The crystalline motion and its generalization

（クリスタライン運動とその一般化）

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論文の内容の要旨

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1. はじめに. 物質の状態が1つの相から他の相に移ることを相転移と呼ぶ．相異なる2つの相が共存しているとき，それらの相と相の境界面が時間とともに動くことがしばしばある．この境界面 – 或は単に「界面」という – の運動を研究するのが相転移の動的理論である．

氷と水の界面の運動を記述する Stefan 問題はその一例である．Stefan 問題では，界面の動きは周囲の氷や水の物理状態に依存している．しかし一方で，材料科学に現れる種々の複合物質に見られるように，界面が周囲の物理状態にほとんど影響されずに自律的な運動をする場合も少なくない．そのとき，界面が動く速さは，自分自身の幾何学的形状 (例えば，曲率) の関数として表されることが多い．以後，平面内の運動についてのみ言及していく．

曲率流方程式は，界面の自律的な運動を表す方程式の1つの典型例である．これは滑らかな曲線上の各点での変形速度 V が，その点での曲率 K に等しいことを要請する方程式 $V=K$ で，微分幾何学の分野ではおなじみの方程式であり，1980 年代後半の Gage，Hamilton，Grayson 等の研究以降，精力的に研究がなされている．曲率流方程式は，また，金属の焼きなまし時における粒界の運動を記述するモデルとして，1950 年代に材料科学者 Mullins の論文に登場した．その際，界面の運動を律する界面エネルギーが等方的であるという仮定のもとにこの方程式が導かれている．

ところが結晶成長などの問題では，表面エネルギー f がその界面の方向に依存しており，この異方性そのものが，結晶の成長過程を考える上で重要である．さて，表面エネルギーの異方性が，$f=f(\boldsymbol{n})$ という形で表されたとする．ただし，$\boldsymbol{n}=\boldsymbol{n}(\theta)$ は界面の各点での法線ベクトルであり，θ は法線と $\boldsymbol{x}$ 軸のなす角度である．すると，熱力学的要請から，重み付き曲率流 $V=g(\theta)K$ が導かれる．ここで，$g=f(\theta)+f''(\theta)$ である ($f(\theta):=f(\boldsymbol{n}(\theta))$)．ところでこの重み付き曲率流方程式 $V=(f+f'')K$ は，数学的には，総界面エネルギー (すなわち上記の g を界面全体で積分した量) の勾配流として導出されることを注意しておく．とくに $f\equiv 1$ の場合が通常の曲率流 $V=K$ であり，この場合は，総界面エネルギーは，界面を表す曲線の全長に他ならない．

さて，結晶成長に現れるような特別なクラスの界面エネルギーを考えると，その逆数 $f(\theta)^{-1}$ の極形式 (Frank 図形と呼ばれる) の凸包が多角形になることがある．このような界面エネルギーを**クリスタライン・エネルギー**と呼ぶ．この場合，エネルギーはもはや滑らかでなく，通常のやり方で重み付き曲率流を導出することはできない．それどころか，重み付き曲率の満たす方程式は，

形式的には Dirac の δ 関数を含む形になり，古典的な枠組では意味をもたない．このため，f がクリスタライン・エネルギーの場合は，重み付き曲率流の解の概念を，通常とは違った枠組の中で考える必要がある．現在，次の 3 つの枠組が知られている：(a) 曲線のクラスを制限する方法，(b) 非線形半群論を用いる方法 (劣微分方程式，Fukui，Giga，Elliott，Kobayashi 等)，(c) 比較原理に基づく方法 (粘性解的概念，Giga，Giga M.H. 等)．(a) による解は，(b) や (c) の意味での解でもあることが知られている．本論文では，(a) の方法を基盤とした，異方的曲率流の運動について考察する．

(a) の方法とは，曲線の運動を特殊なクラスの折れ線の運動に制限する方法である．具体的には，各々の隣接辺のなす角度が一定であるような折れ線を考える．このような折れ線を**許容折れ線**と呼ぶ．許容折れ線のクラスにおいては，総クリスタライン・エネルギーの勾配流なるものが自然に定式化でき，その勾配流が表す折れ線の運動法則は常微分方程式系に帰着する．この事実は，1980 年代終りから 1990 年初頭にかけて，Taylor と Angenent，Gurtin 等によって独立に発見された．この運動を**クリスタライン運動**という．通常の曲率が曲線の全長の第一変分で与えられるように，許容折れ線の全長 (或はもっと一般に許容折れ線上の総クリスタライン・エネルギー) の第一変分で与えられる量を**クリスタライン曲率**と呼ぶ．本論文では従来のクリスタライン運動をさらに一般化した折れ線の運動方程式を考察するとともに，それらの方程式の解の漸近挙動を論じる．

2. 方程式の設定と主目的.　本論文では，平面内の閉凸多角形の運動に考察の対象を絞る．以後，隣接する辺のなす角が $\pi - \Delta\theta$ である閉凸多角形を許容折れ線と呼ぶことにする．ここで，$\Delta\theta := 2\pi/n$ であり，n は多角形の辺の数である．

さて，平面内に許容折れ線 $\mathcal{P}_0$ が与えられたとする．$\mathcal{P}_0$ を初期値とする折れ線の運動で，次のような常微分方程式系で記述されるものを考えよう．これは従来のクリスタライン運動を一般化したものであるが，今後，本論文ではこの種の運動をクリスタライン運動と総称する．

クリスタライン運動

多角形の族 $\mathcal{P} = \bigcup_{0\le t<T}(\mathcal{P}_t \times \{t\})$ は，次の常微分方程式系を満たす．

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_j(t) = v_j(t)\boldsymbol{n}_j, \quad 0 \le j < n, \quad 0 < t < T, \\ \mathcal{P} \cap \{t = 0\} = \mathcal{P}_0. \end{cases} \tag{E}$$

