学位論文

Applied Conformal Bootstrap

(応用共形ブートストラップ)

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Abstract

In this thesis, I will discuss the application of the bootstrap program to conformal field theories (CFTs) in $d = 3$ Euclid space-time dimensions, with particular emphasis on the hypothetical models with $O(n) \times O(2)$-symmetry, the existence of which have been somewhat controversial.

These hypothetical CFTs correspond to the scale invariant fixed points, if any, of $O(n) \times O(2)$-symmetric Landau-Ginzburg-Wilson (LGW) models formulated in $d = 3$ Euclid space-time, and are candidates for the thermal effective theories of various physical systems at their criticality. These models are of great physical interest, including geometrically frustrated spin systems on triangular lattice and massless two-flavor quantum chromodynamics in the scenario where the axial $U(1)_A$ is restored above the chiral phase transition temperature. The phenomenological relevance of these hypothetical CFTs then follows: if they are absent, the corresponding systems must undergo discontinuous (first order) phase transitions, while the presence allows continuous transition with the prediction of universal critical exponents.

Despite their importance, however, these models are notoriously hard to analyze. The serious problem is that the answer is method-dependent. Notwithstanding their common origin, while ε-expansion based scheme for dealing with β-functions and functional renormalization group analysis predict the absence of these CFTs (first order transitions), resummed β-functions indicate the presence of such models.

We employ a technique called the *conformal bootstrap program* to study CFTs. The benefit of the method is the rigorous bounds (with easily controllable errors) on the parameters (including various critical exponents) characterizing CFTs, and these bounds must be met by all the unitary CFTs. What is quite intriguing is that these numerically derived bounds often seem to be saturated by the actual models, which have been located so far by other methods, with characteristic behaviors called "kinks". The models cornered in this way include nontrivial ones in $d = 3$ dimensions, like the Ising, XY, Heisenberg models and their $O(n)$ descendants.

Given the success of the bootstrap program for these simple LGW-models, we here carry out the bootstrap studies for $O(n) \times O(2)$-symmetric general CFTs following the state of the art bootstrap technologies to obtain any information about the controversial fixed points. We numerically compute the bounds for various operators contained in these models, and it will turn out that some of these bounds too are strong enough to be saturated by the hypothetical CFTs stated above, with some characteristic behaviors observed as in the previous examples of simple LGW models. We will also provide non-trivial checks for our scenario, i.e., the saturation of the bootstrap bounds by these hypothetical CFTs, existence of which we are led to believe in.

As a by-product, we will simply argue that the ramification of $U(1)_A$-restoration scenarios are required to conclude the order of massless two-flavor QCD chiral phase transition.

Contents

Chapter 1

Introduction

Because it's all logic and reason now. Science. Progress. Laws of hydraulics. Laws of social dynamics. Laws of this, that, and the other. No place for three-legged cyclops in the South Seas. No place for cucumber trees and oceans of wine. No place for me.

– The Adventures of Baron Munchausen

After the introduction of renormalization group (RG) by Wilson, it was realized that scale invariant quantum field theories are ubiquitous objects in theoretical physics, as they describe the theories at the endpoints of RG flows. Although it is easy to maintain scale invariance at classical level, the task becomes surprisingly nontrivial once an ultraviolet cutoff (or some other renormalization scale) is introduced to define quantum theory. In fact, without supersymmetry, scale invariance emerges only at some special points in the space of theories, and the candidates for non-trivial such points are rather limited. This observation led him to the explanation of the universality of critical phenomena observed earlier for various condensed-matter systems – there, certain quantities, that is, critical exponents somehow exhibit an agreement between apparently quite different systems.

A further surprising observation is that scale invariance, combined with other assumptions like unitarity and Lorentz invariance, leads to an enhanced amount of symmetry, i.e., conformal symmetry. While scale invariance states that the theory is invariant under constant scale shift of metric $g_{\mu\nu} \to \lambda g_{\mu\nu}$, conformal invariance requires the invariance under local scale transformation $g_{\mu\nu}(x) \to \lambda(x) g_{\mu\nu}(x)$. Although it looks a much more stringent requirement, this enhancement is in fact the case for $d = 2$ scale invariant theories under the assumption of unitary and rotational invariance [1]. Much effort and ingenious arguments have been made so far [2][3] but it still lacks the complete non-perturbative proof in the case of most interest, $d = 4$ (for odd space-time dimensions the situation is even worse). See [4] and reference therein for the aspects of this interesting conjecture and attempts for a proof.

Regardless of the general validity of the equivalence between scale and conformal invariance, we already have numerous important examples of quantum field theories which have enough evidence to be realized as CFTs: those dual to quantum (super)gravity theories in AdS backgrounds, various supersymmetric gauge theories, and effective field theories which describe continuous phase transitions. Investigation of CFTs has thus been one of the primary concerns for theoretical physicists.

Although the task of understanding CFTs looks easier compared to non-scale invariant ones due to the presence of extended symmetry, this bonus has not been fully appreciated in conventional schemes of quantum field theory like perturbation series. Even worse, in strongly coupled region (where we are sometimes obliged to employ numerical techniques) studying theories at the fixed points is in some sense harder, since numerical techniques are not capable of handling an infinite number of lattice sites, which however is mandatory to have scale invariance in the infrared (IR).

Already in early 1970s, an alternative approach called the *conformal bootstrap* has been proposed [5][6]. The hope is that, since CFTs are constrained more tightly than ordinary field theories, we might be able to start from the space of all potentially possible parameters characterizing CFTs and gradually solve the consistency conditions to classify all possible CFTs, independently of their (if any) Lagrangian description. Such a Lagrangian-independent approach should be welcomed both

phenomenologically and theoretically, as some important class of CFTs like $d = 6$ $(2, 0)$ theories are speculated to possess no description by Lagrangian. Even more, throughout this process every quantity is finite and the constraints can be stated with mathematical rigor. However, this approach too had been considered a quite challenging one. It turns out that both the number of unknown parameters and the dimensionality of the constraints are infinitely many. Except in the case of $d = 2$ (where the further enhancement of the symmetry to the Virasoro algebra takes place) with $c < 1$ (where the unitarity representation theory is highly constraining) [7], again the full task seems to be unwieldy.

This is not a mere evanescent dream, however. What I want to tell you in sections 2 through 4 is the surprising success in this direction after the ground-breaking paper [8]. Now we know that highly nontrivial information can be derived along the conformal bootstrap philosophy. The information has the form of bounds for the parameters characterizing CFTs, which must be respected by all unitary CFTs formulated in specific space-time dimensions d, irrespective of their Lagrangian description (if any). One might ask to what extent these bounds are strong. In fact, they turn out to be **so strong for certain models**. The most dramatic success is for the $d = 3$ Ising model [9]. Despite its simple nature, we believe this model is not integrable and we have to resort to approximation methods like lattice Monte-Carlo or high temperature expansion or higher-order resummed perturbative series. The conformal bootstrap is believed to capture this models in a completely different way – it is located on a corner in the space of all possible unitary CFTs, and numerical evidence has been given. Once you accept this conjecture, you can produce the best prediction for critical exponents in a computationally quite cheap way [10]. Even more dramatically, we can almost prove the universality of the $d = 3$ Ising model with the assumption on the number of relevant operators in the spectrum [11][12], if we adopt the scale $\to$ conformal enhancement.

Now that we have found the surprising power of the conformal bootstrap approach to these models (the properties of which have been well-known already by other methods), it is quite natural to ask whether we can say something about more non-trivial models where the conventional techniques of field theories can be poorly applied. In later Chapters 5 and 6, we discuss the application of the conformal bootstrap to quite controversial but "real-world" problems. These models include

- Anomalous $U(1)_A$-restored chiral symmetry breaking phase transition of $N_f = 2$ flavor massless QCD,
- Anti-ferromagnetic Heisenberg n-component spin-system placed on stacked triangular lattice.

Universal properties of these phase transition can be, if they are of second order, described by the fixed point of $O(n) \times O(2)$-symmetric Landau-Ginzburg-Wilson(LGW) models in $d = 3$ dimensions. The mechanism of these emergent symmetries are reviewed in Chapter 5. Although the $O(n) \times O(2)$-LGW models are direct generalization of the $O(n)$-LGW (vector) models, they have been known to possess dynamically richer structure of RG flows. First, when $\varepsilon = 4 - d$ is reasonably small, one-loop beta functions already indicate that there exists a critical value $n_c(\varepsilon)$ for n, above which non-Heisenberg type fixed points can exist. When ε is around physical value $\varepsilon \sim 1$, these fixed points do not seem to exist for phenomenologically preferable values $n = 2, 3, 4$ (although $n_c(1)$ is much smaller). However, an alternative scenario is suggested: for these relatively small values of n, new fixed points which are *absent* if $\varepsilon \ll 1$ show up. This means that the conventional ε-expansion to seek for the zeroes of beta functions fails to capture these fixed points. If this conjecture is the case, these properties bring us a striking consequence for the above-stated problems – such models might undergo phase transitions which are not of the conventional $O(n)$-Heisenberg type. Note that $\varepsilon = 1$ and small n (recall that the smaller n, the easier it is to realize in experiment) is the hardest regime to apply conventional field theory technique like ϵ-expansion and $1/n$-expansion, and the conjecture has been based on higher-order (with typical loop order 5, 6) perturbation series with elaborate use of resummation technique. However, in other approaches like functional (also called "exact" or "non-perturbative") RG method, the presence of conjectured fixed points has not been implied, so these transitions are predicted to be first-order. Experiments (for spin systems) and Monte-Carlo studies too have been inconclusive to decide whether the scenario proposed from higher-loop series is correct or not.

These controversies urge us to apply the conformal bootstrap program to these models, given the success of the program applied for the $d = 3$ Ising model and its $O(n)$ descendants, which too

are summarized as the LGW models. We discuss our numerical studies of $O(n) \times O(2)$ symmetric CFTs in Chapter 6. This will be carried out in two-steps. We first discuss how the unfamiliar fixed points are indicated in the bootstrap bounds for a non-controversial region, i.e., the one where the large n-predictions are believed to be reliable. Along with the conventional Heisenberg-type fixed point CFTs, non-Heisenberg type interacting fixed points will be discovered in a somewhat different way. The presence of the conformal window (existence of which is agreed in all other approach) is also indicated. Then, the next step is to apply the methods to controversial region. The results have two aspects - one is scientifically rigorous and the other is empirical. On the rigorous side, bounds are so strong for proposed fixed and exclude vast regions of these predictions. On the empirical side the surviving regions of the predictions are concentrated around the bounding curves, and around those regions, the bootstrap bounds develop corners just as the Ising and $O(n)$-model family did. Moreover, around those regions, other quantities (which are not constrained so-tightly) derived from the bootstrap outputs show a non-trivial agreement with the values predicted on resummed perturbative series. Such an agreement between Lagrangian-based diagrammatic approach and the one without Lagrangian is a miracle without the presence of these fixed points. As a by-product, we will show that the validity of the conjecture will lead to an interesting conclusion about the QCD chiral phase transition.

The organization of this thesis is as follows.

In chapter 2 and 3, we collect features of CFTs required to formulate them without any reference to Lagrangian descriptions. Since there are already nice references for these topics like [13][14][15][16], I tried to make them as brief as possible, yet the essential ingredients for the up-to-date bootstrap studies are included.

Chapter 4 introduces an exciting development after [8], and illustrate how nontrivial models like Ising and its $O(n)$-descendants are cornered.

Chapter 5 explains the relevance of phase transitions related to the $O(n)\times O(2)$-symmetric Landau-Ginzburg-Wilson models, with a very brief summary of earlier RG studies.

In Chapter 6, we will discuss our applications of the bootstrap program to the $O(n)\times O(2)$-models.

Finally in Chapter 7, we conclude the discussion with summary and some future prospects.

The discussion and the lines of argument of the chapter 6 are re-worked version of the following two papers with Yu Nakayama:

- "Approaching the conformal window of $O(n) \times O(m)$ symmetric Landau-Ginzburg models using the conformal bootstrap", Phys. Rev. D89, 126009[17],
- "Bootstrapping phase transitions in QCD and frustrated spin systems", Phys. Rev. D91,021901[18].

Some of the results presented here are new with different symmetry groups and improved value of parameters. Also, the original conclusion of the latter paper for the QCD chiral phase transitions are a bit refined.

Chapter 2

CFT Hilbert space and radial quantization

This Chapter is devoted to define the objects of interest in this thesis:*radially quantized conformal field theories*. To this end, we first review the structure of conformally invariant quantum theories focusing on their Hilbert spaces. The algebra of the conformal symmetry in the flat space, together with its representation theory, makes this subject interesting and rather constraining. In particular, the natural and convenient way to label states in a Hilbert space turns out to be somewhat different from usual massive QFTs: although it is a good idea to use d-momenta eigenvalues as the label when particle interpretation of the states is valid, in CFTs a state cannot be interpreted as a collection of particles (in particular, S-matrix cannot be defined). Fortunately, conformal invariance provides us with a way to label the Hilbert space by discrete eigenvalues while keeping everything covariant. This is achieved through an interpretation of conformal correlators by different quantization scheme, which is called *radial quantization*. Moreover, radial quantization urges us to make a requirement known as *state-operator correspondence*, which asserts that in CFTs the states are in one-to-one correspondence with the set of local operators in radial quantization.

Since this is a quite concise introduction to the topic, those who are interested in the subject are advised to consult more detailed lectures like [13] and [15] (but the lines of discussion might differ). Although our focus is on $d > 2$ space-time dimensions, some features are common in $d = 2$ (where the conformal symmetry is much larger!) as well, and it will also be helpful to refer to $d = 2$ literatures like [19][20][21].

2.1 Conformal invariance and its algebraic counterparts in flat space

Conformal transformations are coordinate transformations after which the metric retains its original form up to a scalar factor which can be position-dependent, hence preserving the mutual angle of two vectors defined on an identical point. Thus, if a diffeomorphism $x \to y(x)$ is a conformal transformation on a flat d-dimensional space with flat metric η, it must satisfy

$$\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \eta_{\mu\nu} = e^{\sigma(x)} \eta_{\mu'\nu'} \tag{2.1}$$

In a flat space where the metric (either Euclidean or Minkowskian) is constant for an appropriate coordinate system x, it is not hard to classify these diffeomorphisms: suppose an infinitesimal transformation $x \to x + \xi$ satisfies the infinitesimal version of (2.1),

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \sigma(x) \eta_{\mu\nu}. \tag{2.2}$$

Then the solution ξ for $d > 2$ is a linear combination of the following:

$$\text{Lorentz Rotation} \ : \ -\omega^{\mu}{}_{\nu} x^{\nu} \tag{2.3}$$

$$\text{Translation} \ : \ a^{\mu} \tag{2.4}$$

$$\text{Dilatation} \ : \ -\lambda x^{\mu} \tag{2.5}$$

$$\text{Special Conformal} \ : \ b^{\nu}(\delta^{\mu}{}_{\nu} x^2 - 2x^{\mu} x_{\nu}). \tag{2.6}$$

Here $\omega^{\mu}{}_{\nu}$, a^{μ}, λ and b^{ν} are real infinitesimal parameters. As usual $\omega^{\mu}{}_{\nu}$ must satisfy anti-symmetry condition $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$. In this way, we have in total $\frac{(d+2)(d+1)}{2}$ parameters as the transformation group.

This classification is easy: acting on (2.2) $\partial_\rho \partial_\sigma$, we have

$$\partial_\rho \partial_\sigma \partial_\mu \xi_\nu + \partial_\rho \partial_\sigma \partial_\nu \xi_\mu = \eta_{\mu\nu} \partial_\rho \partial_\sigma \sigma(x). \tag{2.7}$$

Contraction with $\eta^{\mu\rho}$ and symmetrization with respect to $\nu \leftrightarrow \sigma$ yields

$$\begin{aligned} &\frac{\partial^2}{2}(\partial_\nu \xi_\sigma + \partial_\sigma \xi_\nu) + \partial_\nu \partial_\sigma \partial \cdot \xi = \partial_\nu \partial_\sigma \sigma(x) \\ = \ &\frac{1}{2} \eta_{\nu\sigma} \partial^2 \sigma(x) + \partial_\nu \partial_\sigma \partial \cdot \xi. \end{aligned} \tag{2.8}$$

On the other hand, contraction of (2.7) with $\eta^{\mu\nu}$ leads to

$$2\partial_\rho \partial_\sigma \partial \cdot \xi = d \partial_\rho \partial_\sigma \sigma(x). \tag{2.9}$$

Substituting this expression for (2.8), we obtain

$$\left(\frac{d}{2} - 1\right) \partial_\mu \partial_\nu \sigma(x) + \frac{1}{2} \eta_{\mu\nu} \partial^2 \sigma(x) = 0. \tag{2.10}$$

When $d > 2$, this means

$$\partial_\mu \partial_\nu \sigma(x) = 0, \tag{2.11}$$

hence

$$\sigma(x) = \lambda + b^{\mu} x_{\mu}, \tag{2.12}$$

where b_μ and λ are integration constant. Next we want to calculate ξ_ν. For this purpose, note

$$\begin{aligned} \partial_\rho \partial_\mu \xi_\nu &= \partial_\mu \partial_\rho \xi_\nu \\ &= \partial_\mu(-\partial_\nu \xi_\rho + \eta_{\nu\rho} \sigma(x)) \\ &= -\partial_\nu \partial_\mu \xi_\rho + \eta_{\nu\rho} \partial_\mu \sigma(x) \\ &= -\partial_\nu(-\partial_\rho \xi_\mu + \eta_{\mu\rho} \sigma(x)) + \eta_{\nu\rho} \partial_\mu \sigma(x) \qquad (2.13) \\ &= \partial_\rho \partial_\nu \xi_\mu - \eta_{\mu\rho} \partial_\nu \sigma(x) + \eta_{\nu\rho} \partial_\mu \sigma(x), \qquad (2.14) \end{aligned}$$

and

$$\partial_\rho(\partial_\mu \xi_\nu - \partial_\nu \xi_\mu) = \partial_\rho(2\partial_\mu \xi_\nu - \eta_{\mu\nu} \sigma(x)) = \eta_{\nu\rho} \partial_\mu \sigma(x) - \eta_{\mu\rho} \partial_\nu \sigma(x). \tag{2.15}$$

In this way, we have

$$\begin{aligned} 2\partial_\rho \partial_\mu \xi_\nu &= \eta_{\nu\rho} \partial_\mu \sigma(x) - \eta_{\mu\rho} \partial_\nu \sigma(x) + \eta_{\mu\nu} \partial_\rho \sigma(x) \qquad (2.16) \\ &= \eta_{\nu\rho} b_\mu - \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho. \qquad (2.17) \end{aligned}$$

Integrating this twice, we reach

$$2\xi_\mu = x_\mu (b \cdot x) - \frac{x^2}{2} b_\mu + \Omega_{\mu\nu} x^{\nu} + a_\nu, \tag{2.18}$$

where $\Omega_{\mu\nu}$ and a_μ are integration constant. Plugging this back into (2.2), we have additional constraints on $\Omega_{\mu\nu}$:

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 2\eta_{\mu\nu}\lambda. \tag{2.19}$$

In this way, our final expression for ξ_μ is (after adding a prefactor),

$$\xi_\mu = a_\mu + b^\nu(x^2\eta_{\mu\nu} - 2x_\mu x_\nu) - \lambda x_\mu + \omega_{\mu\nu}x^\nu, \tag{2.20}$$

where $\omega_{\mu\nu}$ is anti-symmetric. A simple mnemonic for special conformal transformation rule $x^\mu \to x^\mu + (b^\mu x^2 - 2x^\mu(x\cdot b))$ is to regard it as the translation in the *conformal inversion* of x:

$$y^\mu := \frac{x^\mu}{x^2}. \tag{2.21}$$

Although this transformation does not lie in the identity-connected component of the conformal group (hence its realization as a quantum mechanical symmetry is not guaranteed), this itself keeps the metric up to a Weyl factor. Then, pulling back the translation $y \to y + b$ results in the special conformal transformation rule (2.6). Given this interpretation, exponentiated form of the special conformal transformation follows to be

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2x^2}. \tag{2.22}$$

Quantum mechanically, these symmetries manifest themselves as hermitian operators acting on the Hilbert space. We denote the operators corresponding to symmetry generators by

$$\begin{aligned} \text{Lorentz Rotation} &: M_{\mu\nu} \\ \text{Translation} &: P_\mu \\ \text{Dilatation} &: D \\ \text{Special Conformal} &: K_\mu. \end{aligned}$$

Their commutation relations can be read off from the infinitesimal transformation rules (2.3) - (2.6). In the convention of [22], non-zero commutators are

$$[M_{\mu\nu}\,,\,M_{\rho\sigma}] = \frac{1}{i}\left\{(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma}) - (\eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho})\right\}, \tag{2.23}$$

$$[M_{\mu\nu}\,,\,P_\rho] = \frac{1}{i}\left(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\right), \tag{2.24}$$

$$[M_{\mu\nu}\,,\,K_\rho] = \frac{1}{i}\left(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\right), \tag{2.25}$$

$$[D\,,\,P_\mu] = \frac{1}{i}P_\mu, \tag{2.26}$$

$$[D\,,\,K_\mu] = \frac{1}{i}K_\mu, \tag{2.27}$$

$$[P_\mu\,,\,K_\nu] = \frac{1}{i}\left(2\eta_{\mu\nu}D + 2M_{\mu\nu}\right). \tag{2.28}$$

Those not listed like $[P_\mu\,,\,P_\nu]$ vanish. Unlike the Poincaré symmetry $SO(d-1,1)\ltimes\mathbb{R}^d$, the algebra here is simple, which means that there is no proper subalgebra invariant under the adjoint action of the whole algebra. The algebra is $SO(d,2)$ if the space is flat Minkowskian, and $SO(d+1,1)$ if Euclidean. To see this, we define $SO(d,2)$-generator $S_{AB} = -S_{AB} \quad (A,B = -1,0,\cdots,d)$ by

$$S_{\mu\nu} = M_{\mu\nu}, \tag{2.29}$$

$$S_{-1\nu} = (P_\nu + K_\nu)/2, \tag{2.30}$$

$$S_{d\nu} = (P_\nu - K_\nu)/2, \tag{2.31}$$

$$S_{-1d} = D. \tag{2.32}$$

Defining $\mathbb{R}^{2,d}$ metric η_{AB} by $\eta_{-1,-1} = \eta_{0,0} = -1$ and $\eta_{a,b} = \delta_{ab}$, it is straightforward to check

$$[S_{AB}\,,\,S_{CD}] = \frac{1}{i}\left[\{-\eta_{AC}S_{BD} + (A\leftrightarrow B)\} - \{C\leftrightarrow D\}\right]. \tag{2.33}$$

2.2 Radial quantization interpretation of $SO(d,2)$ algebra

Having identified the conserved quantities, the next task is to find a convenient way to label states in Hilbert states. As stated, labelling them by momenta-eigenvalues is rather inconvenient due the failure of particle interpretation of states. Here we introduce a trick called *radial quantization* which regulates IR-issues without destroying any symmetry. A nice bonus is that we have a covariant way to have $\mathfrak{so}(d,2)$ symmetry on Hilbert space while working on the Euclidean field theory (where the conformal algebra is $\mathfrak{so}(d+1,1)$).

Before this, it is convenient to rotate $\mathfrak{so}(d,2)$ algebra into $\mathfrak{so}(d+1,1)$ algebra by adding extra phases,

$$\begin{aligned} S'_{ab} &= S_{ab}, \\ S'_{0b} &= iS_{0b}, \\ S'_{-1b} &= S_{-1b}, \\ S'_{-10} &= iS_{-10}, \end{aligned} \tag{2.34}$$

where a, b run from 1 to d. At the cost of obtaining $\mathfrak{so}(d+1,1)$, hermiticity of the generators is lost, i.e., S'_{0b} and S'_{-10} are now anti-hermitian. In other notation,

$$P'_a \equiv S'_{-1a} + S'_{0a}, \tag{2.35}$$

$$K'_a \equiv S'_{-1a} - S'_{0a}, \tag{2.36}$$

$$D' \equiv S'_{-10}, \tag{2.37}$$

with the commutator

$$\left[P'_a\,,\,K'_b\right] = -2i\left(2M'_{ab} + \delta_{ab}D'\right). \tag{2.38}$$

Non-hermiticity appears as

$$(P'_a)^\dagger = K'_a, \tag{2.39}$$

$$(D')^\dagger = -D'. \tag{2.40}$$

Now to the point: we start from a Euclidean CFT on $\mathbb{R}^d$, where the metric is, in terms of spherical coordinate,

$$\mathrm{d}s^2 = \sum_{i=1}^{d}(\mathrm{d}x^i)^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\Omega_{d-1})^2. \tag{2.41}$$

Here $\mathrm{d}\Omega_{d-1}$ represents the line element on S^{d-1}. Writing $r = Re^\tau$ for some arbitrary $R > 0$, we obtain

$$\mathrm{d}s^2 = R^2e^{2\tau}\left\{\mathrm{d}\tau^2 + (\mathrm{d}\Omega_{d-1})^2\right\}. \tag{2.42}$$

Therefore, $\mathbb{R}^d$ is conformally equivalent to $\mathbb{R}\times S^{d-1}$, with the Weyl rescaling factor $e^{2\tau}$. Given this equivalence, we automatically possess the conformal group $SO(d+1,1)$ as the isometry (up to Weyl factor) of $\mathbb{R}\times S^{d-1}$. We parametrize $\mathbb{R}$ by τ and S^{d-1} by $\mathbf{n}$ with $||\mathbf{n}||^2 = 1$ (and setting $R = 1$ for convenience), hence

$$x = e^\tau\mathbf{n}. \tag{2.43}$$

In this interpretation of the space-time, it is natural to consider $\mathbb{R}$-factor as the time-direction, and S^{d-1} as the spatial-direction. In the Euclid space, this corresponds to its foliation by "equal-time surfaces", S^{d-1}, each of which is centered at the origin. Then, under hermitian conjugation, the generators of $\mathfrak{so}(d+1,1)$ in this space-time behave exactly as the brute-force rotated generators defined above. First of all, time translation operator in this picture is simply given by the dilatation operator D', which is <u>anti</u>-hermitian because it represents Euclidean time-translation operator. For

$SO(d)$ rotation, since it maps the equal-time onto itself, it is guaranteed to be hermitian. However, due to our choice of equal-time surface which is not preserved under translation or special conformal transformation, P' and K' do not need to be hermitian: rather, these operators are flipped under conjugation. To see this, observe that D'-eigenvalue is flipped under the conjugation,

$$\begin{aligned}\left[D'\,,\,P'\right]^\dagger &= -\left[D'^\dagger\,,\,P'^\dagger\right] = \left[D'\,,\,P'^\dagger\right] \\ &= (-iP')^\dagger = +iP'^\dagger.\end{aligned}$$

This means that $P'^\dagger$ has the D'-eigenvalue of K', which leads to the relation[1] $P'^\dagger = K'$, assuming the absence of further symmetry.

In the following sections, we will study eigenvalues of D' (i.e., "energy" in radial quantization) in more detail. To avoid cluttering the notation, until the end of this thesis we will drop the primes of the conformal group generators and let the Greek indices like μ, ν, ρ, σ run through 1 to d.

2.3 Unitary irreducible representations of the conformal algebra

2.3.1 Highest weight representations of $SO(d,2)$ – general remarks

We have seen that the algebra of conformal transformations is (locally isomorphic to) the group $SO(d,2)$, and thus the Hilbert space of CFTs must develop certain representations of this group. If the theory under consideration is unitary, the space must be a direct sum of unitary irreducible representations. Thus seeking for unitary irreducible representations of $SO(d,2)$ is a building block in the investigation of CFTs, but one immediately confront the problem: the conformal group $SO(d,2)$ is non-compact! Thus we do not have finite-dimensional unitary representation except for the trivial case[2], and have to look for an infinite-dimensional ones instead.

Compared to the compact group representation theory, the tools in such cases are rather limited and indeed the task of classifying unitary irreducible representations (or "unitary dual of the group" in mathematical literature) is one of the unsolved problems in representation theory. To the author's knowledge, the full list is known only for special cases like $SO(4,2) \simeq SU(2,2)$ for $d = 4$ [23]. However, if we restrict ourselves to limited cases – which is sufficient for CFT physics – of highest weight modules, we have complete answers [24]. Indeed we should have the highest weights for our modules. Take the Cartan subalgebra to be that of maximal compact subgroup $SO(d) \times SO(2)$ where the $SO(2)$ factor is spanned by D. According to the radial quantization interpretation, this is the Hamiltonian along the radial direction and it is natural to require them to be bounded from below to have sensible dynamics. Thus, any unitary irreducible component must have lowest D-eigenstates, which we call *primary states*.

Primary states are annihilated by K^μ-action: otherwise $K^\mu|\text{primary}\rangle$ will have D-eigenvalue smaller than the primaries. One observes that the entire representation is spanned by the states of the form

$$P^{\mu_1}P^{\mu_2}\cdots P^{\mu_n}|\text{primary}\rangle. \tag{2.44}$$

The number of Ps acting on primary states is referred to as the level.

The primary states must furnish an $SO(d)$-unitary representation, as $SO(d)$ and D commute. Moreover, in order for the entire $\mathfrak{so}(d,2)$-module to be irreducible, the $SO(d)$-representation by primary states must be irreducible as well, as the orthogonality at the level 0 will be preserved at higher levels. In this way, primary states appear in the $SO(d)$-irreducible multiplet with highest weight $(h_1, h_2, \cdots, h_{[\frac{d}{2}]})$.

[1] We should not have the opposite sign $P'^\dagger = -K'$, because it will lead to unreasonable spectrum of D' (see next Section).

[2] If there is some unitary representation of finite-dimensionality n, we see that a certain quotient of the group can be embedded into $U(n)$, which is impossible. The only and readily seen exception to the argument is the case of abelian group like $\mathbb{R}$, a certain quotient of which ($U(1)$) is compact.

