学位論文

Long-time asymptotic states of periodically driven open quantum systems
(時間周期駆動量子開放系の定常状態)

平成 27 年 12 月博士(理学)申請

東京大学大学院理学系研究科

物理学専攻

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Doctoral thesis

Long-time asymptotic states in periodically driven open quantum systems

December, 2015

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Abstract

In this thesis, we study long-time asymptotic states of periodically driven quantum systems in a dissipative environment. In order to describe the subclass of the systems, we introduce the Floquet-Gibbs state, i.e., a state whose density matrix is diagonal in the basis of the Floquet states with diagonal elements given by a Boltzmann distribution over the quasienergies. We obtain sufficient conditions for the realization of the Floquet-Gibbs state in a system with an infinitesimal system-bath coupling strength, and find that these conditions severely restrict a class of suitable systems attaining this form. With the aid of a truncated Floquet Hamiltonian in the Floquet-Magnus expansion and without the rotating wave approximation, we lift the condition of the infinitesimal coupling strength and extend the idea of the Floquet-Gibbs state to a broader subclass of open quantum system with a finite dissipation effect. We also study cooperative phenomena of a periodically driven cavity system surrounded by a dissipative environment. We found a novel-type of symmetry breaking phenomenon, which originates from a synergistic effect of the microscopic dynamical effects of the driving field and interaction effects. We show that a part of this phenomenon can be understood from the concept of the Floquet-Gibbs state.

Acknowledgment

It is great pleasure to express my gratitude to all the people who help, guide, and encourage me during my time as a graduate student. Without their help I definitely could not write up this thesis. First of all, I would like to thank my supervisor Professor Seiji Miyashita and Dr. Takashi Mori. Their energetic attitudes towards physics and personality continuously stimulate me. I acknowledge Professor Peter Hänggi to join the activity in Augsburg university for three months. It was a short time, but the discussions with him, Professor Sergey Denisov, Professor Tobias Brandes, Dr. Juzar Thingna and Dr. Victor Bastidas were exciting experiences for me. I am also grateful to the referees of this thesis, Professor Masaki Oshikawa, Professor Yasunobu Nakamura, Professor Ryo Shimano, Professor Akira Shimizu, and Professor Shinji Todo, for their insightful comments extending my scientific knowledge. I thank all the members of Miyashita group and others, Dr. Keiji Saito, Dr. Shu Tanaka, Dr. Chihiro Matsui, Dr. Sergio Andraus, Dr. Sasmita Mohakud, Dr. Eriko Kaminishi, Dr. Tomotaka Kuwahara, Mr. Taro Nakada, Mr. Shun Kamatsuka, Mr. Yusuke Endo, Mr. Takafumi Suzuki, Mr. Futoshi Futami, Mr. Adrien Bolens, Mr. Hiroki Ikeuchi, Mr. Takuya Hatomura, Mr. Moriaki Sugita, Mr. Kaoru Yamamoto, and Mr. Haw Jing Yang. I had a great experience through the valuable discussion with them, and I passed a comfortable physical life thanks to their attention. I also extend my gratitude to secretaries in Miyashita group, Ms. Keiko Yashima and Ms. Hiromi Okuzawa for their support. Finally I thank my family and my friends who push and encourage me to pursue my interests. I acknowledge supports from JSPS (Grant No. 258794) and Advanced Leading Graduate Course for Photon Science (ALPS).

Tatsuhiko Shirai

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List of Publications

The results of this thesis have been reported in the article: For Chap. 3,

- Tatsuhiko Shirai, Takashi Mori, and Seiji Miyashita, “Condition for emergence of the Floquet-Gibbs state in periodically driven systems”, Phys. Rev. E **91**, 030101(R) (2015) (see also in arXiv:1410.0464).

For Chap. 4,

- Tatsuhiko Shirai, Juzar Thingna, Takashi Mori, Sergey Denisov, Peter Hänggi and Seiji Miyashita, “Floquet-Gibbs states for dissipative quantum systems”, arXiv:1511.06864 (2015).

For Chap. 5,

- Tatsuhiko Shirai, Takashi Mori, and Seiji Miyashita, “Novel symmetry-broken phase in a driven cavity system in the thermodynamic limit”, J.Phys. B: At. Mol. Opt. Phys. **47**, 025501 (2014) (see also in arXiv:1204.5516).

Chapter 1

Introduction and periodically driven closed systems

1.1 Aim of thesis

Surrounded by a heat bath, the system with a time independent Hamiltonian relaxes into an equilibrium state [1]. When the coupling strength between the system and the heat bath is infinitesimal, the state is described by a canonical Gibbs distribution, $\rho \propto e^{-\beta H}$, where H is the Hamiltonian of the system and β is the inverse temperature of the heat bath [2]. The closed-form solution allows us to evaluate the properties of the equilibrium state without details of the heat bath except for its temperature. The mechanism behind the universal closed-form solution and its emergence from the complicated dynamics remain in the focus of active studies for a long time [3, 4, 5, 6].

When a system is subjected to a periodic driving, no universal closed-form expression is known for the long-time asymptotic state, and hence we have to explicitly take the heat bath into account and analyze the dissipative dynamics numerically or analytically. In a coherent quantum dynamics, a system presents a variety of nontrivial dynamical properties. We review here dynamics observed in driven two-level systems, which show peculiar phenomena, e.g., Rabi oscillations [7] and suppression of quantum tunneling [8]. Some applications of these properties include the control of chemical reaction rates [9], the control of electric transport [9], ion-trap experiments [10], and the photo-induced phase transitions [11]. In cold atom optics, recent experimental progress enables us to control macroscopic properties by using off-resonant and strong oscillating field. An effective Hamiltonian, which is obtained by a time-evolution operator over one period of the driving field and a perturbative method, is an efficient tool to engineer these properties theoretically, which is called "Floquet engineering". This Hamiltonian depends on the amplitude and frequency of the driving field, and new phases or topological structures, absent in equilibrium, have been found; see the recent review by Bukov *et al.* [12], references therein.

In this thesis, we will discuss the meaning of the effective Hamiltonian from a thermodynamical viewpoint. When we try to control some system by using a periodically driving field, the system is usually surrounded by a dissipative environment, and then relaxes into a steady state. If the dissipative environment is characterized by a heat bath at very low temperature, is the steady state described by a ground state of the effective Hamiltonian? In general, when the system interacts with a heat bath with finite temperature, is the steady state described by a Gibbs form of the effective Hamiltonian?

In order to give a basis to answer this issue, in Chap. 2, we plan to explain a master equation formalism, which is one of the conventional methods to treat the dissipation effects. In this framework, it is known that the long-time asymptotic state of a system under a periodic driving field generally does not have a closed-form solution due to the lack of the conserved energy owing to the breakdown of the time-translational symmetry [13]. However, it does not always deny the existence of the closed-form solution, and then we may ask a question: Which system can acquire the closed-form solution?

Indeed, a closed-form solution arises in some systems: a single particle subjected to a modulated harmonic oscillator [14, 15] and systems where the time dependence can be eliminated through a unitary transformation [16]. In a non-integrable system we are tempting to use the Floquet basis and see whether its diagonal elements of the asymptotic density matrix are the Boltzmann distribution or not. But before doing so, we have to determine what quantity should take place of the energy E_i. One of the natural ideas that this role could be played by is an averaged energy of the Floquet state, i.e., the expectation value of the system Hamiltonian averaged over one period of the driving. This idea has been tested in [14, 17]. They demonstrate that in some region the Boltzmann factor with an "effective temperature", which is different from the actual temperature of the heat bath, appears, while in other region so-called infinite temperature states appear. This separation is related to coexistence of semi-classically chaotic and regular Floquet states.

These previous studies imply the existence of parameter regimes where the long-time asymptotic states can be described in a closed-form solution. The aim of this thesis is to find the conditions without limiting to some specific model. In Chap. 2, we introduce two types of quantum mater equation: Floquet Lindblad equation [18] and Redfield equation [19]. The Lindblad equation has a much simpler structure than the Redfield equation, but via the derivation it needs the rotating wave approximation (RWA) or secular approximation. We discuss the range of the applicability of these master equations, and explicitly show that the long-time asymptotic state in the Lindblad formalism is independent of the dissipation strength and only applicable to systems with infinitesimal system-bath coupling strength.

In Chap. 3, we try to find the conditions that the long-time asymptotic state is given in a closed-form solution in the framework of the Lindblad equation. The simple structure of the Lindblad equation and the Kubo-Martin-Schwinger relation [20], which holds for the correlation functions of the bath operators, allows us to introduce the notion of the Floquet-Gibbs state, i.e., a state whose density matrix is diagonal in the basis of the Floquet state with diagonal elements given by the Boltzmann distribution of the quasienergies. Thus, the Floquet states and quasienergies take place of the eigenenergy states and eigenenergies in the conventional canonical Gibbs state. Here, we try to show that under what conditions the Floquet-Gibbs state appears in the long-time asymptotic state.

The obtained result in Chap. 3 is, however, only applicable to systems with infinitesimal system-bath coupling. Then, we ask a following question: Is the long-time asymptotic state independent of the details of the coupling to the heat bath, e.g., system-bath coupling strength and timescales of the bath dynamics? The answer for the system with a time-independent Hamiltonian is "yes", because the asymptotic properties of the system only depend on the temperature and/or chemical potential of the heat bath. However, the answer for systems subjected to a periodic driving field is not obvious. In order to answer this question, in Chap. 4, we use the Redfield equation, which allows us to investigate the effects of finite system-bath coupling. Then, we study the dependence of the conditions for the Floquet-Gibbs state on the details of system-bath coupling.

In Chap. 5, we study a cooperative phenomenon in a driven cavity system. We adopt the Dicke model in which multiple two-level systems interacts with a single quantized mode of photon field. Owing to the effects of the atom-photon coupling and the dynamical effects induced by a periodic driving field, we find a new type of dynamical phase transition. We discuss the mechanism of this phenomenon and relation to the Floquet-Gibbs state. For the study of the long-time asymptotic states, we introduce a phenomenological master equation in Chap. 2.

Finally, in Chap. 6, we give a conclusion and future prospect.

1.2 Floquet state, quasienergy, and Floquet Hamiltonian

The primary interest of this thesis is in a system subjected to an intense oscillating field. The system can be investigated in a semiclassical treatment [21]; A quantum system interacts with a time-dependent classical field. We study a system under a periodic driving field at frequency Ω whose Hamiltonian reads

$$H(t) = H(t+T), \tag{1.1}$$

where T is a period of the driving field ($T = 2\pi/\Omega$).

The time-evolution of the system obeys the Schrödinger equation,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle . \tag{1.2}$$

The periodicity of the Hamiltonian enables us to expand the solution of the Schrödinger equation into eigenmodes, called "Floquet states" $|\psi_n(t)\rangle$.

In order to define the Floquet states, we introduce the time evolution operator from time t_0 to time t_1,

$$U(t_1, t_0) = \mathcal{T} e^{-\frac{i}{\hbar}\int_{t_0}^{t_1} H(\tau) d\tau}, \tag{1.3}$$

where $\mathcal{T}$ denotes the time-ordering operator. The Floquet state is given by the eigenstate of the time evolution operator over one period,

$$U(t_0+T,t_0)\,|\psi_n(t_0)\rangle = a_n(t_0)\,|\psi_n(t_0)\rangle\,. \tag{1.4}$$

Owing to the unitarity of $U(t_0+T,t_0)$, the absolute value of eigenvalues, $\{a_n(t_0)\}$, is one, and the set of $|\psi_n(t_0)\rangle$ can be chosen to form the complete orthonormal basis. Because $|\psi_n(t_0)\rangle$ obeys the Schödinger equation by definition,

$$|\psi_n(t_1)\rangle = U(t_1,t_0)\,|\psi_n(t_0)\rangle\,, \tag{1.5}$$

by using the property of the unitary operator, $U(t_1+T,t_0+T)=U(t_1,t_0)$, we obtain from Eq. (1.4)

$$U(t_1+T,t_1)\,|\psi_n(t_1)\rangle = a_n(t_0)\,|\psi_n(t_1)\rangle\,. \tag{1.6}$$

The independence of the eigenvalue on time leads us to a quasienergy ϵ_n,

$$a_n(t_0)=a_n(t_1)=e^{-\frac{i}{\hbar}\epsilon_n T}, \tag{1.7}$$

which is defined within the region $-\hbar\Omega/2 \le \epsilon_n < \hbar\Omega/2$. The Floquet state then satisfies the following relation,

$$|\psi_n(t+T)\rangle = e^{-\frac{i}{\hbar}\epsilon_n T}\,|\psi_n(t)\rangle\,, \tag{1.8}$$

from which we can divide the Floquet state into two parts,

$$|\psi_n(t)\rangle = e^{-\frac{i}{\hbar}\epsilon_n t}\,|u_n(t)\rangle\,. \tag{1.9}$$

Here, $|u_n(t)\rangle$ is called Floquet mode, which is periodic in time,

$$|u_n(t+T)\rangle = |u_n(t)\rangle\,. \tag{1.10}$$

The solution of the Schrödinger equation with initial condition $|\psi(t_0)\rangle$ is therefore given by

$$|\psi(t)\rangle = \sum_n c_n\,|\psi_n(t)\rangle = \sum_n c_n\,|u_n(t)\rangle\, e^{-\frac{i}{\hbar}\epsilon_n t}, \tag{1.11}$$

where $c_n = \langle\psi_n(t_0)\,|\psi(t_0)\rangle$. By substituting this form, Eq. (1.9), into the Schrödinger equation, Eq. (1.2), we obtain the equation for the Floquet mode to satisfy;

$$\left(H(t) - i\hbar\frac{\partial}{\partial t}\right)|u_n(t)\rangle = \epsilon_n\,|u_n(t)\rangle\,. \tag{1.12}$$

In atomic laser physics, laser field is often treated as a single mode of a quantized photon field. In this treatment, since the system Hamiltonian including the laser field becomes time independent, the eigenstates called dressed states can be introduced. The idea of the dressed states gives us simple interpretations of the resonance effects such as Rabi oscillation, AC stark shifts, and Bloch Siegert shifts, and so on [22].

Here, although the effects of photon fluctuations are intriguing, we limit ourselves to situations where the driving field can be treated as a classical field, in which the dressed states are identical to the Floquet states.

Finally, we introduce the Floquet Hamiltonian $H_{\mathrm{F}}^{[t]}$, which is defined by the time-evolution operator over one period;

$$e^{-\frac{i}{\hbar}H_{\mathrm{F}}^{[t]}T} = U(t+T,t). \tag{1.13}$$

This Hamiltonian describes dynamics at stroboscopic times at $t+nT$ with n integers. The Floquet Hamiltonian depends on the initial time of the unitary operator $U(t+T,t)$ while its energy spectrum is independent of t because quasienergies are independent of time.

1.3 Perturbative methods

The set of the Floquet states and quasienergies completely gives a time evolution of the coherent quantum systems under a periodic driving field. However, there are only a few systems in which their exact solutions can be analytically obtained. The systems are for example a parametrically driven harmonic oscillator [23] and a spin system under a specific driving field [7]. We hence resort to perturbative methods or numerical calculation in order to obtain the Floquet states and quasienergies.

In order to obtain them by solving Eq. (1.12), it is convenient to introduce an extended Hilbert space, which is given by the product space of a state space of a quantum system and a space of time-periodic function [24]. A state in this extended Hilbert space includes its time dependence from $t=0$ to $t=T$, and the inner product $\langle\langle\cdot|\cdot\rangle\rangle$ is defined by

$$\langle\langle u|v\rangle\rangle = \int_0^T \langle u(t)|\, v(t)\rangle \frac{dt}{T}, \tag{1.14}$$

where $|u(t)\rangle$ and $|v(t)\rangle$ belong to the extended Hilbert space.

In the extended Hilbert space, i times derivative with respect to the time, $i\partial/\partial t$, is a Hermite operator,

$$\langle\langle u|i\frac{\partial}{\partial t}|v\rangle\rangle = \int_0^T \langle u(t)|\, i\frac{\partial}{\partial t}\,|v(t)\rangle\,\frac{dt}{T} = (\langle\langle v|i\frac{\partial}{\partial t}|u\rangle\rangle)^*, \tag{1.15}$$

and hence Eq. (1.12) is nothing but an eigenvalue problem for an Hermite operator $H(t) - i\hbar\partial/\partial t$. We can employ the methods for the system with time-independent Hamiltonian: perturbative methods for the stationary states [24], which will be explained in the following subsection 1.3.2, variational method, and adiabatic theorem [25], and so on.

1.3.1 Weak perturbation: non-degenerate case

In this section we study the system subjected to a weak periodic driving field, and then obtain the Floquet modes and quasienergies perturbatively [24]. The Hamiltonian of the system reads

$$H(t) = H_0 + \xi H_{\rm ex}(t), \tag{1.16}$$

where ξ represents the driving amplitude. We treat the second term as a perturbation. In this section, we assume that there is no degeneracy in quasienergy spectrum. For simplicity, we assume time independence of the non-perturbative Hamiltonian H_0, but its extension to a time-periodic case is straightforward.

The eigenmode problem for non-perturbative Floquet mode is given by

$$\left(H_0 - i\hbar\frac{\partial}{\partial t}\right)|u_i^{(0)}(t)\rangle = \epsilon_i^{(0)}\,|u_i^{(0)}(t)\rangle\,, \tag{1.17}$$

where the label i is chosen so that $\lim_{\xi\to 0}|u_i(t)\rangle = |u_i^{(0)}(t)\rangle$. This Floquet state and its quasienergy are related to the eigenstate $|\psi_i\rangle$ and its eigenenergy E_i of H_0 as follows

$$\begin{cases} |u_i^{(0)}(t)\rangle = e^{-im_i\Omega t}\,|\psi_i\rangle\,, \\ \qquad \epsilon_i^{(0)} = E_i - m_i\hbar\Omega, \end{cases} \tag{1.18}$$

where m_i is chosen so that $-\hbar\Omega/2 \le \epsilon_i^{(0)} < \hbar\Omega/2$.

The perturbative method aims to approximate the genuine Floquet state $|u_i(t)\rangle$ by using the set of these non-perturbative Floquet states $|u_i^{(0)}(t)\rangle$ or eigenenergy states $|\psi_i\rangle$. First, we fix the phase of the Floquet states at $t=0$ as in the conventional perturbative method [26] so that the inner product of $|u_i^{(0)}(0)\rangle$ and $|u_i(0)\rangle$ is real;

$$\langle u_i^{(0)}(0)|\,u_i(0)\rangle = \langle u_i(0)|\,u_i^{(0)}(0)\rangle. \tag{1.19}$$

Since $|u_i(t)\rangle$ and $|u_i^{(0)}(t)\rangle$ obey the first-order differential equation of t, Eq, (1.12) and Eq. (1.17), respectively, it is sufficient to fix the phase at a certain moment. However, because they obey different equations, the inner product $\langle u_i^{(0)}(t)|\,u_i(t)\rangle$ is not generally real for $0 < t < T$. We then introduce a real and periodic variable $\theta_i(t,\xi)$, $\theta_i(t,\xi) \in \mathbf{R}$ and $\theta_i(t,\xi) = \theta_i(t+T,\xi)$, as

$$|u_i(t)\rangle = |v_i(t)\rangle\, e^{-\frac{i}{\hbar}\theta_i(t,\xi)}, \tag{1.20}$$

so that the inner product between $|u_i^{(0)}(t)\rangle$ and $|v_i(t)\rangle$ is real,

$$\langle u_i^{(0)}(t)|\,v_i(t)\rangle = \langle v_i(t)|\,u_i^{(0)}(t)\rangle. \tag{1.21}$$

Here, $\theta_i(t,\xi)$ is chosen to satisfy $\theta_i(0,\xi) = \theta_i(t,0) = 0$. Due to the periodicity of $|u_i(t)\rangle$ and $\theta_i(t,\xi)$, $|v_i(t)\rangle$ is also periodic in time.

We find from Eq. (1.12) and Eq. (1.20) that $|v_i(t)\rangle$ obeys

$$\left(H_0 - i\hbar\frac{\partial}{\partial t} - \frac{\partial\theta_i(t,\xi)}{\partial t} + \xi H_{\rm ex}(t)\right)|v_i(t)\rangle = \epsilon_i\,|v_i(t)\rangle\,. \tag{1.22}$$

From Eq. (1.17) and Eq. (1.22), we find for $\theta_i(t,\xi)$

$$\begin{aligned}&\frac{\partial\theta_i(t,\xi)}{\partial t}\\ =&\frac{\langle u_i^{(0)}(t)|\,H_0 - i\hbar\frac{\partial}{\partial t} + \xi H_{\rm ex}(t)\,|v_i(t)\rangle}{\langle u_i^{(0)}(t)|\,v_i(t)\rangle} - \frac{\langle v_i(t)|\,H_0 - i\hbar\frac{\partial}{\partial t}\,|u_i^{(0)}(t)\rangle}{\langle v_i(t)|\,u_i^{(0)}(t)\rangle} - (\epsilon_i - \epsilon_i^{(0)}),\\ =&\epsilon_i(t) - \epsilon_i,\end{aligned} \tag{1.23}$$

where

$$\epsilon_i(t) = \epsilon_i^{(0)} + \xi{\rm Re}\left[\frac{\langle u_i^{(0)}(t)|\,H_{\rm ex}(t)\,|v_i(t)\rangle}{\langle u_i^{(0)}(t)|\,v_i(t)\rangle}\right]. \tag{1.24}$$

Thus, $\theta_i(t,\xi)$ is obtained by integrating the differential equation for the interval 0 to t with the boundary condition, $\theta_i(0,\xi) = 0$. Owing to the periodicity of $\theta_i(t,\xi)$ and Eq. (1.23), the quasienergy is given by

$$\epsilon_i = \int_0^T \epsilon_i(t)\frac{dt}{T}. \tag{1.25}$$

Substituting Eq. (1.23) into Eq. (1.22), we obtain the differential equation for $|v_i(t)\rangle$,

$$\left(H_0 - i\hbar\frac{\partial}{\partial t} + \xi H_{\rm ex}(t)\right)|v_i(t)\rangle = \epsilon_i(t)\,|v_i(t)\rangle\,, \tag{1.26}$$

which is perturbatively solved under the following conditions. First, we impose orthonormal condition on $|u_i(t)\rangle$;

$$\langle\langle u_i|u_j\rangle\rangle = \delta_{i,j} \leftrightarrow \langle v_i(t)|\,v_j(t)\rangle = \delta_{i,j}, \tag{1.27}$$

where δ represents the Kronecker delta. Next, we assume that $|v_i(t)\rangle$ can be expanded into the power series of ξ as

$$|v_i(t)\rangle = |u_i^{(0)}(t)\rangle + \xi\,|v_i^{(1)}(t)\rangle + \cdots\,. \tag{1.28}$$

Here we use $\lim_{\xi\to 0}|v_i(t)\rangle = |u_i^{(0)}(t)\rangle$. Finally, we assume that the equations (1.19) and (1.27) hold for each order of ξ. These conditions uniquely give the perturbative solution of $|v_i(t)\rangle$.

We obtain the solution up to the first order of ξ. In the order of ξ, we obtain two constraints on $|v_i^{(1)}(t)\rangle$:

$$\begin{aligned}&\langle v_i^{(1)}(t)|\,u_i^{(0)}(t)\rangle = \langle u_i^{(0)}(t)|\,v_i^{(1)}(t)\rangle \quad \text{(Eq. (1.19))},\\ &\langle v_i^{(1)}(t)|\,u_j^{(0)}(t)\rangle + \langle u_i^{(0)}(t)|\,v_j^{(1)}(t)\rangle = 0 \quad \text{(Eq. (1.27))},\end{aligned} \tag{1.29}$$

which indicate that

$$\langle u_i^{(0)}(t)|\, v_i^{(1)}(t)\rangle = 0. \tag{1.30}$$

We find from Eq. (1.24) that $\epsilon_i(t)$ is given up to the order of ξ by

$$\epsilon_i(t) = \epsilon_i^{(0)} + \xi \langle u_i^{(0)}(t)|\, H_{\rm ex}(t)\, |u_i^{(0)}(t)\rangle\,. \tag{1.31}$$

The equation for $|v_i(t)\rangle$, Eq. (1.26), up to the order of ξ thus reads

$$\left(H_0 - \epsilon_i^{(0)} - i\hbar\frac{\partial}{\partial t}\right)|v_i^{(1)}(t)\rangle = \left(\langle u_i^{(0)}(t)|\, H_{\rm ex}(t)\, |u_i^{(0)}(t)\rangle - H_{\rm ex}(t)\right)|u_i^{(0)}(t)\rangle\,, \tag{1.32}$$

and we obtain for $j \neq i$

$$\left[i\hbar\frac{\partial}{\partial t} + (\epsilon_i^{(0)} - \epsilon_j^{(0)})\right]\langle u_j^{(0)}(t)|\, v_i^{(1)}(t)\rangle = \langle u_j^{(0)}(t)|\, H_{\rm ex}(t)\, |u_i^{(0)}(t)\rangle\,. \tag{1.33}$$

The restriction Eq. (1.30) leads to

$$|v_i^{(1)}(t)\rangle = \sum_{\substack{j\\(j\neq i)}} \sum_{m=-\infty}^{\infty} \frac{\tilde{H}_{jim}}{E_i - E_j + m\hbar\Omega} e^{-i(m+m_i)\Omega t}\, |\psi_j\rangle\,, \tag{1.34}$$

where $\tilde{H}_{jin}$ is a Fourier component of the driving Hamiltonian in the basis of the energy eigenstates of H_0,

$$\tilde{H}_{jin} = \int_0^T \langle\psi_j|\, H_{\rm ex}(t)\, |\psi_i\rangle\, e^{in\Omega t}\frac{dt}{T}. \tag{1.35}$$

The Floquet states and quasienergies up to O(ξ) are as follows: Since we obtain $\theta_i(t,\xi)$ from Eq. (1.23) by

$$\theta_i(t,\xi) = \int_0^t (\epsilon_i(\tau) - \epsilon_i)d\tau = \sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty} \frac{-i\xi\tilde{H}_{iin}(1-e^{-im\Omega t})}{m\hbar\Omega}, \tag{1.36}$$

the Floquet states are given by

$$\begin{aligned} |u_i(t)\rangle =& |v_i(t)\rangle\, e^{-\frac{i}{\hbar}\theta_i(t,\xi)} \\ =& e^{-im_i\Omega t}\left[\left(1 - \sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}\frac{\xi\tilde{H}_{iim}}{m\hbar\Omega}\right)|\psi_i\rangle + \sum_{\substack{j\\(j\neq i)}}\sum_{m=-\infty}^{\infty}\frac{\xi\tilde{H}_{jim}e^{-im\Omega t}}{E_i - E_j + m\hbar\Omega}\,|\psi_j\rangle\right]. \end{aligned} \tag{1.37}$$

The quasienergies are given by

$$\epsilon_i = \epsilon_i^{(0)} + \xi\int_0^T \langle u_i^{(0)}(t)|\, H_{\rm ex}(t)\, |u_i^{(0)}(t)\rangle\,\frac{dt}{T} = \epsilon_i^{(0)} + \xi\tilde{H}_{ii0}. \tag{1.38}$$

1.3.2 Weak perturbation: Degenerate case

We can employ the degenerate perturbation theory for the system with time-independent Hamiltonian in order to obtain the Floquet modes and quasienergies when the unperturbed quasienergies are degenerate [24]. The Hamiltonian of the system is divided into two parts,

$$H(t) = H_0(t) + \xi H_1(t), \tag{1.39}$$

where ξ is sufficiently small. We treat the second term as a perturbation. The Floquet modes and quasienergies are assumed to be expanded by power series of ξ;

$$\begin{aligned} |u_j(t)\rangle &= |u_j^{(0)}(t)\rangle + \xi\,|u_j^{(1)}(t)\rangle + \cdots, \\ \epsilon_j &= \epsilon_j^{(0)} + \xi\epsilon_j^{(1)} + \cdots. \end{aligned} \tag{1.40}$$

We study the case where some Floquet states for $H_0(t)$ are n-fold degenerate. The corresponding unperturbed Floquet modes $|v_{jk}(t)\rangle$ satisfy for $k = 1, \cdots, n$,

$$\left(H_0(t) - i\hbar\frac{\partial}{\partial t}\right)|v_{jk}(t)\rangle = \epsilon_j^{(0)}\,|v_{jk}(t)\rangle\,, \tag{1.41}$$

where j labels the different unperturbed quasienergies $\epsilon_j^{(0)}$. Here, we choose $|v_{jk}(t)\rangle$ to be orthonormal with each other at each instant,

$$\langle v_{jk}(t)\,|v_{jl}(t)\rangle = \delta_{k,l}. \tag{1.42}$$

In order to obtain the first-order quasienergy $\epsilon_j^{(1)}$ and the corresponding appropriate lowest-order Floquet mode $|u_j^{(0)}(t)\rangle$, we write the lowest-order Floquet mode as a superposition of the unperturbed Floquet modes,

$$|u_j^{(0)}(t)\rangle = \sum_k c_{jk}\,|v_{jk}(t)\rangle\,. \tag{1.43}$$

The equation (1.12) in the order of ξ reads

$$\left(H_0(t) - i\hbar\frac{\partial}{\partial t}\right)|u_j^{(1)}(t)\rangle + H_1(t)\,|u_j^{(0)}(t)\rangle = \epsilon_j^{(0)}\,|u_j^{(1)}(t)\rangle + \epsilon_j^{(1)}\,|u_j^{(0)}(t)\rangle\,. \tag{1.44}$$

By multiplying $\langle v_{jk}(t)|$ from the left and taking average over one period, we obtain

$$\begin{aligned} &\langle\langle v_{jk}|H_1(t)|u_j^{(0)}\rangle\rangle = \epsilon_j^{(1)}\langle\langle v_{jk}|u_j^{(0)}\rangle\rangle, \\ \leftrightarrow &\sum_l \langle\langle v_{jk}|H_1(t)|v_{jl}\rangle\rangle c_{jl} = \epsilon_j^{(1)} c_{jk} \end{aligned} \tag{1.45}$$

where we have used the fact that $H_0(t) - i\hbar\partial/\partial t$ is an Hermite operator. By solving this eigenvalue problem, we obtain $|u_j^{(0)}(t)\rangle$ and $\epsilon_j^{(1)}$ as in the case of degenerate perturbation theory for the system with time-independent Hamiltonian. Resolution of the degeneracy in this order is proportional to ξ. If the degeneracy is not resolved within the lowest-order perturbation, we have to calculate higher-order contribution in order to obtain non-degenerate lowest-order Floquet modes $|u_j^{(0)}(t)\rangle$. If the degeneracy is resolved when the mth-order perturbation is performed, the resolution is proportional to ξ^m, which describes, e.g., a transition rate induced by a weak driving field between two energy eigenstates with eigenvalues E and $E + m\hbar\Omega$.

1.3.3 High frequency case: Floquet-Magnus expansion

There are situations where the system is subjected to a rapidly oscillating field, and hence well-separated two timescales exist. If the prime interest is in the dynamics of the long-time scale, the system can be effectively described by a time-independent Hamiltonian with renormalized coefficients due to the rapidly oscillating field. If the rapidly oscillating term is time periodic, the Floquet Hamiltonian plays a role of the time-independent Hamiltonian, and the renormalized coefficients are obtained in a systematic way by the Floquet-Magnus expansion [27].

