{V}$ to distinguish it from the dynamical string field for S_η. The actions are given by

$$S_\eta[V] = \frac{1}{2}\langle V, Q\eta V\rangle + \frac{\kappa}{3!}\langle V, Q[V,\eta V]^\eta\rangle + \frac{\kappa^2}{4!}\langle V, Q\big([V,\eta V,\eta V]^\eta + [V,[V,\eta V]^\eta]^\eta\big)\rangle + \cdots, \tag{6.103}$$

$$S_{\text{WZW}}[\widetilde{V}] = \frac{1}{2}\langle \eta\widetilde{V}, Q\widetilde{V}\rangle + \frac{\kappa}{3!}\langle \eta\widetilde{V}, [\widetilde{V}, Q\widetilde{V}]\rangle + \frac{\kappa^2}{4!}\langle \eta\widetilde{V}, \big([\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}] + [\widetilde{V},[\widetilde{V},Q\widetilde{V}]]\big)\rangle + \cdots. \tag{6.104}$$

Since $\mathbf{L}_2^\eta = -\mathbf{L}_2^{\mathrm{B}}$, we can identify $\widetilde{V} = V + O(\kappa^2)$ and the first nontrivial order is κ^2, the quartic interaction. Let us see how the equivalence can be shown at this order.

We can show that $\mathbf{L}_3^\eta$ and $\mathbf{L}_3^{\mathrm{B}}$ are made from the same gauge product $\lambda_3^{[1]}$:

$$\mathbf{L}_3^\eta = -\frac{1}{2}[\![\mathbf{Q}, \boldsymbol{\lambda}_3^{[1]}]\!], \qquad \mathbf{L}_3^{\mathrm{BOS}} = \frac{1}{2}[\![\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{[1]}]\!]. \tag{6.105}$$

The latter comes from its construction, and the former will be derived in section 9.1. Utilizing them, the first terms of the quartic interactions in both actions can be written in term of $\lambda_3^{[1]}$ as

$$\langle V, Q[V, \eta V, \eta V]^\eta \rangle = -\frac{1}{2}\langle QV, \lambda_3^{[1]}(QV, \eta V, \eta V)\rangle + \langle QV, \lambda_3^{[1]}(V, Q\eta V, \eta V)\rangle, \tag{6.106}$$

$$\langle \eta\widetilde{V}, [\widetilde{V}, Q\widetilde{V}, Q\widetilde{V}]\rangle = -\frac{1}{2}\langle \eta\widetilde{V}, \lambda_3^{[1]}(\eta\widetilde{V}, Q\widetilde{V}, Q\widetilde{V})\rangle - \langle \eta\widetilde{V}, \lambda_3^{[1]}(\widetilde{V}, Q\eta\widetilde{V}, Q\widetilde{V})\rangle. \tag{6.107}$$

The difference between the quartic interactions comes from the second terms of (6.106) and (6.107):

$$S_{\eta,4}[V] - S_{\mathrm{WZW},4}[\widetilde{V}] = \frac{\kappa^2}{4!}\langle Q\eta V, 2\lambda_3^{[1]}(V, \eta V, QV)\rangle + O(\kappa^3), \tag{6.108}$$

which can be compensated by the following field redefinition:

$$\widetilde{V} = V + \frac{2\kappa^2}{4!}\lambda_3^{[1]}(V, \eta V, QV) + O(\kappa^3). \tag{6.109}$$

Thus, if we identify the string fields $\widetilde{V}$ and V by the field redefinition (6.109), two actions $S_\eta[V]$ and $S_{\mathrm{WZW}}[\widetilde{V}]$ are shown to be equivalent up to κ^2.

Part III

Construction of heterotic string field theory including the Ramond sector

In part III, we provide a construction of an action of heterotic string field theory including the Ramond sector. First, we briefly review the construction of complete action of open superstring field theory, which is quadratic order in the Ramond string field. Then, in section 8, we naturally extend it to the heterotic string, and provide an action of heterotic string field theory up to quadratic order in the Ramond string field, starting with the dual WZW-like action for the NS sector. For the heterotic string, interaction terms of higher order in the Ramond string field will be necessary. In section 9, we construct an action of heterotic string field theory at quartic order in the Ramond string field. Our action is all-order in the NS string field at each order in the Ramond string field.

7 Complete action of open superstring field theory

Recently in the work by Kunitomo and Okawa [20], the first construction of a complete action of open superstring field theory, including both the NS sector and the Ramond sector, is provided. It is known that one can write an appropriate kinetic term for the Ramond string field which is restricted to certain subspace of the small Hilbert space [45–50]. In [20], starting with the Berkovits action in the $\mathbb{Z}_2$-dual description for the NS sector, a complete action which contains the full interaction including the Ramond string field is constructed. Later, in [51], the gauge invariance of the action in [20] is understood through the WZW-like relation including the Ramond sector, and the relation between the equation of motion in [20] and that in A_∞-formulation [40] is discussed. In this section we briefly review [20] and [51].

7.1 Kinetic term for the Ramond sector

XY-projection

The string field for the Ramond sector of the open string Ψ is Grassmann odd, carries ghost number 1 and picture number $-1/2$, and belongs to the small Hilbert space: $\eta\Psi = 0$. The superconformal ghost can be described by $\beta(z)$ and $\gamma(z)$, which is related to the description by $\xi(z)$, $\eta(z)$, and $\phi(z)$ as follows:

$$\beta(z) = \partial\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = e^{\phi(z)}\eta(z). \tag{7.1}$$

The string field Ψ can be expanded based on the zero modes as

$$\Psi = \sum_{n=0}^{\infty}(\gamma_0)^n(\phi_n + c_0\psi_n), \tag{7.2}$$

where

$$b_0\phi_n = 0, \quad \beta_0\phi_n = 0, \quad b_0\psi_n = 0, \quad \beta_0\psi_n = 0. \tag{7.3}$$

In [45–47], it is shown that the physical state condition can be written as $Q\Psi = 0$ with the string field Ψ which is restricted to the following form:

$$\Psi = \phi - (\gamma_0 + c_0 G)\psi, \tag{7.4}$$

where $G = G_0 + 2b_0\gamma_0$. It is pointed out in [50] that the restriction (7.4) is given by the projection by XY, where X and Y are Grassmann even operators with ghost numbers 0 and picture numbers 1 and -1, respectively, which are given by

$$X = -\delta(\beta_0)G_0 + b_0\delta'(\beta_0), \tag{7.5}$$

$$Y = -c_0\delta'(\gamma_0). \tag{7.6}$$

In particular, the string field Ψ in the restricted form (7.4) satisfies the projection invariance

$$XY\Psi = \Psi. \tag{7.7}$$

We say Ψ is in the restricted space when Ψ satisfies (7.7).

Properties of X and Y

The operators X and Y satisfy the following properties:

$$XYX = X, \quad YXY = Y, \quad [\![Q, X]\!] = 0, \quad \eta X\eta = 0, \quad \eta Y\eta = 0, \quad XYQXY = QXY. \tag{7.8}$$

In particular, the first means that any state of the form $A = XB$ satisfies $XYA = A$, and the last means that if Ψ belongs to the restricted space, $Q\Psi$ also belongs to it.

Since X is BRST-closed, it can be written as a BRST-exact operator in the large Hilbert space

$$X = [\![Q, \Xi]\!], \tag{7.9}$$

where Ξ is Grassmann odd and carries ghost number -1 and picture number 1. One can take Ξ to be BPZ-even[19)]

$$\langle \Xi A, B\rangle = (-)^{\epsilon(A)}\langle A, \Xi B\rangle, \tag{7.10}$$

and satisfies

$$[\![\eta, \Xi]\!] = 1. \tag{7.11}$$

In [20], Ξ in [76] is used, which is defined by

$$\Xi = \Theta(\beta_0), \tag{7.12}$$

where Θ is the Heaviside step function.

Since (7.11), we can use Ξ to relate the inner products in the small Hilbert space and in the large Hilbert space:

$$\langle\!\langle A, B\rangle\!\rangle = \langle \xi A, B\rangle = \langle \Xi A, B\rangle. \tag{7.13}$$

19) This assumption is not necessary, see [20].

The properties of the inner product is preserved: for a pair of string fields A and B in the small Hilbert space,

$$\langle\langle A, B\rangle\rangle = (-)^{\epsilon(A)\epsilon(B)}\langle\langle B, A\rangle\rangle, \quad \langle\langle QA, B\rangle\rangle = (-)^{\epsilon(A)+1}\langle\langle A, QB\rangle\rangle. \tag{7.14}$$

For the inner product in the small Hilbert space, X is BPZ-even

$$\langle\langle XA, B\rangle\rangle = \langle\langle A, XB\rangle\rangle. \tag{7.15}$$

The appropriate inner product for a pair of string fields A and B in the restricted space is given by

$$\langle\langle A, YB\rangle\rangle. \tag{7.16}$$

The following properties follow from $XYA = A$, $XYB = B$, (7.8), and (7.15) :

$$\langle\langle YA, B\rangle\rangle = \langle\langle A, YB\rangle\rangle, \quad \langle\langle QA, YB\rangle\rangle = (-)^{\epsilon(A)+1}\langle\langle A, YQB\rangle\rangle. \tag{7.17}$$

Kinetic term

The kinetic term for Ramond open string field Ψ is given by

$$S_2 = -\frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle. \tag{7.18}$$

The variation of the action can be taken as follows:

$$\delta S_2 = -\langle\langle\delta\Psi, YQ\Psi\rangle\rangle. \tag{7.19}$$

The equation of motion derived from this action is

$$Q\Psi = 0. \tag{7.20}$$

The action is invariant under the gauge transformation

$$\delta\Psi = Q\lambda, \tag{7.21}$$

where the gauge parameter λ is Grassmann even, carries ghost number 0 and picture number $-1/2$, and belongs to the small and restricted space.

The operator Y in the kinetic term (7.18) can be replaced with Y_{mid}, an insertion of $Y(z) = -c(z)\delta'(\gamma(z))$ at the midpoint of the open string. Utilizing $XY_{\text{mid}}X = X$[20], (7.18) becomes

$$S_2 = -\frac{1}{2}\langle\langle\Psi, Y_{\text{mid}}Q\Psi\rangle\rangle. \tag{7.22}$$

It coincides with the kinetic term in the Witten theory [21] for Ψ in the restricted space. The necessity of the restriction (7.4), or equivalently the XY-projection (7.7), was also pointed out in [48–50].

[20] See [20] for more details.

7.2 Complete action

In this subsection, we introduce the complete action constructed in [20] and its WZW-like structure including the Ramond sector pointed out in [51]. We only show results and essences. For the detail and explicit computations, see [20] and [51]. One may refer to section 8 where the computations for the heterotic string are given: by truncating the higher products of $\mathbf{L}^{\mathrm{B}}$ (or $\mathbf{M}^{\mathrm{B}}$), which was discussed in section 6.6.1, they are reduced to those for the open string.

NS sector

As the action for the NS sector, the Berkovits action is used in [20]. Note that the Berkovits action can be written in the η-based WZW-like form :

$$-\int_0^1 dt \langle A_t(t), Q A_\eta(t) \rangle, \tag{7.23}$$

where $A_\eta(t)$ is the pure-gauge-like field which satisfies

$$\eta A_\eta(t) - A_\eta(t) * A_\eta(t) = 0, \tag{7.24}$$

and $A_{\mathbb{X}}(t)$ is the associated field which satisfies

$$(-)^{\mathbb{X}} \mathbb{X} A_\eta(t) = D_\eta(t) A_{\mathbb{X}}(t), \tag{7.25}$$

where $D_\eta(t)$ is given by

$$D_\eta(t) B = \eta B - [\![A_\eta(t), B]\!]^*. \tag{7.26}$$

We do not need to specify their parameterization, since the gauge invariance follows from these relations. We assume the dynamical string field is parameterized by t and vanishes at $t = 0$. Since A_η satisfies the Maurer-Cartan equation for $\mathbf{M}^\eta = \boldsymbol{\eta} - \mathbf{m}_2$, where $m_2(A, B) = (-)^{\deg(A)} A * B = (-)^{\epsilon(A)+1} A * B$, $D_\eta(t)$ and the star product $*$[21] satisfy A_∞-relations, namely $D_\eta(t)$ is nilpotent and act as a derivation with respect to the star product:

$$(D_\eta(t))^2 = 0, \quad D_\eta(t)(A * B) = (D_\eta(t) A) * B + (-)^{\epsilon(A)} A * D_\eta(t) B. \tag{7.27}$$

$D_\eta(t)$ is BPZ-odd, which follows from the cyclicity of $\mathbf{M}^\eta$:

$$\langle A, D_\eta(t) B \rangle = (-)^{\epsilon(A)+1} \langle D_\eta(t) A, B \rangle. \tag{7.28}$$

Complete action

With the restricted Ramond string field $XY\Psi = \Psi$ and the kinetic term (7.18), the complete action of open superstring field theory is constructed in [20] as

$$S = -\frac{1}{2} \langle\!\langle \Psi, Y Q \Psi \rangle\!\rangle - \int_0^1 dt \langle A_t(t), Q A_\eta(t) + F(t)\Psi * F(t)\Psi \rangle, \tag{7.29}$$

[21] Consider the dual products based on the star product $\mathbf{M}^\eta = \boldsymbol{\eta} - \mathbf{m}_2$, and their shifted structure. Since $\mathbf{M}^\eta_{n\geq 3}$ are zero, the shifted 2-product M^η_{2,A_η} is just the star product m_2. See also (6.93). The shifted structures of $\mathbf{M}^\eta$, namely $M^\eta_{1,A_\eta} = D_\eta$ and $M^\eta_{2,A_\eta} = m_2$, satisfy the A_∞-relations, since A_η is a solution for the Maurer-Cartan equation for $\mathbf{M}^\eta$.

where $F(t)$ is an operator defined by

$$F(t) = \frac{1}{1 + \Xi(D_\eta(t) - \eta)} = 1 + \sum_{n=1}^{\infty} \Big(- \Xi(D_\eta(t) - \eta) \Big)^n. \tag{7.30}$$

We may write $D_\eta(t) - \eta = -[\![A_\eta(t), \ \cdot \ ]\!]^*$. $F(t)$ satisfies $F(t = 0) = 1$ since $A_\eta(t = 0) = 0$. Note that the Ramond string field Ψ does not depend on t.

This $F(t)$ is invertible and $F^{-1}(t)$ is given by $F^{-1}(t) = 1+\Xi(D_\eta(t)-\eta)$. $F(t)$ satisfies the following properties:

$$F(t)\eta = D_\eta(t)F(t), \quad [\![D_\eta(t), F(t)\Xi]\!] = 1, \quad \Xi F(t) = \Xi, \quad \langle F(t)\Xi A, B\rangle = (-)^A \langle A, F(t)\Xi B\rangle. \tag{7.31}$$

$F(t)\Psi$ is annihilated by $D_\eta(t)$ since it is $D_\eta(t)$-exact:

$$F(t)\Psi = F(t)\eta\Xi\Psi = D_\eta(t)F(t)\Xi\Psi. \tag{7.32}$$

Utilizing these properties the variation of the action is taken as

$$\delta S = -\langle A_\delta, QA_\eta + F\Psi * F\Psi\rangle - \langle\!\langle \delta\Psi, Y(Q\Psi + \eta XF\Psi)\rangle\!\rangle. \tag{7.33}$$

Here we write $A_{\mathbb{X}}(1) = A_{\mathbb{X}}$ and $F(1) = F$. The equation of motion can be read from it as

$$0 = QA_\eta + F\Psi * F\Psi, \qquad 0 = Q\Psi + \eta XF\Psi. \tag{7.34}$$

which is equivalence to that in [36] under the suitable field redefinition, see section 5 of [20]. The action is invariant under the gauge transformations

$$A_\delta = D_\eta\Omega + Q\Lambda + [\![F\Psi, F\Xi([\![F\Psi, \Lambda]\!]^* - \lambda)]\!]^*, \tag{7.35}$$

$$\delta\Psi = Q\lambda + X\eta F\Xi D_\eta([\![F\Psi, \Lambda]\!]^* - \lambda). \tag{7.36}$$

In [51] the action (7.29) was transformed into the following form by introducing the t-parameterized string field $\Psi(t)$ satisfying $\Psi(0) = 0$ and $\Psi(1) = \Psi$:

$$S = -\int_0^1 dt \Big(\langle \Xi Y \partial_t \Psi(t), QF(t)\Psi(t)\rangle + \langle A_t(t), QA_\eta(t) + F(t)\Psi(t) * F(t)\Psi(t)\rangle \Big). \tag{7.37}$$

Note that we used

$$Q\Psi + X\eta F(t)\Psi = \eta\Xi QF(t)\Psi. \tag{7.38}$$

The variation of the action can be taken as

$$\delta S = -\langle \Xi Y \delta\Psi, QF\Psi\rangle - \langle A_\delta, QA_\eta + F\Psi * F\Psi\rangle. \tag{7.39}$$

WZW-like structure including the Ramond sector

The properties of $F(t)$ of (7.30), which is necessary for the gauge invariance of the action (7.29), can be encoded into a WZW-like structure including the Ramond sector [51]. Introducing a Ramond pure-gauge-like field $A^R_\eta(t)$ by

$$A^R_\eta(t) = F(t)\Psi(t), \tag{7.40}$$

the action (7.37) can be written as follows:

$$S = -\int_0^1 dt\Big(\langle \Xi Y \partial_t(\mathcal{P}_s A_\eta^R(t)), QA_\eta^R(t)\rangle + \langle A_t^{NS}(t), QA_\eta^{NS}(t) + A_\eta^R(t) * A_\eta^R(t)\rangle\Big), \tag{7.41}$$

where we write $A_\eta^{NS} = A_\eta$ and $A_{\mathbb{X}}^{NS} = A_{\mathbb{X}}$, and $\mathcal{P}_s = \eta\xi$ is a projector to the small Hilbert space. Note that $\eta\Xi F\Psi = \eta\Xi\Psi = \Psi$. The variation of the action and the gauge transformations are given by

$$\delta S = -\langle \Xi Y \delta(\mathcal{P}_s A_\eta^R(t)), QA_\eta^R\rangle - \langle A_\delta^{NS}, QA_\eta^{NS} + A_\eta^R * A_\eta^R\rangle, \tag{7.42}$$

$$A_\delta^{NS} = D_\eta\Omega + Q\Lambda^{NS} + [\![A_\eta^R, \Lambda^R]\!]^*, \tag{7.43}$$

$$\delta(\mathcal{P}_s A_\eta^R) = -\mathcal{P}_s Q(D_\eta\Lambda^R - [\![A_\eta^R, \Lambda^{NS}]\!]^*), \tag{7.44}$$

where the gauge parameters are given by $\Lambda^{NS} = \Lambda$ and $\Lambda^R = F\Xi(-\lambda + [\![A_\eta^R, \Lambda]\!]^*)$. Note that $\delta(\mathcal{P}_s A_\eta^R)$ has to be in the restricted space: $XY\delta(\mathcal{P}_s A_\eta^R) = \delta(\mathcal{P}_s A_\eta^R)$. In this representation, the properties of $F(t)$ of (7.30) is encoded into the following relations which is called the WZW-like relations including the Ramond sector in [51]:

$$D_\eta A_\eta^R = 0, \qquad XY(\mathcal{P}_s A_\eta^R) = \mathcal{P}_s A_\eta^R, \tag{7.45}$$

$$(-)^{\mathbb{X}}\mathbb{X}A_\eta^R + [\![A_\eta^R, A_{\mathbb{X}}^{NS}]\!]^* = (D_\eta\text{-exact}) \equiv D_\eta A_{\mathbb{X}}^R, \tag{7.46}$$

where $A_{\mathbb{X}}^R$ is called the Ramond associated field. Using F of (7.30), it can be parameterized as

$$(-)^{\mathbb{X}}A_{\mathbb{X}}^R = F\Xi\big(\mathbb{X}\Psi - (-)^{\mathbb{X}}[\![A_{\mathbb{X}}^{NS}, F\Psi]\!]^* + \eta[\![\Xi, \mathbb{X}]\!]F\Psi\big). \tag{7.47}$$

It is pointed out in [51] that the variation of the action and the gauge invariance (7.42), (7.43), and (7.44) can be derived from these relations (7.45) and (7.46).

Also in [51], the unified notation using the concept of the Ramond number proposed in [40] is introduced. The pure-gauge-like field and the associated field including both sectors are introduced as follows:

$$\mathcal{A}_\eta = A_\eta^{NS} + A_\eta^R, \quad \mathcal{A}_{\mathbb{X}} = A_{\mathbb{X}}^{NS} + A_{\mathbb{X}}^R, \tag{7.48}$$

and their defining equations are written as

$$\eta\mathcal{A}_\eta - \frac{1}{2}[\![\mathcal{A}_\eta, \mathcal{A}_\eta]\!]^*_{|0} = 0, \quad (-)^{\mathbb{X}}\mathbb{X}\mathcal{A}_\eta = \mathcal{D}_\eta\mathcal{A}_{\mathbb{X}}, \tag{7.49}$$

where

$$\mathcal{D}_\eta = \eta - [\![\mathcal{A}_\eta,\ \cdot\ ]\!]^*_{|0}. \tag{7.50}$$

The subscript $2r$ of $_{|2r}$ denotes the Ramond number: the number of the Ramond inputs minus the number of the Ramond outputs. Since the number of the Ramond outputs are always 0 or 1, $b_{2|0}$ vanishes when both inputs are the Ramond inputs, and $b_{2|2}$ is nonvanishing only when both inputs are the Ramond inputs. The action and its variation are written in terms of $\mathcal{A}_\eta$ and $m_{2|2}$ as

$$S = -\int_0^1 dt\langle \mathcal{A}_t^*(t), Q\mathcal{A}_\eta(t) + m_{2|2}(\mathcal{A}_\eta(t), \mathcal{A}_\eta(t))\rangle, \tag{7.51}$$

$$\delta S = -\langle \mathcal{A}_\delta^*, Q\mathcal{A}_\eta + m_{2|2}(\mathcal{A}_\eta, \mathcal{A}_\eta)\rangle, \tag{7.52}$$

where $\mathcal{A}^*_{\mathbb{X}}$ for $\mathbb{X} = \partial_t, \delta$ is defined by

$$\mathcal{A}^*_{\mathbb{X}} = A^{NS}_{\mathbb{X}} + \Xi Y \mathbb{X}(\mathcal{P}_s A^R_\eta). \tag{7.53}$$

The equation of motion is give by

$$Q\mathcal{A}_\eta + m_{2|2}(\mathcal{A}_\eta, \mathcal{A}_\eta) = 0. \tag{7.54}$$

In [51], the relation between this equation of motion and that in [40] is discussed.

8 Heterotic string field theory up to quadratic order in the Ramond string field

In the previous section we see the construction of the complete action of open superstring field theory without stubs. The aim of this section and the following section is its extension to the heterotic string, the construction of a complete action of heterotic string field theory. This section and the following section are based a collaboration with H. Kunitomo.