ここで，ベクトル $\boldsymbol{x}_j(t)$ は解多角形 $\mathcal{P}_t$ の第 j 辺 (を含む直線) に原点から下ろした垂線の足の位置ベクトル，ベクトル $\boldsymbol{n}_j$ は第 j 辺の内向き法線ベクトル，v_j は第 j 辺の内向き法線方向の速度 (法速度) である．法速度 v_j は一般に，法線ベクトル $\boldsymbol{n}_j$ とクリスタライン曲率 κ_j の関数で与えられるものとする．

$$v_j(t) = F(\boldsymbol{n}_j, \kappa_j(t)), \quad 0 \le j < n, \quad 0 \le t < T. \tag{V}$$

ここで，解多角形 $\mathcal{P}_t$ の第 j 辺のクリスタライン曲率 κ_j とは

$$\kappa_j(t) = \frac{2n\tan(\Delta\theta/2)}{d_j(t)}, \quad 0 \le j < n, \quad 0 \le t < T, \tag{K}$$

で定義される量で，$d_j(t)$ は第 j 辺の長さである．

本論文の主たる目的は，以下の点について考察することである．

- 上の方程式系の解多角形 $\mathcal{P}_t$ が存在する最大時間区間 $[0,T)$ は有限か無限か．
- 時刻 t が T に近付いたときに，$\mathcal{P}_t$ はどのような形状に変形，或は漸近していくのか．
- 速度 $v_j(t)$ は，特に $t \nearrow T$ のとき，t のどのような関数になっているのか．

3. 主結果. 本論文では，法速度を定める (V) 式の関数 F として，異なる 4 つのタイプを考え，それぞれのタイプについて，解の性質を論じる．

3.1 第 1 章. 式 (V) の関数 F が κ_j について 1 次で増加関数であるならば，第 j 法速度は

$$v_j(t) = a(\boldsymbol{n}_j)\kappa_j(t) - b(\boldsymbol{n}_j), \quad 0 \le j < n, \quad 0 \le t < T, \tag{V_1}$$

となる．ここで，a は S^1 上の正値関数，b は S^1 上の実数値関数である．この式は，Taylor や Gurtin が提唱した物理モデル方程式をやや一般化した方程式である．この問題 (E)-(V_1)-(K) の解多角形の運動は，初期多角形 $\mathcal{P}_0$ の大きさと関数 b の値，特に符号に大きく左右される．ある状況下では，$\mathcal{P}_t$ は有限時間 $T_\star$ で一点に収縮し (定理 A, 定理 B)，また別の状況下では無限時間かかって無限遠方に膨張していく (定理 C)．定理 A と定理 B では $T_\star$ の上からの評価を与え，定理 D と定理 E では，Schwarz の不等式を 2 回適用することにより，下からの評価を与えている．また，定理 C のケースにおいて，特に関数 b が定数であった場合，解多角形は無限に膨張する正多角形に Hausdorff の距離の意味で近付くことが，等周不等式の多角形版 (下記第 3 章の第 3 節において証明される) を用いて示される．尚，本章の付録の節では，閉曲線上の表面エネルギーの勾配流についての一般的注意を述べるとともに，曲率，離散曲率，重み付き曲率，及びクリスタライン曲率の関係について説明する．

3.2 第 2 章. 関数 F が κ の正巾に比例する場合について考える．即ち，第 j 法速度は

$$v_j(t) = g(\boldsymbol{n}_j)\kappa_j(t)^\alpha, \quad 0 \le j < n, \quad 0 \le t < T, \tag{V_2}$$

で与えられる．ここで，g は S^1 上の正値関数，α は正のパラメータである．任意の初期多角形 $\mathcal{P}_0$ から出発したとき，問題 (E)-(V_2)-(K) の解多角形は，有限時間 $T_\star$ で，少なくとも 1 辺が消滅する (定理 A)．特に，$\alpha \ge 1$ のときは，多角形全体が 1 点に消滅する (定理 B)．しかし，$\alpha < 1$ のときは，線分の形につぶれる例がある (第 4 節)．定理 C では $T_\star$ の下からの評価を Hölder の不等式を 2 回用いて示している．

ところで，解多角形が 1 点に収縮する場合，クリスタライン曲率は無限大に発散する．その爆発の速さは，形状が時間によって変化しない自己相似解の場合は，関数 g の如何によらず $(T_\star - t)^{-1/(1+\alpha)}$ というオーダーになることが簡単な計算からわかる．一般に，1 点に消滅する多角形のクリスタライン曲率が高々このオーダーで爆発するとき，この 1 点消滅は「タイプ I」であるという．パラメータ $\alpha \ge 1$ のときは，解多角形の 1 点消滅は常にタイプ I であることが証明できる (定理 D)．また，任意のパラメータ $\alpha > 0$ に対して，解多角形の 1 点消滅とタイプ I を仮定すると，解多角形は漸近的に自己相似解に近付いて行くことも示される (定理 E)．これより，自己相似解の存在がわかる (系 E.1)．なお，任意の正多角形は，$g \equiv 1$ のときは自己相似解であるが，$g \not\equiv 1$ のときは

必ずしもそうではない．また，$g \equiv 1$ のときでも自己相似解は正多角形だけではない．実際，第4節では，$g \equiv 1$ かつ α が小さい場合について，正多角形以外の自己相似解を具体的に与えている．

3.3 第3章. 第2章とは逆に次のような負巾の成長法則を考える．

$$v_j(t) = -\kappa_j(t)^{-\beta}, \quad 0 \leq j < n, \quad 0 \leq t < T. \tag{V_3}$$

ここで，β は正のパラメータである．また，v_j は負なので，$-\boldsymbol{n}_j$ 方向に解多角形は膨張していく(なお，第3章の本文中では，話をわかりやすくするため，外向き法線を $\boldsymbol{n}_j$ としている)．問題 (E)-(V_3)-(K) の解多角形は，パラメータ $\beta < 1$ のときは多項式的に，$\beta = 1$ のときは指数的に無限大に膨張していく．また，$\beta > 1$ のときは，有限時間で無限大に膨張する．いずれの場合も，解多角形は膨張する正多角形に Hausdorff の距離で近付いて行く (定理 A)．この結果の証明には，第3節で証明する離散版等周不等式と，第5節で証明する離散版 Aleksandrov の反射法を用いる．