The idea of the Floquet-Magnus expansion is as follows. First, take the logarithm of both sides in Eq. (1.13),

$$H_{\mathrm{F}}^{[t_0]} = \frac{i\hbar}{T} \ln[\mathcal{T} e^{-\frac{i}{\hbar}\int_{t_0}^{t_0+T} H(\tau)d\tau}]. \tag{1.46}$$

Next, we expand the right hand side in power series of the inverse of the driving frequency Ω^{-1} by using the Baker-Campbell-Hausdorff formula,

$$H_{\mathrm{F}}^{[t_0]} = \frac{1}{T}\sum_{n=1}^{\infty} \Omega^{(n)[t_0]}(T), \tag{1.47}$$

where $\Omega^{(n)[t_0]}(T)$ is the order of Ω^{-n}. The first order term is given by

$$\Omega^{(1)[t_0]}(t) = \int_{t_0}^{t_0+t} H(\tau)d\tau, \tag{1.48}$$

and n-th order term for $n \geq 2$ is given by

$$\begin{aligned} \Omega^{(n)[t_0]}(t) =& \sum_{j=1}^{k-1} \frac{(-i)^j B_j}{\hbar^j j!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_j=1}^{\infty} \delta_{\sum_{i=1}^{j} s_i, k-1} \\ &\times \int_{t_0}^{t_0+t} [\Omega^{(s_1)[t_0]}(\tau) \cdots [\Omega^{(s_j)[t_0]}(\tau), H(\tau)] \cdots] d\tau. \end{aligned} \tag{1.49}$$

The first two terms are explicitly given by

$$\begin{aligned} \Omega^{(1)[t_0]}(T) =& \int_{t_0}^{t_0+T} H(t)dt, \\ \Omega^{(2)[t_0]}(T) =& \frac{-i}{2!\hbar} \int_{t_0}^{t_0+T} dt_1 \int_{t_0}^{t_0+t_1} dt_2 [H(t_1), H(t_2)]. \end{aligned} \tag{1.50}$$

The first-order term is simply the time-averaged Hamiltonian over one period. The higher-order term gives the correction for the time-averaged Hamiltonian. We call the sum of the term up to nth-order term nth-order-truncated Floquet Hamiltonian, which is given by

$$H_{\mathrm{F}}^{(n)[t_0]} = \frac{1}{T} \sum_{k=1}^{n} \Omega^{(k)[t_0]}(T). \tag{1.51}$$

1.4 Periodically driven two-level system

Even when the system itself is simple, the dynamics under a time-periodic driving field shows various peculiar dynamical features. We take up a two-level system, where the Hamiltonian is given by

$$H_0 = h^z S^z, \tag{1.52}$$

where S^z is a usual spin operator along z-axis and h^z is the corresponding energy gap.

This model describes diverse systems whose Hilbert space can be effectively restricted to a two-dimensional space. The simplest system is a particle with one-half spin in a static magnetic field along z-axis. Around the z-axis, the magnetic moment precesses at Larmor frequency $h^z/\hbar$. An electron system hopping between double quantum dots [28] in a Coulomb blockade regime is another realization of this system. Even when the system consists of multiple eigenstates, there are some situations where truncation to the two-level system is possible. For example, in quantum optics the transition due to the interaction with light can occur mainly between two atomic states, e.g., the ground state and an excited state. The situation is also observed in a system with two locally stable potential wells such as a superconducting quantum bit [29]. If only the two lowest doublets should be concerned, the system can be regarded as a two-level system.

We apply a time-periodic driving field to the two-level system. The Hamiltonian is given by

$$H(t) = H_0 + \xi H_{\mathrm{ex}}(t). \tag{1.53}$$

The response to the driving field has been extensively studied in the above various systems, which leads to the control of an electron transport through double quantum dots by an external oscillating electric field [30] and the control of a superconducting artificial atom in a strong microwave pulses [31].

In this chapter, we overview the phenomena of this two-level system under two different types of periodic driving fields:

$$\begin{cases} \xi H_{\mathrm{ex}}(t) = \xi(\cos(\Omega t) S^x + \sin(\Omega t) S^y), \\ \xi H_{\mathrm{ex}}(t) = 2\xi \cos(\Omega t) S^x, \end{cases}$$

In the first case, a circularly polarized driving field is applied to the two-level system, in which the dynamics is exactly solvable. In the second case, a linearly polarized driving field is applied to the two-level system. Although the model is simple, the dynamics cannot be solved exactly which leads us to the approximation based on the perturbation methods.

1.4.1 Circularly polarized driving field: Rabi oscillation

When a two-level system is under a circularly polarized driving field, the well-known phenomenon called Rabi oscillation occurs [7]. We can follow the dynamics exactly because the time-dependent Hamiltonian can be mapped into a time-independent one by using a unitary transformation. The unitary transformation changes a state in the static frame $|\psi(t)\rangle$ into the state in the rotating frame $|\psi^{\mathrm{R}}(t)\rangle$,

$$|\psi^{\mathrm{R}}(t)\rangle = e^{i\Omega S^z t}\,|\psi(t)\rangle = V(t)\,|\psi(t)\rangle\,. \tag{1.54}$$

In the following, we use the notation $V(t)$ in order to denote the unitary operator which transforms a state in the static frame into the state in the rotating frame. The time evolution of $|\psi^{\mathrm{R}}(t)\rangle$ obeys the Schödinger equation with the time-independent Hamiltonian,

$$H^{\mathrm{R}} = V(t)\left(H(t) - i\hbar\frac{\partial}{\partial t}\right)V^{\dagger}(t) = \Delta_d S^z + \xi S^x. \tag{1.55}$$

Here, Δ_d is defined by the detuning energy between the energy gap and the driving frequency, $\Delta_d = h^z - \hbar\Omega$. Since the Hamiltonian is time independent, we obtain the dynamics by solving the eigenvalue problem of H^{R};

$$H^{\mathrm{R}}\,|\pm\rangle = \pm\frac{\sqrt{\Delta_d^2+\xi^2}}{2}\,|\pm\rangle\,, \tag{1.56}$$

where

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{(\Delta_d+\sqrt{\Delta_d^2+\xi^2})^2+\xi^2}}\left[(\Delta_d+\sqrt{\Delta_d^2+\xi^2})\,|\uparrow\rangle + \xi\,|\downarrow\rangle\right], \\
|-\rangle &= \frac{1}{\sqrt{(\Delta_d+\sqrt{\Delta_d^2+\xi^2})^2+\xi^2}}\left[-\xi\,|\uparrow\rangle + (\Delta_d+\sqrt{\Delta_d^2+\xi^2})\,|\downarrow\rangle\right].
\end{aligned} \tag{1.57}$$

Here, $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of S^z with positive eigenvalue and negative eigenvalue, respectively.

For simplicity, we study the dynamics when an initial state at $t=0$ is prepared in $|\downarrow\rangle$. We try to obtain the probability, $\mathrm{P}_{\uparrow}(t)$, that the system is in $|\uparrow\rangle$ as a function of time t. Since the initial state is expanded into

$$|\psi^{\mathrm{R}}(0)\rangle = \frac{1}{\sqrt{(\Delta_d+\sqrt{\Delta_d^2+\xi^2})^2+\xi^2}}\left[\xi\,|+\rangle + (\Delta_d+\sqrt{\Delta_d^2+\xi^2})\,|-\rangle\right], \tag{1.58}$$

the system evolves in time as

$$|\psi^{\mathrm{R}}(t)\rangle = \frac{\xi e^{-i\frac{\sqrt{\Delta_d^2+\xi^2}}{2\hbar}t}\,|+\rangle + (\Delta_d + \sqrt{\Delta_d^2+\xi^2})e^{i\frac{\sqrt{\Delta_d^2+\xi^2}}{2\hbar}t}\,|-\rangle}{\sqrt{(\Delta_d + \sqrt{\Delta_d^2+\xi^2})^2 + \xi^2}}. \tag{1.59}$$

The probability $\mathrm{P}_\uparrow(t)$ is therefore given by

$$\begin{aligned}\mathrm{P}_\uparrow(t) =& |\langle\uparrow|\psi(t)\rangle|^2 = |\langle\uparrow|\psi^{\mathrm{R}}(t)\rangle|^2, \\ =& \frac{\xi^2}{\Delta_d^2+\xi^2}\sin^2\left(\frac{\sqrt{\Delta_d^2+\xi^2}}{2\hbar}t\right).\end{aligned} \tag{1.60}$$

This equation is validated for any driving frequency and strength. When the detuning energy Δ_d is large compared to the driving strength, the system almost remains to be in the initial state. The system is not heated by off-resonant driving field. On the other hand, when the system is resonant with the driving field, that is, $\Delta_d = h^z - \hbar\Omega = 0$, the system goes back and forth between $|+\rangle$ and $|-\rangle$. This phenomenon is called the Rabi oscillation and the frequency between two states are proportional to the driving amplitude ξ.

1.4.2 Linearly polarized weak driving field: Bloch-Siegert shift

From hereon, we study the second case where the system is subjected to the linearly polarized driving field, Eq. (1.54). Depending on the driving frequency and the driving amplitude, the dynamics of the two-level system are qualitatively different. We first study the case where a weak driving field is nearly resonant with the energy gap of the two-level system;

$$\xi \ll \hbar\Omega \text{ and } |\Delta_d| = |h^z - \hbar\Omega| \ll \hbar\Omega. \tag{1.61}$$

We will find a similar dynamical feature to the two-level system under a circularly polarized driving field, but the energy gap h^z is modified by the driving field.

In this case, it is also convenient to consider the rotating frame defined by the unitary operator $V(t)$, Eq. (1.54). The corresponding Hamiltonian in the rotating frame reads

$$H^{\mathrm{R}}(t) = \Delta_d S^z + \xi S^x + \frac{\xi}{2}(S^+ e^{2i\Omega t} + S^- e^{-2i\Omega t}). \tag{1.62}$$

The remaining time dependence avoids us to solve the dynamics exactly, and we have to resort to the approximation.

Since under the above conditions, Eq. (1.61), the time-dependent terms in this Hamiltonian rapidly oscillate, we can obtain the approximated Floquet Hamiltonian by using the Floquet-Magnus expansion. The first-order-truncated Floquet Hamiltonian is just the time-averaged Hamiltonian (see Eq. (1.50)),

$$H_{\mathrm{F}}^{\mathrm{R}(1)[0]} = \Delta_d S^z + \xi S^x, \tag{1.63}$$

which is the same as the Hamiltonian in the rotating frame for the two-level system under the circularly polarized driving field (see Eq. (1.55)). The second-order-truncated Hamiltonian is also obtained by calculating the commutator in Eq. (1.50), which renormalizes the coefficients as

$$H_{\mathrm{F}}^{\mathrm{R}(2)[0]} = \left(\Delta_d + \frac{\xi^2}{\hbar\Omega}\right) S^z + \xi \left(1 + \frac{\Delta_d}{2\hbar\Omega}\right) S^x. \tag{1.64}$$

The renormalization changes the resonant frequency;

$$\Delta_d + \frac{\xi^2}{\hbar\Omega} = 0 \leftrightarrow \Omega \simeq \frac{1}{\hbar}\left(h^z + \frac{\xi^2}{h^z}\right). \tag{1.65}$$

This shift of the resonance frequency is called Bloch-Siegert shift [32].

1.4.3 Linearly polarized strong driving field: Coherent destruction of tunneling

When the driving field becomes stronger, the dynamical features are qualitatively different from previous cases. In order to study the system under a strong driving field, $\xi \gg h^z$, we regard H_0 as a perturbation term [33]. There are two linearly independent solutions for the Schrödinger equation of $H_{\mathrm{ex}}(t)$ given by

$$|\psi_\pm(t)\rangle = \frac{1}{\sqrt{2}} e^{\mp \frac{i\xi \sin \Omega t}{\hbar\Omega}} (|\uparrow\rangle \pm |\downarrow\rangle). \tag{1.66}$$

These solutions which are time periodic correspond to the unperturbative Floquet modes $|v_\pm(t)\rangle$ (corresponding to $|v_{jk}(t)\rangle$ in subsection 1.3.2) with degenerate quasienergies, $\epsilon_\pm^{(0)} = 0$.

In a perturbation theory for the degenerate Floquet modes, what we have to do is solve the eigenvalue problem of the following matrix;

$$\begin{pmatrix} \langle\langle v_+(t)|H_0|v_+(t)\rangle\rangle & \langle\langle v_+(t)|H_0|v_-(t)\rangle\rangle \\ \langle\langle v_-(t)|H_0|v_+(t)\rangle\rangle & \langle\langle v_-(t)|H_0|v_-(t)\rangle\rangle \end{pmatrix} = \begin{pmatrix} 0 & \frac{h^z}{2} J_0(\frac{2\xi}{\hbar\Omega}) \\ \frac{h^z}{2} J_0(\frac{2\xi}{\hbar\Omega}) & 0 \end{pmatrix}, \tag{1.67}$$

where $J_0(\cdot)$ is the zeroth-order Bessel function. The eigenstates of this matrix are the appropriate lowest-order Floquet modes,

$$\begin{aligned} |u_1^{(0)}(t)\rangle =& \frac{1}{\sqrt{2}}(|v_+(t)\rangle + |v_-(t)\rangle) = \frac{1}{\sqrt{2}}(|\psi_+(t)\rangle + |\psi_-(t)\rangle), \\ |u_2^{(0)}(t)\rangle =& \frac{1}{\sqrt{2}}(|v_+(t)\rangle - |v_-(t)\rangle) = \frac{1}{\sqrt{2}}(|\psi_+(t)\rangle - |\psi_-(t)\rangle), \end{aligned} \tag{1.68}$$

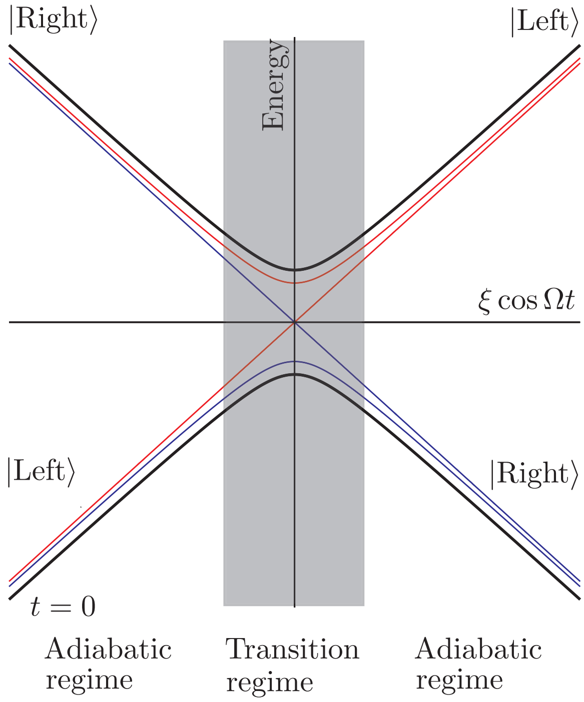

Figure 1.1: Two instantaneous eigenvalues of $H(t)$ are plotted. We divide the dynamic into two regimes, adiabatic regime and transition regime. In the adiabatic regime, two instantaneous eigenstates of $H(t)$ are denoted by $|\mathrm{Right}\rangle$ and $|\mathrm{Left}\rangle$. Gray stripe denotes the transition regime.

and the eigenvalues of this matrix are the quasienergies up to the order of h^z

$$\epsilon_1 = -\epsilon_2 = \frac{h^z}{2} J_0\left(\frac{2\xi}{\hbar\Omega}\right). \tag{1.69}$$

When we observe the dynamics at stroboscopic times $t = nT$ with integers n, the long-time dynamics is described by the Floquet Hamiltonian,

$$H_{\mathrm{F}}^{[0]} \simeq h^z J_0\left(\frac{2\xi}{\hbar\Omega}\right) S^z, \tag{1.70}$$

where we have used $|u_1^{(0)}(0)\rangle = |\uparrow\rangle$ and $|u_2^{(0)}(0)\rangle = |\downarrow\rangle$. If the system is a particle with one-half spin, the Larmor frequency is modified by the factor $J_0(2\xi/(\hbar\Omega))$. Especially when the ratio $2\xi/(\hbar\Omega)$ is one of the zeros of the zeroth-order Bessel function, the system returns back to the same state at every period, which is known as the coherent destruction of tunneling (CDT).

The origin of this phenomenon is a quantum interference effect [34, 35]. In Fig. 1.1, we plot two instantaneous eigenvalues of $H(t)$ by black curves. When the driving amplitude is sufficiently large, we can divide the dynamics into two regimes: adiabatic regime and transition regime (see Fig. 1.1). We define the transition regime where the separation of two instant energies is small, $\xi H_{\rm ex}(t) = 2\xi \cos(\Omega t) \simeq h^z$.

In the transition regime, the dynamics is regarded as a non-adiabatic transition process described by the Landau-Zener-Stückelberg process [36]. The transition probability between $|\mathrm{Left}\rangle$ and $|\mathrm{Right}\rangle$ is given by

$$\mathrm{P_{LZS}} = 1 - e^{-\frac{\pi (h^z)^2}{4\xi\hbar\Omega}}. \tag{1.71}$$

As the driving amplitude increases, the transition probability becomes small. In a diabatic limit $\xi(\hbar\Omega) \gg (h^z)^2$, the state acquires the phase shift of $\pi/4$ through the transitions between $|\mathrm{Left}\rangle$ and $|\mathrm{Right}\rangle$ [34]. In the adiabatic regime, owing to a large energy separation between two instant energy eigenstates, the dynamics is regarded as a time evolution without non-adiabatic transition, and hence we call this regime adiabatic regime.

We evaluate the probability, $\mathrm{Prob_{L\to R}}$, that the system is in $|\mathrm{Right}\rangle$ after one sweep of AC external field when we prepare the initial state at $t = 0$ in $|\mathrm{Left}\rangle$. From the path integral method, the transition probability is obtained by calculating all the probability amplitudes of paths from $|\mathrm{Left}\rangle$ at $t = 0$ and $|\mathrm{Right}\rangle$ at $t = T$. There exist two paths which give a significant contribution to the transition probability: One is the path denoted by red curve in Fig. 1.1 in which the state goes to $|\mathrm{Left}\rangle$ at $t = T/2$ and then goes to $|\mathrm{Right}\rangle$ at $t = T$. The other is the path denoted by blue curve in the figure in which the state goes to $|\mathrm{Right}\rangle$ at $t = T/2$ and then goes to $|\mathrm{Right}\rangle$ at $t = T$. Because of the symmetry of the paths, the magnitude of the transition amplitudes is the same, $\sqrt{\mathrm{P_{LZS}}} \simeq \sqrt{\pi(h^z)^2/4\xi\Omega}$. The relative phase of the transition amplitudes of two paths is approximately given by

$$\Delta\phi = 2\left(\int_{\frac{T}{4}}^{\frac{3T}{4}} \frac{\xi}{\hbar}\cos\Omega t dt + \frac{\pi}{4}\right) = -2\left(\frac{2\xi}{\hbar\Omega} - \frac{\pi}{4}\right). \tag{1.72}$$

We therefore obtain the probability by

$$\mathrm{Prob_{L\to R}} = \mathrm{P_{LZS}}|1 + e^{i\Delta\phi}|^2, \tag{1.73}$$

$$= \frac{\pi(h^z)^2}{\xi\hbar\Omega}\cos^2\left(\frac{2\xi}{\hbar\Omega} - \frac{\pi}{4}\right). \tag{1.74}$$

We compare the dynamics under a periodic driving field to a precession along z-axis at the Larmor frequency $\omega(> 0)$, and estimate the Larmor frequency ω from the transition probability $\mathrm{Prob_{L\to R}}$. The corresponding probability in the Larmor precession is the probability of the system in one of the eigenstate of S^x, $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, at time T when the state is initially prepared in other eigenstate of S^x at $t = 0$. The probability is given by $\sin^2(\pi\omega/\Omega)$. We therefore estimate the Larmor frequency in the driven two-level system as

$$\sin^2\left(\frac{\pi\omega}{\Omega}\right) = \frac{\pi(h^z)^2}{\hbar\xi\Omega}\cos^2\left(\frac{2\xi}{\hbar\Omega}-\frac{\pi}{4}\right),$$
$$\leftrightarrow \hbar\omega \simeq h^z\sqrt{\frac{\hbar\Omega}{\pi\xi}}\cos\left(\frac{2\xi}{\hbar\Omega}-\frac{\pi}{4}\right) \simeq h^z J_0\left(\frac{2\xi}{\hbar\Omega}\right), \tag{1.75}$$

which is consistent with the previous result, Eq. (1.70). This observation clearly indicates that the CDT originates from the quantum interference effect of two paths [35].

1.4.4 Linearly polarized high-frequency field: Coherent destruction of tunneling

Finally we study the case where a strong and fast driving field is applied, $\xi \gg h^z$ and $\hbar\Omega \gg h^z$, and the driving amplitude is scaled by the driving frequency as $\xi \sim \hbar\Omega$. In this case, it is convenient to transform into a rotating frame where a state in the rotating frame $|\psi^{\mathrm{R}}(t)\rangle$ is related to the state in the static frame $|\psi(t)\rangle$ by

$$|\psi^{\mathrm{R}}(t)\rangle = e^{\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau)d\tau}|\psi(t)\rangle = e^{\frac{2i\xi}{\hbar\Omega}\sin\Omega t S^x}|\psi(t)\rangle. \tag{1.76}$$

The Hamiltonian in the rotating frame is given by

$$H^{\mathrm{R}}(t) = e^{\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau)d\tau}\left(H(t) - i\hbar\frac{\partial}{\partial t}\right)e^{-\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau)d\tau},$$
$$= h^z\left[\cos\left(\frac{2\xi}{\hbar\Omega}\sin\Omega t\right)S^z + \sin\left(\frac{2\xi}{\hbar\Omega}\sin\Omega t\right)S^y\right]. \tag{1.77}$$

Since the time-dependent terms in this Hamiltonian show a rapid oscillation, we can employ the Floquet-Magnus expansion. The lowest-order term is a time average of the rotating Hamiltonian,

$$H_{\mathrm{F}}^{\mathrm{R}[0]} \simeq \int_0^T H^R(t)\frac{dt}{T} = h^z J_0\left(\frac{2\xi}{\hbar\Omega}\right)S^z, \tag{1.78}$$

which is the same form as the previous case (see Eq. (1.70)). Thus, when the ratio $2\xi/(\hbar\Omega)$ is one of the zeros of the Bessel function, the CDT occurs. This phenomenon is originally proposed in a particle system with a double-well potential [8] and is now extended to a two-level system [37, 35] and a system with double quantum dots [38].

If we take the high-frequency limit keeping ξ fixed, the effect of the driving field disappears in $H_{\mathrm{F}}^{\mathrm{R}[0]}$ because the Bessel function approaches one. In the limit of $\Omega \to \infty$, the simultaneous scaling of the driving amplitude and the driving frequency is thus necessary to obtain the non-trivial Floquet Hamiltonian, which is different from the undriven Hamiltonian H_0. This scaling is used in the field of the Floquet engineering [12], which is explained in next section.

1.5 Floquet engineering

As mentioned before, the two-level system describes double quantum dots where an electron tunnels between adjacent two dots. The tunneling strength determined by h^z can be modified into $h^z J_0(2\xi/\hbar\Omega)$ by using, e.g., a fast and strong driving field. These observations lead us to control the dynamical properties of lattice systems, e.g. tight-binding model [39], Hubbard model [40], and graphene [41], and so on by using the driving field. With the progress of cold atomic systems, the driving field is now a new knob to engineer effective Hamiltonians with novel properties which are absent in equilibrium systems or difficult to realize. We overview recent achievements in this direction: quantum phase transition using the periodic driving field.

1.5.1 Dynamical localization and superfluid-insulator transition

We first consider a tight-binding model subjected to an external driving field. The Hamiltonian is given by

$$\begin{aligned} H(t) =& H_0 + \xi H_{\mathrm{ex}}(t), \\ H_0 =& h \sum_{n=-\infty}^{\infty} [|n+1\rangle \langle n| + |n\rangle \langle n+1|], \\ \xi H_{\mathrm{ex}}(t) =& 2\xi \cos(\Omega t) \sum_{n=-\infty}^{\infty} n \, |n\rangle \langle n| \, , \end{aligned} \tag{1.79}$$

where n denotes each cite and h denotes a tunneling strength between adjacent cites.

In this system, we can obtain an explicit form of the Floquet Hamiltonian. It is also convenient to move to a rotating frame. Again, the state in the rotating frame $|\psi^{\mathrm{R}}(t)\rangle$ is related to the state in the static frame $|\psi(t)\rangle$ by

$$|\psi^{\mathrm{R}}(t)\rangle = e^{\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau) d\tau} |\psi(t)\rangle = e^{i\frac{2\xi}{\hbar\Omega}\sin\Omega t \sum_{n=-\infty}^{\infty} n|n\rangle\langle n|} |\psi(t)\rangle \, . \tag{1.80}$$

The state in the rotating frame obeys the Schrödinger equation of the following Hamiltonian,

$$\begin{aligned} H^{\mathrm{R}}(t) =& e^{\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau) d\tau} H_0 e^{-\frac{i}{\hbar}\int_0^t \xi H_{\mathrm{ex}}(\tau) d\tau} \\ =& h \sum_{n=-\infty}^{\infty} [|n\rangle \langle n+1| \, e^{-i\frac{2\xi}{\hbar\Omega}\sin\Omega t} + |n+1\rangle \langle n| \, e^{i\frac{2\xi}{\hbar\Omega}\sin\Omega t}]. \end{aligned} \tag{1.81}$$

Since the following two operators are commutable [42],

$$\left[\sum_{n=-\infty}^{\infty} |n\rangle \langle n+1| \, , \sum_{m=-\infty}^{\infty} |m+1\rangle \langle m| \right] = 0, \tag{1.82}$$

we can evaluate the time evolution over one period as

$$
\begin{aligned}
|\psi(T)\rangle =& e^{-\frac{i}{\hbar}\int_0^T \xi H_{\rm ex}(\tau)d\tau} |\psi^{\rm R}(T)\rangle ,\\
=& e^{-\frac{i}{\hbar}\int_0^T h\sum_{n=-\infty}^{\infty}|n\rangle\langle n+1|e^{-i\frac{2\xi}{\hbar\Omega}\sin\Omega t}dt} e^{-\frac{i}{\hbar}\int_0^T h\sum_{n=-\infty}^{\infty}|n+1\rangle\langle n|e^{i\frac{2\xi}{\hbar\Omega}\sin\Omega t}dt} |\psi^{\rm R}(0)\rangle ,\\
=& e^{-\frac{i}{\hbar}hJ_0\left(\frac{2\xi}{\hbar\Omega}\right)\sum_{n=-\infty}^{\infty}[|n\rangle\langle n+1|+|n+1\rangle\langle n|]T} |\psi(0)\rangle .
\end{aligned}
\tag{1.83}
$$

The Floquet Hamiltonian is therefore given by

$$
H_{\rm F}^{[0]} = hJ_0\left(\frac{2\xi}{\hbar\Omega}\right)\sum_{n=-\infty}^{\infty}[|n\rangle\langle n+1| + |n+1\rangle\langle n|]. \tag{1.84}
$$

We control the tunneling strength by adjusting the driving amplitude and frequency. Especially, when $2\xi/\hbar\Omega$ is one of zeros of the zeroth-order Bessel function, an electron localized in a cite at $t = 0$ returns back to the same cite at stroboscopic times $t = nT$ with integers n. This phenomenon is called the Dynamical localization [39], and it is observed in a system of cold atoms in an optical lattice [43].

Next, we consider a one-dimensional softcore Bose Hubbard model with on-site potential U. The Hamiltonian reads

$$
\begin{aligned}
H(t) &= H_0 + \xi H_{\rm ex}(t),\\
H_0 &= h\sum_{j=-\infty}^{\infty}(c_{j+1}^\dagger c_j + c_{j+1}c_j^\dagger) + U\sum_j n_j(n_j-1),\\
\xi H_{\rm ex}(t) &= 2\xi\cos(\Omega t)\sum_{j=-\infty}^{\infty} jn_j,
\end{aligned}
\tag{1.85}
$$

where $c_j^\dagger$ and c_j are creation and annihilation bosonic operator on site j, respectively, and $n_j = c_j^\dagger c_j$ is a number operator of site j. We study the system under a driving field with large amplitude and high frequency, $\xi \gg (h, U)$, $\hbar\Omega \gg (h, U)$, and $\xi \sim \hbar\Omega$ [40].

We also move to a rotating frame by using a similar unitary transformation to the previous case, in which the Hamiltonian reads

$$
\begin{aligned}
H^{\rm R}(t) =& e^{\frac{i}{\hbar}\int_0^t H_{\rm ex}(\tau)d\tau} H_0 e^{-\frac{i}{\hbar}\int_0^t H_{\rm ex}(\tau)d\tau}\\
=& h\sum_{j=-\infty}^{\infty}[c_j^\dagger c_{j+1}e^{-i\frac{2\xi}{\hbar\Omega}\sin\Omega t} + c_{j+1}^\dagger c_j e^{i\frac{2\xi}{\hbar\Omega}\sin\Omega t}] + U\sum_{j=-\infty}^{\infty} n_j(n_j-1).
\end{aligned}
\tag{1.86}
$$

We approximately evaluate the Floquet Hamiltonian by using the Floquet-Magnus expansion. The first-order-truncated Floquet Hamiltonian is the time average of $H^{\rm R}(t)$;

$$
H_{\rm F}^{\rm R(1)[0]} = \int_0^T H^R(t)\frac{dt}{T} = hJ_0\left(\frac{2\xi}{\hbar\Omega}\right)\sum_{j=-\infty}^{\infty}[c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j] + U\sum_{j=-\infty}^{\infty} n_j(n_j-1). \tag{1.87}
$$

The effects of driving field are renormalized into the tunneling strength as observed in the driven tight binding model.

The ground state of this system with large tunneling strength $hJ_0(2\xi/\hbar\Omega)$ is a superfluid phase, while the ground state with large onsite potential U is a Mott-insulator phase. Between these qualitatively different ground states, a quantum phase transition occurs. Since the tunneling strength can be tuned by sweeping the driving amplitude, we can control the macroscopic features of the system by applying the driving field [40].

This phase transition has been observed in a cold atom system [44]. Owing to the adiabatic theorem on the Floquet state [25], the slow sweeping of the driving amplitude keeps the system to be in the ground state of the instantaneous Floquet Hamiltonian. This procedure makes it possible to change the state between a superfluid phase and a Mott-insulator phase repeatedly.

1.5.2 The role of truncated Floquet Hamiltonians

So far we have observed that the driving field induces peculiar dynamical features, and it now plays a role to control the macroscopic properties of the system. The off-resonant and strong oscillating field is used to create these properties in experiment, and their properties are investigated by the truncated Floquet Hamiltonian [12], which is obtained by truncating up to the first or second order of the Floquet-Magnus expansion.

However, the validity of the truncated Floquet Hamiltonian is still now under debating, because the convergence of the Floquet-Magnus expansion is not ensured for macroscopic systems. It has been known that the sufficient condition for the convergence is [27]

$$\max_t \|H(t)\| < \frac{\hbar\Omega}{2}, \tag{1.88}$$

but this condition is useless for a large system because the norm of the Hamiltonian is proportional to the system size. When the system Hamiltonian including driving fields is written by a bilinear form of fermionic operators or bosonic operators, we can divide the Hilbert space into product of independent Hilbert spaces, whose size is independent of system size. In this case, the condition Eq. (1.88) can be applied to each small Hilbert space [12], but this situation is limited to such an integrable system.

There exist other conditions which can be applied to large systems. Recent successive papers have shown that there exists an truncated Floquet Hamiltonian with optimal order that accurately describes long-time transient states when $\hbar\Omega$ is much larger than the energy to flip a single spin [45, 46, 47]. Since the truncated Floquet Hamiltonian is quasi-conserved quantity in this timescale, the transient states can be described by the microcanonical ensemble for the corresponding energy of the truncated Hamiltonian [48, 49]. Thus, the eigenbasis of the truncated Floquet Hamiltonian is better to describe this transient states than that of the Floquet Hamiltonian when the Floquet-Magnus expansion is not converged.