Towards the construction of the complete action of heterotic string field theory, we construct a gauge-invariant action order by order in the Ramond string field. To begin with, let us explain the notation used in this section and the following section. We denote the power of the Ramond string field Ψ by superscript[22], and expand the action as

$$S = S^{(0)} + S^{(2)} + S^{(4)} + \cdots . \tag{8.1}$$

We define $E^{(k)}$ as an left-hand side of equation of motion with k Ramond string fields:

$$\delta S^{(2n)} = -\langle\!\langle \delta\Psi, YE^{(2n-1)}\rangle\!\rangle + \langle A_\delta, E^{(2n)}\rangle. \tag{8.2}$$

In this section and the following section, we omit the parameter dependence if $t = 1$, for example, we write $A_{\mathbb{X}}(1) = A_{\mathbb{X}}$. Also, the gauge transformations are expanded as

$$A_\delta = A_\delta^{(0)} + A_\delta^{(2)} + A_\delta^{(4)} + \cdots , \tag{8.3}$$

$$\delta\Psi = \delta\Psi^{(1)} + \delta\Psi^{(3)} + \cdots . \tag{8.4}$$

Note that the superscript also counts the gauge parameter λ that we will introduce soon. We determine the action and the gauge transformations order by order in the Ramond string field so that the action is invariant under the gauge transformations in each power of the Ramond string field:

$$0 = -\sum_{k=1}^{n}\langle\!\langle \delta\Psi^{(2n-2k+1)}, YE^{(2k-1)}\rangle\!\rangle + \sum_{k=0}^{n}\langle A_\delta^{(2n-2k)}, E^{(2k)}\rangle. \tag{8.5}$$

In this section, after summarizing the properties of the dual products $\mathbf{L}^\eta$ and introducing the action for the NS sector $S^{(0)}$, we provide $S^{(2)}$ and the gauge transformation $A_\delta^{(2)}$ and $\delta\Psi^{(1)}$ so that the action is gauge invariant at quadratic order in the Ramond string field:

$$0 = -\langle\!\langle \delta\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \langle A_\delta^{(2)}, E^{(0)}\rangle + \langle A_\delta^{(0)}, E^{(2)}\rangle. \tag{8.6}$$

This $S^{(2)}$ can be taken in almost the same form as that of open string [20], with replacing the commutator $[\![A, B]\!]^*$ with the shifted 2-product $[A, B]^\eta_{A_\eta}$. We explicitly show the gauge invariance at quadratic order in the Ramond string field in almost the same manner as [20]. We also present $S^{(2)}$ in various forms: a form without a parameter integration, a form with a topological parameter dependence of the Ramond string field, and the WZW-like form including the Ramond sector, as in [51].

The same construction works also for open string with stubs, by replacing $\mathbf{L}^\eta$ with $\mathbf{M}^\eta$, or more basically $\mathbf{L}^{\mathrm{B}}$ with $\mathbf{M}^{\mathrm{B}}$. The interpretation of the higher shifted products is given by (6.86). See also section 6.6.1.

[22] Although we denoted the picture number of string products in the same manner in section 4.2.2, hereafter we do not use it, and there will not arise any confusion about the notation.

8.1 Action for the NS sector

For the NS sector we use the η-based WZW-like action, which we construct in section 6. In this subsection we review what is necessary in the rest of this section: the properties of dual products and the WZW-like action.

Dual products $\mathbf{L}^\eta$

The dual products $\mathbf{L}^\eta = \sum_{n=1} \mathbf{L}^\eta_n$ are degree odd products satisfying the L_∞-relations $[\![\mathbf{L}^\eta, \mathbf{L}^\eta]\!] = 0$ and the Q-derivation properties $[\![\mathbf{Q}, \mathbf{L}^\eta]\!] = 0$. The n-product L^η_n carries ghost number $3-2n$ and picture number $n-2$. We write

$$\pi_1 \mathbf{L}^\eta_n (B_1 \wedge ... \wedge B_n) = [B_1, ..., B_n]^\eta, \tag{8.7}$$

and $[B]^\eta = \eta B$. The L_∞-relations and the commutativity with the operators $\mathbb{X} = \{Q, \partial_t, \delta\}$ can be written as

$$\sum_\sigma \sum_{k=1}^{n} \frac{1}{k!(n-k)!} (-)^{|\sigma|} \big[[B_{i_{\sigma(1)}}, \dots, B_{i_{\sigma(k)}}]^\eta, B_{i_{\sigma(k+1)}}, \dots, B_{i_{\sigma(n)}}\big]^\eta = 0, \tag{8.8}$$

$$\mathbb{X}\big[B_1, \dots, B_n\big]^\eta = \sum_{i=1}^{n-1} (-)^{\mathbb{X}(B_1 + \dots + B_{k-1} + 1)} \big[B_1, \dots, \mathbb{X}B_k, \dots, B_n\big]^\eta. \tag{8.9}$$

In addition, we require $\mathbf{L}^\eta$ to be cyclic,

$$\langle B_1, [B_2, \cdots, B_{n+1}]^\eta \rangle = (-)^{B_1 + B_2 + \dots + B_n} \langle [B_1, \cdots, B_n]^\eta, B_{n+1} \rangle, \tag{8.10}$$

where $\langle A, B \rangle$ is the c_0^--inserted BPZ inner product, which satisfies

$$\langle A, B \rangle = (-)^{(A+1)(B+1)} \langle B, A \rangle, \quad \langle QA, B \rangle = (-)^A \langle A, QB \rangle. \tag{8.11}$$

In this section we only use these properties and not an explicit form of $\mathbf{L}^\eta$. One can construct such products by $\mathbf{L}^\eta = \mathbf{G} \boldsymbol{\eta} \mathbf{G}^{-1}$, where $\mathbf{G}$ is the cohomomorphism used in defining the NS products $\mathbf{L}^{\rm EKS} = \mathbf{G}^{-1} \mathbf{Q} \mathbf{G}$. See section 6.1 or appendix D.

Let A_η be a solution for the Maurer-Cartan equation for L^η:

$$0 = \pi_1 \mathbf{L}^\eta (e^{\wedge A_\eta}) = \eta A_\eta + \frac{1}{2} [A_\eta, A_\eta]^\eta + \frac{1}{3!} [A_\eta, A_\eta, A_\eta]^\eta + \cdots . \tag{8.12}$$

The A_η-shifted dual products, which are defined by

$$[B_1, ..., B_n]^\eta_{A_\eta} = \sum_{m=0}^{\infty} \frac{1}{m!} [\underbrace{A_\eta, A_\eta, \cdots, A_\eta}_{m}, B_1, ..., B_n]^\eta, \tag{8.13}$$

satisfy the L_∞-relations

$$\sum_\sigma \sum_{k=1}^{n} \frac{1}{k!(n-k)!} (-)^{|\sigma|} \big[[B_{i_{\sigma(1)}}, \dots, B_{i_{\sigma(k)}}]^\eta_{A_\eta}, B_{i_{\sigma(k+1)}}, \dots, B_{i_{\sigma(n)}}\big]^\eta_{A_\eta} = 0. \tag{8.14}$$

We write $[B]^{\eta}_{A_\eta} = D_\eta B$. The lowest two L_∞-relations mean that D_η is nilpotent and that D_η acts as a derivation with respect to $[A,B]^{\eta}_{A_\eta}$:

$$(D_\eta)^2 = 0, \tag{8.15}$$
$$D_\eta[A,B]^{\eta}_{A_\eta} = -[D_\eta A, B]^{\eta}_{A_\eta} - (-)^A [A, D_\eta B]^{\eta}_{A_\eta}. \tag{8.16}$$

The next lowest L_∞-relation reads

$$\begin{aligned} D_\eta[A,B,C]^{\eta}_{A_\eta} =& -[D_\eta A, B, C]^{\eta}_{A_\eta} - (-)^A[A, D_\eta B, C]^{\eta}_{A_\eta} - (-)^{A+B}[A,B,D_\eta C]^{\eta}_{A_\eta} \\ & - [[A,B]^{\eta}_{A_\eta}, C]^{\eta}_{A_\eta} - (-)^{BC}[[A,C]^{\eta}_{A_\eta}, B]^{\eta}_{A_\eta} - (-)^A[A,[B,C]^{\eta}_{A_\eta}]^{\eta}_{A_\eta}. \end{aligned} \tag{8.17}$$

The shifted dual products are cyclic, which follows from that of the dual products:

$$\langle B_1, [B_2, \cdots, B_{n+1}]^{\eta}_{A_\eta}\rangle = (-)^{B_1+B_2+\cdots+B_n}\langle [B_1,\cdots,B_n]^{\eta}_{A_\eta}, B_{n+1}\rangle. \tag{8.18}$$

Because of the shift, the operators $\mathbb{X} = \{Q, \partial_t, \delta\}$ are not derivations with respect to the shifted product:

$$\mathbb{X}\big[B_1,\ldots,B_n\big]^{\eta}_{A_\eta} = \sum_{i=1}^{n-1}(-)^{\mathbb{X}(B_1+\cdots+B_{k-1}+1)}\big[B_1,\ldots,\mathbb{X}B_k,\ldots,B_n\big]^{\eta}_{A_\eta} + (-)^{\mathbb{X}}[\mathbb{X}A_\eta, B_1,\ldots,B_n]^{\eta}_{A_\eta}. \tag{8.19}$$

In particular,

$$\mathbb{X}D_\eta B = (-)^{\mathbb{X}} D_\eta \mathbb{X}B + (-)^{\mathbb{X}}[\mathbb{X}A_\eta, B]^{\eta}_{A_\eta}, \tag{8.20}$$
$$\mathbb{X}[B,C]^{\eta}_{A_\eta} = (-)^{\mathbb{X}}[\mathbb{X}B, C]^{\eta}_{A_\eta} + (-)^{(1+B)\mathbb{X}}[B, \mathbb{X}C]^{\eta}_{A_\eta} + (-)^{\mathbb{X}}[\mathbb{X}A_\eta, B, C]^{\eta}_{A_\eta}. \tag{8.21}$$

Action for the NS sector $S^{(0)}$

As in [20], we use the η-based WZW-like action[23] as the action of the NS sector $S^{(0)}$:

$$S^{(0)} = \int_0^1 dt \langle A_t(t), QA_\eta(t)\rangle. \tag{8.22}$$

A_η(t) is the pure-gauge-like field which satisfies the Maurer-Cartan equation for the products $\mathbf{L}^\eta$,

$$0 = \pi_1 \mathbf{L}^\eta(e^{A_\eta(t)}) = \eta A_\eta(t) + \frac{1}{2}[A_\eta(t), A_\eta(t)]^\eta + \frac{1}{3!}[A_\eta(t), A_\eta(t), A_\eta(t)]^\eta + \cdots, \tag{8.23}$$

and $A_{\mathbb{X}}(t)$ for $\mathbb{X} = \{Q, \partial_t, \delta\}$ is the associated field which satisfies

$$(-)^{\mathbb{X}}\mathbb{X}A_\eta(t) = D_\eta(t) A_{\mathbb{X}}(t). \tag{8.24}$$

For a pair of derivations $\mathbb{X}, \mathbb{Y} = \{Q, \partial_t, \delta\}$, the following relation can be derived from (8.24):

$$0 = D_\eta\big(\mathbb{X}A_Y(t) - (-)^{XY}\mathbb{Y}A_X(t) + (-)^{YX+Y}[A_Y(t), A_X(t)]^{\eta}_{A_\eta(t)}\big). \tag{8.25}$$

[23] In [20], the Berkovits action is used as the action for the NS sector. In particular, they use the Berkovits action as the η-based WZW-like action. See section 7. As its natural extension to the heterotic string, we use the η-based WZW-like action constructed in [62], which is the subject of the section 6.

Utilizing these properties, the variation of the action can be taken as

$$\delta S^{(0)} = \big\langle A_{\delta}, QA_{\eta}\big\rangle, \tag{8.26}$$

We define $E^{(0)}$ by $\delta S^{(0)} = \langle A_{\delta}, E^{(0)}\rangle$:

$$E^{(0)} = QA_{\eta}. \tag{8.27}$$

The equation of motion without the Ramond string fields is given by $E^{(0)} = QA_{\eta} = 0$. The gauge transformations can be derived from the nilpotency of D_{η} and Q:

$$A^{(0)}_{\delta_{\Omega}} = D_{\eta}\Omega, \qquad A^{(0)}_{\delta_{\Lambda}} = Q\Lambda. \tag{8.28}$$

We do not need to specify the parameterizations of A_{η} and $A_{\mathbb{X}}$, since the gauge invariance follows from the WZW-like relations (8.23) and (8.24). We assumed that the dynamical string field for the NS sector is parameterized by t and vanishes at $t = 0$, then $A_{\eta}(t = 0) = 0$ and $A_{\mathbb{X}}(t = 0) = 0$.

8.2 Kinetic term for the Ramond sector

XY-projection

The kinetic term for the Ramond sector of heterotic string field theory can be constructed in the same manner as open superstring field theory [20]. The string field for the Ramond sector of the heterotic string Ψ is Grassmann even, carries ghost number 2 and picture number $-1/2$, and satisfies

$$\eta\Psi = 0, \quad b_0^-\Psi = 0, \quad L_0^-\Psi = 0. \tag{8.29}$$

As in the case of the open string, we impose XY-projection condition on Ψ. We define X by

$$X =[\![Q, \Xi]\!], \tag{8.30}$$

where Ξ is an operator which is Grassmann odd and BPZ-even,

$$\langle \Xi A, B\rangle = (-)^{A+1}\langle A, \Xi B\rangle, \tag{8.31}$$

carries ghost number -1 and picture number 1, and satisfies

$$[\![\eta, \Xi]\!] = 1, \quad [\![b_0^-, \Xi]\!] = 0, \quad [\![L_0^-, \Xi]\!] = 0. \tag{8.32}$$

X is Grassmann even and BPZ-even,

$$\langle XA, B\rangle = \langle A, XB\rangle, \tag{8.33}$$

carries ghost numbers 0 and picture number 1, and satisfies

$$[\![b_0^-, X]\!] = 0, \quad [\![L_0^-, X]\!] = 0, \quad [\![Q, X]\!] = 0, \quad [\![\eta, X]\!] = 0. \tag{8.34}$$

We also require that, when acting on a state with picture number $-3/2$, X is given by

$$X = -\delta(\beta_0)G_0 + b_0\delta'(\beta_0). \tag{8.35}$$

Then, we impose XY-projection condition on Ψ:

$$XY\Psi = \Psi, \tag{8.36}$$

where Y is a Grassmann even operator with ghost numbers 0 and picture number -1, which is given by

$$Y = -c_0^+ \delta'(\gamma_0). \tag{8.37}$$

Note that X and Y commute with b_0^- and L_0^-: $[\![b_0^-, Y]\!] = 0$, $[\![L_0^-, Y]\!] = 0$. We say Ψ is in the restricted space when Ψ carries picture number $-1/2$ and satisfies (8.36). The operators X of (8.35) and Y satisfy the following properties:

$$XYX = X, \quad YXY = Y, \quad \eta Y \eta = 0, \quad XYQXY = QXY. \tag{8.38}$$

In particular, the first means that any state of the form $A = XB$ satisfies $XYA = A$. and the last means that if Ψ belongs to the restricted space, $Q\Psi$ also belongs to it.

Since $[\![\eta, \Xi]\!] = 1$, we can use Ξ to relate the inner products in the small Hilbert space and in the large Hilbert space:

$$\langle\!\langle A, B\rangle\!\rangle = \langle \xi A, B\rangle = \langle \Xi A, B\rangle. \tag{8.39}$$

For the fields A and B in the small Hilbert space, the inner product satisfies

$$\langle\!\langle A, B\rangle\!\rangle = (-)^{(A+1)(B+1)}\langle\!\langle B, A\rangle\!\rangle, \quad \langle\!\langle QA, B\rangle\!\rangle = (-)^A\langle\!\langle A, QB\rangle\!\rangle. \tag{8.40}$$

The operator X is BPZ even:

$$\langle\!\langle XA, B\rangle\!\rangle = \langle\!\langle A, XB\rangle\!\rangle. \tag{8.41}$$

The appropriate inner product for A and B in the restricted space is given by

$$\langle\!\langle A, YB\rangle\!\rangle. \tag{8.42}$$

The following properties follow from $XYA = A$, $XYB = B$, (8.38), and (8.41) :

$$\langle\!\langle YA, B\rangle\!\rangle = \langle\!\langle A, YB\rangle\!\rangle, \quad \langle\!\langle QA, YB\rangle\!\rangle = (-)^A\langle\!\langle A, YQB\rangle\!\rangle. \tag{8.43}$$

For a general state A and a state B_r in the restricted space,

$$\langle A, B_r\rangle = \langle\!\langle \eta XA, YB_r\rangle\!\rangle, \qquad \langle B_r, A\rangle = (-)^{B+1}\langle\!\langle YB_r, \eta XA\rangle\!\rangle. \tag{8.44}$$

Kinetic term for the Ramond sector

The kinetic term for the Ramond sector of the heterotic string field Ψ is given by

$$S_2^{(2)} = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi\rangle\!\rangle, \tag{8.45}$$

where we denote the order of the field by subscript. The variation of the $S_2^{(2)}$ can be taken as follows:

$$\delta S_2^{(2)} = -\langle\!\langle \delta\Psi, YQ\Psi \rangle\!\rangle. \tag{8.46}$$

The equation of motion is given by

$$E_1^{(1)} = Q\Psi = 0. \tag{8.47}$$

The kinetic term $S_2^{(2)}$ is invariant under the gauge transformation

$$\delta_\lambda \Psi_1^{(1)} = Q\lambda, \tag{8.48}$$

where the gauge parameter λ is Grassmann odd, carries ghost number 1 and picture number $-1/2$, and belongs to the small and restricted space. Note that the superscript is taken to count not only the Ramond string field Ψ but also the Ramond gauge parameter λ.

8.3 Gauge-invariant action at quadratic order in the Ramond string field

In this subsection, we provide the action at quadratic order in the Ramond string field $S^{(2)}$, by a natural extension of the action of the open string [20]:

$$S^{(2)} = -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi \rangle\!\rangle + \int_0^1 dt \big\langle A_t(t), \frac{1}{2}[F(t)\Psi, F(t)\Psi]^\eta_{A_\eta(t)} \big\rangle, \tag{8.49}$$

where $F(t)$ is the linear operator defined by

$$F(t) = \frac{1}{1 + \Xi(D_\eta(t) - \eta)} = 1 + \sum_{n=1}^{\infty} \Big(- \Xi(D_\eta(t) - \eta) \Big)^n. \tag{8.50}$$

In what follows, we see that the variation of the action $S^{(2)}$ is taken as

$$\delta S^{(2)} = -\langle\!\langle \delta\Psi, Y(E^{(1)}) \rangle\!\rangle + \big\langle A_\delta, E^{(2)} \big\rangle = -\langle\!\langle \delta\Psi, Y(\eta\Xi QF\Psi) \rangle\!\rangle + \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle, \tag{8.51}$$

and show that the action $S^{(0)} + S^{(2)}$ is invariant at quadratic order in the Ramond string field:

$$0 = \delta(S^{(0)} + S^{(2)})\big|^{(2)} = -\langle\!\langle \delta\Psi^{(1)}, YE^{(1)} \rangle\!\rangle + \big\langle A_\delta^{(0)}, E^{(2)} \big\rangle + \big\langle A_\delta^{(2)}, E^{(0)} \big\rangle, \tag{8.52}$$

under the gauge transformations at this order

$$A_\delta^{(0)} = D_\eta \Omega + Q\Lambda, \tag{8.53}$$

$$A_\delta^{(2)} = -[F\Xi[\Lambda, F\Psi]^\eta_{A_\eta}, F\Psi]^\eta_{A_\eta} + \frac{1}{2}[\Lambda, F\Psi, F\Psi]^\eta_{A_\eta} - [F\Xi\lambda, F\Psi]^\eta_{A_\eta}, \tag{8.54}$$

$$\delta^{(1)}\Psi = \eta\Xi QF\Xi[D_\eta\Lambda, F\Psi]^\eta_{A_\eta} + \eta\Xi QF\lambda. \tag{8.55}$$

Properties of F

We first summarize the properties of $F(t)$ which we use in this section and the following section. $F(t)$ satisfies $F(t=0) = 1$ since $F(t)$ depends on t only through $A_\eta(t)$ and $A_\eta(t=0) = 0$. $F(t)$ is invertible and $F^{-1}(t) = 1 + \Xi(D_\eta(t) - \eta) = \eta\Xi + \Xi D_\eta(t)$. $F(t)$ satisfies

$$F(t)\eta = D_\eta(t)F(t), \quad [\![D_\eta(t), F(t)\Xi]\!] = 1, \quad \Xi F(t) = \Xi, \quad \langle F(t)\Xi A, B \rangle = (-)^{A+1}\langle A, F(t)\Xi B \rangle. \tag{8.56}$$

The first can be derived by acting with $F(t)$ from the right and the left of $\eta F^{-1}(t) = F^{-1}(t)D_\eta(t)$, which follows from the expression of $F^{-1}(t)$ and nilpotency of η and $D_\eta(t)$. The second can be derived as follows:

$$\begin{aligned} D_\eta(t)F(t)\Xi + F(t)\Xi D_\eta(t) &= F(t)\eta\Xi + F(t)\Xi D_\eta(t) \\ &= F(t) - F(t)(-\Xi(D_\eta(t)-\eta)) \\ &= \sum_{n=0}^{\infty}\Big(-\Xi(D_\eta(t)-\eta)\Big)^n - \sum_{n=1}^{\infty}\Big(-\Xi(D_\eta(t)-\eta)\Big)^n \\ &= 1. \end{aligned} \tag{8.57}$$

The commutator of $F(t)$ and $\mathbb{X} = Q, \partial_t, \delta$ which are derivations with respect to $\mathbf{L}^\eta$ is given by

$$\begin{aligned} [\![\mathbb{X}, F(t)]\!]B &= -F(t)[\![\mathbb{X}, F^{-1}(t)]\!]F(t)B \\ &= -F(t)[\![\mathbb{X}, \Xi]\!](D_\eta(t)-\eta)F(t)B - F(t)\Xi[\mathbb{X}A_\eta(t), F(t)B]^\eta_{A_\eta(t)}. \end{aligned} \tag{8.58}$$

For later uses, we also summarize the properties of $F(t)\Psi$. Since $\eta\Psi = 0$ and $F(t)\eta = D_\eta(t)F(t)$, $F(t)\Psi$ is $D_\eta(t)$-exact:

$$F(t)\Psi = F(t)\eta\Xi\Psi = D_\eta(t)F(t)\Xi\Psi. \tag{8.59}$$

For $\mathbb{X} = Q, \partial_t, \delta$, utilizing $[\![\mathbb{X}, D_\eta(t)]\!]B = (-)^{\mathbb{X}}[\mathbb{X}A_\eta(t), B]^\eta_{A_\eta(t)}$, $\mathbb{X}F(t)\Psi$ can be written as

$$\begin{aligned} \mathbb{X}F(t)\Psi &= (D_\eta(t)F(t)\Xi + F(t)\Xi D_\eta(t))\mathbb{X}F(t)\Psi \\ &= D_\eta(t)F(t)\Xi(\eta\Xi\mathbb{X}F(t)\Psi) - F(t)\Xi[\mathbb{X}A_\eta(t), F(t)\Psi]^\eta_{A_\eta(t)}. \end{aligned} \tag{8.60}$$

In particular for $\mathbb{X} = \partial_t, \delta$ which commute with Ξ, it can be transformed into the following form:

$$\begin{aligned} \mathbb{X}F(t)\Psi &= D_\eta(t)F(t)\Xi\mathbb{X}\Psi + (-)^{\mathbb{X}}F(t)\Xi D_\eta(t)[A_{\mathbb{X}}, F(t)\Psi]^\eta_{A_\eta(t)} \qquad (8.61) \\ &= D_\eta(t)F(t)\Xi\mathbb{X}\Psi + (-)^{\mathbb{X}}[A_{\mathbb{X}}(t), F(t)\Psi]^\eta_{A_\eta(t)} - (-)^{\mathbb{X}}D_\eta(t)F(t)\Xi[A_{\mathbb{X}}(t), F(t)\Psi]^\eta_{A_\eta(t)}, \qquad (8.62) \end{aligned}$$

where we used $\Xi F(t) = \Xi$, $\eta\Psi = 0$, $\mathbb{X}A_\eta(t) = (-)^{\mathbb{X}}D_\eta(t)A_{\mathbb{X}}(t)$, and $[\![D_\eta(t), F(t)\Xi]\!] = 1$.