To summarize, every irreducible representation relevant for CFT Hilbert space comprises primary states, which have D-weight Δ and transforms under $SO(d)$ as unitary irreducible representation specified by its highest weight $\left(h_1, \cdots, h_{[\frac{d}{2}]}\right)$, and their *descendants*, which are of the form $P^{\mu_1} \cdots P^{\mu_n}|\text{primary}\rangle$.

2.3.2 Necessary Conditions for Unitarity

Let us examine what conditions are imposed in order for the representation to be unitary. As a first step, we normalize primary states (which we assume to have D-eigenvalue Δ) by

$$\langle \Delta; m|\Delta; n\rangle = \delta_m^{\ n}, \tag{2.45}$$

where m and n label the states in an $SO(d)$-irreducible representation R to which they belong. Then consider the states at the first level

$$P^\rho|\Delta; n\rangle. \tag{2.46}$$

For these states to have non-negative norm, the elements

$$\langle \Delta; m|K^\rho P^\sigma|\Delta; n\rangle, \tag{2.47}$$

seen as a matrix with index $_{\{m\rho:n\sigma\}}$, i.e., seen as an operator on the space $R \otimes \mathrm{v}$ where v refers to the vector representation of $SO(d)$, must be positive (semi-)definite. One can compute these matrix elements by

$$[K_\rho\,,\, P_\sigma] = -2iM_{\rho\sigma} + 2i\delta_{\rho\sigma}D, \tag{2.48}$$

and the primary property

$$K_\rho|\Delta; n\rangle = 0, \tag{2.49}$$

resulting in

$$\langle \Delta; m|K_\rho P_\sigma|\Delta; n\rangle = -2i(T^{(R)}_{\rho\sigma})_m^{\ n} + 2\Delta\delta_{\rho\sigma}\delta_m^{\ n}, \tag{2.50}$$

where $(T^{(R)}_{\rho\sigma})_m^{\ n}$ is the representation matrix of $M_{\rho\sigma}$, i.e., $M_{\rho\sigma}|\Delta; n\rangle = |\Delta; m\rangle(T^{(R)}_{\rho\sigma})_m^{\ n}$.

Thus the unitarity condition is

$$\Delta \geq -\text{the smallest eigenvalue of the operator } (-iT^{(R)}_{\rho\sigma})_m^{\ n} \text{ acting on } R \otimes \mathrm{v}. \tag{2.51}$$

The right-hand side can be written as

$$-i(T^{(R)}_{\rho\sigma})_m{}^n = \frac{1}{2}(T^{(R)}_{\mu\nu})_m{}^n(T^{(\mathrm{v})}_{\mu\nu})_{\rho\sigma}, \tag{2.52}$$

as the vector representation matrix is given by

$$(T^{(\mathrm{v})}_{\mu\nu})_{\rho\sigma} = -i(\delta_{\mu\rho}\delta_{\nu\sigma} - \{\mu \leftrightarrow \nu\}). \tag{2.53}$$

If we define

$$(T^{(R_1)} \cdot T^{(R_2)})_{m_1 m_2}{}^{n_1 n_2} \equiv \frac{1}{2}(T^{(R_1)}_{\mu\nu})_{m_1}{}^{n_1}(T^{(R_2)}_{\mu\nu})_{m_2}{}^{n_2} \tag{2.54}$$

or

$$(T^{(R_1)} \cdot T^{(R_2)}) = \frac{1}{2}\, T^{(R_1)}_{\mu\nu} \otimes T^{(R_2)}_{\mu\nu}, \tag{2.55}$$

then

$$-i(T^{(R)}_{\rho\sigma})_m^{\ n} = \left(T^{(R)} \cdot T^{(\mathrm{v})}\right)_{\{\rho m:\sigma n\}} \tag{2.56}$$

$$= \frac{1}{2}\left\{(T^{(R)} \otimes 1 + 1 \otimes T^{(\mathrm{v})})^2 - (T^{(R)} \otimes 1)^2 - (1 \otimes T^{(\mathrm{v})})^2\right\}_{\{\rho m:\sigma n\}}. \tag{2.57}$$

Here,

$$(T^{(R)})^2 \equiv T^{(R)} \cdot T^{(R)} \tag{2.58}$$

is the quadratic Casimir for R, and $(T^{(R)} \otimes 1 + 1 \otimes T^{(\mathrm{v})})^2$ takes constant value if we decompose $R \otimes \mathrm{v}$ into irreducible representations.

Thus, we have replaced the problem

" Find the smallest eigenvalue of $(T^{(R)}_{\mu\nu})_m^{\ n}$."

with

" Decompose $R \otimes \mathrm{v}$ into irreducible representation
and find one with the smallest quadratic Casimir."

This is an easy exercise of finite-dimensional Lie-algebra representation theory (for the answer, see [22] Appendix 2). Letting $\{h_i\}_{i=1}^{[d/2]}$ be the highest weight which characterize R, the result is

$$\frac{1}{2}(c_2(R) + c_2(\mathrm{v}) - c_2(R')) = |h_i| + d - i - 1, \tag{2.59}$$

where R' is the irreducible representation that appears in $R \otimes \mathrm{v}$ with the smallest Casimir and i is the least integer such that

$$h_i \geq h_{i+1} + 1. \tag{2.60}$$

If there are no such an index i, i.e., $h_1 = h_2 = \cdots h_{[d/2]-1} = |h_{[d/2]}|$, we have to consider them separately. When $|h_{[d/2]}| \geq 1$, substitute $i = [d/2]$ in the above formula. When $h_{[d]/2]} = 1/2$, R' is again a spinor (with chirality flip when d is even) and

$$\frac{1}{2}(c_2(R) + c_2(\mathrm{v}) - c_2(R')) = \frac{d-1}{2}. \tag{2.61}$$

When all h_i are 0, of course

$$\frac{1}{2}(c_2(R) + c_2(\mathrm{v}) - c_2(R')) = 0. \tag{2.62}$$

For example, vector representation has $h_1 = 1$ and others 0, so $i = 1$ and

$$\Delta \geq d - 1, \tag{2.63}$$

which is precisely the dimension of conserved currents (see next Section for the state-operator correspondence). The bounds for spinor states implies canonical dimension of free spinor theory. In fact this is a general feature: unitarity bounds are saturated if the corresponding operator obeys conservation condition or equation of motion. However, for scalar operator, the bound and the canonical dimension $(d-2)/2$ do not agree (except for the identity operator). The condition $\Delta \geq (d-2)/2$ can be derived from second level constraints, i.e., the requirement of the positive semi-definiteness of

$$\langle\phi|K_\rho K_\sigma P^\rho P^\sigma|\phi\rangle. \tag{2.64}$$

Although it is tedious to handle all the tensor indices, the scalar e.o.m suggests that $P^2|\phi\rangle$ becomes 0 for free theory and we require the non-negativity only of

$$\langle\phi|K^\rho K_\rho P^\sigma P_\sigma|\phi\rangle. \tag{2.65}$$

This element can be reduced again by moving K toward $|\phi\rangle$,

$$\begin{aligned}
\langle\phi|K^\rho K_\rho P^\sigma P_\sigma|\phi\rangle &= \langle\phi|K^\rho P^\sigma K_\rho P_\sigma|\phi\rangle + \langle\phi|K^\rho(-2iM_\rho{}^\sigma + 2i\delta_\rho{}^\sigma D)P^\nu|\phi\rangle && (2.66)\\
&= \langle\phi|K^\rho P^\sigma(-2iM_{\rho\sigma} + 2i\delta_{\rho\sigma}D)|\phi\rangle + \langle\phi|K^\rho(-2P_\rho\delta^\sigma{}_\sigma + 2P_\sigma\delta^\sigma{}_\rho + 2\delta_{\rho\sigma}P^\sigma)|\phi\rangle \\
&+ \langle\phi|K^\rho P^\sigma(-2iM_{\rho\sigma} + 2iD\eta_{\rho\sigma})|\phi\rangle && (2.67)\\
&= (4\Delta - 2d + 4)\underbrace{\langle\phi|K^\rho P_\rho|\phi\rangle}_{2d\Delta} && (2.68)\\
&= 8d\Delta\left(\Delta - \frac{d-2}{2}\right). && (2.69)
\end{aligned}$$

Thus, the region $(0,(d-2)/2)$ is excluded as desired. To summarize according to the highest weight $\{h_i\}_{i=1}^{[d/2]}$,

- If an index j with

$$h_j \geq |h_{j+1}| + 1 \tag{2.70}$$

  exists, the bound is

$$\Delta \geq h_i + d - i - 1, \tag{2.71}$$

  where i is the least one satisfying (2.70). As an important example, for the traceless-symmetric tensor representation of rank $l > 1$, since its highest weight is $(l, 0, \cdots, 0)$, we have $i = 1$ and

$$\Delta \geq l + d - 2. \tag{2.72}$$

  Note that the finite-dimensional lie algebra representation theory tells us that the non-existence of such an index implies $h_1 = h_2 = \cdots h_{[d/2]-1} = |h_{[d/2]}|$.

- For $h_1 = \cdots h_{[d/2]-1} = |h_{[d/2]}| \geq 1$, put $i = [d/2]$ in (2.71).

- For $h_1 = \cdots h_{[d/2]-1} = |h_{[d/2]}| = 1/2$,

$$\Delta \geq \frac{d-1}{2}. \tag{2.73}$$

- For $h_1 = \cdots h_{[d/2]} = 0$,

$$\Delta \geq \frac{d-2}{2}. \tag{2.74}$$

The argument made so far has been to derive necessary conditions for unitarity, and says nothing about these sufficiency. However, according to the construction of [25], for $d = 4$ the conditions derived here actually turn out to be also sufficient. For general dimensions, the sufficiency of these conditions is derived in [26] by adapting the result of [24]. The proof makes extensive use of the determinant formula derived by Jantzen [27]. See also [28][29] for the discussion of the results of [27].

2.4 State-operator correspondence and the properties of local operators

So far we have concentrated solely on the representation theoretical aspects of the CFT Hilbert space. Although Hilbert spaces are essential ingredient in any quantum theories, what concerns us is usually not the Hilbert space itself, but space-time dependent operators acting on it and expectation values thereof. In generic quantum field theories, there is no way to guess the local operator spectra from given Hilbert space. However, in CFTs, the story greatly simplifies: there is a one-to-one correspondence between the operator and Hilbert space spectra, known as the *state-operator correspondence*.

This idea is readily motivated by the structure of highest weight representations discussed above. We have seen that an irreducible component is spanned by primary states and its descendants by P^μ-action. Thus, an irreducible representation component of the Hilbert space looks exactly like the Taylor-expansion of some space-time dependent quantity, which presumably reminds us of local operators.

Indeed, one way of this correspondence is straightforward to declare. From a local operator $O(x)$, we can directly construct the associated state by taking the limit,

$$\lim_{x\to 0} O(x)|0\rangle := |O\rangle. \tag{2.75}$$

Beware that such states can be finite-norm thanks to the radial-quantization: in usual quantization scheme in Minkowski signature, operators can be hermitian $O(x)^\dagger = O(x)$, and $||O(0)|0\rangle||^2 = \langle 0|O(0)O(0)|0\rangle$, which generally diverges. This suggests that local operators are composed quite differently in radial quantization. We will have something to say in the next subsection about hermitian conjugate of local operators in radial quantization. In this way, we have a map from operators to states.

The map for the other direction, i.e., state→operator is much harder to construct and we have to resort to an additional requirement. For example, when the path-integral formulation of the theory is available, the correspondences can be proven using the wave-functional representation of general state (see e.g. [21] section 2 for an elegant exposition). If this is not the case, to the author's knowledge, a definite axiom which leads to the complete proof for this claim is not known for $d > 2$ (see e.g.[30] for a discussion in $d = 2$). Although it might be better to require the correspondence as an axiom to characterize CFTs, to grasp the essence, here we very roughly argue for it assuming a fork of the Reeh-Schlieder theorem to be valid in the present context[3]. The Reeh-Schlider theorem in axiomatic quantum field theory asserts that the actions of sufficient number of local operators on the vacuum can create any states in the theory: if this is not the case, there exists a decoupled sector in the Hilbert space. Then, given a primary state $|\Phi\rangle$, there exists a string of local operators satisfying

$$\phi_1(x_1)\cdots\phi(x_i)|0\rangle = |\Phi\rangle.$$

Now send the operator toward infinite past of the cylinder (the origin of the Euclid spaces) by the action of $\exp(iD\tau)$, while keeping the norm of RHS by a factor of $\exp(\Delta_\Phi\tau)$. We see

$$\phi_1(e^\tau x_1)\cdots\phi_i(e^\tau x_i)|0\rangle \propto |\Phi\rangle.$$

In the limit $\tau \to -\infty$, in $\mathbb{R}^d$, the string of operators is confined into a tiny ball centered at the origin and we expect it to behave as a local operator, so we have (though very loosely) constructed the desired operator. The crucial point in the argument is the ability of the conformal transformation to shrink arbitrary equal radial-time surface into a single point, the infinite past on the cylinder.

The local operators at arbitrary position is obtained from operators at the origin by P translation

$$O(x) = e^{-iP\cdot x}O(0)e^{iP\cdot x}. \tag{2.76}$$

Note that the translation operator $e^{-iP\cdot x}$ is not unitary due to unusual hermitian conjugation rule in radial quantization, $P^\dagger = K$.

Using (2.76), one can deduce various actions of conformal generators on primary operator O_m.

First, trivially,

$$\begin{aligned}[P_\rho\,,\,O_m(x)] &= e^{-iP\cdot x}P^\rho O(0)e^{iP\cdot x} - e^{-iP\cdot x}O(0)P^\rho e^{iP\cdot x}\\ &= i\partial_\rho O(x).\end{aligned}$$

[3]The theorem itself has been proven only in the context of causal Minkowskian QFT, and therefore, strictly speaking, does not apply here.

Commutators with other conformal generators such as K_ρ can also be read off via

$$\begin{aligned} e^{iP\cdot x}M_{\rho\sigma}e^{-iP\cdot x} &= M_{\rho\sigma} + [iP\cdot x\,,\, M_{\rho\sigma}] \\ &= M_{\rho\sigma} + (x_\rho P_\sigma - x_\sigma P_\rho), \end{aligned} \tag{2.77}$$

$$\begin{aligned} e^{iP\cdot x}De^{-iP\cdot x} &= D + [iP\cdot x\,,\, D] \\ &= D - x^\rho P_\rho, \end{aligned} \tag{2.78}$$

$$\begin{aligned} e^{iP\cdot x}K^\rho e^{-iP\cdot x} &= K^\rho + [iP\cdot x\,,\, K_\rho] + \frac{1}{2}\left[iP\cdot x\,,\, [iP\cdot x\,,\, K_\rho]\right], \\ &= K_\rho + 2(x^\sigma M_{\sigma\rho} + x_\rho D) + (x^2\delta_\rho{}^\sigma - 2x_\rho x^\sigma)P_\sigma. \end{aligned} \tag{2.79}$$

These result in

$$[P_\mu\,,\, O_m(x)] = i\partial_\mu O(x), \tag{2.80}$$

$$[M_{\rho\sigma}\,,\, O_m(x)] = i(x_a\partial_\sigma - x_\sigma\partial_\rho)O_m(x) - (M_{\rho\sigma})^n{}_m O_n(x), \tag{2.81}$$

$$[D\,,\, O_m(x)] = i(-x\cdot\partial)O_m(x) - i\Delta_O O_m(x), \tag{2.82}$$

where Δ_O is the (iD)-eigenvalue of the corresponding state, which we also call "the operator dimension of O" by an obvious reason. The action of K_ρ for general operators is much complicated. Special simplification occurs for *primary operators*, which are the corresponding operators for primary states (likewise, the operators corresponding to descendant states are called *descendant operators*). Since such operators $O_m(x)$ satisfies $[K_\rho\,,\, O_m(0)] = 0$, we have

$$[K_\rho\,,\, O_m(x)] = i(x^2\delta_\rho{}^\sigma - 2x_\rho x^\sigma)\partial_\sigma O_m(x) - 2x^\sigma(M_{\sigma\rho})^n{}_m - 2i\Delta_O x_\rho O_m(x). \tag{2.83}$$

The commutators (2.80)–(2.83) define differential operators $(\mathcal{L}_{\mathcal{AB}})_m{}^n$, which represents $\mathfrak{so}(d+1,1)$ and impose strong constraints on the correlation functions of local operators, as we will see in the next Chapter.

2.4.1 Note on hermitian conjugate in radial quantization

In Minkowski QFTs, taking hermitian conjugate of local operators is a simple task. For an operator ϕ to be real, we just have to require

$$\phi(x)^\dagger = \phi(x).$$

The story is not as easy for radial quantization, due to the unusual hermiticity property of $SO(d+1,1)$-generators, (2.39) and (2.40). Recalling the interpretation of D as the time-translation operator for radial-direction, we define scalar operator on the cylinder as $\phi(\tau,\mathbf{n}) = |x|^{-\Delta_\phi}\phi(x)$. Here, additional factor $|x|^{-\Delta_\phi}$ is to ensure $[D\,,\,\phi] = -i\partial_\tau\phi$. Then, $\phi(\tau,\mathbf{n})$ is given by time-translation

$$\phi(\tau,\mathbf{n}) = \exp(iD\tau)\phi(\tau,0)\exp(-iD\tau),$$

but due to $D^\dagger = -D$, its conjugate is translated by $-\tau$:

$$(\phi(\tau,\mathbf{n}))^\dagger = \exp(-iD\tau)\ (\phi(\tau,0))^\dagger\ \exp(iD\tau).$$

This suggests the correct form of the hermitian conjugation to be

$$\phi(x)^\dagger = \frac{1}{|x|^{2\Delta_\phi}}\phi(y), \tag{2.84}$$

where $y = x^\mu/x^2$. The conjugation rule thus defined is now consistent with both hermiticity (2.39)–(2.40) and operator transformation rules (2.80)-(2.83).

For operators with spin, additional issues arise. For vector operator $V_\mu(x)$, the answer is

$$V_\mu(x)^\dagger = \frac{1}{|x|^{2\Delta_\phi}}I_{\mu\nu}(x)V_\nu(y), \tag{2.85}$$

where a tensor $I_{\mu\nu}(x)$ is defined by

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}. \tag{2.86}$$

Again, one can check its compatibility with (2.80)–(2.83). There is a simple explanation for the appearance of $I_{\mu\nu}$: on a cylinder it is natural to decompose a vector into components tangent to the sphere, V_σ, and the radial component, V_τ. Since radial direction is a Wick-rotated one, for V_τ, an additional factor of $-i$ is imposed compared to the Minkowskian one (where we can impose trivial hermiticity). Therefore, $V_\tau^\dagger = -V_\tau$, and the tensor I works for reversing the τ-component while keeping V_σ fixed.

For operators with a representation which is a tensor-product of vectors, simply act $I_{\mu\nu}$ on each index. Then, symmetry property, irreducibility against taking trace, and self-duality with ϵ-tensor are preserved by

$$I^2 = \mathbf{1}$$
$$\det I = -1.$$

For operators in Dirac representations, the answer turns out to be

$$\psi(x)^\dagger = \frac{x_\mu \gamma^\mu}{|x|^{2\Delta_\psi+1}}\psi(y), \tag{2.87}$$

where γ-matrices are chosen so that $\{\gamma^\mu\,,\,\gamma^\nu\} = \delta^{\mu\nu}$ and $(\gamma^\mu)^\dagger = \gamma^\mu$. For its tensor product with vector indices, irreducibility with respect to the contraction by $(\gamma^\mu)_\alpha{}^\beta$ can be again checked easily.

Given these definitions for hermitian conjugate, we can now state the condition for reflection-positivity in radially quantized CFTs. In unitary CFTs, given local operators of distinct positions $\{\phi_i(x_i)\}$, a correlation function of the following form must be positive,

$$\langle 0|(\phi_n(x_n))^\dagger \cdots (\phi_1(x_1))^\dagger \phi_1(x_1)\cdots\phi_n(x_n)|0\rangle > 0, \tag{2.88}$$

provided that it is defined (see the beginning of the next Chapter for radial ordering).

2.4.2 Operator product expansion

Operator product expansion (OPE) is a statement that a product of two local operator can be, if their arguments are sufficiently close, expanded into the complete set of local operators present in the theory. Although in usual massive field theory the proof is only perturbatively available (even worse, they are known to be only asymptotic), this can be made very precise in CFTs thanks to the state-operator correspondence (see [31] for details). Consider a product of two operators (suppressing spins for notational convenience) acting on the vacuum,

$$\phi_1(x)\phi_2(0)|0\rangle = \phi_1(x)|\phi_2\rangle.$$

This state has a finite norm-squared if x is close enough to o, as its norm is given by a 4-point function

$$y^{2\Delta_{\phi_1}}\langle\phi_2|\phi_1(y)\phi_1(x)|\phi_2\rangle \quad (y^\mu = x^\mu/x^2),$$

which has non-clashing arguments if $|x| < 1$. Trivially, finite norm state can be expanded into orthonormal basis of the Hilbert space $\{O_i\}$, so with c-number coefficient functions $c_i(x)$, we can write

$$\phi_1(x)|\phi_2\rangle = \sum_i c_i(x)|O_i\rangle,$$

but according to the state-operator correspondence, each $|O_i\rangle$ is of the form $O_i(0)|0\rangle$. From invariance of the vacuum under translation, at least on the vacuum we have an expansion

$$\phi_1(x)\phi_2(y) = \sum_i c_i(x-y)O_i(y). \tag{2.89}$$

Note that the right-hand side should include both the primary and descendant operators.

Two comments follow:

- From conformal invariance, the form of $c_i(x)$ is severely restricted. In principle they are fully determined up to small number of parameters once the spins and scaling dimensions of $\phi_{1,2}$ and O_i are known. In particular, the prominent feature is that the knowledge of OPE coefficients between primary operators fully determines all the OPE. We can directly investigate the issue using the invariance of the vacuum under S_{AB} and differential operator representation of their actions on local operators (2.80)–(2.83), but things will be much more transparent using correlators.

- So far, the operator equality holds only when it acts on the vacuum. In causal Minkowski theory, the equality on the vacuum is enough to ensure operator equality itself (2.89). In this case of radially quantized CFTs, the operator equality holds as a consequence of crossing relation, to be discussed in Chapter 4.

Chapter 3

Scalar conformal correlators

As I mentioned, most import observables in quantum field theories are correlation functions of local operators like

$$\langle O_1(x_1)O_2(x_2)\cdots O_n(x_n)\rangle.$$

Following the radial quantization argument of Chapter.2, these correlator can be interpreted as those on $S^{d-1}\times\mathbb{R}$, with equal time surfaces taken to be spheres S^{d-1} centered at the origin and (the logarithms of) their radii correspond to time direction, whose translation is generated by the dilatation generator D. To make the correlators sensible, ordering of the operator matters: to see this, consider operator products written in terms of cylinder coordinate (2.43),

$$\begin{aligned} O_1(x_1)O_2(x_2) &= O_1(\tau_1,\mathbf{n}_1)O_2(\tau_2,\mathbf{n}_2) \\ &= \exp^{iD\tau_1} O_1(0,\mathbf{n}_1)\exp^{-iD(\tau_1-\tau_2)} O_2(0,\mathbf{n}_2)\exp^{-iD\tau_2}\,. \end{aligned}$$

To avoid divergence coming from the factor $\exp(-(\tau_1-\tau_2)iD)$ (recall that iD has a spectrum bounded from below), we have to have $\tau_1>\tau_2$ if the order of operator is $O_1(x_1)O_2(x_2)$. That is, operators inside the correlator must be sorted according to the radii of their arguments. This operation R is an analogue of time-ordering in Minkowski signature QFT and called *radial ordering*. The precise form of the correlator should thus be understood as

$$\langle 0|\mathrm{R}\left\{O_1(x_1)\cdots O_n(x_n)\right\}|0\rangle.$$

Like Lorentz boost in Minkowski QFT, there is a finite symmetry transformation which can exchange the operator ordering. Henceforth we impose

$$[O_1(x_1)\,,\,O_2(x_2)]=0, \tag{3.1}$$

if $|x_1|=|x_2|$ and $x_1\neq x_2$ to make this action continuous.

Given these definitions, conformal correlators enjoy differential equations following from the invariance of vacuum $S_{AB}|0\rangle=0$,

$$\begin{aligned} &\langle 0|\left[S_{AB}\,,\,\mathrm{R}\left\{O_1(x_1)\cdots O_n(x_n)\right\}\right]|0\rangle \\ =&\sum_{i=1}^{n}\langle 0|\mathrm{R}\left\{O_1(x_1)\cdots O_{i-1}(x_{i-1})\left[S_{AB}\,,\,O_i(x_i)\right]O_{i+1}(x_{i+1})\cdots O_n(x_n)\right\}|0\rangle \\ =&(\sum_{i=1}^{n}\mathcal{L}_{AB})\langle 0|\mathrm{R}O_1(x_1)\cdots O_n(x_n)|0\rangle=0, \end{aligned} \tag{3.2}$$

where the differential operators $\mathcal{L}_{AB}$ was defined in (2.80)–(2.83). From this differential equation, we can in principle write down an expression for all 2-point and 3-point correlators up to small number of parameters (like iD-eigenvalues of operators and OPE coefficients among three operators). This is one of the most prominent feature of CFTs. In other words, the dependence of these physical observables are achieved only through these parameters.

Although such an expression is not available for 4-point correlators, we have a neat and rapidly converging infinite sum representation for them known as *conformal partial wave* or *conformal block* decomposition. What this decomposition tells you is that 4-point functions too are severely restricted (but here not from the conformal symmetry alone) and determined solely once all the three-point functions are known.

Here we review various aspects of conformal correlators with $n = 2, 3, 4$ points, restricting our attentions to the cases where at least two of the inserted operators are scalar, which will be of repeated use in the following chapters.

3.1 Correlators from Embedding Space Formalism

As mentioned, conformal invariance strictly restricts the form of correlation functions. This is because any two points can be reached by some fixed, canonical points, say $x^\mu = 0$ and $y^\mu = (1, 0, \cdots, 0)$. The same procedure works for three-point functions: conformal invariance allows one to locate any three points on

$$(0, \cdots, 0), \quad (1, 0, \cdots, 0), \quad \infty. \tag{3.3}$$

To do this, first put one of three points on the origin by translation. Then using special conformal transformation (2.22), send one of the remaining two points to the infinity[1]. Recall that the special conformal transformation preserves the location of the origin. Finally, by rotation and dilatation, one can put the remaining one position to $(1, 0, \cdots, 0)$. Since every configuration of three-point functions can be transformed into this particular choice, if we fix the correlator at this configuration to some value, we can unambiguously determine the correlator at any other configurations by conformal transformations up to constants. In particular, it suffices to know these constants for primary three-point functions, where all three operators are primary, because those for descendants can be directly obtained by the differentiation.

In principle one can proceed in the above steps and derive an expression for all three-point functions. However, the operator transformation rule under conformal transformations are rather messy to trace. In particular, if the operators have non-trivial spin representation, the task of the classification becomes devastatingly complicated and the closed expression for operators in arbitrary spins and in arbitrary dimensions have not been known at present.

Here we use a formalism called *Embedding Space* method [32], which is a trick to realize the operator transformation rules in a quite simple fashion. For simplicity we restrict our attentions to correlators with at least two scalars. For recent discussions of the generalization, see [33][34].

3.1.1 Embedding space lift of scalars

Here we construct a $d+1$-dimensional space to which the conformal field is lifted, which is called "projective null-cone". The benefit is that the action of $SO(d+1,1)$ linearizes for the lifted operator, thus making conformally invariant quantities quite transparent. The construction is as follows;

1. Denote the standard coordinate in $\mathbb{R}^{d+1,1}$ by $X^{\check{A}}, Y^{\check{A}} \cdots$ where $\check{A} = -1, 0, 1, \cdots, d$. From this coordinate we compose light-cone coordinate X^A, Y^A, and a non-checked index A runs $-, +, 1, \cdots, d$. Define the null-cone in $\mathbb{R}^{d+1,1}$ by

$$\eta_{AB} X^A X^B = -X^- X^+ + X^i X^i = 0 \quad \text{with} \quad X^\pm > 0, \tag{3.4}$$

   where η_{AB} is the metric for $\mathbb{R}^{d+1,1}$ written in light-cone terminology. Note that $SO(d+1,1)$ acts transitively on this null-cone (with the origin omitted).

2. This null-cone is projected to our physical Euclidean space by equivalence relation

$$X \sim Y \iff \exists\, \lambda > 0, \quad X = \lambda Y. \tag{3.5}$$

[1] An operator at infinity should be understood as the conjugate of an operator at the origin, i.e., factor $y^{2\Delta_O}$ is accompanied (see (2.84)). Special conformal transformation generates such a prefactor.

3. We take a convenient representative of the above equivalence class to be[2]

$$P_x^A = (1, x^2, x^\mu). \tag{3.6}$$

Following [33], we call P_x as "Poincaré section".