It has been extensively studied on the long-time asymptotic states of the coherent system subjected to a driving field. The notion of eigenstate thermalization hypothesis (ETH) [50, 51] has been extended to systems subjected to a periodic driving field. The ETH states that the expectation values of local observables with respect to eigenstates have thermal values in the thermodynamic limit. Since there is no conserved energy in a periodically driven system, each Floquet state is expected to be indistinguishable with an infinite temperature state as far as we observe local observables. It is still an open problem to give conditions for this hypothesis to hold; Numerical studies show the heating up to the infinite temperature state in some systems [52, 53, 54], but it does not occur in other systems and/or situations [55, 56].

Chapter 2

Periodically driven open systems

In this section we overview the master equation formalism which describes dynamics of a system of interest under a periodic driving field in a dissipative environment. First, in section 2.1, we review the derivation of two types of master equations: Redfield equation and Floquet Lindblad equation. Next, in section 2.2, we introduce a phenomenological master equation for the study of a driven cavity system in a dissipative environment.

2.1 Master equation formalism

The dissipation effects appear through the interaction between a large system. The total system is modeled by the Hamiltonian,

$$H(t) = H_{\mathrm{S}}(t) + H_{\mathrm{B}} + \lambda H_{\mathrm{I}}, \tag{2.1}$$

where $H_{\mathrm{S}}(t)$ and H_{B} are the Hamiltonians for the system of interest and the heat bath, respectively. For a while, we consider a generic time-dependent system Hamiltonian, and thus we do not impose the time-periodicity on $H_{\mathrm{S}}(t)$. The interaction Hamiltonian between the system of interest and a heat bath is given by H_{I} with the interaction strength λ, which is assumed to be small. Without loss of generality, we can decompose the interaction Hamiltonian into sum of product of Hermite operators of the system of interest and the heat bath [18];

$$H_{\mathrm{I}} = \sum_{\mu} X^{\mu} \otimes Y^{\mu}. \tag{2.2}$$

The Hermite operators for the system of interest and the heat bath are denoted by X^{μ} and Y^{μ}, respectively.

The Liouville equation gives a time evolution of the total system,

$$\frac{dW(t)}{dt} = -\frac{i}{\hbar}L(t)W(t) = -\frac{i}{\hbar}(L_{\mathrm{S}}(t) + L_{\mathrm{B}} + \lambda L_{\mathrm{I}})W(t), \tag{2.3}$$

where $L_\alpha(\{\alpha\} = \mathrm{S}, \mathrm{B}, \mathrm{I})$ denote Liouville operators for the system of interest, the heat bath, and interaction between them, respectively. The Liouville operators are defined by a commutator such as

$$L(t)\cdot = [H(t), \cdot]. \tag{2.4}$$

The density matrix of the whole system including the heat bath is defined by $W(t)$. We assume that at initial time $t = t_0$, there is no correlation between the system of interest and the heat bath;

$$W(t_0) = \rho(t_0) \otimes \rho_{\mathrm{B}}. \tag{2.5}$$

The state of the heat bath is described by the Boltzmann distribution,

$$\rho_{\mathrm{B}} = \frac{e^{-\beta H_{\mathrm{B}}}}{\mathrm{Tr}_{\mathrm{B}} e^{-\beta H_{\mathrm{B}}}}, \tag{2.6}$$

where the symbol Tr_{B} stands for the trace operation over the bath degrees of freedom and β is the inverse temperature of the heat bath. We assume that this canonical state has a Gaussian property on correlation functions of the operator Y^μ, e.g., a four-body correlation function can be decoupled into the sum of the product of two-body correlation functions. We also assume that the time correlation function, $\mathrm{Tr}_{\mathrm{B}}(Y^\mu(t) Y^\nu \rho_{\mathrm{B}})$, decays at a certain time τ_{bath} which denotes the typical timescale of the bath dynamics. Without loss of generality we can set the constant force due to the interaction Hamiltonian to be zero,

$$\lambda \langle H_{\mathrm{I}} \rangle_{\mathrm{B}} \equiv \lambda \mathrm{Tr}_{\mathrm{B}}(H_{\mathrm{I}} \rho_{\mathrm{B}}) = 0, \tag{2.7}$$

because we can eliminate it by redefining the system Hamiltonian as $H_{\mathrm{S}}(t) + \lambda \langle H_{\mathrm{I}} \rangle_{\mathrm{B}} \to H_{\mathrm{S}}(t)$. Here, $\langle \cdot \rangle_{\mathrm{B}}$ represents the average of the operator over the Boltzmann distribution of the heat bath,

If we focus on the observables of the system of interest, the reduced density matrix $\rho(t) = \mathrm{Tr}_{\mathrm{B}} W(t)$ is sufficient to evaluate them. Under the above conditions, we obtain the Markov equation of $\rho(t)$, which is called the Born-Markov master equation. There are two types of master equations, the Redfield equation and the Floquet Lindblad equation. We overview the derivation of the Redfield equation in Sec. 2.1.1 and the Floquet Lindblad equation in Sec. 2.1.2, and then we compare two types of the master equations in Sec. 2.1.3. Finally, we review the reason why the long-time asymptotic states subjected to a periodic driving field does not have a universal description in the framework of the Lindblad formulation.

2.1.1 Derivation of Redfield equation (finite system-bath coupling)

We first define the projection operators P and Q [57], which act as

$$\mathrm{P} f = (\mathrm{Tr}_{\mathrm{B}} f) \otimes \rho_{\mathrm{B}}, \quad \mathrm{Q} = 1 - \mathrm{P}. \tag{2.8}$$

where f is an arbitrary operator defined in the total Hilbert space. From the Liouville equation, Eq. (2.3), we obtain sets of equations:

$$\begin{cases} \dfrac{d\mathrm{P}W(t)}{dt} &= -\dfrac{i}{\hbar}\mathrm{P}L(t)\mathrm{P}W(t) - \dfrac{i}{\hbar}\mathrm{P}L(t)\mathrm{Q}W(t), \\ \dfrac{d\mathrm{Q}W(t)}{dt} &= -\dfrac{i}{\hbar}\mathrm{Q}L(t)\mathrm{P}W(t) - \dfrac{i}{\hbar}\mathrm{Q}L(t)\mathrm{Q}W(t), \end{cases} \tag{2.9}$$

where we have used $[\mathrm{P}, d/dt] = 0$. From the second equation, we can solve $\mathrm{Q}W(t)$ as

$$\mathrm{Q}W(t) = -\frac{i}{\hbar}\int_{t_0}^{t} \mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}\mathrm{Q}L(\tau)\mathrm{Q}d\tau}\mathrm{Q}L(t_1)\mathrm{P}W(t_1)dt_1, \tag{2.10}$$

where we have used the assumption that there is no correlation between the system of interest and the heat bath in the initial state, $\mathrm{Q}W(t_0) = 0$. By substituting this form into the first one of Eqs. (2.9), we obtain the closed equation for the reduced density matrix $\rho(t)$ as

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}L_{\mathrm{S}}(t)\rho(t) - \frac{\lambda^2}{\hbar^2}\int_{t_0}^{t}\langle L_{\mathrm{I}}\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}\mathrm{Q}L(\tau)\mathrm{Q}d\tau}L_{\mathrm{I}}\rho(t_1)\rangle_{\mathrm{B}}dt_1, \tag{2.11}$$

which is called Nakajima-Zwanzig equation [58, 57].

By using the following relation,

$$\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}\mathrm{Q}L(\tau)\mathrm{Q}d\tau} = \mathcal{T}e^{-\frac{i}{\hbar}\mathrm{Q}(L_{\mathrm{S}}(\tau)+L_{\mathrm{B}})d\tau}\mathcal{T}e^{-\frac{i}{\hbar}\lambda\int_{t_1}^{t}\mathrm{Q}L_{\mathrm{I}}(t_1,\tau)\mathrm{Q}d\tau}, \tag{2.12}$$

where

$$L_{\mathrm{I}}(t,t') = \mathcal{T}e^{-\frac{i}{\hbar}\int_{t'}^{t}(L_{\mathrm{S}}(\tau)+L_{\mathrm{B}})d\tau}L_{\mathrm{I}}, \tag{2.13}$$

and by expanding the second exponential in the right hand side of Eq. (2.12) into the power series of λ, Eq. (2.11) up to the order of λ^4 reads

$$\begin{aligned} \frac{d\rho(t)}{dt} &= -\frac{i}{\hbar}L_{\mathrm{S}}(t)\rho(t) \\ &- \frac{\lambda^2}{\hbar^2}\int_{t_0}^{t}dt_1\langle L_{\mathrm{I}}L_{\mathrm{I}}(t,t_1)\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}L_{\mathrm{S}}(\tau)d\tau}\rho(t_1)\rangle_{\mathrm{B}} \\ &+ i\frac{\lambda^3}{\hbar^3}\int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2\langle L_{\mathrm{I}}\mathrm{Q}L_{\mathrm{I}}(t,t_2)\mathrm{Q}L_{\mathrm{I}}(t,t_1)\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}L_{\mathrm{S}}(\tau)d\tau}\rho(t_1)\rangle_{\mathrm{B}} \\ &+ \frac{\lambda^4}{\hbar^4}\int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2\int_{t_2}^{t}dt_3\langle L_{\mathrm{I}}\mathrm{Q}L_{\mathrm{I}}(t,t_3)\mathrm{Q}L_{\mathrm{I}}(t,t_2)\mathrm{Q}L_{\mathrm{I}}(t,t_1)\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^{t}L_{\mathrm{S}}(\tau)d\tau}\rho(t_1)\rangle_{\mathrm{B}}. \end{aligned} \tag{2.14}$$

The λ^2 term is the order of $\lambda^2\tau_{\mathrm{bath}}/\hbar^2$ because the time correlation function of Y^μ has a significant contribution only in the regime, $t - \tau_{\mathrm{bath}} < t_1 < t$. The λ^3 term including three-body bath correlation function of Y^μ is zero, because the correlation function can be decomposed into the product of two-body correlation function and expectation value of Y^μ. The λ^4 term includes the product of four bath operators, Y^μ. The representative term consists of

$$\langle Y^\mu(t) Q Y^{\mu'}(t_3) Q Y^\nu(t_2) Q Y^{\nu'}(t_1)\rangle_{\rm B}, \tag{2.15}$$

where $Y^\mu(t) = e^{\frac{i}{\hbar}L_{\rm B}t}Y^\mu$. Owing to the existence of the projection operator Q and gaussian property of the bath state, we can express it by

$$\langle Y^\mu(t) Y^{\nu'}(t_1)\rangle_{\rm B}\langle Y^{\mu'}(t_3) Y^\nu(t_2)\rangle_{\rm B} \pm \langle Y^\mu(t) Y^\nu(t_2)\rangle_{\rm B}\langle Y^{\mu'}(t_3) Y^{\nu'}(t_1)\rangle_{\rm B}, \tag{2.16}$$

where $\pm$ depends on whether the heat bath consists of boson particles (+) or fermion particles (-). It is important to note that these combinations of the correlation functions contribute to Eq. (2.14) only when $t - t_1 \sim {\rm O}(\tau_{\rm bath})$. This argument holds also in other terms, and hence the λ^4 term is the order of $\lambda^4\tau_{\rm bath}^3/\hbar^4$. Now, we use the assumption that the system of interest weakly couples to a heat bath $\lambda^2\tau_{\rm bath}^2/\hbar^2 \ll 1$, and perform the Born approximation, i.e., we keep terms up to the order of λ^2;

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}L_{\rm S}(t)\rho(t) - \frac{\lambda^2}{\hbar^2}\int_{t_0}^t dt_1 \langle L_{\rm I} L_{\rm I}(t,t_1)\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_1}^t L_{\rm S}(\tau)d\tau}\rho(t_1)\rangle_{\rm B}. \tag{2.17}$$

In order to perform the Markov approximation, it is convenient to move to a rotating frame in which the reduced density operator in the rotating frame $\sigma(t)$ is given by

$$\rho(t) = \mathcal{T}e^{-\frac{i}{\hbar}\int_{t_0}^t L_{\rm S}(\tau)d\tau}\sigma(t). \tag{2.18}$$

The equation (2.17) is transformed as

$$\frac{d\sigma(t)}{dt} = -\frac{\lambda^2}{\hbar^2}\int_{t_0}^t dt_1 \langle L_{\rm I}(t_0,t) L_{\rm I}(t_0,t_1)\sigma(t_1)\rangle_{\rm B}. \tag{2.19}$$

Because the right hand side is the order of $\lambda^2\tau_{\rm bath}/\hbar^2$ and the integrand contributes only in the regime, $t - \tau_{\rm bath} < t_1 < t$, for this time region of t_1,

$$\sigma(t_1) = \sigma(t) + O(\lambda^2\tau_{\rm bath}^2/\hbar^2). \tag{2.20}$$

Furthermore, if we focus on the long-time dynamics $t - t_0 \gg \tau_{\rm bath}$, we can take the integration range from $[t_0, t]$ to $[-\infty, t]$. These observations lead to the Markov equation, by putting t_1 by $t - \tau$,

$$\frac{d\sigma(t)}{dt} = -\frac{\lambda^2}{\hbar^2}\int_0^\infty d\tau \langle L_{\rm I}(t_0,t) L_{\rm I}(t_0,t-\tau)\sigma(t)\rangle_{\rm B}. \tag{2.21}$$

When we transform back to the static frame, the equation is given by

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}L_{\rm S}(t)\rho(t) - \frac{\lambda^2}{\hbar^2}\int_0^\infty d\tau \langle L_{\rm I} L_{\rm I}(t,t-\tau)\rho(t)\rangle_{\rm B} d\tau. \tag{2.22}$$

We call this equation the "Redfield equation". Up to here, we have not used the periodicity of the system Hamiltonian $H_\mathrm{S}(t)$

In order to derive the Redfield equation, we perform the Born-Markov approximation on the Nakajima-Zwanzig equation, Eq. (2.11). This approximation is justified when $\lambda^2\tau_\mathrm{bath}^2/\hbar^2$ is sufficiently small. The relaxation timescale of the system of interest approaching a long-time asymptotic state is determined by the order of the right hand side of Eq. (2.21) given by

$$\tau_\mathrm{relax} = \left(\frac{\lambda^2\tau_\mathrm{bath}}{\hbar^2}\right)^{-1}, \tag{2.23}$$

and τ_bath represents the timescale of the bath dynamics. This approximation is therefore justified when the relaxation timescale of the system is much larger the timescale of the heat bath.

It is noted that this approximation excludes higher-order processes within a heat bath, e.g., Raman process. These processes may become important when the system-bath coupling is stronger and/or observation timescale is much longer than $\mathrm{O}(\lambda^{-2})$ such as $\mathrm{O}(\lambda^{-4})$. Here, we restrict ourselves to systems with weak system-bath coupling strength, and discuss the long-time asymptotic state within the Born-Markov approximation.

2.1.2 Derivation of Floquet Lindblad equation (infinitesimal system-bath coupling)

In order to obtain simpler equation, it is necessary to resort to a further approximation called the rotating wave approximation (RWA) or secular approximation. From hereon, we impose the time periodicity on the system Hamiltonian,

$$H_\mathrm{S}(t) = H_\mathrm{S}(t+T). \tag{2.24}$$

We denote the Floquet modes and quasienergies by $|u_a(t)\rangle$ and ϵ_a, respectively. Here, we assume the non-resonance condition of the quasienergy;

$$\text{if } \epsilon_a - \epsilon_b + n\hbar\Omega = \epsilon_c - \epsilon_d + n'\hbar\Omega \text{ and } a \neq b \text{ then } \epsilon_a = \epsilon_c, \epsilon_b = \epsilon_d, \text{ and } n = n', \tag{2.25}$$

which also imposes no degeneracy on quasienergy spectrum.

We first introduce an operator,

$$\Pi^\mu_{abn}(t) = \left(\int_0^T \langle u_a(\tau)| X^\mu |u_b(\tau)\rangle e^{-in\Omega(\tau-t)} \frac{d\tau}{T}\right) |u_a(t)\rangle \langle u_b(t)|. \tag{2.26}$$

By using this operator, we can rewrite Eq. (2.21) as

$$\frac{d\sigma(t)}{dt} = -\frac{\lambda^2}{\hbar^2} \sum_{a,b,c,d} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} e^{-i(\omega_{abn}-\omega_{cdn'})(t-t_0)} \sum_{\mu,\nu} \times \{G_+^{\mu\nu}(\omega_{cdn'})[\Pi^{\mu\dagger}_{abn}(t_0), \Pi^\nu_{cdn'}(t_0)\sigma(t)] - G_-^{\nu\mu}(-\omega_{cdn'})[\Pi^{\mu\dagger}_{abn}(t_0), \sigma(t)\Pi^\nu_{cdn'}(t_0)]\}, \tag{2.27}$$

where $\hbar\omega_{abn} = \epsilon_a - \epsilon_b + n\hbar\Omega$ and

$$G_{\pm}^{\mu\nu}(\omega) = \pm \int_0^{\pm\infty} \langle Y^\mu(\tau) Y^\nu \rangle_{\mathrm{B}} e^{-i\omega t} dt. \tag{2.28}$$

In the RWA we neglect the terms except for the elements $(a, b, c, d, n, n') \in \mathcal{A}$ satisfying

- $\epsilon_a = \epsilon_c, \epsilon_b = \epsilon_d, n = n', \epsilon_a \neq \epsilon_b,$
- $\epsilon_a = \epsilon_b, \epsilon_c = \epsilon_d, n = n',$

for which the phase factor in Eq. (2.27) proportional to $\omega_{abn} - \omega_{cdn'}$ is zero. Among the elements $(a, b, c, d, n, n') \notin \mathcal{A}$, we define

$$\omega_{\min} = \min_{(a,b,c,d,n,n') \notin \mathcal{A}} |\omega_{abn} - \omega_{cdn'}|, \tag{2.29}$$

which describes the rate of the slowest mode of the density matrix for the system of interest. The RWA is justified when the timescale of this slowest mode is much smaller than the relaxation timescale of the system,

$$\omega_{\min}^{-1} \ll \tau_{\mathrm{relax}}. \tag{2.30}$$

This approximation leads to the "Floquet Lindblad equation";

$$\begin{aligned}
\frac{d\rho(t)}{dt} =& -\frac{i}{\hbar}[H_{\mathrm{S}}(t) + H_{\mathrm{Lamb}}(t), \rho(t)] \\
&- \frac{\lambda^2}{\hbar^2} \sum_{\substack{a,b \\ (a \neq b)}} \sum_{n=-\infty}^{\infty} \sum_{\mu,\nu} G^{\mu\nu}(\omega_{abn}) \left[\frac{1}{2} \{ \Pi_{abn}^{\mu\dagger}(t) \Pi_{abn}^{\nu}(t), \rho(t) \} - \Pi_{abn}^{\nu}(t) \rho(t) \Pi_{abn}^{\mu\dagger}(t) \right] \\
&- \frac{\lambda^2}{\hbar^2} \sum_{a,c} \sum_{n=-\infty}^{\infty} \sum_{\mu,\nu} G^{\mu\nu}(n\Omega) \left[\frac{1}{2} \{ \Pi_{aan}^{\mu\dagger}(t) \Pi_{ccn}^{\nu}(t), \rho(t) \} - \Pi_{ccn}^{\nu}(t) \rho(t) \Pi_{aan}^{\mu\dagger}(t) \right],
\end{aligned} \tag{2.31}$$

where $\{\cdot, \cdot\}$ is an anticommutator and

$$G^{\mu\nu}(\omega) = G_+^{\mu\nu}(\omega) + G_-^{\mu\nu}(\omega) = \int_{-\infty}^{\infty} \langle Y^\mu(t) Y^\nu \rangle_{\mathrm{B}} e^{-i\omega t} dt. \tag{2.32}$$

The Hamiltonian $H_{\mathrm{Lamb}}(t)$ represents the Lamb shift, which is explicitly given by

$$\begin{aligned}
H_{\mathrm{Lamb}}(t) =& -i \frac{\lambda^2}{2\hbar^2} \sum_{n=-\infty}^{\infty} \sum_{\mu,\nu} \Big\{ \sum_{\substack{a,b \\ (a \neq b)}} (G_+^{\mu\nu}(\omega_{abn}) - G_-^{\mu\nu}(\omega_{abn})) \Pi_{abn}^{\mu\dagger}(t) \Pi_{abn}^{\nu}(t) \\
&+ \sum_{a,c} (G_+^{\mu\nu}(n\Omega) - G_-^{\mu\nu}(n\Omega)) \Pi_{aan}^{\mu\dagger}(t) \Pi_{ccn}^{\nu}(t) \Big\},
\end{aligned} \tag{2.33}$$

which is an Hermite operator and renormalized into the system Hamiltonian. It is also noted that this operator commutes with the Floquet Hamiltonian of the system of interest at time t, i.e., $[H_{\mathrm{F}}^{[t]}, H_{\mathrm{Lamb}}(t)] = 0$, and hence the Lamb shift Hamiltonian does not contribute to the transition among the Floquet states of the system of interest.

We expand the Floquet Lindblad equation by using the basis of the Floquet states of the system of interest. We then obtain the set of equations for the elements of density matrix in this basis given by

$$\rho_{ij}(t) = \langle u_i(t)| \rho(t) |u_j(t)\rangle . \tag{2.34}$$

Particularly for the diagonal elements $\{\rho_{ii}(t)\}$, the set of equations is closed;

$$\frac{d\rho_{ii}(t)}{dt} = \sum_{\substack{j\\(i\neq j)}} [T_{j\to i}\rho_{jj}(t) - T_{i\to j}\rho_{ii}(t)]. \tag{2.35}$$

Here, $T_{i\to j}$ is the transition probability from the Floquet state $|u_i(t)\rangle$ to $|u_j(t)\rangle$ given by

$$T_{i\to j} = \frac{\lambda^2}{\hbar^2} \sum_{n=-\infty}^{\infty} \sum_{\mu,\nu} G^{\mu\nu}(\omega_{jin}) X^{\mu *}_{jin} X^{\nu}_{jin}, \tag{2.36}$$

where X^{μ}_{jin} denotes the Fourier components of the matrix elements of X^{μ},

$$X^{\mu}_{jin} = \int_0^T \langle u_j(t)| X^{\mu} |u_i(t)\rangle e^{-in\Omega t} \frac{dt}{T}. \tag{2.37}$$

This type of equation Eq. (2.35) is often referred to as the Pauli's master equation [14, 15] or the rate equation [13].

The off-diagonal elements of density matrix $\rho_{ij}(t)$ for $i \neq j$ obey

$$\frac{d\rho_{ij}(t)}{dt} = [-i(\epsilon_i - \epsilon_j) - \Gamma_{ij}]\rho_{ij}(t). \tag{2.38}$$

Since the real part of Γ_{ij} is positive, each off-diagonal element decays to be zero while oscillating at frequency, $\epsilon_i - \epsilon_j + \mathrm{Im}\Gamma_{ij}$. The off-diagonal element disappears in the long-time asymptotic state [59].

As a result, the density matrix for the long-time asymptotic state in the Lindblad formalism, $\rho^{A}_{\mathrm{Lind}}(t)$, is given in the diagonalized form of the Floquet state, [14, 13]

$$\rho^{A}_{\mathrm{Lind}}(t) = \sum_i \mathrm{P}_i |u_i(t)\rangle \langle u_i(t)| , \tag{2.39}$$

in which the Floquet occupations $\mathrm{P}_i (= \rho_{ii})$ are obtained by putting the right hand side of Eq. (2.35) to be zero;

$$0 = \sum_{\substack{j\\(i\neq j)}} [T_{j\to i}\mathrm{P}_j - T_{i\to j}\mathrm{P}_i]. \tag{2.40}$$

The long-time asymptotic state is periodic in time, $\rho^A_{\rm Lind}(t+T) = \rho^A_{\rm Lind}(t)$. Since the sum of P_i over i is normalized to be one, we can interpret the set of $\{\mathrm{P}_i\}$ as the probability distribution.

2.1.3 Comparison between the Redfield equation and the Floquet Lindblad equation

We have derived the Redfield equation Eq. (2.22) and the Floquet Lindblad equation Eq. (2.31). First we discuss again the region of the applicability of these two master equations. In the derivation of the Redfield equation, we have used the Born-Markov approximation. This approximation is justified when the relaxation timescale of the system of interest is much longer than the bath correlation times, $\tau_{\rm relax} \gg \tau_{\rm bath}$. In order to derive the Floquet Lindblad equation, the non-resonance condition and the RWA are necessary. The RWA is justified when the relaxation timescale of the system is also much longer than the timescale of the slowest mode of the system, $\tau_{\rm relax} \gg \omega_{\rm min}^{-1}$. These arguments on conditions are listed in the following table.

master equation	conditions
Redfield equation Eq. (2.22)	$\tau_{\rm relax} \gg \tau_{\rm bath}$
Floquet Lindblad equation Eq. (2.31)	non-resonance condition $\tau_{\rm relax} \gg \tau_{\rm bath}$, $\tau_{\rm relax} \gg \omega_{\rm min}^{-1}$

We discuss the relation of the long-time asymptotic states obtained by the Lindblad equation $\rho^{\rm A}_{\rm Lind}(t)$ and the Redfield equation $\rho^{\rm A}_{\rm Red}(t)$. The Pauli's master equation, Eq. (2.35), gives the long-time asymptotic state for the Floquet Lindblad equation. Since we can eliminate λ-dependence of the Pauli's master equation by rescaling time as $t \to \tau = \lambda^2 t$, the long-time asymptotic state is independent of the system-bath coupling strength λ. On the other hand, the asymptotic state for the Redfield equation depends on the coupling strength. Actually, the long-time asymptotic state obtained by the Pauli's master equation is identical to that of the Redfield equation only in the weak coupling limit, $\lambda \to 0$.

In order to see this relation, we define the long-time asymptotic state of the Redfield equation by imposing the time pediodicty on the density matrix,

$$\rho^{\rm A}_{\rm Red}(t+T) = \rho^{\rm A}_{\rm Red}(t). \qquad (2.41)$$

In this subsection, since we will address only $\rho^{\rm A}_{\rm Red}(t)$, the label "A" and "Red" are dropped. We expand the asymptotic density matrix into the power series of λ^2,

$$\rho(t) = \rho^{(0)}(t) + \lambda^2 \rho^{(2)}(t) + \mathrm{O}(\lambda^4), \qquad (2.42)$$

where each term is expanded into Fourier components, for $l = 0$ and 2,

$$\rho^{(l)}_{ijn} = \int_0^T \langle u_i(t)|\, \rho^{(l)}(t)\, |u_j(t)\rangle\, e^{-in\Omega t} \frac{dt}{T}. \qquad (2.43)$$

Substituting this form into the Redfield equation Eq. (2.22) and comparing the order of λ^0, we obtain

$$0 = -i\omega_{ijn}\rho_{ijn}^{(0)}. \tag{2.44}$$

Using the assumption that there exists no degeneracy in quasienergy spectrum, we evaluate the Fourier components of the 0th-order density matrix, except for $i = j$ and $n = 0$,

$$\rho_{ijn}^{(0)} = 0. \tag{2.45}$$

Thus, the asymptotic density matrix in the Floquet basis is diagonalized and time independent in the weak coupling limit ($\lambda \to 0$).

In order to obtain the diagonal elements, we evaluate the order of the λ^2. The equations for the asymptotic density matrix in the order of λ^2 reads

$$\begin{aligned}
0 = &- i\omega_{ijn}\rho_{ijn}^{(2)} \\
&- \frac{1}{\hbar^2}\sum_{\mu,\nu}\sum_{k}\sum_{n'=-\infty}^{\infty}\{G_+^{\mu\nu}(\omega_{kjn'})X_{kjn'}^{\nu}X_{ki(n'-n)}^{\mu*}\rho_{jj0}^{(0)} + G_-^{\nu\mu}(-\omega_{ikn'})X_{ikn'}^{\nu}X_{jk(n'-n)}^{\mu*}\rho_{ii0}^{(0)} \\
&- [G_+^{\mu\nu}(\omega_{ikn'})X_{ikn'}^{\nu}X_{jk(n'-n)}^{\mu*} + G_-^{\nu\mu}(-\omega_{kjn'})X_{kjn'}^{\nu}X_{ki(n'-n)}^{\mu*}]\rho_{kk0}^{(0)}\}.
\end{aligned} \tag{2.46}$$

For the set $i = j$ and $n = 0$, the first term of right hand side is zero, and thus we obtain equations for $\{\rho_{ii0}^{(0)}\}$,

$$0 = \sum_i [T_{i\to j}\rho_{ii0}^{(0)} - T_{j\to i}\rho_{jj0}^{(0)}]. \tag{2.47}$$

Since this expression is identical to Eq. (2.40), $\rho_{ii0} = \mathrm{P}_i$. Thus, the asymptotic density matrix for the Redfield equation approaches that for the Floquet Lindblad equation, as the system-coupling strength decreases; $\lim_{\lambda\to 0}\rho_{\mathrm{Red}}^{\mathrm{A}}(t) = \rho_{\mathrm{Lind}}^{\mathrm{A}}(t)$. In other words, the long-time asymptotic state in the Lindblad formalism is only applicable to systems with infinitesimal system-bath coupling strength.

The Equation (2.46) gives the λ^2-order term of the asymptotic density matrix, $\rho_{ijn}^{(2)}$, except for $i = j$ and $n = 0$. In order to evaluate the components $\rho_{ii0}^{(2)}$, we generally resort to the fourth-order master equation. The hierarchical structure has been discussed in the system with time-independent system Hamiltonian coupled to a single bath [5, 6] or multiple baths [60, 61].

It is more importantly noted that the Redfield equation can describe the long-time asymptotic states even when the use of the Floquet Lindblad equation is no more valid, e.g., $\omega_{\mathrm{min}}^{-1} > \tau_{\mathrm{relax}}$. In this case, the perturbative expansion of the asymptotic density operator, Eq. (2.42), is not allowed, and we have to solve it in a non-perturbative way (see Chap. 4).

2.1.4 Long-time asymptotic states of systems with infinitesimal system-bath coupling

In this subsection, we overview the generic properties of the long-time asymptotic density matrix obtained by the Floquet Lindblad equation. If a system Hamiltonian is time independent, the system coupled to a dissipative environment relaxes into a canonical state. On the other hand, under a periodic driving field, it is known that the long-time asymptotic state can not be generally described in a universal way [13].

First, we consider a system whose Hamiltonian is time independent. The eigenenergy states $|\psi_i\rangle$ and eigenvalues E_i of the time-independent system Hamiltonian H_0 are related to the Floquet states and quasienergies by

$$
\begin{aligned}
|u_i(t)\rangle &= |\psi_i\rangle\, e^{-im_i\Omega t}, \\
\epsilon_i &= E_i - m_i\hbar\Omega,
\end{aligned} \tag{2.48}
$$

where m_i is chosen so that the quasienergy ϵ_i is within the regime, $-\hbar\Omega/2 \leq \epsilon_i < \hbar\Omega/2$. By substituting this form into the matrix elements of X^μ, Eq. (2.37), we obtain

$$
X^\mu_{ijn} = \langle\psi_i|\, X^\mu\, |\psi_j\rangle\, \delta_{n,m_i-m_j}, \tag{2.49}
$$

which gives a transition probability from the eigenenergy state $|\psi_i\rangle$ to $|\psi_j\rangle$ as

$$
T_{i\to j} = \frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu} G^{\mu\nu}([E_j - E_i]/\hbar)\, \langle\psi_j|\, X^\nu\, |\psi_i\rangle\, \langle\psi_i|\, X^\mu\, |\psi_j\rangle\,. \tag{2.50}
$$

The role of the heat bath that thermalizes the attached system into a canonical state is played by the Kubo-Martin-Schwinger (KMS) relation [20],

$$
G^{\mu\nu}(\omega) = G^{\nu\mu}(-\omega)e^{-\beta\hbar\omega}, \tag{2.51}
$$

which holds independently of the details of the heat bath at its temperature, $\beta^{-1}/k_{\mathrm{B}}$ though the function $G^{\mu\nu}(\omega)$ itself depends on the details such as the bath Hamiltonian and the bath operator interacting with the system of interest (see Eq. (2.32)). The KMS relation leads to the detailed balance condition between transition probabilities, $T_{i\to j}$ and $T_{j\to i}$;

$$
\begin{aligned}
T_{i\to j} =& \frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu} G^{\nu\mu}([E_i - E_j]/\hbar)e^{-\beta(E_j-E_i)}\, \langle\phi_j|\, X^\nu\, |\phi_i\rangle\, \langle\phi_i|\, X^\mu\, |\phi_j\rangle = T_{j\to i}e^{-\beta(E_j-E_i)}, \\
\leftrightarrow \frac{T_{i\to j}}{T_{j\to i}} =& \frac{e^{-\beta E_j}}{e^{-\beta E_i}}.
\end{aligned} \tag{2.52}
$$

If the ergodic property on the transition probability is satisfied [1], the asymptotic density matrix is diagonalized by the energy eigenstates of H_0 and its diagonal elements are given by the Boltzmann distribution at the temperature of the heat bath $\propto e^{-\beta E_i}$. The detailed balance condition thus ensures the universal description of the long-time asymptotic state.