3.4 第4章. 本章では，面積を保存するクリスタライン運動，即ち，法速度が次の式で与えられる運動を考える．

$$v_j(t) = \kappa_j(t) - \frac{2n\tan(\Delta\theta/2)}{|\mathcal{P}_t|}, \quad 0 \leq j < n, \quad 0 \leq t < T. \tag{V_4}$$

ここで，$|\mathcal{P}_t|$ は解多角形 $\mathcal{P}_t$ の周長である．この発展方程式は，面積が一定の許容多角形全体のなす空間の中で周長の勾配流を考えることにより得られる．問題 (E)-(V_4)-(K) の解多角形の形状は，$t \to \infty$ のとき，正多角形に指数的オーダーで近付く (定理 A)．この定理は，等周不等式の発展方程式版である，Bonnesen の不等式や Gage の不等式を用いて証明される．

Contents

Acknowledgements

The author would like to express his gratitude to Professor Hiroshi Matano for valuable advice and continued encouragement. He also thanks Dr. Takeo Ushijima for stimulating discussions, especially on the so-called crystalline algorithm for the motion by curvature. This work was partly supported by the JSPS Research Fellowships for Young Scientists. Chapters 1, 2 and 3 were done while the author was visiting the National Tsing Hua University, TAIWAN, during the summer of 1999; this visit was sponsored by the National Center for Theoretical Sciences.

Chapter 1

Point-extinction and geometric expansion of solutions to a crystalline motion

Outline: We consider the asymptotic behavior of solutions to a generalized crystalline motion which describes evolution of plane curves driven by nonsmooth interfacial energy. Our main results say that solution polygonal curves shrink to a single point, or expand to infinity depending on the size of initial data and the sign of the driving force term. We also give lower and upper bounds of the extinction time for the shrinking case. An isoperimetric ratio is estimated for the expansion case. Moreover, we show that if the driving force term is a constant, then any solution polygon approaches to a regular polygon even if the motion is anisotropic. In the appendix, we shall expalin the notion of a discrete curvature and crystalline curvature from a numerical point of view.

Key Words: crystalline motion, crystalline curvature, discrete curvature, motion by curvature, curve-shortening, point-extinction, geometric expansion, estimates of blow-up time, entropy estimate, comparison principle, isoperimetric ratio.

1 Introduction and main results

1.1 The aim of this chapter

Let $\mathcal{P}_0$ be a convex closed polygon in the plane $\boldsymbol{R}^2$ with the angle between two adjacent sides of $\mathcal{P}_0$ being $\pi - \Delta\theta$, where $\Delta\theta := 2\pi/n$ and n is the number of sides of the polygon. We consider the evolution problem of finding a family of polygons $\mathcal{P} = \bigcup_{0 \le t < T}(\mathcal{P}_t \times \{t\})$ satisfying

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_j(t) = v_j(t)\boldsymbol{n}_j, \quad 0 \le j < n, \quad 0 < t < T, \\ \mathcal{P} \cap \{t = 0\} = \mathcal{P}_0, \end{cases} \tag{1.1a}$$

where the vector $\boldsymbol{n}_j$ is the inward normal of the jth side of the polygon $\mathcal{P}_t$ and the vector $\boldsymbol{x}_j(t)$ denotes the point of intersection between the line containing the jth side of the polygon $\mathcal{P}_t$ and the line spanned by $\boldsymbol{n}_j$. The function v_j is the inward normal velocity of the jth side which will be specified later. Throughout this paper the interval $[0, T)$, with $T \in (0, \infty]$, will be understood to be the maximal time interval of existence for each solution polygon. We note that the angle between two adjacent sides of $\mathcal{P}_t$ always equals $\pi - \Delta\theta$ as long as the solution polygons exist.

In this chapter we consider a generalized crystalline motion of the form

$$v_j(t) = a(\boldsymbol{n}_j)\kappa_j(t) - b(\boldsymbol{n}_j), \quad 0 \le j < n, \tag{1.1b}$$

where $a > 0$ and b are smooth functions defined on S^1, and κ_j is the crystalline curvature:

$$\kappa_j(t) = \frac{2\tan(\Delta\theta/2)}{d_j(t)}, \quad 0 \le j < n. \tag{1.1c}$$

Here $d_j(t)$ is the length of the jth side of polygon $\mathcal{P}_t$.

We introduce the set

$$\mathcal{N}_\star := \{\boldsymbol{n}_j = -{}^t(\cos\theta_j, \sin\theta_j) \,|\, \theta_j = j\Delta\theta,\ \Delta\theta = 2\pi/n,\ 0 \le j < n\}.$$

$\mathcal{N}_\star$ is the set of orientations that appear on the Wulff shape (see below) being an regular n-polygon. A convex polygon $\mathcal{P}$ is *$\mathcal{N}_\star$-admissible* if the normal vector of each side of $\mathcal{P}$ is the element of $\mathcal{N}_\star$. We can then translate Problem (1.1) into the problem of finding an $\mathcal{N}_\star$-admissible polygon evolved by the crystalline flow (1.1b) with (1.1c). On a general admissibility, we touch upon later.

The aim of this chapter is to study the asymptotic behavior of solutions to Problem (1.1). Our main results say that solution polygons shrink to a single point, or expand to infinity depending on the size of initial data $\mathcal{P}_0$ and the sign of driving force term b. Roughly speaking, if the initial polygon is sufficiently small (resp., large), then a solution polygon shrinks to a single point (resp., expands toward to infinity). We also give lower and upper bounds of the extinction time for the shrinking case. An isoperimetric ratio is estimated for the expansion case. Moreover, if the driving force term b is isotropic (that is, independent of j), then any solution polygon approaches to a regular polygon even if the coefficient $a = a(\boldsymbol{n}_j)$ is anisotropic. In the appendix, we shall explain the notion of a discrete curvature and crystalline curvature from a numerical point of view.