4. Define the lift of scalar operators on $\mathbb{R}^d$ to the null-cone by

$$O(X) \equiv (X^+)^{-\Delta} O\left(\frac{X^\mu}{(X^+)}\right), \tag{3.7}$$

where Δ is the operator dimension. The crucial property of this definition is

$$O(\lambda X) = \lambda^{-\Delta} O(X). \tag{3.8}$$

It is easy to check that various $SO(d+1,1)$ actions on the embedding space actually reproduces those in physical space. Linear $SO(d+1,1)$ action induces

$$X^{\check{A}} \to X^{\check{A}} + \delta X^{\check{A}} = X^{\check{A}} + \epsilon^{\check{A}}_{\ \check{B}} X^{\check{B}} \tag{3.9}$$

where $\epsilon_{\check{A}\check{B}}$ is anti-symmetric infinitesimal parameter for $SO(d+1,1)$-Lorentz rotation. In component,

$$\delta X^{-1} = +\epsilon^{-10} X^0 + \epsilon^{-1\mu} X_\mu \tag{3.10}$$

$$\delta X^0 = +\epsilon^{-10} X^{-1} + \epsilon^{0\mu} X_\mu \tag{3.11}$$

$$\delta X^\mu = +\epsilon^{-1\mu} X^{-1} - \epsilon^{0\mu} X^0 + \epsilon^{\mu\mu} X_\nu \tag{3.12}$$

In light-cone frame,

$$(X + \delta X) = \begin{pmatrix} X^+ - \epsilon^{-10} X^+ + (\epsilon^{-1\mu} - \epsilon^{0\mu}) X^\mu \\ X^- + \epsilon^{-10} X^- + (+\epsilon^{-1\mu} + \epsilon^{0\mu}) X^\mu \\ X_\mu + \epsilon^{-1\mu} \frac{(X^+ + X^-)}{2} - \epsilon^{0\mu} \frac{(-X^+ + X^-)}{2} + \epsilon^{\mu\nu} X^\nu \end{pmatrix}. \tag{3.13}$$

Dividing by the + component so that this makes the representative (3.6), we conclude that the transformation corresponds to

$$\delta x^\mu = -\epsilon^{\nu\rho}(x_\nu \partial_\rho - x_\rho \partial_\nu) - \frac{1}{2}(\epsilon^{-1\nu} + \epsilon^{0\nu})\partial_\nu - \frac{1}{2}(\epsilon^{-1\nu} - \epsilon^{0\nu})(x^2 \partial_\nu - 2x_\nu x \cdot \partial) - (\epsilon^{-10})(-x \cdot \partial) \tag{3.14}$$

which reduces to the conformal transformation (2.3)–(2.6).

So we have identified the transformation rule for the lifted operator $O(X)$ to be

$$\left[\frac{\epsilon^{AB}}{2} S_{AB}\,,\, O(X)\right] = -\,i\frac{\epsilon^{AB}}{2}\left\{X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\right\} O(X). \tag{3.15}$$

This simple linearized transformation rule is the fruits of the embedding tensor formalism, enabling us to classify the conformal invariants quite easily. First consider the 2-point correlator of O,

$$\langle O(X) O(Y) \rangle := F(X, Y). \tag{3.16}$$

This has to be an $SO(d+1,1)$-invariant function of X and Y. Since $X^2 = Y^2 = 0$, the only available quantity is $(-2X \cdot Y)$. So $F(X,Y) = f(-2X \cdot Y)$. However, we have to require the homogeneity equation,

$$f(-\lambda 2X \cdot Y) = \langle O(\lambda X) O(Y) \rangle = \lambda^{-\Delta} \langle O(X) O(Y) \rangle = \lambda^{-\Delta} f(-2X \cdot Y). \tag{3.17}$$

[2] Points with $X^- = 0$ do not lie in this orbit. Actually, this forces $X^i = 0$ by null-cone condition, so the points correspond to (0,1,0). This represents the point at infinity.

This requires

$$f(-2X\cdot Y)=\frac{C}{(-2X\cdot Y)^{\Delta}}. \tag{3.18}$$

Finally, substituting $X=(1,x^2,x^\mu)$ and $Y=(1,y^2,y^\mu)$, we obtain

$$\langle O(x)O(y)\rangle=\frac{C}{((x-y)^2)^{\Delta}} \tag{3.19}$$

The value of C is just a matter of convention. Below, we always normalize $O(X)$ by

$$\langle O(x)O(y)\rangle=\frac{1}{((x-y)^2)^{\Delta}}. \tag{3.20}$$

This operator normalization is equivalent to require that $O(0)$ creates a unit-norm states according to (2.84). Three point functions can also be fixed in this way to be

$$\langle O_1(X)O_2(Y)O_3(Z)\rangle=\frac{\lambda}{(-2X\cdot Y)^{\alpha}(-2Y\cdot Z)^{\beta}(-2Z\cdot X)^{\gamma}}. \tag{3.21}$$

Homogeneity in X,Y,Z completely determines these exponents: it turns out that

$$\begin{aligned}\alpha&=\frac{\Delta_1+\Delta_2-\Delta_3}{2},\\ \beta&=\frac{\Delta_2+\Delta_3-\Delta_1}{2},\\ \gamma&=\frac{\Delta_3+\Delta_1-\Delta_2}{2}.\end{aligned}$$

Thus the expression for 3-point function is

$$\langle O_1(X)O_2(Y)O_3(Z)\rangle=\frac{\lambda_{123}}{(-2X\cdot Y)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(-2Y\cdot Z)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(-2Z\cdot X)^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}, \tag{3.22}$$

or in d-dimensional language,

$$\langle O_1(x)O_2(y)O_3(z)\rangle=\frac{\lambda_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|y-z|^{\Delta_2+\Delta_3-\Delta_1}|z-x|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{3.23}$$

The constant λ is called OPE coefficient: if scalar-scalar OPE $O_1(x)O_2(0)$ contains $c_{1,2,3}(x)O_3(0)$ as in (2.89), we can extract $c_{1,2,3}(x)$ by the projection with

$$\{O_3(0)|0\rangle\}^\dagger O_1(x)O_2(0)|0\rangle=\lim_{y\to\infty}|y|^{2\Delta_3}\langle 0|O_3(y)O_1(x)O_2(0)|0\rangle.$$

So for scalar-scalar to scalar OPE, we can relate

$$c_{1,2,3}(x)=\frac{\lambda_{123}}{x^{\Delta_1+\Delta_2-\Delta_3}}.$$

Proceeding in this way, it is also possible to determine how descendants of O_3 appear in the OPE by projecting with $\{P^\mu|O_3\rangle\}^\dagger=\langle O_3|K^\mu$.

Finally, let us consider the 4-point function of scalars with identical dimension $\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$. This time the form of the correlator cannot be determined by conformal invariance alone. Indeed, all we can do by conformal invariance is to set

$$\begin{aligned}x_1&=0,\\ x_2&=(x,y,0,\cdots,0),\\ x_3&=(1,0,\cdots,0),\\ x_4&=\infty.\end{aligned} \tag{3.24}$$

To do this first put x_1, x_3 and x_4 on their desired position (just as we did for three-point functions) and then employ residual $SO(d-1)$ symmetry to set $x_2^\mu = 0$ for $\mu > 2$. We call such configurations as standard configurations. In this way, we have two remaining degrees of freedom which cannot be fixed by the symmetry.

The embedding space formalism allows us to restrict the form of the four-point function to be

$$\langle O(X_1)O(X_2)O(X_3)O(X_4)\rangle = \frac{1}{X_{12}^{\Delta_O}X_{34}^{\Delta_O}}F(X_1,X_2,X_3,X_4), \tag{3.25}$$

where we have defined $X_{ij} := -2X_i \cdot X_j$ (which reduces to $(x_i - x_j)^2$ in the d-dimensional language) and F is a function invariant under both $SO(d+1,1)$ and rescaling. Unlike three-point function, any function made out of the following variables (called "conformal cross ratio")

$$u = \frac{X_{12}X_{34}}{X_{13}X_{24}} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \tag{3.26}$$

$$v = \frac{X_{14}X_{23}}{X_{13}X_{24}} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{3.27}$$

respects the condition (3.2). These variables are related to the x_2 coordinate in the standard configuration by

$$u = z\bar{z}, \tag{3.28}$$

$$v = (1-z)(1-\bar{z}), \tag{3.29}$$

$$\text{where } z = x + iy. \tag{3.30}$$

Using u and v (or z, $\bar{z}$,equivalently), we have

$$\langle O(X_1)O(X_2)O(X_3)O(X_4)\rangle = \frac{1}{X_{12}^{\Delta_O}X_{34}^{\Delta_O}}F(u,v). \tag{3.31}$$

As will be discussed in the next section, compatibility of the four-point function with three-point functions together with the state-operator correspondence puts further constraints on the form of $F(u,v)$.

3.1.2 Embedding Space Formalism for Symmetric Tensors

Before plunging into the study of four-point functions, let us briefly consider the generalization of the embedding formalism to operators with $SO(d)$ spin l, i.e., symmetric traceless tensor of rank l [3]. The highest weight of this representation is

$$h_1 = l\ ,\ h_2 = h_3 = \cdots = h_{[d/2]} = 0.$$

In this case, to lift the tensor indices in a covariant manner, one should also lift the index to $\mathbb{R}^{d+1,1}$. This can be achieved as follows (see [33])

1. Consider traceless symmetric tensor $F_{A_1\cdots A_l}$ on the null cone.
2. To reduce the degrees of freedom to that of $\mathbb{R}^d$, impose the homogeneity condition

$$F_{A_1\cdots A_l}(\lambda X) = \lambda^{-\Delta}F_{A_1\cdots A_l}(X) \tag{3.32}$$

3. We have to reduce the number of independent spins - to do this, impose transversality condition

$$X^{A_1}F_{A_1\cdots A_l}(X) = 0. \tag{3.33}$$

To further reduce the expression, we require gauge invariance,

$$F_{A_1\cdots A_l} \sim X_{A_1}G_{A_2\cdots A_l} + (\text{symmetrization}) \tag{3.34}$$

[3] For $d = 3$ this sufficient for bosonic representations. For an attempt with non-symmetric representation in other dimensions (especially in $d = 4$), see [35][36] and [34].

4. Components in physical space is given as a pull back by Poincaré section;

$$f_{\mu_1\cdots\mu_l} = \frac{\partial P_x^{A_1}}{\partial x^{\mu_1}}\cdots\frac{\partial P_x^{A_l}}{\partial x^{\mu_l}} F_{A_1\cdots A_l}(P_x). \tag{3.35}$$

One can directly confirm that $f_{\mu_1\cdots\mu_l}$ produced in this way does behave in a desired way.

Note also that these constraints $X^A F_{AB\cdots C} = 0$ and gauge invariance $F_{AB\cdots C} \sim X_A G_{B\cdots C}$ are analogous to Lorentz gauge condition $k^\mu A_\mu(k)$ and residual (on-shell) gauge invariance $A_\mu(k)+k_\mu c(k)$.

One can also treat tensor indices conveniently [33] by noting the one-to-one correspondence

$$\begin{gathered} d\text{-dimensional rank } l \text{ symmetric traceless tensor } f_{\mu_1\cdots\mu_l}(x) \\ \Longleftrightarrow \text{ homogeneous polynomial } f(x,z) \text{ of degree } l \text{ in } d \text{ variables } (z_1\cdots z_d) \textit{ modulo } z^2 := z^\mu z_\mu \ . \end{gathered}$$

The $\Rightarrow$ correspondence is given by

$$f(x,z) = f_{\mu_1\cdots\mu_l}(x) z^{\mu_1}\cdots z^{\mu_l} \quad (\mathrm{mod}(z^2)). \tag{3.36}$$

To give the $\Leftarrow$ direction, first choose a representative $\tilde{f}$ and put

$$\tilde{f}_{\mu_1\cdots\mu_l} = \frac{1}{l!}\frac{\partial}{\partial z^{\mu_1}}\cdots\frac{\partial}{\partial z^{\mu_l}}\tilde{f}(z) \tag{3.37}$$

and then subtract traces from $\tilde{f}$. That this procedure does not depend on the choice of representative is obvious. In a similar way one can construct a homogeneous polynomial $F(Z)$ of $d+2$ variables $Z_-, Z_+, Z_1, \cdots Z_d$ modulo $Z^2 = -Z_-Z_+ + Z^\mu Z_\mu$, which represents the lifted tensor. The crucial advantage is that $F(Z)$ is invariant under

$$F(X,Z) = F(Z+tX) \quad \forall t \in \mathbb{R}, \tag{3.38}$$

which represents constraints (3.33). Then, one can easily confirm that

$$f(x,z) = F(X|x, Z|_{z,x}) \tag{3.39}$$

relates $f(x,z)$ and $F(X,Z)$. $X|_x$ is the Poincaré section defined in (3.6) and $Z_{z,x}$ are defined by

$$(Z|_{z,x})_- = 0 \ , \ (Z|_{z,x})_+ = 2x\cdot z \ , \ (Z|_{z,x})_\mu = z_\mu. \tag{3.40}$$

These are chosen so that

$$X|_x \cdot Z|_{z,x} = 0, \tag{3.41}$$

which is consistent with gauge invariance $Z \to Z + tX$.

Examples

Let us now enjoy the advantage of the index-free notation above. First consider the two-point function

$$\langle O_{\mu_1\cdots\mu_l}(x_1) O_{\nu_1\cdots\nu_l}(x_2)\rangle. \tag{3.42}$$

Our strategy is to calculate

$$F(X_1,X_2,Z_1,Z_2) = \langle O(X_1,Z_1)O(X_2,Z_2)\rangle \tag{3.43}$$

instead. This function has the homogeneity

$$F(X_1,X_2,Z_1,Z_2) = \lambda^\Delta F(\lambda X_1,X_2,Z_1,Z_2) = \lambda^\Delta F(X_1,\lambda X_2,Z_1,Z_2) \tag{3.44}$$

$$= \lambda^{-l} F(X_1,X_2,\lambda Z_1,Z_2) = \lambda^{-l} F(X_1,X_2,Z_1,\lambda Z_2), \tag{3.45}$$

and partial translation invariance

$$F(X_1, X_2, Z_1, Z_2) = F(X_1, X_2, Z_1 + tX_1, Z_2) = F(X_1, X_2, Z_1, Z_2 + tX_2). \tag{3.46}$$

To maintain the property (3.46), F must be composed out of the combination

$$C_{i;AB} = X_{iA}Z_{iB} - X_{iB}Z_{iA}. \tag{3.47}$$

Although

$$Z_i \cdot X_i \tag{3.48}$$

also respects translation invariance, this vanishes after the substitution (3.39). By the same reason, we can omit $X_i^A C_{i;AB}$ and $C_{i;AB}C^{i;AC}$ from F. Thus,

$$F \propto \frac{(C_{1;AB}C_2^{AB})^l}{(-2X_1 \cdot X_2)^{\Delta+l}} \propto \frac{(X_1 \cdot X_2\ Z_1 \cdot Z_2 - X_2 \cdot Z_1\ X_1 \cdot Z_2)^l}{(-2X_1 \cdot X_2)^{\Delta+l}} \tag{3.49}$$

According to our dictionary (3.40), in $d-$dimensional expression this takes the form

$$\frac{\left\{x_{12}^2(z_1 \cdot z_2) - (z_1 \cdot x_{12})(z_2 \cdot x_{12})\right\}^l}{x_{12}^{2(\Delta+l)}}. \tag{3.50}$$

Recovering the index, we conclude

$$\langle O_{\mu_1\cdots\mu_l}(x_1)O_{\nu_1\cdots\nu_l}(x_2)\rangle = \frac{I^{\mu_1\nu_1}(x_{12})\cdots I^{\mu_l\nu_l}(x_{12})}{x_{12}^{2\Delta}} - (\text{trace}), \tag{3.51}$$

where the tensor $I^{\mu\nu}(x) = \delta^{\mu\nu} - 2x^\mu x^\nu/x^2$ is the one introduced in (2.86). This factor maintains the hermitian conjugation relation of tensor operators, e.g., for vectors,

$$\delta_{\mu\nu} = \langle V_\mu|V_\nu\rangle = \lim_{x\to 0}\langle 0|(V_\mu(x))^\dagger V_\nu(0)|0\rangle = \lim_{y\to\infty} I_{\mu\rho}(y)|y|^{2\Delta_V}\langle 0|V_\rho(y)V_\nu(0)|0\rangle.$$

For scalar-scalar-tensor correlation function

$$\langle O_1(x_1)O_2(x_2)O^{\mu_1\cdots\mu_l}(x_3)\rangle, \tag{3.52}$$

we prepare the function

$$F(X_1, X_2, X_3, Z) = \langle O_1(X_1)O_2(X_2)O(X_3, Z)\rangle \tag{3.53}$$

As in the two-point function example, we have to construct it using

$$C_{AB} = X_{3A}Z_B - X_{3B}Z_A. \tag{3.54}$$

Indices of C_{AB} must be contracted with X_1 and X_2, so along with homogeneity,

$$F(X_1, X_2, X_3, Z) \propto \frac{(X_1^A X_2^B C_{AB})^l}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+l}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+l}{2}} X_{13}^{\frac{\Delta_3+\Delta_1-\Delta_2+l}{2}}} \tag{3.55}$$

$$\propto \frac{\left\{Z^A(X_{1A}X_{23} - X_{2A}X_{13})\right\}^l}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+l}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+l}{2}} X_{13}^{\frac{\Delta_3+\Delta_1-\Delta_2+l}{2}}} \tag{3.56}$$

and in d-dimensional language,

$$\langle O_1(x_1)O_2(x_2)O^{\mu_1\cdots\mu_l}(x_3)\rangle = \lambda_{12O}\frac{v^{\mu_1}(x_{13}, x_{23})\cdots v^{\mu_l}(x_{13}, x_{23}) - (\text{trace})}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+l}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+l}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2+l}}, \tag{3.57}$$

where

$$v^\mu(x_{13}, x_{23}) = x_{13}^\mu x_{23}^2 - x_{23}^\mu x_{13}^2. \tag{3.58}$$

A great simplification for such scalar-scalar-tensor correlator is that there is only a single OPE coefficient. This is not the case when two of three operators have spin.

3.2 Conformal Block Decomposition of four-point Functions

3.2.1 Definition

So far we have seen how conformal symmetry works to fix two and three-point functions, but it does not do the same job for four-point correlators: due to the presence of conformal invariant cross ratios u, v defined as (3.26), any functions made up of u, v are perfectly compatible with the constraint (3.2).

However, not every function of u, v is allowed to occur for correlation function. Indeed, the compatibility with three-point functions and Hilbert space structure determines the natural basis for three-point function compatible functions. To see this, let us consider a correlator with four identical real scalars (in the sense of (2.84)) ϕ with dimension Δ_ϕ, and fix the operator ordering to be

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle 0|\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0\rangle$$

by assuming the radial order

$$|x_4| > |x_3| > |x_2| > |x_1|. \tag{3.59}$$

Recall that our Hilbert space is decomposed into irreducible representations spanned by primary and descendants. In particular, denote the projector into a irreducible subspace with a primary O by Π_O. Summing over all primaries yields $1 = \sum_O \Pi_O$.

Now the four-point function can be written as

$$\sum_O \langle 0|\phi(x_4)\phi(x_3)\Pi_O\phi(x_2)\phi(x_1)|0\rangle \tag{3.60}$$

where the summation is over all irreducible representations in the Hilbert space labeled by primaries O. Each component is called *conformal partial wave* from analogy with the decomposition of scattering wave function into rotation-group representations.

The virtue of the decomposition in terms of conformal partial waves is that they are determined solely from the information of three-point functions. The first thing to note is that projectors are obtained by summing over primaries and descendants in the multiplet,

$$\Pi_O = \sum_{i,j} |\Phi_i\rangle\langle\Phi_j|(\mathcal{N}^{-1})^i{}_j. \tag{3.61}$$

Here Φ_i runs over all states in the multiplet O, so they are of the form $O, P^\mu O, P^\mu P^\nu O, \cdots$, and the matrix $\mathcal{N}_i{}^j$ represents the inner product $\langle\Phi_j|\Phi_i\rangle$ (note that this is block-diagonal where each block comprises the states of equal D-eigenvalue and is finite-dimensional). So long as we project out null-states, $\mathcal{N}$ is invertible and responsible for $\Pi_O^2 = \Pi_O$. To compute the matrix element, first consider when $\Phi = O$ (i.e., primary states) and O is scalar. By the state-operator correspondence and operator conjugation rule (2.84), we have

$$\begin{aligned}
\langle O|\phi(x_2)\phi(x_1)|0\rangle &= \lim_{x\to 0}\{O(x)|0\rangle\}^\dagger\,\phi(x_2)\phi(x_1)|0\rangle \\
&= \lim_{y\to\infty}|y|^{2\Delta_O}\langle 0|O(y)\phi(x_2)\phi(x_1)|0\rangle \\
&= \lambda_{\phi\phi O}\frac{1}{|x_{12}|^{2\Delta_\phi-\Delta_O}},
\end{aligned}$$

where we have used the three-point function (3.23). Similar relation holds for the paired matrix elements $\langle 0|\phi(x_4)\phi(x_3)|O\rangle$. For the descendants of O like $P^{\mu_1}\cdots P^{\mu_n}$, matrix elements are of the form

$$\langle O|K^{\mu_1}\cdots K^{\mu_n}\phi(x_1)\phi(x_2)|0\rangle$$

and can be also computed from primary three-point function $\langle O\phi\phi\rangle$ by the differential operator action (2.83). Note that the descendants contribute as terms with higher powers of z when $z \to 0$ limit is taken: a scaling analysis tells that the matrix elements behave as $1/|z|^{2\Delta_\phi-(\Delta_O+n)}$.

While the defining sum for partial waves involves complicated factors of internal operator dimension Δ_O and Δ_ϕ (but the latter dependence turns out to be somewhat simpler than naïve expectation), the dependence on the primary OPE coefficient $\lambda_{\phi\phi O}$ is common among the summation. So the conformal partial wave has actually the form

$$\langle 0|\phi(x_4)\phi(x_3)\Pi_O\phi(x_2)\phi(x_1)|0\rangle = \lambda^2_{\phi\phi O}\frac{g(\Delta_O,\Delta_\phi;u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}.$$

where the prefactor $x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}$ is added to make g an invariant function.

Entirely analogous argument goes through for intermediate multiplet with non-scalar primary. An important remark is that only multiplet of spin l-representation (i.e., totally symmetric traceless representation of rank l) can contribute to scalar four-point functions. To see this it suffices to consider scalar-scalar OPE to some non-scalar operator O

$$\phi(x)\phi(0) = \cdots + \mathcal{I}^{\mu_1\cdots\mu_n}(x)O_{\mu_1\cdots\mu_n}(0) + \cdots .$$

Every bosonic irreducible representation is made up by proper symmetrization condition for indices $\mu_1\cdots\mu_n$ and trace subtraction. On the other hand, the only ingredients to compensate for indices in $\mathcal{I}^{\mu_1\cdots\mu_n}(x)$ are x^μ and $\delta^{\mu\nu}$, but the latter is projected out because of $SO(d)$-irreducibility of O. Thus,

$$\phi(x)\phi(0) = x^{\mu_1}\cdots x^{\mu_n}O_{\mu_1\cdots\mu_n}(0) \tag{3.62}$$

which can be non-vanishing only when O is in a spin-n representation. In particular, when the primary operators in the multiplet are not of spin-l type, the entire contribution must vanish as the contribution from descendants to conformal partial wave is obtained by the differential operator action on the primary contribution. Furthermore, since we are only considering identical scalars, going to more symmetric configuration $\phi(x/2)\phi(-x/2)$, we see that n must be even[4]. The partial wave decomposition is now

$$\langle 0|\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0\rangle = \sum_{O\text{: even spin primaries}} \lambda^2_{\phi\phi O}\frac{g(\Delta_O,l_O,\Delta_\phi;u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \tag{3.63}$$

where λ-independent function g is called the *conformal block*. Thus we have a way to decompose four-point functions. Again the theory dependence of the function enters only through parameters, i.e., OPE coefficients and operator dimensions.

The defining summation for the conformal block is quite complicated. An exception occurs when $d=2$ with $SL(2,\mathbb{C})$ (not Virasoro) conformal algebra, thanks to the factorization of the entire algebra into holomorphic and anti-holomorphic subalgebra. The result factorizes as well,

$$g(\Delta_O,\Delta_l,\Delta_\phi;u,v) = \{k_{\Delta_O+l}(z)k_{\Delta_O-l}(\bar{z}) + (z\leftrightarrow\bar{z})\} \tag{3.64}$$

$$k_\alpha(z) = z^{\alpha/2}\ {}_2F_1\left(\frac{\alpha}{2},\frac{\alpha}{2},\alpha\ ;z\right) \tag{3.65}$$

Surprisingly enough, in $d=4$, Dolan and Osborn found that the exact expression is available and analogous to $d=2$ one,

$$g_O(\Delta_O,\Delta_l,\Delta_\phi;u,v) = \frac{z\bar{z}}{z-\bar{z}}\{k_{\Delta_O+l}(z)k_{\Delta_O-l-2}(\bar{z}) - (z\leftrightarrow\bar{z})\} \tag{3.66}$$

by summing up the series directly in [37] for $l=0$. Later in [38] they derived the expression for general l by solving the Casimir differential equation, which will be briefly explained below. The $k\alpha(z)$ in the $d=4$ expression is exactly the same as in the one for $d=2$ expression (3.65). This explicit expression is the foundation of many earlier conformal bootstrap investigations [8][39][40][41][42].

One thing to note is that both (3.64) and (3.66) are independent of Δ_ϕ. This is the case for every space-time dimensions d when all the scalar operators under consideration have an identical dimension.

[4]Later we will deal with correlation function of $O(n)$-multiplet of scalars. Then odd spin operators contribute as well.

3.2.2 Convergence

The summation in the decomposition (3.63) is in fact taken over an infinite number of primary operators (this too is a consequence of conformal bootstrap). So one might be afraid of the issue of convergence. It turns out that the expansion is absolutely convergent, by a simple argument [31]. The only requirement to prove the assertion is the finiteness of correlation function at non-colliding points. To see this first move to a configuration with $x_1 = 0$ and $x_2 = \infty$,

$$\langle 0| \{\phi(0)\}^\dagger \phi(x_3)\phi(x_2)\phi(0)|\rangle = \langle\phi|\phi(x_3)\phi(x_2)|\phi\rangle.$$

By conformal transformation we can always assume $|x_2| < 1 < |x_3| = |x_2|^{-1}$. In particular we can take $x_2 = \frac{1}{\sqrt{|z|}}(x, y, \cdots, 0)$ and $|x_3| = \frac{1}{\sqrt{|z|}}(1, 0, \cdots)$. Again x, y and $z = x + iy$ is the standard configuration coordinate for x_2 (3.24). The point is that the norm of the state $\phi(x_2)|\phi\rangle$ takes the form of correlation function and hence is finite:

$$||\phi(x_2)|\phi\rangle||^2 = |y_2|^{2\Delta_\phi}\langle\phi|\phi(y_2)\phi(x_2)|\phi\rangle \tag{3.67}$$

where $y_2^\mu = x_2^\mu/x_2^2$ and we used (2.84). Similarly $||\langle\phi|\phi(x_3)||^2$ is also finite. Then consider the decomposition of of the four-point function

$$\sum_{\substack{i:\text{ all } SO(d)\\ \text{multiplets}}} \langle\phi|\phi(x_3)\pi_i\phi(x_2)|\phi\rangle, \tag{3.68}$$

where

$$\pi_i = \sum_\alpha |i;\alpha\rangle\langle i;\alpha|.$$

Here α labels the orthonormal basis for $SO(d)$-representation labelled by i, so π_i works as the projector. For a moment we are ignoring primary/descendants structure in the complete set summation and regarding the Hilbert space as a direct sum of $SO(d)$-representations, each of which is finite-dimensional. Each π_i-projected contribution is a finite sum (in the representation i) of real analytic function[5]with respect to x_2, and it reaches the maxima at $x_3 = y_2 := x_2/x_2^2$ within the constraint $|x_2| = |x_3|^{-1} = |\sqrt{z}|$,

$$\begin{aligned}\langle\phi|\phi(x_3)\pi_i\phi(x_2)|\phi\rangle &\le \langle\phi|\phi(y_2)\pi_i\phi(x_2)|\phi\rangle\\ &= |x_2|^{2\Delta_\phi}||\pi_i\phi(x_2)|\phi\rangle||^2,\end{aligned}$$

which sums up to the norm-defining correlation function (3.67). Thus we have obtained the positive upper bound series and hence the summation is absolutely convergent with respect to i. The conformal blocks are partial (infinite) summation of these, so it also converges as a summation over conformal blocks.

The Norm formula (3.67) becomes divergent as $|x_2| \to 1$ as the operators tend not to be radially ordered, and the above argument becomes invalid if $z > 1$. As the exact expression in $d = 2$ suggests, however, the region of convergence is larger (recall that the hypergeometric function has branch cut with branch point starting from $z = 1$). In fact, except for real z with $z > 1$, we can always take a sphere surrounding x_1 and x_2 with a large enough radius (but now it's not necessarily centered at the origin), and the above argument goes through. What is invalidated is the expansion of the conformal

[5]More explicitly, from the argument similar to the one around (3.62) i must be a traceless symmetric tensor representation, and we can write the explicit form of the matrix element to be

$$\langle i;\mu_1\cdots\mu_n|\phi(x)|\phi\rangle = c_{\phi\phi i}\frac{1}{|x|^{2\Delta_\phi-\Delta_i}}\frac{(x^{\mu_1}\cdots x^{\mu_n} - \text{trace})}{|x|^n}.$$

where $c_{\phi\phi i}$ can be (in principle) determined with primary-correlator differentiation. In particular, $\langle\phi|\phi(x_3)\pi_i\phi(x_2)|\phi\rangle$ can be written in terms of Gegenbauer polynomials [43].

blocks in terms of z with $|z| > 1$, but the function still exists and the decomposition of the four-point function is still absolutely convergent.

The benefit of this fact can be best explained by a new choice of conformal kinematics representative

$$\begin{aligned} x_1 &= (-r\cos\theta, -r\sin\theta, 0, \cdots, 0), \\ x_2 &= (r\cos\theta, r\sin\theta, 0, \cdots, 0), \\ x_3 &= (1, 0, \cdots 0), \\ x_4 &= (-1, 0, \cdots 0). \end{aligned}$$

The cross ratio can be matched by setting

$$\rho := re^{i\theta} = \frac{z}{(1+\sqrt{1-z})^2}. \tag{3.69}$$

Entire region of z-plane except for $z > 1$ real axis is mapped to the unit circle. Repeating the above argument, we immediately see that the expansion is absolutely converging again, this time over in entire unit circle, hence in the entire z-plane except for x-axis with $x \geq 1$.