This property originates from the existence of the conserved energy in the total system. Through the transition process from $|\psi_i\rangle$ to $|\psi_j\rangle$, the change of the energy in the heat bath denoted by the argument of $G(\omega)$ is $E_i - E_j$. Thus, the sum of the energy of the system of interest and the energy of the heat bath is conserved.

On the other hand, under a periodic driving field, the summation of n appears in the transition probabilities, Eq. (2.36), which avoids the existence of the conserved energy. The detailed balance condition is thus not generally satisfied;

$$\begin{aligned}\frac{T_{i\to j}}{T_{j\to i}} =&\frac{\sum_{n=-\infty}^{\infty}\sum_{\mu,\nu}G^{\mu\nu}(\omega_{jin})X^{\mu *}_{jin}X^{\nu}_{jin}}{\sum_{n=-\infty}^{\infty}\sum_{\mu,\nu}G^{\mu\nu}(\omega_{ijn})X^{\mu *}_{ijn}X^{\nu}_{ijn}},\\ =&\frac{\sum_{n=-\infty}^{\infty}\sum_{\mu,\nu}G^{\mu\nu}(\omega_{ijn})X^{\mu *}_{ijn}X^{\nu}_{ijn}e^{\beta\omega_{ijn}}}{\sum_{n=-\infty}^{\infty}\sum_{\mu,\nu}G^{\mu\nu}(\omega_{ijn})X^{\mu *}_{ijn}X^{\nu}_{ijn}}.\end{aligned} \tag{2.53}$$

The long-time asymptotic states generally depend on the details of the heat bath, $G^{\mu\nu}(\omega)$ [13], and hence its universal description does not exist.

2.2 The master equation in cavity systems

The effects of interaction between photons and atoms have attracted much attention for a long time. A cavity system is introduced to enhance the interaction by confining photons in a finite region with mirrors. A cavity system, consisting of a two-level atom coupled to a single mode of photon field, is described by Jaynes-Cummings model [62, 63] or Rabi model. The effects of interaction has been observed in experiment as cavity ringing phenomena [64, 65] and vacuum Rabi splitting [66]. Recently, this coupling has been utilized to control quantum information [67, 68, 69, 70, 71, 72, 73].

When the cavity system includes multiple atoms, cooperative phenomena due to the interaction occur, which has been studied since Dicke pointed out the importance of the coherent coupling between multiple atoms and photons in the study of superradiance [74]. An optical bistability occurs in the strong coupling regime where the atom-photon coupling strength is larger than the decay rate of photons and atoms [75, 76]. This phenomenon is a discontinuous transition of nonequilibrium steady states by changing an amplitude or a frequency of a laser field, which was observed in experiment [77, 78]. When the interaction increases and it is comparable with the energy scales of two-level atoms and photons, the regime is called the ultra-strong coupling (USC) regime. In this regime, the phase transition called the Dicke transition appears in equilibrium state [79, 80, 81]. In the ordered phase, the photon number of the ground state is not zero and the dipole moment of atoms is spontaneously polarized. This phase is called the "superradiant phase" [79].

Recent experimental progress allows us to study the USC regime. A cold atom system in an optical cavity shows a phase transition corresponding to the Dicke transition [82], and phenomena induced by a parametric resonance were proposed [83, 84]. Other systems such as semiconductor cavities [85, 86, 87, 88] and circuit QED systems [89, 90] show the ultra-strong coupling, which is expected to be utilized to study the cooperative phenomena in the USC regime.

The cavity system is thus an attractive system to study non-equilibrium steady states subjected to a periodic driving field in a dissipative environment. In Chap. 5, we will show the cooperative phenomena induced by the dynamical effects and interaction effects.

2.2.1 Dressed master equation

As a preparation to study the long-time asymptotic state of the driven cavity system, in this subsection we introduce a phenomenological master equation. In order to study the USC regime, we have to extend the master equation which was conventionally used to study the optical bistability [76]. In a single two-level atom system coupled to a quantized photon mode, it has been pointed out that the incorporation of the atom-photon coupling into the dissipative terms is important to give a correct equilibrium value [91, 92]. For such a treatment, it is necessary to obtain all the eigenvalues and eigenmodes of the system, which is possible in a system with small degrees of freedom [91, 92, 93, 94, 95] or a simple system such as a harmonic chain [96]. However, it is difficult for macroscopic systems because the system consists of many degrees of freedom. In order to overcome this difficulty, we use the property of the cavity system that all the atoms uniformly couple to the photon field. This property enables us to study the many-body problem by using a mean-field (MF) treatment, which was actually justified in the thermodynamic limit [97]. In this section, we first introduce the model of the cavity system. Next, we give a master equation in terms of photons and atoms dressed by the mean field, and compare it with the master equation for the study of the optical bistability.

We adopt the Dicke model [74] as a model of the cavity system, which describes an ensemble of two-level atoms coupled to a single quantized mode of photon field. The Hamiltonian reads

$$H_0 = \omega_{\rm p} a^\dagger a + \sum_{j=1}^{N} \omega_{\rm a} S_j^z + \sum_{j=1}^{N} \frac{2g}{\sqrt{N}} S_j^x (a + a^\dagger). \tag{2.54}$$

The first term describes the single quantized photon mode for which the annihilation and creation bosonic operators, a and $a^\dagger$, are assigned. The energy of the photon is denoted by $\omega_{\rm p}$. The second term represents the ensemble of two-level atoms with excitation energy $\omega_{\rm a}$. Since the dimension of the Hilbert space for each atom is two, they are described by a usual one-half spin operators $\boldsymbol{S}_j$, where j labels each atom, $j = 1, \cdots, N$. The last term describes the interaction among the photons and atoms with the strength being g. Here, we study the USC regime where g is comparable to $\omega_{\rm p}$ and $\omega_{\rm a}$.

We apply a periodic driving field which enhances photon field inside the cavity. The Hamiltonian is given by

$$\xi H_{\rm ex}(t) = 2\sqrt{N}\xi \cos(\Omega t)(a + a^\dagger), \tag{2.55}$$

where ξ and Ω describes the amplitude and frequency of the driving field.

The Hamiltonian of the cavity system is thus expressed by

$$H_{\rm S}(t) = H_0 + \xi H_{\rm ex}(t). \tag{2.56}$$

We study its property in the thermodynamic limit $N \to \infty$, in which the expectation values of photon numbers $a^\dagger a$ and excitations of atoms $\sum_{j=1}^N S_j^z$ is the order of N. Thus, we adopt the rescaled parameters g and ξ for the atom-photon coupling and the amplitude of the driving field, respectively.

In the MF treatment, the density operator of the system is given in the product form;

$$\rho(t) = \rho_{\rm p}(t) \otimes \underbrace{\rho_{\rm a}(t) \otimes \cdots \otimes \rho_{\rm a}(t)}_{N} = \rho_{\rm p}(t) \otimes \rho_{\rm a}(t)^{\otimes N}, \tag{2.57}$$

where $\rho_{\rm p}(t)$ and $\rho_{\rm a}(t)$ are the density operators for photons and atoms, respectively. Here, we assume that all the atoms are in the same states, and hence we will use $\boldsymbol{S}$ instead of $\boldsymbol{S}_j$. Since there is no correlation between photons and each atom, MF Hamiltonians $H_{\rm p}(t)$ and $H_{\rm a}(t)$ govern the time evolution of photon density matrix $\rho_{\rm p}(t)$ and atom density matrix $\rho_{\rm a}(t)$, respectively. They are obtained by trancing out other degrees of freedom. The MF Hamiltonian of photons reads

$$\begin{aligned} H_{\rm p}(t) =& {\rm Tr}_{\rm atoms}[H_{\rm S}(t)\rho_{\rm a}(t)^{\otimes N}], \\ =& \omega_{\rm p} a^\dagger a + 2\sqrt{N}[g\langle S^x(t)\rangle + \xi\cos\Omega t](a + a^\dagger) + \zeta(t), \end{aligned} \tag{2.58}$$

where $\langle \boldsymbol{S}(t)\rangle = {\rm Tr}_{\rm atom}[\boldsymbol{S}\rho_{\rm a}(t)]$ and $\zeta(t)$ is a scalar which does not affect the dynamics. We introduce a bosonic operator $\tilde{a}(t)$,

$$\tilde{a}(t) = a + \frac{2\sqrt{N}}{\omega_{\rm p}}[g\langle S^x(t)\rangle + \xi\cos\Omega t], \tag{2.59}$$

by which the MF Hamiltonian is written in the bilinear form;

$$H_{\rm p}(t) = \omega_{\rm p}\tilde{a}^\dagger(t)\tilde{a}(t) + \zeta'(t), \tag{2.60}$$

where $\zeta'(t)$ is a scalar. This operators $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ thus describe the dressed photon which incorporates the mean field of atoms. The MF Hamiltonian for one of the atoms reads

$$\begin{aligned} H_{\rm a}(t) =& {\rm Tr}_{\rm photon}{\rm Tr}_{\text{other atoms}}[H_{\rm S}(t)\rho_{\rm p}(t) \otimes \hat{1} \otimes \rho_{\rm a}(t)^{\otimes N-1}] \\ =& \omega_{\rm a} S^z + \frac{2g}{\sqrt{N}}(\langle a(t)\rangle + \langle a(t)\rangle^*)S^x + \zeta''(t), \end{aligned} \tag{2.61}$$

where $\langle a(t)\rangle = {\rm Tr}_{\rm photon}[a\rho_{\rm p}(t)]$ and $\zeta''(t)$ is a scalar. We introduce a unitary operator $U_{\rm atom}(t)$ so that the MF Hamiltonian is expressed by

$$H_{\rm a}(t) = \tilde{\omega}_{\rm a}(t)\tilde{S}^z(t) + \zeta''(t), \tag{2.62}$$

where $\tilde{\boldsymbol{S}}(t) = U^{\dagger}_{\rm atom}(t)\boldsymbol{S}U_{\rm atom}(t)$. Here $\tilde{\boldsymbol{S}}(t)$ describes the dressed atom which incorporates the MF of photon fields and $\tilde{\omega}_{\rm a}(t)$ reads

$$\tilde{\omega}_{\rm a}(t) = \left[\omega_{\rm a}^2 + \frac{4g^2}{N}(\langle a(t)\rangle + \langle a(t)\rangle^*)^2\right]^{\frac{1}{2}}. \tag{2.63}$$

In order to study the nonequilibrium steady state in a dissipative environment, we adopt the "dressed Lindblad equations" in terms of the dressed quantities: dressed photons denoted by $\tilde{a}(t)$ and a dressed atom denoted by $\tilde{\boldsymbol{S}}(t)$. For photon fields, it is given by

$$\frac{d\rho_{\rm p}(t)}{dt} = -\frac{i}{\hbar}[H_{\rm p}(t), \rho_{\rm p}(t)] - \frac{\gamma_{\rm p}}{\hbar}(\{\tilde{a}^\dagger(t)\tilde{a}(t), \rho_{\rm p}(t)\} - 2\tilde{a}(t)\rho_{\rm p}(t)\tilde{a}^\dagger(t)), \tag{2.64}$$

where $\gamma_{\rm p}$ describes the decay rate of the dressed photons. For each atom, it is given by

$$\frac{d\rho_{\rm a}(t)}{dt} = -\frac{i}{\hbar}[H_{\rm a}(t), \rho_{\rm a}(t)] - \frac{\gamma_{\rm a}(t)}{\hbar}(\{\tilde{S}^+(t)\tilde{S}^-(t), \rho_{\rm a}(t)\} - 2\tilde{S}^-(t)\rho_{\rm a}(t)\tilde{S}^+(t)), \tag{2.65}$$

where $\gamma_{\rm a}(t)$ describes the decay rate of the excited state of the dressed atom into its ground state. Here, $\tilde{S}^{\pm}(t)$ is defined by $\tilde{S}^x(t) \pm i\tilde{S}^y(t)$. Now, we assume that the system interacts with a thermal bath via dipole moment of each atom, which is represented by S^x. The time dependence of $\gamma_{\rm a}(t)$ is given by

$$\gamma_{\rm a}(t) = 4|\langle\tilde{\downarrow}(t)|\, S^x \,|\tilde{\uparrow}(t)\rangle|^2\gamma_{\rm a}, \tag{2.66}$$

where $|\tilde{\uparrow}(t)\rangle$ and $|\tilde{\downarrow}(t)\rangle$ are the eigenenergy states of $\tilde{S}^z(t)$ with the positive eigenvalue and the negative eigenvalue, respectively. Here, $\gamma_{\rm a}$ denotes the decay rate of each atom when there is no interaction between photon fields and atoms.

The dressed Lindblad equation incorporates the effects of atom-photon coupling and driving field into the dressed quantities $\tilde{a}(t)$ and $\tilde{\boldsymbol{S}}(t)$. When g and ξ is much smaller than $\omega_{\rm a}$ and $\omega_{\rm p}$, this equation approaches the "bare Lindblad equation",

$$\begin{cases} \dfrac{d\rho_{\rm p}(t)}{dt} = -\dfrac{i}{\hbar}[H_{\rm p}(t), \rho_{\rm p}(t)] + \dfrac{\gamma_{\rm p}}{\hbar}(2a\rho_{\rm p}(t)a^\dagger - \{a^\dagger a, \rho_{\rm p}(t)\}), \\ \dfrac{d\rho_{\rm a}(t)}{dt} = -\dfrac{i}{\hbar}[H_{\rm a}(t), \rho_{\rm a}(t)] + \dfrac{\gamma_{\rm a}}{\hbar}(2S^-\rho_{\rm a}(t)S^+ - \{S^+S^-, \rho_{\rm a}(t)\}), \end{cases} \tag{2.67}$$

in which dissipative effects relax bare photons inside cavity and excited states of bare atoms, $|\uparrow\rangle$. The bare Lindblad equation is applicable to study the optical bistability [76] because this phonomenon occurs in the strong coupling regime, $(\gamma_{\rm a}, \gamma_p) < (g, \xi) \ll (\omega_{\rm a}, \omega_{\rm p})$. However, it is expected that the application is not extended to the USC regime since the bare Lindblad equation does not take the effect of the interaction into account.

Finally, we demonstrate that bare Lindblad equation does not give the correct ground state for the Dicke transition at $\xi = 0$. In Fig. 2.1, we plot the ordered

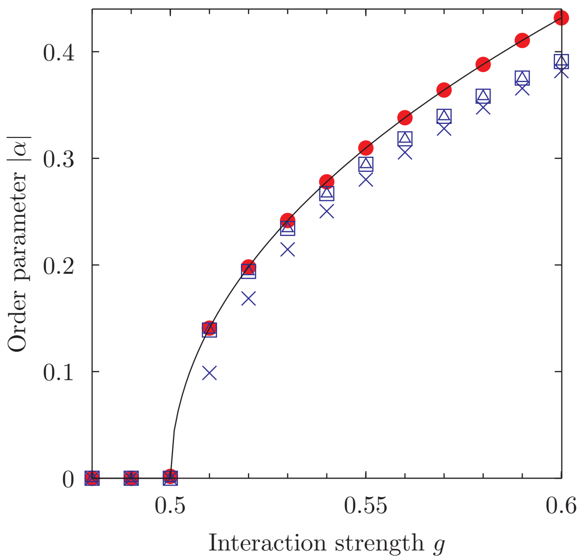

Figure 2.1: Dependence of order parameter of photon field as a function of the interaction strength g. The data obtained by the bare Lindblad equations (2.67) are denoted by crosses ($\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.1$), squares ($\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.01$), and triangles ($\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.001$), respectively. The data obtained by the dressed Lindblad equations (2.64) and (2.65) are denoted by bullets. They agree with the curve in the ordered phase ($g > 0.5$), $|\alpha| = \frac{1}{2}[4g^2/\omega_{\mathrm{p}}^2 - \omega_{\mathrm{a}}^2/(4g^2)]^{1/2}$, which is obtained from the equilibrium statistical mechanics [81]. We set parameters, $\omega_{\mathrm{a}} = \omega_{\mathrm{p}} = 1$.

component of the photon field $|\alpha| = \lim_{N\to\infty} |\langle a \rangle / \sqrt{N}|$ as the function of the atom-photon coupling strength g. We show the stationary states obtained by the bare Lindblad equations for $\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.1$ (crosses), $\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.01$ (squares), $\gamma_{\mathrm{p}} = \gamma_{\mathrm{a}} = 0.001$ (triangles). The stationary states show a phase transition, but they do not give the correct values of the order parameter in the ordered phase. This deviation is not owing to the finite system-bath coupling but owing to the form of the master equation. Indeed, as the system-bath coupling decreases, the data converges to the limiting values which are different from the correct values. This observation indicates that the bare Lindblad equations are inadequate to study the USC regime. On the other hand, the dressed Lindblad equations reproduce the correct values which are obtained by the equilibrium statistical physics [81]. Thus, the present formalism of dissipation terms satsfies the minimal condition in order to study the USC regime subjected to a periodic driving field, which will be discussed in Chap. 5.

In this section, we employed a mean field approach and introduced the phenomenological dressed master equations. It is noted that the dressed master equations at time t depend only on the mean field at time t in spite of the finite correlation time of the heat bath, τ_{bath}. We therefore implicitly assume that the mean field does not change much during the correlation time of the heat bath. When the system Hamiltonian is time periodic, the dynamics during the bath correlation time can be incorporated by expanding it into the Floquet modes, Eq. (2.27), which is not applicable to our case since the mean field is not time periodic in general. However, it is plausible that there exists a parameter regime where the mean field becomes time periodic in the long-time asymptotic state. In Chap. 5, in order to study the high-frequency regime, in which the mean field also rapidly oscillates, we propose a method to find the long-time asymptotic state by using the Floquet Lindblad equation and the MF approach.

Chapter 3

Floquet-Gibbs state in open systems with infinitesimal system-bath coupling

3.1 Floquet-Gibbs state

In the previous chapter, we have shown that under an appropriate condition, the dissipation dynamics can be described by the Floquet Lindblad equation. The simple structure of the Floquet Lindblad equation and the KMS relation naturally leads us to the notion of the Floquet-Gibbs state. The strategy that we employ is as follows. In the framework of the Floquet Lindblad equation, the probability distribution $\{\mathrm{P}_i\}$ for the long-time asymptotic state is obtained by solving the following equations,

$$0 = \sum_j T_{j\to i}\mathrm{P}_j - T_{i\to j}\mathrm{P}_i, \tag{3.1}$$

where

$$T_{i\to j} = \sum_{m=-\infty}^{\infty} T_{i\to j,m} = \frac{\lambda^2}{\hbar^2}\sum_{m=-\infty}^{\infty}\sum_{\mu,\nu} G^{\mu\nu}(\omega_{jim})X^{\mu *}_{jim}X^{\nu}_{jim}. \tag{3.2}$$

As shown previously, the asymptotic states are not generally given in the Gibbs form owing to the summation of the m. However, if for each transition process, there exists an integer m_{ij} for which $T_{i\to j,m_{ij}}$ gives a dominant contribution of $T_{i\to j}$ and the integer m_{ij} can be expressed in the form[1],

$$m_{ij} = m_i - m_j, \tag{3.3}$$

then the detailed balance condition recovers owing to the KMS relation, Eq. (2.51);

$$T_{i\to j}e^{-\beta(\epsilon_i + m_i\hbar\Omega)} = T_{j\to i}e^{-\beta(\epsilon_j + m_j\hbar\Omega)}. \tag{3.4}$$

[1]There is an ambiguity to add a certain integer to every m_i owing to this definition, but it does not matter because the origin of m_i is not important

If the ergodic property is satisfied, the long-time asymptotic state is given in the Gibbs form as

$$\rho^{\mathrm{A}}_{\mathrm{Lind}}(t) = \sum_i \frac{e^{-\beta(\epsilon_i + m_i\hbar\Omega)}}{\sum_j e^{-\beta(\epsilon_j + m_j\hbar\Omega)}} |u_i(t)\rangle \langle u_i(t)| = \frac{e^{-\beta H_{\mathrm{F}}^{[t]}}}{\mathrm{Tr} e^{-\beta H_{\mathrm{F}}^{[t]}}}, \tag{3.5}$$

where

$$H_{\mathrm{F}}^{[t]} = \sum_i (\epsilon_i + m_i\hbar\Omega) |u_i(t)\rangle \langle u_i(t)| \,. \tag{3.6}$$

It is noted that this state is obtained by replacing the eigenstates and eigenenergies in the canonical state by the Floquet states and their quasienergies, and thus we call it Floquet-Gibbs state.

The qualitative explanation why the Gibbs distribution of the Floquet Hamiltonian is realized is as follows. The introduction of m_i allows us to assign well-defined "energies", $\epsilon_i + m_i\hbar\Omega$, for each Floquet state, and there exists conserved total energy consisting of this energy and the energy of the heat bath through the transition process. The existence of the conserved total energy ensures the Gibbs description of the Floquet Hamiltonian as in a time-independent system. In this chapter, we show that the above situation actually occurs in some physically relevant cases.

The Hamiltonian for the total system is given in the same form as Eq. (2.1). We divide the system Hamiltonian $H_{\mathrm{S}}(t)$ into two parts;

$$H_{\mathrm{S}}(t) = H_0 + \xi H_{\mathrm{ex}}(t), \tag{3.7}$$

where the driving Hamiltonian is determined to satisfy

$$\int_0^T H_{\mathrm{ex}}(t) dt = 0. \tag{3.8}$$

3.2 Emergence of the Floquet-Gibbs state in the linear response (weakly driven) regime

Here we study the linear response regime where the driving amplitude ξ is sufficiently weak. We show that in this regime the probability distribution of the asymptotic state is given by the Floquet-Gibbs state up to the first order of ξ. This result is plausible because the asymptotic state is expected to be close to the thermal equilibrium state as discussed in [98], but the explicit derivation has not be done before.

We expand the Floquet mode perturbatively into the basis of the eigenenergy states of H_0, $H_0 |\psi_i\rangle = E_i |\psi_i\rangle$. By using a perturbation method [24] (See details in section 1.3.1), the Floquet mode up to the order of ξ is given by

$$|u_i(t)\rangle e^{im_i\Omega t} = \left[1 - \xi \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \frac{\tilde{H}_{iim}}{m\hbar\Omega}\right] |\psi_i\rangle - \xi \sum_{j(\neq i)} \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \frac{\tilde{H}_{jim} e^{-im\Omega t}}{E_j - E_i - m\hbar\Omega} |\psi_j\rangle \,, \tag{3.9}$$

where $\tilde{H}_{jim}$ is a Fourier component of the driving Hamiltonian in the basis of the Floquet states, Eq. (1.35), and m_i is defined so that $-\frac{\hbar\Omega}{2} < E_i - m_i\hbar\Omega \leq \frac{\hbar\Omega}{2}$. In order to use this perturbative method, we have assumed that there is no near degeneracy in quasi-energy spectrum,

$$|E_i - E_j + m\hbar\Omega| \gg \xi \max_t |\langle\psi_i| H_{\mathrm{ex}}(t) |\psi_j\rangle|, \tag{3.10}$$

for any set of (i, j, m) except $i = j$ and $m = 0$. This condition is not satisfied when there exist two eigenenergy states of H_0 which are in resonance with the driving field.

We evaluate X^{μ}_{ijn} in the basis of eigenstates of H_0 up to the order of ξ. For $n = m_i - m_j$,

$$X^{\mu}_{ijn} = \left[1 + \xi \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \frac{\tilde{H}_{jjm} - \tilde{H}_{iim}}{m\hbar\Omega}\right] \langle\psi_i| X^{\mu} |\psi_j\rangle, \tag{3.11}$$

and for $n \neq m_i - m_j$,

$$X^{\mu}_{ij(m_i-m_j-n)} = \xi \sum_l \left(\frac{\tilde{H}_{iln} \langle\psi_l| X^{\mu} |\psi_j\rangle}{E_i - E_l - n\hbar\Omega} - \frac{\langle\psi_i| X^{\mu} |\psi_l\rangle \tilde{H}_{ljn}}{E_l - E_j - n\hbar\Omega} \right). \tag{3.12}$$

The transition probability from the Floquet mode $|u_i(t)\rangle$ to $|u_j(t)\rangle$ is given by (see next section for the bound of $\mathrm{O}(\xi^2)$ term),

$$T_{i\to j} = \frac{\lambda^2}{\hbar^2} \sum_{\mu,\nu} G^{\mu\nu}(\omega_{ji[m_j-m_i]}) \langle\psi_i| X^{\mu} |\psi_j\rangle \langle\psi_j| X^{\nu} |\psi_i\rangle + \mathrm{O}(\xi^2), \tag{3.13}$$

where $\omega_{ijn} = (\epsilon_i - \epsilon_j)/\hbar + n\Omega$. An integer m_i is assigned for each Floquet mode $|u_i(t)\rangle$ in the transition process (see Eq. (3.3)), which ensures the detailed balance condition among the Floquet states. The KMS relation, Eq. (2.51), indeed leads to

$$T_{i\to j} e^{-\beta(\epsilon_i + m_i\hbar\Omega)} = T_{j\to i} e^{-\beta(\epsilon_j + m_j\hbar\Omega)} + \mathrm{O}(\xi^2). \tag{3.14}$$

If an ergodic property is satisfied in the limit $\xi \to 0$, then the long-time asymptotic state is given by the Floquet-Gibbs state,

$$\rho^{\mathrm{A}}_{\mathrm{Lind}}(t) = \frac{e^{-\beta H^{[t]}_{\mathrm{F}}}}{\mathrm{Tr} e^{-\beta H^{[t]}_{\mathrm{F}}}} + \mathrm{O}(\xi^2), \tag{3.15}$$

where

$$H^{[t]}_{\mathrm{F}} = \sum_i (\epsilon_i + m_i\hbar\Omega) |u_i(t)\rangle \langle u_i(t)|. \tag{3.16}$$

Owing to the definition of the Hamiltonian for the driving field, Eq. (3.8), the quasienergy is related to the eigenenergy of H_0 as $E_i = \epsilon_i + m_i\hbar\Omega + \mathrm{O}(\xi^2)$ (See the derivation in Eq. (1.38)), and thus the distribution P_i does not change in the first order of ξ. However, because the Floquet state changes in the order of ξ, the expectation values of observables in the long-time asymptotic state are different from the equilibrium values in this order.

3.2.1 Estimation of the smallness of transition probabilities that break the detailed balance condition

Here we show that $\mathrm{O}(\xi^2)$ term in Eq. (3.13) is bounded by

$$\frac{\lambda^2\xi^2}{\hbar^2}\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}(\Delta_i(m)+\Delta_j(-m))^2\sum_{\mu,\nu}\|X^\mu\|\|X^\nu\||G_{\mu\nu}(\omega_{jim})|, \tag{3.17}$$

where $\Delta_i(m)$ is given by

$$\Delta_i(m)=\sqrt{\sum_l\frac{|\tilde{H}_{ilm}|^2}{(E_i-E_l-m\hbar\Omega)^2}}. \tag{3.18}$$

In this subsection, we derive this bound. The $\mathrm{O}(\xi^2)$ term, which breaks the detailed balance condition among Floquet states, is given by

$$\frac{\lambda^2}{\hbar^2}\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}\sum_{\mu,\nu}G^{\mu\nu}(\omega_{jim})X^{\mu}_{ij[m_i-m_j-m]}X^{\nu}_{ji[m_j-m_i+m]}. \tag{3.19}$$

The absolute value of $X^{\mu}_{ij(m_i-m_j-m)}$ is bounded as

$$\begin{aligned}|X^{\mu}_{ij[m_i-m_j-m]}|\leq&\xi\left(\left|\sum_l\frac{\tilde{H}_{ilm}\langle\psi_l|X^\mu|\psi_j\rangle}{E_i-E_l-m\hbar\Omega}\right|-\left|\sum_l\frac{\langle\psi_i|X^\mu|\psi_l\rangle\tilde{H}_{ljm}}{E_l-E_j-m\hbar\Omega}\right|\right),\\ \leq&\xi(\Delta_i(m)+\Delta_j(-m))\sqrt{\langle\psi_j|X^{\mu\dagger}X^\mu|\psi_j\rangle},\\ \leq&\xi(\Delta_i(m)+\Delta_j(-m))\|X^\mu\|.\end{aligned} \tag{3.20}$$

In a similar way, $|X^{\nu}_{ji[m_j-m_i+m]}|$ is bounded by

$$|X^{\nu}_{m_j-m_i+m}|\leq\xi(\Delta_i(m)+\Delta_j(-m))\|X^\nu\|. \tag{3.21}$$

Substituting these forms into Eq. (3.19),

$$\begin{aligned}\text{Eq. (3.19)}\leq&\frac{\lambda^2}{\hbar^2}\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}\sum_{\mu,\nu}|G^{\mu\nu}(\omega_{jim})||X^{\mu}_{ij(m_i-m_j-m)}||X^{\nu}_{m_j-m_i+m}|,\\ \leq&\frac{\lambda^2\xi^2}{\hbar^2}\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}(\Delta_i(m)+\Delta_j(-m))^2\sum_{\mu,\nu}|G_{\mu\nu}(\omega_{jim})|\|X^\mu\|\|X^\nu\|.\end{aligned} \tag{3.22}$$

It is noted that the $\Delta_i(m)$ is finite only when the driving Hamiltonian has a corresponding Fourier component, $\int_0^T H_{\mathrm{ex}}(t)e^{im\Omega t}dt\neq 0$.

3.3 Emergence of the Floquet-Gibbs state beyond the linear response regime

Next, we study the case beyond the linear response regime. We will show that under the following three conditions,

$$\begin{cases} \text{(i) } \hbar\Omega \gg \|H_0\|, \\ \text{(ii) for all } t_1 \text{ and } t_2, [H_{\rm ex}(t_1), H_{\rm ex}(t_2)] = 0, \\ \text{(iii) } [H_{\rm ex}(t), H_{\rm I}] = 0, \end{cases}$$

the long-time asymptotic state is described by the Floquet-Gibbs state,

$$\rho_{\rm FG}(t) = \sum_i \frac{e^{-\beta\epsilon_i}}{\sum_j e^{-\beta\epsilon_j}} |u_i(t)\rangle \langle u_i(t)|, \tag{3.23}$$

up to the order of Ω^{-1}. The first condition states that the system is subjected to a high frequency driving field. The second condition restricts the form of the driving field; The driving Hamiltonians $H_{\rm ex}(t)$ (see Eq. (3.7)) at different times commute with each other. The third condition requires that the interaction Hamiltonian $H_{\rm I}$ commutes with the driving Hamiltonian. It is noted that these conditions are independent of driving strength ξ, and therefore the Floquet-Gibbs state can be realized even when the Floquet quasienergy spectrum is far from the eigenenergy spectrum of H_0.