1.2 Background

Problem (1.1) is a typical model equation for crystal growth in the plane. In this context the solution polygon represents the boundary curve between two different materials. Such a boundary curve is called the interface, or free boundary. The motion of interfaces or free boundaries fascinates many researchers in the fields of applied mathematics, material sciences, physics, biology and so on. The notion of interfacial energy plays an important role in those contexts. As we shall show below, the gradient flow of a total interfacial energy provides a curvature-dependent motion.

Now let us explain how one derives Problem (1.1) in the context of curvature dependent motion of curves. Let Γ_t be a closed curve parametrized by θ, the angle between the outward normal of Γ_t and the fixed axis. Let f be an interfacial energy defined on Γ_t. If the interfacial energy $f = f(\boldsymbol{n})$ is positively homogeneous of degree one, then the gradient flow of total interfacial energy with respect to the L^2-metric provides the weighted curvature flow $v = \omega := (f(\theta) + f''(\theta))\kappa$. Here we set $f(\theta) = f(\boldsymbol{n}(\theta))$, and $\kappa = \kappa(\theta, t)$ is the curvature of Γ_t. See Elliott [E] and appendix A.

We note that $f + f''$ is the inverse of the curvature of the boundary of the Wulff shape $\mathcal{W}_f$: a region enclosed by a solution to the problem of finding a closed embedded plane curve Γ that minimizes the total interfacial energy $\int_\Gamma f\,ds$ at fixed enclosed area in the plane. It is not difficult to see that the solution is uniquely determined and the Wulff shape is described by

$$\mathcal{W}_f = \{\boldsymbol{x} \in \boldsymbol{R}^2 \,|\, \langle \boldsymbol{x}, -\boldsymbol{n}(\theta)\rangle \le f(\theta) \ \text{ for all } \theta \in \boldsymbol{R}\}.$$

See, e.g., Gurtin [Gu1] for about properties of the Wulff shape.

If the Wulff shape $\mathcal{W}_f$ is a polygon, we call f the crystalline energy (see Angenent-Gurtin [AGu]). Let f be a crystalline energy with $\mathcal{W}_f$ being an n-polygon (n-gon in short) and $\boldsymbol{n}(\theta_j)$ being the normal of the jth side, called facet, of $\partial\mathcal{W}_f$. We can then define the finite set

$$\mathcal{N} := \{\boldsymbol{n}(\theta_j) \,|\, 0 \le \theta_0 < \theta_1 < \cdots < \theta_{n-1} < 2\pi\}.$$

For such an energy, Taylor [T2] and Angenent–Gurtin [AGu] restrict the curve Γ_t to the class of $\mathcal{N}$*-admissible* piecewise linear curves $\mathcal{P}_t$, in which (1) each normal vector is the element of $\mathcal{N}$ and (2) normal vectors of two adjacent sides of $\mathcal{P}_t$ are the adjacent in $\mathcal{N}$ (see, e.g., Giga–Gurtin [GGu]). Note that we do not need the condition (2) if $\mathcal{P}_t$ is convex (see the definition of $\mathcal{N}_*$-admissible above). The evolution equation of $\mathcal{P}_t$ is then reduced to the ordinary differential equations $v_j(t) = \omega_j(t)$. This equation is called the crystalline motion, or crystalline flow. Here v_j is the velocity of the jth side and ω_j is the jth crystalline curvature defined by $\omega_j(t) = \chi_j l(\boldsymbol{n}_j)/d_j(t)$. Here $l(\boldsymbol{n}_j)$ is the length of the side of $\partial\mathcal{W}_f$ that has orientation $\boldsymbol{n}_j \in \mathcal{N}$, χ_j is the transition number which has the constant value $+1$, -1, or 0 depending on whether the polygon is strictly convex, strictly concave, or neither near the jth side of $\mathcal{P}_t$, d_j is the length of the jth side of $\mathcal{P}_t$. In fact, the jth crystalline curvature can be decomposed as follows (see appendix C):

$$\omega_j(t) = (f + \Delta_\theta f)_j \kappa_j(t), \quad \kappa_j(t) = \chi_j \frac{\gamma_j}{d_j(t)}.$$

Here $\gamma_j := \tan(\Delta\theta_{j+1}/2) + \tan(\Delta\theta_j/2)$, and Δ_θ is a kind of difference operator defined by

$$(\Delta_\theta(\cdot))_j := \frac{(\mathrm{D}_+(\cdot))_j - (\mathrm{D}_+(\cdot))_{j-1}}{\gamma_j}, \quad (\mathrm{D}_+(\cdot))_j := \frac{(\cdot)_{j+1} - (\cdot)_j}{\sin\Delta\theta_{j+1}} \tag{1.2}$$

with $\Delta\theta_j = \theta_j - \theta_{j-1}$. We call κ_j the "discrete curvature," which is an approximation of the real curvature $\kappa(\theta_j)$ (see appendix B). We note that the discrete curvature and the crystalline curvature are equivalent when the Wulff shape is a regular polygon.

Remark 1.1 In this chapter we consider the asymptotic behavior of an $\mathcal{N}_\star$-admissible convex n-gon. Although $\mathcal{N}_\star$ is a special case of $\mathcal{N}$, the set $\mathcal{N}_\star$ is better than $\mathcal{N}$ from a numerical point of view. See Remark 1.8 below and appendix B.

1.3 Generalized crystalline motion and its application

Angenent–Gurtin [AGu] proposed a generalized crystalline motion:

$$\beta(\boldsymbol{n}_j)v_j(t) = \omega_j(t) - U, \tag{1.3}$$

where $\beta(\boldsymbol{n}_j)$ is the kinetic modulus, U is the constant bulk energy. Independently, Taylor [T2] derived the planar crystalline motion under the assumption: $\beta = \text{const} \times f^{-1}$ and $U \equiv 0$. For the further detail and background of a crystalline flow and a weighted curvature flow, see the papers [AlT, RT, T1, T4], the papers including a survey [T3, TCH, GirK2, GMHG4, Gu2] and the book [Gu1]. Recently, the three dimensional crystalline flow is analyzed in [GGuM]. In [Ry], a Stefan-type problem which has the crystalline interfacial energy is studied. In [IIU], they apply the crystalline motion for the shrinking spiral problem. A numerical simulation is proposed for a curvature-dependent motion with a crystalline type anisotropy in [GP]. Structure and existence of stationary finger of two-dimensional solidification for crystalline energy are investigated in [Al].