In [31], the authors further argued to establish the exponentially fast convergence of the series. This means that if we make a sufficiently large cutoff Δ_* in the decomposition (3.63), i.e., approximating it by

$$\sum_{O:\Delta_O \leq \Delta_*} \lambda^2_{\phi\phi O} \frac{g(\Delta_O, l_O; z, \bar{z})}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}},$$

the error term is exponentially suppressed:

$$\sum_{O:\Delta > \Delta_*} \lambda^2_{\phi\phi O}\, g_{\Delta_O, l_O} \sim (\Delta_\phi\text{-depending factor}) \times \rho(z)^{\Delta_*}. \tag{3.70}$$

The key in their argument is the asymptotics of the correlation function in taking $\rho \to 1$ ($z \to 1$) limit, which we expect to be

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \sim \frac{1}{(1-z)^{2\Delta_\phi}} \sim \frac{1}{(1-\rho)^{4\Delta_\phi}}, \tag{3.71}$$

because $\phi(x_2)$ is colliding with $\phi(x_3)$ and most singular contribution to it comes from $\phi \times \phi \to 1$ OPE. Then the classical theorem by Hardy and Littlewood can be neatly applied to give the estimate (3.71). See also [44] for the estimate of the convergence rate.

3.2.3 Conformal Block as the Solution to Casimir equation

There is a neat way to characterize conformal partial waves. Note that Π_O is conformally invariant:

$$[S_{AB}\,,\, \Pi_O] = 0. \tag{3.72}$$

A conformal partial wave can be characterized as an eigenfunction for the conformal Casimir equation as follows. First we note operator equality

$$[S_{AB}\,,\, \phi_1(x_1)\phi_2(x_2)] = (\mathcal{L}_{1AB} + \mathcal{L}_{2AB})\phi_1(x_1)\phi_2(x_2)|0\rangle$$

Acting S_{AB} twice,

$$\begin{aligned} [S_{CD}\,,\, [S_{AB}\,,\, \phi_1(x_1)\phi_2(x_2)]] &= [S_{CD}\,,\, (\mathcal{L}_{1AB} + \mathcal{L}_{2AB})\phi_1(x_1)\phi_2(x_2)] && (3.73) \\ &= (\mathcal{L}_{1AB} + \mathcal{L}_{2AB})\,[S_{CD}\,,\, \phi_1(x_1)\phi_2(x_2)] && (3.74) \\ &= (\mathcal{L}_{1AB} + \mathcal{L}_{2AB})(\mathcal{L}_{1CD} + \mathcal{L}_{2CD})\phi_1(x_1)\phi_2(x_2) && (3.75) \end{aligned}$$

Hence inside the vacuum and Π_O, by the invariance of vacuum under conformal transformation,

$$\Pi_O\left[S_{CD}\,,\,[S_{AB}\,,\,\phi_1(x_1)\phi_2(x_2)]\right]|0\rangle = (\mathcal{L}_{1AB}+\mathcal{L}_{2AB})(\mathcal{L}_{1CD}+\mathcal{L}_{2CD})\Pi_O\phi_1(x_1)\phi_2(x_2)|0\rangle \tag{3.76}$$

$$= \Pi_O S_{CD}\left[S_{AB}\,,\,\phi_1(x_1)\phi_2(x_2)\right]|0\rangle \tag{3.77}$$

$$= \Pi_O S_{CD} S_{AB}\phi_1(x_1)\phi_2(x_2)|0\rangle \tag{3.78}$$

Contracting the indices,

$$(\mathcal{L}_{1AB}+\mathcal{L}_{2AB})(\mathcal{L}_1^{AB}+\mathcal{L}_2^{AB})\Pi_O\phi_1(x_1)\phi_2(x_2)|0\rangle = \Pi_O S^{AB}S_{AB}\phi_1(x_1)\phi_2(x_2)|0\rangle, \tag{3.79}$$

but $S^{AB}S_{AB}$ is the quadratic Casimir element of the conformal algebra and therefore constant when projected to the irreducible subspace by Π_O. We have thus established the second order differential equation that characterizes conformal partial waves. The approach of [38] was to solve this equation directly, with boundary condition

$$g(\Delta, l; z, \bar{z}) \to |z|^{\Delta} \qquad (z \to 0 \text{ along real axis}).$$

A similar equation holds for quartic Casimir elements like ${S_A}^B{S_B}^C{S_C}^D{S_D}^A$ and $({S_A}^B{S_B}^A)^2$, which lead to other partial differential equation and impose further constraints constraints on the blocks. In [45], based on these quadratic and quartic differential equation, a very efficient way to evaluate conformal block and its derivatives along the real axis (that is, $z = \bar{z} = x$) was proposed:

1. Using quadratic equation, express $\partial/\partial y$-derivative of the conformal block at the real axis in terms of its $\partial/\partial x$-derivative. Plugging the expression in to quartic-Casimir equation, we obtain an *ordinary* differential equation for $g_{\Delta,l}(z = \bar{z} = x)$.

2. This equation can be solved by means of series expansion. Moreover, the convergence can be made drastically accelerated by changing the variable to (3.69),

$$\rho = \frac{x}{(1+\sqrt{1-x})^2}$$

   and rewriting the differential equation by ρ. The acceleration of the convergence is expected because it now converges in entire physical $\rho < 1$.

3. Re-expressing the conformal block x-derivative is just a simple linear algebra if we know the derivative table of ρ. Now we (approximately but almost exactly) know the $g(\Delta, l; z = \bar{z} = x)$ and its x-derivatives.

4. Using the quadratic Casimir equation again, we can also derive y-derivatives along the real axis.

In later chapters, we compute the conformal blocks with this procedure.

3.2.4 Recursion relation for conformal block

Another way to approach the conformal blocks is invented long ago by A.B.Zamolodchikov [46] in the context of $d = 2$ (Virasoro) conformal block and utilized for $d \geq 3$ first in [47]. This approach characterizes conformal blocks in more representation-theoretical way.

Let us consider the conformal block as a function of internal operator dimension Δ. Its defining equation includes the projector into irreducible component Π_O, defined in (3.61)

$$\Pi_O = \sum_{i,j} |\Phi_i\rangle\langle\Phi_j|(\mathcal{N}^{-1})^i{}_j$$

If the unitary condition (2.71)–(2.74) are met for Δ, the norm-matrix $\mathcal{N}$ will be regular thanks to the positive-definiteness of the descendant states.[6] However, if we let O to have non-unitary dimension,

[6]Strictly speaking, null states also appear in physical region at the edge of unitarity bounds. However, such an unitarity-bound saturating operator enjoys the field equation, which means that such pole-creating descendant like $\partial_\mu j^\mu$ will becomes zero in the correlator. Hence such contribution was absent in the result in [47]. However, conservation condition or equation of motion can be consistent only when two operator dimensions are identical (differentiate (3.57)), and if it is not the case, conserved current term too contribute as a pole [11].

this does not necessarily hold because null-states might be present in the multiplet, i.e., the descendant states with vanishing norm could appear at $\Delta = \Delta_i$ for some Δ_i. Such null states then contribute to the conformal block as a pole in $\Delta \to \Delta_i$ as the factor $\mathcal{N}$ develops a vanishing eigenvalue. Then, the representation multiplet at $\Delta = \Delta_i$ is no longer irreducible – there exists a proper submodule defined as the set of all the null states. In particular there exists the highest weight state in this submodule, denoted by $|O^{(i)}\rangle$ and satisfying the primary condition $K_\mu|O^{(i)}\rangle = 0$. In this way, when Δ approaches the value Δ_i, all the states in this submodule descending from $|O^{(i)}\rangle$ has the norm decaying polynomially (recall how the norm of descendant states are computed),

$$\langle O^{(i)}|K^{\mu_1}\cdots K^{\mu_n}P^{\nu_1}\cdots P^{\nu_n}|O^{(i)}\rangle \sim (\Delta - \Delta_i)^{k_i}$$

with some integer exponent k_i. Then the residue at $\Delta_O = \Delta_*$ is another conformal block with primary state $O^{(i)}$. The analytic structure around $\Delta \sim \Delta_*$ is

$$g(\Delta, l) \sim c_i \frac{g(\Delta_{O^{(i)}}, l_{O^{(i)}})}{(\Delta - \Delta_*)^{k_i}}$$

with some representation theoretical constant c_i.

On the other hand, the asymptotic behavior of the conformal block with $|\Delta| \to \infty$ limit can be derived from the quadratic Casimir equation (3.79) to be [47]

$$g(\Delta \to \infty, l) \sim (4|\rho|)^{\Delta} h_\infty(l)$$

with h_∞ independent of Δ,

$$h_\infty(\rho) = \frac{G_l^{(d-2)/2}(\cos\theta)}{(1-r^2)^{(d-2)/2}\sqrt{(1+r^2)^2 - 4r^2\cos^2\theta}} \frac{l!}{(d-2)_l}.$$

Splitting off the factor of $(4|\rho|)^{-\Delta_\phi}$ to define

$$h(\Delta, l; z, \bar{z}) := (4|\rho|)^{-\Delta_\phi} g(\Delta, l; z, \bar{z}),$$

$h(\Delta)$ behaves regularly as $\Delta \to \infty$. Elimination of singularity in $\Delta \to \infty$ and poles leaves entirely holomorphic function of Δ, which obligatorily is a constant (w.r.t. Δ) h_∞, so

$$h(\Delta, l) = h_{\infty,l} + \sum_i c_i \frac{(4\rho)^{n_i} h(\Delta_{O^{(i)}}, l_{O^{(i)}})}{(\Delta - \Delta_i)^{k_i}}. \tag{3.80}$$

Here $n_i = \Delta_{O^{(i)}} - \Delta_i$, i.e., the level at which $O^{(i)}$ appears. For scalar-conformal blocks, the actual location of poles Δ_i, primary-descendants $O^{(i)}$, and coefficient c_i were first guessed numerically in [47] when all the external scalars are identical. It has been generalized in [11] to the case with non-identical scalars. Representation theoretical derivation has been carried out in [28] for $d = 3$.

Chapter 4

The bootstrap equation and positivity argument

Now we are ready to introduce the dramatic success of the conformal bootstrap program after the breakthrough in 2008 by Rattazzi, Rychkov, Tonni and Vichi [8]. This chapter illustrates how their ingenious argument imposes highly non-trivial constraints in the operator spectrum, and even "solve" particular models.

4.1 The bootstrap equation

As stated in the beginning of the last chapter, the precise form of the conformal four-point correlator

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$$

is radially-ordered one,

$$\langle 0|\mathrm{R}\left\{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\right\}|0\rangle.$$

While we have considered its decomposition in the configuration (3.24) or (3.69) so that the operator ordering is

$$\langle 0|\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0\rangle,$$

we can make a conformal transformation to send the arguments to align differently. For example, we can rotate and translate the standard configuration to set x_3 to the origin and $x_1 = (1, 0, \cdots)$ (with x_4 fixed at the infinity), getting $x_2 = 1 - z$ on the plane.

We also have the conformal block decomposition in this region, but this time converging in the region everywhere except for real z with $(1 - z) > 1$, which has an overlap with the above one. These two expression must agree on this overlapping region. This results in the celebrated *crossing relation* for scalar four point function,

$$\sum_{O\in\phi\times\phi} \lambda^2_{\phi\phi O}\frac{g(\Delta_O, l_O; z, \bar{z})}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} = \sum_{O\in\phi\times\phi} \lambda^2_{\phi\phi O}\frac{g(\Delta_O, l_O; 1-z, 1-\bar{z})}{x_{23}^{2\Delta_\phi}x_{14}^{2\Delta_\phi}} \tag{4.1}$$

We can also consider other operator orderings, e.g, $\phi(x_4)\phi(x_2)\phi(x_3)\phi(x_1)$, this time resulting in

$$\sum_{O\in\phi\times\phi} \lambda^2_{\phi\phi O}\frac{g(\Delta_O, l_O; z, \bar{z})}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} = \sum_{O\in\phi\times\phi} \lambda^2_{\phi\phi O}\frac{g(\Delta_O, l_O; 1/z, 1/\bar{z})}{x_{23}^{2\Delta_\phi}x_{13}^{2\Delta_\phi}} \tag{4.2}$$

In this case of identical scalar function (4.2) is satisfied once (4.1), but in general this equation of give rise to another constraints (e.g., when scalar is charged under $SU(n)$ [41]).

Of course, similar requirements must be made for all four-point functions made up by all operators in the theory, possibly with spins. The proposal made in early 1970s [48] [6] was that we define CFTs with the *CFT data*

- Operator multiplets specified by primary, i.e., operator dimensions and spins
- OPE coefficients among operators

subject to all the non-trivial crossing relations. Along with an infinite number of CFT data parameters, as we will investigate in a moment, the crossing relation requires a non-trivial functional equality to hold. This means that the constraint too is an infinite-dimensional one. Although this system of degrees of freedom and constraints look devastatingly complicated one, from the single equation (4.1) alone, we can extract surprisingly non-trivial information. This argument is called the *positivity argument*.

4.2 Positivity argument

To introduce the method we first slightly rewrite the equation (4.1) to a more convenient form by multiplying $\left\{(x_{12}^2 x_{34}^2 x_{14}^2 x_{23}^2)/(x_{13}^2 x_{24}^2)\right\}^{\Delta_\phi}$. Now every quantity in the equation is covariant:

$$\begin{aligned} 0 &= \sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}^2 \left\{ v^{\Delta_\phi} g(\Delta_O, l_O, z, \bar{z}) - u^{\Delta_\phi} g(\Delta_O, l_O, 1-z, 1-\bar{z}) \right\} \\ &= \sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}^2 \; F(\Delta_\phi, \Delta_O, l_O; z, \bar{z}), \end{aligned} \tag{4.3}$$

where F is defined by

$$F(\Delta_\phi, \Delta, l; z, \bar{z}) := (1-z)^{\Delta_\phi}(1-\bar{z})^{\Delta_\phi} g(\Delta, l; z, \bar{z}) - z^{\Delta_\phi}\bar{z}^{\Delta_\phi} g(\Delta, l; 1-z, 1-\bar{z}), \tag{4.4}$$

and is anti-symmetric in exchanging $z \leftrightarrow (1-z)$. The function is non-zero away from exchange-symmetric point $z = 1/2$, so the functional equality is non-trivial.

The argument in [8] can be essentially stated in the following manner [42].

positivity argument

Suppose there exists a real number Δ_h and a linear functional α, whose domain is the function of z and $\bar{z}$ anti-symmetric in $z \leftrightarrow (1-z)$ with the following properties:

$$\begin{aligned} &\alpha(F(\Delta_\phi, 0, 0; *)) = 1, \\ &\alpha(F(\Delta_\phi, \Delta, l; *)) \geq 0 \text{ whenever } \Delta \geq l + d - 2, \\ &\alpha(F(\Delta_\phi, \Delta, 0; *)) \geq 0 \text{ whenever } \Delta \geq \Delta_h. \end{aligned} \tag{4.5}$$

Apply this functional to the equation (4.3). Linearity assumption allows us to write

$$0 = \sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}^2 \alpha(F(\Delta_\phi, \Delta_O, l_O; *)).$$

Then divide the summation in the following way: noting that the identity operator always contribute with $\lambda_{\phi\phi 1} = 1$, we have

$$\begin{aligned} &= \alpha(F(0,0;*)) + \sum_{\substack{O\in\phi\times\phi \\ l_O=0,\quad \Delta_O\leq\Delta_h}} \lambda_{\phi\phi O}^2 \; \alpha(F(\Delta_\phi, \Delta_O, 0; *) \\ &+ \sum_{\substack{O\in\phi\times\phi \\ l_O=0,\quad \Delta_O>\Delta_h}} \lambda_{\phi\phi O}^2 \; \alpha(F(\Delta_\phi, \Delta, 0; *)) + \sum_{l\geq 2} \sum_{\substack{O\in\phi\times\phi \\ l_O=l}} \lambda_{\phi\phi O}^2 \; \alpha(F(\Delta_\phi, \Delta_O, l; *)). \end{aligned}$$

The unitarity of the theory tells us $\Delta_O \geq l_O + d - 2$ is always satisfied (2.72). The reflection positivity tells $\lambda_{\phi\phi O}^2 > 0$. Thus these facts together with the above assumptions about α make the first, third, and forth term sum up to positive number ≥ 1.

In order for the equality to hold, the second sum must be non-empty. There must exist an operator in $\phi\times\phi$ OPE with dimension below Δ_h.

4.2.1 Truncation method 1: discretization and linear programming

The presence of such an α together with some value Δ_h brings us a quite nontrivial constraint on the spectrum, which holds for general unitary CFTs, whatever their contents are.

How can we find such a linear functional α and Δ_h? To obtain better bounds, we would like to seek for Δ_h as small as possible with fixed Δ_ϕ . We denote such an optimal value by Δ^c.

Mainly there are two hurdles in carrying this search out if we want to let the computer solve the problem numerically.

The first difficulty is that the space of such linear functionals is infinite-dimensional. We usually compromise by satisfying ourselves with some finite-dimensional subspace to search. Currently, numerically most reasonable way is to take this subspace to be derivative expansions up to finite order,

$$\alpha_{\text{candidate}}(F) = \sum_{\substack{m,n\geq 0\\ m+n\leq\Lambda}} a_{m,n}\partial_z^m\partial_{\bar{z}}^n F|_{z=\bar{z}=\frac{1}{2}}, \tag{4.6}$$

with $a_{m,n}$ real and anti-symmetric under $m \leftrightarrow n$ exchange. With a fixed cutoff Λ, such functionals make finite-dimensional vector space. Notice that a functional found within this truncated vector space still gives a completely rigorous bound provided the positivity condition (4.5) is met.

The second difficulty is that the number of inequality to be checked in (4.5) is continuously infinite. In earlier papers [8][39][41][49][40][42], this difficulty was truncated by the discretization over values of Δ with some finite cutoff $\Delta_{\max}$. That is, we check the inequalities

$$\alpha(F_{\Delta,l}) \geq 0$$

on discretized intervals

$$\Delta = \Delta_{\min},\ \Delta_{\min}+\epsilon,\ \Delta_{\min}+2\epsilon,\ \cdots\ ,\Delta+N\epsilon.$$

Here $\Delta_{\min} = \Delta_h$ if $l = 0$ and $d+l-2$ otherwise, ϵ is a sufficiently small interval, and N is taken to be large enough so that $N\epsilon >> 1$. We also have to have a cutoff $l_{\max}$ for l in order to make the number of checked inequality finite.

Though not rigorous, from the smoothness of the conformal block dependence on Δ, we expect a sufficiently good approximation can be obtained when we take ϵ small enough. Also, for individual CFTs, recalling that the exponential suppression of high-dimensional operator [31][44] with

$$\rho(z = 1/2) = 3 - 2\sqrt{2} \simeq 0.17,$$

it does not sound like a bad idea to have a cutoff in the inequality for high-operator dimensions.

After these truncations, the problem is

$$\begin{aligned}
&\text{find } a_{m,n} \text{with}\\
&\sum_{\substack{m,n\geq 0\\ m+n\leq\Lambda}} a_{m,n}\partial_z^m\partial_z^n F(0,0;z,\bar{z})|_{z=\bar{z}=\frac{1}{2}} = 1,\\
&\sum_{\substack{m,n\geq 0\\ m+n\leq\Lambda}} a_{m,n}\partial_z^m\partial_z^n F(\Delta,l;z,\bar{z})|_{z=\bar{z}=\frac{1}{2}} \geq 0 \quad \text{for discretized values of } (\Delta,l),
\end{aligned}$$

which has precisely the form of linear programming, and can be solved by computers efficiently. Now the task of finding smallest possible Δ_h can be performed with a try-and-error approach (usually by binary-search starting from presumed large Δ_h). Anyway, one can check that, beyond sufficiently small value of ϵ and large cutoff for (Δ, l), the output is converging.

4.2.2 Truncation method 2: translation into semi-definite programming

An alternative to the discretization method was invented in [50]. This is achieved through approximating conformal block $g(\Delta, l; z, \bar{z})$ regarded as a function of Δ. This always exists due to the recursion formula discussed in Section 3.2.4. To recap, the approximation takes the form

$$g(\Delta, l; z, \bar{z}) = (4|\rho|)^{\Delta} \left\{ h(\infty, l, ; z, \bar{z}) + \sum_i c_i \frac{(4|\rho|)^{n_i} h(\Delta_i + n_i, l_i)}{(\Delta - \Delta_i)^{k_i}} \right\}$$

where $\{n_i, l_i, c_i, k_i\}$ are constants. The very fact that the asymptotic solution $h(\infty)$ exist assures that the summation tends to 0 along $\Delta \to \infty$. Thus, we can truncate the sum by setting cutoff for n_i, which is called $\nu_{\max}$ in the literature [11][12]. After this, the conformal blocks factors into rational and irrational part with respect to Δ,

$$\begin{aligned} g(\Delta, l; z, \bar{z}) &\simeq (4|\rho|)^{\Delta} \left\{ h(\infty, l, ; z, \bar{z}) + \sum_{i:\ n_i \leq \nu_{\max}} c_i \frac{(4|\rho|)^{n_i} h(\Delta_i + n_i, l_i)}{(\Delta - \Delta_i)^{k_i}} \right\} \\ &= \frac{(4|\rho|)^{\Delta}}{\prod_{i: n_i \leq \nu_{\max}} (\Delta - \Delta_i)^{k_i}} \times (\text{polynomial of } \Delta)\,. \end{aligned} \tag{4.7}$$

Notice that taking derivatives with respect respect to $z, \bar{z}$ changes the form of the polynomial, but the approximation retains its form. So we write

$$\partial_z^m \partial_{\bar{z}}^n g(\Delta, l; z, \bar{z})|_{z=\bar{z}=1/2} = \frac{(4\rho_*)^{\Delta}}{\prod_{i: n_i \leq \nu_{\max}} (\Delta - \Delta_i)^{k_i}} \times\ p_{m,n}(l; \Delta) \tag{4.8}$$

with $\rho_* = \rho(z = 1/2) = 3 - 2\sqrt{2}$ and $p_{m,n}(l; \Delta)$ is the result of the differentiation. This property descends to F-function (4.4),

$$\partial_z^m \partial_{\bar{z}}^n F(\Delta_\phi, \Delta_\phi, l, z, \bar{z})|_{z=\bar{z}=1/2} \simeq \frac{(4\rho_*)^{\Delta}}{\prod_{i: n_i \leq \nu_{\max}} (\Delta - \Delta_i)^{k_i}} \times q_{m,n}(\Delta_\phi, l; \Delta) \tag{4.9}$$

Now we go back to the positivity argument. Assuming the form of α to be a differential operator as in (4.6), the inequality conditions (4.5) reads

$$\begin{aligned} &\sum_{\substack{m,n \geq 0 \\ m+n \leq \Lambda}} a_{m,n} \partial_z^m \partial_{\bar{z}}^n F|_{z=\bar{z}=\frac{1}{2}} \simeq \sum_{\substack{m,n \geq 0 \\ m+n \leq \Lambda}} \frac{(4\rho_*)^{\Delta}}{\prod_{i: n_i \leq \nu_{\max}} (\Delta - \Delta_i)^{k_i}}\ a_{m,n} q_{m,n}(\Delta_\phi, l; \Delta) \\ &= \frac{(4\rho_*)^{\Delta}}{\prod_{i: n_i \leq \nu_{\max}} (\Delta - \Delta_i)^{k_i}} \sum_{m,n} a_{m,n} q_{m,n}(\Delta_\phi, l; \Delta) \geq 0 \end{aligned}$$

for $\Delta \geq \Delta_{\min}$. Since the poles of the conformal blocks lie in the non-unitary region and $\Delta_{\min} \geq \Delta_i$ holds for all the poles, the prefactor is positive in the region under consideration. Now eliminating this away, we have translated the inequality constraints by polynomial conditions

$$\sum_{m,n} a_{m,n}\, q(\Delta_\phi, l; \Delta) \geq 0 \qquad (\text{ for } \Delta \geq \Delta_{\min}).$$

Making cutoff $l_{\max}$ again for l, we have a finite number of such polynomial inequalities on half-lines. The task of finding such $\{a_{m,n}\}$ can again be handled with computers, this time by using the solver for *semi-definite programming*(SDP), a generalization to linear programming. After creating tables of polynomials $q_{m,n}$, it can be passed to a general-SDP solver like `SDPA-GMP` [51] or more bootstrap-specialized one `SDPB` [12].

SDP is more natural language for bootstrap

Though elegant, the SDP truncation with polynomial approximation might look somewhat artificial way to approximate bootstrap problems. However, in more general situations e.g.

- Require consistency conditions for a multiple number of four-point correlation functions [11][12][52][53],
- External operators are not necessarily scalar and multiple types of OPE coefficients can appear in three-point functions [54],

it is mandatory to use SDP to carry out the generalization of the above-stated argument [41][11].

4.2.3 Bounds for single correlator without any further assumption

$d = 4$

We here reproduce the original result of [8] for $d = 4$ CFTs, where several phenomenological model of walking technicolor has been proposed. To carry out the computation, we implemented in the open-source general purpose mathematics software `sage-math` [55] a code to create a polynomial table, which then can be solved by `SDPB` [12]. Part of the codes written for this purpose is available at `https://sites.google.com/site/tomokuohtsky/cboot` (in Japanese).

The result shown in Fig.4.1 is a newly obtained one using slightly improved parameters over past works [8][39][50] with $\Lambda = 23$ and $\nu_{\max} = 22$ as truncation parameters, where Λ is the derivative-order cutoff in (4.6) and $\nu_{\max}$ is the pole-inclusion cutoff in (4.7). To recap, this bounding curve means that

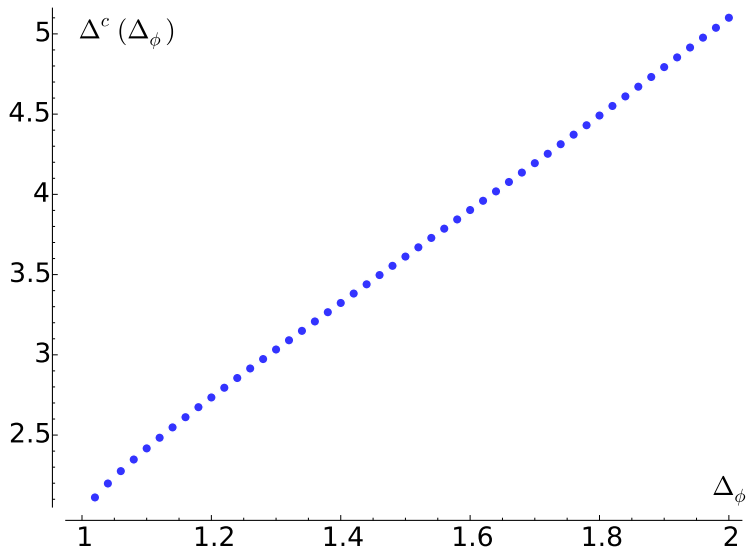

Figure 4.1: Bounds for general $d = 4$ CFTs with scalar field ϕ derived from SDP-truncation method. $\nu_{\max} = 22$ with $\Lambda = 23$ is taken. Spins included in the optimization problem are $l = 0, 2, \cdots, 20, 24, 28 \cdots, 48$. Binary search precision goal is generically 10^{-3}. The parameter for SDPB solver can be the default one in [12]. The value of Δ_ϕ for which Δ^c cross the marginality line $\Delta^c = 4$ is about $\Delta_\phi \sim 1.618$, which is only slightly improved over the one in [50] derived with $\Lambda = 21$.

the first non-trivial (non-identity) operator appearing in $\phi \times \phi$ must have dimension below $\Delta^c(\Delta_\phi)$. The result of [8] improved in [39] ,[49], and [50] was strong enough to exclude almost all possibility of conformal-technicolor proposed in [56].

Such a rigorous statement is a precious one in quantum field theories without supersymmetry. From the current perspective, however, this was merely the starting point of the surprising adventure of the bootstrap program.

$d = 2$

Already in [8], to verify the validity of the method, they carried out a rough (small derivative cutoff Λ) estimate for the bound for $d = 2$, where certain exact information is available like the critical exponents of minimal model sequence [7]. They used $d = 2$ $SL(2, \mathbb{C})$-conformal block – forgetting the actual Virasoro module structure of the Hilbert space [1].

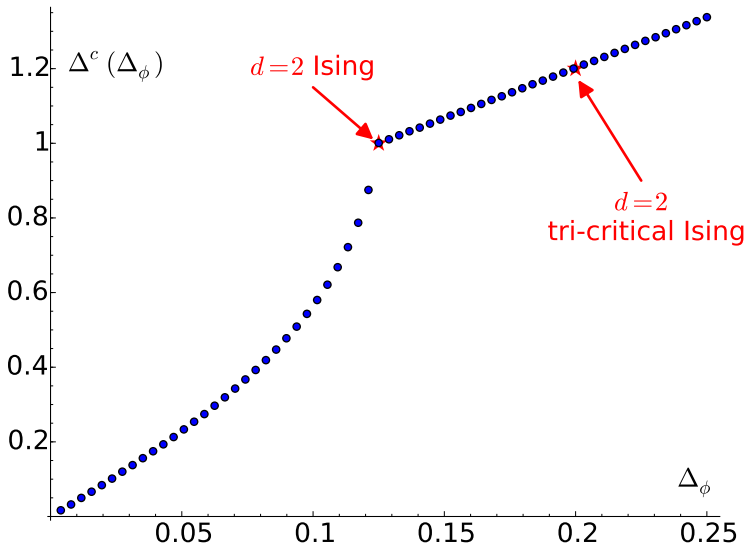

Figure 4.2: Bounds for general $d = 2$ CFT with scalar field ϕ, derived from SDP-truncation method. $\nu_{\max} = 20$ with $\Lambda = 23$ is taken. Intermediate operator spins taken into account are $l = 0, 2, \cdots, 20, 24, 28 \cdots, 48$. Binary search precision goal is generically 10^{-3}. The parameter for SDPB solver can be the default one in [12].