8.3.1 Equations of motion

Variation of $S^{(2)}_{int}$

Let us consider the variation of the interaction vertex with two Ramond string fields:

$$S^{(2)}_{int} \equiv \int_0^1 dt \big\langle A_t(t), \frac{1}{2}[F(t)\Psi, F(t)\Psi]^\eta_{A_\eta(t)}\big\rangle. \tag{8.63}$$

From here to (8.68), we omit the t-dependence for notational brevity. We first consider the variation of the integrand:

$$\delta\big\langle A_t, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle = \big\langle \delta A_t, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle + \big\langle A_t, [\delta F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle + \big\langle A_t, \frac{1}{2}[\delta A_\eta, F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle. \tag{8.64}$$

For the first term on the right-hand side of (8.64), since $\frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}$ is D_η-exact, we can use (8.25) and obtain

$$(1st) = \big\langle \partial_t A_\delta + [A_\delta, A_t]^{\eta}_{A_\eta}, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle. \tag{8.65}$$

For the second term on the right-hand side of (8.64), utilizing (8.62), we find

$$\begin{aligned}
(2nd) &= -\big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, \delta F\Psi \big\rangle \\
&= -\big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, D_\eta F\Xi\delta\Psi + [A_\delta, F\Psi]^{\eta}_{A_\eta} - D_\eta F\Xi[A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= -\big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle - \big\langle F\Xi D_\eta [A_t, F\Psi]^{\eta}_{A_\eta}, \delta\Psi - [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= -\big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle - \big\langle \partial_t F\Psi, \delta\Psi - [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= -\big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle \delta\Psi, \partial_t F\Psi\big\rangle + \big\langle A_\delta, [\partial_t F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle.
\end{aligned} \tag{8.66}$$

From the third line to the fourth line, we used (8.61) and $\partial_t\Psi = 0$. For the third term on the right-hand side of (8.64), utilizing the L_∞-relation of A_η-shifted $\mathbf{L}^\eta$, we obtain

$$\begin{aligned}
(3rd) &= \frac{1}{2}\big\langle A_t, [D_\eta A_\delta, F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= \frac{1}{2}\big\langle A_t, -D_\eta[A_\delta, F\Psi, F\Psi]^{\eta}_{A_\eta} - 2[[A_\delta, F\Psi]^{\eta}_{A_\eta}, F\Psi]^{\eta}_{A_\eta} + [A_\delta, [F\Psi, F\Psi]^{\eta}_{A_\eta}]^{\eta}_{A_\eta}\big\rangle \\
&= \frac{1}{2}\big\langle \partial_t A_\eta, [A_\delta, F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle + \frac{1}{2}\big\langle [A_t, A_\delta]^{\eta}_{A_\eta}, [F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= \frac{1}{2}\big\langle A_\delta, [\partial_t A_\eta, F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle [A_t, F\Psi]^{\eta}_{A_\eta}, [A_\delta, F\Psi]^{\eta}_{A_\eta}\big\rangle - \frac{1}{2}\big\langle [A_\delta, A_t]^{\eta}_{A_\eta}, [F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle.
\end{aligned} \tag{8.67}$$

The second term on the right-hand side of (8.65) and the first term on the right-hand side of (8.66) cancel with the third term and the second term on the right-hand side of (8.67) respectively. Then the total variation is given by

$$\begin{aligned}
&\delta\big\langle A_t, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= \big\langle \partial_t A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle A_\delta, [\partial_t F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle \delta\Psi, \partial_t F\Psi\big\rangle + \frac{1}{2}\big\langle A_\delta, [\partial_t A_\eta, F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= \partial_t\Big(\big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle \delta\Psi, F\Psi\big\rangle\Big).
\end{aligned} \tag{8.68}$$

Integrating it by t, we obtain

$$\begin{aligned}
&\int_0^1 dt\partial_t\Big(\big\langle A_\delta(t), \frac{1}{2}[F(t)\Psi, F(t)\Psi]^{\eta}_{A_\eta(t)}\big\rangle + \big\langle \delta\Psi, F(t)\Psi\big\rangle\Big) \\
&\qquad = \big\langle A_\delta(1), \frac{1}{2}[F(1)\Psi, F(1)\Psi]^{\eta}_{A_\eta(1)}\big\rangle + \big\langle \delta\Psi, F(1)\Psi\big\rangle.
\end{aligned} \tag{8.69}$$

Note that $A_{\mathbb{X}}(0) = 0$, and that $\langle\delta\Psi, F(0)\Psi\rangle = 0$ since $F(0) = \mathbb{1}$ and there is no Ξ insertion. Then the variation of the interaction with two Ramond string fields is given by

$$\delta S^{(2)}_{int} = \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle + \big\langle \delta\Psi, F\Psi\big\rangle = \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle - \big\langle\!\big\langle \delta\Psi, Y(X\eta F\Psi)\big\rangle\!\big\rangle. \tag{8.70}$$

For the open string with and without stubs, the variation can be taken in the same manner. In particular, for the open string without stubs the shifted dual products are truncated, as seen in (6.94).

Concretely, the shifted 2-product $[A,B]^{\eta}_{A_\eta}$ and the shifted higher products $[A,B,...]^{\eta}_{A_\eta}$ are replaced with $-(-)^{\deg(A)}[\![A,B]\!]^*$ and 0, respectively. Then, the third term on the right-hand side of (8.64) does not appear, and the second term on the right-hand side of (8.65) cancels with the first term on the right-hand side of (8.66) because of the associativity of the star product.

Equations of motion

Since $S^{(2)} = S_2^{(2)} + S_{int}^{(2)}$, $\delta S^{(2)}$ is given by

$$\delta S^{(2)} = -\langle\!\langle \delta\Psi, Y(Q\Psi + X\eta F\Psi)\rangle\!\rangle + \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle. \tag{8.71}$$

The equations of motion $E^{(1)}$ and $E^{(2)}$ can be read from it:

$$E^{(1)} = Q\Psi + X\eta F\Psi, \tag{8.72}$$

$$E^{(2)} = \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}. \tag{8.73}$$

We can check the consistency with the XY-projection $XYE^{(1)} = E^{(1)}$, by $XYQXY = QXY$ and $XYX = X$. Utilizing $\eta XF = \eta\Xi QF - Q\eta\Xi$ which follows from $[\![Q,\Xi]\!] = X$ and $\Xi F = \Xi$, we obtain

$$Q\Psi + X\eta F\Psi = Q\eta\Xi\Psi + X\eta F\Psi = \eta\Xi QF\Psi, \tag{8.74}$$

and the variation of the action $S^{(2)}$ and the equations of motion $E^{(1)}$ can be written as

$$\delta S^{(2)} = -\langle\!\langle \delta\Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle + \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle, \tag{8.75}$$

$$E^{(1)} = \eta\Xi QF\Psi. \tag{8.76}$$

8.3.2 Gauge invariance

We can determine $\delta\Psi^{(1)}$ and $A_\delta^{(2)}$ by requiring the gauge invariance at quadratic order in the Ramond string field:

$$0 = \delta(S^{(0)} + S^{(2)})\big|^{(2)} = -\langle\!\langle \delta\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle A_\delta^{(0)}, E^{(2)}\big\rangle + \big\langle A_\delta^{(2)}, E^{(0)}\big\rangle. \tag{8.77}$$

Transformation with Ω

For the invariance under the transformation with Ω

$$0 = -\langle\!\langle \delta_\Omega\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle A_{\delta_\Omega}^{(0)}, E^{(2)}\big\rangle + \big\langle A_{\delta_\Omega}^{(2)}, E^{(0)}\big\rangle, \tag{8.78}$$

since $A_{\delta_\Omega}^{(0)} = D_\eta\Omega$ and $D_\eta E^{(2)} = 0$, we obtain $\langle A_{\delta_\Omega}^{(0)}, E^{(2)}\rangle = 0$. Then, we can set

$$A_{\delta_\Omega}^{(2)} = 0, \quad \delta_\Omega\Psi^{(1)} = 0. \tag{8.79}$$

Transformation with Λ

Let us consider the invariance under the transformation with Λ:

$$0 = -\langle\!\langle \delta_\Lambda\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle A_{\delta_\Lambda}^{(0)}, E^{(2)}\big\rangle + \big\langle A_{\delta_\Lambda}^{(2)}, E^{(0)}\big\rangle. \tag{8.80}$$

The second term can be transformed as follows:

$$\big\langle A^{(0)}_{\delta_\Lambda}, E^{(2)}\big\rangle = \big\langle Q\Lambda, \frac{1}{2}[F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle = \big\langle \Lambda, -[QF\Psi, F\Psi]^{\eta}_{A_\eta} - \frac{1}{2}[QA_\eta, F\Psi, F\Psi]^{\eta}_{A_\eta}\big\rangle. \tag{8.81}$$

Utilizing (8.60), $QF\Psi$ is written as

$$QF\Psi = D_\eta F\Xi(\eta\Xi QF\Psi) - F\Xi[QA_\eta, F\Psi]^{\eta}_{A_\eta} = D_\eta F\Xi(E^{(1)}) - F\Xi[E^{(0)}, F\Psi]^{\eta}_{A_\eta}. \tag{8.82}$$

Then all terms on the right-hand side of (8.81) contain $E^{(0)}$ or $E^{(1)}$, and therefore one can determine $A^{(0)}_{\delta_\Lambda}$ and $\delta_\Lambda\Psi^{(1)}$ so that the gauge invariance at quadric order in the Ramond string field (8.80) holds.

Explicitly, the first term on the right-hand side of (8.81) becomes

$$\begin{aligned}
(1st) &= -\big\langle [\Lambda, F\Psi]^{\eta}_{A_\eta}, QF\Psi\big\rangle \\
&= -\big\langle [\Lambda, F\Psi]^{\eta}_{A_\eta}, D_\eta F\Xi(\eta\Xi QF\Psi) - F\Xi[QA_\eta, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= -\big\langle F\Xi D_\eta[\Lambda, F\Psi]^{\eta}_{A_\eta}, (\eta\Xi QF\Psi)\big\rangle + \big\langle F\Xi[\Lambda, F\Psi]^{\eta}_{A_\eta}, [QA_\eta, F\Psi]^{\eta}_{A_\eta}\big\rangle \\
&= -\big\langle\!\big\langle X\eta F\Xi D_\eta[\Lambda, F\Psi]^{\eta}_{A_\eta}, Y(\eta\Xi QF\Psi)\big\rangle\!\big\rangle + \big\langle [F\Xi[\Lambda, F\Psi]^{\eta}_{A_\eta}, F\Psi]^{\eta}_{A_\eta}, QA_\eta\big\rangle,
\end{aligned} \tag{8.83}$$

and then the right-hand side of (8.81) becomes

$$\begin{aligned}
\big\langle A^{(0)}_{\delta_\Lambda}, E^{(2)}\big\rangle = &-\big\langle\!\big\langle X\eta F\Xi D_\eta[\Lambda, F\Psi]^{\eta}_{A_\eta}, Y(\eta\Xi QF\Psi)\big\rangle\!\big\rangle \\
&+ \big\langle [F\Xi[\Lambda, F\Psi]^{\eta}_{A_\eta}, F\Psi]^{\eta}_{A_\eta} - \frac{1}{2}[\Lambda, F\Psi, F\Psi]^{\eta}_{A_\eta}, QA_\eta\big\rangle.
\end{aligned} \tag{8.84}$$

From (8.80) and (8.84) we can determine the gauge transformations at this order as

$$A^{(2)}_{\delta_\Lambda} = -[F\Xi[\Lambda, F\Psi]^{\eta}_{A_\eta}, F\Psi]^{\eta}_{A_\eta} + \frac{1}{2}[\Lambda, F\Psi, F\Psi]^{\eta}_{A_\eta}, \tag{8.85}$$

$$\delta_\Lambda\Psi^{(1)} = -X\eta F\Xi D_\eta[\Lambda, F\Psi]^{\eta}_{A_\eta} \tag{8.86}$$

Utilizing $-X\eta F\Xi D_\eta[\Lambda, F\Psi]^{\eta}_{A_\eta} = \eta\Xi QF\Xi[D_\eta\Lambda, F\Psi]^{\eta}_{A_\eta}$, which follows from $\eta XF = \eta\Xi QF - Q\eta\Xi$, $\delta_\Lambda\Psi^{(1)}$ can be written as

$$\delta_\Lambda\Psi^{(1)} = \eta\Xi QF\Xi[D_\eta\Lambda, F\Psi]^{\eta}_{A_\eta}. \tag{8.87}$$

Transformation with λ

Finally, let us consider the invariance under the transformation with λ:

$$0 = -\langle\!\langle \delta_\lambda\Psi^{(1)}, YE^{(1)}\rangle\!\rangle + \big\langle A^{(2)}_{\delta_\lambda}, E^{(0)}\big\rangle. \tag{8.88}$$

Note that since the superscript counts λ, $A^{(0)}_{\delta_\lambda}$ never appears. We write $\delta_\lambda\Psi^{(1)} = \delta_\lambda\Psi^{(1)}_1 + \delta_\lambda\Psi^{(1)}_{int}$. Since $\delta_\lambda\Psi^{(1)}_1 = Q\lambda$, $E^{(1)} = \eta\Xi QF\Psi = Q\Psi + \eta XF\Psi$, and $\langle\!\langle Q\lambda, Y(Q\Psi)\rangle\!\rangle = 0$, the following equation holds:

$$-\langle\!\langle \delta_\lambda\Psi^{(1)}_1, YE^{(1)}\rangle\!\rangle = -\big\langle\!\big\langle Q\lambda, Y(\eta XF\Psi)\big\rangle\!\big\rangle = \big\langle Q\lambda, F\Psi\big\rangle = -\big\langle \lambda, QF\Psi\big\rangle. \tag{8.89}$$

Then, (8.88) becomes

$$0 = -\big\langle \lambda, QF\Psi\big\rangle - \langle\!\langle \delta_\lambda\Psi^{(1)}_{int}, YE^{(1)}\rangle\!\rangle + \big\langle A^{(2)}_{\delta_\lambda}, E^{(0)}\big\rangle. \tag{8.90}$$

As seen in (8.82), $QF\Psi$ consists of two terms which contain $E^{(0)}$ and $E^{(1)}$, and then one can determine $\delta_\lambda\Psi^{(1)}_{int}$ and $A^{(2)}_{\delta_\lambda}$ so that the gauge invariance (8.88) holds.

Explicitly, utilizing (8.60), (8.89) can be transformed as follows:

$$\begin{aligned}
-\big\langle\lambda, QF\Psi\big\rangle &= -\big\langle\lambda, D_\eta F\Xi(\eta\Xi QF\Psi) - F\Xi[QA_\eta, F\Psi]^\eta_{A_\eta}\big\rangle \\
&= -\big\langle F\Xi D_\eta\lambda, \eta\Xi QF\Psi\big\rangle + \big\langle F\Xi\lambda, [QA_\eta, F\Psi]^\eta_{A_\eta}\big\rangle \\
&= -\big\langle\!\big\langle X\eta F\Xi D_\eta\lambda, Y(\eta\Xi QF\Psi)\big\rangle\!\big\rangle + \big\langle [F\Xi\lambda, F\Psi]^\eta_{A_\eta}, QA_\eta\big\rangle. \qquad (8.91)
\end{aligned}$$

From (8.91) and (8.88), we can determine the gauge transformations as

$$\delta_\lambda\Psi^{(1)}_{int} = -X\eta F\Xi D_\eta\lambda, \qquad (8.92)$$

$$A^{(2)}_{\delta_\lambda} = -[F\Xi\lambda, F\Psi]^\eta_{A_\eta}. \qquad (8.93)$$

$\delta_\lambda\Psi^{(1)} = \delta_\lambda\Psi^{(1)}_1 + \delta_\lambda\Psi^{(1)}_{int}$ is given by

$$\delta_\lambda\Psi^{(1)} = Q\lambda - X\eta F\Xi D_\eta\lambda = \eta\Xi QF\lambda, \qquad (8.94)$$

where we used $Q\lambda - X\eta F\Xi D_\eta\lambda = \eta\Xi QF\lambda$ which follows from $[\![D_\eta, F\Xi]\!] = 1$ and $D_\eta F = F\eta$.

8.4 Various forms of the action in this order

To conclude this section, let us represent $S^{(2)}$ in various forms.

Action $S^{(2)}$ in the integrated form

We can perform the t-integration in the action $S^{(2)}_{int}$. Utilizing $F(t)\Psi = D_\eta(t)F(t)\Xi\Psi$ and $D_\eta(t)A_t(t) = \partial_t A_\eta(t)$, we can transform the integrand of the interaction term $S^{(2)}_{int}$ as follows:

$$\begin{aligned}
&\langle A_t(t), [F(t)\Psi, F(t)\Psi]^\eta_{A_\eta(t)}\rangle = -\langle A_t(t), D_\eta(t)[F(t)\Xi\Psi, F(t)\Psi]^\eta_{A_\eta(t)}\rangle = \langle \partial_t A_\eta(t), [F(t)\Xi\Psi, F(t)\Psi]^\eta_{A_\eta(t)}\rangle \\
&\quad = \langle F(t)\Xi\Psi, [\partial_t A_\eta(t), F(t)\Psi]^\eta_{A_\eta(t)}\rangle = -\langle\Psi, F(t)\Xi[\partial_t A_\eta(t), F(t)\Psi]^\eta_{A_\eta(t)}\rangle = \langle\Psi, \partial_t F(t)\Psi\rangle. \qquad (8.95)
\end{aligned}$$

In the last line, we use $\partial_t F(t)\Psi = -F(t)\Xi[\partial_t A_\eta(t), F(t)\Psi]^\eta_{A_\eta(t)}$ which follows from (8.60). Integrating it by t, we obtain

$$S^{(2)}_{int} = \frac{1}{2}\int_0^1 dt\langle A_t(t), [F(t)\Psi, F(t)\Psi]^\eta_{A_\eta(t)}\rangle = \frac{1}{2}\int_0^1 dt\langle\Psi, \partial_t F(t)\Psi\rangle = \frac{1}{2}\langle\Psi, F(1)\Psi\rangle. \qquad (8.96)$$

Note that $\langle\Psi, F(0)\Psi\rangle = \langle\Psi, \Psi\rangle$ vanishes since there is no Ξ insertion. Then $S^{(2)}$ can be written as

$$S^{(2)} = -\frac{1}{2}\langle\!\langle\Psi, YQ\Psi\rangle\!\rangle + \frac{1}{2}\langle\Psi, F\Psi\rangle = -\frac{1}{2}\langle\!\langle\Psi, Y(Q\Psi + \eta XF\Psi)\rangle\!\rangle = -\frac{1}{2}\langle\!\langle\Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle. \qquad (8.97)$$

Utilizing $\eta\Xi F\Psi = \eta\Xi\Psi = \Psi$, it can also be written as

$$S^{(2)} = -\frac{1}{2}\langle\!\langle\eta\Xi F\Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle. \qquad (8.98)$$

Action $S^{(2)}$ with the topological t-dependence of both the NS and Ramond string fields

If string fields for both the NS sector and the Ramond sector are parameterized by t, the action of δ and ∂_t are the same: they commute with $\mathbf{L}^\eta$, are Grassmann-even, and act only on the string fields. Then, since we have derived

$$\delta S^{(2)} = \delta\big(-\frac{1}{2}\langle\!\langle \Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle\big) = -\langle\!\langle \delta\Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle + \big\langle A_\delta, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle, \tag{8.99}$$

the same computation holds for ∂_t:

$$\partial_t\big(-\frac{1}{2}\langle\!\langle \Psi(t), Y(\eta\Xi QF(t)\Psi(t))\rangle\!\rangle\big) = -\langle\!\langle \partial_t\Psi(t), Y(\eta\Xi QF(t)\Psi(t))\rangle\!\rangle + \big\langle A_t(t), \frac{1}{2}[F(t)\Psi, F(t)\Psi(t)]^\eta_{A_\eta(t)}\big\rangle. \tag{8.100}$$

Then we can write $S^{(2)}$ in terms of the NS and Ramond string fields which are parameterized by t as follows:

$$\begin{aligned} S^{(2)} &= \int_0^1 dt\partial_t\big(-\frac{1}{2}\langle\!\langle \Psi(t), Y(\eta\Xi QF(t)\Psi)\rangle\!\rangle\big) \\ &= \int_0^1 dt\Big(-\langle\!\langle \partial_t\Psi(t), Y(\eta\Xi QF(t)\Psi(t))\rangle\!\rangle + \big\langle A_t(t), \frac{1}{2}[F(t)\Psi(t), F(t)\Psi(t)]^\eta_{A_\eta(t)}\big\rangle\Big). \end{aligned} \tag{8.101}$$

Action $S^{(2)}$ in the WZW-like form

As in [51], the gauge invariance at the quadratic order in the Ramond string field does not depend on the specific expression of F. We write $A^{NS}_\eta = A_\eta$, $A^{NS}_{\mathbb{X}} = A_{\mathbb{X}}$, $\mathcal{P}_s = \eta\Xi$ and

$$A^R_\eta(t) = F(t)\Psi(t). \tag{8.102}$$

Note that $\mathcal{P}_s A^R_\eta = \eta\Xi A^R_\eta = \eta\Xi F\Psi = \eta\Xi\Psi = \Psi$. The action can be written in terms of A^{NS}_η, $A^{NS}_{\mathbb{X}}$, and A^R_η:

$$S^{(2)} = \int_0^1 dt\Big(-\langle\!\langle \partial_t(\mathcal{P}_s A^R_\eta), Y(\eta\Xi QA^R_\eta)\rangle\!\rangle + \big\langle A^{NS}_t, \frac{1}{2}[A^R_\eta, A^R_\eta]^\eta_{A^{NS}_\eta}\big\rangle\Big). \tag{8.103}$$

The gauge invariance can be shown by the following relations

$$D_\eta A^R_\eta = 0, \tag{8.104}$$

$$(-)^{\mathbb{X}}\mathbb{X}A^R_\eta - [A^{NS}_{\mathbb{X}}, A^R_\eta]^\eta_{A^{NS}_\eta} = (D_\eta\text{-exact}) \equiv D_\eta A^R_{\mathbb{X}}. \tag{8.105}$$

The property (8.60) which is crucial in the computations of the variation of the action and the gauge invariance can be written as

$$\mathbb{X}F\Psi - (-)^{\mathbb{X}}[A_{\mathbb{X}}, F\Psi]^\eta_{A_\eta} = D_\eta F\Xi(\eta\Xi\mathbb{X}F\Psi) - (-)^{\mathbb{X}}D_\eta F\Xi[A_{\mathbb{X}}, F\Psi]^\eta_{A_\eta}, \tag{8.106}$$

and it is equivalent to (8.105) if we define

$$(-)^{\mathbb{X}}A^R_{\mathbb{X}} = F\Xi\big(\eta\Xi\mathbb{X}F\Psi - (-)^{\mathbb{X}}[A^{NS}_{\mathbb{X}}, F\Psi]^\eta_{A^{NS}_\eta}\big). \tag{8.107}$$

The computations of the variation of the action and the gauge invariance at quadratic order in the Ramond string field based on (8.104) and (8.105) are the same as the open string case [51].