First of all, let us qualitatively explain why these conditions lead the asymptotic state to the Floquet-Gibbs state. As we have shown in Chap. 1 in order to obtain the approximated Floquet Hamiltonian when a strong and high-frequency field is subjected, it is convenient to transform into a rotating frame by using a unitary operator,

$$V(t) = \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t \xi H_{\rm ex}(\tau)d\tau}. \tag{3.24}$$

The total Hamiltonian, which gives a time evolution in the rotating frame, reads

$$H^{\rm R}(t) = H^{\rm R}_{\rm S}(t) + \lambda H^{\rm R}_{\rm I}(t) + H_{\rm B}, \tag{3.25}$$

where

$$\begin{aligned} H^{\rm R}_{\rm S}(t) =& V^\dagger(t) H_0 V(t), \\ H^{\rm R}_{\rm I}(t) =& V^\dagger(t) H_{\rm I} V(t). \end{aligned} \tag{3.26}$$

When a system meets all the conditions (i), (ii), and (iii), the time-dependent Hamiltonian $H^{\rm R}(t)$ can be effectively replaced by a time-independent one. Here, it is noted that the bath Hamiltonian does not change because the unitary operator acts only on the Hilbert space of the system of interest.

Conditions (i) and (ii) play a role of replacing $H^{\rm R}_{\rm S}(t)$ by a time-independent one. The condition (ii) allows us to drop the time-ordering operator in the unitary operator $V(t)$, Eq. (3.24). This property with the definition of the driving Hamiltonian, Eq. (3.8), ensures the time periodicity of this operator, $V(t) = V(t+T)$. As a result, $H_{\rm S}^{\rm R}(t)$ is also periodic in time,

$$H_{\rm S}^{\rm R}(t+T) = H_{\rm S}^{\rm R}(t), \tag{3.27}$$

which can be expressed by the Fourier expansion,

$$H_{\rm S}^{\rm R}(t) = \sum_{n=-\infty}^{\infty} \tilde{H}^{(n)} e^{-in\Omega t}. \tag{3.28}$$

Since the norm of $H_{\rm S}^{\rm R}(t)$ is independent of the driving amplitude,

$$\|H_{\rm S}^{\rm R}(t)\| = \|H_0\|, \tag{3.29}$$

the amplitude of the oscillating terms in $H_{\rm S}^{\rm R}(t)$, $\tilde{H}^{(n)}$, is also bounded by a value independent of ξ;

$$\|\tilde{H}^{(n)}\| \leq \int_0^T \|H_{\rm S}^{\rm R}(t) e^{in\Omega t}\| \frac{dt}{T} = \|H_0\|. \tag{3.30}$$

We thus find from Eq. (3.27) and Eq. (3.30) that at high frequency (condition (i)) the Hamiltonian in the rotating frame consists of rapidly oscillating terms with amplitude bounded by $\|H_0\|$. We then have an intuitive picture that this rotating Hamiltonian $H_{\rm S}^{\rm R}(t)$ can be replaced by the time-averaged one (effective Hamiltonian),

$$H_{\rm S}^{\rm R}(t) \to H_{\rm eff} = \tilde{H}^{(0)}. \tag{3.31}$$

The effective Hamiltonian is actually close to the Floquet Hamiltonian in the rotating frame owing to the Floquet-Magnus expansion [27] (see Sec. 1.3.3),

$$H_{\rm eff} = H_{\rm F}^{[0]} + {\rm O}\left(\frac{\|H_0\|T}{\hbar}\right). \tag{3.32}$$

Under the conditions (i) and (ii), we thus introduce the time-independent system Hamiltonian in the rotating frame.

The third condition (iii) ensures time independence of the interaction Hamiltonian in this rotating frame, $H_{\rm I}^{\rm R}(t)$. When the condition (iii) is satisfied, unitary operator $V(t)$ and the interaction Hamiltonian $H_{\rm I}$ commutes with each other, and thus

$$H_{\rm I}^{R}(t) = H_{\rm I}. \tag{3.33}$$

Therefore, we can effectively introduce the time-independent Hamiltonian for the total system in the rotating frame, $H_{\rm F}^{[0]} + \lambda H_{\rm I} + H_{\rm B}$. The time independence ensures the existence of the conserved energy for a total system, which allows us to describe the asymptotic density matrix in a Gibbs form.

We now confirm this scenario in the framework of the Floquet Lindblad equation. For this purpose, we introduce Floquet modes $|u_i^{\rm R}(t)\rangle$ and quasienergies $\epsilon_i^{\rm R}$ for $H_{\rm S}^{R}(t)$,

$$\left(H_{\rm S}^{\rm R}(t) - i\hbar\frac{\partial}{\partial t}\right) |u_i^{\rm R}(t)\rangle = \epsilon_i^{\rm R} |u_i^{\rm R}(0)\rangle \tag{3.34}$$

Here we have used the periodicity of $H_{\mathrm{S}}^{\mathrm{R}}(t)$ due to the condition (ii). The quasienergies and the Floquet states are related to the ones in the static frame by

$$|u_i^{\mathrm{R}}(t)\rangle = V(t)\,|u_i(t)\rangle\,, \quad \epsilon_i^{\mathrm{R}} = \epsilon_i. \tag{3.35}$$

The unitary operator $V(t)$ commutes with the interaction Hamiltonian H_{I} owing to the condition (iii),

$$[V(t), H_{\mathrm{I}}] = 0 \rightarrow [V(t), X^{\mu}] = 0, \tag{3.36}$$

which leads to the expression of X_{ijn}^{μ} in Eq. (2.37) using the Floquet basis in the rotating frame;

$$\begin{aligned} X_{ijn}^{\mu} &= \int_0^T \langle u_i^{\mathrm{R}}(t)|\, V(t) X^{\mu} V^{\dagger}(t)\, |u_j^{\mathrm{R}}(t)\rangle\, e^{-in\Omega t} \frac{dt}{T} \\ &= \int_0^T \langle u_i^{\mathrm{R}}(t)|\, X^{\mu}\, |u_j^{\mathrm{R}}(t)\rangle\, e^{-in\Omega t} \frac{dt}{T}. \end{aligned} \tag{3.37}$$

We expand the Floquet mode in the rotating frame up to the first order of T. Using the Magnus expansion for $0 \leq t \leq T$,

$$\begin{aligned} |u_i^{\mathrm{R}}(t)\rangle &= \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t (H_{\mathrm{S}}^{\mathrm{R}}(\tau)-\epsilon_i)d\tau}\, |u_i^{\mathrm{R}}(0)\rangle\,, \\ &= \left(e^{-\frac{i}{\hbar}\int_0^t (H_{\mathrm{S}}^{\mathrm{R}}(\tau)-\epsilon_i)d\tau - \frac{1}{2\hbar^2}\int_0^t d\tau_1 \int_0^{\tau_1} [H_{\mathrm{S}}^{\mathrm{R}}(\tau_1), H_{\mathrm{S}}^{\mathrm{R}}(\tau_2)] + \cdots} \right) |u_i^{\mathrm{R}}(0)\rangle\,, \\ &= \left[e^{-\frac{i}{\hbar}\int_0^t (H_{\mathrm{S}}^{\mathrm{R}}(\tau)-\epsilon_i)d\tau - \mathrm{O}\left(\frac{\|H_0\| t^2}{\hbar^2}\right)} \right] |u_i^{\mathrm{R}}(0)\rangle\,, \\ &= \left[1 - \frac{i}{\hbar}\int_0^t (H_{\mathrm{S}}^{\mathrm{R}}(\tau) - \epsilon_i) d\tau + \mathrm{O}\left(\frac{\|H_0\| t^2}{\hbar^2}\right) \right] |u_i^{\mathrm{R}}(0)\rangle\,. \end{aligned} \tag{3.38}$$

We use the fact that the operator norm of $H_{\mathrm{S}}^{\mathrm{R}}(t)$ is $\|H_0\|$.

Owing to the periodicity of the Floquet state, $|u_i^{\mathrm{R}}(T)\rangle = |u_i^{\mathrm{R}}(0)\rangle$, we find from Eq. (3.38) that

$$(\tilde{H}^{(0)} - \epsilon_i) T\, |u_i^{\mathrm{R}}(0)\rangle = \mathrm{O}\left(\frac{\|H_0\|^2 T^2}{\hbar^2}\right) |u_i^{\mathrm{R}}(0)\rangle\,. \tag{3.39}$$

Substituting this form into Eq. (3.38), the Floquet state is given by

$$|u_i^{\mathrm{R}}(t)\rangle = \left[1 + \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} \frac{\tilde{H}^{(m)}(e^{-im\Omega t} - 1)}{m\hbar\Omega} + \mathrm{O}\left(\frac{\|H_0\|^2 T^2}{\hbar^2}\right) \right] |u_i^{\mathrm{R}}(0)\rangle\,. \tag{3.40}$$

The independence of $\frac{\|H_0\| T}{\hbar}$ on ξ reflects the fact that the amplitude of the oscillating terms in $H_{\mathrm{S}}^{\mathrm{R}}(t)$ is bounded by $\|H_0\|$.

We then evaluate X^{μ}_{ijn} for $n=0$,

$$X^{\mu}_{ij0} = \langle u^{\mathrm{R}}_i(0)|\, X^{\mu}\, |u^{\mathrm{R}}_j(0)\rangle + \mathrm{O}\left(\frac{\|H_0\|T}{\hbar}\right), \tag{3.41}$$

and for $n \neq 0$,

$$X^{\mu}_{ijn} = \frac{\langle u^{\mathrm{R}}_i(0)|\, [\tilde{H}^{(-n)}, X^{\mu}]\, |u^{\mathrm{R}}_j(0)\rangle}{n\hbar\Omega} + \mathrm{O}\left(\frac{\|H_0\|^2T^2}{\hbar^2}\right). \tag{3.42}$$

The transition probability is written as(see Sec. 3.3.1 for the bound of $\mathrm{O}(T^2)$ term)

$$\begin{aligned} T_{i\to j} =& \frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu}\left[\langle u^{\mathrm{R}}_i(0)|\, X^{\mu}\, |u^{\mathrm{R}}_j(0)\rangle \langle u^{\mathrm{R}}_j(0)|\, X^{\nu}\, |u^{\mathrm{R}}_i(0)\rangle + \mathrm{O}\left(\frac{\|H_0\|T}{\hbar}\right)\right] G^{\mu\nu}(\omega_{ji0}) \\ &+ \mathrm{O}\left(\frac{\|H_0\|^2T^2}{\hbar^2}\right). \end{aligned} \tag{3.43}$$

Since an integer $m_i = 0$ is assigned for each Floquet state, the KMS relation Eq. (2.51) leads to the detailed balance condition,

$$T_{i\to j}e^{-\beta\epsilon_i} = T_{j\to i}e^{-\beta\epsilon_j} + \mathrm{O}\left(\frac{\|H_0\|^2T^2}{\hbar^2}\right). \tag{3.44}$$

If the ergodicity is satisfied in the limit of $T\to 0$ with ξT held fixed, the long-time asymptotic state is expressed by the Floquet-Gibbs state,

$$\rho^{\mathrm{A}}_{\mathrm{Lind}}(t) = \frac{e^{-\beta H^{[t]}_{\mathrm{F}}}}{\mathrm{Tr}e^{-\beta H^{[t]}_{\mathrm{F}}}} + \mathrm{O}\left(\frac{\|H_0\|^2T^2}{\hbar^2}\right), \tag{3.45}$$

where

$$H^{[t]}_{\mathrm{F}} = \sum_i \epsilon_i \,|u_i(t)\rangle\langle u_i(t)|\,. \tag{3.46}$$

In this way, under three conditions (i), (ii), and (iii), the long-time asymptotic state is described by the Floquet-Gibbs state.

This result is compatible with the idea of the effective Hamiltonian. In the rotating frame, the asymptotic density matrix is obtained by replacing $H^{[t]}_{\mathrm{F}}$ by

$$H^{\mathrm{R}[t]}_{\mathrm{F}} = \sum_i \epsilon_i \,|u^{\mathrm{R}}_i(t)\rangle\langle u^{\mathrm{R}}_i(t)|\,, \tag{3.47}$$

which is close to the effective Hamiltonian owing to the Floquet-Magnus expansion,

$$H^{\mathrm{R}[t]}_{\mathrm{F}} = \int_t^{t+T} H^{\mathrm{R}}_{\mathrm{S}}(\tau)\frac{d\tau}{T} + \mathrm{O}\left(\frac{\|H_0\|^2T}{\hbar}\right) = H_{\mathrm{eff}} + \mathrm{O}\left(\frac{\|H_0\|^2T}{\hbar}\right). \tag{3.48}$$

The long-time asymptotic state in the rotating frame is therefore approximately given by the canonical state of the effective Hamiltonian.

3.3.1 Estimation of the smallness of transition probabilities that break the detailed balance condition

In this section we show that the $\mathrm{O}(T^2)$ term in Eq. (3.43) is bounded by

$$\frac{2\lambda^2}{\hbar^2}\frac{\|H_0\|^2}{(\hbar\Omega)^2}\sum_{\mu,\nu}\overline{G^{\mu\nu}}\|X^\mu\|\|X^\nu\|, \tag{3.49}$$

where

$$\overline{G^{\mu\nu}} = \max_{i,j}\sqrt{\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{|G^{\mu\nu}(\omega_{ijn})|^2}{n^4}}. \tag{3.50}$$

The $\mathrm{O}(T^2)$ term, which breaks the detailed balance condition between the Floquet states $|u_i(t)\rangle$ and $|u_j(t)\rangle$, is given by

$$\frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu}\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}G^{\mu\nu}(\omega_{jim})X^\mu_{ij[-m]}X^\nu_{jim}. \tag{3.51}$$

By using the explicit form of X^μ_{ijm} in Eq. (3.42), this term is bounded by

$$\begin{aligned}
&\frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu}\left|\sum_{\substack{m=-\infty\\(m\neq 0)}}^{\infty}\frac{G^{\mu\nu}(\omega_{jim})}{(m\hbar\Omega)^2}\langle\psi_i|\,[X^\mu,\tilde{H}^{(m)}]\,|\psi_j\rangle\,\langle\psi_j|\,[X^\nu,\tilde{H}^{(-m)}]\,|\psi_i\rangle\right|,\\
\leq&\frac{\lambda^2}{\hbar^2}\sum_{\mu,\nu}\frac{\overline{G^{\mu\nu}}}{(\hbar\Omega)^2}\sqrt{\sum_{m=-\infty}^{\infty}|\,\langle\psi_i|\,[X^\mu,\tilde{H}^{(m)}]\,|\psi_j\rangle\,|^2|\,\langle\psi_j|\,[X^\nu,\tilde{H}^{(-m)}]\,|\psi_i\rangle\,|^2}.
\end{aligned} \tag{3.52}$$

The quantity inside the square root is written as

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty}\left(\prod_{l=1}^{4}\int_0^T\frac{dt_l}{T}\right)e^{im\Omega[(t_1-t_2)-(t_3-t_4)]}\,\langle\psi_i|\,[X^\mu,H^{\mathrm{R}}_{\mathrm{S}}(t_1)]\,|\psi_j\rangle\,,\\
&\times(\langle\psi_i|\,[X^\mu,H^{\mathrm{R}}_{\mathrm{S}}(t_2)]\,|\psi_j\rangle)^*\,\langle\psi_j|\,[X^\nu,H^{\mathrm{R}}_{\mathrm{S}}(t_3)]\,|\psi_i\rangle\,(\langle\psi_j|\,[X^\nu,H^{\mathrm{R}}_{\mathrm{S}}(t_4)]\,|\psi_i\rangle)^*,\\
\leq&(\max_t|\,\langle\psi_i|\,[X^\mu,H^{\mathrm{R}}_{\mathrm{S}}(t)]\,|\psi_j\rangle\,|)^2(\max_t|\,\langle\psi_i|\,[X^\nu,H^{\mathrm{R}}_{\mathrm{S}}(t)]\,|\psi_j\rangle\,|)^2,\\
\leq&4\|X^\mu\|^2\|X^\nu\|^2\|H_0\|^4,
\end{aligned} \tag{3.53}$$

where we have used $\|H^{\mathrm{R}}_{\mathrm{S}}(t)\| = \|H_0\|$. Substituting this into Eq. (3.52), we obtain the bound Eq. (3.49).

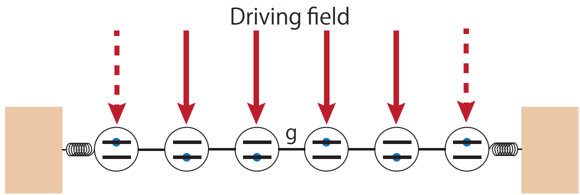

Figure 3.1: Illustration of the spin chain model subjected to a periodic driving in a dissipative environment. The spin sites at edges are connected to independent baths. Both heat baths possess the same thermal properties.

3.3.2 Numerical simulation in a spin-chain model

We demonstrate the Floquet-Gibbs state in a spin-chain model. As shown in Fig. 3.1, the system consists of six spins and the only spins at edges are in contact with heat baths. The static Hamiltonian is given by

$$H_0 = \sum_{i=1}^{6}(h^z S_i^z + h^x S_i^x) - g\sum_{i=1}^{6}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y). \tag{3.54}$$

Here h^z and h^x are the static magnetic field applying along the z and x direction, respectively, and g is the coupling strength between nearest neighbor spins. We use the following set of parameters: $h^z = 1.0$, $h^x = 0.01$, and $g = 0.2$. In addition, there is a time-periodic driving field denoted by $H_{\rm ex}(t)$. In order to test the conditions in various situations, we study the following three cases:

$$\begin{cases} \text{Case A: } \xi H_{\rm ex}(t) = & \xi\sum_{i=1}^{6}\cos(\Omega t)S_i^x \\ \text{Case B: } \xi H_{\rm ex}(t) = & \xi\sum_{i=2}^{5}\cos(\Omega t)S_i^x \\ \text{Case C: } \xi H_{\rm ex}(t) = & \xi\sum_{i=2}^{5}[\cos(\Omega t)S_i^x + r\sin(\Omega t)S_i^y] \end{cases} \tag{3.55}$$

For case A, it is necessary to fine-tune the system-bath coupling to satisfy condition (iii), while for case B and case C, the fine-tuning is not necessary because the spins in contact with heat bath and the spins driven by the external field are different. For case C, the condition (ii) is broken when $r \neq 0$. Throughout this section, we fix the ratio of driving amplitude and the frequency to be $\xi/\Omega = 2\hbar$. The spins at edges are coupled to independent baths denoted by L and R through

$$X^{\mathrm{L}} = \sum_{k=\{x,y,z\}} \alpha^k S_1^k,$$
$$X^{\mathrm{R}} = \sum_{k=\{x,y,z\}} \alpha^k S_6^k. \tag{3.56}$$

The properties of the heat baths are determined by temperature β^{-1} and the Fourier transform of bath correlations, $G^{\mathrm{LL}}(\omega) = G^{\mathrm{RR}}(\omega) = G(\omega)$. The temperature of the both heat baths is set to be $\beta^{-1} = 0.1$ and $G(\omega)$ is taken to satisfy the KMS relation Eq. (2.51);

$$G(\omega) = \begin{cases} \dfrac{\omega_c^2}{\omega^2+\omega_c^2} & (\omega < 0), \\ \dfrac{\omega_c^2}{\omega^2+\omega_c^2} e^{-\beta\hbar\omega} & (\omega > 0), \end{cases} \tag{3.57}$$

with cut-off frequency $\omega_c = 5000/\hbar$. It is noted that the long-time asymptotic states are independent of the system-bath coupling strength within the Lindblad formalism as discussed in Chap. 2.

As a measure of the difference between the long-time asymptotic state and the Floquet-Gibbs state, we calculate

$$\Delta\mathrm{Prob} = \sum_i |\left\langle u_i(t)\right| \rho^{\mathrm{A}}_{\mathrm{Lind}}(t) \left|u_i(t)\right\rangle - \left\langle u_i(t)\right| \rho^{\mathrm{FG}}(t) \left|u_i(t)\right\rangle|,$$
$$= \sum_i \left| \mathrm{P}_i - \frac{e^{-\beta\epsilon_i}}{\sum_j e^{-\beta\epsilon_j}} \right|. \tag{3.58}$$

In Fig. 3.2(a) we study the case A for various values of the ratio α^x/α^y. Here we set $\alpha^z = 0$. For $\alpha^y = 0$ when $[H_{\mathrm{ex}}(t), H_{\mathrm{I}}] = 0$, the asymptotic states approache the Floquet-Gibbs state as the period of the driving field is short and ΔProb is proportional to T^2 (squares in figure), which agrees with the result, Eq. (3.45). When the ratio α^y/α^x increases, ΔProb deviates from the T^2 dependence due to the violation of condition (iii). We also depict the probability distribution P_i in Fig. 3.2(b). The asymptotic distribution is described in the Floquet-Gibbs form at the temperature of the heat bath for $\alpha^y = 0$ (squares in figure), while it is substantially heated up for other cases.

In contrast, in the case B depicted in Fig. 3.2(c), the asymptotic state approaches the Floquet-Gibbs state with the T^2 dependencies independently of the couplings.

Finally, in the case C depicted in Fig. 3.2(d), the asymptotic state is deviated from the Floquet-Gibbs state when $r \neq 0$ due to the violation of condition (ii).

The numerical studies demonstrate that the long-time asymptotic states can be described by the Floquet-Gibbs state under the three conditions (i), (ii), and (iii), and these three conditions are necessary in order to realize the Floquet-Gibbs state in the above spin chain model.

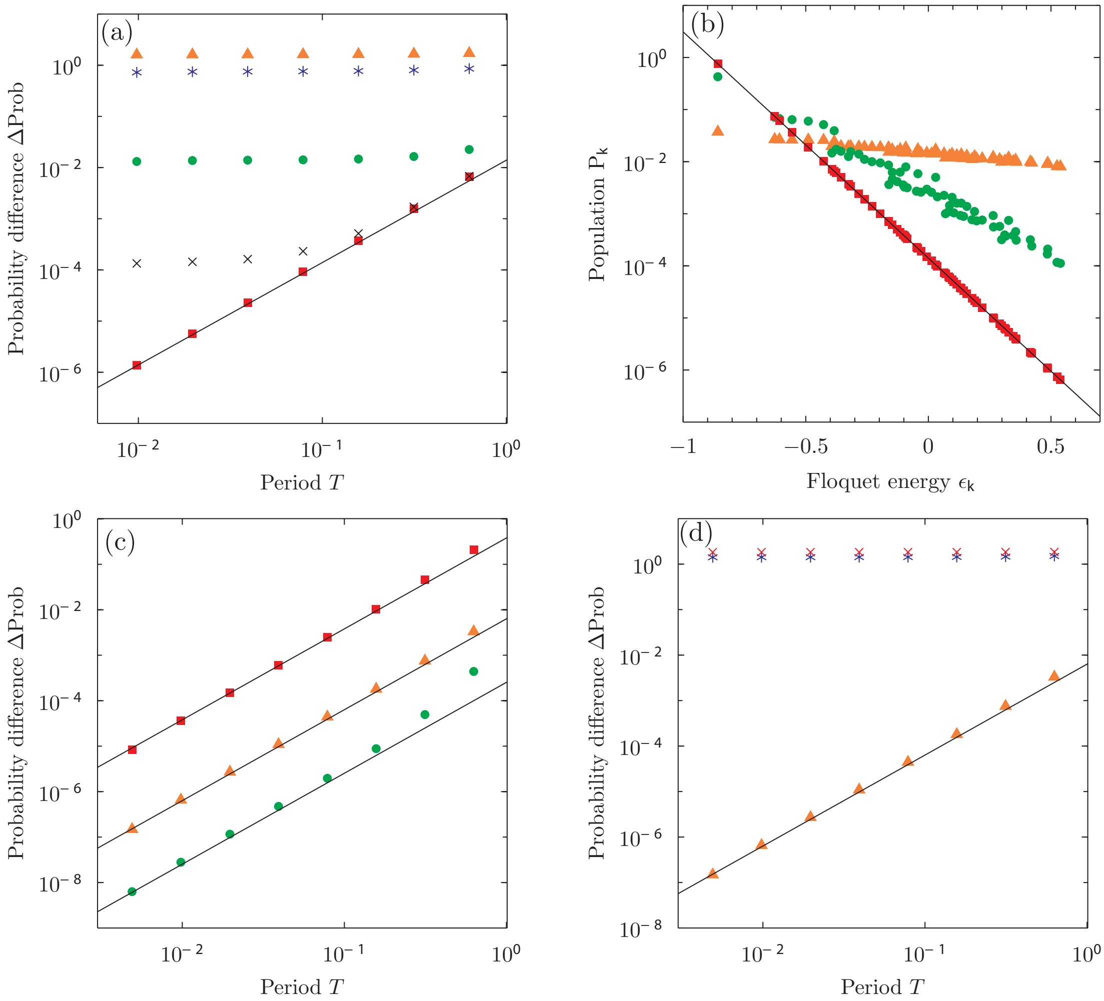

Figure 3.2: (a) ΔProb versus the period of the driving field T for various realizations of $\alpha^y/\alpha^x = \{0$ (squares), 0.01 (crosses), 0.1 (circles), 1 (asterisks), ∞(triangles) $\}$, and $\alpha^z = 0$. When $[H_{\rm ex}(t), H_{\rm I}] = 0$ (squares), the asymptotic state approaches the Floquet-Gibbs state as the period is short. The larger α^y/α^x is, the larger ΔProb is. The black line shows $\Delta\mathrm{Prob} \propto T^2$. (b) Probability distribution $\{\mathrm{P}_i\}$ for the shortest period in figure (a). We use the same symbols to denote different values of α^y/α^x. The black line shows that $\mathrm{P}_i \propto e^{-\beta\epsilon_i}$. (c) Like (a), but the driven spins and the spins coupled to the thermal bath are separated. We study the set $(\alpha^x, \alpha^y, \alpha^z) = \{(0.1, 0.1, 1)$ (filled squares), $(0.1, 1, 0.1)$ (circles), and $(1, 0.1, 0.1)$ (triangles)$\}$. The asymptotic state approaches the Floquet-Gibbs state independently of the set as the period is short. The black line shows $\Delta\mathrm{Prob} \propto T^2$. (d) Like (c), but the driving field on $S_i^y (i = 2, \cdots, 5)$ is applied with strength $r\xi$. We study various realization of $r = \{0$ (triangles), 0.1 (asterisks), 0.5 (crosses)$\}$. The asymptotic state is deviated from the Floquet-Gibbs state when $r \neq 0$. The black line shows $\Delta\mathrm{Prob} \propto T^2$.

3.4 Conclusion and discussion on experiments

We studied the long-time asymptotic state of a periodically driven system by using the Floquet Lindblad equation. We gave the sufficient conditions under which the asymptotic state can be described by the Floquet-Gibbs state even when the system goes beyond the linear response regime. This work reveals the applicability of the Floquet-Gibbs state to an asymptotic state of a driven system surrounded by a heat bath. It has been argued that the Floquet Hamiltonian $H_{\mathrm{F}}^{[0]} \simeq H_{\mathrm{eff}}$ realizes new phases not accessible without driving field as introduced in Chap. 1. It is naively expected that the effective Hamiltonian H_{eff}, Eq. (3.31), can be used for the description of the system in a high-frequency driving field whether the dissipative environment is coupled to the system or not, but this argument is incorrect. The system in a high-frequency field coupled to a heat bath is not necessarily described by the Floquet-Gibbs state. The conditions obtained here restrict the applicability of the Floquet-Gibbs state. First, condition (i) restricts the system with a relatively small Hilbert space. In addition, the condition (iii) is not expected to be realized in experiment, which leads to non-Floquet-Gibbs form as demonstrated in Fig. 3.2(a). The Floquet-Gibbs state is relevant for the description of the long-time asymptotic states when (A) the operator coupled to a dissipative environment can be specified or (B) the subsystems in contact with heat baths and the subsystems driven by the external field are different as in Fig. 3.2(c). We expect that driven two-level systems, e.g., an atom in a laser field [22] and a superconducting qubit under microwaves [31] are candidates to realize the first setup (A). In these systems, the dipole moment coupled to the driving field also becomes the origin of the dissipation, for example, by additionally couping the atom to a bad cavity. We also expect that the second setup (B) is realized in systems consisting of quantum dots [28] or molecular wires under irradiation of microwaves, in which quantum transport of the system with the edge coupled to electrodes (heat baths) has been studied [99].

3.5 Comparison to other works

We have shown two cases where the long-time asymptotic state is expressed by the Floquet-Gibbs state. In addition to these cases, it has been reported that in some specific models the Gibbs form gives an appropriate description. We discuss the following three cases:

- Case 1: Time dependence of the total Hamiltonian can be eliminated by a unitary transformation only on the system of interest,
- Case 2: Time dependence of the total Hamiltonian can be eliminated by a unitary transformation on the system of interest and the heat bath,
- Case 3: A linearly driven harmonic oscillator.

In the first and third case, the asymptotic state is given by the Floquet-Gibbs state, while in the second case it is not in general.

- Case 1

We consider a periodically driven open system with a total Hamiltonian $H(t) = H_{\rm S}(t) + \lambda H_{\rm I} + H_{\rm B}$, which can be transformed into a time-independent Hamiltonian by using a unitary operator acting only on the Hilbert space of the system of interest, $V_{\rm S}(t)$. The time independent Hamiltonian reads

$$H^{\rm R} = V_{\rm S}^{\dagger}(t)\left(H(t) - i\hbar\frac{\partial}{\partial t}\right)V_{\rm S}(t), \tag{3.59}$$

where $V_{\rm S}(t)$ is chosen to be time periodic. The simple system satisfying this property is a particle with one-half spin under a static and circularly polarized magnetic fields,

$$H_{\rm S}(t) = h^z S^z + \xi(S^x \cos\Omega t + S^y \sin\Omega t), \tag{3.60}$$

which interacts with a heat bath through S^z. Here, $\boldsymbol{S}$ is a usual one-half spin operator and h^z denotes the strength of the static magnetic field. In this case, the unitary operator given by $V_{\rm S}(t) = e^{-i\Omega t(S^z+1/2)}$ transforms the total Hamiltonian to the time-independent one. This is also realized in more sophisticated system, e.g. graphene with time- and site-dependent driving field [100, 16].

First, we derive the relation between the Floquet mode in the static frame and the eigenenergy state in the rotating frame. The time-independent system Hamiltonian in the rotating frame reads

$$H_{\rm S}^{\rm R} = V_{\rm S}^{\dagger}(t)\left(H_{\rm S}(t) - i\hbar\frac{\partial}{\partial t}\right)V_{\rm S}(t), \tag{3.61}$$

whose eigenstates and eigenenergies are given by $H_{\rm S}^{\rm R}\left|\psi_i^{\rm R}\right\rangle = E_i^{\rm R}\left|\psi_i^{\rm R}\right\rangle$. Now, we show that $V_{\rm S}(t)\left|\psi_i^{\rm R}\right\rangle$ is the Floquet mode. The Floquet mode $|u_i(t)\rangle$ with a quasienergy ϵ_i obeys a following differential equation (see Eq. (1.12)),

$$\left(H_{\rm S}(t) - i\hbar\frac{\partial}{\partial t}\right)|u_i(t)\rangle = \epsilon_i\,|u_i(t)\rangle\,. \tag{3.62}$$

Since

$$\left(H_{\rm S}(t) - i\hbar\frac{\partial}{\partial t}\right)V_{\rm S}(t)\left|\psi_i^{\rm R}\right\rangle = V_{\rm S}(t)H_{\rm S}^{\rm R}\left|\psi_i^{\rm R}\right\rangle = \epsilon_i V_{\rm S}(t)\left|\psi_i^{\rm R}\right\rangle, \tag{3.63}$$

$V_{\rm S}(t)\left|\psi_i^{\rm R}\right\rangle$ is the Floquet mode with quasienergy $\epsilon_i = E_i^{\rm R}$ [2].