Improved bounds derived in [39] looked quite intriguing, for the two-dimensional Ising model, which has scalar σ with dimension 1/8 and ε with dimension 1, seemed to saturate the bound in a peculiar way (with slope-changing behavior). Of course this is a numerical approximation and discontinuity is not a real one[2], and saturation of the bound is only approximate (the ideal value for the bound at $\Delta_\phi = 0.125$ is $1 + 4.4 \times 10^{-7}$ here at $\Lambda = 23$). However, the figure is so impressive that we are obliged to assume the saturation and actual discontinuity at the $\Lambda \to \infty$ limit. What cannot be emphasized enough is that such a "kink" emerged from almost no input – the only one input was the presence of scalar field ϕ. The figure is derived by changing the values of Δ_ϕ, and polynomials $q_{m,n}$ change their coefficient continuously, but the result looks (for our poor eyes) singular. The result encouraged the authors of [9] to carry out the same analysis but in $d = 3$.

By the way, although it seems that the tri-critical Ising model, who has $\Delta_\sigma = 1/5$ and $\Delta_\epsilon = 6/5$ also comes close to saturate the bound, but the convergence (if any) is not as impressive as for the Ising model. At $\Lambda = 23$, the best bound is $\Delta^c(0.2) = 1.2016$. See also [10] for this numerology.

$d = 3$

For the $d = 3$ CFTs, at this stage, no exact expression for the conformal blocks is available[3]and the authors of [9] had to solve numerically the intricate recursion relation written down in [58] to derive the conformal block values of interest. The result looked like Fig. 4.3, again showing the kink. The $d = 3$ Ising model critical exponents computed by Monte-Carlo [59] again seems to saturate the bootstrap

[1] The main reason might be that the full Virasoro conformal blocks are harder to access, due to the absence of analytic expressions unlike the ones in $SL(2, \mathbb{C})$-block in (3.64), and one has to resort to the Zamolodchikov's recursion formula [46] in order to numerically estimate it. Also, to specify the Virasoro-block, one has to specialize to a certain value of central charge. Such a type of study has been recently implemented in supersymmetric context [57].

[2] A resolution will reveal that everything is continuous. Indeed, in the original figure of [39], due to the smallness of Λ compared the one in here, the kink is not as sharp as the one in here.

bounds around the kink location $(\Delta_\phi, \Delta^c) \sim (0.518, 1.41)$. Given this result, further computation with increased cutoff Λ and more elaborate method was applied in [10] [12], and the current most precise estimates for the $d = 3$ Ising critical exponents using the bootstrap method beat all the other predictions.

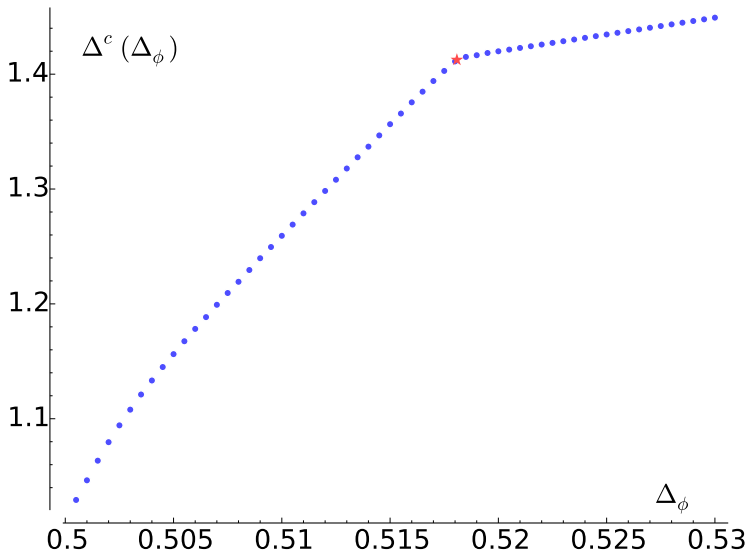

Figure 4.3: Bounds for general $d = 3$ CFT with scalar field ϕ, derived from SDP-truncation method. $\nu_{\max} = 20$ with $\Lambda = 23$ is taken. Intermediate operator spins taken into account are $l = 0, 2, \cdots, 20, 24, 28 \cdots, 48$. The parameter for SDPB solver can be the default one in [12].

4.3 Incorporation of global symmetry : $O(n)$ example

So far we have been seeking for the constraints for CFTs which must be respected by all CFTs in specific dimensions. Since many interesting models are believed to exist with further global symmetries, given the surprising power of the bootstrap study for the $2 \leq d < 4$ Ising models, the next natural task is then to derive the bounds for theories with global symmetries, especially the simplest descendants of the Ising model (which has $\mathbb{Z}_2 \simeq O(1)$ as its symmetry), $O(n)$-LGW (vector) models. Let us illustrate what we can do in such cases in this section.

4.3.1 Structured conformal block decomposition

We assume the presence of a CFT with $O(n)$-symmetry, and scalar operators $\phi_i(x)$ behaving as an $O(n)$-vector representation. Two and three-point functions are fixed in the same way, but additional structure appears. For example, two-point functions of ϕs are fixed to be

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_\phi}}.$$

In order for a three-point function $\langle \phi_i \phi_j O\rangle$ to be non-zero, from the invariance of the vacuum under $O(n)$-symmetry, O must behave as an irreducible representation in the tensor product vector $\otimes$ vector. This decomposes into scalar (S), traceless-symmetric tensor (T), and anti-symmetric tensor (A) rep-

[3]See, however, [44] for the compact exact expression in $d = 3$ along $z = \bar{z}$ axis

resentations[4]. They respectively have the three-point functions of the forms

$$\langle \phi_i(x_1)\phi_j(x_2)O^{(\mathrm{S})}_{\mu_1\cdots\mu_l}(x_3)\rangle \propto \delta_{ij}, \tag{4.10}$$

$$\langle \phi_i(x_1)\phi_j(x_2)O^{(\mathrm{T})}_{kl;\mu_1\cdots\mu_l}(x_3)\rangle \propto (\delta_{li}\delta_{kj}+\delta_{lj}\delta_{ki}-\frac{2}{n}\delta_{ij}\delta_{kl}), \tag{4.11}$$

$$\langle \phi_i(x_1)\phi_j(x_2)O^{(\mathrm{A})}_{kl;\mu_1\cdots\mu_l}(x_3)\rangle \propto (\delta_{li}\delta_{kj}-\delta_{lj}\delta_{ki}), \tag{4.12}$$

where the kinematical factors are given by (3.57). An important remark follows: under the exchange of $x_1 \leftrightarrow x_2$, the kinematical factor acquires the factor of $(-1)^l$ (this is why only spin even operator appeared in the previous computation), so it must be compensated by the sign from $i \leftrightarrow j$ exchange in order for the three point function to be invariant under the simultaneous exchange of $x_1 \leftrightarrow x_2$ and $i \leftrightarrow j$. So the operators in S and T-representations must have even spin (e.g. the identity operator and the energy-momentum tensor), while those in A have spin odd (e.g. $O(n)$-symmetry current).

Conformal partial wave decomposition for four-point functions is performed similarly, but here the three point function structures must be taken into account. The result for $\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle$ in the radial ordering by $|x_1|<|x_2|<|x_3|<|x_4|$ is [5]

$$\begin{aligned} x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle &= \delta_{ij}\delta_{kl}\sum_{\substack{O\in\phi\times\phi\\ \text{and S}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;z,\bar z)\\ &+\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}-\frac{2}{n}\delta_{ij}\delta_{kl}\right)\sum_{\substack{O\in\phi\times\phi\\ \text{and T}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;z,\bar z)\\ &+(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})\sum_{\substack{O\in\phi\times\phi\\ \text{and A}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;z,\bar z). \end{aligned} \tag{4.13}$$

4.3.2 Vectorial bootstrap equations and positivity arguments

Then the crossing relation for four-point functions are required in an equal fashion. In the kinematical region $|x_4|>|x_1|>|x_2|>|x_3|$, the decomposition is

$$\begin{aligned} x_{23}^{2\Delta_\phi}x_{14}^{2\Delta_\phi}\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle &= \delta_{jk}\delta_{il}\sum_{\substack{O\in\phi\times\phi\\ \text{and S}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;1-z,1-\bar z)\\ &+\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}-\frac{2}{n}\delta_{jk}\delta_{il}\right)\sum_{\substack{O\in\phi\times\phi\\ \text{and T}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;1-z,1-\bar z)\\ &+(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl})\sum_{\substack{O\in\phi\times\phi\\ \text{and A}}}\lambda^2_{\phi\phi O}\; g(\Delta_O,l_O;1-z,1-\bar z), \end{aligned} \tag{4.14}$$

which must agree with (4.13). Matching all the coefficients of Kronecker's deltas imposes three functional equalities

$$\begin{aligned} S(z,\bar z)-\frac{2}{n}T(z,\bar z) &= T(1-z,1-\bar z)+A(1-z,1-\bar z),\\ T(z,\bar z)-A(z,\bar z) &= T(1-z,1-\bar z)-A(1-z,1-\bar z),\\ S(z,\bar z)-\frac{2}{n}T(z,\bar z) &= S(1-z,1-\bar z)-\frac{2}{n}T(1-z,1-\bar z), \end{aligned}$$

where $R(z,\bar z)$ represents the sum $(1-z)^{\Delta_\phi}(1-\bar z)^{\Delta_\phi}\sum_{O\in\phi\times\phi;\ O\text{ in }R}\; g(\Delta_O,l_O;z,\bar z)$. The third equation is actually redundant in the sense that it can be obtained from the first one by substituting $z\to(1-z)$.

[4]Strictly speaking, for $n=2$ A is actually equivalent to S, but this does not affect the bootstrap equation.

[5] The sign for A-sector contribution might be a bit confusing. This is due to the normalization convention of conformal block $g(\Delta,l;z,\bar z)>0$ along the positive side of real axis. The positivity of each summand at the reflection positive configuration like $\langle 0|\phi_1(\infty)\phi_2(x_2/x_2^2)\phi_2(x_2)\phi_1(0)|0\rangle$ then fixes this sign.

It is customary to separate the functional equations into symmetric parts and anti-symmetric parts with respect to $z \leftrightarrow (1-z)$ exchange. The result can be conveniently summarized in terms of vectorial bootstrap equation:

$$\vec{0} = \sum_{\substack{O\in\phi\times\phi \\ O \text{ in S}}} \lambda^2_{\phi\phi O}\vec{V}_{\mathrm{S}}(\Delta_\phi,\Delta_O,l_O;z,\bar{z}) + \sum_{\substack{O\in\phi\times\phi \\ O \text{ in T}}} \lambda^2_{\phi\phi O}\vec{V}_{\mathrm{T}}(\Delta_\phi,\Delta_O,l_O;z,\bar{z}) + \sum_{\substack{O\in\phi\times\phi \\ O \text{ in T}}} \lambda^2_{\phi\phi O}\vec{V}_{\mathrm{A}}(\Delta_\phi,\Delta_O,l_O;z,\bar{z}), \tag{4.15}$$

where

$$\vec{V}_{\mathrm{S}}(*;*) := \begin{pmatrix} 0 \\ F^{(-)}(*;*) \\ F^{(+)}(*;*) \end{pmatrix}, \tag{4.16}$$

$$\vec{V}_{\mathrm{T}}(*;*) := \begin{pmatrix} F^{(-)}(*;*) \\ \left(1-\frac{2}{n}\right) F^{(-)}(*;*) \\ \left(-1-\frac{2}{n}\right) F^{(+)}(*;*) \end{pmatrix}, \tag{4.17}$$

$$\vec{V}_{\mathrm{A}}(*;*) := \begin{pmatrix} -F^{(-)}(*;*) \\ F^{(-)}(*;*) \\ -F^{(+)}(*;*) \end{pmatrix}, \tag{4.18}$$

with the (anti-)symmetrized function

$$F^{(\pm)}(\Delta_\phi,*;z,\bar{z}) := (1-z)^{\Delta_\phi}(1-\bar{z})^{\Delta_\phi} g(*;z,\bar{z}) \pm (z)^{\Delta_\phi}(\bar{z})^{\Delta_\phi} g(*;1-z,1-\bar{z}) \tag{4.19}$$

The positivity argument developed in Section 4.2 can be almost parallelized[41]. This time the linear functional α takes its domain to be 3-dimensional vector-valued functions of $z,\bar{z}$ with the first and second column being anti-symmetric in $z \leftrightarrow (1-z)$ exchange and the third component symmetric.

First consider the following positivity conditions for α:

- $$\alpha(\vec{V}_{\mathrm{S}})(\Delta_\phi,0,0;z,\bar{z}) = 1.$$

  Note that $\vec{V}_{\mathrm{S}}(\Delta_\phi,0,0;*)$ is the identity operator contribution to the bootstrap equation 4.15, which is included in the S-channel.

- $$\alpha(\vec{V}_{\mathrm{S}})(\Delta_\phi,\Delta,0;z,\bar{z}) \geq 0 \qquad (\text{ for } \Delta \geq \Delta_h),$$

  where Δ_h plays an analogous role of that in (4.5).

- $$\alpha(\vec{V}_{\mathrm{T}})(\Delta_\phi,\Delta,0;z,\bar{z}) \geq 0 \qquad (\text{ for } \Delta \geq \frac{d-2}{2}).$$

  For now we do not insert artificial parameters, instead the unitarity bound for the scalar operator (2.74).

- $$\alpha(\vec{V}_{R})(\Delta_\phi,\Delta,l;z,\bar{z}) \geq 0 \qquad (\text{ for } l>0,\ \Delta \geq l+d-2, \text{ and } R=\mathrm{S},\mathrm{T},\mathrm{A}).$$

  Again for these sectors, we require the positivity for all unitary region of Δ.

If we find such a functional α, an argument entirely analogous to that in (4.2) tells you the following: there must be an $O(n)$-singlet scalar operator in $\phi \times \phi$ OPE with dimension below Δ_h.

A very important remark is that we do not necessarily have to insert hypothetical gap parameter in the S spin 0 sector. For example, we can equally impose

$$\alpha(\vec{V}_{\mathrm{S}})(\Delta_\phi, \Delta, 0, z; \bar{z}) \geq 0 \qquad (\text{ for } \Delta \geq \frac{d-2}{2})$$

and

$$\alpha(\vec{V}_{\mathrm{T}})(\Delta_\phi, \Delta, 0; z, \bar{z}) \geq 0 \qquad (\text{ for } \Delta \geq \Delta_h),$$

and this time the argument tells you that there must be an $O(n)$-T scalar operator in $\phi \times \phi$ OPE with dimension below Δ_h.

4.3.3 $O(n)$-bootstrap results

The first computations was performed for $d = 4$ dimensions [41] with discretization scheme employed to truncate the problem. Those bounds have been improved in the first SDP-based paper [50]. For $d = 4$ model, as in Fig.4.1, no kink was observed. Below we will see what happens in $d < 4$.

$d = 3$ $O(2)$

In dimension $2 < d < 4$, $O(n)$-LGW models, which are direct generalizations of the ϕ^4 model, are believed to flow into IR fixed points describing the continuous phase transitions of various spin systems (see e.g., [60] [61]). Since the $d = 3$ Ising models are cornered by means of the conformal bootstrap, it is natural to ask what will happen for the bounds which must be respected by those LGW models.

The result in [47][6]for the $O(2)$-model looks like Fig.4.4. Again there showed up a kink. Here the

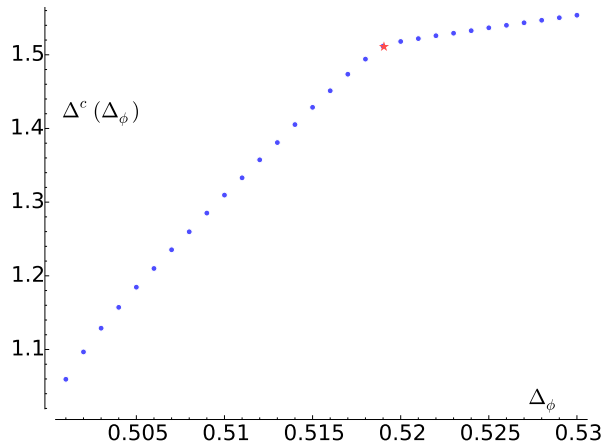

Figure 4.4: $d = 3$ bounds derived from SDP-truncation method for $O(2)$-singlet operator dimensions in vector×vector OPE. Here $\nu_{\max} = 20$ with $\Lambda = 23$ is taken. Spins included in the matrix $l = 0, 1, 2, \cdots, 25, 28, 29, 32, 33, \cdots, 48, 49$. Binary search precision goal is 10^{-3}. The parameter for SDPB solver can be the default one in [12] as well.

star represents the hybrid lattice-MC result of [62], which is indistinguishable in this figure from the prediction of space-shuttle experiment result in [63]. These two results however disagree (see [64] for a discussion). Unfortunately, at this stage, even with increased precision and inevitably increased numerical cost, bootstrap method is not powerful enough to resolve this discrepancy[52].

[6]In [47], to increase the numerical efficiency of the rational approximation of conformal blocks, a bit stronger condition $\alpha(\vec{V}_{\mathrm{T}}(\Delta, 0; *)) > 0$ for $\Delta \geq 1$ is imposed, which is not completely general. Here we did not impose this condition, at the cost of increasing $\nu_{\max}$.

As remarked, we can also find bounds for operators transforming in T-representation. The result at $\Lambda = 23$ is shown in Fig.4.5. The rectangle there is a combined prediction for the XY-model lowest T-operator dimension, horizontal value of which is from [62] and vertical value from [65]. In this example of $O(2)$-CFTs, two bounds (for S and T-sectors) computed independently seemed to exhibit kinks corresponding to the same theory (critical XY-model). However, this is not always the case – see Chapter 6. Note also the dictionary between $SO(2) \to U(1)$. The bound for singlet in $\phi \times \phi$ can be equivalently stated as the bound for charge 0 operator dimensions appearing in the OPE of charge 1 and its conjugate. Meanwhile, that for T-representation operator is the bound for charge 2 operator dimension contained in a charge 1 operator with itself. This restatement will be useful in the discussion of 6.3.3.

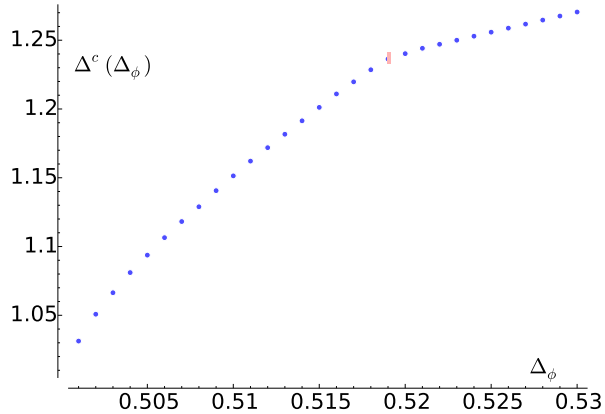

Figure 4.5: $d = 3$ bounds derived from SDP-truncation method for $O(2)$-T representation operator dimensions in vector×vector OPE. Here $\nu_{\max} = 20$ with $\Lambda = 23$ is taken. Spins included in the matrix $l = 0, 1, 2, \cdots, 25, 28, 29, 32, 33, \cdots, 48, 49$. Binary search precision goal is 5×10^{-4}. The parameter for SDPB solver can be the default one in [12] as well.

Fun with $d = 2$ $O(n)$-bounds

The situation in $d = 2$-dimensions is quite different. While the ϕ^4-family of the Wilson-Fisher fixed points continue to exist for $d = 2$, the $O(n)$-vector models with $n \geq 2$ no-longer have fixed points (non-linear sigma models exhibit the confinement). To have some fun, we derived bootstrap bounds in $d = 2$ for $O(n)$-singlet, which looks like Fig.4.6. The first feature is that the bounds for $O(2)$ model seems to have a plateau starting from $\Delta_\phi = 0$ with value= 2. This is as it should be: in two-dimensions, (periodic) free scalar field X can be exponentiated (with normal ordering) to give a well-defined scalar field (see Section 2.2 of [21])

$$: e^{ikX} :$$

with dimension k^2. The translation invariance of the target torus act as a $U(1)$-action on this field, so these fields must obey the bounds derived above. Then, $O(2)$-invariant spectrum is contained in

$$: e^{ikX(z)} :: e^{-ikX(0)} :\sim \frac{1}{|z|^{2k^2}}(1 + k^2|z|^2 : \partial X \bar{\partial} X(0) : + \cdots)$$

The singlet operator with smallest nontrivial dimension is thus $: \partial X \partial X :$, which has dimension 2. The numerical bound should not exclude such a possibility.[7]

Another peculiar feature is that $O(n > 2)$ bounds have peaks: at some point they start to decrease. In particular, for $n = 4$, the peak look singular, despite the absence for non-linear sigma model fixed points with target S^3. We conjecture that the bound will be saturated in $\Lambda \to \infty$

[7] I thank Yu Nakayama for pointing this out to me.

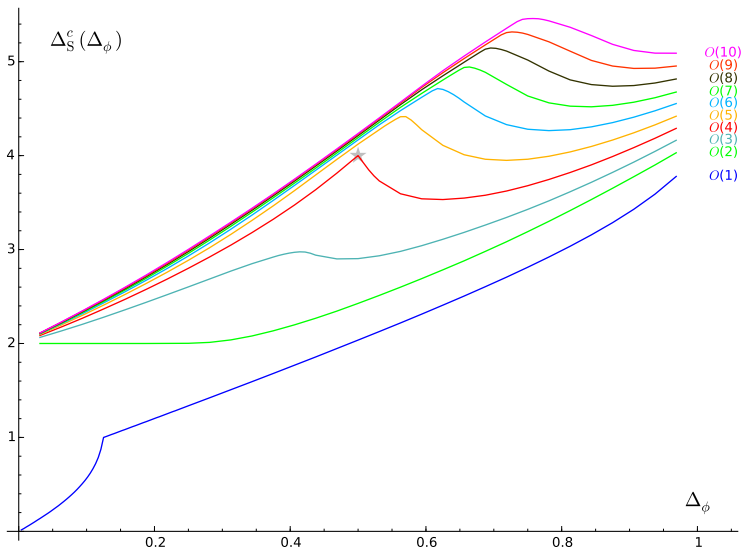

Figure 4.6: Bounds derived from SDP-truncation method for $O(N)$-singlet operator dimensions in vector×vector OPE. Here $\nu_{\max} = 22$ with $\Lambda = 23$ is taken. Spins included in the matrix $l = 0, 1, 2, \cdots, 25, 28, 29, 32, 33, \cdots, 48, 49$. Although the original data was obtained for discrete values of Δ_ϕ, with horizontal step of 0.03125 at generic points and 0.0078125 around the peak, we interpolated the results to help the readers' eyes. Binary search precision goal is 10^{-3}. The parameter for SDPB solver can be the default one in [12] as well. The gray star represents the location of the first level $SU(2)$ WZW-model.

limit by the $SU(2)$ first-level Wess-Zumino-Witten model, which has a scalar field charged under $SU(2)_L \times SU(2)_R \simeq SO(4)$-fundamental representation with dimension 1/2. Somehow $SL(2,\mathbb{C})$-bootstrap equation "knows" about current algebras.

Chapter 5

Zoo of $O(n) \times O(2)$-symmetric Landau-Ginzburg-Wilson models

In this chapter, two very different type of $3+1$-dimensional thermal systems are introduced. One is a chromodynamical system with $N_f = 2$ massless flavors or Dirac fermions, which is a suitable idealization of the real QCD. The other system (potentially less familiar to high-energy physicists) is a frustrated spin system on stacked triangular lattice, where a significantly more complicated dynamics than usual spin system appears. Both are believed to undergo phase transition with some spontaneous symmetry breaking pattern. Although their transition temperature differs by a magnitude of 10^{10}, potentially relevant (in this case both emergent at the transition) symmetry of these systems can be summarized to be $O(n) \times O(2)$, and this is why we treat them together. In Section 5.1 and 5.2, we will review why such a symmetry emerges at the transition point of these systems.

Following the philosophy of Landau-Ginzburg-Wilson(LGW), we write a general super-renormalizable Lagrangian by the order parameter fluctuation ϕ to seek a candidate of critical point. Unlike the Ising and its $O(n)$-generalizations, however, these systems are extremely hard to study by any conventional schemes like Monte-Carlo, epsilon-expansion, and functional-renormalization group. In 5.3, we will concisely review several aspects of LGW models and summarize earlier studies and controversies regarding the presence of IR-stable fixed point.

5.1 Linear σ model analysis of $N_f = 2$ QCD chiral phase transition

5.1.1 Pisarski-Wilczek argument

The most dramatic success of the notion of spontaneous symmetry breaking in the 1960s was the explanation of the relatively small masses of pions by Nambu. He pointed out that the theory of strong interaction (which we now believe is QCD) has approximate symmetry

$$SU(2)_L \times SU(2)_R \tag{5.1}$$

and it breaks down to diagonal subgroup $SU(2)_V$ at some non-zero temperature, T_c. Since then, one of the primary concerns of the theory of hadrons is to clarify the nature of this phase transition, like the transition temperature and its order.

To discuss its order, Pisarski and Wilczek followed a classical argument by Landau-Ginzburg-Wilson[66]. Since we know that after the confinement the effective degrees of freedom comprises those of mesons, it is natural to attempt writing an effective Lagrangian out of them. They are created by quark bilinear

$$\Phi_i^j = \bar{\Psi}^j \left(\frac{1+\gamma_5}{2} \right) \Psi_i$$

and transform as a bi-fundamental of $SU(2)_L \times SU(2)_R$ or a fundamental of $SO(4) \simeq SU(2)_L \times SU(2)_R$. Under the diagonal unbroken subgroup $SU(2)_V$, they decompose as

$$\Phi(x) = \frac{1}{2}(\sigma(x) + i\eta(x))1_{2\times 2} + \frac{1}{2}(i\vec{\pi}(x) + \vec{\delta}(x)) \cdot \vec{\tau}, \tag{5.2}$$

where $\vec{\tau}$ is the Pauli's matrix. After integrating out modes with non-zero thermal frequency, the most general (three-dimensional) quartic action made from this field invariant under the $SO(4)$ symmetry is

$$\begin{aligned}\mathrm{tr}(\partial_\mu\Phi\partial^\mu\Phi^\dagger)+c(T)\mathrm{tr}(\Phi\Phi^\dagger)+c_A(T)(\det\Phi+\det\Phi^\dagger)+\lambda(T)(\mathrm{tr}(\Phi\Phi^\dagger))^2\\ +\lambda_x\mathrm{tr}(\Phi\Phi^\dagger)(\det\Phi+\det\Phi^\dagger)+\lambda_y\left\{(\det\Phi)^2+(\det\Phi^\dagger)^2\right\},\end{aligned} \tag{5.3}$$

under the assumption of parity $\Phi\to\Phi^\dagger$. Here all the coefficients are real and temperature dependent and as usual, we expect thermal effective mass $c(T)$ to decrease (from some positive value) as we cool down the system. The term proportional to c_A represents the effect of chiral anomaly: the instanton configuration induces a $2N_f-$point interaction of quark of this form. In the case of interest $N_f=2$, it also works as a mass term: while $c(T)$ represents uniform mass

$$\frac{c(T)}{2}\left\{\sigma^2+\vec{\pi}^2+\eta^2+\vec{\delta}^2\right\}, \tag{5.4}$$

$c_A(T)$ gives rise to the mass-splitting

$$\frac{c_A(T)}{2}\left\{\sigma^2+\vec{\pi}^2-\eta^2-\vec{\delta}^2\right\}. \tag{5.5}$$

In order for all the mesons to be massless, we have to tune both $c(T)=c_A(T)=0$, which is generically impossible. Thus at the critical temperature, if $c_A(T)\neq 0$, only half of meson fields are massless and the effective Lagrangian for scale invariant physics should be that of $SO(4)$-vector model, and there is an IR-stable fixed point of Heisenberg type. Of course we cannot conclude the actual order of the transition solely from this universality argument as it depends on the bare coupling, but if it is of second order, various critical exponents must be those of $O(4)$ universality class.

5.1.2 possibility of $U(1)_A$ restoration and Aoki-Fukaya-Taniguchi "theorem"

This is not the end of the story, however. In the above discussion, we have assumed that the $U(1)_A$-breaking mass term survives beyond chiral symmetry breaking point. Although the anomaly manifests itself in our world (which is nearly in absolute-zero temperature environment) as the mass of η' meson, at finite temperature environment the situation becomes quite subtle due to thermal compactification (or more physically, Debye screening). Indeed at the high-temperature limit, the partition function is computed by dimensionally reduced path integral and it is free of anomaly.

What really happens for the chiral phase transition point (which is of order $O(10^2\mathrm{MeV})$) has been quite controversial, in both theoretical and lattice QCD simulation perspective, depending on the assumption or the method we use. See e.g. [67][68] for earlier discussions. Recently, a quite elaborate use of Ward identity for $SU(2)_L\times SU(2)_R$ symmetry restored at $T\geq T_c$ led the authors of [69] to conclude the following striking assertion.

Aoki-Fukaya-Taniguchi "theorem"

$$c_A(T)=0$$

for $T\geq T_c$, under the assumptions to be stated.

Below we review the point of the assumptions they made and the argument which led to $c_A(T)=0$ for $T\geq T_c$, following the simplified discussion of [70].

Setup

To avoid nuisance coming from UV and IR-divergence, we start with finite spatial volume V_3 with some UV-regularization preserving $SU(2)_L\times SU(2)_R$ (like lattice with overlap fermions, which was the original setup in [69]). For convenience we also add a mass term $M^i{}_j\bar{\Psi}^j(1+\gamma_5)\Psi_i/2$.