It is clear that any circle shrinks to a point self-similarly under the isotropic flow $v = \kappa$. In general, we call a solution curve which does not change shape a *self-similar* solution. We can easily check that the boundary of the Wulff shape is a self-similar solution of the weighted curvature flow $v = f\omega = f(f + f'')\kappa$. In [GL], they show the existence and uniqueness of self-similar solution to the anisotropic flow $v = a(\theta)\kappa$. The assumption on $a(\cdot)$ is relaxed to just boundness in [DGM]. Stancu [S1, S2, S3] shows the existence and uniqueness, under a symmetric assumption, of self-similar solution to the crystalline flow $v_j = a(\theta_j)\kappa_j$. Recently in [Y1] and [Y2], the author studies the asymptotic behavior of solutions to a motion by a power of crystalline curvature $v_j = a(\theta_j)\kappa_j^\alpha$ $(\alpha > 0)$ and $v_j = -\kappa_j^{-\beta}$ $(\beta > 0)$, respectively.

Remark 1.2 Let $\mathcal{P}_t$ be a convex $\mathcal{N}$-admissible polygon with a crystalline energy f. We consider the crystalline motion $v_j = f_j(\omega_j - U)$. Then we can find a self-similar solution $\mathcal{P}_t = \lambda(t)\partial\mathcal{W}_f$ with $\mathcal{P}_0 = \lambda_0\partial\mathcal{W}_f$. Here λ is the solution of

$$\frac{d}{dt}\lambda(t) = -\frac{1}{\lambda(t)} + U, \quad \lambda(0) = \lambda_0.$$

When $U = 0$, it is easy to obtain the exact solution $\lambda(t) = \sqrt{\lambda_0^2 - 2t}$. In general, we have the followings:

- If $U \le 0$, then the polygon shrinks to a single point;
- If $U > 0$ and $\lambda_0 < U^{-1}$, then the polygon shrinks to a single point;
- If $U > 0$ and $\lambda_0 > U^{-1}$, then the polygon expands into infinity.

Angenent–Gurtin [AGu] extend Remark 1.2 to the following three cases for the evolution equation (1.3) of $\mathcal{N}$-admissible piecewise linear curve. Let $T > 0$ be a duration of solution polygon of equation (1.3), $\mathcal{L}(t)$ the length and $\mathcal{A}(t)$ the enclosed area. Here and hereafter, we use the term "duration" for the maximal existence time of solution polygons.

- If $U \le 0$, then $\mathcal{A}(t) \to 0$ as $t \to T < \infty$;
- If $U > 0$ and $\mathcal{L}(0)$ is small enough, then $\mathcal{A}(t) \to 0$ as $t \to T < \infty$;
- If $U > 0$ and $\mathcal{A}(0)$ is large enough, then $\mathcal{A}(t) \to \infty$ as $t \to T = \infty$. Even so, isoperimetric ratio remains bounded: $\limsup_{t\to\infty} \mathcal{L}(t)^2/(4\pi\mathcal{A}(t)) < \infty$.

1.4 Main results

The goal of this chapter is to extend Remark 1.2 and the above results of [AGu] for the motion of convex $\mathcal{N}_\star$-admissible n-gon with general a and b. We assume one of the following:

(A1) $b \le 0$ is a constant.

(A1)' $b \le 0$ is not constant and $\displaystyle\min_{0\le j<n} \kappa_j(0) > \frac{\max_{0\le j<n} b(\boldsymbol{n}_j) - \min_{0\le j<n} b(\boldsymbol{n}_j)}{\min_{0\le j<n} a(\boldsymbol{n}_j)}$.

(A2) $b > 0$, and $\displaystyle\min_{0\le j<n} \kappa_j(0) \ge \frac{2\max_{0\le j<n} b(\boldsymbol{n}_j)}{\min_{0\le j<n} a(\boldsymbol{n}_j)}$.

$$\text{(A3)}\quad b>0,\ \text{ and }\ \max_{0\le j<n}\kappa_j(0)\le\frac{\min_{0\le j<n}b(\boldsymbol{n}_j)}{\max_{0\le j<n}a(\boldsymbol{n}_j)(1+\delta)}\ \text{ for any fixed }\ \delta>0.$$

Assumptions (A1)' and (A2) mean that the initial polygon $\mathcal{P}_0$ is sufficiently small and (A3) means that $\mathcal{P}_0$ is sufficiently large. We note that if $b\le 0$ is not constant, then $\min_{0\le j<n} b(\boldsymbol{n}_j)<0$.

Our main results are the following.

Theorem A (point-extinction) *Let $n\ge 4$. Assume* (A1), *or* (A1)'. *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1) *with a duration $T_\star$. Then any solution polygon $\mathcal{P}_t$ shrinks to a single point as $t\to T_\star$ and it holds that*

$$T_\star\le\frac{1}{2\min_{0\le j<n}a(\boldsymbol{n}_j)}\left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)}\right)^2.$$

No side of the polygon vanishes before t reaches $T_\star$. Here $\mathcal{L}(0)$ is the initial length of $\mathcal{P}_0$.

