論文の内容の要旨

論文題目　On the Weak Value and Uncertainty Relations based on Quasi-Joint-Probabilities in Quantum Mechanics

(量子力学における擬同時確率分布から見る弱値と不確定性関係)

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1　General Introductions

Quasi-probabilities of Quantum Observables　Since the discovery of quantum mechanics in the beginning of the last century, our classical understanding of the concept of 'observables' has undergone a drastic change. It is now a widely accepted and established fact that, in the microscopic world, outcomes of a quantum observable behave intrinsically randomly, and that certain combinations of quantum observables do not admit coexistence, as typically exemplified by position and momentum of a free particle. Such remarkable characteristics of quantum observables impose certain limitation to the mathematical framework to be employed for describing their probabilistic behaviour, namely, that it is not always possible to assign probability spaces for the description of the 'joint behaviour' of their arbitrary combinations in the classical sense. There had, nonetheless, long been various attempts to construct a mathematical framework for the probabilistic description of the combination of quantum observables that resembles the Kolmogorovian style of formulation of classical probability theory, and extending the notion of 'probability' had thus been one of the major trends. Such extended notion of probabilities are generally termed 'quasi-probabilities' or 'pseudo-probabilities', and among the most celebrated classical proposals are the Wigner-Ville distribution (WV distribution) and the Kirkwood-Dirac distribution (KD distribution), the former of which is alleged to describe the 'joint behaviour' of a canonically conjugate pair of quantum observables, whereas the latter can be defined for arbitrary pairs. Historically, both the WV and KD distributions, along with the various other proposals, are said to be discovered more or less in a heuristic manner, and as such, the general mathematical framework for the study, including a general prescription for the concrete construction of such 'quasi-joint-probability' (QJP) distributions of a pair of arbitrary quantum observables and a transparent overview of how each of them relate to each other, is rather vague. In comparison to classical probability theory, both the WV and KD distributions retain similar properties to the standard joint-probability distributions defined for a pair of classical random variables. On the other hand, they also retain their own outstanding queerness, in that the former generally admits negative numbers to be assigned whereas the latter even takes complex numbers, which has occasionally been considered a serious impediment to their physical interpretation.

The theme of this thesis revolves around the concept of QJP distributions of quantum observables, and the first objective of our analysis would thus be to construct a mathematically solid framework to address some of their problems in a more transparent and systematic manner.

Aharonov's Weak Value The novel physical quantity in quantum mechanics called the *weak value*

$$A_w := \frac{\langle \psi', A\psi \rangle}{\langle \psi', \psi \rangle} \tag{1.1}$$

was proposed in 1988 by Aharonov and co-workers. Despite its growing attention, the positioning of the weak value in the framework of quantum mechanics along with its physical interpretation has yet to come to an agreement. One of the recent strategies in addressing this question has been to investigate its relations to quasi-probabilities, specifically those to the KD distributions.

As an application to the findings in the study of QJP distributions in the thesis, we shall follow this line of the study of weak values and intend to provide its geometric/statistical interpretation based on the general framework of QJP distributions.

Uncertainty Relations Uncertainty relations lie at the heart of quantum mechanics, characterising the indeterministic nature of microscopic phenomena stemming from the incompatibility of simultaneous measurement of two non-commuting observables, as typically exemplified by position and momentum. Soon after the celebrated exposition of Heisenberg's tradeoff relation between error and disturbance, there appeared a revised form known as the Robertson-Kennard (RK) inequality which refers to the relation in standard deviation in independently performed measurements on the two observables. Due to its mathematical clarity and universal validity, the latter has now become a standard textbook material. On the other hand, the uncertainty relation between time and energy has to be dealt with quite independently from these, due to the lack of a genuine time operator conjugate to the Hamiltonian. For this, several ingenious frameworks have been proposed, including the one devised by Mandelshtam and Tamm and that by Helstrom, where the uncertainty relation is shown to be identified with a quantum version of the Cramér-Rao inequality in estimation theory.

As another application to the findings in the study of QJP distributions, we shall conduct a closer analysis on the quantum analogue of 'correlations' of a given pair of observables, and thus present a novel inequality of uncertainty relations, providing its geometric/statistical interpretations alongside.

2 Main Themen of the Thesis

The main theme of this thesis is thus to obtain a more coherent understanding to the formalism of QJP distributions of quantum observables, and subsequently to apply the findings in some areas of the foundational problems of quantum mechanics, in which the problem of interpretation of the weak value and a proposal of novel uncertainty relations are the two main choices. Now, the key problems regarding QJP distributions include, on top of the other,

(i) providing a solid and rigorous mathematical framework for the study of QJP distributions based on measure and integration theory, and possibly on the theory of generalised functions,

(ii) presenting a systematic scheme to address the inherent indefiniteness/arbitrariness to the possible candidates for QJP distributions of non-commuting pairs of quantum observables, a methodical way for their constructions, and the relation between each of the candidates,

(iii) devising a concrete method in measuring such various candidates of QJP distributions in a systematic manner.

We shall address this problem from two complementary approaches: one from a bottom-up and strictly operational construction by carefully examining the mathematical framework of the conditioned measurement scheme, and the other from a top-down viewpoint by applying the results of spectral theorem for normal operators and its Fourier transforms.