Setting mass matrix M to be real diagonal and integrating out fermions, the action for pure gauge field A is

$$P_M(A) = e^{-S_{YM}(A)}(\det M)^{N_L^A}(\det M^\dagger)^{N_R^A} \times (\text{non-zero modes contributions}). \tag{5.6}$$

More specifically, when we take mass to be real and diagonal, this is.

$$P_M(A) = e^{-S_{YM}(A)}(m_u m_d)^{N_L^A+N_L^A} \prod_{i=u,d} \prod_{n>0} (\lambda_n^2 + m_i^2). \tag{5.7}$$

Here $N_{L(R)}$ refers to the number of left (right) handed zero-modes for given gauge configuration A, $m_{u,d}$ up and down quark masses, and λ_n the n-th positive eigenvalue of the Dirac operator. Recall that λ_n-eigenstate u_n is accompanied by $-\lambda_n$-eigenstate $\gamma_5 u_n$, due to the anti-commutativity of the Dirac operator and γ_5.

Assumption 1

The weight function $P_M(A)$ is real analytic with respect to M, and at finite-volume, since the gauge field integration is just a integration over compact set, this property descends to the partition function. Taking $V_3 \to \infty$ or IR-cutoff $\to 0$ limit is subtle, but if we are in $T \geq T_c$, all the effective degrees of freedom are massive, and we are led to assume[1]:

assumption 1

Expectation value of observables made out purely of gauge field

$$\frac{\int \mathcal{D}A\ \mathcal{O}(A)\ P_M(A)}{\int \mathcal{D}A\ P_M(A)} := \langle \mathcal{O}(A) \rangle_M \tag{5.8}$$

is an (real) analytic function of $m_{u,d}$ in $V_3 \to \infty$ limit if $T \geq T_c$.

Of course we also assumed the free-energy per volume f defined from partition function

$$\langle \mathbf{1} \rangle_M = \int \mathcal{D}A P_M(A) = \exp\left(-V_4\ f(T, V_3, M)\right), \tag{5.9}$$

where $V_4 = V_3/T$ is the space-time volume, has a Taylor-expansion in terms of M. Though there is already no dynamical field, just as an analogy we call $f(T, V_3, M)$ "dual effective action" for M.

The dual effective action takes a form analogous to (5.3) from invariance under background field transformation. At $T \geq T_c$, the general form up to quadratic order (compatible with parity transformation $M \to M^\dagger$) is

$$f(T, V, M) = f_0 - f_1 \mathrm{tr}(MM^\dagger) + f_A(\det M + \det M^\dagger) + O(M^4). \tag{5.10}$$

Again these coefficients are temperature dependent. c_A vanishes precisely when f_A does. To see this, it suffices to prove effective mass difference between π^3 and δ^3 vanishes at $M = 0$. A convenient quantity to measure mass difference is chiral susceptibility defined by

$$\begin{aligned}
&\lim_{p\to 0} \left\{ \langle \pi^3(p)\pi^3(-p) \rangle - \langle \delta^3(p)\delta^3(-p) \rangle \right\} \\
=&\frac{1}{V_3} \int \mathrm{d}^3x\ \left\{ \langle \pi^3(x)\pi^3(0) \rangle - \langle \delta^3(x)\delta^3(0) \rangle \right\} \\
=&\frac{1}{c(T) + c_A(T)} - \frac{1}{c(T) - c_A(T)},
\end{aligned}$$

[1]This is a slightly stronger requirement compared to the original one in [69], where the analyticity with respect to uniform mass $m^2 = m_u^2 = m_d^2$ was required.

where we used (5.4) and (5.5) to express thermal effective mass for π^3 and δ^3. The propagator integration can be neatly expressed by dual effective action. From decomposition (5.2), $(1/V_3)\int\langle\delta^3(x)\delta^3(0)\rangle$ can be calculated setting $M = \mathrm{diag}(b, -b)$ with real b and differentiation:

$$\frac{1}{V_3}\int \mathrm{d}^3x\ \langle\delta^3(x)\delta^3(0)\rangle = \frac{\partial^2}{\partial b^2}f \tag{5.11}$$

$$= f_1 + f_A. \tag{5.12}$$

In turn, setting $M = \mathrm{dial}(ib, -ib)$ equates

$$\frac{1}{V_3}\int \mathrm{d}^3x\ \langle\pi^3(x)\pi^3(0)\rangle = \frac{\partial^2}{\partial b^2}f \tag{5.13}$$

$$= f_1 - f_A. \tag{5.14}$$

Thus, $c_A(T) = 0$ if and only if $f_A = 0$. Below we argue f_A vanishes under one more assumption.

Assumption 2

Topologically nontrivial configuration of gauge fields and accompanying fermion (exact) zero modes were the key to solve the $U(1)_A$ puzzle. However, this is highly suppressed in $T \geq T_c$ and $V_3 \to \infty$ limit. To see this consider an expectation value of $\Sigma := \bar{\Psi}^i\Psi_i$, which is invariant under diagonal $SU(2)_V$ but transforms non-trivially in the full $SU(2)_L \times SU(2)_L$. As we are in an $SU(2)_L \times SU(2)_R$ restored phase, when we take $M \to 0$ and $V \to \infty$ limit, we have

$$\lim_{M\to 0}\lim_{V\to\infty}\frac{1}{V}\langle\Sigma\rangle_M = 0. \tag{5.15}$$

The expectation value in the left-hand side can be neatly expressed as a integral of quark propagator:

$$\frac{1}{V}\sum_{i=u,d}\left(\frac{N_L^A + N_R^A}{m_i} + \sum_n \frac{m_i}{\lambda_n^2 + m_i^2}\right). \tag{5.16}$$

Here m_i is the diagonal entry of M and note that N_L is the number of zero modes with respect to the massless Dirac operator, not $\not{D} + M$, and their m-dependence is provided solely by $P_m(A)$. Taking $m_i \to +0$, we see that both zero and non-zero modes contribution tend to be 0 as both are positive. Thus we arrive at

$$\frac{\langle N_L^A + N_R^A\rangle_M}{m_i V} = O(m).$$

Indeed, according to [69], we can further prove

$$\frac{\langle N_L^A + N_R^A\rangle_M}{V} = 0$$

independently of small but nonzero mass M. Although their proof is elaborate and dependent also on another assumption to be made shortly, it can be directly inferred from (5.7) [69]. If $N_L^A + N_R^A$ is of order V_3, then such contribution of gauge configuration is suppressed by the factor $|\det M|^{O(V_3)}$. Thus in the presence of fermion and at large V_3-limit, such configuration is measure-zero.

The vanishing of second term implies nontrivial consequence for *Dirac eigenvalue distribution*, defined as

$$\rho(\lambda) := \frac{1}{V_4}\sum_n \delta(\lambda - \lambda_n). \tag{5.17}$$

Them the second term in (5.16) can be written as

$$\int \mathrm{d}\lambda \frac{m_i}{\lambda^2 + m_i^2}\rho(\lambda).$$

If $V \to \infty$ makes ρ continuous, then $m_i \to 0$ limit produces $\pi\langle\rho(0)\rangle_M$ and this must tend to 0. Whether it is continuous or not, or even analytic around $\lambda = 0$ or not, is still quite nontrivial, but above argument prohibits overpopulation of near-zero modes. We are encouraged to assume:

Assumption 2

$\rho(\lambda)$ defined in (5.17) is analytic around $\lambda = 0$.

Under these assumptions we have shown that

$$\langle\rho(0)\rangle_M = O(M^2) \tag{5.18}$$

Note that the right-hand side is $O(M^2)$, not $O(M)$ because of the background $SU(2)_L \times SU(2)_R$ invariance.

Proof of $f_A = 0$

Now it is straightforward to prove $f_A = 0$ using the dual effective action (5.10). Setting $M = \mathrm{diag}(m_u, m_d)$ again, consider the derivative of partition function with respect to m_u,

$$\frac{1}{V_4}\frac{\partial}{\partial m_u} Z_M(T, V_3, M). \tag{5.19}$$

From the expression of $P_M(A)$ (5.7), the microscopic expression for (5.19) is

$$\sum_n \langle \frac{m_u}{\lambda_n^2 + m_u} \rangle_M = \int_0^{\Lambda_{UV}} \frac{2m_u \ \langle\rho(\lambda)\rangle_M}{\lambda^2 + m_u^2} \mathrm{d}\lambda, \tag{5.20}$$

where Λ_{UV} is the UV-cutoff. Macroscopically, from (5.10), this is expressed as

$$2f_1 m_u + 2f_A m_d + O(m^3). \tag{5.21}$$

Taking $m_u \to 0$ limit reduces (5.20) to $\pi\langle\rho(0)\rangle_M$, which according to (5.18) is of $O(m_d^2)$, while macroscopically derived expression reduces to $2f_A m_d$. Thus,

$$2f_A m_d = O(m_d^2). \tag{5.22}$$

This completes the proof of Aoki-Fukaya-Taniguchi "theorem".

The assumption 2 on the analyticity of the Dirac-eigenvalue spectrum $\langle\rho(\lambda)\rangle$ around $\lambda \sim 0$ is physically less persuading than the assumption 1. Much effort has been made by lattice studies to determine the behavior of $\langle\rho(\lambda)\rangle$ [71, 72, 73, 74, 75, 76, 77, 78, 79], but there are obvious difficulties. First it was critical to employ a UV-regularization which maintain the full chiral symmetry $SU(2)_L \times SU(2)_R$, which had been unreachable before the introduction of overlap fermion. See [80] for a recent discussion of the relation between the Dirac eigenvalues and violation of Ginsparg-Wilson relation. Secondly and trivially, taking $V \to \infty$ limit is also required in making $\rho(\lambda)$ continuous. At this stage, enlarging lattice size with overlap fermion seems to be the only plausible way to actually guess the true nature of $\langle\rho\rangle$.

The relation between the absence of c_A and fully $U(1)_A$-restored LGW model is rather subtle [81]. Even with $f_A = 0$, there are still $U(1)_A$ violating interactions in (5.3). Apparently, since λ_x-term has exactly the same symmetry as c_A mass term, it can easily generate a c_A mass term under renormalization, which according to the above discussion is forbidden. To circumvent this, it is natural to assume the presence of symmetry, e.g. $\mathbb{Z}_4$ which rotates meson field (or $\mathbb{Z}_8$ for quark field), or irrelevance of such an interaction in IR. For λ_y-term, at this stage, there seems to be no principle to forbid its appearance. Hence, if this is indeed non-vanishing, the corresponding operator $\det \Phi^2 + \mathrm{c.c.}$ at fixed point (if any) must be an irrelevant one to achieve emergent $O(4) \times O(2)$. We will reconsider this point later.

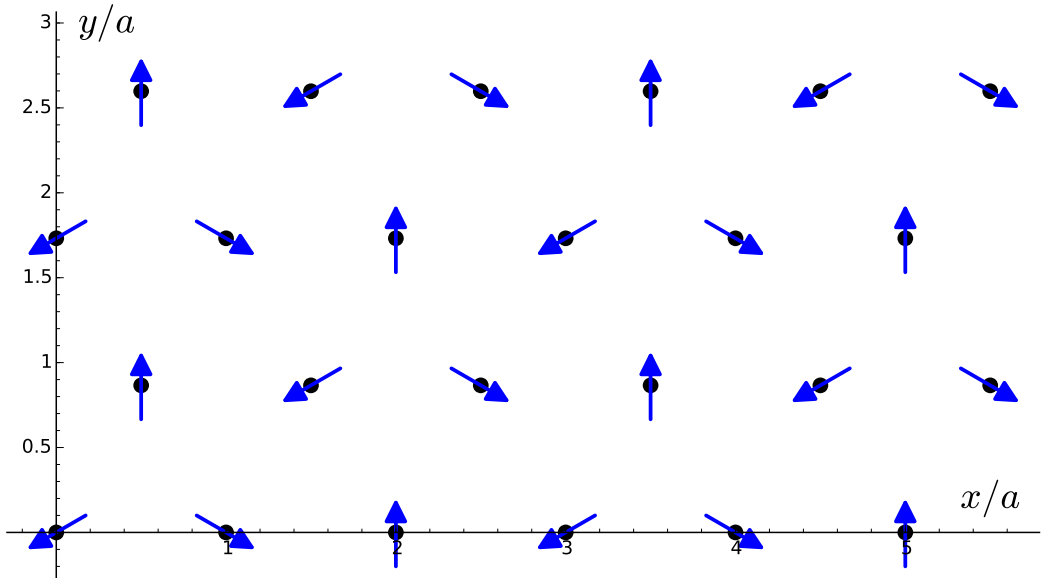

Figure 5.1: A two dimensional slice of stacked triangular lattice. The ground states are "120° configurations".

5.2 Frustrated spin system on triangular lattices

Introducing frustration is a simple but intriguing twist for spin systems: much complicated structure of ground-states brings much tough but interesting dynamics of the phase transition. Here we discuss the most simple class of frustrated spin systems, which is called *stacked triangular anti-ferromagnets* (STA). In this system, as its name indicates, anti-ferromagnetic n-component spins are placed on the two-dimensional triangular lattice (Fig. 5.1), which is piled up into the third direction to form a three-dimensional one. We denote the lattice spacing of triangular lattice as a and take fundamental lattice vectors to be

$$r_1 = (a, 0)\,,$$
$$r_2 = \left(\frac{a}{2}.\frac{\sqrt{3}a}{2}\right)$$

Whether interlayer interaction is ferromagnetic or anti-ferromagnetic will be irrelevant for the following discussion. Perhaps $n = 1$ ($\mathbb{Z}_2$-spin) is the simplest example of spin frustration, where for each triangle we can only have at most two anti-parallel pairings of spins. In this case, we have an infinite number of degenerate ground states in the large-volume limit. For $n > 1$, thanks to relaxation by the continuity of spin direction, situation is totally different.

5.2.1 Ground states

Here we follow [82] to discuss how ground states look like. To do this it is convenient to perform Fourier transform on the triangular lattice. Hamiltonian

$$J \sum_{\langle xj \rangle} S(x) \cdot S(y) \quad (J > 0)$$

will then be

$$JN \sum_k \tilde{S}(k) \cdot \tilde{S}(-k) \left\{ \cos(k \cdot r_1) + \cos(k \cdot r_2) + \cos(k \cdot (r_1 + r_2) \right\}, \tag{5.23}$$

where N is the number of sites (which at moment taken to be finite) and $\tilde{S}(k) = \sum_x e^{ik\cdot x} S(x)$ subjects to

$$\tilde{S}(-k) = \tilde{S}(k)^*, \tag{5.24}$$

$$\sum_{k,k'} \tilde{S}(k) \cdot \tilde{S}(k') e^{i(k+k')\cdot x} = 1. \tag{5.25}$$

In minimizing (5.23), since the condition (5.25) is a bit complicated, we replace the condition by the integrated form,

$$\begin{aligned} 1 &= \frac{1}{N} \sum_x \sum_{k,k'} \tilde{S}(k) \cdot \tilde{S}(k') e^{i(k+k')\cdot x} \\ &= \sum_k \tilde{S}(k)\tilde{S}(-k), \end{aligned} \tag{5.26}$$

and then check that the minimizing solution satisfies the original condition (this will be met if there is a sort of translational invariance). Now the task is rudimentary: introducing Lagrange multiplier λ for (5.26), we yield

$$\tilde{S}(k) \left\{ \cos(k \cdot r_1) + \cos(k \cdot r_2) + \cos(k \cdot (r_1 + r_2)) - \lambda \right\}. \tag{5.27}$$

Dotting (5.27) with $\tilde{S}(-k)$ and summing over k, we see that the total ground states energy should be $JN\lambda$, hence we have to minimize λ. If $\cos(k \cdot r_1) + \cos(k \cdot r_2) + \cos(k \cdot (r_1 + r_2))$ does not reach its minima, then $\tilde{S}(k) = 0$ by (5.27). Thus, the critical modes are at

$$k = \pm Q = \pm \frac{2\pi}{a}(2/3, 0), \tag{5.28}$$

and the ground states are of the form

$$S(x) = v_1 \cos(Q \cdot x) + v_2 \sin(Q \cdot x), \tag{5.29}$$

where $v_{1,2}$ are mutually orthogonal unit vectors and then (5.25) is satisfied. An example of these configuration for $n = 2$ is delineated in Fig.5.1.

5.2.2 emergent $O(2)$

The degeneracy of the ground states (5.29) is somewhat different from non-frustrated systems. Along with $O(n)$-rotation of entire spin, we can rotate $(v_1, v_2) \to (v_1, v_2)g$ by $g \in O(2)$. The connected part of $O(2)$ can be simply understood as the phase shift of the trigonometric function in (5.29). Since $O(n-2) \times O(2)_{\text{diag}}$ (the latter is a mixed subgroup of $O(n)$ and $O(2)$) preserves a particular choice of ground states, the moduli space is

$$O(n) \times O(2)/(O(n-2) \times O(2)_{\text{diag}}). \tag{5.30}$$

In this way, in writing the non-linear σ model description of the phase transition, new factor $O(2)$ emerges as a symmetry of its target space.

We can also explain the emergence of $O(2)$ in a more renormalization group theoretical fashion [83]. The first step is to employ the Hubbard-Stratonovich transformation trick to replace rigid spins to unconstrained ones. The Hamiltonian in terms of new spin variable $\phi(x)$ is

$$\int \mathrm{d}k \; P(k)\tilde{\phi}(k)\tilde{\phi}(-k) + \sum_x V(\phi(x)), \tag{5.31}$$

where $P(k) = \{r + \cos(k \cdot r_1) + \cos(k \cdot r_2) + \cos(k \cdot (r_1 + r_2))\}^{-1}$ is the inverse of kinetic operator in (5.23) with regularization parameter r, and the potential term $V(\phi) = \int_{S \in S^{n-1}} \mathrm{d}S \; \exp(-S \cdot \phi)$ appears

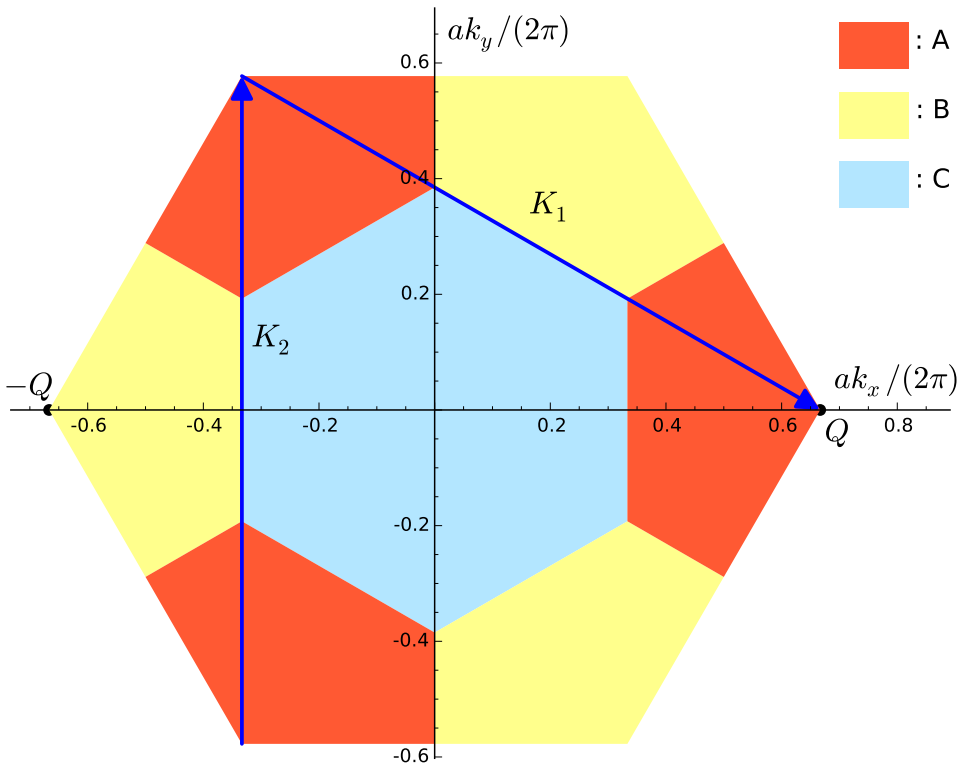

Figure 5.2: The first Brillouin zone of triangular lattice

in integrating out the original variable. Again the ground states are realized not on $k = 0$, but $k = \pm Q$ where Q is defined in (5.28). To explain the shell-momentum integration procedure, it is crucial to recognize the first Brillouin zone of the triangular lattice. Fundamental reciprocal lattice vectors

$$K_1 = 2\pi/a\left(1, -\frac{1}{\sqrt{3}}\right)$$
$$K_2 = 2\pi/a\left(0, \frac{2}{\sqrt{3}}\right)$$

also form a triangular lattice and the resulting quotient is hexagonal as in Fig.5.2. Under the identification by inverse lattice vectors, this hexagon can be neatly divided into three independent sub-hexagons A, B, and C, the center of which are $Q, -Q, 0$, respectively. The vector $Q = (2K_1 + K_2)/3$ defined in (5.28) describes the ground states, while the parallel configuration $k = 0$ corresponds to that of *largest* kinetic energy. Thus we want to retain the modes with momentum k around $\pm Q$, not around $k \sim 0$. Especially, in the RG flow into IR, entire region of C is integrated out in the procedure. In the final form, the effective Hamiltonian should look like

$$\int P'(k)(A(k)\cdot B(-k)) + \lambda_1 \int A(k_1)\cdot A(k_2)B(k_3)\cdot B(k_4)$$
$$+\lambda_2 \int A(k_1)\cdot B(k_2)A(k_3)\cdot B(k_4). \tag{5.32}$$

Here $A(k) = \tilde{\phi}(Q+k)$, and $B(k) = \tilde{\phi}(-Q+k)$ hence $A(k)^* = B(-k)$. In (5.32), each k-integration is performed in a tiny region centered at the origin and in the second and third terms, they are subject to $\sum_i k = 0$. The absence of A^4, A^3B, AB^3, B^4-like term is due to momentum-conservation rule in the original momentum variable: for example, A^4 like term has momentum $4Q + \sum k_i$, which cannot be proportional to any inverse-lattice vectors provided that the left region for momenta is small enough. At quartic order of spin variables, the action now has an additional symmetry $A(k) \to e^{i\theta}A(k)$ and $B(k) \to B(k)e^{-i\theta}$, resulting in $O(n) \times O(2)$-symmetry. Although $A^6(k)$-like term is compatible with

momentum-conservation, we expect such terms to be suppressed compared to the super-renormalizable interactions in (5.32).

5.3 Aspects of $O(n) \times O(m)$ Landau-Ginzburg-Wilson models

5.3.1 The model

Here ϕ transforms as a bi-fundamental representation of $O(n) \times O(2)$ indexed as ϕ_{ia}, where $i = 1, 2, \cdots, n$ and $a = 1, 2$. The most general Lagrangian is

$$\sum_{i=1}^{n} \sum_{a=1,2} \frac{1}{2} \partial_\mu \phi_{ia} \partial_\mu \phi ia + \frac{1}{2} m^2 \phi_{ia} \phi_{ia} + u \left(\sum_{i=1}^{n} \sum_{a=1,2} \phi_{ia} \phi_{ia} \right)^2 + v \sum_{i,j=1}^{n} \sum_{a,b=1,2} \left(\phi_{ia} \phi_{ja} \phi_{jb} \phi_{ib} - \phi_{ia} \phi_{ia} \phi_{jb} \phi_{jb} \right). \tag{5.33}$$

Note that the second contribution of the v-term is identical with the u-term. Thanks to this choice of v, the symmetry breaking pattern is directly connected to the sign of v. To discuss ground states, it is convenient to diagonalize the positive-semidefinite symmetric 2×2 matrix $\Phi := \phi_{ia} \phi_{ib}$ to $\mathrm{diag}(\Phi_1, \Phi_2)$ with $\Phi_1 \geq \Phi_2 \geq 0$. It is now elementary to minimize the potential in terms of Φ_i,

$$\frac{m^2}{2} (\Phi_1 + \Phi_2) + u (\Phi_1 + \Phi_2)^2 - 2v \Phi_1 \Phi_2. \tag{5.34}$$

First of all, to bound the potential from below, we require $u > 0$ and $2u \geq v$. The sign of v determines the symmetry-breaking pattern: when $v > 0$, vacuum occurs for $\Phi_1 = \Phi_2$ and unbroken symmetry is $O(n-2) \times O(2)$, and when $v < 0$, we have to set $\Phi_2 = 0$ and the unbroken symmetry is $O(n-1)$.

Thus, the LGW model relevant for $U(1)_A$-restored QCD chiral phase transition, the symmetry breaking pattern of which is

$$SU(2)_L \times SU(2)_R \times U(1)_A \simeq O(4) \times O(2) \to SU(2)_V \simeq O(3),$$

should be $n = 4$ with $v < 0$ model, while the one for n-component triangular lattice model has to have $v > 0$.

5.3.2 The status of RG studies: a chronological overview

Although the model introduced above is a straightforward generalization of $O(n)$-LGW models, its property under the RG flow is much more interesting. The phenomenologically relevant question is whether an IR-stable RG fixed point exists or not with the desired symmetry breaking pattern explained above. If it does, the phase transition can be (but do not have to be) of second order with the universality class characterized by the RG-fixed point, while the absence of such a fixed point concludes with a prediction of the first order transition. Thus the renormalization group theoretical aspects of the models are intensively studied. Here we briefly summarize the pre-bootstrap history of the issue. For a more complete summary and the discussion of experimental results for spin systems, we refer the reader to [84][85].

In [86] the beta functions of at two-loop order of the $\overline{\mathrm{MS}}$ scheme has been computed. Kawamura [83] analyzed these functions and found that depending on the value of n, there are four possible-types of RG flows in $d = 4 - \varepsilon$:

1. $n > n_{\mathrm{I}}(d) = 21.8 - 23.4\varepsilon + O(\varepsilon^2)$: In this case there are two additional fixed points with $v > 0$ to Gaussian and Heisenberg ones. The striking property is that one of them called the *chiral fixed point* is IR-**stable**. When n hits $n_I(d)$, the chiral fixed point co-annihilates (starting to acquire imaginary part) with the other unstable fixed point, which is called the *anti-chiral fixed point*. See Fig.5.3 for a plot of this type of flow. The beta functions used in the plots are the 3-loop results taken from [87].

2. $n_{\mathrm{I}}(d) > n > n_{\mathrm{II}}(d) = 2.20 - 0.57\varepsilon + O(\varepsilon^2)$:

   In this region there are only Gaussian and Heisenberg fixed points, and both are IR-unstable. The Gaussian one is unstable against both u, v while the Heisenberg one against v. See Fig.5.4.

3. $n_{\mathrm{II}}(d) > n > n_{\mathrm{III}}(d) = 2 - \varepsilon + O(\varepsilon^2)$:

   Again additional fixed points appear, but with $v < 0$, and one is stable and the other is unstable. The stable one is called *sinusoidal* or *collinear* fixed point, while the unstable one acquires the "anti" prefix, prosaically. See Fig.5.5.

4. $n_{\mathrm{III}}(d) > n$: As n hits n_{III}, the collinear fixed point moves towards the u-axis and cross it colliding with the Heisenberg fixed point. In the collision, the stabilities of the Heisenberg and the collinear fixed points are exchanged, making the Heisenberg fixed point stable, i.e, the v-term becomes irrelevant around the Heisenberg point. See Fig.5.6.

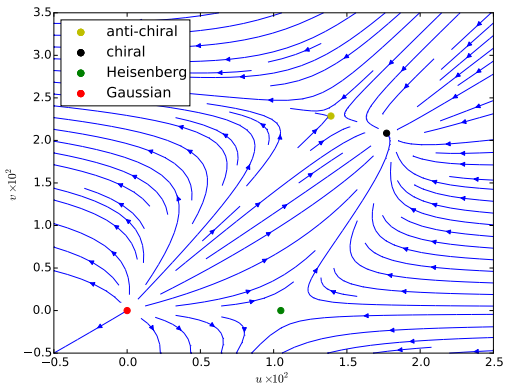

Figure 5.3: The RG flow of $d = 3.9$ (i.e., $\varepsilon = 0.1$) $O(25) \times O(2)$-LGW model. The arrows head toward IR.

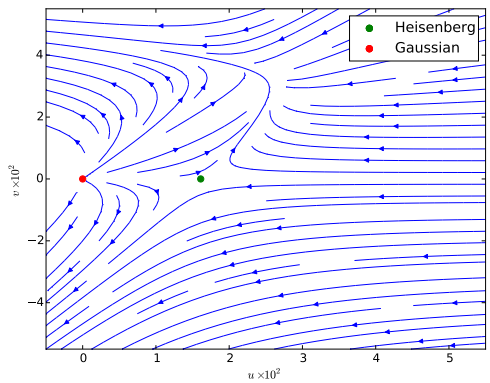

Figure 5.4: The RG flow of $d = 3.9$ (i.e., $\varepsilon = 0.1$) $O(15) \times O(2)$-LGW model. The arrows head toward IR.

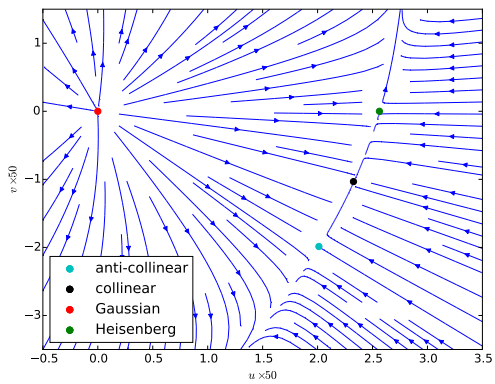

Figure 5.5: The RG flow of $d = 3.9$ (i.e., $\varepsilon = 0.1$) $O(2.14) \times O(2)$-LGW model. The arrows head toward IR.