Theorem B (point-extinction) *Let $n\ge 4$. Assume* (A2). *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1) *with a duration $T_\star$. Then any solution polygon $\mathcal{P}_t$ shrinks to a single point as $t\to T_\star$. Moreover*

$$T_\star\le\min\{T_1,\,T_2,\,T_3\},\quad \textit{where}\quad T_1=\frac{(\mathcal{L}(0)/2n\tan(\Delta\theta/2))^2}{\min_{0\le j<n}a(\boldsymbol{n}_j)},$$

$$T_2=\frac{\mathcal{L}(0)}{2\tan(\Delta\theta/2)\sum_{0\le j<n}b(\boldsymbol{n}_j)},\quad T_3=T_2-\nu+\sqrt{(\nu-T_2)^2+\nu T_1},$$

and $\nu=n^2(\sum_{0\le j<n}b(\boldsymbol{n}_j)\sum_{0\le j<n}b(\boldsymbol{n}_j)/a(\boldsymbol{n}_j))^{-1}$. No side of the polygon vanishes before t reaches $T_\star$.

Remark 1.3 We call $T_\star$ the "extinction time," or the "blow-up time" (see section 2.4).

Remark 1.4 If $b\equiv 0$, then the point-extinction holds and the solution is asymptotic self-similar (see [S3]). Let $\mathcal{A}(t)$ be the area of region enclosed by $\mathcal{P}_t$. We can easily check $d\mathcal{A}(t)/dt=-2\tan(\Delta\theta/2)\sum_{0\le j<n}a(\boldsymbol{n}_j)$, hence we have

$$T_\star=T_{\star\star}=\frac{\mathcal{A}(0)}{2\tan(\Delta\theta/2)\sum_{0\le j<n}a(\boldsymbol{n}_j)}$$

since point-extinction holds.

For a convex $\mathcal{N}_\star$-admissible polygon $\mathcal{P}_t$, we define the isoperimetric ratio by

$$\mathcal{I}(t)=\frac{\mathcal{L}(t)^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)}. \tag{1.4}$$

It is not difficult to see that the inequality $\mathcal{I}(t)\ge 1$ holds. The equality $\mathcal{I}(t)=1$ holds if and only if the polygon $\mathcal{P}_t$ is a regular polygon. See [Y2], especially section 3.

Theorem C (geometric expansion) *Let $n \geq 4$. Fix $\delta > 0$ and assume* (A3). *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1). *Then the length $\mathcal{L}(t)$ and the enclosed area $\mathcal{A}(t)$ of the polygon $\mathcal{P}_t$ diverge to infinity as t tends to infinity. Every side of the polygon is finite if t is finite. Moreover, the isoperimetric ratio $\mathcal{I}(t)$ remains bounded:*

$$\limsup_{t\to\infty} \mathcal{I}(t) \leq \left(\frac{\max_{0\leq j<n} b(\boldsymbol{n}_j)}{\min_{0\leq j<n} b(\boldsymbol{n}_j)}\right)^2 .$$

Consequently if b is a positive constant, then any solution polygon $\mathcal{P}_t$ expands to infinity approaching a regular polygon in the Hausdorff metric as t tends to infinity.

Remark 1.5 For the evolution equation (1.3) Angenent–Gurtin[AGu](section 11) conjectures that a solution polygon is asymptotic to the Wulff shape for β^{-1} as $t \to \infty$ and remarks that if $\beta = const.$, then $\mathcal{L}(t)^2/(4\pi\mathcal{A}(t)) \to n\tan(\Delta\theta/2)/\pi$, that is, $\mathcal{I}(t) \to 1$ as $t \to \infty$. Although we do not give a complete answer to this conjecture, Theorem C asserts that if $b = const.$, the asymptotic shape is a regular polygon even if $a(\boldsymbol{n}_j)$ is not constant. We also note that the result does not depend on δ in Assumption (A3).

Theorem D (lower bound of the blow-up time) *Assume $b \not\equiv 0$. Under the same assumption of Theorem* A, *the blow-up time $T_\star$ is estimated as follows:*

$$T_\star \geq \frac{\max_{0\leq j<n} a(\boldsymbol{n}_j)}{8\left(\min_{0\leq j<n} b(\boldsymbol{n}_j)\right)^2}\left(1-\sqrt{1-\frac{8\min_{0\leq j<n} b(\boldsymbol{n}_j)\mathcal{A}(0)}{\max_{0\leq j<n} a(\boldsymbol{n}_j)\mathcal{L}(0)}}\right)^2 .$$

Here $\mathcal{A}(0)$ is the area of the region enclosed by $\mathcal{P}_0$.

Remark 1.6 Let $\mathcal{P}_0$ be a regular polygon. Suppose $a \equiv 1$ and $b \equiv \text{const} < 0$. We denote the upper bound in Theorem A by T_u, and the lower bound in Theorem D by T_ℓ. If we set $b = \mu\kappa(0)$ ($\kappa_j(0) \equiv \kappa(0)$), then we have $\mu < 0$ and $\lim_{\mu\to 0-} T_\ell = T_u = \kappa(0)^{-2}/2 = T_{\star\star}$.

Theorem E (lower bound of the blow-up time) *Under the same assumption of Theorem* B, *the blow-up time $T_\star$ is estimated as follows:*

$$T_\star \geq \frac{\mathcal{L}(0)}{8\tan(\Delta\theta/2)\sum_{0\leq j<n} b(\boldsymbol{n}_j)}\left(\sqrt{1+\frac{32\tan(\Delta\theta/2)\sum_{0\leq j<n} b(\boldsymbol{n}_j)\mathcal{A}(0)^2}{\max_{0\leq j<n} a(\boldsymbol{n}_j)\mathcal{L}(0)^3}}-1\right).$$

Remark 1.7 Let $\mathcal{P}_0$ be a regular polygon. Suppose $a \equiv 1$, $b \equiv \text{const} > 0$ and the Assumption (A2) holds. If we set $b = \mu\kappa(0)$ ($\kappa_j(0) \equiv \kappa(0)$), then we have $\mu \leq 1/2$. We denote the lower bound in Theorem E by T_ℓ. It holds that $T_2 > T_1 > T_3 > T_\ell$ and that $\lim_{\mu\to 0+} T_\ell = \lim_{\mu\to 0+} T_3 = \kappa(0)^{-2}/2 = T_{\star\star}$.