The findings of the study shall be subsequently applied to two specific areas among the problems of foundation of quantum mechanics, namely, weak values and uncertainty relations. To this end, we first concentrate on the L^2 structures which QJP distributions naturally induce, and see that they provide 'statistical' interpretation of the geometric structures introduced on the space of observables on the underlying Hilbert space, in analogue to those introduced on the space of random variables in classical probability theory. Geometric concepts such as orthogonal projections and inner products are accordingly endowed statistical interpretation as 'conditionings' and 'correlations', respectively. These observation form a solid starting place to perform further study on weak values and uncertainty relations. Weak values A_w are thus given a geometric and statistical interpretation, either as the orthogonal projection of an observable A on the subspace generated by another observable B, in geometric terminology, or equivalently in statistical expressions, as the conditioning of A given B with respect to the QJP distribution under consideration. On the other hand, application of the Cauchy-Schwartz inequality to the correlation of A and B is found to yield novel inequalities interpreted as uncertainty relations of approximation/estimation, and thus providing connections between uncertainty relation for correlations (the Robertson-Kennard type) and the time-energy uncertainty relations, in particular.

3 Main Results

Systematic Construction of QJP distributions We first provide a general prescription for the construction of QJP distributions intended to describe the 'joint behaviour' of an arbitrary pair of quantum observables. Inspired by the observations made on the Fourier transform of the product spectral measure of two simultaneously measurable observables A and B, we introduce

$$\#(s,t) := \text{a mixture of the disintegrated components of } e^{-isA} \text{ and } e^{-itB} \tag{3.1}$$

for arbitrary pairs of (generally non-commuting) observables, and define the QJP distribution of the pair by the inverse Fourier transform of the distribution $(s,t) \mapsto \langle\psi, \#(s,t)\psi\rangle/\|\psi\|^2$ given a quantum state $|\psi\rangle$. Here, each of the QJP distributions is found to possess reasonable properties to be qualified as what its name suggests it to be, and that both the WV distribution and the KD distribution belong to this class. The inherent indefiniteness/arbitrariness to the possible candidates for QJP distributions is then understood as the possible variety of the way one could 'mix' the disintegrated components of the unitary operators, which originates directly from the non-commutative nature of the pair A and B. A concrete measurement scheme for members of a specific subfamily of QJP distributions is further proposed.

Geometric and Statistical Interpretation of the Weak Value As distributions, each QJP distribution naturally induces an L^2 structure. It is found that the QJP distributions provide convenient methods of representing geometric structures, specifically inner products of the form

$$\langle\!\langle B, A \rangle\!\rangle_{\psi,\alpha} := \frac{1+\alpha}{2}\langle B\psi, A\psi\rangle + \frac{1-\alpha}{2}\langle A\psi, B\psi\rangle, \quad -1 \le \alpha \le 1, \tag{3.2}$$

which can be introduced on the space of operators on the underlying Hilbert space. Endowed with the inner product, we then specifically focus on orthogonal projections onto the subspaces $\mathfrak{C}(B)$ of operators generated by self-adjoint operators B, and find that the orthogonal projections can be interpreted as 'conditioning' given B with respect to the QJP distributions under consideration. The projection

$$P_\alpha(A|B;\psi) = \int_{\mathbb{R}} \left(\frac{1+\alpha}{2}\frac{\langle b, A\psi\rangle}{\langle b, \psi\rangle} + \frac{1-\alpha}{2}\frac{\langle \psi, Ab\rangle}{\langle \psi, b\rangle} \right) |b\rangle\langle b| db \tag{3.3}$$

of the observable A on the subspace $\mathfrak{C}(B)$ is further found to be described by the weak value, providing us its geometric and statistical interpretation (Proposition 7.5).

Proposal of novel Uncertainty Relations We then subsequently investigate the inner product (3.2) of observables in depth. The representation of the inner product by integration with respect to the QJP distribution under consideration is thus found to provide statistical interpretation to the quantities: the inner products are examples of the possible definitions for quantum analogues of 'correlations' between observables, in imitation to those between random variables in classical probability theory. Now, in studying the correlation, we start from the inequality

$$\|A - f(B)\|_\psi \cdot \|g(B)\|_\psi \ge \frac{1}{2}\left| \langle [A, g(B)] \rangle_\psi \right|, \tag{3.4}$$

by introducing the inner product $\langle Y, X\rangle_\psi := \langle\!\langle Y, X\rangle\!\rangle_{\psi,0}$ and the semi-norm $\|X\|_\psi := \langle X, X\rangle_\psi^{\frac{1}{2}}$, where the notation $f(A)$ represents the operator defined from a function $f(a)$ of a through the spectral decomposition, $f(A) = \int f(a)|a\rangle\langle a|\, da$. The inequality admits interpretation as uncertainty relations for analysing the error of approximating an observable based on the measurement of another observable through an appropriate choice of proxy functions, and thus directly addresses the effect of non-commutativity as

$$\min_f \|A - f(B)\|_\psi \ge \max_{\bar{g}} \frac{1}{2}\left|\langle [A, \bar{g}(B)]\rangle_\psi\right|, \tag{3.5}$$

by introducing $\bar{g}(B) = g(B)/\|g(B)\|_\psi$ (Theorem 7.9). Since the standard deviation may be regarded as a special case of the approximation error, the inequality can formally be considered as an extension of the RK inequality. Moreover, instead of approximating an observable, we may also choose to estimate a physical parameter pertinent to the observable by considering the unitary evolution $|\psi(t)\rangle := e^{-itA}|\psi\rangle$ so that the time-energy relation

$$\|H - f(B)\|_{t_0} \cdot \|t_0 - g(B)\|_{t_0} \ge \frac{\hbar}{2}, \tag{3.6}$$

where we introduced the abbreviation $\|X\|_t := \|X\|_{\psi(t)}$, can be treated along with the position-momentum relation (Theorem 7.10). Interestingly, in both approximation and estimation, Aharonov's weak value of the concerned observable arises as a key geometric ingredient, deciding the optimal choice for the proxy functions.