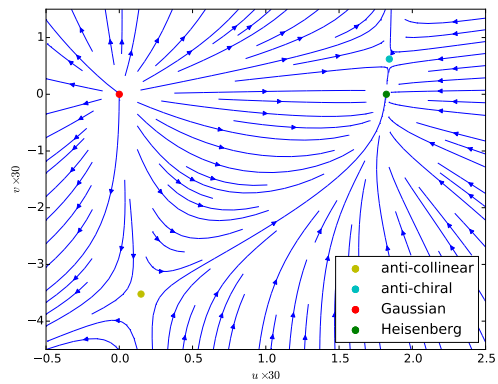

Figure 5.6: The RG flow of $d = 3.9$ (i.e., $\varepsilon = 0.1$) $O(1.5) \times O(2)$-LGW model. The arrows head toward IR.

Note that the terminology "chiral" for $v > 0$ fixed points is a conventional one. Rather it is its $v < 0$ counterpart that works for the QCD chiral phase transition.

So, at least when d is close enough to 4 and $n > n_{\mathrm{I}}$, there exists an IR-stable fixed point with symmetry breaking pattern $O(n) \times O(2) \to O(n-2) \times O(2)$, which could be a candidate to explain the phase transition of STA system. Kawamura [83], extrapolating the two-loop result $n_{\mathrm{I}} = 21.8 - 23.4\varepsilon$ to $\varepsilon = 1$, conjectured that the chiral fixed point continues to exist around $n \sim 2, 3$ (i.e., experimentally

accessible values) in $d = 3$ spatial dimensions. In particular, he explained unconventional type of phase transition reported experimentally through this fixed point.

In contrast, for the fixed point with $v < 0$ (which is suitable for QCD purpose), the allowed region of n for it to exist is $n_{\rm II}(d) > n > n_{\rm III}$, which is quite narrow if $\varepsilon \sim 0$. Extrapolation $\epsilon \to 1$ does not seem to help so much for broadening the region. Based on the RG result (though the beta functions are of one-loop and computed in the context of $U(n) \times U(n)$-LGW models), Pisarski-Wilczek [66] concluded that fully $U(1)_A$-restored chiral phase transition is of first order.

These conclusions were, however, just the beginnings of controversies for these systems. First of all, perturbative computation at even higher loops has been performed in [88][89]. The state of the art, five-loop result for $n_{\rm I}$ is

$$n_{\rm I}(\varepsilon) = 21.80 - 23.43\varepsilon + 7.09\varepsilon^2 - 0.03\varepsilon^3 + 4.26\varepsilon^4 + O(\varepsilon^5).$$

Although these expansion are only asymptotic in $\varepsilon = 4 - d$, we see that the higher-order terms contribute positively and extrapolation $\varepsilon = 1$ tends to imply $n_{\rm I} > 3$. Meanwhile the functional renormalization group approach for these system initiated in [90][91] for $n = 2, 3$ with $v > 0$ also imply the *absence* of the corresponding fixed points around these physical points. All these methods predict this value to be $n_{\rm I} = 5 \sim 6$.

On the contrary, in [92], the fixed-dimension $d = 3$ perturbation series (proposed in [93]) has been worked out for the model and they predicted fixed points for $n = 2, 3$, but seemingly without Borel-summability. The point is that these fixed points are disconnected to the one present in larger n (they also agree that these disappear somewhere around $n \sim 6$), and cannot be captured by the conventional scheme of ε-expansion. In later analysis, the use of beta-functions computed via $\overline{\rm MS}$ scheme also turned out to predict such fixed points with better Borel-summability, marginally agreeing with fixed-dimension series (see the tables in the next chapter) [94]. Again, in order to find these points, we must not persist in seeking the zeros of beta-function by expanding in ε, but rather should plug first $\varepsilon = 1$ in the $\overline{\rm MS}$ beta-function, and then solve it after the resummation.

For the analysis of $v < 0$, the situation is almost parallel. The functional RG studies for the $O(4) \times O(2)$-model was initiated in [95][96] and developed in various settings e.g., [97][98]. They tend to support the original conclusions of Pisarski-Wilczek (see however [99]) – that is, the system undergoes a first-order phase transition. The method has been applied also for $O(3) \times O(2)$ with $v < 0$ in [100], in the context of the ^{3}He-phase transition. Again on the contrary, $v < 0$ fixed points have been found within the perturbative scheme for $v > 0$ [101] in the condensed-matter context. Later in [102], the model was re-investigated in the context of QCD and stability has been verified within the fixed-dimension and $\varepsilon = 1$ $\overline{\rm MS}$ scheme.

Both the perturbative and functional RG schemes have their drawbacks. For the functional RG studies, although it is originally formulated as the one-loop exact ordinary differential equation formulated on the infinite-dimensional space of all possible action functional, in order to trace the flow numerically, we have to truncate this space to a finite-dimensional one. Indeed, it is reported in [99] that the existence of the fixed point is rather unstable against the change of truncation for $O(4) \times O(2)$ $v < 0$ theories[2]. Perturbative series too have serious problem from the beginning – they are only asymptotic and one has to go through the resummation procedure to obtain a sensible answer, in which we have to manually select real parameters. This point, arbitrariness of these real parameters, was critically examined in [104][105] and found to affect the final result in a significant way.

Thus, all what is clear is the necessity of other non-perturbative argument – like the conformal bootstrap.

[2] Very recently, in [103] the analysis for $O(3) \times O(2)$ with $v > 0$ has been reworked with higher-dimensional truncation space, again concluding the absence of the fixed points.

Chapter 6

Bootstrapping controversies

In this chapter we work out the bootstrap study for the CFTs with $O(n) \times O(2)$ symmetry, the motivation of which has been explained in the last chapter. What is physically important and controversial is the (non-)existence of the $d = 3$ fixed points which are not of $O(nm)$-Heisenberg or Gaussian type.

To carry this out we obtain numerical bounds by the methods described in Chapter 4. This is achieved in several steps. We first perform the group theoretical classification of relevant OPEs and the bootstrap equations in 6.1. After this setup we test the validity of the methods and check whether it works for non-Heisenberg type fixed points in a less-controversial region. To this end, we chose $n \gg 2$ models in $d = 3$ dimensions. The choice is made so that we have better control of critical exponents by means of large-n expansion, which has been computed to order $1/n^2$ in [87]. Note that while ε-expansion tends to be invalid for $\varepsilon = 1$, the large n method is legitimate (though asymptotic) whenever n is large. We will observe the intriguing behaviors of the bounds as in those witnessed in Chapter 4, but in a more elaborate way. After confirming the power of the bootstrap program for these non-controversial cases, we finally carry out the study of controversial region, $n = 3, 4$ models formulated in $d = 3$.

6.1 Setup : the bootstrap equation

The first step in carrying out the bootstrap analysis is to write down the bootstrap equation as in Section 4.3. With LGW-order parameter bi-fundamental field ϕ_{ia} of Section 5.3 in mind, we expect a natural extension of the bounds obtained for the Ising and $O(n)$-models to be those from

$$\langle \phi_{ia}(x_1)\phi_{jb}(x_2)\phi_{kc}(x_3)\phi_{ld}(x_4)\rangle. \tag{6.1}$$

Recall that $i, j, k, l = 1, 2, \cdots, n$ and $a, b, c, d = 1, 2$. In carrying out the conformal block decomposition of this four-point function, it is crucial to classify the additional structures labelled by the irreducible representations contained in the OPE. In this case, there are nine representations inside the OPE,

$$(\text{bi-fundamental}) \otimes (\text{bi-fundamental}) = \text{SS} \oplus \text{ST} \oplus \text{SA} \oplus \text{TS} \oplus \text{TT} \oplus \text{TA} \oplus \text{AS} \oplus \text{AT} \oplus \text{AA}. \tag{6.2}$$

Here for each label, the first letter represents the $O(n)$-representation under which the field is charged, and the second letter the $O(2)$-representation. So for example, SS is neutral under both $O(n)$ and $O(2)$, ST represents $O(n)$-singlet but traceless-symmetric tensor of $O(2)$ (or charge 2 in $U(1)$-terminology). Again index-exchange symmetry of a representation is correlated with the operator spins which it can include. It follows then that SS, ST, TS, TT, AA sectors contain the even spin contributions, while SA, AS, AT, TA odd ones.

The bootstrap equation too can be derived in a fashion entirely analogous to the $O(n)$-example, matching all the coefficients of Kronecker's deltas. In this case the possible independent Kronecker's delta is a product of $O(n)$ and $O(2)$ ones, so nine conditions appear. A straightforward computation leads to

$$\sum_{R\in(\text{bi-fund})^2} \sum_{\substack{O\in\phi\times\phi \\ O \text{ in } R}} \lambda^2_{\phi\phi O}\vec{V}_R(\Delta_\phi, \Delta_O, l_O; z, \bar{z}) = (0,0,0,0,0,0,0,0,0)^t,$$

where

$$\vec{V}_{\rm SS} := \begin{pmatrix} F^{(-)} \\ 0 \\ 0 \\ 0 \\ 0 \\ F^{(+)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{V}_{\rm ST} := \begin{pmatrix} \frac{-2}{m} F^{(-)} \\ F^{(-)} \\ F^{(-)} \\ 0 \\ 0 \\ -\frac{2}{m} F^{(+)} \\ F^{(+)} \\ F^{(+)} \\ 0 \end{pmatrix}, \qquad \vec{V}_{\rm SA} := \begin{pmatrix} 0 \\ -F^{(-)} \\ F^{(-)} \\ 0 \\ 0 \\ 0 - F^{(+)} \\ F^{(+)} \\ 0 \end{pmatrix}$$

$$\vec{V}_{\rm TS} := \begin{pmatrix} -\frac{2}{n} F^{(-)} \\ 0 \\ F^{(-)} \\ F^{(-)} \\ 0 \\ 0 \\ -\frac{2}{n} F^{(+)} \\ 0 - F^{(+)} \\ F^{(+)} \end{pmatrix}, \qquad \vec{V}_{\rm TT} := \begin{pmatrix} \left(1 + \frac{4}{nm}\right) F^{(-)} \\ \left(1 - \frac{2}{n}\right) F^{(-)} \\ \left(-\frac{2}{n} - \frac{2}{m}\right) F^{(-)} \\ \left(1 - \frac{2}{m}\right) F^{(-)} \\ F^{(-)} \\ \left(1 + \frac{4}{nm}\right) F^{(+)} \\ \left(-1 - \frac{2}{n}\right) F^{(+)} \\ \left(-\frac{2}{n} + \frac{2}{m}\right) F^{(+)} \\ \left(-1 - \frac{2}{m}\right) F^{(+)} \end{pmatrix}, \qquad \vec{V}_{\rm TA} := \begin{pmatrix} F^{(-)} \\ \left(-1 + \frac{2}{n}\right) F^{(-)} \\ -\frac{2}{n} F^{(-)} \\ F^{(-)} \\ -F^{(-)} \\ -F^{(+)} \\ \left(1 + \frac{2}{n}\right) F^{(+)} \\ -\frac{2}{n} F^{(+)} \\ -F^{(+)} \end{pmatrix}$$

$$\vec{V}_{\rm AS} := \begin{pmatrix} 0 \\ 0 \\ F^{(-)} \\ -F^{(-)} \\ 0 \\ 0 \\ 0 \\ -F^{(+)} \\ -F^{(+)} \end{pmatrix}, \qquad \vec{V}_{\rm AT} := \begin{pmatrix} F^{(-)} \\ F^{(-)} \\ -\frac{2}{m} F^{(-)} \\ \left(-1 + \frac{2}{m}\right) F^{(-)} \\ -F^{(-)} \\ -F^{(+)} \\ -F^{(+)} \\ \frac{2}{m} F^{(+)} \\ \left(1 + \frac{2}{m}\right) F^{(+)} \end{pmatrix}, \qquad \vec{V}_{\rm AA} := \begin{pmatrix} F^{(-)} \\ -F^{(-)} \\ 0 \\ -F^{(-)} \\ F^{(-)} \\ -F^{(+)} \\ F^{(+)} \\ 0 \\ F^{(+)} \end{pmatrix}. \tag{6.3}$$

Since this bootstrap equation is quite lengthy, the solver requires about $O(100)$-times as long runtime as for the Ising model bootstrap equation, if the cutoff parameter Λ is equal.

A technical remark is that for $O(4) \simeq SU(2) \times SU(2)$, the anti-symmetric representation ("A") is in fact a direct sum of two irreducible representations, each of which satisfies self-duality condition. For the model of interest, $O(4) \times O(2) = SU(2)_L \times SU(2)_R \times U(1)_A$-symmetric LGW model, however, we expect the discrete symmetry exchanging $SU(2)_L$ and $SU(2)_R$ (which is a parity transformation, microscopically) to be present, so there is a relation among the operator contents and OPE coefficients. Encapsulating these related channel reduces the crossing relation to (6.3).

6.2 $O(n) \times O(2)$ with $n \gg 2$: uncontroversial region

Before directly plunging into the study of theories with great physical interest $n = 3, 4$, we first check the validity of the bootstrap program to the problems under better-control, yet non-trivial physics emerges. We choose $O(n) \times O(2)$ models with $n \gg 2$ as the starter[1], as we expect $1/n$-expansion becomes precise in such a region. According to the large n-ansatz, the fixed points found around $d = 4 - \varepsilon$ in Section 5.3 continue to exist in $d = 3$. There two additional fixed points to Heisenberg and Gaussian ones are present, i.e., the chiral and anti-chiral fixed point. Natural question is then, *can we find these additional fixed points by means of the conformal bootstrap, as in the case of the Ising and the $O(n)$ models?* Below we carry out this theoretical experiment.

[1] In our original paper [17], the symmetry group $O(n) \times O(3)$ was chosen, because $O(n) \times O(2)$ seemed too controversial then.

6.2.1 $O(10) \times O(2)$

Let us begin with the symmetry group with $n = 10$, i.e., $O(10) \times O(2)$. The choice $n = 10$ has no physical meaning but lies well beyond the existence limit of these fixed points $n \sim 6$ predicted by the other methods. First we consider the bounds for $O(10) \times O(2)$-singlet operator (i.e., dimension of the mass term if the model under consideration is a LGW model). The result is plotted in Fig.6.1 as blue dots. They turned out to be equal (within our precision goal of 5×10^{-4}) with those plotted as the

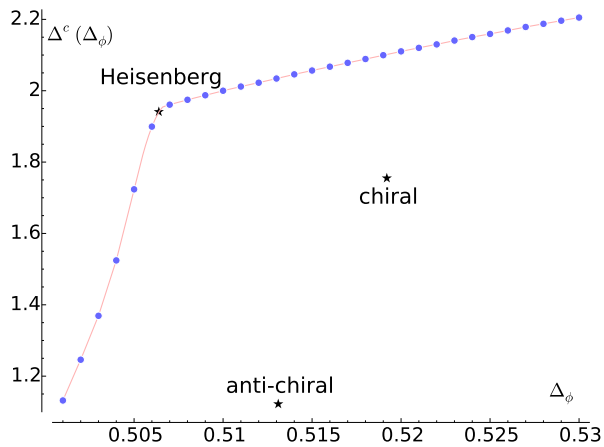

Figure 6.1: Blue dots: The upper bound for the dimension of the first operator appearing in SS-channel of $O(10) \times O(2)$ bi-fundamental OPE with itself. Red line: The bound for $O(20)$-S operator in $O(20)$ vector OPE with itself. For $O(20)$, the original data are discrete and step is 0.001, but they are refined to be 0.0002 around kink location $\Delta_\phi \sim 0.506$. The binary precision goal is 5×10^{-4}. Considered spins are $0, 1, \cdots, 21, 24, 25, 28, 29, \cdots, 40, 41$, and $\nu_{\max} = 18$ is taken. Parameters can again be the SDPB default set in [12].

red line, which is the bound for $O(20) = O(10 \times 2)$-singlet operator dimension in $O(20)$ vector$\times$vector OPE (this has been already considered in [47], by the way). Since $O(20)$ can be artificially decomposed in such a way that $O(20)$-vector becomes $O(10) \times O(2)$bi-fundamental, any consistent CFT spectra of $O(20)$ lead to those for $O(10) \times O(2)$, and it trivially follows that $\Delta^c_{O(20),\mathrm{S}} \leq \Delta^c_{O(10)\times O(2),\mathrm{SS}}$. However, the equality $\Delta^c_{O(20),\mathrm{S}} = \Delta^c_{O(10)\times O(2),\mathrm{SS}}$ has not been proven analytically. Such an agreement was also observed between the bound of the singlet operators for $SU(N)$ and $SO(2N)$ [50].

Anyway, we are obliged to conjecture that the equality is valid at least in this case, and although It has a sharp kink around $\Delta_\phi = 0.506$, it should represent the $O(20)$-Heisenberg fixed point.

On the other hand, large $1/n$-expansion for $O(n) \times O(2)$ models predicts the critical exponents for chiral fixed points to be

$$\begin{aligned}
\Delta_\phi &= \frac{1}{2} + \frac{2}{\pi^2}\frac{1}{n} + \frac{-32}{3\pi^4}\frac{1}{n^2} + O(1/n^3), \\
\Delta_{\mathrm{SS}} &= 2 - \frac{16}{\pi^2}\frac{1}{n} - \left(\frac{56}{\pi^2} + \frac{128}{3\pi^4}\right)\frac{1}{n^2} + O(1/n^3),
\end{aligned} \tag{6.4}$$

and anti-chiral

$$\begin{aligned}
\Delta_\phi &= \frac{1}{2} + \frac{4}{3\pi^2}\frac{1}{n} - \frac{112}{27\pi^4}\frac{1}{n^2} + O(1/n^3), \\
\Delta_{\mathrm{SS}} &= 1 + \frac{32}{3\pi^2}\frac{1}{n} + \frac{4096}{27\pi^4}\frac{1}{n^2} + O(1/n^3),
\end{aligned} \tag{6.5}$$

where we have used the conversion rule for correlation-length exponent ν and Δ_{SS},

$$\Delta_{\mathrm{SS}} = 3 - \frac{1}{\nu} \tag{6.6}$$

and η and Δ_ϕ,

$$\Delta_\phi = \frac{d-2+\eta}{2}. \tag{6.7}$$

These predictions to order $1/n^2$ for $n = 10$ are also depicted in Fig.6.1, and both are well-beneath the bounding line. As a matter of course, this bound is to be saturated by all $O(10) \times O(2)$-symmetric CFTs including these, and this result is perfectly consistent. However, what surprised us in Chapter 4 is saturation of bounds by actual LGW-model fixed points, and from this perspective, the result is not satisfactory.

spin 1 operator in the TA sector

This is not the end of the adventure, however. We have a lot more to compute, i.e., the bounds for operators in the other channels of global symmetry representations, like we did for $O(2)$ T-representation in Section 4.3. In this case, to gain some information about the extent of symmetry-breaking, we have first computed the operator dimension bound in TA-spin 1 sector. Note that lowest-dimensional operator in this sector has dimension exactly 2 when the full $O(20)$-symmetry is maintained, but has positive anomalous dimension on proper $O(10) \times O(2)$-symmetric CFTs. The numerical result is shown in Fig.6.2. There is a slope-changing behavior around $\Delta_\phi = 0.513$, as Fig.6.10 will suggest. The hori-

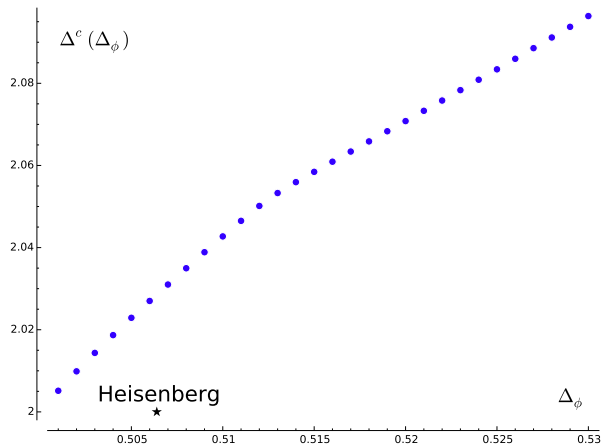

Figure 6.2: The upper bound for the dimension of the first operator appearing in the TA-sector spin 1 channel. The same parameter as in Fig.6.1 setting works. $O(20)$ Heisenberg fixed points has the dimension exactly 2 for this operator, because on the point, TA spin 1 operators combine with the ones in TA, SA and AS sector to form A-representation of $O(20)$, lowest of which is the $O(20)$-conserved current.

zontal location of this "kink" looks reasonably close to the large-n prediction of the anti-chiral fixed points for $n = 10$ in 6.5,

$$\Delta_\phi^{(\text{large } n)} = 0.513, \tag{6.8}$$

$$\Delta_{\text{SS}}^{(\text{large } n)} = 1.14. \tag{6.9}$$

Let us provide a further piece of evidence for the saturation of the bounds by the anti-chiral fixed point.

Spectral study for the kink

In [106], a clever use of the bootstrap output linear functional is proposed and applied to $d = 2$ Ising model, to reproduce the entire low-lying spectrum contained in the $\phi \times \phi$ OPE. This argument makes the full-use of the singular property of CFT spectra *saturating* the bootstrap bounds. To do this,

consider the variant of the positivity argument (without any global symmetry, for simplicity), which is equivalent to the formulation of Section 4.2. We seek for α satisfying

$$\alpha(F(\Delta, l = \text{artificially chosen value above the unitarity bound})) = 1, \tag{6.10}$$
$$\alpha(F(\Delta_\phi, \Delta, l; *)) \geq 0 \text{ whenever } \Delta \geq l + d - 2,$$
$$\alpha(F(\Delta_\phi, \Delta, 0; *)) \geq 0 \text{ whenever } \Delta \geq \Delta_h, \tag{6.11}$$

where the first condition serves merely as the normalization condition for α, and the search for the one which maximizes the identity contribution

$$\text{obj} := \max(\alpha(F(0,0))). \tag{6.12}$$

If we find such an α with obj > 0, again we conclude the presence of operator under Δ_h. Equivalence of the previous formulation is just a rescaling then. As we try to choose the smaller Δ_h, obj trivially decreases as the search space for α shrinks, and at some point, obj becomes 0. This is the optimal value for the bootstrap dimension bound Δ^c.

Now, assume that a CFT is saturating the bound. Then, action of optimal α on the crossing relation $\sum_O \lambda^2_{\phi\phi O} F(\Delta_O, l_O; z, \bar{z}) = 0$ is,

$$\alpha(F(0,0)) + \sum_{\Delta_O \geq \Delta^c} \lambda^2_{\phi\phi O} \alpha(F(\Delta_O, l_O)) = \text{obj} + \sum_{\Delta_O \geq \Delta^c} \lambda^2_{\phi\phi O} \alpha(F(\Delta_O, l_O)) = 0$$

From the definition, the first term is 0 and the second contribution is non-negative. Thus, if a bound-saturating CFT exists, such a CFT is highly-constrained, in a sense that whenever $\lambda_{\phi\phi O} \neq 0$ $\alpha(F(\Delta_O, l_O)) = 0$. In other ways, if we start from α with these properties, we can list up the potential candidate for operators which are allowed to appear in $\phi \times \phi$ OPE by the condition $\alpha(F(\Delta_O, l_O)) = 0$. This method was extensively used to derive a spectrum (including those which have not been accessible before the pre-bootstrap era) in the $d = 3$ Ising model [10]. The generalization to the case with global symmetry is just an easy exercise.

A caveat in the argument is that we are assuming the ideal linear functional out of the infinite-dimensional search space is available there. Since we can only handle the finite-dimensional search space, in order to apply the method in a numerical fashion, we have to resort to the assumption that the functional obtained with a finite-cutoff Λ is appropriately capturing the character of the ideal one.

Let us apply the above method (called "extremal functional method") to the hypothetical CFT located on the kink. After a binary search for the best bound at $\Lambda = 15$ with goal 5×10^{-5}, we collected the SDPB output and applied the output functional to polynomially-approximated $\vec{V}_{\text{SS}}(\Delta, 0)$. An example of resulting polynomial is depicted in the Fig.6.3, where we observe a sharp minimum. Locations of such minima are depicted in Fig.6.4 with varying Δ_ϕ around the kink.

Unlike the rigorous bound for the TA-sector, from which we have read the functional off, these values do not tell us anything exact. The source of error comes from the horizontal-impreciseness in locating the kink (about ± 0.001) and the finite-Λ effect, and we are forced to guess the latter value from the lower-Λ result[2]. In this way, we predict AT sector kink-saturating CFT have an SS scalar operator with dimension

$$\Delta^{\text{SS}} = 1.15(3),$$

to be compared reasonably with the large-n prediction in (6.9).

6.2.2 Other sectors

We have then tried the bounds for other sectors. We tried ST, TS, TT, with spin 0 and AT with spin 1 (SA and AS spin 1 include the conserved current as their lowest-dimensional operator, so we did not try it). Out of them, ST and TS looked intriguing. First for ST, the bound is shown in Fig.6.5. The bound shows a slope change (though not as sharp in that in TA-sector, and we expect this to be

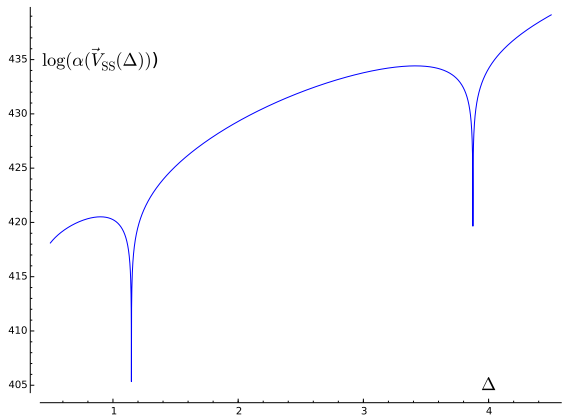

Figure 6.3: Linear functional out put of TA-sector bounds at $\Delta_\phi = 0.513$ (Fig.6.2) acting on $\vec{V}_{\rm SS}(\Delta, 0)$ to read off the SS spectrum. Rather large exponents are due to the polynomial approximation (with prefactor omitted for simplicity). Note that the second minima exceeds the $\Delta^c_{\rm SS}$ around Δ_ϕ in Fig.6.1, so the first minimum must correspond to an actual operator.

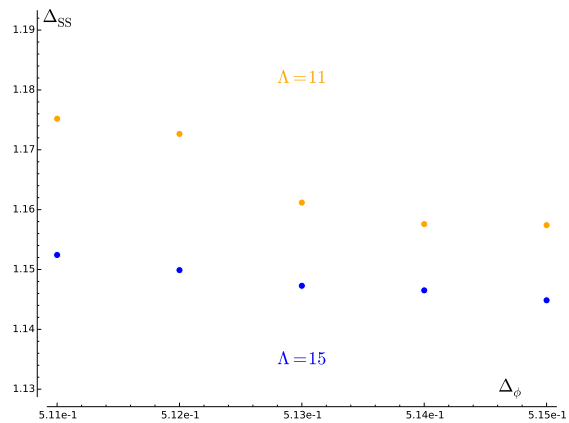

Figure 6.4: The location of the first minima of $\alpha(\vec{V}_{\rm SS}(\Delta, 0))$, with varying Δ_ϕ. For the estimation of error, we have also performed binary-search for $\Lambda = 11$, where the parameter is the same as that in $\Lambda = 15$ except $\nu_{\rm max} = 16$.

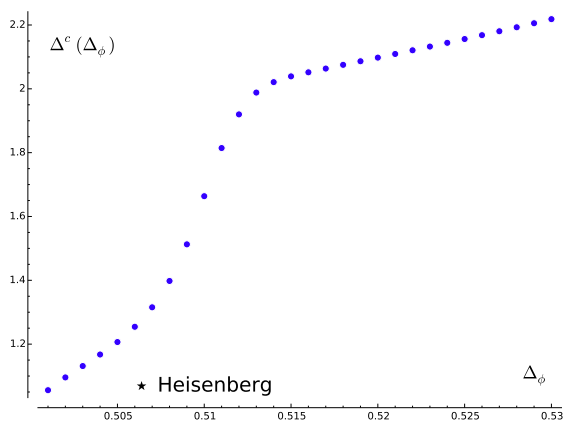

Figure 6.5: ST-sector bounds for $O(10) \times O(2)$. The parameter settings are the same as in Fig.6.1.

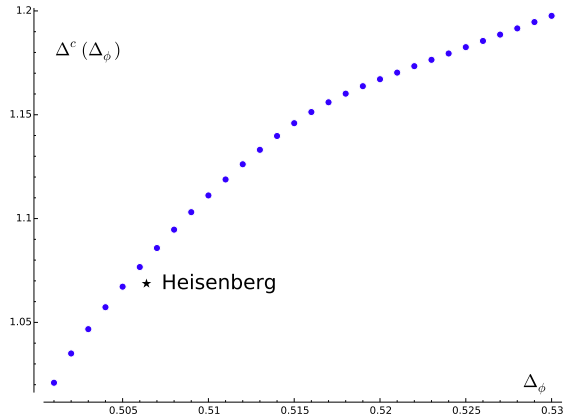

Figure 6.6: TS-sector bounds for $O(10) \times O(2)$. The parameter settings are the same as in Fig.6.1.

sharpen as we take larger Λ) in the region $\Delta_\phi = 0.513 \pm 0.002$. Then from the result of the spectral study Fig.6.7, we estimate Δ^{SS} to be

$$\Delta_{\mathrm{SS}} = 1.17(10).$$

Though consistent with (6.9), this is less impressive than the result from TA-sector.

Meanwhile the bounds for TS-sector spin 0 operator shows an intriguing but not as sharp behavior around the region $\Delta_\phi = 0.518 \pm 0.03$, from which it starts growing linearly. Such a behavior is also observed for $O(n)$-T sector bounds for large n in Figure 3 of [47]. If we assume the around the presence of a CFT saturating this bound, such one should have

$$\Delta_{\mathrm{SS}} = 1.79(10)$$

as is implied from Fig.6.8. Plugging $n = 10$ to the large-n expression (6.4) produces

$$\Delta_\phi^{(\text{large } n)} = 0.519, \tag{6.13}$$

$$\Delta_{\mathrm{SS}}^{(\text{large } n)} = 1.75, \tag{6.14}$$

and is included well in our prediction for TS bound-saturating CFTs $(\Delta_\phi, \Delta_{\mathrm{SS}}) = (0.518(2), 1.79(10))$.