9 Heterotic string field theory at quartic order in the Ramond string field

Since $S^{(0)} + S^{(2)}$ is not invariant at quartic order in the Ramond string field under the gauge transformations $A_\delta^{(0)} + A_\delta^{(2)}$ and $\delta\Psi^{(1)}$,

$$\big\langle A_\delta^{(2)}, E^{(2)} \big\rangle \neq 0, \tag{9.1}$$

we need interaction terms in higher order in the Ramond string fields and corrections for the gauge transformations. In this section we construct the action in quartic order in the Ramond string field $S^{(4)}$ and the corrections in the gauge transformations $A_\delta^{(4)}$ and $\delta\Psi^{(3)}$ so that the action $S^{(0)} + S^{(2)} + S^{(4)}$ is invariant at quartic order in the Ramond string field under the gauge transformations $A_\delta^{(0)} + A_\delta^{(2)} + A_\delta^{(4)}$ and $\delta\Psi^{(1)} + \delta\Psi^{(3)}$:

$$0 = -\big\langle\!\big\langle \delta\Psi^{(1)}, YE^{(3)} \big\rangle\!\big\rangle - \big\langle\!\big\langle \delta\Psi^{(3)}, YE^{(1)} \big\rangle\!\big\rangle + \big\langle A_\delta^{(0)}, E^{(4)} \big\rangle + \big\langle A_\delta^{(2)}, E^{(2)} \big\rangle + \big\langle A_\delta^{(4)}, E^{(0)} \big\rangle, \tag{9.2}$$

where $E^{(3)}$ and $E^{(4)}$ are equations of motion derived from $S_{(4)}$:

$$\delta S^{(4)} = -\big\langle\!\big\langle \delta\Psi, Y(E^{(3)}) \big\rangle\!\big\rangle + \big\langle A_\delta, E^{(4)} \big\rangle. \tag{9.3}$$

Note that for the open string without stubs, where the dual 2-product is associative, the shifted dual products are truncated, as seen in (6.94). Then, $S^{(0)} + S^{(2)}$ is invariant at quartic order in the Ramond string field under the gauge transformations $A_\delta^{(0)} + A_\delta^{(2)}$ and $\delta\Psi^{(1)}$: $\langle A_\delta^{(2)}, E^{(2)} \rangle = 0$, and the action quadratic in the Ramond string field $S^{(0)} + S^{(2)}$ provides the complete action [20].

To construct $S^{(4)}$, we first determine $E^{(3)}$ by requiring the gauge invariance under the transformation with the gauge parameter λ:

$$\begin{aligned} 0 = &- \big\langle\!\big\langle \eta\Xi QF\lambda, Y(E^{(3)}) \big\rangle\!\big\rangle - \big\langle\!\big\langle \delta_\lambda\Psi^{(3)}, Y(\eta\Xi QF\Psi) \big\rangle\!\big\rangle \\ &+ \big\langle -[F\Xi\lambda, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle + \big\langle A^{(4)}_{\delta_\lambda}, QA_\eta \big\rangle. \end{aligned} \tag{9.4}$$

For the gauge invariance, it is crucial that the dual products $\mathbf{L}^\eta$ are Q-exact and can be written as commutators of Q and some products $\boldsymbol{\rho}$ which we call dual gauge products:

$$\mathbf{L}^\eta = [\![\mathbf{Q}, \boldsymbol{\rho}]\!]. \tag{9.5}$$

In section 9.1 we define $\boldsymbol{\rho}$ and its shifted structure, and explain their properties. Then, in section 9.2, we determine $E^{(3)}$ and $S^{(4)}$ in terms of this $\boldsymbol{\rho}$, and derive $E^{(4)}$ from $S^{(4)}$. In section 9.3, we determine $A_\delta^{(4)}$ and $\delta\Psi^{(3)}$ so that the gauge invariance (9.2) holds. First we determine $\delta_\lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\lambda}$ by (9.4). Next, we see that the gauge invariance under the transformation with the gauge parameter Ω trivially holds since $E^{(4)}$ is D_η-exact,

$$0 = - \big\langle\!\big\langle \delta_\Omega\Psi^{(3)}, Y(\eta\Xi QF\Psi) \big\rangle\!\big\rangle + \big\langle D_\eta\Omega, E^{(4)} \big\rangle + \big\langle A^{(4)}_{\delta_\Omega}, QA_\eta \big\rangle, \tag{9.6}$$

and that the transformations with the gauge parameter Ω do not need corrections at this order:

$$A^{(4)}_{\delta_\Omega} = 0, \qquad \delta_\Omega\Psi^{(3)} = 0. \tag{9.7}$$

Finally we show that one can construct the transformations $\delta_\Lambda \Psi^{(3)}$ and $A^{(4)}_{\delta_\Lambda}$ which leave the action invariant at this order:

$$\begin{aligned} 0 = & -\big\langle\!\big\langle \eta\Xi Q F\Xi[D_\eta \Lambda, F\Psi]^\eta_{A_\eta}, Y(E^{(3)})\big\rangle\!\big\rangle - \big\langle\!\big\langle \delta_\Lambda\Psi^{(3)}, Y(\eta\Xi QF\Psi)\big\rangle\!\big\rangle \\ & + \big\langle Q\Lambda, E^{(4)}\big\rangle + \big\langle -[F\Xi[\Lambda, F\Psi]^\eta_{A_\eta}, F\Psi]^\eta_{A_\eta} + \frac{1}{2}[\Lambda, F\Psi, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta}\big\rangle + \big\langle A^{(4)}_{\delta_\Lambda}, QA_\eta\big\rangle. \end{aligned} \tag{9.8}$$

We also provide explicit forms of $\delta_\Lambda \Psi^{(3)}$ and $A^{(4)}_{\delta_\Lambda}$, while we omit the detail of the derivation. In section 9.4, we provide the observations that the equations of motion and the gauge transformations are related which might be important for construction of a complete action. This section are based on collaboration with H. Kunitomo.

9.1 Dual gauge products and the shifted structure

In the construction of the action at quartic order in the Ramond string field Ψ and for the gauge invariance at this order, the Q-exactness of dual products $\mathbf{L}^\eta$ plays an important role. For the preparation for the following subsections, we define the products $\boldsymbol{\rho}$ which we call *dual gauge products* satisfying $\mathbf{L}^\eta = [\![\mathbf{Q}, \boldsymbol{\rho}]\!]$, and explain their properties and the relation to the gauge products $\boldsymbol{\lambda}$. We also define the shifted structure of the dual gauge products and summarize its properties. More details will be provided in appendix D.

Dual gauge products $\boldsymbol{\rho}$

Since $\mathbf{L}^\eta$ commutes with $\mathbf{Q}$, $\mathbf{L}^\eta$ itself is written as commutator of $\mathbf{Q}$ and some product $\boldsymbol{\rho}$,

$$\mathbf{L}^\eta(\tau) = [\![\mathbf{Q}, \boldsymbol{\rho}(\tau)]\!], \tag{9.9}$$

where we call $\boldsymbol{\rho}$ *dual gauge product*, and it can be expanded as

$$\boldsymbol{\rho}(\tau) = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\rho}_{n+1}. \tag{9.10}$$

$\boldsymbol{\rho}_n$ is a coderivation derived from a n-fold map ρ_n: in the present context, $\boldsymbol{\rho}_n$ defines ρ_n by

$$\rho_n(B_1, ..., B_n) = \pi_1 \boldsymbol{\rho}_n(B_1 \wedge ... \wedge B_n). \tag{9.11}$$

The definition (9.9) reads

$$Q\rho_n(B_1, ..., B_n) = [B_1, ..., B_n]^\eta + \sum_{k=1}^{n} (-)^{B_1+...+B_{k-1}} \rho_n(B_1, ..., QB_k, ..., B_n). \tag{9.12}$$

The n-th dual gauge product ρ_n is Grassmann-even, carries ghost number $2-2n$ and picture number $n-2$, and is cyclic:

$$\langle B_1, \rho_n(B_2, \cdots, B_{n+1})\rangle = (-)^{B_2+\cdots+B_n+1} \langle \rho_n(B_1, \cdots, B_n), B_{n+1}\rangle. \tag{9.13}$$

The ρ commute with ∂_t and δ, but not with η.

Hereafter we take a choice of $\mathbf{L}^{\eta}(\tau) = \mathbf{G}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau)$ where $\mathbf{G}(\tau) = \overleftarrow{\mathcal{P}}\exp\big(\int_0^{\tau} d\tau' \boldsymbol{\lambda}^{[0]}(\tau')\big)$ is a cohomomorphism used in defining the NS string products $\mathbf{L}^{\mathrm{NS}} = \mathbf{G}^{-1}\mathbf{Q}\mathbf{G}$. $\mathbf{L}^{\eta}$ satisfies the following differential equation:

$$\partial_{\tau}^2 \mathbf{L}^{\eta}(\tau) = -[\![\mathbf{Q}, \mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau)]\!]. \tag{9.14}$$

For the derivation, see (D.13). One can find that the dual gauge product $\boldsymbol{\rho}$ and the gauge product $\boldsymbol{\lambda}$ are related by

$$\partial_{\tau}^2 \boldsymbol{\rho}(\tau) = -\mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau). \tag{9.15}$$

Since the left-hand side of (9.15) is expanded in powers of τ as

$$\partial_{\tau}^2 \boldsymbol{\rho}(\tau) = \sum_{n=0}^{\infty}(n+1)(n+2)\tau^n \boldsymbol{\rho}_{3+n} = 2\boldsymbol{\rho}_3 + 6\tau\boldsymbol{\rho}_4 + 12\tau^2\boldsymbol{\rho}_5 + 20\tau^3\boldsymbol{\rho}_6 + \cdots , \tag{9.16}$$

(9.15) determines $\boldsymbol{\rho}_{n\geq 3}$ in terms of $\boldsymbol{\lambda}^{[0]}$ and $\boldsymbol{\lambda}^{[1]}$, which consist of $\mathbf{L}^{\mathrm{B}}$, ξ, and η.[24] Note that the cyclicity of $\boldsymbol{\rho}$ follows from those of $\boldsymbol{\lambda}^{[0]}$ and $\boldsymbol{\lambda}^{[1]}$. Note also that in the following subsections we do not use $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$: we only use $\boldsymbol{\rho}_{n\geq 3}$ and they can be constructed in terms of ξ, η, and $\mathbf{L}^{\mathrm{B}}$. Expanding the right-hand side of (9.15) in powers of τ, the explicit form of $\boldsymbol{\rho}_n$ in lower order are obtained as follows:

$$\boldsymbol{\rho}_3 = -\frac{1}{2}\boldsymbol{\lambda}_3^{[1]}, \tag{9.17}$$

$$\boldsymbol{\rho}_4 = -\frac{1}{6}\Big(\boldsymbol{\lambda}_4^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]\Big), \tag{9.18}$$

$$\boldsymbol{\rho}_5 = -\frac{1}{12}\Big(\boldsymbol{\lambda}_5^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]\Big), \tag{9.19}$$

$$\begin{aligned}\boldsymbol{\rho}_6 = -\frac{1}{20}\Big(&\boldsymbol{\lambda}_6^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_5^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!]]\!] \\ &+ \frac{1}{3!}\big(2[\![\boldsymbol{\lambda}_4^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + 2[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_3^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]]\!]\big)\Big).\end{aligned} \tag{9.20}$$

Shifted dual gauge products ρ_{n,A_η}

We define the *shifted dual gauge products* ρ_{n,A_η} by

$$\rho_{n,A_\eta}(B_1, ..., B_n) = \sum_{m=0}^{\infty}\frac{1}{m!}\rho_{n+m}(A_\eta^m, B_1, ..., B_n). \tag{9.21}$$

Note that ρ_{n,A_η} contains all $\rho_{m\geq n}$. The shifted dual products ρ_{n,A_η} are cyclic, which follows from the cyclicity of ρ:

$$\langle B_1, \rho_{n,A_\eta}(B_2, \cdots, B_{n+1})\rangle = (-)^{B_2+\cdots+B_n+1}\langle \rho_{n,A_\eta}(B_1, \cdots, B_n), B_{n+1}\rangle. \tag{9.22}$$

[24] To define $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ the so-called *homotopy operator* R satisfying $[\![Q, R]\!] = 1$ is necessary, see appendix D.

They are related to the shifted dual products $[B_1, ..., B_n]^\eta_{A_\eta}$ as follows:

$$\begin{aligned}
[B_1, ..., B_n]^\eta_{A_\eta} = \sum_{m=0}^{\infty} \frac{1}{m!} \Big(& Q\rho_{m+n}(\underbrace{A_\eta, \cdots, A_\eta}_{m}, B_1, ..., B_n) - m\rho_{m+n}(\underbrace{A_\eta, \cdots, A_\eta}_{m-1}, QA_\eta, B_1, ..., B_n) \\
& - \sum_{k=1}^{n} (-)^{B_1+...+B_{k-1}} \rho_{m+n}(\underbrace{A_\eta, \cdots, A_\eta}_{m}, B_1, ..., B_{k-1}, QB_k, B_{k+1}, ..., B_n) \Big) \\
= Q\rho_{n,A_\eta}(B_1, ..., B_n) & - \rho_{n+1,A_\eta}(QA_\eta, B_1, ..., B_n) \\
& - \sum_{k=1}^{n} (-)^{B_1+...+B_{k-1}} \rho_{n,A_\eta}(B_1, ..., B_{k-1}, QB_k, B_{k+1}, ..., B_n).
\end{aligned} \tag{9.23}$$

The shifted dual gauge product ρ_{n,A_η} does not commute with $\mathbb{X} = \partial_t, \delta$:

$$\mathbb{X}\rho_{n,A_\eta}(B_1, ..., B_n) = \sum_{k=1}^{n} \rho_{n,A_\eta}(B_1, ..., \mathbb{X}B_k, ..., B_n) + \rho_{n+1,A_\eta}(\mathbb{X}A_\eta, B_1, ..., B_n). \tag{9.24}$$

9.2 Action at quartic order in the Ramond string field

To determine the action at quartic order in the Ramond string field, let us focus on the gauge invariance under the transformation with the gauge parameter λ:

$$\begin{aligned}
0 = & - \langle\!\langle \eta\Xi QF\lambda, Y(E^{(3)}) \rangle\!\rangle - \langle\!\langle \delta_\lambda \Psi^{(3)}, Y(\eta\Xi QF\Psi) \rangle\!\rangle \\
& + \big\langle - [F\Xi\lambda, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle + \big\langle A^{(4)}_{\delta_\lambda}, QA_\eta \big\rangle.
\end{aligned} \tag{9.25}$$

In this subsection, we show that the third term on the right hand side of (9.25) can be written as the sum of terms with $\eta\Xi QF\lambda$, terms with $\eta\Xi QF\Psi$, and terms with QA_η, and therefore we can determine $E^{(3)}$, $A^{(4)}_{\delta_\lambda}$, and $\delta_\lambda\Psi^{(3)}$ so that the gauge invariance (9.25) holds. Then, we explicitly determine $E^{(3)}$ from (9.25), and construct $S^{(4)}$ which reproduces $E^{(3)}$:

$$\delta S^{(4)} = -\langle\!\langle \delta\Psi, Y(E^{(3)}) \rangle\!\rangle + \langle A_\delta, E^{(4)} \rangle. \tag{9.26}$$

We also derive $E^{(4)}$ from $S^{(4)}$.

Gauge invariance

Utilizing the L_∞-relations of the shifted dual products $[B_1, ..., B_n]^\eta_{A_\eta}$, the third term on the right-hand side of (9.25) can be transformed as follows:

$$\begin{aligned}
(3rd) = \big\langle - [F\Xi\lambda, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle &= -\frac{1}{2}\big\langle F\Xi\lambda, [F\Psi, [F\Psi, F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta} \big\rangle \\
= \frac{1}{6}\big\langle F\Xi\lambda, D_\eta[F\Psi, F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle &= \frac{1}{6}\big\langle F\lambda, [F\Psi, F\Psi, F\Psi]^\eta_{A_\eta} \big\rangle.
\end{aligned} \tag{9.27}$$

Here we used $D_\eta F\Psi = 0$ and $D_\eta F\Xi\lambda = F\lambda$. Utilizing the property of the shifted dual gauge product (9.23), $[F\Psi, F\Psi, F\Psi]^\eta_{A_\eta}$ can be transformed as

$$[F\Psi, F\Psi, F\Psi]^\eta_{A_\eta} = Q\rho_{3,A_\eta}(F\Psi, F\Psi, F\Psi) - 3\rho_{3,A_\eta}(QF\Psi, F\Psi, F\Psi) - \rho_{4,A_\eta}(QA_\eta, F\Psi, F\Psi, F\Psi). \tag{9.28}$$

Then the third term on the right-hand side of (9.25) becomes

$$(3rd) = -\frac{1}{6}\big\langle QF\lambda, \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle - \frac{1}{2}\big\langle F\lambda, \mathsf{\rho}_{3,A_\eta}(QF\Psi, F\Psi, F\Psi)\big\rangle \\ -\frac{1}{6}\big\langle F\lambda, \mathsf{\rho}_{4,A_\eta}(QA_\eta, F\Psi, F\Psi, F\Psi)\big\rangle. \tag{9.29}$$

Utilizing $1 = D_\eta F\Xi + F\Xi D_\eta$, $QF\Psi$ and $QF\lambda$ can be transformed into a following form:

$$QF\Psi = (D_\eta F\Xi + F\Xi D_\eta)QF\Psi = D_\eta F\Xi(\eta\Xi QF\Psi) - F\Xi[QA_\eta, F\Psi]^\eta_{A_\eta}, \tag{9.30}$$

$$QF\lambda = (D_\eta F\Xi + F\Xi D_\eta)QF\lambda = D_\eta F\Xi(\eta\Xi QF\lambda) - F\Xi[QA_\eta, F\lambda]^\eta_{A_\eta}. \tag{9.31}$$

We find that the third term on the right-hand side of (9.25) consists of three types of terms: terms with $\eta\Xi QF\lambda$, terms with $\eta\Xi QF\Psi$, and terms with QA_η, and therefore we can determine $E^{(3)}$, $A^{(4)}_{\delta_\lambda}$, and $\delta_\lambda\Psi^{(3)}$ so that the gauge invariance (9.25) holds.

Determination of the action

Picking up the $\eta\Xi QF\lambda$ term in (9.25),

$$\begin{aligned}0 &= -\langle\!\langle \eta\Xi QF\lambda, Y(E^{(3)})\rangle\!\rangle - \frac{1}{6}\big\langle D_\eta F\Xi(\eta\Xi QF\lambda), \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle \\ &= -\langle\!\langle \eta\Xi QF\lambda, Y(E^{(3)})\rangle\!\rangle - \frac{1}{6}\big\langle (\eta\Xi QF\lambda), F\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle \\ &= -\langle\!\langle \eta\Xi QF\lambda, Y(E^{(3)})\rangle\!\rangle + \frac{1}{6}\langle\!\langle Y(\eta\Xi QF\lambda), \eta X F\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\rangle\!\rangle,\end{aligned} \tag{9.32}$$

one can determine $E^{(3)}$:

$$E^{(3)} = \frac{1}{6}\eta X F\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi). \tag{9.33}$$

Note that, utilizing $XF\Xi = \Xi QF\Xi$ which follows from $\Xi F\Xi = 0$, one can write

$$E^{(3)} = \frac{1}{6}\eta\Xi QF\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi). \tag{9.34}$$

To construct the action in quartic order in the Ramond string field $S^{(4)}$ which reproduces $E^{(3)}$, $\delta S^{(4)}\big|_{A_\delta=0} = -\langle\!\langle \delta\Psi, Y(E^{(3)})\rangle\!\rangle$, let us transform $\delta S^{(4)}\big|_{A_\delta=0}$ as follows:

$$\begin{aligned}\delta S^{(4)}\big|_{A_\delta=0} &= -\frac{1}{6}\langle\!\langle \delta\Psi, Y\eta X F\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\rangle\!\rangle \\ &= \frac{1}{6}\big\langle \delta\Psi, F\Xi D_\eta \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle \\ &= \frac{1}{6}\big\langle F\delta\Psi, \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle.\end{aligned} \tag{9.35}$$

In the last line we used $D_\eta F\Xi\delta\Psi = F\delta\Psi$. Since $\delta(F\Psi)\big|_{A_\delta=0} = F\delta\Psi$, $\delta(A_\eta)\big|_{A_\delta=0} = 0$, and $\mathsf{\rho}_{3,A_\eta}$ is cyclic, the following $S^{(4)}$ reproduces (9.35) and therefore $E^{(3)}$:

$$S^{(4)} = \frac{1}{24}\big\langle F\Psi, \mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle. \tag{9.36}$$

Let us derive $E^{(4)}$ by taking the variation of $S^{(4)}$ by the NS field: $\delta S^{(4)}\big|_{\delta\Psi=0} = \langle A_\delta, E^{(4)}\rangle$. It is given by

$$\begin{aligned}
\delta S^{(4)}\big|_{\delta\Psi=0} =& + \frac{1}{6}\big\langle [\![\delta, F]\!]\Psi, \mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle + \frac{1}{24}\big\langle F\Psi, \mathsf{p}_{4,A_\eta}(\delta A_\eta, F\Psi, F\Psi, F\Psi)\big\rangle \\
=& + \frac{1}{6}\big\langle -F\Xi[\delta A_\eta, F\Psi]^\eta_{A_\eta}, \mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\big\rangle + \frac{1}{24}\big\langle F\Psi, \mathsf{p}_{4,A_\eta}(\delta A_\eta, F\Psi, F\Psi, F\Psi)\big\rangle \\
=& - \frac{1}{6}\big\langle \delta A_\eta, [F\Psi, F\Xi\mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta}\big\rangle + \frac{1}{24}\big\langle \delta A_\eta, \mathsf{p}_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi)\big\rangle \\
=& + \frac{1}{6}\big\langle A_\delta, D_\eta[F\Psi, F\Xi\mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta}\big\rangle - \frac{1}{24}\big\langle A_\delta, D_\eta\mathsf{p}_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi)\big\rangle,
\end{aligned} \tag{9.37}$$

where we used (8.58), the cyclicity of p_{n,A_η}, and $\delta A_\eta = D_\eta A_\delta$. Then, $E^{(4)}$ is obtained as follows:

$$E^{(4)} = +\frac{1}{6}D_\eta[F\Psi, F\Xi\mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} - \frac{1}{24}D_\eta\mathsf{p}_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi). \tag{9.38}$$

9.3 Gauge transformations

Transformation with λ

We have seen that one can determine $\delta_\lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\lambda}$ so that (9.25) holds. Picking up the $\eta\Xi QF\Psi$ terms and QA_η terms in (9.25), we can determine $\delta_\lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\lambda}$ as follows:

$$\delta_\lambda\Psi^{(3)} = \frac{1}{2}\eta XF\Xi D_\eta\mathsf{p}_{3,A_\eta}(F\lambda, F\Psi, F\Psi), \tag{9.39}$$

$$\begin{aligned}
A^{(4)}_{\delta_\lambda} =& -\frac{1}{6}[F\lambda, F\Xi\mathsf{p}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} + \frac{1}{2}[F\Psi, F\Xi\mathsf{p}_{3,A_\eta}(F\lambda, F\Psi, F\Psi)]^\eta_{A_\eta} \\
& - \frac{1}{6}\mathsf{p}_{4,A_\eta}(F\lambda, F\Psi, F\Psi, F\Psi).
\end{aligned} \tag{9.40}$$

Transformation with Ω

The gauge invariance under the transformation with the gauge parameter Ω is written as

$$0 = -\big\langle\!\big\langle \delta_\Omega\Psi^{(3)}, Y(\eta\Xi QF\Psi)\big\rangle\!\big\rangle + \big\langle D_\eta\Omega, E^{(4)}\big\rangle + \big\langle A^{(4)}_{\delta_\Omega}, QA_\eta\big\rangle. \tag{9.41}$$

Since $E^{(4)}$ derived in (9.38) is D_η-exact, the second term vanishes: $\big\langle D_\eta\Omega, E^{(4)}\big\rangle = 0$. Then, the transformations with the gauge parameter Ω do not need corrections at this order:

$$A^{(4)}_{\delta_\Omega} = 0, \qquad \delta_\Omega\Psi^{(3)} = 0. \tag{9.42}$$

Note that, since the NS string fields are contained in $S^{(4)}$ only through A_η, D_η-exactness of $E^{(4)}$ is automatic: it follows from $\delta A_\eta = D_\eta A_\delta$.