Remark 1.8 (approximation) Many authors recently studied an approximation of curvature-dependent motions by using crystalline motions. In both [GirK1] and [FG], the convergence results are shown for graph-like curves. In [EGS], the properties of a solution in the sense of [FG] are investigated, and several numerical examples are presented in order to visualize their results. The new notion of solutions to a fully nonlinear equation including crystalline motion is introduced and analyzed in [GMHG1, GMHG2, GMHG4]. Its notion is in the realm of viscosity solution theory, and so is based on comparison principle which is an extension of [GGu]. The convergence results are discussed in [GMHG3, GMHG5] for the solutions in its notion.

Let the Wulff shape be a regular polygon. Girão [Gir] showed that the crystalline motion $v_j = \omega_j$ approximates the weighted curvature flow $v = \omega$ if the curve is closed and convex. This result was extended by [UY1] for the motion by a power of curvature $v = \kappa^\alpha$ $(\alpha > 0)$. Moreover, they constructed a crystalline algorithm to the equation $v = |\kappa|^{\alpha-1}\kappa$ for nonconvex curves in [UY2]. Implicit crystalline algorithm is treated in [UY3] for an area-preserving motion by curvature $v = \kappa - 2\eta\pi/\mathcal{L}$ ($\eta \geq 1$ is a winding number of curve). In [IS], the curve-shortening equation $v = \kappa$ is approximated by the crystalline motion $v_j = \kappa_j$ if the Wulff shape is a regular polygon. See the survey [E] for more general information about an approximation of curvature-dependent motion.

The organization of this chapter is as follows: in section 2, we give several fundamental properties of solutions to Problem (1.1). In section 3, we present a point-extinction property of solutions via entropy estimates, and prove Theorems A and B. In section 4, we prove Theorem C by the super- and subsolution method, or the comparison principle. By using Schwarz inequality twice, we give a lower bound of the extinction time, and the proof of Theorems D and E in section 5. In appendix A, we give a brief summary on the gradient flow for a total interfacial energy. In appendices B and C, we explain the notion of the discrete curvature and the crystalline curvature, respectively.

2 Properties of solutions to Problem (1.1)

In this section we first give an equivalent formulation of Problem (1.1). Secondly, we present comparison principle, and evolution of the length and the area. Finally, we show a finite time blow-up of solution.

Throughout this chapter we use the notation $\sum_j u_j$, $u_{\max}$, $u_{\min}$ and $\dot{u}(t)$ for $\sum_{0\leq j<n} u_j$, $\max_{0\leq j<n} u_j$, $\min_{0\leq j<n} u_j$ and $du(t)/dt$, respectively. Hereafter we denote $a_j := a(\boldsymbol{n}_j)$ and $b_j := b(\boldsymbol{n}_j)$ for simplicity, and assume $n \geq 4$. We note again $\theta_j = j\Delta\theta$.

2.1 A formulation equivalent to Problem (1.1)

Let $\mathcal{P}_t$ be a solution of Problem (1.1). The jth vertex $\boldsymbol{B}_j(t)$ of $\mathcal{P}_t$ is given as the following:

$$\begin{aligned} \boldsymbol{B}_j(t) &= \langle \boldsymbol{x}_{j-1}(t) - \boldsymbol{x}_j(t), \boldsymbol{t}_j + \boldsymbol{n}_j \cot \Delta\theta \rangle \, \boldsymbol{t}_j + \boldsymbol{x}_j(t), \\ &= \boldsymbol{B}_0(t) + \sum_{0 \le m < j} d_m(t) \boldsymbol{t}_m, \quad 1 \le j \le n, \quad 0 \le t < T \end{aligned} \tag{2.1}$$

with $\boldsymbol{B}_0(t) \equiv \boldsymbol{B}_n(t)$, where $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j)$ is the tangent vector, since the position vector $\boldsymbol{x}_j$ is on the line containing the jth side (n.b. $\boldsymbol{x}_j$ is not necessarily on the jth side), and $\langle \cdot, \cdot \rangle$ is the usual inner product. Then the time evolution of the length of the jth side $d_j(t)$ is given as the following (cf. Figure 10C in [AGu]):

$$\frac{d}{dt} d_j(t) = \frac{d}{dt} |\boldsymbol{B}_{j+1}(t) - \boldsymbol{B}_j(t)| = -2 \tan \frac{\Delta\theta}{2} (\Delta_\theta v + v)_j. \tag{2.2}$$

Here the operator Δ_θ is defined as

$$(\Delta_\theta (\cdot))_j := \frac{(\cdot)_{j+1} - 2(\cdot)_j + (\cdot)_{j-1}}{2(1 - \cos\Delta\theta)}$$

which is a kind of central difference operator (this is a special version of (1.2)). Then we get a discretized version of the equation (2.20) in the book [Gu1]:

$$\frac{d}{dt} \kappa_j(t) = \kappa_j^2 (\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T.$$

Therefore we can restate Problem (1.1) as follows.

Problem 1 *Let $n \ge 4$. Find a function $v(t) = (v_0, v_1, ..., v_{n-1}) \in [C^1(0,T)]^n$, and a duration $T \in (0, \infty]$ satisfying*

$$\frac{d}{dt} v_j(t) = a(\boldsymbol{n}_j)^{-1} (v_j + b(\boldsymbol{n}_j))^2 (\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T, \tag{2.3a}$$

$$v_j(0) = a(\boldsymbol{n}_j) \kappa_j(0) - b(\boldsymbol{n}_j), \quad 0 \le j < n, \tag{2.3b}$$

$$v_{-1}(t) = v_{n-1}(t), \quad v_n(t) = v_0(t), \quad 0 \le t < T, \tag{2.3c}$$

where $\kappa_j(0)$ is the jth initial crystalline curvature of $\mathcal{P}_0$.