It would be interesting to carryout the large-n analysis for the dimensions of operators in the sector other than SS, to see whether they agree.

6.2.3 Predicting conformal window

So far everything is fine – the bootstrap approach reproduces the critical exponents of $O(10) \times O(2)$ LGW model's non-Heisenberg fixed points, so the answer to the above stated question seems "yes".

Let us then lower the value n to see what happens there. All theoretical approaches (ϵ-expansion, fixed-dimension series, $\overline{\mathrm{MS}}$ without ϵ-expansion and functional RG) predict the lower existence limit for (anti-)chiral fixed point, on which anti-chiral and chiral merge into single fixed point and below which they no longer exist.

To see how this fact is reflected in the bootstrap outputs, we use the bounds for TA-sector as a prove because it was the most sharp one for $n = 10$. We computed TA bounds for $n = 9, 8, 7, 6, 5$. The results is presented in Fig.6.9. To see how slope is changing for each n, we also present in Fig.6.10 the plot of its first derivatives interpolating the plot with piecewise linear polynomial class in `scipy.interpolate`. As we decrease n, while for $n = 9, 8$ there still is a definite slope changing, for $n = 7, 6$ it is not clear to us whether they should be regarded as the "kink". For $n = 5$ the kink is completely absent. Thus we predict that the edge of the conformal window for the anti-chiral

[2]However, the author doesn't know an example in which this procedure bowls a wide.

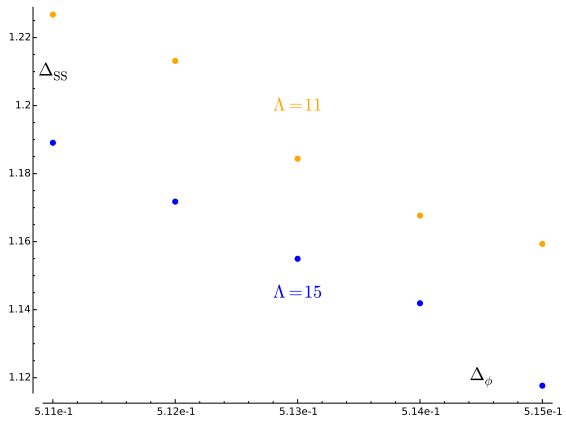

Figure 6.7: Spectra read off from the bounds for ST-sector. The setting is equal to that in Fig.6.4.

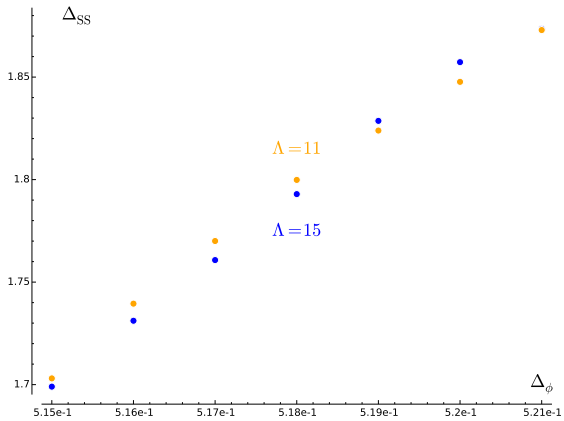

Figure 6.8: Spectra read off from the bounds for ST-sector. The setting is equal to that in Fig.6.4.

fixed points is somewhere between $5 \sim 7$. This is to be compared with the Table I of [89], which summarizes the earlier values for the predictions from the other approaches, all higher-order analysis lies within our prediction. It would be interesting to further pursue the value of existence limit using the bootstrap program with increased cutoff Λ.

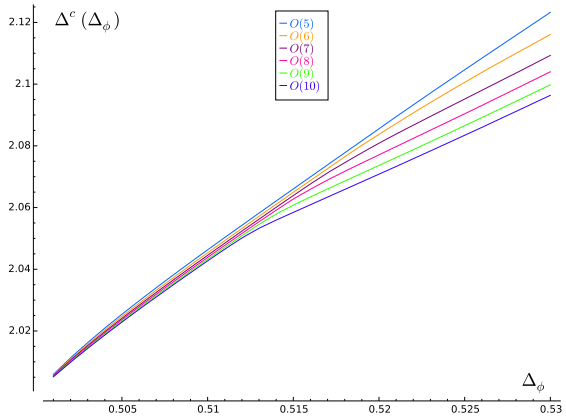

Figure 6.9: TA-sector bounds for $O(n) \times O(2)$ with varying n. Horizontal step 0.001 is hidden to help the reader's eye. The parameter settings are the same as in Fig.6.1.

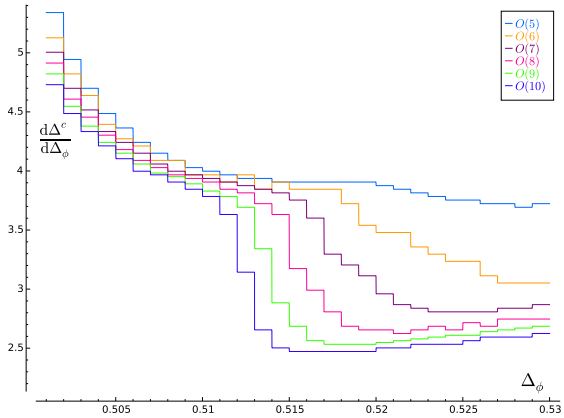

Figure 6.10: The slope of of Fig.6.9.

6.3 $O(3) \times O(2)$ and $O(4) \times O(2)$: controversial region

6.3.1 $O(3) \times O(2)$: implications for frustrated spin systems

We first present our results for $O(3)\times O(2)$-symmetric CFTs, which would describe the anti-ferromagnetic Heisenberg spin systems placed on stacked-triangular lattices, as discussed in Section 5.2. For this, resummed perturbation series studied in [92][94],[107], and [101] predict the presence of two stable fixed points called chiral ($v > 0$ in (5.33)) and collinear ($v < 0$ in (5.33)) fixed points in addition to Gaussian and Heisenberg ones. On the other hand, the functional RG analyses in [90] (for chiral) and [100](for collinear) predict the absence of such fixed points.

In Figs.6.11 and 6.12, we present the bounds for ST and AA-sector. In both plots we observe kinks, the location of which is determined as the center point of the domain of rapid slope change. Another intriguing feature of the kinks is the rapidity of the convergence to optimal bounds around them: in generic region, the larger Δ_ϕ, the more slowly the bound approaches its optimal value, but around the kinks they tend to converge to the optimal result fastest in their neighborhood. From these we read off the kink location for ST to be $\Delta_\phi = 0.539(3)$ and for AA $\Delta_\phi = 0.536(3)$, where we claim that interacting CFTs are located.

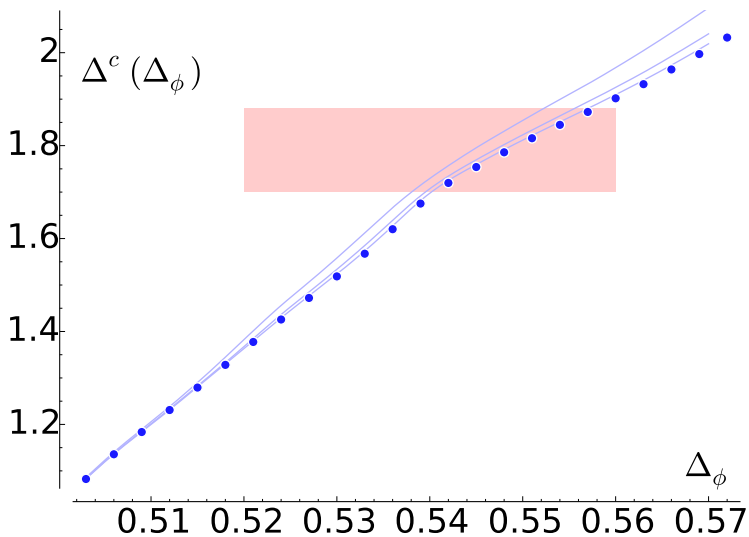

Figure 6.11: The bounds for ST-sector operator dimensions in $O(3) \times O(2)$-CFTs. The blue dots and the lower-most line correspond to the bounds obtained with $\Lambda = 27, 23$ using SDPB, while the other lines represent those for $\Lambda = 15, 19$ obtained via sdpa-gmp in [18]. The light red rectangle represents the combined prediction of [107] and [101] for these exponents using the $\overline{\text{MS}}$-series.

As in the previous section, we can read off the spectrum of the bound-saturating operator from the bootstrap output. We computed the low-lying scalar spectra in all the intermediate channels which we denote by Δ_R, and summarize the values in Tables 6.1 and 6.2.

While the two kinks that appeared in the two sectors are close in the values of Δ_ϕ, given the spectra of the first operators in various OPE channels, we conclude that they represent two distinct CFTs. In the tables, we also quote the values of Δ_ϕ and Δ_Rs for each representation channel. The converting formulae for Δ_{SS}, Δ_ϕ are again given by equations(6.6-6.7). The values of other channel operators are taken from Table III of [101][3]. Our spectra for the kink in the bounds for TS-sector agree with the resummed 5-loop expansions in $\overline{\text{MS}}$ scheme (without ε-expansion) for the chiral fixed point within the systematic error, and those for AA also agree with their results for the collinear fixed point. Although the comparison with the six-loop expansions in MZM scheme is less impressive, it was anticipated in [101] because of the better Borel-summability of $\overline{\text{MS}}$ series. Our spectra do not show further relevant operators in the SS sector in agreement with the claim that these fixed points are

[3] The exact correspondence is $\Delta_{\text{AA}} = 3 - y_1$, $\Delta_{\text{TT}} = 3 - y_2$, $\Delta_{\text{TS}} = 3 - y_3$ and $\Delta_{\text{ST}} = 3 - y_4$.

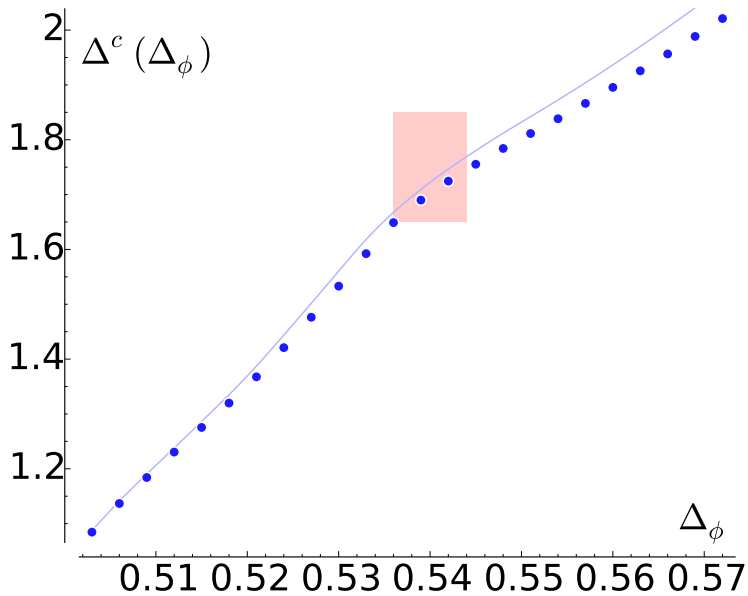

Figure 6.12: The bounds for AA-sector operator dimensions in $O(3) \times O(2)$-CFTs. The blue dots correspond to the bounds obtained with $\Lambda = 19$, while the line represents those for $\Lambda = 15$, both of which were obtained via sdpa-gmp in [18]. The light red rectangle represents the prediction for these exponents by the MZM-series [101].

stable. We, however, do not find any indication that the chiral fixed point has a focus point behavior reported in [94].

While the correspondence between kinks and actual CFTs has not been proven yet, we emphasize that our results (as in all the conformal bootstrap studies) are obtained without any reference to the RG analysis based on the Lagrangian (5.33). We believe that the most natural explanation for such an agreement is the actual existence of these CFTs. We are therefore led to the conjecture: *both the chiral and collinear fixed points for $O(3) \times O(2)$ LGW model exist and saturate the bound for ST and AA-sector, respectively.* The presence of the chiral fixed point with the symmetry breaking pattern $O(3) \times O(2) \to O(2)$ implies that the phase transition in frustrated spin systems can be continuous, as was first conjectured by Kawamura[83].

In the other channels, we do not see any interesting behaviors. The kink present in TA-spin 1 channel for $n = 10$ but disappeared around $n \sim 6$ still do not show up again for $n = 3, 4$. On the other hand, in $\Delta^{c,\mathrm{AA},0}$, where we observed a kink for $O(3) \times O(2)$, we do not find any interesting behaviors for $O(10) \times O(2)$. It is plausible that such a "switching" behavior of the sectors showing kinks is a reflection of the picture that the anti-chiral fixed point merges into chiral one when n is below the value n_{I} continued from $4 - \varepsilon$ dimension, and a qualitatively different fixed point (i.e., collinear fixed point) emerges for n below another critical value.

6.3.2 $O(4) \times O(2)$: kinks and spectral studies

We present our results showing the bounds for ST-sector and AA-sector in Fig.6.13 and 6.13. In Tables 6.3 and 6.4, we list the spectra at the kink ($\Delta_\phi \simeq 0.530$ for ST and $\Delta_\phi \simeq 0.558$ for AA). From this comparison we find it reasonable to regard the kink in the ST-sector bounds as the chiral fixed point and that in AA as the collinear fixed point. Hence our non-perturbative results in agreement with the higher-loop analysis in RG provides a strong support for the existence of the chiral as well as collinear fixed point, and the latter, in particular, suggests the possibility of continuous chiral phase transition in QCD once the $U(1)_A$ is restored at the level of chiral effective Lagrangian (5.3).

Note that, unlike the earlier prediction, our prediction for the $\Delta_{\mathrm{SS}} \simeq 1.52(5)$ significantly differs from $\Delta_{\mathrm{S}} \simeq 1.67(1)$ of the $O(4)$ Heisenberg universality class and could offer a clue for judging $U(1)_A$

	Δ_ϕ	$\Delta_{\rm SS}$	$\Delta_{\rm ST}$	$\Delta_{\rm TS}$	$\Delta_{\rm TT}$	$\Delta_{\rm AA}$
Bootstrap	0.539(3)	1.42(4)	1.68(6)	1.39(3)	1.113(3)	0.89(2)
$\overline{\rm MS}$	0.54(2)	1.41(12)	1.79(9)	1.46(8)	1.04(11)	0.75(12)
MZM	0.55(1)	1.18(10)	1.91(5)	1.49(3)	1.01(4)	0.65(13)

Table 6.1: The low-lying spectra read off around the $O(3) \times O(2)$ ST-bound saturating CFT with $\Lambda = 27, 23$ using SDPB. The spectra for the $O(3) \times O(2)$ chiral fixed point are taken from [107] (Δ_ϕ and $\Delta_{\rm SS}$) and [101] (others).

	Δ_ϕ	$\Delta_{\rm SS}$	$\Delta_{\rm ST}$	$\Delta_{\rm TS}$	$\Delta_{\rm TT}$	$\Delta_{\rm AA}$
Bootstrap	0.536(3)	1.43(4)	0.90(2)	1.003(5)	1.228(3)	1.65(5)
$\overline{\rm MS}$	0.543(12)	1.43(20)	0.9(2)	1.0(1)	1.25(5)	1.8(1)
MZM	0.540(4)	1.31(10)	0.95(15)	1.0(2)	1.25(10)	1.75(10)

Table 6.2: The low-lying spectra of the $O(3)\times O(2)$ AA-bound saturating CFT read off from $\Lambda = 19, 15$ result obtained with sdpa-gmp out put in [18]. The spectra for the $O(3) \times O(2)$ collinear fixed point are taken from [108] (Δ_ϕ and $\Delta_{\rm SS}$) and [101] (others).

restoration scenario itself in the future lattice and experimental studies of the correlation length.

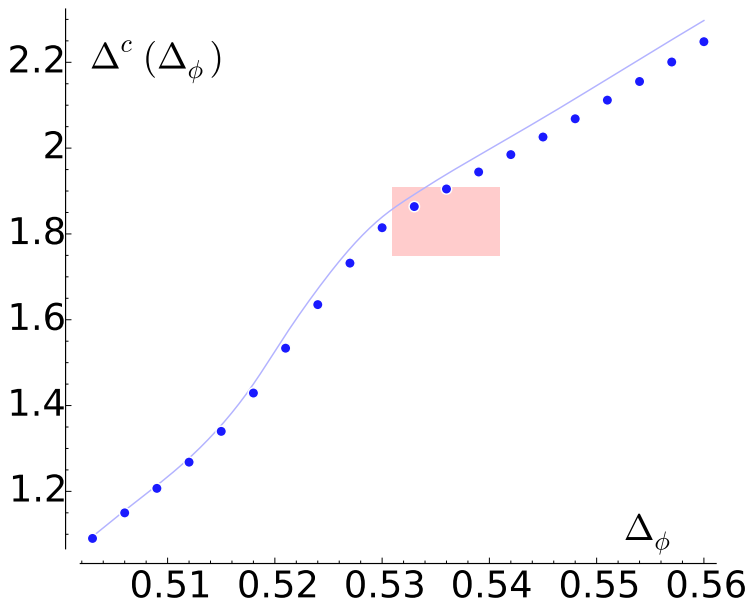

Figure 6.13: The bounds for ST-sector operator dimensions in $O(4) \times O(2)$-CFTs. The blue dots correspond to the bounds obtained with $\Lambda = 19$, while the lines represent those for $\Lambda = 15$, both of which were obtained via sdpa-gmp in [18]. The prediction from MZM-series in [101] is plotted in a light red rectangle.

6.3.3 Bootstrap^2: Refined scenarios for the chiral phase transitions

As remarked at the end of Section 5.2, when the $U(1)_A$ non-neutral interactions proportional to $\lambda_{x,y}$-terms in the effective action (5.3) are present, the RG evolution process does not necessarily end up with the $O(4) \times O(2)$-fixed point, existence of which we have just argued. Realization of this fixed points requires either the vanishing of $\lambda_{x,y}$ (from the analysis of QCD) or the RG-irrelevance of such interactions at the fixed point. Note that the higher-order perturbative studies in [101][102] have not produced the answer for this operators.

Here we exclude the latter possibility using the bootstrap output of the spectra listed in Table 6.4,

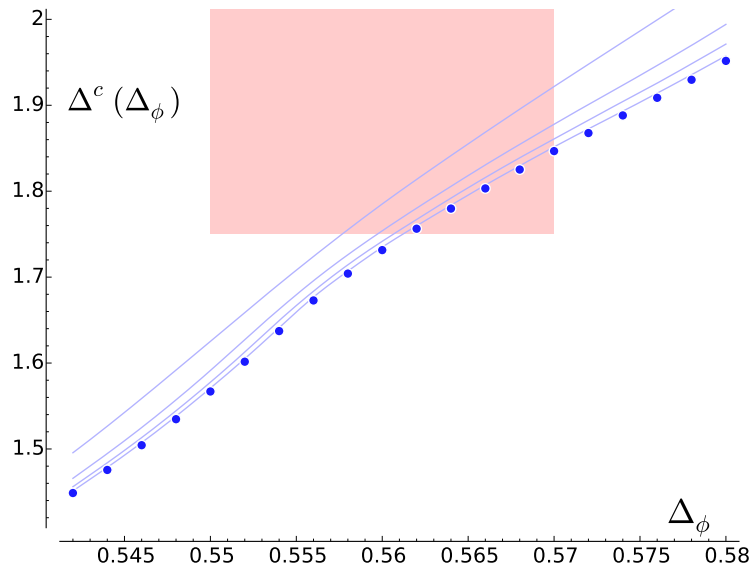

Figure 6.14: The bounds for AA-sector operator dimensions in $O(4) \times O(2)$-CFTs. The blue dots correspond to the bounds obtained with $\Lambda = 31$ using SDPB, while the other lines represent those for $\Lambda = 15, 19, 23$ obtained via sdpa-gmp in [18] and $\Lambda = 27$ via SDPB. The combined prediction of [101] and [102] by MZM-series is plotted in a light red rectangle.

	Δ_ϕ	Δ_{SS}	Δ_{ST}	Δ_{TS}	Δ_{TT}	Δ_{AA}
bootstrap	0.530(3)	1.35(4)	1.80(6)	1.31(2)	1.085(3)	0.90(1)
$\overline{\mathrm{MS}}$	0.536(5)	1.44(10)	1.83(8)	1.35(3)	1.06(10)	0.83(10)
MZM	0.533(3)	1.04(12)	1.94(7)	1.36(5)	0.96(20)	0.71(8)

Table 6.3: The low-lying spectra read off around the kink in Fig.6.13 $\Lambda = 19, 15$ result with sdpa-gmp result in [18]. The spectra for the $O(4) \times O(2)$ chiral fixed point are taken from [107] (Δ_ϕ and Δ_{SS}) and [101] (others).

	Δ_ϕ	Δ_{SS}	Δ_{ST}	Δ_{TS}	Δ_{TT}	Δ_{AA}
bootstrap	0.558(4)	1.52(5)	0.82(2)	1.045(3)	1.26(1)	1.70(6)
$\overline{\mathrm{MS}}$	0.56(3)	1.68(17)	1.0(3)	1.10(1k)	1.35(10)	1.9(1)
MZM	0.56(1)	1.59(14)	0.95(15)	1.25(10)	1.34(5)	1.90(15)

Table 6.4: The low-lying spectra read off around the kink in Fig.6.14 with $\Lambda = 31, 27$ SDPB output. The spectra for the $O(4) \times O(2)$ collinear fixed point are taken from [102] (for Δ_ϕ and Δ_{SS}), [101].

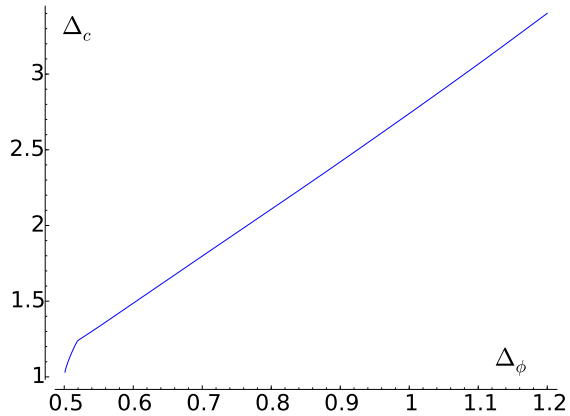

Figure 6.15: The same setting as in Fig.4.5, but with large values of Δ_ϕ. The horizontal step is actually 0.05 after $\Delta_\phi = 0.53$.

by showing that the dimension of the operator $y := (\det(\Phi)^2 + \det(\Phi^\dagger)^2)$ is forced to be below $d = 3$.

First of all, note that the four point function does not contain $U(1)_A$-charge four operators (like y)[4]as intermediate states, and we will not obtain a bound for this operator directly. Meanwhile a charge 2 operator $\det(\Phi)$ is in the $\Phi \times \Phi$ OPE, and translates as an ST operator in the $O(4) \times O(2)$-terminology.

In the operator language, then, the operator y is the lowest-dimensional operator contained in the OPE of $\det \Phi \times \det \Phi$ with charge 4. Denominating the charges by the factor of 2 and forgetting the $O(4)$-structure (as $\det \Phi$ is $O(4)$-singlet), this is a charge 1 $\times$ charge 1 $\to$ charge 2 OPE. Thus, forgetting completely about the presence of original field Φ and $O(4)$-structure, the operator dimension bounds of Fig.4.5 obtained in Section 4.3 perfectly applies here to $\det \Phi \times \det \Phi \to y$ OPE.

In Fig.6.15, we compute the numerical bounds in the same setting as in Fig.4.5, but this time extending it to larger values of Δ_ϕ. This shows $(\det \Phi)^2$ must be relevant unless $\Delta_{\det \Phi} > 1.08$, contradicting our prediction in Table 6.4, $\Delta_{\det \Phi} = \Delta_{\mathrm{ST}} \simeq 0.82(2)$. From this we conclude the relevance of the operator y, and $O(4) \times O(2)$-fixed point found above is realized only when $\lambda_y = 0$.

[4] Although it might be confusing, we are assigning charge 1 to the quark bilinear field Φ, so 1/2 to quark field.

Chapter 7

Conclusions and future directions

In this dissertation, we applied the conformal bootstrap program to $O(n) \times O(2)$-symmetric CFTs, to gain insight for the phase transitions represented by the fixed points of corresponding LGW models. These bounds are rigorous and completely general ones to be respected by all the CFTs with this symmetry, in particular by LGW-fixed points theories.

Though empirically, we argued that these bounds are actually saturated by these LGW-fixed points CFTs. This is confirmed by the comparison with large-n expansion around $n \sim 10$ by the comparison of the critical exponents Δ_ϕ, $\Delta_{\rm SS}$.

When n is small, in particular for $n \leq 4$, the presence of these fixed points has been quite controversial within other approaches so far: the presence of the fixed points are conjectured only through the perturbative computation with resummation. Our result for these fixed points, marginal saturation of the bounds by these fixed points in ST and AA sectors, together with an agreement of other critical exponents read-off from bootstrap output, strongly suggests that these fixed points are indeed present.

As a bonus of the precise determination of critical exponents, for $O(4) \times O(2)$-"collinear" fixed point, we also showed that the fixed points is **unstable** against certain kind of partial $U(1)_A$-breaking term in the LGW-model, by another use of the conformal bootstrap. To summarize, we argued that the universal properties of the chiral phase transition must be classified into three patterns according to the extent of partial $U(1)_A$-restoration. These are

1. Complete $U(1)_A$-breaking: In this case, as originally argued in [66], half of the meson fields becomes massive and integrated out before the RG endpoint, and the transition can be either of first order or second order. In the latter case, the critical exponents are those of $O(4)$.

2. $U(1)_A$-restored completely at the effective-Lagrangian level: In this case, although the conclusion in [66] is widely believed, we argued that the $SU(2)_L \times SU(2)_R \times U(1)_A$-symmetric LGW-model has a fixed point, from the agreement with the higher-order perturbative expansions. The transition can be either of first order or second order. In the latter case, the critical exponents are those listed in Table 6.4. In particular, $\Delta_{\rm SS}$ (equivalently ν-exponent) is significantly different from that of $O(4)$.

3. $U(1)_A$-restored only partially for the effective-Lagrangian: In this case, we have shown that at least one of the quartic terms, y, in the most general effective Lagrangian is relevant at $SU(2)_L \times SU(2)_R \times U(1)_A$-fixed point, and this CFT cannot contribute as a critical dynamics of the model. The phase transition is likely to be of first order then, unless there is an IR-stable, non-trivial fixed point with $\lambda_y \neq 0$.

The prediction for the third case is a new one.

There are several possible future directions related to our works.

First of all, we have to understand the meaning of kinks – even the saturation of the bounds for the $d = 2, 3$ Ising models are not rigorously proven. Some speculation is made in [10], where it is observed that there is some jump in the spectra read off via the extremal functional method, when Δ_ϕ cross the Ising model location, i.e., the kink.

Another promising direction for an understanding of the mystery is simultaneous consideration of consistency conditions coming from multiple number of correlators [11]. There general CFTs with $\mathbb{Z}_2$ symmetry with $\mathbb{Z}_2$-odd scalar ϕ (like the Ising model spin field) and $\mathbb{Z}_2$-even scalar S (like the Ising energy operator) was considered with non-vanishing correlators $\langle\phi\phi\phi\phi\rangle$, $\langle\phi\phi SS\rangle$, and $\langle SSSS\rangle$, and the generalization of positivity argument is applied to extract the constraint coming from the simultaneous requirement of crossing relation for all these three correlators. In particular for the Ising model, we expect it to have only two relevant parameters, temperature and magnetic-field. In operator language, this means that the only relevant primary operators are ϕ and S. Imposition of this constraint is so strong that it almost sweeps out the possible parameter region, except for very tiny region around the kink. It is generalized in [52] to $O(n)$-case, assuming there is only one relevant primary operator in vector and singlet channel, respectively, and again all but small (but significantly larger than that of the Ising) region is excluded.

This does not generalize to $O(n)\times O(m)$ directly. To see why, recall that the source of the relevant primary vector operator isolation is the equation of motion: the candidate of relevant primary vector operator $\phi_j\phi_j\phi_i$ is, from the naïve equation of motion for $O(n)$-vector model

$$\partial^2\phi_i = \lambda\phi_j\phi_j\phi_i,$$

expected to be proportional to the descendants of ϕ_j. In our case of $O(n)\times O(m)$, however, there are two candidates for such bi-fundamental relevant operator,

$$\phi_{jb}\phi_{jb}\phi_{ia}, \quad \phi_{jb}\phi_{ib}\phi_{ja},$$

and we can expect only one of them to be a descendant of the fundamental field. Therefore, it would be interesting to find a proper conditions to isolate $O(n)\times O(m)$-kinks.

Our conclusion for $O(4)\times O(2)$-fixed points clearly demands even more delicate study for $U(1)_A$ restoration scenario than in [69]. In particular, we have shown that the presence of λ_y-term in (5.3) is likely to affect the order of the chiral phase transition as a relevant operator. Analytic argument to show the (non)presence of this term or a numerical method to measure the presence of this operator (involving 8-th order of fermion) is desired. At the level of the effective sigma-model analysis, we cannot exclude an exotic possibility of the fixed point of (5.3) with $c_A = 0$ with $\lambda_y \neq 0$. It would also be interesting to pursue the possibility of the fixed points in such a model via the conformal bootstrap or other methods.

The author believes that the conformal bootstrap program will revolutionize (or is already revolutionizing) our understanding of nature. In particular, once the complete proof for the correspondence between "kink" and actual CFT is given, the method will take over the building block of quantum field theories. Then, the final resolution of the long-standing controversies regarding the presence of $O(n)\times O(2)$-LGW models will be only a tiny tip of the iceberg.

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