Transformation with Λ

We first show that one can determine $\delta_\Lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\Lambda}$ so that the gauge invariance at quartic order in the Ramond string field under the transformations with the gauge parameter Λ, which is given by the following equation, holds:

$$0 = -\,\langle\!\langle \eta\Xi QF\Xi[D_\eta\Lambda, F\Psi]^\eta_{A_\eta}, Y\big(\frac{1}{6}\eta XF\Xi D_\eta \rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big)\rangle\!\rangle - \langle\!\langle \delta_\Lambda\Psi^{(3)}, Y(\eta\Xi QF\Psi)\rangle\!\rangle$$
$$+\big\langle Q\Lambda, \frac{1}{6}D_\eta[F\Psi, F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)]^\eta_{A_\eta} - \frac{1}{24}D_\eta\rho_{4,A_\eta}(F\Psi,F\Psi,F\Psi,F\Psi)\big\rangle$$
$$+\big\langle -[F\Xi[\Lambda,F\Psi]^\eta_{A_\eta}, F\Psi]^\eta_{A_\eta} + \frac{1}{2}[\Lambda, F\Psi, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle + \big\langle A^{(4)}_{\delta_\Lambda}, QA_\eta\big\rangle. \tag{9.43}$$

If there are no terms which can not be compensated by $-\langle\!\langle\delta_\Lambda\Psi^{(3)}, Y(\eta\Xi QF\Psi)\rangle\!\rangle$ and $\langle A^{(4)}_{\delta_\Lambda}, QA_\eta\rangle$, one can determine $\delta_\Lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\Lambda}$ so that the gauge invariance (9.43) holds. Since terms containing $\eta\Xi QF\Psi$ and QA_η can be compensated by $-\langle\!\langle\delta_\Lambda\Psi^{(3)}, Y(\eta\Xi QF\Psi)\rangle\!\rangle$ and $\langle A^{(4)}_{\delta_\Lambda}, QA_\eta\rangle$, respectively, what we have to show is that the right-hand side of (9.43) vanishes up to terms containing $\eta\Xi QF\Psi$ and QA_η:

$$0 \cong -\,\langle\!\langle \eta\Xi QF\Xi[D_\eta\Lambda, F\Psi]^\eta_{A_\eta}, Y\big(\frac{1}{6}\eta XF\Xi D_\eta \rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big)\rangle\!\rangle$$
$$+\big\langle Q\Lambda, \frac{1}{6}D_\eta[F\Psi, F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)]^\eta_{A_\eta} - \frac{1}{24}D_\eta\rho_{4,A_\eta}(F\Psi,F\Psi,F\Psi,F\Psi)\big\rangle$$
$$+\big\langle -[F\Xi[\Lambda,F\Psi]^\eta_{A_\eta}, F\Psi]^\eta_{A_\eta} + \frac{1}{2}[\Lambda, F\Psi, F\Psi]^\eta_{A_\eta}, \frac{1}{2}[F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle, \tag{9.44}$$

where we write $A \cong B$ to denote that A equals to B up to terms containing $\eta\Xi QF\Psi$ and QA_η. Obviously one finds $\eta\Xi QF\Psi \cong 0$ and $QA_\eta \cong 0$. The following properties hold:

$$QF\Psi \cong 0, \tag{9.45}$$
$$Q[B_1,...,B_n]^\eta_{A_\eta} \cong \sum_{k=1}^{n}(-)^{1+B_1+...+B_{k-1}}[B_1,...,QB_k,...,B_n]^\eta_{A_\eta}, \tag{9.46}$$
$$[\![Q, D_\eta]\!] \cong 0, \tag{9.47}$$
$$Q\rho_{n,A_\eta}(F\Psi,...,F\Psi) \cong [F\Psi,...,F\Psi]^\eta_{A_\eta}. \tag{9.48}$$

Utilizing them, the first term on the right-hand side of (9.44) becomes

$$\begin{aligned}(1st) &= -\frac{1}{6}\big\langle \Xi QF\Xi[D_\eta\Lambda, F\Psi]^\eta_{A_\eta}, \eta F\Xi D_\eta\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle\\
&= \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, D_\eta F\Xi QF\Xi D_\eta\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle\\
&= \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, D_\eta F\Xi Q(1-D_\eta F\Xi)\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle\\
&\cong \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, D_\eta F\Xi[F\Psi,F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle - \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, QD_\eta F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle\\
&= \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, (1-F\Xi D_\eta)[F\Psi,F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle - \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, QD_\eta F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle\\
&= \frac{1}{6}\big\langle \Lambda, [F\Psi,[F\Psi,F\Psi,F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta}\big\rangle - \frac{1}{6}\big\langle F\Xi[\Lambda, F\Psi]^\eta_{A_\eta}, D_\eta[F\Psi,F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle\\
&\quad - \frac{1}{6}\big\langle [\Lambda, F\Psi]^\eta_{A_\eta}, QD_\eta F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)\big\rangle.\end{aligned} \tag{9.49}$$

The second term and the third term on the right-hand side of (9.44) become

$$(2nd) \cong \frac{1}{6}\big\langle \Lambda, [F\Psi, QD_\eta F\Xi\rho_{3,A_\eta}(F\Psi,F\Psi,F\Psi)]^\eta_{A_\eta}\big\rangle + \frac{1}{24}\big\langle \Lambda, D_\eta[F\Psi,F\Psi,F\Psi,F\Psi]^\eta_{A_\eta}\big\rangle, \tag{9.50}$$
$$(3rd) \cong \frac{1}{4}\big\langle \Lambda, [F\Psi,F\Psi,[F\Psi,F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta}\big\rangle - \frac{1}{2}\big\langle F\Xi[\Lambda,F\Psi]^\eta_{A_\eta}, [F\Psi,[F\Psi,F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta}\big\rangle. \tag{9.51}$$

Then we can show that the right-hand side of (9.44) vanishes by the L_∞-relations of A_η-shifted dual products:

$$\begin{aligned}(r.h.s) \cong\ & \frac{1}{24}\big\langle \Lambda, D_\eta[F\Psi, F\Psi, F\Psi, F\Psi]^\eta_{A_\eta} + 4[F\Psi, [F\Psi, F\Psi, F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta} + 6[F\Psi, F\Psi, [F\Psi, F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta}\big\rangle \\ & - \frac{1}{6}\big\langle F\Xi[\Lambda, F\Psi]^\eta_{A_\eta}, D_\eta[F\Psi, F\Psi, F\Psi]^\eta_{A_\eta} + 3[F\Psi, [F\Psi, F\Psi]^\eta_{A_\eta}]^\eta_{A_\eta}\big\rangle \\ =\ & 0. \end{aligned} \tag{9.52}$$

Recall that $F\Psi$ is annihilated by D_η.

We can explicitly determine $\delta_\Lambda\Psi^{(3)}$ and $A^{(4)}_{\delta_\Lambda}$ by picking up the terms with $\eta\Xi QF\Psi$ and QA_η, respectively. Here we omit the computation and present the results:

$$\begin{aligned}\delta_\Lambda\Psi^{(3)} =& -\frac{1}{6}\eta XF\Xi D_\eta[D_\eta\Lambda, F\Xi\mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} \\ & + \frac{1}{2}\eta XF\Xi D_\eta\mathsf{\rho}_{3,A_\eta}(F\Xi[D_\eta\Lambda, F\Psi]^\eta_{A_\eta}, F\Psi, F\Psi) \\ & - \frac{1}{6}\eta XF\Xi D_\eta\mathsf{\rho}_{4,A_\eta}(D_\eta\Lambda, F\Psi, F\Psi, F\Psi), \end{aligned} \tag{9.53}$$

$$\begin{aligned}A^{(4)}_{\delta_\Lambda} =& + \frac{1}{24}\mathsf{\rho}_{5,A_\eta}(D_\eta\Lambda, F\Psi, F\Psi, F\Psi, F\Psi) \\ & - \frac{1}{24}[\Lambda, \mathsf{\rho}_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} \\ & - \frac{1}{6}\mathsf{\rho}_{4,A_\eta}(F\Xi[D_\eta\Lambda, F\Psi], F\Psi, F\Psi, F\Psi) \\ & - \frac{1}{6}[F\Psi, F\Xi\mathsf{\rho}_{4,A_\eta}(D_\eta\Lambda, F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} \\ & - \frac{1}{6}[F\Xi[D_\eta\Lambda, F\Psi], F\Xi\mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} \\ & - \frac{1}{6}[\Lambda, [F\Psi, F\Xi\mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta}]^\eta_{A_\eta} \\ & - \frac{1}{6}[F\Psi, F\Xi[D_\eta\Lambda, F\Xi\mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta}]^\eta_{A_\eta} \\ & + \frac{1}{6}[D_\eta\Lambda, F\Psi, F\Xi\mathsf{\rho}_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta},]^\eta_{A_\eta} \\ & + \frac{1}{2}[F\Psi, F\Xi\mathsf{\rho}_{3,A_\eta}(F\Xi[D_\eta\Lambda, F\Psi], F\Psi, F\Psi)]^\eta_{A_\eta}. \end{aligned} \tag{9.54}$$

9.4 Towards all-order construction

To conclude with this section, we present observations from the results in section 8 and section 9: the equations of motion and the gauge transformations are related by replacing fields with gauge parameters. Then one can find that the following relation holds:

- NS-EOM and A_{δ_Λ} For $k = 0, 1$, [25]

$$\Big(\Lambda\frac{\delta}{\delta A_\eta}\Big)E^{(2k)} = A^{(2k)}_{\delta_\Lambda}. \tag{9.55}$$

[25] We have not confirmed $k = 2$ yet.

- <u>NS-EOM and A_{δ_λ}</u> For $k = 1, 2$,

$$-\Big(\lambda \frac{\delta}{\delta \Psi}\Big) E^{(2k)} = D_\eta A_{\delta_\lambda}^{(2k)} \tag{9.56}$$

  Note that $\big(\lambda \frac{\delta}{\delta \Psi}\big)$ is degree odd.

- <u>R-EOM and $\delta_\lambda \Psi$</u> For $k = 0, 1$

$$-\Big(\lambda \frac{\delta}{\delta \Psi}\Big) E^{(2k+1)} = \delta_\lambda \Psi^{(2k+1)}. \tag{9.57}$$

- <u>R-EOM and $\delta_\Lambda \Psi$</u> For $k = 0, 1$

$$\Big((D_\eta \Lambda) \frac{\delta}{\delta A_\eta}\Big) E^{(2k+1)} = \delta_\Lambda \Psi^{(2k+1)}. \tag{9.58}$$

  Note that $\big((D_\eta \Lambda) \frac{\delta}{\delta A_\eta}\big)$ is degree odd.

These relations might be an appearance of an L_∞-structure of the action constructed in the previous section and this section: in formulations based on the L_∞-products, the gauge transformation is given by a functional differentiation of the equation of motion. For example, in closed bosonic string field theory, the following relation holds:

$$\delta \Phi = \pi_1 \mathbf{L}^{\mathrm{B}} (\Lambda^{\mathrm{B}} \wedge e^{\wedge \Phi}) = -\Big(\Lambda^{\mathrm{B}} \frac{\delta}{\delta \Phi}\Big) \pi_1 \mathbf{L}^{\mathrm{B}} (e^{\wedge \Phi}) = -\Big(\Lambda^{\mathrm{B}} \frac{\delta}{\delta \Phi}\Big) (EOM). \tag{9.59}$$

To elucidate the role of the relations (9.55), (9.56), (9.57), and (9.58) in detail remains as future works which may provide an insight for a construction of an action to all orders.

10 Conclusion

In the present thesis, we have constructed the action of heterotic string field theory up to quartic order in the Ramond string field, and to all orders in the NS string field. First, we have constructed the alternative action for the NS sector S_η in terms of the pure-gauge-like field A_η which satisfies the Maurer-Cartan equation for the dual L_∞-products $\mathbf{L}^\eta$ and its associated fields $A_{\mathbb{X}}$ satisfying $(-)^{\mathbb{X}}\mathbb{X}A_\eta = D_\eta A_{\mathbb{X}}$. This action is dual to the conventional WZW-like action, and is equivalent to the L_∞-(or A_∞-) action. Then, we have demonstrated that, starting with $S^{(0)} = S_\eta$, an natural extension of the complete action of open superstring field theory [20] provides an action of heterotic string field theory in quadratic order in the Ramond string field $S^{(2)}$. We also discussed that the gauge invariance follows from the WZW-like structure including the Ramond sector, as in [51]. Finally, we have constructed the action in quartic order in the Ramond string field $S^{(4)}$ in terms of the A_η-shifted structure of the dual gauge products $\boldsymbol{\rho}$ satisfying $\mathbf{L}^\eta = [\![\mathbf{Q}, \boldsymbol{\rho}]\!]$. The action up to quartic order in the Ramond string field which is constructed in this thesis is given by

$$S = \int_0^1 dt \langle A_t(t), QA_\eta(t)\rangle - \frac{1}{2}\langle\!\langle \eta\Xi F\Psi, Y(\eta\Xi QF\Psi)\rangle\!\rangle + \frac{1}{24}\langle F\Psi, \rho_{3,A_\eta}(F\Psi, F\Psi, F\Psi)\rangle + \cdots . \tag{10.1}$$

The equations of motion derived from this action are

$$\begin{aligned} E^{NS} &= QA_\eta + \frac{1}{2}[F\Psi, F\Psi]^\eta_{A_\eta} + \frac{1}{24}D_\eta\Big(4[F\Psi, F\Xi\rho_{3,A_\eta}(F\Psi, F\Psi, F\Psi)]^\eta_{A_\eta} \\ &\qquad\qquad - \rho_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi)\Big) + \cdots , \end{aligned} \tag{10.2}$$

$$E^R = \eta\Xi QF\Psi + \frac{1}{6}\eta\Xi QF\Xi D_\eta \rho_{3,A_\eta}(F\Psi, F\Psi, F\Psi) + \cdots . \tag{10.3}$$

This action is gauge invariance up to quartic order in the Ramond string field.

Discussions

It remains as an important future works to construct an action to all orders in the Ramond string field and to understand the relation to other formulations. There may be some possible approaches to these problems.

WZW-like relations including the Ramond sector

For the open string, the relation between the complete action [20] and the equation of motion including the Ramond sector in the A_∞-formulation [40] is discussed through the WZW-like structure including the Ramond sector, in [51]. The WZW-like relations including the Ramond sector are expected to be important also for the heterotic string.

In the heterotic string, the string products $\mathbf{L}^{\mathrm{B}}$ is not truncated, and the WZW-like structure will receive corrections at higher orders in the Ramond string field. Let us consider the generalization of the pure-gauge-like field of the following form:

$$\mathcal{A}_\eta = \sum_{n=0}^{\infty} A_\eta^{(n)} = \pi_1 \mathbf{G}(e^{\wedge(\Phi+\Psi)}), \tag{10.4}$$

where Φ and Ψ are string fields in the NS sector and the Ramond sector, respectively, and belong to the small Hilbert space. $\mathbf{G}$ is a cohomomorphism which we do not specify.[26] As seen in the section 6, the η-constraint on the sting fields naturally induces the constraint on $\mathcal{A}_\eta$ as

$$0 = \pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(e^{\wedge \mathcal{A}_\eta}), \tag{10.5}$$

which is expanded in power of the Ramond string field to provide the constraint on $A_\eta^{(n)}$ in the following way:

$$0 = \pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(e^{\wedge A_\eta^{(0)}}), \tag{10.6}$$
$$0 = D_\eta A_\eta^{(1)}, \tag{10.7}$$
$$0 = D_\eta A_\eta^{(2)} + \frac{1}{2}[A_\eta^{(1)}, A_\eta^{(1)}]', \tag{10.8}$$
$$0 = D_\eta A_\eta^{(3)} + [A_\eta^{(2)}, A_\eta^{(1)}]' + \frac{1}{6}[A_\eta^{(1)}, A_\eta^{(1)}, A_\eta^{(1)}]', \tag{10.9}$$
$$0 = D_\eta A_\eta^{(4)} + [A_\eta^{(3)}, A_\eta^{(1)}]' + \frac{1}{2}[A_\eta^{(2)}, A_\eta^{(2)}]' + \frac{1}{2}[A_\eta^{(2)}, A_\eta^{(1)}, A_\eta^{(1)}]' + \frac{1}{24}[A_\eta^{(1)}, A_\eta^{(1)}, A_\eta^{(1)}, A_\eta^{(1)}]', \tag{10.10}$$

where $D_\eta B = \pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(B \wedge e^{\wedge A_\eta^{(0)}})$ and $[B_1, ..., B_n]' = \pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(B_1 \wedge ... \wedge B_n \wedge e^{\wedge A_\eta^{(0)}})$. One can also define the associated field $\mathcal{A}_\mathbb{X}$ for $\mathbb{X}$ which commutes with $\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1}$ by

$$(-)^{\mathbb{X}}\mathbb{X}\mathcal{A}_\eta = \mathcal{D}_\eta \mathcal{A}_\mathbb{X}, \tag{10.11}$$

where $\mathcal{D}_\eta B = \pi_1(\mathbf{G}\boldsymbol{\eta}\mathbf{G}^{-1})(B \wedge e^{\wedge \mathcal{A}_\eta})$. Expanding $\mathcal{A}_\mathbb{X}$ in power of the Ramond string field as $\mathcal{A}_\mathbb{X} = \sum_{n=0}^\infty A_\mathbb{X}^{(n)}$, the defining equations for $A_\mathbb{X}^{(n)}$ at lower orders read

$$(-)^{\mathbb{X}}\mathbb{X}A_\eta^{(0)} = D_\eta A_\mathbb{X}^{(0)}, \tag{10.12}$$
$$(-)^{\mathbb{X}}\mathbb{X}A_\eta^{(1)} = D_\eta A_\mathbb{X}^{(1)} + [A_\eta^{(1)}, A_\mathbb{X}^{(0)}]', \tag{10.13}$$
$$(-)^{\mathbb{X}}\mathbb{X}A_\eta^{(2)} = D_\eta A_\mathbb{X}^{(2)} + [A_\eta^{(1)}, A_\mathbb{X}^{(1)}]' + [A_\eta^{(2)}, A_\mathbb{X}^{(0)}]' + \frac{1}{2}[A_\eta^{(1)}, A_\eta^{(1)}, A_\mathbb{X}^{(0)}]'. \tag{10.14}$$

The gauge invariance at quadratic order can be understood by the WZW-like structure (10.7) and (10.13), as in [51]. It will be important to understand the gauge invariance at quartic order in the Ramond string field by the WZW-like structure (10.8), (10.14), and possibly higher order relations.

Relation to the equations of motion derived in [57, 58]

The equations of motion including Ramond sector are constructed in [57,58] in the context of the conventional WZW-like formulation, as

$$\eta\Psi_Q + \frac{1}{2}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}]_{\Psi_Q} + Q_{\Psi_Q}B_{-1} = 0, \qquad Q_{\Psi_Q}B_{-\frac{1}{2}} = 0, \tag{10.15}$$

where B_p is a functional with picture number p, and they are determined by the consistency conditions given by

$$\eta\Big(\eta\Psi_Q + \frac{1}{2}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}]_{\Psi_Q} + Q_{\Psi_Q}B_{-1}\Big) = 0, \qquad \eta\Big(Q_{\Psi_Q}B_{-\frac{1}{2}}\Big) = 0, \tag{10.16}$$

[26] In appendix D we introduce dual products with picture deficit. A similar consideration is possible if we use $\mathbf{G}$ with the Ramond number projection [40].

under the equations of motion. These conditions are equivalent, under the equations of motion, to

$$0 = \eta B_{-\frac{1}{2}} + [B_{-\frac{1}{2}}, B_{-1}]_{\Psi_Q} + \frac{1}{6}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}, B_{-\frac{1}{2}}]_{\Psi_Q} + Q_{\Psi_Q} B_{-\frac{3}{2}}, \tag{10.17}$$

$$\begin{aligned}0 = \eta B_{-1} &+ [B_{-\frac{1}{2}}, B_{-\frac{3}{2}}]_{\Psi_Q} + \frac{1}{2}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}]_{\Psi_Q} + \frac{1}{2}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}, B_{-1}]_{\Psi_Q} \\ &+ \frac{1}{24}[B_{-\frac{1}{2}}, B_{-\frac{1}{2}}, B_{-\frac{1}{2}}, B_{-\frac{1}{2}}]_{\Psi_Q} + Q_{\Psi_Q} B_{-2},\end{aligned} \tag{10.18}$$

where $B_{-\frac{3}{2}}$ and B_{-2} are determined by new consistency conditions, which can be solved using $B_{-\frac{5}{2}}$ and B_{-3}. This sequence of consistency conditions does not terminate but produces an infinite number of equations with an infinite number of functionals $B_{-\frac{n}{2}}$ to be determined. The resultant equations can be written in a simple form

$$(Q+\eta)\widehat{B} + \sum_{m=2}^{\infty} \frac{1}{m!}[\widehat{B}^m] = 0, \tag{10.19}$$

where $\widehat{B} = \sum_{n=0}^{\infty} B_{-\frac{n}{2}}$ and $B_0 = \Psi_Q$.

It will be interesting to discuss the relation between the dual description of the above equations of motion and those derived in this thesis. Let us consider their (naive) $\mathbb{Z}_2$-dual description by replacing $(\mathbf{L}^{\mathrm{B}}, \eta, B_p)$ with $(\mathbf{L}^{\eta}, Q, \widetilde{B}_{1-p})$. The equations of motion (10.15) are mapped to

$$QA_\eta + \frac{1}{2}[\widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_{-\frac{1}{2}}]^{\eta}_{A_\eta} + D_\eta \widetilde{B}_0 = 0, \tag{10.20}$$

$$D_\eta \widetilde{B}_{-\frac{1}{2}} = 0, \tag{10.21}$$

where $\widetilde{B}_p$ is a functional with picture number p and is determined by the consistency condition which is the dual of (10.17):

$$0 = Q\widetilde{B}_{-\frac{1}{2}} + [\widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_0]^{\eta}_{A_\eta} + \frac{1}{6}[\widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_{-\frac{1}{2}}]^{\eta}_{A_\eta} + D_\eta \widetilde{B}_{\frac{1}{2}}. \tag{10.22}$$

As in the conventional description, there are an infinite number of consistency conditions, but they are summarized in a simple form:

$$(\eta+Q)\widetilde{B} + \sum_{m=2}^{\infty} \frac{1}{m!}[\widetilde{B}^m]^{\eta} = 0, \tag{10.23}$$

where $\widetilde{B} = \sum_{n=0}^{\infty} \widetilde{B}_{\frac{n-2}{2}}$ and $\widetilde{B}_{-1} = A_\eta$.

One can find that (10.20) is related to (10.2) by the following identification:

$$\widetilde{B}_{-\frac{1}{2}} = F\Psi - \frac{1}{6} D_\eta F \Xi \rho_{3,A_\eta}(F\Psi, F\Psi, F\Psi) + \cdots, \quad \widetilde{B}_0 = -\frac{1}{24}\rho_{4,A_\eta}(F\Psi, F\Psi, F\Psi, F\Psi) + \cdots. \tag{10.24}$$

In the identification, while (10.21) is satisfied trivially, (10.3) is equivalent to (10.22) up to quartic order in the Ramond string field:

$$\eta\Xi\Big(Q\widetilde{B}_{-\frac{1}{2}} + \frac{1}{6}[\widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_{-\frac{1}{2}}, \widetilde{B}_{-\frac{1}{2}}]^{\eta}_{A_\eta}\Big) = 0. \tag{10.25}$$

This equivalence implies that the equations of motion derived in this thesis can be understood in the form of (10.23), and higher order corrections can be determined by consistency conditions on $\widetilde{B}$.

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Appendices

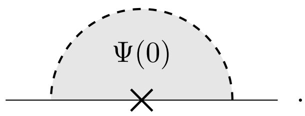
A String products defining bosonic string field theories

In this appendix, we provide the definitions of the BPZ-inner product and the string products defining bosonic string field theories, via conformal maps. Note that, in oscillator representation, string products are given by the Neumann coefficients, which differ depending on the theories.