[2] It is noted that the quasienergies may not be within the region, $-\hbar\Omega/2 \le \epsilon_i < \hbar\Omega/2$. However it is always possible to define the quasienergies outside of the region by multiplying appropriate phase factors on the Floquet modes so that the Floquet states are kept in the same form.

Next, we discuss the long-time asymptotic state of this case. Because the time-independent system couples to a thermal bath, the asymptotic state in the rotating frame is given by the canonical distribution,

$$\rho_{\mathrm{Lind}}^{\mathrm{R,A}} = \sum_i \frac{e^{-\beta E_i^{\mathrm{R}}}}{\sum_j e^{-\beta E_j^{\mathrm{R}}}} |\psi_i^{\mathrm{R}}\rangle \langle \psi_i^{\mathrm{R}}| \,. \tag{3.64}$$

In the static frame, because the density matrix is transformed as

$$\rho_{\mathrm{Lind}}^{\mathrm{A}}(t) = V_{\mathrm{S}}(t)(\rho_{\mathrm{Lind}}^{\mathrm{R,A}})V_{\mathrm{S}}^{\dagger}(t), \tag{3.65}$$

the asymptotic state is expressed by

$$\rho_{\mathrm{Lind}}^{\mathrm{A}}(t) = \sum_i \frac{e^{-\beta E_i^{\mathrm{R}}}}{Z} V_{\mathrm{S}}(t) |\psi_i^{\mathrm{R}}\rangle \langle \psi_i^{\mathrm{R}}| V_{\mathrm{S}}^{\dagger}(t) = \sum_i \frac{e^{-\beta \epsilon_i}}{Z} |u_i(t)\rangle \langle u_i(t)| \,, \tag{3.66}$$

which is a form of the Floquet-Gibbs state.

- Case 2

Next, we study the case where the time dependence of the total Hamiltonian $H(t) = H_{\mathrm{S}}(t) + \lambda H_{\mathrm{I}} + H_{\mathrm{B}}$ can be eliminated by using a unitary operator on the Hilbert space of the system of interest and heat bath,

$$H^{\mathrm{R}} = V_{\mathrm{S}}^{\dagger}(t)V_{\mathrm{B}}^{\dagger}(t)\left(H(t) - i\hbar\frac{\partial}{\partial t}\right)V_{\mathrm{S}}(t)V_{\mathrm{B}}(t), \tag{3.67}$$

where $V_{\mathrm{S}}(t)$ is an operator on the system of interest and $V_{\mathrm{B}}(t)$ is an operator on the heat bath. Even when the total Hamiltonian is time independent, the long-time asymptotic state is not generally given by the Floquet-Gibbs state.

It is important to take note of the fact that the bath Hamiltonian also changes in the rotating frame as

$$H_B \to H_B^{\mathrm{R}} = V_{\mathrm{B}}^{\dagger}(t)\left(H_B - i\hbar\frac{\partial}{\partial t}\right)V_{\mathrm{B}}(t). \tag{3.68}$$

Thus, the time evolution of the bath dynamics in the rotating frame is described by $H_{\mathrm{B}}^{\mathrm{R}}$. On the other hand, even in the rotating frame, the state of the heat bath is described by the canonical distribution of H_{B}. As a result, the bath correlation function does not satisfy the KMS relation, Eq. (2.51), and hence the detailed balance condition is generally violated.

Although it has been reported that when a nonlinear resonator driven by this type of driving field is coupled to a low-temperature heat bath, the concept of effective temperature is useful for the description of the asymptotic state [101], the state is not generally described in a Gibbs form.

- Case 3

Finally, we study a linearly driven harmonic oscillator coupled to a heat bath. In this system, the long-time asymptotic state is given in the Floquet-Gibbs state nevertheless the time dependence of the total Hamiltonian is not completely excluded by the unitary transformation as in the first case.

First, we derive the eigenmodes of a linearly driven harmonic oscillator following [23, 102, 103]. The Hamiltonian is given by

$$H_{\mathrm{S}}(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 - x\xi(t), \tag{3.69}$$

where a particle with mass m is confined in a harmonic potential. The particle is driven by a time-dependent force $\xi(t)$. For a while we study a general case, and thus we do not impose the periodicity on $\xi(t)$.

This system is reduced to a time-independent harmonic oscillator by moving to a rotating frame using a unitary transformation,

$$V(t) = e^{-\frac{i}{\hbar}\int_0^t L(\zeta,\dot{\zeta},\tau)d\tau} e^{-\frac{i}{\hbar}m\dot{\zeta}(t)x} e^{\frac{i}{\hbar}p\zeta(t)}, \tag{3.70}$$

where $L(\zeta, \dot{\zeta}, t)$ is a classical Lagrangian,

$$L(\zeta, \dot{\zeta}, t) = \frac{1}{2}m\dot{\zeta}^2 - \frac{1}{2}m\omega_0^2\zeta + \zeta\xi(t), \tag{3.71}$$

and ζ obeys a classical equation of motion,

$$m\ddot{\zeta} = -m\omega_0^2\zeta + \xi(t). \tag{3.72}$$

The state in the rotating frame obeys the dynamics for the Hamiltonian given by

$$H_{\mathrm{S}}^{\mathrm{R}} = V(t)\left(H_{\mathrm{S}}(t) - i\hbar\frac{\partial}{\partial t}\right)V^\dagger(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2. \tag{3.73}$$

As a result, the general solutions $|\psi(t)\rangle$ can be expanded using eigenmodes $|\psi_n(t)\rangle$,

$$|\psi(t)\rangle = \sum_n \langle\phi_n| V^\dagger(0) |\psi(0)\rangle |\psi_n(t)\rangle, \tag{3.74}$$

where

$$|\psi_n(t)\rangle = V(t)e^{-iE_nt} |\phi_n\rangle. \tag{3.75}$$

Here $H_{\mathrm{S}}^{\mathrm{R}} |\phi_n\rangle = E_n |\phi_n\rangle$ and $E_n = \hbar\omega_0(n + \frac{1}{2})$.

From hereon, we assume that the driving field is time periodic, $\xi(t) = \xi(t + T)$, and the frequency is not resonant with the harmonic oscillator, $\omega_0 \neq \Omega$. In this case, we find that $|\psi_n(t)\rangle$ can be the Floquet state. In order to see this, we set $\zeta_S(t)$ as a time-periodic solution of Eq. (3.72), $\zeta_S(t) = \zeta_S(t+T)$, which is uniquely determined.

By using $\zeta_S(t)$ for the definition of the unitary operator $V(t)$, we find the following relation,

$$|\psi_n(T)\rangle = V(T)e^{-iE_nT}V^\dagger(0)\,|\psi_n(0)\rangle = e^{-i(E_n+\overline{L})T}\,|\psi_n(0)\rangle\,, \tag{3.76}$$

where $\overline{L}$ is a one-period average of $L(\zeta_S(t), \dot{\zeta}_S(t), t)$. From this equation, we can choose $|\psi_n(t)\rangle$ to be the Floquet state, and corresponding quasienergy ϵ_n and the Floquet mode $|u_n(t)\rangle$ are expressed by

$$\epsilon_n = E_n + \overline{L}, \tag{3.77}$$

and

$$|u_n(t)\rangle = e^{i(E_n+\overline{L})t}\,|\psi_n(t)\rangle = V(t)\,|\phi_n\rangle\, e^{i\overline{L}t}, \tag{3.78}$$

respectively.

Here we study the asymptotic state of the driven harmonic oscillator weakly coupled to a heat bath through x, that is, the interaction Hamiltonian is given by

$$H_{\mathrm{I}} = x \otimes Y, \tag{3.79}$$

where Y is an operator of the heat bath. To evaluate the transition probabilities among the Floquet states, we calculate

$$x_{ijm} = \int_0^T \langle u_i(t)|\, x\, |u_j(t)\rangle\, e^{-im\Omega t}\frac{dt}{T}, \tag{3.80}$$

which is simply given by

$$\begin{aligned} x_{ijm} &= \int_0^T \langle \phi_i|\, V^\dagger(t) x V(t)\, |\phi_j\rangle\, e^{-im\Omega t}\frac{dt}{T} \\ &= \langle \phi_i|\, x\, |\phi_j\rangle\, \delta_{m,0} - \delta_{i,j}\int_0^T \zeta_S(t) e^{-im\Omega t}\frac{dt}{T}. \end{aligned} \tag{3.81}$$

Because only one term denoted by $m = 0$ in Eq. (3.2) contributes to the transition probability, the asymptotic state is described by the Floquet-Gibbs state [14],

$$\rho^{\mathrm{A}}_{\mathrm{Lind}}(t) = \sum_i \frac{e^{-\beta\epsilon_i}}{\sum_j e^{-\beta\epsilon_j}}\,|u_i(t)\rangle\,\langle u_i(t)|\,. \tag{3.82}$$

Let us reconsider this result from another aspect. Total Hamiltonian in the rotating frame is given by

$$H^{\mathrm{R}} = V(t)\left(H_{\mathrm{S}}(t) + H_{\mathrm{I}} + H_{\mathrm{B}} - i\hbar\frac{\partial}{\partial t}\right)V^\dagger(t) = H^{\mathrm{R}}_{\mathrm{S}} + \lambda(x - \zeta_S(t))Y + H_{\mathrm{B}}, \tag{3.83}$$

where H_{B} is a bath Hamiltonian. The time-dependent term remains in the interaction Hamiltonian, but since $\zeta_S(t)$ is a scalar, this term does not play any role in a transition probability between different Floquet states within the lowest-order perturbation of the interaction Hamiltonian (Fermi's golden rule). As a result, the system of interest relaxes into the Floquet-Gibbs state. In contrast to the first case, it has been known that this time-dependent term induces finite energy flow from the system of interest to the thermal bath [15].

Chapter 4

Floquet-Gibbs state in open systems with finite system-bath coupling

4.1 Introduction and brief summary

In the last chapter, we introduced the notion of the Floquet-Gibbs states, and showed that within the Lindblad formalism the states are realized under the following three conditions:
(i) the driving frequency is much faster than the width of the eigenenergy spectrum of the system Hamiltonian,
(ii) the driving Hamiltonians at different times commute,
(iii) the driving Hamiltonian and the interaction Hamiltonian for the system-bath coupling commute.

As we mentioned, the above conditions severely restrict the class of suitable models attaining the Floquet-Gibbs state. The condition (i) restricts the system with a relatively small Hilbert space. The condition (iii) requires a fine tuning on the system-bath coupling operators. This work thus indicates that the long-time asymptotic state is different from the Floquet-Gibbs states except for the restrictive situation.

As mentioned in Chap. 2, the Lindblad formalism imposes an extra condition. Its use is justified when the relaxation timescale of the system τ_{relax} is much longer than any other timescales of the system. In addition, the typical timescale of the bath dynamics τ_{bath} is often assumed to be smaller than other timescales of the system.

These assumptions are not always satisfied, and difficult to meet especially when the system is subjected to a high frequency field. At high frequency, the heating rate $\tau_{\mathrm{heat}}^{-1}$ is extremely small [45, 46, 47, 48], and hence this rate $\tau_{\mathrm{heat}}^{-1}$ can be smaller than the relaxation rate $\tau_{\mathrm{relax}}^{-1}$. Thus, a finite system-bath coupling can strongly change the long-time dynamics and there is no guarantee that the Lindblad equation correctly describes the long-time asymptotic state. In addition, the period of the driving field T can be smaller than the timescales of the heat bath τ_{bath}.

In this chapter, we study generic situations of a finite system-bath coupling by carefully considering timescales $\tau_{\mathrm{relax}}, \tau_{\mathrm{heat}}, \tau_{\mathrm{bath}}$, and T. We focus on the dependence of the long-time asymptotic states on the system-bath coupling strength and timescales of the heat bath. In order to achieve our purpose, we go beyond the RWA and use the Redfield equation, which allows us to capture the effects of the finite system-bath coupling strength on the long-time asymptotic state. We then use the effective (truncated) Floquet Hamiltonian, which can be obtained by the Floquet-Magnus expansion. This step allows us to circumvent the condition (i). In addition, the condition (iii) becomes irrelevant when the period of the driving field is smaller than the timescale of the bath dynamics, $T < \tau_{\mathrm{bath}}$. Thus, the Floquet-Gibbs state can be observed in a broader subclass of the driven open systems without imposing the strict conditions (i) and (iii) on the system operators. We illustrate this hypothesis by using a non-integrable spin chain model, and we demonstrate that the Floquet-Gibbs state is realized even when conditions (i) and/or (iii) are not satisfied.

This chapter is organized as follows; In the next section 4.2, we explain the outline how to obtain the long-time asymptotic state in the Redfield formulation. In Sec. 4.3, we redefine the Floquet-Gibbs state using the effective Floquet Hamiltonian, and in Sec. 4.4 we introduce the spin chain model. In Sec. 4.5, we demonstrate how condition (i) is lifted via the finite system-bath coupling. In Sec. 4.6, we show that condition (iii) is not necessary when the bath dynamics is longer than the periodic driving field, $\tau_{\mathrm{bath}} > T$. Finally, in Sec. 4.7 we conclude with a discussion of the physical implications of our results.

4.2 Long-time asymptotic states of systems with finite system-bath coupling strength

In this section, we show how to obtain the long-time asymptotic state of the Redfield equation, Eq. (2.22). An element of the density matrix in the basis of the Floquet state, $\rho_{ij}(t) = \langle u_i(t)| \rho(t) |u_j(t)\rangle$, obeys

$$\begin{aligned}\frac{d\rho_{ij}(t)}{dt} =& -\frac{i}{\hbar}(\epsilon_i - \epsilon_j)\rho_{ij}(t) - \frac{\lambda^2}{\hbar^2}\sum_{\mu}\sum_{l,m}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} e^{i(n+n')\Omega t} \\ &\times \{G_+^{\mu}(\omega_{lmn})X_{lmn}^{\mu}X_{iln'}^{\mu}\rho_{mj}(t) - G_+^{\mu}(\omega_{imn})X_{imn}^{\mu}X_{ljn'}^{\mu}\rho_{ml}(t) \\ &- G_-^{\mu}(-\omega_{ljn})X_{ljn}^{\mu}X_{imn'}^{\mu}\rho_{ml}(t) + G_-^{\mu}(-\omega_{lmn})X_{lmn}^{\mu}X_{mjn'}^{\mu}\rho_{il}(t)\},\end{aligned} \tag{4.1}$$

where the frequency differences are given by $\omega_{lmn} = (\epsilon_l - \epsilon_m)/\hbar + n\Omega$ and X_{lmn}^{μ} is given by Eq. (2.37). Here, we assume that there is no correlation between Y^{μ} and Y^{ν} for $\mu \neq \nu$, and we write the Fourier components of the bath correlation function as

$$G_{\pm}^{\mu}(\omega) = \int_0^{\pm\infty} \langle Y^{\mu}(t)Y^{\mu}\rangle_{\mathrm{B}}\, e^{-i\omega t}dt. \tag{4.2}$$

It is quite natural to assume that the long-time asymptotic behavior converges to a periodic motion,

$$\rho^{\mathrm{A}}_{\mathrm{Red}}(t+T) = \rho^{\mathrm{A}}_{\mathrm{Red}}(t). \tag{4.3}$$

Below we will address this asymptotic solution only, and hence the label "A" and "Red" are dropped from hereon.

The periodic asymptotic state can be expressed by the Fourier expansion of the elements of the density matrix,

$$\rho_{ij}(t) = \sum_{n=-\infty}^{\infty} \rho_{ijn} e^{in\Omega t}. \tag{4.4}$$

By substituting this form into Eq. (4.1), we obtain the equations for the asymptotic solution; for the set of (i, j, n)

$$\begin{aligned} 0 =& i\omega_{ijn}\rho_{ijn} + \frac{\lambda^2}{\hbar^2}\sum_{\mu}\sum_{l,m}\sum_{n'=-\infty}^{\infty}\sum_{n''=-\infty}^{\infty}\Big\{G^{\mu}_{+}(\omega_{lmn'})X^{\mu}_{lmn'}X^{\mu}_{il[n-(n'+n'')]}\rho_{mjn''} \\ & - G^{\mu}_{+}(\omega_{imn'})X^{\mu}_{imn'}X^{\mu}_{lj[n-(n'+n'')]}\rho_{mln''} - G^{\mu}_{-}(-\omega_{ljn'})X^{\mu}_{ljn'}X^{\mu}_{im[n-(n'+n'')]}\rho_{mln''} \\ & + G^{\mu}_{-}(-\omega_{lmn'})X^{\mu}_{lmn'}X^{\mu}_{mj[n-(n'+n'')]}\rho_{iln''}\Big\}. \end{aligned} \tag{4.5}$$

The second term of the right hand side of this equation is of the order of λ^2, and in order to fulfill this equation, the corresponding element ρ_{ijn} has to be small (by absolute value) when ω_{ijn} is large [98]. The argument leads to the following approximation,

$$\forall|\omega_{ijn}| > \hbar^{-1}, \quad \rho_{ijn} = 0. \tag{4.6}$$

This approximation means that we set to be zero those Fourier components of the density matrix element for which the absolute value of $\hbar\omega_{ijn}$ is greater than one. The number of elements with $\hbar|\omega_{ijn}| \leq 1$ decreases as the driving frequency increases, and hence at high frequency this approximation enables us to study the system with large Hilbert space.

The approximation allows us to go beyond the Lindblad formalism, which is reproduced by putting $\rho_{ijn} = 0$ except when $\omega_{ijn} = 0$. The corresponding asymptotic state in the Lindblad formalism is independent of the coupling strength [13, 98, 17]. In contrast, our scheme shows the dependence on λ of the long-time asymptotic state. It also allows us to go beyond the so-called "moderate rotating wave approximation" [59], in which only the $n = 0$ mode is kept [104, 105].

4.3 Redefinition of the Floquet-Gibbs state

In this section we redefine the notion of the Floquet-Gibbs state. In the last chapter, we introduced the Floquet-Gibbs state in the Lindblad formulation by using the

Floquet Hamiltonian $H_{\mathrm{F}}^{[t]}$, which is explicitly given by

$$\rho(t) = \frac{e^{-\beta H_{\mathrm{F}}^{[t]}}}{\mathrm{Tr} e^{-\beta H_{\mathrm{F}}^{[t]}}}. \tag{4.7}$$

However this form is not suitable when the condition (i) is violated. This is because the energies of the Floquet Hamiltonian are not uniquely determined; The quasienergy of the Floquet states, being a mere phase factor, can be shifted by multiple of $\hbar\Omega$. Thus, there is no natural order to sort the quasienergies. This ambiguity restricts the idea of the Floquet-Gibbs state to the case where condition (i) is satisfied.

Here we introduce the truncated Floquet Hamiltonian $H_{\mathrm{F}}^{(n)[t]}$ which is defined via the Floquet-Magnus expansion (see Sec. 1.3.3). This truncated Floquet Hamiltonian overcomes the deficiency of the Floquet Hamiltonian. Energies of the truncated Floquet Hamiltonian are uniquely defined (see Eq. (1.48)-(1.51)). In addition, this Hamiltonian provides an efficient basis to express the asymptotic density matrix.

While it is known that the Floquet-Magnus expansion may not converge when the condition (i) is not satisfied, there exists a truncated Floquet Hamiltonian with an optimal n that accurately describes long-lived transient states of the decoupled dynamics [49, 45, 46, 47, 48] as far as the driving frequency is much higher than energy scale of a *single*-site spin. When a periodically driven system is coupled to a dissipative environment, the interaction with the environment suppresses the continuous "heating process" up to an infinite temperature state, which usually appears in decoupled coherent systems [52, 53, 54]. Since we will study the high frequency regime, the rate of the energy absorption rate and thus the heating rate is slow. When the system-bath coupling is weak but finite, the dissipation rate can be larger than the heating rate. Therefore, the system will not heat up to the infinite temperature state even in the long-time limit. In this situation, the eigenbasis of the truncated Floquet Hamiltonian, corresponding to the transient metastable states, prefers to the Floquet basis. It is reasonable to probe the idea of the asymptotic density matrix having the Gibbs form in the basis of the effective (truncated) Floquet Hamiltonian with an optimal value of n_{eff}. The redefined Floquet-Gibbs state is given by

$$\rho_{\mathrm{FG}}(t) = \frac{e^{-\beta H_{\mathrm{F}}^{(n_{\mathrm{eff}})[t]}}}{\mathrm{Tr} e^{-\beta H_{\mathrm{F}}^{(n_{\mathrm{eff}})[t]}}}. \tag{4.8}$$

We determine n_{eff} by following the previous study [45]; There exists an optimal n which minimizes the deviation of the eigenspectrum of the Floquet Hamiltonian from that of the truncated Floquet Hamiltonian,

$$\Delta = \max_{\substack{|\psi\rangle \\ (\langle\psi|\psi\rangle=1)}} \left\| \left(\mathcal{T} e^{-\frac{i}{\hbar}\int_0^T H_{\mathrm{S}}(t)dt} - e^{-\frac{i}{\hbar} H_{\mathrm{F}}^{(n)[0]} T} \right) |\psi\rangle \right\|. \tag{4.9}$$

The above deviation can be evaluated by diagonalizing the operator in the parenthesis and then taking the maximal eigenvalue. Our idea is that this truncation order n_{eff}

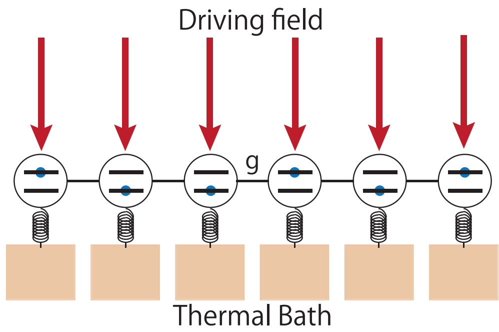

Figure 4.1: Illustration of the spin chain model subjected to a periodic driving in a dissipative environment. Each spin site is connected to an independent bath and a driving field. All baths possess the same thermal properties and are all kept at the same inverse temperature β.

should be used to construct an effective Floquet Hamiltonian, and its eigenstates should be used to express the asymptotic density matrix [1]. When the condition (i) is satisfied, the redefined Floquet-Gibbs state is identical to that defined in the previous chapter, because the Floquet-Magnus expansion converges. The redefined Floquet-Gibbs state is realized under some reasonable conditions, which will be discussed in the following section.

4.4 Spin chain model

In order to verify the existence of the Floquet-Gibbs state, we study a spin chain model (see the schematic picture in Fig. 4.1) with the Hamiltonian,

$$\begin{aligned} H_{\mathrm{S}}(t) =& H_0 + \xi H_{\mathrm{ex}}(t), \\ H_0 =& h^z \sum_{i=1}^{6} S_i^z + h^x \sum_{i=1}^{6} S_i^x - g \sum_{i=1}^{5} S_i^x S_{i+1}^x, \\ \xi H_{\mathrm{ex}}(t) =& \xi \cos(\Omega t) \sum_{i=1}^{6} S_i^x, \end{aligned} \tag{4.10}$$

[1]There exist other Hamiltonians which give the same values of Δ. For instance even if we add the projection operator on a eigenstate of the effective Floquet Hamiltonian times $\hbar\Omega$ to the effective Floquet Hamiltonian, the deviation Δ does not change. However, these other choices than the effective Floquet Hamiltonian is inappropriate for the description because they contain highly nonlocal terms.

where $\{S_i^\alpha\}_{\alpha=x,y,z}$ is a one-half spin operator at site i. Here h^z and h^x denotes the strength of a static magnetic field along z-axis and x-axis, respectively, and g is the coupling strength between neighboring spins. Additionally, a time-periodic driving field is applied along x direction. This model describes a quasi-one-dimensional Ising ferromagnet [106] or a chain of interacting superconducting qubits [107]. Through this chapter we fix the parameters: $h^z = 1$, $g = 0.75$, and $h^x = 0.7$ keeping $\xi/(\hbar\Omega)$ to be $2/3$. These set of parameters fixes the width of the energy spectrum of H_0, $\Delta_0 = 7.6$. With this choice, it is possible to study the regime where the frequency of the driving field Ω is smaller than $\Delta_0/\hbar$, and thus condition (i) is violated. We employ three different values for the frequency of the driving field, $\hbar\Omega = 4.2$, 4.6, and 5.0. These values are smaller than Δ_0, but larger than the characteristic energy scale of a single spin.

In order to introduce the truncated Floquet Hamiltonian when the system is subjected to a fast and strong driving field, it is convenient to transform into the rotating frame. Following the same argument in the last chapter from Eq. (3.24) to Eq. (3.26), we obtain the Hamiltonian in the rotating frame,

$$H_{\mathrm{S}}^{\mathrm{R}}(t) = V^\dagger(t) H_0 V(t), \quad V(t) = e^{\sum_{j=1}^{6}[-i\xi \sin(\Omega t) S_j^x/\hbar\Omega]}, \tag{4.11}$$

which remains time periodic because the condition (ii) is satisfied (see Sec. 3.3). The Floquet Hamiltonian in the rotating frame $H_{\mathrm{F}}^{\mathrm{R}[t]}$ is defined by the time evolution operator over one period, from time t to time $t+T$,

$$e^{-\frac{i}{\hbar} H_{\mathrm{F}}^{\mathrm{R}[t]} T} = \mathcal{T} e^{-\frac{i}{\hbar}\int_t^{t+T} H_{\mathrm{S}}^{\mathrm{R}}(\tau) d\tau}. \tag{4.12}$$

Hereafter we consider in the rotating frame.

In Fig. 4.2, we show the dependence of Δ, Eq. (4.9), on the truncated order n for different frequencies. For each Ω, Δ initially decreases with the increase of n, but then after reaching a minimum, it increases again. The minimal determines n_{eff}. From hereon for notational simplicity, we will suppress the explicit dependence on n_{eff} and assume the values $n_{\mathrm{eff}} = 13, 14$, and 16 for the driving frequency $\hbar\Omega = 4.2, 4.6$, and 5.0, respectively.

4.5 Dependence of asymptotic states on dissipation strength

In this section, we show the dependence of the finite coupling strength on the long-time asymptotic state in the spin chain model. The dissipative effect of the heat bath in the Redfield equation is determined through the correlation function of the bath operators. In order to explicitly give the form, we need to specify the model of the thermal bath. Here we follow the standard prescription and consider a thermal bath consisting of an ensemble of harmonic oscillators [108].

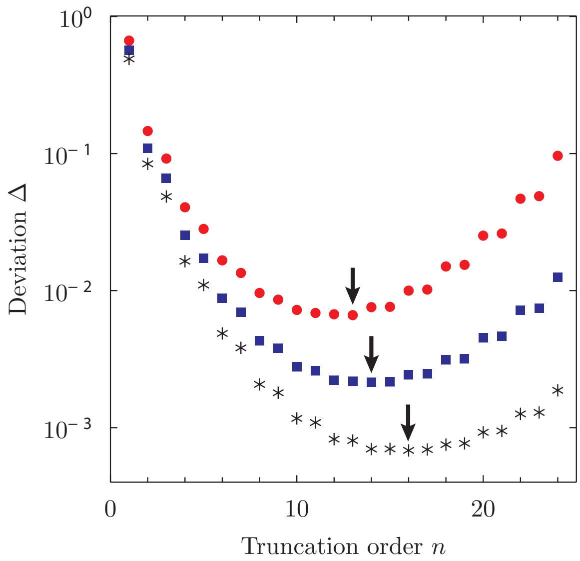

Figure 4.2: Deviation Δ, Eq. (4.9), versus the truncation order n for the spin chain model. Three values of the driving frequency ($\hbar\Omega = 4.2, 4.6$, and 5.0 from the top to the bottom) are used in simulations, while keeping the ratio $\xi/(\hbar\Omega) = 2/3$ fixed. Arrows represent the optimized order $n_{\text{eff}} = 13(\hbar\Omega = 4.2)$, $14(\hbar\Omega = 4.6)$, and $16(\hbar\Omega = 5.0)$

The bath is described by a Hamiltonian,

$$H_{\rm B} = \sum_{i=1}^{6} \sum_{\alpha} \frac{p_i^{\alpha 2}}{2m^\alpha} + \frac{m^\alpha \omega^{\alpha 2} x_i^{\alpha 2}}{2}, \tag{4.13}$$

where x_i^α and p_i^α are canonical variables of α-th mode of the heat bath, and m^α and ω^α are the mass and the frequency of the oscillator, respectively. The index i indicates that each spin is coupled to an independent heat bath, which is uncorrelated with any other heat baths [109].

In this section, we set for the interaction Hamiltonian between the system of interest and the heat bath a form that commutes with $H_{\rm ex}(t)$, i.e.,

$$H_{\rm I} = \lambda \sum_{i=1}^{6} \sum_{\alpha} S_i^x \otimes x_i^\alpha. \tag{4.14}$$

This choice satisfies the condition (iii) (see Sec. 3.3), and hence we study the effects of the finite system-bath coupling when the condition (i) is broken. The heat baths are characterized by a spectral density that we choose to be of the ohmic form,

$$J(\omega) = \sum_{\alpha} \frac{\pi}{2m^\alpha \omega^\alpha} \delta(\omega - \omega_\alpha) = \omega e^{-\frac{\omega}{\omega_c}}, \tag{4.15}$$

where we fix the cutoff frequency to be $\omega_c = 100/\hbar$. As far as the condition (i) is concerned, the value of the cutoff frequency does not play a crucial role, whose effect is strongly related to the condition (iii), which will be shown in the next section. The inverse temperature of the heat baths is set to be $\beta = 1$.

With these settings, we obtain the asymptotic density matrix by solving Eq. (4.5) under the approximation specified in Eq. (4.6). We calculate the probability,

$$\begin{aligned} \mathrm{P}_l^{[t]} &= \langle\phi_l(t)|\, V^\dagger(t)\rho_{\mathrm{Red}}^{\mathrm{A}}(t)V(t)\,|\phi_l(t)\rangle\,, \\ &= \sum_{i,j}\sum_{k=-\infty}^{\infty} \rho_{ij}^k c_{jl}(t)[c_{il}(t)]^* e^{ik\Omega t}, \end{aligned} \tag{4.16}$$

with elements $c_{il}(t) = \langle u_i(t)|\, V(t)\,|\phi_l(t)\rangle$, and $|\phi_l(t)\rangle$ are the eigenstates of the effective Floquet Hamiltonian.

We then compare the probabilities $\mathrm{P}_l^{[t]}$ with the Boltzmann factors by using the norm

$$\Delta\mathrm{Prob}^{[t]} = \sum_l \left| \mathrm{P}_l^{[t]} - \frac{e^{-\beta E_l^{[t]}}}{\sum_j e^{-\beta E_j^{[t]}}} \right|, \tag{4.17}$$

where $E_l^{[t]}$ are the eigenenergies of the effective Floquet Hamiltonian. As the difference $\Delta\mathrm{Prob}^{[t]}$ approaches zero, the asymptotic state is closer to the Floquet-Gibbs state.

In Fig. 4.3, we show the dependence of $\Delta\mathrm{Prob}^{[0]}$ on the system-bath coupling strength λ^2. The values of $\Delta\mathrm{Prob}^{[0]}$ for $\lambda^2 = 10^{-6}$ are nearly identical to those obtained by the Lindblad equation. In addition, $\Delta\mathrm{Prob}^{[t]}$ is almost independent of t, and thus the value at time $t = 0$ gives a good representative of all the instants of t. We also estimate the value of the off-diagonal elements in the basis of the effective Floquet Hamiltonian. For the used set of parameters, their absolute values at largest system-bath coupling in Fig. 4.3 are at most around 10^{-3} and typically the order of 10^{-5} or less. Therefore, the asymptotic density matrix has an almost diagonal form at large system-bath coupling, $\lambda^2 \simeq 10^{-2}$.