A.1 BPZ inner product

Open string

For open string, string field Ψ is a state of world-sheet conformal field theory, which is defined on upper half unit circle. Utilizing the state operator correspondence, it is equivalent to the upper half unit disk with the insertion of a vertex operator $\Psi(z=0)$ at the origin. Schematically,

$\Psi =$ 

. (A.1)

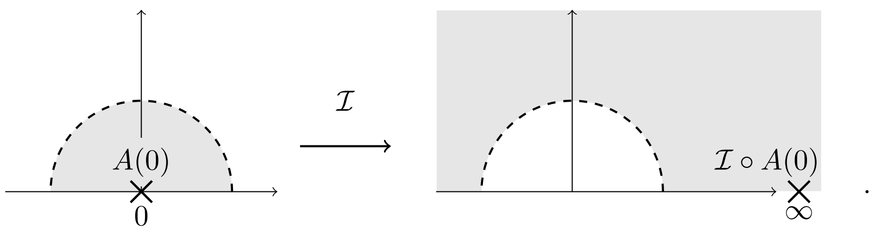
The BPZ inner product can be defined by

$$\langle A|B\rangle = \langle \mathcal{I}\circ A(0)\, B(0)\rangle_{\mathrm{UHP}}, \tag{A.2}$$

where $\langle\cdots\rangle_{\mathrm{UHP}}$ is a correlation function of world-sheet conformal field theory on a upper half plane, and $\mathcal{I}$ is a inversion map defined by

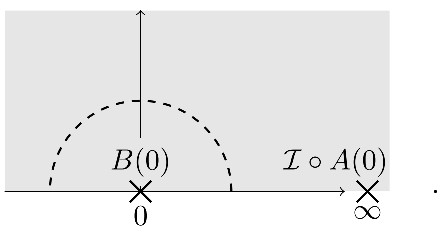
$$\mathcal{I}(z) = -\frac{1}{z}. \tag{A.3}$$

This inversion map can be illustrated as follows:



(A.4)

The BPZ inner product corresponds to the following glueing:

$\langle A|B\rangle =$ 

(A.5)

Closed string

For closed string, the BPZ inner product can be defined similarly. Closed string state is defined by a unit circle, and it is equivalent to a unit disk with insertion of the vertex operator at the origin. The BPZ inner product can be defined by a correlation function in whole complex plane:

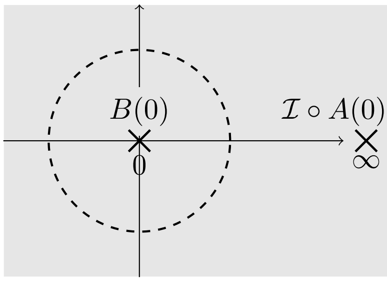
$$\langle A|B\rangle = \langle \mathcal{I} \circ A(0)\, B(0)\rangle_{\mathbb{C}}, \tag{A.6}$$

where the inversion map for closed string is (conventionally) taken as

$$\mathcal{I}(z) = \frac{1}{z}. \tag{A.7}$$

It can be illustrated as follows:

$\langle A|B\rangle =$



. (A.8)

A.2 Star product

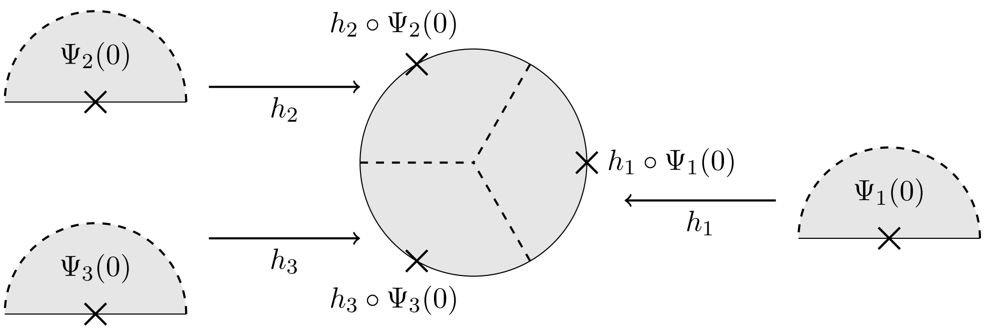
The star product $\Psi_2 * \Psi_3$ of two open string fields Ψ_2 and Ψ_3 is defined to be a glueing of the right half of the first string Ψ_2 and the left half of the second string Ψ_3. It can be represented using conformal maps as follows:

$$\langle \Psi_1|\Psi_2 * \Psi_3\rangle = \langle h_1 \circ \Psi_1(0)\, h_2 \circ \Psi_2(0)\, h_3 \circ \Psi_3(0)\rangle_{disk}, \tag{A.9}$$

where h_k is a conformal map defined by

$$h_k(z) = e^{i\frac{2(k-1)\pi}{3}} \left(\frac{1+iz}{1-iz}\right)^{\frac{2}{3}} = e^{i\frac{2(k-1)\pi}{3} + i\frac{4}{3}\arctan(z)}. \tag{A.10}$$

These conformal maps provide the following glueing of three half disks:



. (A.11)

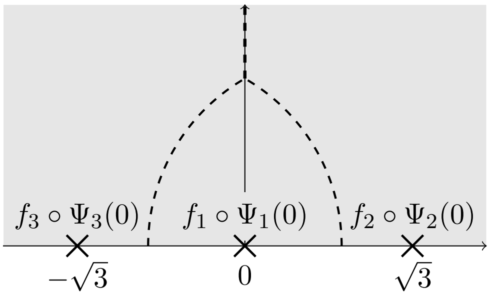
Using a conformal map, we can transform (A.9) into a correlation function in upper half plane

$$\langle \Psi_1|\Psi_2 * \Psi_3\rangle = \langle f_1 \circ \Psi_1(0)\, f_2 \circ \Psi_2(0)\, f_3 \circ \Psi_3(0)\rangle_{UHP}, \tag{A.12}$$

where f_k is given by a composition of a map h_k and a map from a unit disk to upper half plane:

$$f_k(z) = -i\frac{h_k(z)-1}{h_k(z)+1} = \tan\Big(\frac{k-1}{3}\pi + \frac{2}{3}\arctan(z)\Big). \tag{A.13}$$

Schematically, the maps f_1, f_2, f_3 are represented by



. (A.14)

Note that the state $\Psi_2 * \Psi_3$ is defined to be a state satisfying (A.9), or equivalently (A.12).

A.3 Zwiebach's string products

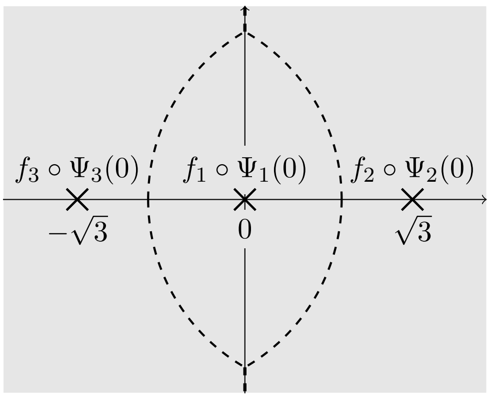
Let us consider a set of closed-string products $\{L_k^{\mathrm{B}}\}_{k\geq 1}$. The k-th product L_k^{B} is a product of k closed string fields $\Psi_1, ..., \Psi_k$ which are annihilated by b_0^- and L_0^-. The output $L_k^{\mathrm{B}}(\Psi_1, ..., \Psi_k)$ is also annihilated by b_0^- and L_0^-.

Two-string product

The 2-string product $L_2^{\mathrm{B}}(\Psi_1, \Psi_2)$ is defined by the same map as the star product:

$$\langle \Psi_1, L_2^B(\Psi_2, \Psi_3)\rangle = \langle f_1 \circ \Psi_1(0)\, f_2 \circ \Psi_2(0)\, f_3 \circ \Psi_3(0)\rangle_{\mathbb{C}}. \tag{A.15}$$

These conformal maps provide the following glueing of three closed string fields :



. (A.16)

Higher products

Let $\mathcal{M}_n$ be a moduli space of Riemann surfaces with n punctures and without genus. Consider the decomposition of $\mathcal{M}_n$ with respect to the number of internal lines:

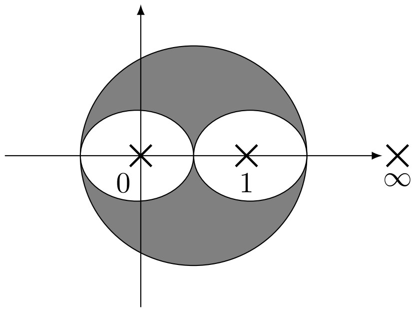
$$\mathcal{M}_n = \mathcal{R}_0^n \cup \mathcal{R}_1^n \cup ... \cup \mathcal{R}_{n-3}^n, \tag{A.17}$$

where we denote by $\mathcal{R}_I^n$ the region of moduli space covered by Feynman diagrams constructed with I internal lines. Higher string products L_k^{B} for $k \geq 3$ are defined by a correlation function of the string states with a integration over $\mathcal{R}_0^{k+1}$, the region which is not covered by Feynman diagrams with lower string products and propagators. It can be written as

$$\langle \Psi_1, L_{n-1}^B(\Psi_2, ..., \Psi_n) \rangle = \int_{\mathcal{R}_0^n} \Big\langle \prod_{I=1}^{2n-6} dm^I \, B_I \, \Psi_1(\zeta_1 = 0)\Psi_2(\zeta_2 = 0)...\Psi_n(\zeta_n = 0) \Big\rangle, \tag{A.18}$$

where ζ_k is a local coordinate for k-th puncture, m^I is a real coordinate parameterizing the moduli space of n-punctured sphere, and B_I is an insertion of b-ghost and $\bar{b}$-ghost providing the correct measure on the integration on moduli space. Thus defined string products $\{L_k^{\mathrm{B}}\}_{k\geq 1}$, where $L_1^{\mathrm{B}} = Q$, are known to naturally satisfy the L_∞-relations. For more details, see [8, 19].

For example, let us consider $\mathcal{R}_0^3$. If we set the position of Ψ_1, Ψ_2, Ψ_3 to $0, 1, \infty$, respectively, the position of Ψ_4 corresponds to the modulus of 4-punctured spheres. Schematically, the integration region $\mathcal{R}_0^3$ in this expression can be illustrated as follows:



. (A.19)

The white regions are covered by the Feynman diagrams with one propagator, which correspond to s, t, u-channel contributions respectively. The gray region is a region which cannot be covered by them. Note that the analytic expressions of the boundary curves are not known, while numerical description is given in [77].

B Path-ordered exponentials

In this appendix, we provide the details of the path-ordered exponential map which we introduce in section 4.2.3.

Definition and properties

We define the path-ordered exponential maps by

$$\begin{aligned}\mathcal{A}_{[\tau_f,\tau_i]} &= \overrightarrow{\mathcal{P}}\exp\left(\int_{\tau_i}^{\tau_f} d\tau' \mathcal{O}_{[\tau']}\right)\\ &= \mathbb{1} + \left(\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right) + \sum_{n=2}^{\infty}\left(\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right)\left(\int_{\tau_i}^{\tau_1} d\tau_2 \mathcal{O}_{[\tau_2]}\right)\cdots\left(\int_{\tau_i}^{\tau_{n-1}} d\tau_n \mathcal{O}_{[\tau_n]}\right),\end{aligned} \tag{B.1}$$

so that it satisfies the following differential equation with the initial condition $\mathcal{A}_{[\tau,\tau]} = \mathbb{1}$:

$$\partial_{\tau_f}\mathcal{A}_{[\tau_f,\tau_i]} = \mathcal{O}_{[\tau_f]}\cdot\mathcal{A}_{[\tau_f,\tau_i]}. \tag{B.2}$$

If $\tau_f \geq \tau_i$ is assumed, the right arrow $\rightarrow$ over $\mathcal{P}$ denotes the ordering of the operations in which the "late-time" operator will act from the left. Equivalently it denotes the direction along which the integration variables τ become small: $\tau_f \geq \tau_1 \geq \tau_2 ... \geq \tau_i$. The definition also works for $\tau_f \leq \tau_i$. In that case, the meaning of "late-time" and "early-time" is reversed: $\tau_f \leq \tau_1 \leq \tau_2 ... \leq \tau_i$. In either case, the operators are "time ordered" from τ_f to τ_i, from the left to the right. We also denote it by the ordering of the argument of $\mathcal{A}_{[\tau_f,\tau_i]}$.

$\mathcal{A}_{[\tau_f,\tau_i]}$ also satisfies the following differential equation:

$$\partial_{\tau_i}\mathcal{A}_{[\tau_f,\tau_i]} = -\mathcal{A}_{[\tau_f,\tau_i]}\cdot\mathcal{O}_{[\tau_i]}. \tag{B.3}$$

It follows from the reparameterization of the range of the integrals:

$$\begin{aligned}\mathcal{A}_{[\tau_f,\tau_i]} &= \mathbb{1} + \left(\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right) + \sum_{n=2}^{\infty}\left(\int_{\tau_i}^{\tau_f} d\tau_1 \int_{\tau_i}^{\tau_1} d\tau_2 \cdots \int_{\tau_i}^{\tau_{n-1}} d\tau_n\right)\mathcal{O}_{[\tau_1]}\mathcal{O}_{[\tau_2]}\cdots\mathcal{O}_{[\tau_n]}\\ &= \mathbb{1} + \left(\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right) + \sum_{n=2}^{\infty}\left(\int_{\tau_i}^{\tau_f} d\tau_n \int_{\tau_n}^{\tau_f} d\tau_{n-1} \cdots \int_{\tau_2}^{\tau_f} d\tau_1\right)\mathcal{O}_{[\tau_1]}\cdots\mathcal{O}_{[\tau_{n-1}]}\mathcal{O}_{[\tau_n]}.\end{aligned} \tag{B.4}$$

We may represent it by

$$\mathcal{A}_{[\tau_f,\tau_i]} = \mathbb{1} + \left(\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right) + \sum_{n=2}^{\infty}\left(\int_{\tau_2}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right)\cdots\left(\int_{\tau_n}^{\tau_f} d\tau_{n-1} \mathcal{O}_{[\tau_{n-1}]}\right)\left(\int_{\tau_i}^{\tau_f} d\tau_n \mathcal{O}_{[\tau_n]}\right). \tag{B.5}$$

Note that in this form the integrations are defined to be performed from the right to the left.

In addition, this path-ordered exponential also satisfies the following "$e^x e^y = e^{x+y}$" property:

$$\mathcal{A}_{[\tau_f,\tau']}\mathcal{A}_{[\tau',\tau_i]} = \mathcal{A}_{[\tau_f,\tau_i]}. \tag{B.6}$$

It can be seen from $\partial_{\tau'}\mathcal{A}_{[\tau_f,\tau']}\mathcal{A}_{[\tau',\tau_i]} = 0$, which follows from (B.2) and (B.3). The left-hand side is independent of τ' and then, $\mathcal{A}_{[\tau_f,\tau']}\mathcal{A}_{[\tau',\tau_i]} = \mathcal{A}_{[\tau_f,\tau_i]}\mathcal{A}_{[\tau_i,\tau_i]} = \mathcal{A}_{[\tau_f,\tau_i]}$.

Explicit form

For a operator $\mathcal{O}$ which can be expanded in powers of τ as $\mathcal{O}[\tau] = \sum_{k=0}^{\infty} \tau^k \mathcal{O}_{k+2}$, let us consider lower-order terms in τ_f and τ_i of $\mathcal{A}[\tau_f,\tau_i] = \overrightarrow{\mathcal{P}} \exp\left(\int_{\tau_i}^{\tau_f} d\tau' \mathcal{O}[\tau']\right)$. For the convenience of the explanation, we represent it as

$$\mathcal{A}[\tau_f,\tau_i] = \overrightarrow{\mathcal{P}} \exp\left(\int_{\tau_i}^{\tau_f} d\tau' \mathcal{O}[\tau']\right) = \overrightarrow{\mathcal{P}} \exp\left(\int_{0}^{\tau_{fi}} d\tau' \mathcal{O}[\tau'+\tau_i]\right) = \overrightarrow{\mathcal{P}} \exp\left(\int_{0}^{\tau_{fi}} d\tau' \mathcal{O}'[\tau']\right), \tag{B.7}$$

where $\tau_{fi} = \tau_f - \tau_i$, and $\mathcal{O}'[\tau']$ is the τ_i-shifted operator:

$$\mathcal{O}[\tau'+\tau_i] = \sum_{n=0}^{\infty} \tau'^n \Big(\frac{1}{n!}(\partial_{\tau_i})^n \mathcal{O}[\tau_i]\Big) = \sum_{n=0}^{\infty} \tau'^n \Big(\sum_{k=n}^{\infty} \frac{k!}{n!(k-n)!} \tau_i^{k-n} \mathcal{O}_{k+2}\Big) \equiv \mathcal{O}'[\tau'] = \sum_{n=0}^{\infty} \tau'^n \mathcal{O}'_{n+2}. \tag{B.8}$$

Note that if $\mathcal{O}_{n+2}$ is an $(n+2)$-product, $\mathcal{O}'_{n+2}$ contains higher products and is not purely an $(n+2)$-product. Utilizing the explicit forms of lower-order operators,

$$\begin{aligned} \mathcal{O}'_2 &= \mathcal{O}[\tau_i] = \mathcal{O}_2 + \tau_i \mathcal{O}_3 + \tau_i^2 \mathcal{O}_4 + \cdots \\ \mathcal{O}'_3 &= \partial \mathcal{O}[\tau_i] = \mathcal{O}_3 + 2\tau_i \mathcal{O}_4 + \cdots \\ \mathcal{O}'_4 &= \tfrac{1}{2} \partial^2 \mathcal{O}[\tau_i] = \mathcal{O}_4 + \cdots, \end{aligned} \tag{B.9}$$

lower terms of $\mathcal{A}[\tau_f,\tau_i]$ can be obtained in the following form:

$$\begin{aligned} \mathcal{A}[\tau_f,\tau_i] =& \mathbb{1} + \tau_{fi} \mathcal{O}'_2 + \frac{\tau_{fi}^2}{2}(\mathcal{O}'_3 + \mathcal{O}'_2\mathcal{O}'_2) + \frac{\tau_{fi}^3}{3!}(2\mathcal{O}'_4 + 2\mathcal{O}'_3\mathcal{O}'_2 + \mathcal{O}'_2\mathcal{O}'_3 + \mathcal{O}'_2\mathcal{O}'_2\mathcal{O}'_2) + \cdots \\ =& \mathbb{1} + (\tau_f - \tau_i)\mathcal{O}_2 + \frac{(\tau_f-\tau_i)^2}{2}\mathcal{O}_2\mathcal{O}_2 + \frac{\tau_f^2 - \tau_i^2}{2}\mathcal{O}_3 \\ &+ \frac{(\tau_f-\tau_i)^3}{3!}\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 + \frac{(\tau_f-\tau_i)^2}{3!}(\tau_f + 2\tau_i)\mathcal{O}_2\mathcal{O}_3 + \frac{(\tau_f-\tau_i)^2}{3!}(2\tau_f + \tau_i)\mathcal{O}_3\mathcal{O}_2 + \frac{\tau_f^3 - \tau_i^3}{3}\mathcal{O}_4 \\ &+ \cdots . \end{aligned} \tag{B.10}$$

If $\mathcal{O}$ is independent of τ, (B.10) becomes an usual exponential.

Inverse map

The inverse of the path-ordered exponential is given by reversing the arguments:

$$\Big(\mathcal{A}[\tau_f,\tau_i]\Big)^{-1} = \mathcal{A}[\tau_i,\tau_f] \tag{B.11}$$

To prove it, consider the differential equation for $\mathcal{A}[\tau_i,\tau_f]\mathcal{A}[\tau_f,\tau_i] - \mathbb{1}$:

$$\partial_{\tau_f}\Big(\mathcal{A}[\tau_i,\tau_f]\mathcal{A}[\tau_f,\tau_i] - \mathbb{1}\Big) = \big(-\mathcal{A}[\tau_i,\tau_f]\mathcal{O}[\tau_f]\big)\mathcal{A}[\tau_f,\tau_i] + \mathcal{A}[\tau_i,\tau_f]\big(\mathcal{O}[\tau_f]\mathcal{A}[\tau_f,\tau_i]\big) = 0. \tag{B.12}$$

Since the initial condition at $\tau_f = \tau_i$ is given by $\mathcal{A}[\tau_i,\tau_i]\mathcal{A}[\tau_i,\tau_i] - \mathbb{1} = 0$, the solution for the differential equation is $\mathcal{A}[\tau_i,\tau_f]\mathcal{A}[\tau_f,\tau_i] - \mathbb{1} = 0$, which leads to

$$\mathcal{A}[\tau_i,\tau_f]\mathcal{A}[\tau_f,\tau_i] = \mathbb{1}. \tag{B.13}$$

Acting with $\mathcal{A}[\tau_f,\tau_i]$ from the left and with $\mathcal{A}^{-1}[\tau_i,\tau_f]$ from the right on (B.13), $\mathcal{A}[\tau_f,\tau_i]\mathcal{A}^{-1}[\tau_i,\tau_f] = \mathbb{1}$ can also be obtained, which proves (B.11).

Note that the inversion of the arguments corresponds to the inversion of the signs of operators $\mathcal{O} \to -\mathcal{O}$ and the reversal of the order of the iterated integrations:

$$\begin{aligned}\mathcal{A}_{[\tau_i,\tau_f]} &= \overleftarrow{\mathcal{P}} \exp\left(-\int_{\tau_i}^{\tau_f} d\tau' \mathcal{O}_{[\tau']}\right) \\ &= \mathbb{1} + \left(-\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right) + \sum_{n=2}^{\infty} \left(-\int_{\tau_i}^{\tau_{n-1}} d\tau_n \mathcal{O}_{[\tau_n]}\right) \cdots \left(-\int_{\tau_i}^{\tau_1} d\tau_2 \mathcal{O}_{[\tau_2]}\right) \left(-\int_{\tau_i}^{\tau_f} d\tau_1 \mathcal{O}_{[\tau_1]}\right),\end{aligned} \tag{B.14}$$

where the left arrow $\leftarrow$ over $\mathcal{P}$ denotes the ordering of the operations: the operators are "time ordered" so that $\tau_i \leq \tau_n \leq ... \leq \tau_2 \leq \tau_1 \leq \tau_f$ for the case of $\tau_i \leq \tau_f$, and $\tau_i \geq \tau_n \geq ... \geq \tau_2 \geq \tau_1 \geq \tau_f$ for the case of $\tau_i \geq \tau_f$.

Response to operations

We compute how operators act on the path-ordered exponentials. For notational convenience, we denote the range of the integrations as follows:

$$\overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_f} d\tau'\Big)^n = \int_{\tau_i}^{\tau_f} d\tau_1 \int_{\tau_i}^{\tau_1} d\tau_2 \int_{\tau_i}^{\tau_2} d\tau_3 \cdots \int_{\tau_i}^{\tau_{n-3}} d\tau_{n-2} \int_{\tau_i}^{\tau_{n-2}} d\tau_{n-1} \int_{\tau_i}^{\tau_{n-1}} d\tau_n. \tag{B.15}$$

In this notation, $\mathcal{A}$ can be represented as

$$\mathcal{A}_{[\tau_f,\tau_i]} = \sum_{n=0}^{\infty} \overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_f} d\tau'\Big)^n \mathcal{O}_{[\tau_1]}\mathcal{O}_{[\tau_2]} \cdots \mathcal{O}_{[\tau_n]}. \tag{B.16}$$

Then consider the commutation relation of $\mathcal{A}$ and an operator $\hat{q}$ which commutes with the integration:

$$[\![\hat{q}, \mathcal{A}_{[\tau_f,\tau_i]}]\!] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_f} d\tau'\Big)^n \mathcal{O}_{[\tau_1]} \cdots \mathcal{O}_{[\tau_{k-1}]} [\![\hat{q}, \mathcal{O}_{[\tau_k]}]\!] \mathcal{O}_{[\tau_{k+1}]} \cdots \mathcal{O}_{[\tau_n]}, \tag{B.17}$$

where we assume $\mathcal{O}$ is even degree. We can change the parameterization of the range of the integrations as follows: $\int_{\tau_I}^{\tau_f} d\tau_k$:

$$\begin{aligned}\overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_f} d\tau'\Big)^n &= \int_{\tau_i}^{\tau_f} d\tau_1 \int_{\tau_i}^{\tau_1} d\tau_2 \int_{\tau_i}^{\tau_2} d\tau_3 \cdots \int_{\tau_i}^{\tau_{n-3}} d\tau_{n-2} \int_{\tau_i}^{\tau_{n-2}} d\tau_{n-1} \int_{\tau_i}^{\tau_{n-1}} d\tau_n \\ &= \left(\int_{\tau_2}^{\tau_f} d\tau_1 \int_{\tau_3}^{\tau_f} d\tau_2 \cdots \int_{\tau_k}^{\tau_f} d\tau_{k-1}\right) \times \int_{\tau_i}^{\tau_f} d\tau_k \times \left(\int_{\tau_i}^{\tau_k} d\tau_{k+1} \int_{\tau_i}^{\tau_{k+1}} d\tau_{k+2} \cdots \int_{\tau_i}^{\tau_{n-1}} d\tau_n\right) \\ &= \overrightarrow{\mathcal{P}}\Big(\int_{\tau_k}^{\tau_f} d\tau'\Big)^{k-1} \times \int_{\tau_i}^{\tau_f} d\tau_k \times \overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_k} d\tau'\Big)^{n-k}.\end{aligned} \tag{B.18}$$

Then we can transform the commutator further:

$$\begin{aligned}[\![\hat{q}, \mathcal{A}_{[\tau_f,\tau_i]}]\!] &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \overrightarrow{\mathcal{P}}\Big(\int_{\tau_k}^{\tau_f} d\tau'\Big)^{k-1} \int_{\tau_i}^{\tau_f} d\tau_k \overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_k} d\tau'\Big)^{n-k} \mathcal{O}_{[\tau_1]} \cdots \mathcal{O}_{[\tau_{k-1}]} [\![\hat{q}, \mathcal{O}_{[\tau_k]}]\!] \mathcal{O}_{[\tau_{k+1}]} \cdots \mathcal{O}_{[\tau_n]} \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overrightarrow{\mathcal{P}}\Big(\int_{\tau_k}^{\tau_f} d\tau' \mathcal{O}_{[\tau']}\Big)^l \int_{\tau_i}^{\tau_f} d\tau_k [\![\hat{q}, \mathcal{O}_{[\tau_k]}]\!] \overrightarrow{\mathcal{P}}\Big(\int_{\tau_i}^{\tau_k} d\tau' \mathcal{O}_{[\tau']}\Big)^m.\end{aligned} \tag{B.19}$$

Finally we obtain

$$[\![\hat{q}, \mathcal{A}_{[\tau_f,\tau_i]}]\!] = \int_{\tau_i}^{\tau_f} d\tau' \mathcal{A}_{[\tau_f,\tau']} \cdot [\![\hat{q}, \mathcal{O}_{[\tau']}]\!] \cdot \mathcal{A}_{[\tau',\tau_i]}. \tag{B.20}$$

C NS string products by path-ordered exponentials

In this appendix, we start with $\mathbf{L}(s,\tau)$ in the form of a similarity transformation of $\mathbf{L}^{\mathrm{B}}(s)$, and derive the condition for the η-derivation properties and the cyclicity, based on the properties of the path-ordered exponentials. Discussions in this appendix are also applicable to the open string. We also discuss the case where we start with the associative star product.

Let us consider the generating functions of $\mathbf{L}^{[d]}$:

$$\mathbf{L}(s,\tau)=\sum_{m=0}^{\infty}s^m\mathbf{L}^{[m]}(\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}s^m\tau^n\mathbf{L}^{[m]}_{m+n+1}. \tag{C.1}$$

At $\tau=0$ and $s=0$, $\mathbf{L}(s,\tau)$ are identified with the string products $\mathbf{L}^{\mathrm{B}}$ and $\mathbf{L}^{\mathrm{NS}}$, respectively:

$$\mathbf{L}(s,0)=\mathbf{L}^{\mathrm{B}}(s)=\sum_{n=0}^{\infty}s^n\mathbf{L}^{\mathrm{B}}_{n+1},\qquad \mathbf{L}(0,\tau)=\mathbf{L}^{[0]}(\tau)=\mathbf{L}^{\mathrm{NS}}(\tau). \tag{C.2}$$

L_∞-relations

We first require $\mathbf{L}(s,\tau)$ to satisfy the L_∞-relations:

$$0=[\![\mathbf{L}(s,\tau),\mathbf{L}(s,\tau)]\!]. \tag{C.3}$$

Such products $\mathbf{L}(s,\tau)$ can be defined by the following differential equations:

$$\partial_\tau\mathbf{L}(s,\tau)=[\![\mathbf{L}(s,\tau),\boldsymbol{\lambda}(s,\tau)]\!], \tag{C.4}$$

where $\boldsymbol{\lambda}(s,\tau)$ is the generating function for the gauge products with deficit picture $\boldsymbol{\lambda}^{[d]}$, defined by

$$\boldsymbol{\lambda}(s,\tau)=\sum_{m=0}^{\infty}s^m\boldsymbol{\lambda}^{[m]}(\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}s^m\tau^n\boldsymbol{\lambda}^{[m]}_{m+n+2}. \tag{C.5}$$

The solution for the differential equation $\partial_\tau\mathbf{L}(s,\tau)=[\![\mathbf{L}(s,\tau),\boldsymbol{\lambda}(s,\tau)]\!]$ can be written as a similarity transformation of $\mathbf{L}(s,0)=\mathbf{L}^{\mathrm{B}}(s)$:

$$\mathbf{L}(s,\tau)=\mathbf{G}^{-1}(s;\tau,0)\mathbf{L}^{\mathrm{B}}(s)\mathbf{G}(s;0,\tau), \tag{C.6}$$

where $\mathbf{G}(s;\tau_i,\tau_f)$ is an invertible cohomomorphism which is defined by the path-ordered exponentials:

$$\mathbf{G}(s;\tau_i,\tau_f)=\overleftarrow{\mathcal{P}}\exp\Big(\int_{\tau_i}^{\tau_f}d\tau'\boldsymbol{\lambda}(s,\tau')\Big),\qquad \mathbf{G}^{-1}(s;\tau_f,\tau_i)=\overrightarrow{\mathcal{P}}\exp\Big(-\int_{\tau_i}^{\tau_f}d\tau'\boldsymbol{\lambda}(s,\tau')\Big). \tag{C.7}$$

Note that in section 4 we omitted the dependence on $\tau_i=0$, for example we wrote $\mathbf{G}(s;\tau)$ instead of $\mathbf{G}(s;0,\tau)$. $\mathbf{G}(s;\tau_i,\tau_f)$ and $\mathbf{G}^{-1}(s;\tau_f,\tau_i)$ satisfy the following differential equations:

$$\partial_{\tau_f}\mathbf{G}(s;\tau_i,\tau_f)=\mathbf{G}(s;\tau_i,\tau_f)\boldsymbol{\lambda}(s,\tau),\qquad \partial_{\tau_f}\mathbf{G}^{-1}(s;\tau_f,\tau_i)=-\boldsymbol{\lambda}(s,\tau)\mathbf{G}^{-1}(s;\tau_f,\tau_i) \tag{C.8}$$

with the initial conditions $\mathbf{G}(s;\tau,\tau)=\mathbb{1}$ and $\mathbf{G}^{-1}(s;\tau,\tau)=\mathbb{1}$. For more details, see appendix B. We can check that (C.6) is a solution for the differential equation $\partial_\tau\mathbf{L}(s,\tau)=[\![\mathbf{L}(s,\tau),\boldsymbol{\lambda}(s,\tau)]\!]$ with the initial condition $\mathbf{L}(s,0)=\mathbf{L}^{\mathrm{B}}(s)$. See also section 4.2.3.

The L_∞-relation follows from only the fact that $\mathbf{L}(s,\tau)$ can be written as a similarity transformation of $\mathbf{L}^{\mathrm{B}}$ which is nilpotent. One can cheek it easily as follows:

$$\begin{aligned}\mathbf{L}(s,\tau)\mathbf{L}(s,\tau) &= \Big(\mathbf{G}^{-1}(s;\tau,0)\mathbf{L}^{\mathrm{B}}(s)\mathbf{G}(s;0,\tau)\Big)\Big(\mathbf{G}^{-1}(s;\tau,0)\mathbf{L}^{\mathrm{B}}(s)\mathbf{G}(s;0,\tau)\Big)\\ &= \mathbf{G}^{-1}(s;\tau,0)\mathbf{L}^{\mathrm{B}}(s)\mathbf{L}^{\mathrm{B}}(s)\mathbf{G}(s;0,\tau)\\ &= 0.\end{aligned} \tag{C.9}$$

That is, if we only require $\mathbf{L}$ to satisfy the L_∞-relations, the gauge products can be taken arbitrary. In what follows, we discuss the conditions on a choice of the gauge products which are required by the cyclicity and η-derivation property of $\mathbf{L}(s,\tau)$.

Cyclicity

If the gauge products $\boldsymbol{\lambda}(s,\tau)$ are BPZ-odd, $\mathbf{G}$ satisfies $(\mathbf{G}(s;\tau,0))^\dagger = \mathbf{G}^{-1}(s;0,\tau)$, which leads to the cyclicity of $\mathbf{L}(s,\tau)$:

$$(\mathbf{L}(s,\tau))^\dagger = (\mathbf{G}(s;0,\tau))^\dagger(\mathbf{L}^{\mathrm{B}}(s))^\dagger(\mathbf{G}^{-1}(s;\tau,0))^\dagger = -\mathbf{L}(s,\tau). \tag{C.10}$$

Note that $(\mathbf{L}^{\mathrm{B}}(s))^\dagger = -\mathbf{L}^{\mathrm{B}}(s)$.

For the cyclicity of the NS product $\mathbf{L}^{[0]} = \mathbf{L}^{\mathrm{NS}}$, it is sufficient to require $\boldsymbol{\lambda}^{[0]}(\tau)$ to be BPZ-odd. Recall that the NS products $\mathbf{L}^{[0]} = \mathbf{L}^{\mathrm{NS}}$ is written as a similarity transformation of $\mathbf{L}^{[0]}(0) = \mathbf{Q}$:

$$\mathbf{L}^{\mathrm{NS}}(\tau) = \mathbf{G}^{-1}(\tau,0)\mathbf{Q}\mathbf{G}(0,\tau), \tag{C.11}$$

where $\mathbf{G}(\tau_i,\tau_f)$ and its inverse are defined by

$$\mathbf{G}(\tau_i,\tau_f) = \overleftarrow{\mathcal{P}}\exp\Big(\int_{\tau_i}^{\tau_f} d\tau'\boldsymbol{\lambda}^{[0]}(\tau')\Big), \qquad \mathbf{G}^{-1}(\tau_f,\tau_i) = \overrightarrow{\mathcal{P}}\exp\Big(-\int_{\tau_i}^{\tau_f} d\tau'\boldsymbol{\lambda}^{[0]}(\tau')\Big). \tag{C.12}$$

If the gauge products $\boldsymbol{\lambda}^{[0]}(\tau)$ are BPZ-odd, $\mathbf{G}$ satisfy $(\mathbf{G}(\tau,0))^\dagger = \mathbf{G}^{-1}(0,\tau)$, and then $\mathbf{L}^{\mathrm{NS}}(\tau)$ in cyclic:

$$(\mathbf{L}^{\mathrm{NS}}(\tau))^\dagger = (\mathbf{G}(0,\tau))^\dagger\mathbf{Q}^\dagger(\mathbf{G}^{-1}(\tau,0))^\dagger = -\mathbf{L}^{\mathrm{NS}}(\tau). \tag{C.13}$$

η-derivation properties

Next, let us consider the conditions for the η-derivation properties of $\mathbf{L}(s,\tau)$:

$$0 = [\![\boldsymbol{\eta},\mathbf{L}(s,\tau)]\!]. \tag{C.14}$$

By the direct calculation using the technique introduced in Appendix B, one can obtain

$$\begin{aligned}[\![\boldsymbol{\eta},\mathbf{L}(s,\tau)]\!] &= [\![\boldsymbol{\eta},\mathbf{G}^{-1}(s;\tau,0)]\!]\mathbf{L}(s;0)\mathbf{G}(s;0,\tau) - \mathbf{G}^{-1}(s;\tau,0)\mathbf{L}(s;0)[\![\boldsymbol{\eta},\mathbf{G}(s;0,\tau)]\!]\\ &= -\int_0^\tau d\tau'\mathbf{G}^{-1}(s;\tau,\tau')[\![\boldsymbol{\eta},\boldsymbol{\lambda}(s;\tau')]\!]\mathbf{G}^{-1}(s;\tau',0)\mathbf{L}(s;0)\mathbf{G}(s;0,\tau)\\ &\qquad\qquad - \mathbf{G}^{-1}(s;\tau,0)\mathbf{L}(s;0)\int_0^\tau d\tau'\mathbf{G}(s;0,\tau')[\![\boldsymbol{\eta},\boldsymbol{\lambda}(s;\tau')]\!]\mathbf{G}(s;\tau',\tau)\\ &= -\int_0^\tau d\tau'\mathbf{G}^{-1}(s;\tau,\tau')\Big[\!\!\Big[[\![\boldsymbol{\eta},\boldsymbol{\lambda}(s;\tau')]\!],\mathbf{G}^{-1}(s;\tau',0)\mathbf{L}(s;0)\mathbf{G}(s;0,\tau')\Big]\!\!\Big]\mathbf{G}(s;\tau',\tau)\\ &= -\int_0^\tau d\tau'\mathbf{G}^{-1}(s;\tau,\tau')\Big[\!\!\Big[[\![\boldsymbol{\eta},\boldsymbol{\lambda}(s;\tau')]\!],\mathbf{L}(s;\tau')\Big]\!\!\Big]\mathbf{G}(s;\tau',\tau).\end{aligned} \tag{C.15}$$

We used (B.20) and $[\![\boldsymbol{\eta}, \mathbf{L}^{\mathrm{B}}(s)]\!] = 0$. We can see that $\big[\![[\![\boldsymbol{\eta}, \boldsymbol{\lambda}(s;t')]\!], \mathbf{L}(s;t')\big]\!] = 0$ is the condition on a choice of $\boldsymbol{\lambda}(s,\tau)$ for the η-derivation of $\mathbf{L}$. If gauge products are given, the identification $[\![\boldsymbol{\eta}, \boldsymbol{\lambda}(s,\tau)]\!] = \partial_s \mathbf{L}(s,\tau)$ leads to the η-derivation property. Or, in reverse direction, if we define the gauge products from $\mathbf{L}(s,\tau)$ by

$$\xi \circ \partial_s \mathbf{L}(s,\tau) = \boldsymbol{\lambda}(s,\tau), \tag{C.16}$$

$\mathbf{L}(s,\tau)$ satisfy the η-derivation property: differentiating (C.15) in τ, one can obtain the homogeneous differential equation (4.67), and then $[\![\boldsymbol{\eta}, \mathbf{L}(s,0)]\!] = 0$ ensures the solution $[\![\boldsymbol{\eta}, \mathbf{L}(s,\tau)]\!] = 0$. See also section 4.2.2.

Note on $\mathbf{L}^{[1]}(\tau) \neq \mathbf{G}(\tau,0)^{-1}\mathbf{L}^{[1]}(0)\mathbf{G}(0,\tau)$

In general, $\mathbf{L}^{[d\geq 1]}(\tau)$ cannot be written as a similarity transformation of $\mathbf{L}^{[d\geq 1]}(0)$, namely $\mathbf{L}^{[1]}(\tau) \neq \mathbf{G}^{-1}(\tau,0)\mathbf{L}^{[1]}(0)\mathbf{G}(0,\tau)$. $\mathbf{L}^{[1]}(\tau)$ corresponds to the s^1 part of $\mathbf{L}(s,\tau)$: $\mathbf{L}^{[1]}(\tau) = \partial_s \mathbf{L}(s,\tau)|_{s=0}$. The s^1 part of $\mathbf{L}(s,\tau)$ can be computed as follows:

$$\begin{aligned}
\partial_s \mathbf{L}(s,\tau) &= \mathbf{G}^{-1}(s;\tau,0)\partial_s \mathbf{L}(s;0)\mathbf{G}(s;0,\tau) - \int_0^\tau d\tau' \mathbf{G}^{-1}(s;\tau,\tau')\partial_s \boldsymbol{\lambda}(s,\tau')\mathbf{G}^{-1}(s;\tau',0)\mathbf{L}(s;0)\mathbf{G}(s;0,\tau) \\
&\qquad + \mathbf{G}^{-1}(s;\tau,0)\mathbf{L}(s;0)\int_0^\tau d\tau' \mathbf{G}(s;0,\tau')\partial_s \boldsymbol{\lambda}(s,\tau')\mathbf{G}(s;\tau',\tau) \\
&= \mathbf{G}^{-1}(s;\tau,0)\partial_s \mathbf{L}(s;0)\mathbf{G}(s;0,\tau) - \int_0^\tau d\tau' \mathbf{G}^{-1}(s;\tau,\tau')[\![\partial_s \boldsymbol{\lambda}(s,\tau'), \mathbf{L}(s;\tau')]\!]\mathbf{G}(s;\tau',\tau).
\end{aligned} \tag{C.17}$$

Taking $s = 0$ and using $\mathbf{G}^{-1}(s;\tau,0) = \mathbf{G}^{-1}(\tau,0) + \mathcal{O}(s)$, $\mathbf{L}^{[1]}(\tau)$ is written as follows:

$$\mathbf{L}^{[1]}(\tau) = \mathbf{G}^{-1}(\tau,0)\mathbf{L}^{[1]}(0)\mathbf{G}(0,\tau) - \int_0^\tau d\tau' \mathbf{G}^{-1}(\tau,\tau')[\![\boldsymbol{\lambda}^{[1]}(\tau'), \mathbf{L}^{[0]}(\tau')]\!]\mathbf{G}(\tau',\tau). \tag{C.18}$$

If L_2^{B} are associative, $\mathbf{L}_2^{\mathrm{B}}\mathbf{L}_2^{\mathrm{B}} = 0$, we can set $\boldsymbol{\lambda}^{[1]} = 0$ and then $\mathbf{L}^{[1]}(\tau) = \mathbf{G}^{-1}(\tau,0)\mathbf{L}^{[1]}(0)\mathbf{G}(0,\tau)$ holds.

Note on the truncated case

In what follows we consider the case of the open string and write $\mathbf{M}$ and $\boldsymbol{\mu}$ instead of $\mathbf{L}$ and $\boldsymbol{\lambda}$. If $\mathbf{M}_2^{\mathrm{B}}$ is associative, $\mathbf{M}_2^{\mathrm{B}}\mathbf{M}_2^{\mathrm{B}} = 0$, higher products are not necessary for A_∞-relations, and can be set to zero: $\mathbf{M}_{N>3}^{\mathrm{B}} = 0$. Then, $\mathbf{M}^{\mathrm{B}}$ consists of $\mathbf{M}_1^{\mathrm{B}}$ and $\mathbf{M}_2^{\mathrm{B}}$:

$$\mathbf{M}^{\mathrm{B}}(s) = \mathbf{M}_1^{\mathrm{B}} + s\mathbf{M}_2^{\mathrm{B}}. \tag{C.19}$$

In this case, one can set $\mathbf{M}^{[d\geq 2]} = 0$ and $\boldsymbol{\mu}^{[d\geq 1]} = 0$. The nonvanishing products are the NS products with deficit picture 0 and 1, $\mathbf{M}^{[0]}$ and $\mathbf{M}^{[1]}$, and the gauge products with deficit picture 0, $\boldsymbol{\mu}^{[0]}$. The generating functions are truncated:

$$\mathbf{M}(s,\tau) = \mathbf{M}^{[0]}(\tau) + s\mathbf{M}^{[1]}(\tau), \qquad \boldsymbol{\mu}(s,\tau) = \boldsymbol{\mu}^{[0]}(\tau). \tag{C.20}$$

In this case, $\mathbf{M}^{[1]}$ is also nilpotent:

$$[\![\mathbf{M}^{[1]}, \mathbf{M}^{[1]}]\!] = 0. \tag{C.21}$$

$\boldsymbol{\mu}(s,\tau)$ does not depend on s, and $\mathbf{G}$ does not acquire the s-dependence:

$$\mathbf{G}(s;0,\tau) = \overleftarrow{\mathcal{P}}\exp\Big(\int_0^\tau d\tau' \boldsymbol{\mu}^{[0]}(\tau')\Big) = \mathbf{G}(0,\tau). \tag{C.22}$$

As in $\mathbf{M}^{[0]}$ which can be written as a similarity transformation of $\mathbf{Q}$, $\mathbf{M}^{[0]} = \mathbf{G}^{-1}(\tau,0)\mathbf{Q}\mathbf{G}(0,\tau)$, $\mathbf{M}(s,\tau)$ is also given by

$$\mathbf{M}(s,\tau) = \mathbf{G}^{-1}(s;\tau,0)\mathbf{M}^{\mathrm{B}}(s)\mathbf{G}(s;0,\tau) = \mathbf{G}^{-1}(\tau,0)\mathbf{M}^{\mathrm{B}}(s)\mathbf{G}(0,\tau). \tag{C.23}$$

Then, expanding in powers of s as

$$\begin{aligned}\mathbf{M}(s,\tau) &= \mathbf{G}^{-1}(\tau,0)\big(\mathbf{M}_1^{\mathrm{B}} + s\mathbf{M}_2^{\mathrm{B}}\big)\mathbf{G}(0,\tau)\\ &= \mathbf{G}^{-1}(\tau,0)\mathbf{M}_1^{\mathrm{B}}\mathbf{G}(0,\tau) + s\mathbf{G}^{-1}(\tau,0)\mathbf{M}_2^{\mathrm{B}}\mathbf{G}(0,\tau)\\ &= \mathbf{M}^{[0]}(\tau) + s\mathbf{M}^{[1]}(\tau),\end{aligned} \tag{C.24}$$

one can find that $\mathbf{M}^{[1]}$ can be written as a similarity transformation of $\mathbf{M}_2^{\mathrm{B}}$:

$$\mathbf{M}^{[1]}(\tau) = \mathbf{G}^{-1}(\tau,0)\mathbf{M}_2^{\mathrm{B}}\mathbf{G}(0,\tau). \tag{C.25}$$

Note that one can derive (C.25) also from (C.18).

D η-based L_∞-products and dual gauge products

In this appendix we define η-based L_∞-products and dual gauge products, and summarize their properties.