We find from this figure that for the driving frequency $\Omega = 4.6/\hbar$, $\Delta\mathrm{Prob}^{[0]}$ reduces as the system-bath coupling becomes stronger. This indicates that the finite system-bath coupling can push the asymptotic state closer to the Floquet-Gibbs state. The large difference observed in the weak system-bath coupling regime at $\hbar\Omega = 4.6$ originates from the coherences between the states due to the resonance, i.e., the energy gap between two eigenenergies of the effective Floquet Hamiltonian is in resonance with the driving frequency, $E_1^{[0]} - E_2^{[0]} \simeq \hbar\Omega = 4.6$. In the inset of Fig. 4.3, we show the dependence of the coherence,

$$\Psi = |\,\langle\phi_1(0)|\,\rho_{\mathrm{Red}}^{\mathrm{A}}(0)\,|\phi_2(0)\rangle\,|, \tag{4.18}$$

as a function of the dissipation strength. The coherence disappears as the system-bath coupling becomes strong which leads to the realization of the Floquet-Gibbs

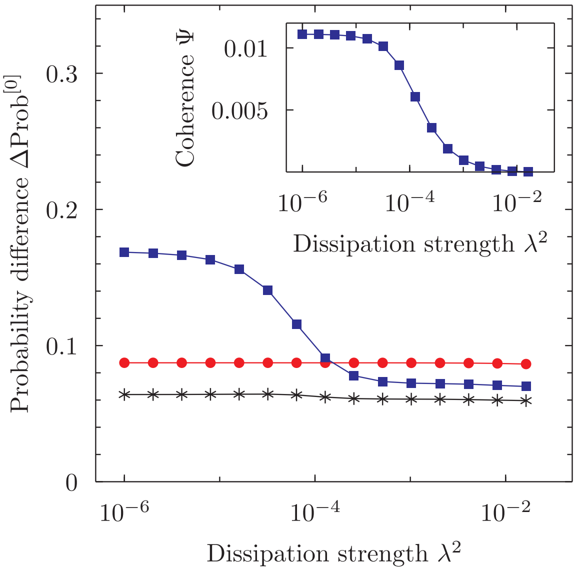

Figure 4.3: The dependence of $\Delta\text{Prob}^{[0]}$ on the dissipation strength λ^2 for three values of driving frequency $\hbar\Omega = 4.2$ (red circles), 4.6 (blue squares), and 5.0 (black asterisks). There is a resonance (see the text) for $\hbar\Omega = 4.6$, which deviates the diagonal elements from the Boltzmann distribution at weak system-bath coupling. The deviation decreases as the system-bath coupling increases. Inset shows the dependence of the coherence Ψ, Eq. (4.18), between the eigenstates in resonance with the driving field as a function of λ^2 for $\hbar\Omega = 4.6$. The cutoff frequency is set to be $\hbar\omega_c = 100$.

state. Since the resonance effect is not observed at $\hbar\Omega = 4.2$ and 5.0, the long-time asymptotic state can be reasonably approximated by the Floquet-Gibbs state even when the system-bath coupling is weak.

In order to see how the asymptotic state is close to the Floquet-Gibbs state for the resonant case $\hbar\Omega = 4.6$, we calculate the populations $\text{P}_k^{[0]}$ as functions of $E_k^{[0]}$ for two values of system-bath coupling strength, $\lambda^2 = 10^{-2}$ and $\lambda^2 = 10^{-6}$; see Fig. 4.4. For the stronger system-bath coupling, the resonance is completely suppressed and the distribution of populations is close to the Boltzmann distribution. However, the linear fit of the log dependence gives the exponent β_{eff} which is smaller than the actual one, ($\beta_{\text{eff}} = 0.930 < \beta = 1$). This is the reason why the difference $\Delta\text{Prob}^{[0]}$ does not approach zero even at large system-bath coupling. This observation agrees with the previous results obtained in the Lindblad formulation [14, 17]. In these works, the "effective temperature" $T_{\text{eff}} = 1/(k_{\text{B}}\beta_{\text{eff}})$ was found to be higher than the actual temperature of the heat bath. The mechanism behind the "effective temperature", although very interesting, is beyond the scope of this thesis. Overall this shows that the Floquet-Gibbs state is realized in a system with a relatively large Hilbert space (condition (i) is not satisfied), and the finite system-bath coupling helps to lead the

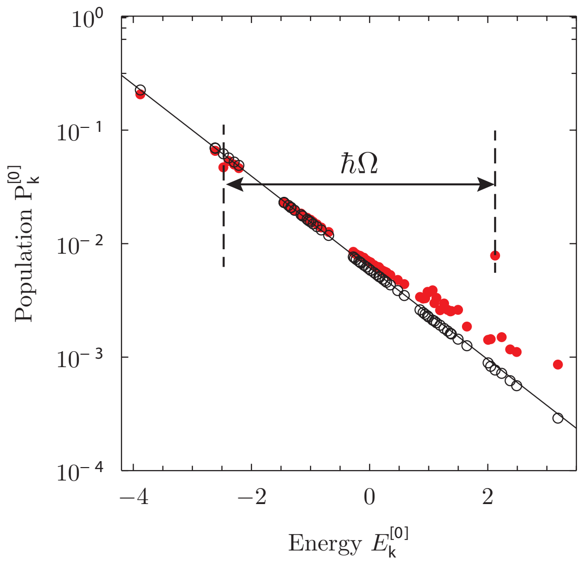

Figure 4.4: The populations $\mathrm{P}_k^{[0]}$ in the eigenbasis of the effective Floquet Hamiltonian of the driving frequency $\Omega = 4.6/\hbar$ for strong system-bath coupling $\lambda^2 = 10^{-2}$ (open black circles) and weak system-bath coupling $\lambda^2 = 10^{-6}$ (closed red circles) . At the weaker coupling, there are two eigenstates which are resonance with the driving field. Their populations strongly deviate from the Boltzmann distribution. Solid black line corresponds to the Boltzmann distribution with $\beta_{\text{eff}} = 0.930 < \beta = 1$. Other parameters are the same in Fig. 4.3.

long-time asymptotic state to this form.

4.6 Dependence of asymptotic states on the thermal bath timescale

Can the asymptotic density matrix be closer to the Floquet-Gibbs state? In other words, what conditions guarantee that the diagonal elements are well described by the Boltzmann factors at the temperature of the heat bath?

In this section we try to answer this question and study the transition to the Floquet-Gibbs state as a function of the timescale of the bath dynamics, τ_{bath}. Thus, We drop the condition (iii) (see Sec. 3.3), and study the general case where the interaction Hamiltonian is not commutable with the driving Hamiltonian. Here, we use the following interaction Hamiltonian:

$$H_{\text{I}} = \sum_{i=1}^{6} \sum_{\alpha} x_i^{\alpha}(S_i^x + S_i^y). \tag{4.19}$$

It is worth noting here that throughout this thesis we have neglected the counter

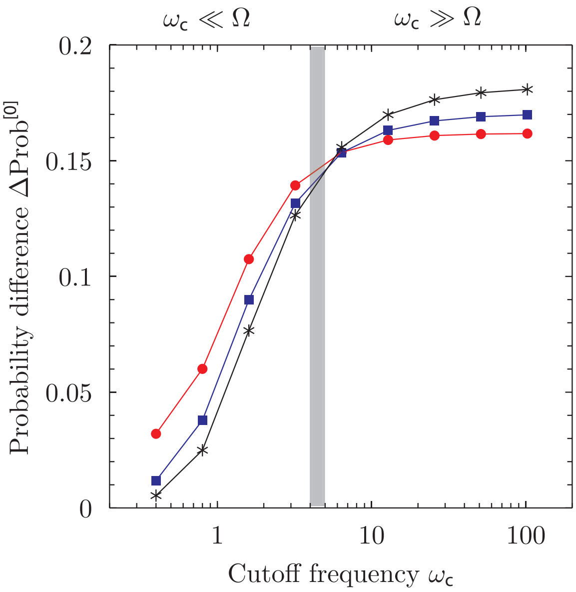

Figure 4.5: The dependence of $\Delta\mathrm{Prob}^{[0]}$, Eq. (4.17), on the bath cutoff frequency ω_c at $\lambda^2 = 10^{-2}$ for three values of driving frequency, $\hbar\Omega = 4.2$ (red circles), 4.6 (blue squares), 5.0 (black asterisks). Gray stripe marks the interval $\omega_c \in [4.2, 5]$.

term that generally appears in the Zwanzig-Caldeira-Leggett model [110, 111, 112]. In the present choice of the interaction Hamiltonian, the counter term does not play any role since it is proportional to $(S_i^x)^2$ and $(S_i^x + S_i^y)^2$ that cause a constant shift in the system Hamiltonian.

When condition (iii) is not satisfied, the fast modes of the heat bath, whose frequencies are close to the integer multiple of the driving frequency, are excited by the periodic driving field. The response of the system to the fast bath dynamics affects the long-time asymptotic states. In our context, this effect will induce further deviation of the asymptotic density matrix from the Floquet-Gibbs state. This is the reason why the condition (iii) needs to be satisfied. However, if the heat bath consists of only slow modes, the energy pumping inside the bath is suppressed, and thus the distortion is weak. The characteristic timescales of the bath dynamics is controlled by the inverse cutoff frequency ω_c^{-1} of the spectral density, Eq. (4.15).

In Fig. 4.5, we study the dependence of $\Delta\mathrm{Prob}^{[0]}$ on the cutoff frequency for different values of the driving frequency at $\lambda^2 = 10^{-2}$. We again observe the almost independence of $\Delta\mathrm{Prob}^{[t]}$ on t. When the timescale of the bath dynamics is short ($\omega_c \approx 100/\hbar \gg \Omega$), the deviation from the Floquet-Gibbs state is large for all frequencies owing to the violation of the condition (iii). On the other hand, when the timescale of the heat bath is long ($\omega_c \ll \Omega$), the long-time asymptotic state is well described by the Floquet-Gibbs state at the temperature of the heat bath.

Figure 4.6 shows the dependencies of $\Delta\mathrm{Prob}^{[0]}$ on λ^2 for the cutoff frequency $\omega_c = 0.4/\hbar$. The dependencies for three values of Ω exhibit similar behavior to that

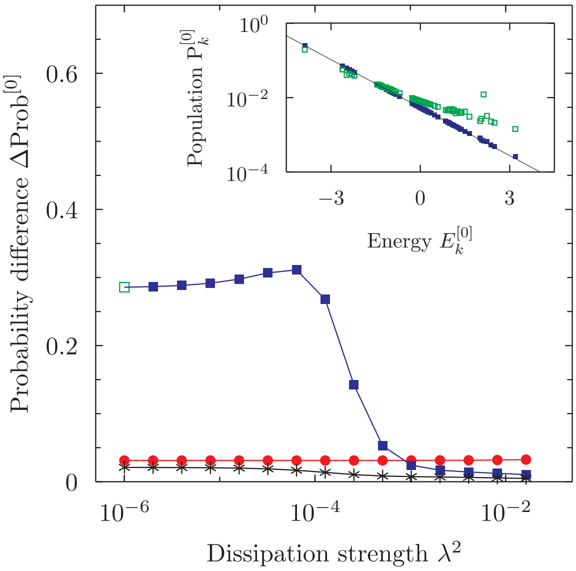

Figure 4.6: The dependence of $\Delta\text{Prob}^{[0]}$ on λ^2 on the dissipation strength λ^2 for $\hbar\Omega = 4.2$ (circles), 4.6 (squares) and 5.0 (asterisks). For $\hbar\Omega = 4.6$ there is a resonant coupling of two eigenstates of the effective Floquet Hamiltonian, and the dependence follows the same scenario observed in Fig. 4.3. The cutoff frequency is set to be $\omega_c = 0.4$. Inset: The populations $\text{P}_k^{[0]}$ for $\hbar\Omega = 4.6$ for two values of dissipation strength $\lambda^2 = 10^{-2}$ (filled squares) and $\lambda^2 = 10^{-6}$ (open squares), respectively. The solid line gives the Boltzmann factors with the exponent $\beta_{\text{eff}} = 0.99 \simeq \beta$.

presented in Fig. 4.3. The resonance present for $\Omega = 4.6/\hbar$ still exists, and it is responsible for the deviation from the Boltzmann distribution in the limit of weak coupling, $\lambda^2 \leq 10^{-4}$. The increase of the system-bath coupling suppresses the effect of resonance, and the distribution of the diagonal elements of the asymptotic density matrix approaches the Boltzmann distribution with $\beta_{\text{eff}} = \beta$; see inset in Fig. 4.6.

4.7 Conclusion and discussion on experiments

Dissipation plays a leading role to shape the long-time asymptotic states of a periodically driven quantum system. If the system meets some conditions, the asymptotic state is characterized by the Floquet-Gibbs state, Eq. (4.8). These conditions are specified by the relations on the timescales of three constituents: the system of interest, the heat bath, and the driving field. Namely, (1) the dissipation rate of the system τ_{relax}^{-1} (controlled by the system-bath coupling strength) is higher than the heating rate τ_{heat}^{-1} (controlled by the driving frequency) and (2) the period of the driving field T is much shorter than the timescale of the bath dynamics τ_{bath} (controlled by the cutoff frequency of the spectral density). Condition (1) suppresses the resonance transitions between the eigenstates of the effective Floquet Hamiltonian, while condition (2) avoids the excitation inside the heat bath due to the periodic driving field.

These observations can be understood by transforming the total Hamiltonian into the rotating frame by using the unitary operator Eq. (3.24). The total Hamiltonian in the rotating frame reads

$$H^{\mathrm{R}}(t) = H_{\mathrm{S}}^{\mathrm{R}}(t) + H_{\mathrm{B}} + \lambda H_{\mathrm{I}}^{\mathrm{R}}(t), \tag{4.20}$$

where $H_{\mathrm{I}}^{\mathrm{R}}(t)$ is given by

$$H_{\mathrm{I}}^{\mathrm{R}}(t) = V^{\dagger}(t) H_{\mathrm{I}} V(t). \tag{4.21}$$

The Floquet-Gibbs state is realized when this time-dependent Hamiltonian can be effectively replaced by a time-independent one. The system Hamiltonian in the rotating frame can be replaced, see e.g. in Ref. [48], in the absence of the resonance effects. Resonances will exist when the condition (i) is broken (in the present model, $\hbar\Omega = 4.6$). The presence of the resonance induces a finite coherence in the system. The coherence leads to the appearance of non-negligible off-diagonal elements in the asymptotic density matrix, and thus causes deviations of the diagonal elements of the density matrix from the Boltzmann distribution. This effect can not be explained by a time-independent Hamiltonian, and thus the corresponding asymptotic state deviates from the Floquet-Gibbs state. However, as the system-bath coupling increases, the coherence decreases, and hence $H_{\mathrm{S}}^{\mathrm{R}}(t)$ can be replaced by the time-independent effective Floquet Hamiltonian. The finite system coupling pushes the asymptotic density matrix into the Floquet-Gibbs state.

Besides the resonance effect, there is another channel for deviations of the asymptotic density matrix from the Floquet-Gibbs state. It is the periodic modulation of the interaction Hamiltonian $H_{\mathrm{I}}^{\mathrm{R}}(t)$. When the system satisfies the condition (iii), this Hamiltonian becomes time independent, and thus there is no excitation due to the periodic motion in the heat bath. This is the reason why the condition (iii) is generally necessary to realize the Floquet-Gibbs state. However, when the spectral properties of the heat bath are such that the fast modes cannot be excited by default (e.g., when the cutoff frequency of the spectral density is much smaller than the frequency of the driving field, $\omega_c \ll \Omega$), condition (iii) is redundant. In this case we can effectively replace $H_{\mathrm{I}}^{\mathrm{R}}(t)$ by the corresponding time-average.

Finally we discuss the experimental implementation of the Floquet-Gibbs state. For the condition (1), although the finite dissipation effect lifts the restriction of the applicability of the Floquet-Gibbs state to a small system (the condition (i)), the frequency is necessary to be much higher than the single site energy of the system, in which the heating rate due to the driving is slow. In addition, strong driving amplitude which is compatible with the driving frequency ($\xi \sim \hbar\Omega$) is necessary to obtain the non-trivial Floquet spectrum. The strong and fast driving field can be realized in cold atomic systems (kilohertz regime), superconducting qubits and quantum dots (gigahertz regime), and semiconductor superlattices and Dirac fermion systems (teraherz regime). The estimation of the relaxation time of the system into a long-time asymptotic state remains to be solved.

For the condition (2) it is necessary to engineer the energy filter so that the excitation of the bath modes around the multiple of the driving frequencies, $n\hbar\Omega(n = 1, 2, \cdots)$, is suppressed. The way of engineering the filter may strongly depend on the frequency of the driving field (lattice oscillation, microwaves) and dissipative environment (electrodes, phonon bath, photon bath). The study of the dependence of, for example, the transport properties for some specific model on the energy filter is an important future problem.

Chapter 5

Periodically driven cavity systems

5.1 Overview of this chapter

In this chapter, we study cooperative phenomena of the driven cavity systems introduced in Sec. 2. We find a novel type of symmetry-breaking phenomenon induced by a strong driving field in the USC regime, and give an explanation of this phenomenon in terms of a dynamical feature observed in a driven two-level system (see Sec. 1.4).

In order to make clear the essence of this phenomenon, we also study other models of the cavity system and/or other situations:

- Driven Tavis-Cummings model (Sec. 5.5),
- Driven Dicke model with the squared electromagnetic vector potential, A^2-term (Sec. 5.6),
- Driven effective spin models for Dicke model subjected to a fast and strong driving field (Sec. 5.7),

which strengthen our findings in the driven cavity systems that dynamical effects due to a periodic driving field are enhanced by the interaction, which appears as a novel type of dynamical phase transition.

5.2 Long-time asymptotic states

We study the expectation values of photon field and polarization of atoms:

$$\alpha(t) = \lim_{N\to\infty} \frac{\mathrm{Tr}(a\rho_{\mathrm{p}}(t))}{\sqrt{N}},$$
$$m^x(t) = \mathrm{Tr}(S^x \rho_{\mathrm{a}}(t)). \tag{5.1}$$

We investigate them by solving the dressed Lindblad equations Eq. (2.64) and Eq. (2.65) by the Runge-Kutta method. We regard the state at $t \sim 10000\pi$ as the long-time asymptotic state.

5.3 Classification of the long-time asymptotic states into three phases

In order to classify the long-time asymptotic states into phases, we focus on two types of symmetries of the system Hamiltonian $H_{\mathrm{S}}(t)$ (See Eq. (2.54)-Eq. (2.56)). One is a discrete time-translation symmetry due to the periodicity of the driving field. Thus, the system Hamiltonian remains to be in the same form under the time translation $t \to t + T$. The other is related to a unitary transformation,

$$U = e^{i\pi\left(a^\dagger a + \sum_{j=1}^{N} S_j^z\right)}, \tag{5.2}$$

which changes the sign of the operators $Y \in \{a, a^\dagger, S^x, S^y\}$, i.e.,

$$U^\dagger Y U = -Y. \tag{5.3}$$

The system Hamiltonian does not change under this unitary transformation with extra time translation, $t \to t + T/2$,

$$H_{\mathrm{S}}(t) = U^\dagger H_{\mathrm{S}}\left(t + \frac{T}{2}\right) U. \tag{5.4}$$

The dynamical features in long-time asymptotic states are qualitatively different whether the states satisfy these two symmetries or not. If both symmetries are satisfied, the system exhibits the following relations,

$$\alpha\left(t + \frac{T}{2}\right) = -\alpha(t), \quad m^x\left(t + \frac{T}{2}\right) = -m^x(t). \tag{5.5}$$

These relations lead to the oscillation over the period T around the origin. We call the state in the "regularly oscillating phase". If the second symmetry is broken, the relations Eq. (5.5) no more hold. In this case, $\alpha(t)$ and $m^x(t)$ oscillate around a finite value with the period T. We call this phase "ordered phase" [113]. If both symmetries are broken, the system shows a periodic oscillation with longer period than T. Such a state is said to be in the "long-periodic phase". The nature of the long-periodic phase is not studied in detail in this thesis. It is noted that since the state satisfying the second symmetry is always time periodic, there are only three phases.

In order to distinghish these three phases, we define two order parameters. First, we calculate a sequence of the time average of $\alpha(t)$ over one period from $jT \le t \le (j+1)T$ denoted by α_j. We define the mean value $\overline{\alpha}$ and fluctuation σ_α of α_j over j as

$$\overline{\alpha} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \alpha_j,$$

$$\sigma_\alpha = \lim_{n\to\infty} \sqrt{\frac{1}{n} \sum_{j=0}^{n-1} (\alpha_j - \overline{\alpha})^2}, \tag{5.6}$$

which play a role of order parameters. We can characterize the phases as follows;

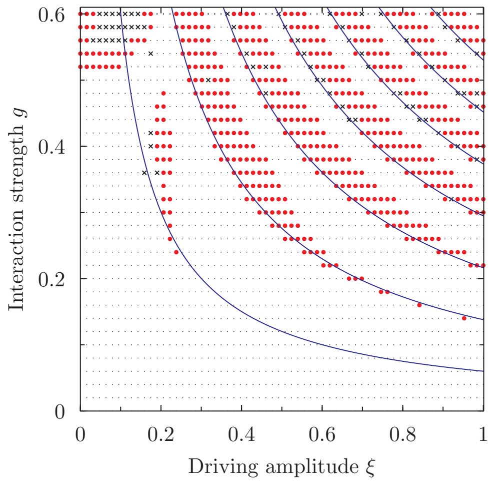

Figure 5.1: Phase diagrams of the driven Dicke model at $\omega_{\rm a} = \omega_{\rm p} = \hbar\Omega = 1$ for the case $\gamma_{\rm p} = \gamma_{\rm a} = 0.1$. The long-time asymptotic state for given g and ξ is labeled by dots (regularly oscillating phase), bullets (ordered phase), and crosses (long-periodic phase). The blue curves are defined by zeros of zeroth order Bessel function, $J_0(4g\xi/\gamma_{\rm p}\hbar\Omega) = 0$, where the CDT occurs. The periodicity of the curves agrees with that of the ordered phase, which indicates the intimate relation between the CDT and this ordered phase.

(i) regularly oscillating phase: $\alpha(t)$ oscillates around zero with the period T; $\overline{\alpha} = 0$ and $\sigma_\alpha = 0$.

(ii) ordered phase: $\alpha(t)$ oscillates around a finite value with the period T; $\overline{\alpha} \neq 0$ and $\sigma_\alpha = 0$.

(iii) long-periodic phase: $\alpha(t)$ shows a long periodic oscillation; $\sigma_\alpha \neq 0$.

5.4 Novel Symmetry-broken phase

In this section, we study the long-time asymptotic states when $\omega_{\rm a} = \omega_{\rm p} = \hbar\Omega = 1$. As an initial state, we choose the equilibrium state of the Dicke model for the given parameters $\omega_{\rm a}, \omega_{\rm p}$, and g. After the system reaching an asymptotic state, we calculate order paremeters $\overline{\alpha}$ and σ_α over sufficiently long time interval.

In Fig. 5.1, we show the overall phase diagram as functions of interaction strength g and driving amplitude ξ. In the phase diagram there are two types of ordered phases denoted by bullets: One is a conventional type of the ordered phase originated from the Dicke transition. It appears when the interaction strength exceeds the critical value $g = 0.5$ along the line $\xi = 0$. The driving field breaks the order $\overline{\alpha}$, and the ordered phase disappears at a certain value of driving amplitude.

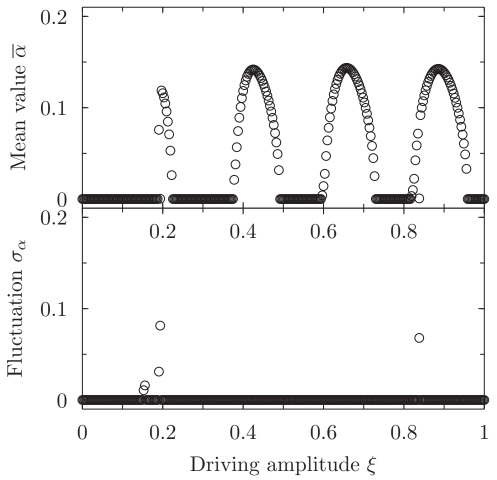

Figure 5.2: Dependence of the order parameters $|\overline{\alpha}|$ (top) and σ_α (bottom) on the driving amplitude. The ordered phase appears repeatedly as a function of ξ.

The other is a novel type of the ordered phases which appear at strong driving field. A characteristic structure repeatedly appears and it extends along the curve given by $g\xi$ =constant. In Figure 5.2, we show the order parameters of $\overline{\alpha}$ and σ_α as a function of ξ for $g = 0.35$. Since the ordered phase appears even when the interaction strength is less than the critical value of the Dicke transition, this phenomenon is not induced only by interaction effects. Thus it is a synergistic phenomenon due to the atom-photon interaction effects and the dynamical effects.

Next, we search for a mechanism of this phenomenon within a simpler model. Since this phenomenon occurs at relatively small g, we treat the interaction effect as a perturbation. The equation for photon field obtained by dressed Lindblad equation (2.64) reads

$$\frac{d}{dt}\alpha(t) = \frac{1}{\hbar}(-i\omega_{\rm p} - \gamma_{\rm p})\left[\alpha(t) + \frac{2}{\omega_{\rm p}}(gm^x(t) + \xi\cos(\Omega t))\right]. \tag{5.7}$$

The perturbative treatment of interaction term which is proportional to g in Eq. (5.7) leads to an effective spin model without degrees of photon field.

As we will see later, the lowest-order term describes the quantum interference effect under a classical photon field, which gives a key ingredient of this nonequilibrium phenomenon. The next-order term exhibits the long-range interaction among atoms, which is also necessary to induce the cooperative phenomenon.

The long-time asymptotic solutions of $\alpha(t)$ in the lowest order is given by

$$\alpha^{(0)}(t) = -i\frac{\xi}{\gamma_{\rm p}}e^{-i\Omega t}, \tag{5.8}$$

where we have used $\omega_{\rm p} = \hbar\Omega$ and $\hbar\Omega \gg \gamma_{\rm p}$. Since the driving frequency is resonant with the frequency of the cavity, the photon field in the cavity system is strongly excited.

By substituting Eq. (5.8) into the MF Hamiltonian of each atom Eq. (2.61), we obtain the Hamiltonian of the effective spin model up to the first order of g,

$$H^{(1)}_{\rm spin} = \sum_{j=1}^{N} \omega_{\rm a} S^z_j - \sum_{j=1}^{N} \frac{4g\xi}{\gamma_{\rm p}} \sin(\Omega t) S^x_j, \tag{5.9}$$

which describes the independent two-level systems under a strong driving field because there is no interaction between spins. As observed in Sec. 1.4.3, the system shows the CDT due to the quantum interference effect when

$$J_0\left(\frac{4g\xi}{\gamma_{\rm p}\hbar\Omega}\right) = 0. \tag{5.10}$$

In the phase diagram, we plot the points for the CDT by blue curves. We find a good agreement between the periodicity of the CDT and that of the ordered phase.

Thus, it is strongly suggested that the CDT gives a key ingredient for this symmetry-breaking phenomenon. However, if we take a dissipation effect into account in this system, Eq. (5.9), the quantum dynamical feature of the CDT disappears and the long-time asymptotic state is simply in the regularly oscillating phase. The lowest order term is thus insufficient to understand the mechanism of the present phase transition.

We then resort to the next-leading-order term of the effective spin model, which describes the long-range interaction among atoms. We find from Eq. (5.7) that time evolution of the photon field in the first order of g denoted by $\alpha^{(1)}(t)$ obeys

$$\frac{d}{dt}\alpha^{(1)}(t) = \left(-i\Omega - \frac{\gamma_{\rm p}}{\hbar}\right)\left(\alpha^{(1)}(t) + \frac{2g}{\hbar\Omega} m^x(t)\right). \tag{5.11}$$

In the long-time asymptotic states, $\alpha^{(1)}(t)$ is given by

$$\alpha^{(1)}(t) = \frac{2g(-i\hbar\Omega - \gamma_{\rm p})}{\hbar^2\Omega} \int_0^\infty m^x(t-\tau) e^{(-i\Omega-\gamma_{\rm p}/\hbar)\tau} d\tau. \tag{5.12}$$

Since the integrand decreases at a decay rate $\gamma_{\rm p}$, we focus on the time regime $0 < \tau < \hbar/\gamma_{\rm p}$. Within this timescale, time evolution of the two-level atom can be regarded as a Hamilton dynamics without dissipative effect, because at strong driving amplitude the decay rate of an atom is much smaller than that of photon field almost all the time, $\gamma_{\rm p} \gg \gamma_{\rm a}(t)$. It is because the MF Hamiltonian for an each atom Eq. (2.61) under the strong driving field is almost commutable with system-bath coupling represented by S^x for almost all the time.

We thus evaluate the dynamics of an atom by using the Schrödinger equation of the lowest-order effective Hamiltonian, Eq. (5.9), which was investigated in section 1.4.3 by replacing h^z by $\omega_{\rm a}$ and 2ξ by $4g\xi/\gamma_{\rm p}$. By using the Floquet modes $|u^{(0)}_1(t)\rangle$ and $|u^{(0)}_2(t)\rangle$ obtained in Eq. (1.68), we evaluate $m^x(t)$ as

$$m^x(t) = \frac{1}{2}(\langle u^{(0)}_1(t)| \rho_{\rm a}(t) |u^{(0)}_2(t)\rangle + \langle u^{(0)}_2(t)| \rho_{\rm a}(t) |u^{(0)}_1(t)\rangle), \tag{5.13}$$

where we have used

$$S^x |u_1^{(0)}(t)\rangle = \frac{1}{2} |u_2^{(0)}(t)\rangle , \quad S^x |u_2^{(0)}(t)\rangle = \frac{1}{2} |u_1^{(0)}(t)\rangle . \tag{5.14}$$

From Eq. (5.13), we obtain

$$m^x(t-\tau) = \frac{1}{2}[e^{\frac{i}{\hbar}\omega_a J_0\left(\frac{4g\xi}{\gamma_p \hbar\Omega}\right)\tau} \langle u_1^{(0)}(t)| \rho(t) |u_2^{(0)}(t)\rangle + e^{-\frac{i}{\hbar}\omega_a J_0\left(\frac{4g\xi}{\gamma_p \hbar\Omega}\right)\tau} \langle u_2^{(0)}(t)| \rho(t) |u_1^{(0)}(t)\rangle]. \tag{5.15}$$

The phase rate is given by the quasienergy difference between two Floquet states (See Eq. (1.69)). Since the asympotic form of the Bessel function $J_0(x)$ at large x is proportional to x^{-1}, at strong driving amplitude such that

$$\omega_a J_0 \left(\frac{4g\xi}{\gamma_p \Omega}\right) \ll \gamma_p, \tag{5.16}$$

we can regard $m^x(t-\tau)$ to be constant in the time region $0 < \tau < \hbar/\gamma_p$.

By substituting $m^x(t-\tau) = m^x(t)$ into Eq. (5.12), we obtain $\alpha^{(1)}(t)$ in the long-time asymptotic state as

$$\alpha^{(1)}(t) = -\frac{2g}{\hbar\Omega} m^x(t) = -\frac{2g}{N\hbar\Omega} \sum_{j=1}^{N} S_j^x. \tag{5.17}$$

The effective spin model up to the order of g^2 thus reads

$$\mathcal{H}_{\text{spin}}^{(2)} = \sum_{j=1}^{N} \omega_a S_j^z - \sum_{j=1}^{N} \frac{4g\xi}{\gamma_p} \sin(\Omega t) S_j^x - \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{4g^2}{\hbar\Omega} S_j^x S_k^x, \tag{5.18}$$

where the second-order term describes the infinite-range interaction among atoms.