D.1 η-based L_∞-products $\mathbf{L}^\eta$

Definitions and basic properties

The dual product $\mathbf{L}^\eta$ is defined using the cohomomorphism $\mathbf{G}$ which appears in the construction of the NS product $\mathbf{L}^{\mathrm{NS}} = \mathbf{G}^{-1}\mathbf{Q}\mathbf{G}$, by

$$\mathbf{L}^\eta(\tau) = \mathbf{G}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau) = \sum_{p=0}^{\infty} \tau^p \mathbf{L}^\eta_{p+1}. \tag{D.1}$$

$\mathbf{L}^\eta$ is degree odd and the n-th dual product $\mathbf{L}^\eta_n$ carries ghost number $3-2n$ and picture number $n-2$. The following basic properties of $\mathbf{L}^\eta$ follow directly from its definition:

1. The initial condition at $\tau = 0$ reads $\mathbf{L}^\eta(0) = \boldsymbol{\eta}$, which follows from $\mathbf{G}(0) = 1$.
2. the L_∞ relation follow from $\mathbf{G}^{-1}\mathbf{G} = 1$ and $\boldsymbol{\eta}^2 = 0$:
$$[\![\mathbf{L}^\eta(\tau), \mathbf{L}^\eta(\tau)]\!] = \frac{1}{2}\mathbf{G}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau)\mathbf{G}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau) = \frac{1}{2}\mathbf{G}(\tau)\boldsymbol{\eta}^2\mathbf{G}^{-1}(\tau) = 0. \tag{D.2}$$
3. The Q-derivation property of $\mathbf{L}^\eta$ follows from $[\![\boldsymbol{\eta}, \mathbf{L}^{\mathrm{NS}}(\tau)]\!] = 0$:
$$[\![\mathbf{Q}, \mathbf{L}^\eta(\tau)]\!] = [\![\mathbf{Q}, \mathbf{G}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau)]\!] = \mathbf{G}(\tau)[\![\mathbf{L}^{\mathrm{NS}}(\tau), \boldsymbol{\eta}]\!]\mathbf{G}^{-1}(\tau) = 0. \tag{D.3}$$
4. The cyclicity of L^η follows from $\mathbf{G}^{-1} = \mathbf{G}^\dagger$, i.e. the BPZ-oddness of gauge products.
5. $\mathbf{L}^\eta$ satisfies the following differential equation:
$$\begin{aligned}\partial_\tau \mathbf{L}^\eta(\tau) &= \mathbf{G}(0,\tau)\boldsymbol{\lambda}^{[0]}(\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau,0) - \mathbf{G}(0,\tau)\boldsymbol{\eta}\boldsymbol{\lambda}^{[0]}(\tau)\mathbf{G}^{-1}(\tau,0)\\ &= [\![\mathbf{G}(0,\tau)\boldsymbol{\lambda}^{[0]}(\tau)\mathbf{G}^{-1}(\tau,0), \mathbf{L}^\eta(\tau)]\!].\end{aligned} \tag{D.4}$$
6. Expanding the path-ordered exponential $\mathbf{G}$, we can obtain the following expression:
$$\begin{aligned}\mathbf{L}^\eta(\tau) =& \boldsymbol{\eta} + \tau[\![\boldsymbol{\lambda}^{[0]}_2, \boldsymbol{\eta}]\!] + \frac{\tau^2}{2}\Big([\![\boldsymbol{\lambda}^{[0]}_3, \boldsymbol{\eta}]\!] + [\![\boldsymbol{\lambda}^{[0]}_2, [\![\boldsymbol{\lambda}^{[0]}_2, \boldsymbol{\eta}]\!]]\!]\Big) +\\ &+ \frac{\tau^3}{3!}\Big(2[\![\boldsymbol{\lambda}^{[0]}_4, \boldsymbol{\eta}]\!] + 2[\![\boldsymbol{\lambda}^{[0]}_2, [\![\boldsymbol{\lambda}^{[0]}_3, \boldsymbol{\eta}]\!]]\!] + [\![\boldsymbol{\lambda}^{[0]}_3, [\![\boldsymbol{\lambda}^{[0]}_2, \boldsymbol{\eta}]\!]]\!] + [\![\boldsymbol{\lambda}^{[0]}_2, [\![\boldsymbol{\lambda}^{[0]}_2, [\![\boldsymbol{\lambda}^{[0]}_2, \boldsymbol{\eta}]\!]]\!]]\!]\Big) + \cdots\\ =& \boldsymbol{\eta} - \tau\mathbf{L}^{[1]}_2 + \frac{\tau^2}{2}\Big(-\mathbf{L}^{[1]}_3 - [\![\boldsymbol{\lambda}^{[0]}_2, \mathbf{L}^{[1]}_2]\!]\Big) +\\ &+ \frac{\tau^3}{3!}\Big(-2\mathbf{L}^{[1]}_4 - 2[\![\boldsymbol{\lambda}^{[0]}_2, \mathbf{L}^{[1]}_3]\!] - [\![\boldsymbol{\lambda}^{[0]}_3, \mathbf{L}^{[1]}_2]\!] - [\![\boldsymbol{\lambda}^{[0]}_2, [\![\boldsymbol{\lambda}^{[0]}_2, \mathbf{L}^{[1]}_2]\!]]\!]\Big) + \cdots .\end{aligned} \tag{D.5}$$

Relation to $\mathbf{L}^{[1]}$

$\mathbf{L}^{\eta}$ carries the same quantum numbers as $\mathbf{L}^{[1]}$. They are related by the following relation:

$$\begin{aligned}\mathbf{L}^{\eta}(\tau) &= \mathbf{G}(0,\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(\tau,0)\\ &= \boldsymbol{\eta} + \mathbf{G}(0,\tau)[\![\boldsymbol{\eta}, \mathbf{G}^{-1}(\tau,0)]\!]\\ &= \boldsymbol{\eta} - \mathbf{G}(0,\tau)\int_0^{\tau} d\tau' \mathbf{G}^{-1}(\tau,\tau')[\![\boldsymbol{\eta}, \boldsymbol{\lambda}^{[0]}(\tau')]\!]\mathbf{G}^{-1}(\tau',0)\\ &= \boldsymbol{\eta} - \int_0^{\tau} d\tau' \mathbf{G}(0,\tau')\mathbf{L}^{[1]}(\tau')\mathbf{G}^{-1}(\tau',0). \end{aligned} \tag{D.6}$$

From this expression, we can derive the differential equation connecting them:

$$\partial_\tau \mathbf{L}^{\eta}(\tau) = -\mathbf{G}(0,\tau)\mathbf{L}^{[1]}(\tau)\mathbf{G}^{-1}(\tau,0). \tag{D.7}$$

Truncated case

If M_2^{B} is associative, $\mathbf{M}^{\mathrm{B}}$ is given by $\mathbf{M}^{\mathrm{B}} = \mathbf{Q} + \mathbf{M}_2^{\mathrm{B}}$ [27], and the second term on the right-hand side of (D.6) becomes

$$-\int_0^{\tau} d\tau' \mathbf{G}(0,\tau')\mathbf{M}^{[1]}(\tau')\mathbf{G}^{-1}(\tau',0) = -\int_0^{\tau} d\tau' \mathbf{M}^{[1]}(0) = -\tau \mathbf{M}_2^{B}. \tag{D.8}$$

Thus, in the associative case, the dual products are also truncated and are given by

$$\mathbf{M}^{\eta}(\tau) = \boldsymbol{\eta} - \tau \mathbf{M}_2^{\mathrm{BOS}}. \tag{D.9}$$

Expression in which Q-derivation property is manifest

Utilizing the relation between $\mathbf{L}^{[1]}(\tau)$ and $\mathbf{L}^{[1]}(0)$, $\mathbf{L}^{\eta}$ can be written as follows:

$$\begin{aligned}\mathbf{L}^{\eta}(\tau) = \boldsymbol{\eta} - \int_0^{\tau} d\tau' \mathbf{G}(0,\tau')\Big(&\mathbf{G}^{-1}(\tau',0)\mathbf{L}^{[1]}(0)\mathbf{G}(0,\tau')\\ &- \int_0^{\tau'} d\tau'' \mathbf{G}^{-1}(\tau',\tau'')[\![\boldsymbol{\lambda}^{[1]}(\tau''), \mathbf{L}^{[0]}(\tau'')]\!]\mathbf{G}(\tau'',\tau')\Big)\mathbf{G}^{-1}(\tau',0)\\ = \boldsymbol{\eta} - \tau\mathbf{L}^{[1]}(0) + \int_0^{\tau} &d\tau''(\tau-\tau'')\mathbf{G}(0,\tau'')[\![\boldsymbol{\lambda}^{[1]}(\tau''), \mathbf{L}^{[0]}(\tau'')]\!]\mathbf{G}^{-1}(\tau'',0). \end{aligned} \tag{D.10}$$

We used $\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' = \int_0^{\tau} d\tau'' \int_{\tau''}^{\tau} d\tau'$. We can transform it in the following way:

$$\begin{aligned}\mathbf{L}^{\eta}(\tau) &= \boldsymbol{\eta} - \tau\mathbf{L}^{[1]}(0) + \int_0^{\tau} d\tau''(\tau-\tau'')\mathbf{G}(0,\tau'')[\![\boldsymbol{\lambda}^{[1]}(\tau''), \mathbf{L}^{[0]}(\tau'')]\!]\mathbf{G}^{-1}(\tau'',0)\\ &= \boldsymbol{\eta} - \tau\mathbf{L}^{[1]}(0) + \int_0^{\tau} d\tau''(\tau-\tau'')[\![\mathbf{G}(0,\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\mathbf{G}^{-1}(\tau'',0), \mathbf{G}(0,\tau'')\mathbf{L}^{[0]}(\tau'')\mathbf{G}^{-1}(\tau'',0)]\!]\\ &= \boldsymbol{\eta} - \tau\mathbf{L}_2^{\mathrm{BOS}} - \int_0^{\tau} d\tau''(\tau-\tau'')[\![\mathbf{Q}, \mathbf{G}(0,\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\mathbf{G}^{-1}(\tau'',0)]\!]. \end{aligned} \tag{D.11}$$

In this expression, the Q-derivation property is manifest. We can also find that the following differential equations hold:

$$\partial_\tau \mathbf{L}^{\eta}(\tau) = -\mathbf{L}_2^{\mathrm{BOS}} - \int_0^{\tau} d\tau'' [\![\mathbf{Q}, \mathbf{G}(0,\tau'')\boldsymbol{\lambda}^{[1]}(\tau'')\mathbf{G}^{-1}(\tau'',0)]\!], \tag{D.12}$$

$$\partial_\tau^2 \mathbf{L}^{\eta}(\tau) = [\![\mathbf{Q}, -\mathbf{G}(0,\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau,0)]\!]. \tag{D.13}$$

[27] We use $\mathbf{M}$ since we consider the open string.

D.2 Dual gauge products $\boldsymbol{\rho}$

Since $[\![\mathbf{Q}, \mathbf{L}^{\eta}(\tau)]\!] = 0$, $\mathbf{L}^{\eta}$ can be written as the commutator of Q and some product $\boldsymbol{\rho}$:

$$\mathbf{L}^{\eta}(\tau) = [\![\mathbf{Q}, \boldsymbol{\rho}(\tau)]\!] = \sum_{n=0}^{\infty} \tau^n [\![\mathbf{Q}, \boldsymbol{\rho}_{n+1}]\!]. \tag{D.14}$$

We will call $\boldsymbol{\rho}$ *dual gauge products*.

$\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can be constructed using the *homotopy operator* R satisfying $[\![Q, R]\!] = 1$. For example, if we require the cyclicity, they can be defined by

$$\rho_1 = R \circ \eta = \frac{1}{2}(R\eta - \eta R), \tag{D.15}$$

$$\rho_2 = R \circ L_2^{\eta} = -\frac{1}{3}\big(RL_2^{\mathrm{B}} - L_2^{\mathrm{B}}(R \wedge \mathbb{I})\big), \tag{D.16}$$

where we write $R\circ$ in the same sense as (4.41) and (4.42).

$\boldsymbol{\rho}_{n\geq 3}$ can be written without R. Recall that $\mathbf{L}^{\eta}$ satisfies $\partial_\tau^2 \mathbf{L}^{\eta}(\tau) = -[\![\mathbf{Q}, \mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau)]\!]$. The dual gauge products $\boldsymbol{\rho}$ are related to the gauge products $\boldsymbol{\lambda}$ as

$$\partial_\tau^2 \boldsymbol{\rho}(\tau) = -\mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau). \tag{D.17}$$

Expanding in powers of τ, the left-hand side becomes

$$\partial_\tau^2 \boldsymbol{\rho}(\tau) = \sum_{n=0}^{\infty}(n+1)(n+2)t^n \boldsymbol{\rho}_{3+n} = 2\boldsymbol{\rho}_3 + 6\tau\boldsymbol{\rho}_4 + 12\tau^2\boldsymbol{\rho}_5 + 20\tau^3\boldsymbol{\rho}_6 + \cdots, \tag{D.18}$$

and then this relation determines $\boldsymbol{\rho}_{n\geq 3}$. The cyclicity of $\boldsymbol{\rho}_{n\geq 3}$ follows from that of $\boldsymbol{\lambda}^{[0]}$ and $\boldsymbol{\lambda}^{[1]}$.

Expanding in powers of τ, the right-hand side of (D.17) becomes

$$\begin{aligned}
&-\mathbf{G}(\tau)\boldsymbol{\lambda}^{[1]}(\tau)\mathbf{G}^{-1}(\tau)\\
&= -\boldsymbol{\lambda}^{[1]}(\tau) - \tau[\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!] - \frac{\tau^2}{2}\Big([\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!] + [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!]]\!]\Big) +\\
&- \frac{\tau^3}{3!}\Big(2[\![\boldsymbol{\lambda}_4^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!] + 2[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!]]\!] + [\![\boldsymbol{\lambda}_3^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!]]\!] + [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}^{[1]}(\tau)]\!]]\!]]\!]\Big) + \cdots\\
&= -\boldsymbol{\lambda}_3^{[1]} - \tau\Big(\boldsymbol{\lambda}_4^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]\Big) - \tau^2\Big(\boldsymbol{\lambda}_5^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]\Big)\\
&\quad - \tau^3\Big(\boldsymbol{\lambda}_6^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_5^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!]]\!]\\
&\qquad + \frac{1}{3!}\big(2[\![\boldsymbol{\lambda}_4^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + 2[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_3^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]]\!]\big)\Big) + \cdots,
\end{aligned} \tag{D.19}$$

then we obtain the following expressions:

$$\boldsymbol{\rho}_3 = -\frac{1}{2}\boldsymbol{\lambda}_3^{[1]}, \tag{D.20}$$

$$\boldsymbol{\rho}_4 = -\frac{1}{6}\Big(\boldsymbol{\lambda}_4^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]\Big), \tag{D.21}$$

$$\boldsymbol{\rho}_5 = -\frac{1}{12}\Big(\boldsymbol{\lambda}_5^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]\Big), \tag{D.22}$$

$$\begin{aligned}
\boldsymbol{\rho}_6 = -\frac{1}{20}\Big(&\boldsymbol{\lambda}_6^{[1]} + [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_5^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!] + \frac{1}{2}[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_4^{[1]}]\!]]\!]\\
&+ \frac{1}{3!}\big(2[\![\boldsymbol{\lambda}_4^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!] + 2[\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_3^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_3^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!] + [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, [\![\boldsymbol{\lambda}_2^{[0]}, \boldsymbol{\lambda}_3^{[1]}]\!]]\!]]\!]\big)\Big).
\end{aligned} \tag{D.23}$$

D.3 η-based L_∞-products with deficit picture $\mathbf{L}^\eta(s)$ and its gauge products $\boldsymbol{\rho}(s)$

Although we do not use these structures in the main text, we can define dual products with deficit picture by the cohomomorphism $\mathbf{G}(s,\tau)$.

Definition and basic properties of $\mathbf{L}^\eta(s)$

We can define η-based L_∞ products with deficit picture by

$$\mathbf{L}^\eta(s;\tau) = \mathbf{G}(s;\tau)\boldsymbol{\eta}\mathbf{G}^{-1}(s;\tau) = \sum_{d=0}^{\infty} s^d \mathbf{L}^{\eta[d]}(\tau) = \sum_{d=0}^{\infty} s^d \sum_{p=0}^{\infty} \tau^p \mathbf{L}^{\eta[d]}_{d+p+1}. \tag{D.24}$$

We denote $[d]$ by the picture deficit relative to $\mathbf{L}^\eta$. As in $\mathbf{L}^\eta$, they satisfy the following properties:

1. As the initial condition at $\tau = 0$, $\mathbf{L}^\eta(s,0) = \boldsymbol{\eta}$. That is, $\mathbf{L}^{\eta[d]}_{d+1} = 0$.
2. $\mathbf{L}^\eta(s,\tau)$ satisfies the L_∞-relations:
$$[\![\mathbf{L}^\eta(s,\tau), \mathbf{L}^\eta(s,\tau)]\!] = 0. \tag{D.25}$$
3. $\mathbf{L}^\eta(s,\tau)$ commutes with $\mathbf{L}^{\mathrm{BOS}}(s)$, which follows from $[\![\boldsymbol{\eta}, \mathbf{L}^{\mathrm{BOS}}(s)]\!] = 0$:
$$[\![\mathbf{L}^{\mathrm{BOS}}(s), \mathbf{L}^\eta(s,\tau)]\!] = 0. \tag{D.26}$$
4. The cyclicity of L^η follows from $\mathbf{G}^{-1} = \mathbf{G}^\dagger$, i.e. the BPZ-oddness of gauge products.
5. $\mathbf{L}^\eta(s;\tau)$ satisfies the following differential equation:
$$\partial_\tau \mathbf{L}^{\eta}(s,\tau) = [\![\mathbf{G}(s;0,\tau)\boldsymbol{\lambda}(s,\tau)\mathbf{G}^{-1}(s;\tau,0), \mathbf{L}^\eta(s,\tau)]\!]. \tag{D.27}$$

Relation to $\mathbf{L}(s,\tau)$

By a calculation parallel to (D.6) for the case of $\mathbf{L}^\eta(\tau)$ without picture deficit, $\mathbf{L}^\eta(s,\tau)$ and $\mathbf{L}(s,\tau)$ are related by the following relation:

$$\mathbf{L}^\eta(s,\tau) = \boldsymbol{\eta} - \int_0^\tau d\tau' \mathbf{G}(s;0,\tau')\big(\partial_s \mathbf{L}(s,\tau')\big)\mathbf{G}^{-1}(s;\tau',0). \tag{D.28}$$

Differentiated in τ, the relation becomes

$$\partial_\tau \mathbf{L}^{\eta}(s,\tau) = -\mathbf{G}(s;0,\tau)\partial_s \mathbf{L}(s,\tau)\mathbf{G}^{-1}(s;\tau,0). \tag{D.29}$$

$\mathbf{L}^{\mathrm{B}}(s)$-closed form

Again, by the calculation parallel to the case of $\mathbf{L}^\eta(\tau)$ without picture deficit, $\mathbf{L}^\eta(s;\tau)$ can be written in the $\mathbf{L}^{\mathrm{B}}(s)$-closed form:

$$\mathbf{L}^\eta(s,\tau) = \boldsymbol{\eta} - \tau\partial_s \mathbf{L}^{\mathrm{B}}(s) - \int_0^\tau d\tau''(\tau-\tau'')[\![\mathbf{L}^{\mathrm{B}}(s), \mathbf{G}(s;0,\tau'')\partial_s\boldsymbol{\lambda}(s,\tau'')\mathbf{G}^{-1}(s;\tau'',0)]\!]. \tag{D.30}$$

Utilizing $[\![\boldsymbol{\eta}, \mathbf{L}^{\mathrm{B}}(s)]\!] = 0$ and $[\![\partial_s \mathbf{L}^{\mathrm{B}}(s), \mathbf{L}^{\mathrm{B}}(s)]\!] = 0$, one can find

$$[\![\mathbf{L}^{\mathrm{B}}(s), \mathbf{L}^{\eta}(s,\tau)]\!] = 0. \tag{D.31}$$

From (D.30), we can find that the following differential equations hold:

$$\partial_\tau \mathbf{L}^{\eta}(s,\tau) = -\partial_s \mathbf{L}^{\mathrm{BOS}}(s) - \int_0^\tau d\tau'' [\![\mathbf{L}^{\mathrm{BOS}}(s), \mathbf{G}(s;0,\tau'')\partial_s \boldsymbol{\lambda}(s,\tau'')\mathbf{G}^{-1}(s;\tau'',0)]\!], \tag{D.32}$$

$$\partial_\tau^2 \mathbf{L}^{\eta}(s,\tau) = -[\![\mathbf{L}^{\mathrm{BOS}}(s), \mathbf{G}(s;0,\tau)\partial_s \boldsymbol{\lambda}(s,\tau)\mathbf{G}^{-1}(s;\tau,0)]\!]. \tag{D.33}$$

Dual gauge products with deficit picture $\boldsymbol{\rho}(s)$

Since $[\![\mathbf{L}^{\mathrm{B}}, \mathbf{L}^{\eta}(s,\tau)]\!] = 0$, $\mathbf{L}^{\eta}(s,\tau)$ can be written in an $\mathbf{L}^{\mathrm{B}}$-exact form:

$$\mathbf{L}^{\eta}(s,\tau) = [\![\mathbf{L}^{\mathrm{B}}(s), \boldsymbol{\rho}(s,\tau)]\!], \tag{D.34}$$

$$\boldsymbol{\rho}(s,\tau) = \sum_{d=0}^{\infty} s^d \boldsymbol{\rho}^{[d]}(\tau) = \sum_{n=0}^{\infty}\sum_{d=0}^{\infty} \tau^n s^d \boldsymbol{\rho}^{[d]}_{d+n+1} = \sum_{n=0}^{\infty}\sum_{d=0}^{\infty} \tau^n s^d \boldsymbol{\rho}^{(n-1)}_{d+n+1} = \sum_{n=0}^{\infty} \tau^n \boldsymbol{\rho}^{(n-1)}(s). \tag{D.35}$$

We call $\boldsymbol{\rho}(s,\tau)$ *dual gauge products with deficit picture*. We identify $\boldsymbol{\rho}^{[0]}(\tau) = \boldsymbol{\rho}(\tau)$. We denoted the picture number p of the product by superscript as (p).

$\boldsymbol{\rho}^{(n\geq 1)}$ can be written without using R. The equation (D.33) provides the relation between $\boldsymbol{\rho}(s,\tau)$ and $\boldsymbol{\lambda}(s,\tau)$:

$$\partial_\tau^2 \boldsymbol{\rho}(s,\tau) = -\mathbf{G}(s;\tau)\partial_s \boldsymbol{\lambda}(s,\tau)\mathbf{G}^{-1}(s;\tau). \tag{D.36}$$

This equation determine $\boldsymbol{\rho}^{[d]}_N$ in the region of ghost and picture numbers where corresponding $\boldsymbol{\lambda}^{[d]}_N$ exist.

$\boldsymbol{\rho}^{(0)}$ can be constructed using R which is the homotopy operator satisfying $[\![Q, R]\!] = 1$. We can represent $\mathbf{L}^{\mathrm{B}}$ as a similarity transformation of $\mathbf{Q}$:

$$\mathbf{L}^{\mathrm{B}}(s) = \mathsf{h}^{-1}(s)\mathbf{Q}\mathsf{h}(s), \qquad \mathsf{h}(s) = \overleftarrow{\mathcal{P}} \exp\Big[\int_0^s ds' \boldsymbol{\sigma}(s')\Big], \tag{D.37}$$

where $\boldsymbol{\sigma}(s) = \sum_{k=0}^{\infty} s^k \boldsymbol{\sigma}_{2+k}$ is a new gauge product. We can define this $\boldsymbol{\sigma}(s)$ from $\mathbf{L}^{\mathrm{BOS}}(s)$ and R recursively by the differential equation for $\mathbf{L}^{\mathrm{B}}(s)$ which is given by $\partial_s \mathbf{L}^{\mathrm{B}}(s) = [\![\mathbf{L}^{\mathrm{BOS}}(s), \boldsymbol{\sigma}(s)]\!]$. Its s^N part reads $(N+1)\mathbf{L}^{\mathrm{B}}_{N+2} = \sum_{n=0}^{N} [\![\mathbf{L}^{\mathrm{B}}_{n+1}, \boldsymbol{\sigma}_{N-n+2}]\!]$, and it determines $\boldsymbol{\sigma}_{N+2}$ from $\boldsymbol{\sigma}_{n\leq N+1}$:

$$\boldsymbol{\sigma}_{N+2} = R \circ \Big((N+1)\mathbf{L}^{\mathrm{B}}_{N+2} - \sum_{n=1}^{N} [\![\mathbf{L}^{\mathrm{B}}_{n+1}, \boldsymbol{\sigma}_{N-n+2}]\!]\Big). \tag{D.38}$$

Since τ^0 part of the equation (D.32) reads $[\![\mathbf{L}^{\mathrm{BOS}}(s), \boldsymbol{\rho}^{(0)}(s)]\!] = -\partial_s \mathbf{L}^{\mathrm{BOS}}(s)$, one can find that $\boldsymbol{\rho}^{(0)}$ is given by

$$\boldsymbol{\rho}^{(0)}(s) = -\boldsymbol{\sigma}(s). \tag{D.39}$$

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