In Fig. 5.3, we present the phase diagrame of the effective spin model. We obtain this phase diagram in the similar way to the driven Dicke model. First, we derive the MF Hamiltonian for each atom from which we obtain the dressed Lindblad equation. Next, we evaluate the order parameters which are defined by replacing photon field $\alpha(t)$ by polarization of atoms $m^x(t)$.

The obtained phase diagram agrees with that of driven Dicke model at strong driving amplitude. Thus, the novel type of symmetry breaking phenomenon observed at strong driving amplitude is governed by the driving strength for each atom $4g\xi/\gamma_p$ and interaction strength between atoms $4g^2/(\hbar\Omega)$

5.5 Driven Tavis-Cummings model

In this section, we study another model of cavity systems called Tavis-Cumming model in a periodic driving field. The Hamiltonian of the Tavis-Cummings model reads [114]

$$H_0 = \omega_p a^\dagger a + \omega_a \sum_{j=1}^{N} S_j^z + \sum_{j=1}^{N} \frac{g}{\sqrt{N}} (S_j^+ a + S_j^- a^\dagger), \tag{5.19}$$

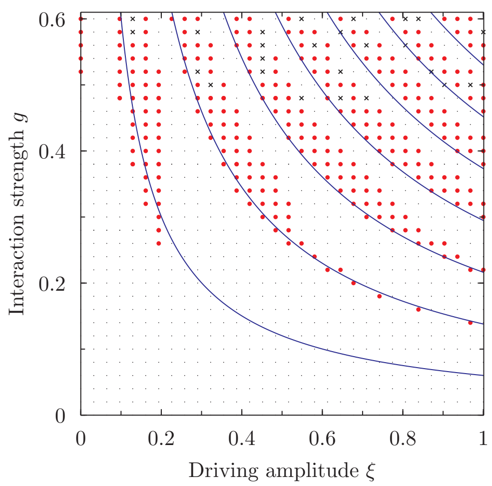

Figure 5.3: Phase diagrams of the effective spin model with the same parameters as the driven Dicke model. The long-time asymptotic state for given g and ξ is labeled by dots (regularly oscillating phase), bullets (ordered phase), and crosses (long-periodic phase). The blue curves are defined by zeros of zeroth order Bessel function, $J_0(4g\xi/\gamma_{\rm p}\hbar\Omega) = 0$, where CDT occurs. The qualitative agreement of the phase diagram with the driven Dicke model at strong driving field justifies the use of the effective spin model to study the novel type of symmetry breaking phenomenon.

in which the interaction term in the Dicke model changes. This model is proposed to be implemented in the system with superconducting qubits [115]. This model shows the Dicke transition at the critical value of the interaction strength twice as large as that of the Dicke model at zero temperature [81, 79, 80].

It is noted that the Tavis-Cummings model has a $U(1)$ symmetry while the Dicke model has only a $Z(2)$ symmetry; In the Tavis-Cummings model the Hamiltonian is invariant under the following unitary transformation,

$$e^{i\theta(\sum_j S_j^z + a^\dagger a)} H_0 e^{-i\theta(\sum_j S_j^z + a^\dagger a)} = H_0, \tag{5.20}$$

for any real value of θ, while in the Dicke model this equality holds only for $\theta = \pi$ or $\theta = 0$ modulo 2π.

We apply the same type of the driving field as the Dicke model, $H_{\rm ex}(t)$ (see Eq. (2.55)), and study the long-time asymptotic states. We derive dressed Lindblad equations in a similar way to the case of the driven Dicke model by using the MF Hamiltonians for the photon field,

$$H_{\rm p}(t) = \omega_{\rm p} a^\dagger a + \sqrt{N}[g(\langle S^+(t)\rangle a + \langle S^-(t)\rangle a^\dagger) + 2\xi \cos\Omega t(a + a^\dagger)] \tag{5.21}$$

and for each atom,

$$H_{\rm a}(t) = \omega_{\rm a} S^z + \frac{g}{\sqrt{N}}(\langle a(t)\rangle S^+ + \langle a(t)\rangle^* S^-), \tag{5.22}$$

We use the same order parmeters in order to classify the asymptotic states into three phases. We find that the difference of the symmetry substantially changes the phase diagram in the USC regime at strong driving amplitude.

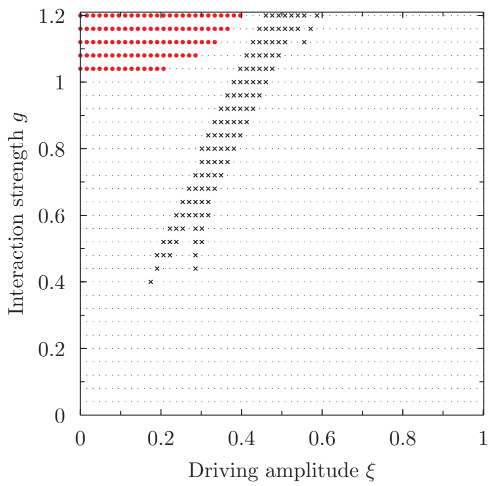

Figure 5.4: Phase diagrams of the Tavis-Cummings model with the same parameters as the driven Dicke model. The long-time asymptotic state for given g and ξ is labeled by dots (regularly oscillating phase), bullets (ordered phase), and crosses (long-periodic phase), respectively.

In Fig. 5.4, we show the phase diagram of the driven Tavis-Cummings model for $\omega_{\rm a} = \omega_{\rm p} = \hbar\Omega = 1$, where the ordered phase does not appear at strong driving field. We obtain an effective spin model in order to investigate why the phase diagram is totally different from that of the Dicke model. Since the lowest order of photon field in the long-time asymptotic state is in the same form as Eq. (5.8), the effective spin model up to the order of g, which is obtained by substituting Eq. (5.8) into Eq. (5.22), reads

$$H_{\rm spin} = \sum_{j=1}^{N} \omega_{\rm a} S_j^z - \sum_{j=1}^{N} \frac{2g\xi}{\gamma_{\rm p}} \left[\sin(\Omega t) S_j^x - \cos(\Omega t) S_j^y\right]. \tag{5.23}$$

In contrast to the driven Dicke model where the effective spin model shows the CDT at a specific value of the driving amplitude, this model shows the Rabi oscillation (see Sec. 1.4.1), in which the xy components of spins rotate at the frequency Ω. Since the localization of the spin state due to the driving field does not occur, we expect that the ordered state does not appear in the driven Tavis-Cummings model as observed in the phase diagram.

Through the study of this model, we have found that the difference of the symmetry in the interaction Hamiltonian strongly changes the dynamical features of the spin system, which substantially changes the phase diagram at strong driving amplitude.

5.6 Driven Dicke model with the squared electromagnetic vector potential, A^2-term

In this section we study the long-time asymptotic state of the driven Dicke model with the A^2-term. The Hamiltonian is given by

$$H_0 = \omega_{\rm p} a^\dagger a + \sum_{j=1}^{N} \omega_{\rm a} S_j^z + \sum_{j=1}^{N} \frac{2g}{\sqrt{N}} S_j^x (a + a^\dagger) + D(a + a^\dagger)^2, \tag{5.24}$$

where the last term is called the A^2-term with strength being D. First, we explain the origin of A^2-term and the relation with the no go theorem of the Dicke transition. Next, we study the system subjected to a periodic driving field, $H_{\rm ex}(t)$ (see Eq. (2.55)).

In order to explain the origin of the A^2-term, we consider the cavity system embedding identical N atoms, whose Hamiltonian is given by

$$H = H_{\rm photon} + \sum_{i=1}^{N} H_{\rm atom}^{(i)}(\{\boldsymbol{x}_j^{(i)}\}, \{\boldsymbol{p}_j^{(i)} + e\boldsymbol{A}(\boldsymbol{x}_j^{(i)})\}), \tag{5.25}$$

where $H_{\rm photon}$ is the Hamiltonian for photon field and $H_{\rm atom}^{(i)}$ is the Hamiltonian for ith atom. The atom Hamiltonian is described by the set of canonical variables of j-th elctron in i-th atom denoted by $\boldsymbol{x}_j^{(i)}$ and $\boldsymbol{p}_j^{(i)}$. We assume that there is no interaction among atoms, and the Hamiltonian of each atom can be described in a non-relativestic way;

$$H_{\rm atom}^{(i)}(\{\boldsymbol{x}_j^{(i)}\}, \{\boldsymbol{p}_j^{(i)}\}) = \sum_j \frac{\boldsymbol{p}_j^{(i)2}}{2m_e} + V(\{\boldsymbol{x}_j^{(i)}\}), \tag{5.26}$$

where m_e is an electron mass and $V(\{\boldsymbol{x}_j^{(i)}\})$ is a coulomb potential for the ith atom. The interaction effects are taken into account by replacing the momentum of an electron in the atom Hamiltonian $\boldsymbol{p}_j^{(i)}$ by $\boldsymbol{p}_j^{(i)} + e\boldsymbol{A}(\boldsymbol{x}_j^{(i)})$, where e is an electron charge and $\boldsymbol{A}(\boldsymbol{x}_j^{(i)})$ is a vector potetial for the cavity photon field.

In order to derive the Dicke model, we perform the following approximations:

- cavity photon field can be described by a single quantized mode with polarization vector $\mathbf{e}$,
- only the transition between the ground state $|0\rangle$ and an excited state $|1\rangle$ occurs,
- the wave length of the cavity photon field is so long that the spacial dependence of the vector potential can be neglected, $\boldsymbol{A}(\boldsymbol{x}_j^{(i)}) = \boldsymbol{A}$.

The atom Hamiltonian includes the term proportional to $(\boldsymbol{p}_j^{(i)} + e\boldsymbol{A})^2$. The term propotional to $e\boldsymbol{p}_j^{(i)} \cdot \boldsymbol{A}$ gives the interaction term of the Dicke model with the strength,

$$g = \sqrt{\frac{\rho}{2\epsilon_0\omega_{\mathrm{p}}}}\omega_{\mathrm{a}} |\langle 0| e \sum_j \boldsymbol{x}_j^{(i)} \cdot \mathbf{e} |1\rangle |, \tag{5.27}$$

where ρ is a density of atoms inside the cavity and ϵ_0 is the vacuum permittivity. In addition, $e^2 \boldsymbol{A} \cdot \boldsymbol{A}$ gives the A^2-term, Eq. (5.24), with the strength,

$$D = \sum_j \frac{\rho e^2}{4\epsilon_0 m_e \omega_{\mathrm{p}}}. \tag{5.28}$$

The Dicke transition was originally investigated in the Dicke model without the A^2-term [81, 79, 80], but after that it has been argued that the A^2-term plays a crucial role of the no-go theorem of the Dicke transition [116]. This argument is based on the sum rule called Thomas-Raiche-Kuhn (TRK) sum rule, which gives a bound between g and D.

In order to see the bound, we use the following identity for $k \geq 0$ (for the derivation, see [117]),

$$\begin{aligned} &\langle \phi_\alpha | [B, [\underbrace{\tilde{H}, [\cdots [\tilde{H}}_{k}, A] \cdots]]] |\phi_\alpha\rangle \\ =& \sum_\beta [(E_\beta - E_\alpha)^k \langle \phi_\alpha | B |\phi_\beta\rangle \langle \phi_\beta | A |\phi_\alpha\rangle - (E_\alpha - E_\beta)^k \langle \phi_\alpha | A |\phi_\beta\rangle \langle \phi_\beta | B |\phi_\alpha\rangle], \end{aligned} \tag{5.29}$$

where $\tilde{H}$ is a self-adjoint operator with eigenvalues E_α and eigenstates $|\phi_\alpha\rangle$, and A and B are hermite operators. By substituting $k = 1$, $\tilde{H} = H_{\mathrm{atom}}^{(i)}$, and

$$A = B = \sum_j e\boldsymbol{x}_j^{(i)} \cdot \mathbf{e}, \tag{5.30}$$

we obtain the TRK sum rule,

$$\sum_j \frac{e^2}{m_e} = 2 \sum_\beta (E_\beta - E_\alpha) |\langle \alpha | \sum_j e\boldsymbol{x}_j^{(i)} \cdot \mathbf{e} |\beta\rangle |^2. \tag{5.31}$$

When we take $|\alpha\rangle$ to be the ground state of the atom Hamiltonian $|0\rangle$, then the right hand side is the sum of positive terms because $E_\beta - E_\alpha \geq 0$. This observation leads to the relation,

$$\sum_j \frac{e^2}{m_e} \geq 2\omega_{\mathrm{a}} |\langle 0| \sum_j e\boldsymbol{x}_j^{(i)} \cdot \mathbf{e} |1\rangle |^2 \leftrightarrow D \geq \frac{g^2}{\omega_{\mathrm{a}}}, \tag{5.32}$$

which bounds the interaction strength g.

The ordered phase appears at zero temperature when

$$\frac{g^2}{\omega_{\mathrm{a}}} > D + \frac{\omega_{\mathrm{p}}}{4}, \tag{5.33}$$

which is incompatible with the bound Eq. (5.32). Thus, the Dicke transition does not occur due to the inclusion of A^2-term, which is called no-go theorem of the Dicke transition [116].

Now we study the system subjected to a periodic driving field. The MF Hamiltonians for photon field is given by

$$H_{\mathrm{p}}(t) = \omega_{\mathrm{p}} a^\dagger a + 2\sqrt{N}[g \langle S^x(t)\rangle + \xi \cos \Omega t](a + a^\dagger) + D(a + a^\dagger)^2. \tag{5.34}$$

In order to simplify the expression, we perform a Bogoliubov transformation; We define annihilation and creation bosonic operators b and $b^\dagger$ as

$$\begin{cases} b = \dfrac{1}{\sqrt{1-p^2}} a - \dfrac{p}{\sqrt{1-p^2}} a^\dagger, \\ b^\dagger = \dfrac{1}{\sqrt{1-p^2}} a^\dagger - \dfrac{p}{\sqrt{1-p^2}} a, \end{cases} \tag{5.35}$$

where

$$p = \frac{-(\omega_{\mathrm{p}} + 2D) + \sqrt{\omega_{\mathrm{p}}(\omega_{\mathrm{p}} + 4D)}}{2D}, \tag{5.36}$$

so that the MF Hamiltonian reads

$$H_{\mathrm{p}}(t) = \tilde{\omega}_{\mathrm{p}} b^\dagger b + 2\sqrt{N}[\tilde{g} \langle S^x(t)\rangle + \tilde{\xi} \cos \Omega t](b + b^\dagger). \tag{5.37}$$

Here, the A^2-term modifies the paremeters as

$$\begin{aligned} \tilde{\omega}_{\mathrm{p}} &= \frac{(1+p^2)\omega_{\mathrm{p}} + 2D(1+p)^2}{1-p^2}, \\ \frac{\tilde{g}}{g} &= \frac{\tilde{\xi}}{\xi} = \sqrt{\frac{1+p}{1-p}}. \end{aligned} \tag{5.38}$$

The MF Hamiltonian for each atom can be written using the modified interaction strength $\tilde{g}$ as

$$H_{\mathrm{a}}(t) = \omega_{\mathrm{a}} S^z + \frac{2\tilde{g}}{\sqrt{N}}(\langle b(t)\rangle + \langle b(t)\rangle^*) S^x, \tag{5.39}$$

Thus, we can map the system with parameters $(\omega_{\mathrm{a}}, \omega_{\mathrm{p}}, g, \xi, \Omega)$ with finite A^2-term into the system with parameters $(\omega_{\mathrm{a}}, \tilde{\omega}_{\mathrm{p}}, \tilde{g}, \tilde{\xi}, \Omega)$ without A^2-term. For example, the phenomenon for system parameters $\omega_{\mathrm{a}} = 1, \omega_{\mathrm{p}} = 0.62, g = 0.4, D = 0.25$ is estimated from the phenomenon for $\omega_{\mathrm{a}} = 1, \omega_{\mathrm{p}} \simeq 1, \tilde{g} \simeq 0.31, D = 0$. Since we find from the phase diagram of the driven Dicke model (see Fig. 5.1) that for the latter parameters, the novel type of symmetry breaking phenomenon occurs as we sweep the driving amplitude, this clearly shows that this phenomenon appears even when the A^2-term exists and $D > g^2/\omega_{\mathrm{a}}$.

5.7 Understanding the non-equilibrium phase transition in terms of the Floquet-Gibbs state

Finally, we investigate the effective spin model subjected to a strong and high frequency field. The Hamiltonian of this model is given by

$$H_{\rm spin}^{(2)}(t) = \sum_{j=1}^{N} \omega_{\rm a} S_j^z - \sum_{j=1}^{N} \frac{4g\xi}{\gamma_{\rm p}} \sin(\Omega t) S_j^x - \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \frac{4g^2}{\hbar\Omega} S_j^x S_k^x, \tag{5.40}$$

where we have used $\omega_{\rm p} = \hbar\Omega$. In order to obtain the meaningful results at high frequency limit $\Omega \to \infty$, we rescale the interaction strength g and driving amplitude ξ as

$$g = \sqrt{\hbar\Omega}\bar{g}, \quad \xi = \sqrt{\hbar\Omega}\bar{\xi}. \tag{5.41}$$

In the high frequency limit, $\Omega \to \infty$, the Floquet-Gibbs state is expected to be realized, since all the conditions (i)-(iii)(see Chap. 3) are satisfied. In this subsection, we aim to compare the result obtained by the Floquet-Gibbs state and that obtained by a master equation. In order to treat the high-frequency regime, where the mean field rapidly oscillates, and thus the use of the dressed Lindblad equations is inappropriate, we use the Floquet Lindblad equation, Eq. (2.31), in this subsection.

First we study what kind of a phenomenon is expected from the Floquet-Gibbs state. When the system is subjected to a strong and high-frequency field, it is convenient to move to a rotating frame, where the state in the static frame $|\psi(t)\rangle$ is transformed by using a unitary operator $V(t)$ into

$$|\psi^{\rm R}(t)\rangle = e^{-i\int_0^t \sum_{j=1}^{N} \frac{4\bar{g}\bar{\xi}\Omega}{\gamma_{\rm p}} \sin\Omega\tau S_j^x d\tau} |\psi(t)\rangle = V(t)|\psi(t)\rangle. \tag{5.42}$$

The time evolution of $|\psi^{\rm R}(t)\rangle$ obeys the Schrödinger equation for the rotating Hamiltonian,

$$\begin{aligned} H_{\rm spin}^{(2)\rm R}(t) =& V^\dagger(t)\left(H_{\rm spin}^{(2)}(t) - i\hbar\frac{\partial}{\partial t}\right)V(t), \\ =& \sum_{j=1}^{N} \omega_{\rm a} V^\dagger(t) S_j^z V(t) - \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{4g^2}{\hbar\Omega} S_j^x S_k^x. \end{aligned} \tag{5.43}$$

As an approximation, we evaluate the lowest-order truncated Floquet Hamiltonian by using the Floquet-Magnus expansion, $H_{\rm spin}^{(2)\rm R}(t)$, which is given by (see Eqs. (1.50) and (1.51))

$$H_{\rm F}^{(1)[0]} = \omega_{\rm a} J_0\left(\frac{4\bar{g}\bar{\xi}}{\gamma_{\rm p}}\right)\sum_{j=1}^{N}\left[S_j^z\cos\left(\frac{4\bar{g}\bar{\xi}}{\gamma_{\rm p}}\right) + S_j^y \sin\left(\frac{4\bar{g}\bar{\xi}}{\gamma_{\rm p}}\right)\right] - \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{4\bar{g}^2}{N} S_j^x S_k^x, \tag{5.44}$$

which is a transverse Ising model with infinite-range coupling. It is known that this model shows a phase transition at a critical value of interaction strength between atoms, e.g., at zero temperature given by

$$\bar{g} = \frac{1}{2}\sqrt{\omega_{\mathrm{a}} J_0\left(\frac{4\bar{g}\bar{\xi}}{\gamma_{\mathrm{p}}}\right)}. \tag{5.45}$$

Since the right hand side is determined by the rescaled driving amplitude, it is expected from the Floquet-Gibbs state that the phase transition occurs when the driving amplitude is swept. In Fig. 5.5, we depict the order parameter $m^x = \sum_{j=1}^{N}\langle S_j^x\rangle/N$ by dotted line.

Next, we study the long-time asymptotic state of this model in a dissipative environment by using the Floquet Lindblad equation and MF approach. The MF Hamiltonian of each atom is given by

$$H_{\mathrm{a}}(t) = \omega_{\mathrm{a}} S^z - \frac{4g\xi}{\gamma_{\mathrm{p}}}\sin(\Omega t) S^x - \frac{8g^2}{\hbar\Omega} m^x(t) S^x. \tag{5.46}$$

In general, because the mean field denoted by $m^x(t)$ is not time periodic, the Floquet Lindblad equation cannot be used. However, it is plausible that there exists a parameter regime, where the mean field in the long-time asymptotic state is time periodic. In this region, the asymptotic solution should satisfy the Floquet Lindblad equation, and thus the solution is obtained by the Pauli's master equation, Eq. (2.40),

$$0 = \sum_{\substack{j \\ (i\neq j)}} T_{j\to i}[m^x(t)]\mathrm{P}_j - T_{i\to j}[m^x(t)]\mathrm{P}_i, \tag{5.47}$$

where we explicitly show the dependence of the transition probabilities on the mean field.

We further assume that during one period of the driving field, the time evolution of each atom can be described by the Hamilton dynamics of $H_{\mathrm{a}}(t)$,

$$\frac{d}{dt}\rho_{\mathrm{a}}(t) = -\frac{i}{\hbar}[H_{\mathrm{a}}(t), \rho_{\mathrm{a}}(t)], \tag{5.48}$$

which is a nonlinear equation on $\rho_{\mathrm{a}}(t)$. Because it is the first-order differential equation, we have only to obtain an appropriate $\rho_{\mathrm{a}}(0)$ so that $\rho_{\mathrm{a}}(t)$ is time periodic and consistent with Eq. (5.47).

In order to find the appropriate $\rho_{\mathrm{a}}(0)$, we employ the following strategy; First, we take some $\rho_{\mathrm{a}}(0)$. Next, From Eq. (5.48), we obtain $m^x(t)$ for $0 \le t < T$, from which we obtain the Floquet modes and quasienergies. By using them, we evaluate the transition probabilities $T_{i\to j}[m^x(t)]$, and then we obtain the asymptotic density matrix for the long-time asymptotic state, $\tilde{\rho}_{\mathrm{a}}(t)$. If $\tilde{\rho}_{\mathrm{a}}(0)$ is inconsistent with $\rho_{\mathrm{a}}(0)$,

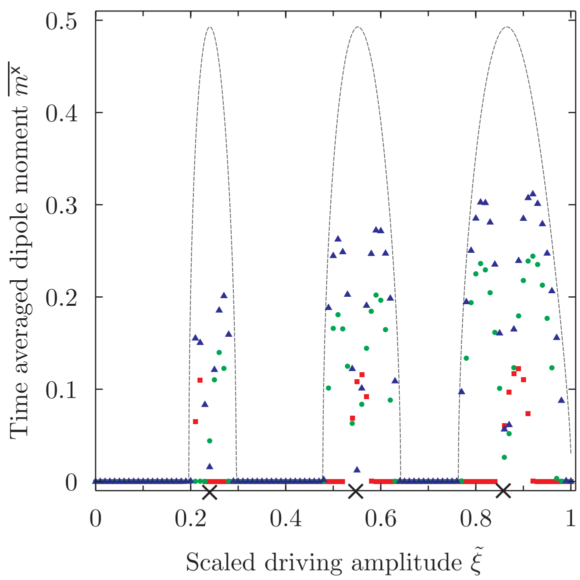

Figure 5.5: The dependence of the time-averaged dipole moment $\overline{m^x}$ on the scaled driving amplitude $\bar{\xi}$. The data for $\hbar\Omega = 1, 3$, and 5 are depicted by red squares, green circles, and blue triangles, respectively. The black dotted line corresponds to the value of the Gibbs state of the lowest-order truncated Floquet Hamiltonian, Eq. (5.44). The crosses denotes the points of the CDT, $\bar{\xi} = 0.24, 0.55$, and 0.87. we set the inverse temperature of the heat bath $\beta = 20$.

we replace $\rho_{\rm a}(0)$ by $(1-\epsilon)\rho_{\rm a}(0) + \epsilon\tilde{\rho}_{\rm a}(0)$ [1]. We repeat this cycle until $\Delta\rho_{\rm a} = \|\tilde{\rho}_{\rm a}(0) - \rho_{\rm a}(0)\| < 10^{-5}$.

An each atom couples to the heat bath through the dipole moment represented by S^x. As the property of the bath, we adopt the ohmic spectral density with Lorentz-Drude cutoff frequency ω_c,

$$J(\omega) = \frac{\omega}{\pi}\frac{\omega_c^2}{\omega^2 + \omega_c^2}, \tag{5.49}$$

where $\omega_c = 10$. We set the temperature of the bath to be $\beta^{-1}/k_{\rm B} = 0.05$.

In Fig. 5.5, we show the time-averaged values of the dipole moment $\overline{m^x}$ in the long-time asymptotic states for the driving frequency, $\hbar\Omega = 1, 3$ and 5. At $\hbar\Omega = 1$, there exist some points where $\Delta\rho_{\rm a}$ does not converge during 10^6 cycles, which are dropped in this figure. The crosses on x-axis denote the point of the CDT, around which the order parameter $\overline{m^x}$ appears. However, at high frequency $\hbar\Omega = 3$ and 5, there are two peaks and the order parameter disappears around the points of the CDT in contrast to the expectation of the Floquet-Gibbs state.

[1] We test various values of $\epsilon(= 0.001, 0.01, 0.1$, and 1). They give the same results for almost all the points except around the points of the CDT, in which smaller ϵ gives a better convergence of $\Delta\rho_{\rm a}$. In Fig. 5.5, we depict the data for $\epsilon = 0.001$.

The deviation from the Floquet-Gibbs state can be explained by the ergodic breaking in the leading-order transition probabilities. At the point of the CDT, the lowest-order truncated Floquet Hamiltonian, Eq. (5.44), commutes with the interaction Hamiltonian H_{I}, and thus the energy exchange between the system and the heat bath does not occur in the order of Ω^0. The detailed analysis of the next-leading-order term is a future problem.

When $\bar{\xi}$ is apart from the points of the CDT, the ergodicity recovers, and thus the Floquet-Gibbs state gives a well description of the long-time asymptotic states at high frequency.

5.8 Conclusion

In this section, we studied cooperative phenomena of the driven cavity systems in a dissipative environment. First we adopted the dressed Lindblad equations, Eq. (2.64) and Eq. (2.65), in order to study the driven Dicke model. In the USC regime subjected to a strong driving field, we found a novel type of symmetry breaking phenomenon. Through the study of the simpler effective spin model, we concluded that this phenomenon originats from the synergestic effects of the driving field and atom-photon coupling; The microscopic dynamical feature of each atom under a periodic driving field is enhanced by the interaction effects.

We also studied other models for comparison. In the Tavis-Cummings model, the difference of the symmetry qualitatively changes the dynamical feature of each atom, leading to the different phase diagram at strong driving amplitude. Since this nonequilibrium phenomenon originates in the dynamical effect, the inclusion of the A^2-term, which prohibits the occurrence of the Dicke transition, does not harm at all.

Finally we used the Floquet Lindblad equation in order to study the long-time asymptotic states under a strong and high-frequency driving field. We found that the Floquet-Gibbs state gives a well description of the long-time asymptotic state as far as the ergodicity is satisfied.

These investigations demonstrated that microscopic dynamical features can induce a cooperative phenomenon due to interaction effect even in the existence of a dissipative environment. Since the Floquet-Gibbs state appears only in a restricted parameter regime, we usually cannot expect directly from the Floquet Hamiltonian or its coherent dynamics what types of phenomena occur in the long-time asymptotic state. However, our study supports the idea that the dynamical features remain to be observed even out of the regime of the Floquet-Gibbs state.

Chapter 6

Summary and future prospects

In the present thesis, long-time asymptotic states of a time-periodically driven system in a dissipative environment was investigated by using quantum master equations. Throughout the thesis we tried to find the condition in order to express the asymptotic states by a closed-form solution (Floquet-Gibbs state).

In Chap. 3, we investigated within the Lindblad formulation the condition for the emergence of the Gibbs state of the Floquet Hamiltonian. We found that even when the driving field is strong, the asymptotic state can be expressed by the Floquet-Gibbs state under the following three conditions:

(i) the driving frequency is much larger than the spectral width of the system Hamiltonian,
(ii) the driving Hamiltonians commute with itself at different instants of time,
(iii) the driving Hamiltonian and the system-bath interaction Hamiltonian should commute.

These conditions severely limit a class of suitable physical models attaining the Floquet-Gibbs state. The condition (i) restricts the system with a relatively small Hilbert space and the condition (iii) requires a fine tuning of the system-bath coupling.

In Chap. 4, we found that the conditions (i) and/or (iii), which severely restrict the emergence of the Floquet-Gibbs state, can be lifted by imposing conditions on timescales of the three constituents, the system, heat bath, driving field. Namely,

(i)' the relaxation rate of the system τ_{relax} is higher than the heating rate τ_{heat},
(iii)' the period of the driving field T is shorter than the timescale of the bath dynamics τ_{bath}.

The condition (i)' suppresses the resonance transition between the eigenenergy state of the effective Floquet Hamiltonian, which allows us to circumvent the condition (i). When the condition (iii)' is satisfied, energy excitation of the heat bath due to the periodic driving is suppressed, in which the condition (iii) is no more necessary. We illustrated these scenarios in a spin-chain model by using the Redfield equation, and demonstrated that the Gibbs state of the effective Floquet Hamiltonian can be realized without imposing strict conditions on the system operators. This result clearly indicates that the long-time asymptotic states of a periodic driven system strongly depend on the detailed property of the coupling to the heat bath in contrast to the system with time-independent Hamiltonian.

In Chap. 5, we studied the cooperative phenomena of the driven cavity systems by combining the Lindblad formalism and the MF approach (dressed Lindblad equation). We found a novel type of symmetry breaking phenomenon subjected to a strong driving field. Through the study of the effective spin model, we found that this phenomenon originates from the synergistic effects of the CDT, which is observed in the driven two-level systems, and the effective interaction among atoms. We also studied the asymptotic state under a fast and strong driving field by using the Floquet Lindblad equation and MF approach. When the system meets the conditions for the Floquet-Gibbs state, we demonstrate that this cooperative phenomenon can be understood in terms of the Floquet-Gibbs state of the truncated Floquet Hamiltonian.

As a future prospect, the realization of the Floquet-Gibbs state in a dissipative environment may allow us to control thermal or electronic transport by using a periodic driving field. Since the Floquet Hamiltonian or the effective Floquet Hamiltonian plays a role of the energy instead of the system Hamiltonian, the quasienergies and the energy of the heat bath are exchanged with each other. Because the Floquet Hamiltonian depends on the amplitude and frequency of the periodic driving field, for example, the slow operation of the amplitude and frequency in time corresponds to an adiabatic process, which may control currents between the system and a heat bath or between two different heat baths. The formulation of, e.g., a thermodynamics including effects of a periodic driving field is one of the future directions.

There remains a broad parameter regime where the Floquet-Gibbs state is not realized. The asymptotic states are not determined only by the Floquet Hamiltonian but also the details of the system-bath coupling. Thus, it is necessary to make clear the role played by the interaction Hamiltonian on the long-time asymptotic states as in Chap. 4. Even in an ideal Bose Einstein condensation, the unique phenomenon has been reported [118], and thus it is expected that future research will unveil novel types of phenomena in this regime.

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