Remark 2.1 (equivalence) Problem (1.1) and Problem 1 are equivalent except the indefiniteness of position of the polygon. Indeed, suppose v is a solution of Problem 1, then we have

$$\frac{1}{2\tan(\Delta\theta/2)} \frac{d}{dt} \sum_j \frac{2 a_j \tan(\Delta\theta/2)}{v_j(t) + b_j} \boldsymbol{t}_j = -\sum_j (\Delta_\theta v + v)_j \boldsymbol{t}_j = -\sum_j (\Delta_\theta \boldsymbol{t} + \boldsymbol{t})_j v_j = \boldsymbol{0}.$$

Here we have used the relation of summation by parts:

$$\sum_j f_j (\Delta_\theta g)_j = -\sum_j (\mathrm{D}_+ f)_j (\mathrm{D}_+ g)_j = \sum_j g_j (\Delta_\theta f)_j, \tag{2.4}$$

and the relation $(\Delta_\theta \boldsymbol{t})_j = -\boldsymbol{t}_j$. Here and hereafter, we define the forward difference such as

$$(\mathrm{D}_+ f)_j := \frac{f_{j+1} - f_j}{2\sin(\Delta\theta/2)}.$$

Hence by equation (2.1), we can construct a closed convex n-gon whose length of the jth side is $2a_j \tan(\Delta\theta/2)/(v_j(t) + b_j) =: d_j(t)$, and the jth normal vector is $\boldsymbol{n}_j$, as long as v is a solution of Problem 1. This n-gon is the very solution polygon of Problem (1.1).

2.2 Comparison principle

The following comparison principle plays an important roll in this chapter.

Lemma 2.2 *Fix $T > 0$. Let $(p_j(t))_{0\le j<n} > 0$ and $(q_j(t))_{0\le j<n}$ be defined on $t \in [0, T]$. If $u = (u_j(t))_{0\le j<n} \in [C^1(0,T) \cap C[0,T]]^n$ is a solution of*

$$\begin{cases} \dfrac{d}{dt}u_j \ge p_j(\Delta_\theta u)_j + q_j u_j, & 0 \le j < n, \quad 0 < t < T, \\ u_{-1}(t) = u_{n-1}(t), \quad u_n(t) = u_0(t), & 0 \le t \le T, \\ u_j(0) \ge 0, \quad 0 \le j < n, \end{cases}$$

then $u_j(t) \ge 0$ holds for $0 \le j < n$ and $0 \le t \le T$.

See, e.g., [Y1] for the proof of this lemma.

As an application of the above lemma, we obtain the next:

Lemma 2.3 *For a solution v of Problem 1, we have the followings.*

(1) If $v_j(0) \ge c$, then $v_j(t) \ge c$ for a constant $c \ge 0$.

(2) If $v_j(0) \le c$, then $v_j(t) \le c$ for a constant $c \le 0$.

Proof. For each proposition, put (1) $u_j = v_j - c$; (2) $u_j = c - v_j$; and apply Lemma 2.2. □

2.3 The length and the area

The (total) length of the polygon is

$$\mathcal{L}(t) := \sum_j d_j = 2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j^{-1} = 2\tan\frac{\Delta\theta}{2}\sum_j \frac{a_j}{v_j + b_j}, \tag{2.5}$$

and the rate of change of $\mathcal{L}(t)$ can be computed by

$$\dot{\mathcal{L}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j v_j(t). \tag{2.6}$$

If $v_j(0) \geq 0$, then $v_j(t) \geq 0$ by Lemma 2.3, $\dot{\mathcal{L}}(t) \leq 0$ and the motion of solution polygons is discretized curve-shortening.

The area enclosed by the polygon is

$$\mathcal{A}(t) := -\frac{1}{2}\sum_j \langle \boldsymbol{x}_j(t), \boldsymbol{n}_j \rangle d_j(t),$$

and the rate of change of $\mathcal{A}(t)$ can be computed by

$$\dot{\mathcal{A}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j \frac{a_j v_j}{v_j + b_j} \tag{2.7}$$

Here we use equations (2.2) and (2.4), definition $\langle \dot{\boldsymbol{x}}_j, \boldsymbol{n}_j \rangle = v_j$ and geometric relation $d_j = -2\tan(\Delta\theta/2)(\Delta_\theta \langle \boldsymbol{x}, \boldsymbol{n} \rangle + \langle \boldsymbol{x}, \boldsymbol{n} \rangle)_j$.

2.4 Finite time blow-up

In this subsection, we give a partial proof of Theorems A and B, namely the statement concerning finite time blow-up.

Lemma 2.4 (finite time blow-up) *Suppose v is a solution of Problem 1. Under the same assumption of Theorem A, there exists a finite time $T_\star > 0$ such that the maximum of $\{\kappa_j = (v_j + b_j)/a_j\}$ blows up to infinity as $t \nearrow T_\star$:*

$$T_\star \leq \frac{1}{2\min_{0 \leq j < n} a(\boldsymbol{n}_j)} \left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} \right)^2.$$

Proof. Since $n^2 = (\sum_j 1)^2 = (\sum_j \kappa_j^{1/2}\kappa_j^{-1/2})^2$, Schwarz inequality and the assumption $b \leq 0$ yields

$$\begin{aligned} \left(2n\tan\frac{\Delta\theta}{2}\right)^2 &\leq -\frac{1}{2a_{\min}}\frac{d}{dt}\mathcal{L}(t)^2 + 2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j \frac{b_j}{a_j} \\ &\leq -\frac{1}{2a_{\min}}\frac{d}{dt}\mathcal{L}(t)^2. \end{aligned} \tag{2.8}$$

By the general argument for ordinary differential equation, a solution v of Problem 1 exists uniquely, and locally in time. Put $T_\star > 0$ such as maximal existing time. Take $0 < t < T_\star$. Integration of the above inequality over $(0, t)$ yields

$$\mathcal{L}(t) \leq \sqrt{\mathcal{L}(0)^2 - 2a_{\min}\left(2n\tan\frac{\Delta\theta}{2}\right)^2 t}.$$

Since $\mathcal{L}(t) \geq 2n\tan(\Delta\theta/2)/\kappa_{\max}$, we have

$$\kappa_{\max} \geq 2n\tan\frac{\Delta\theta}{2}\left(\mathcal{L}(0)^2 - 2a_{\min}\left(2n\tan\frac{\Delta\theta}{2}\right)^2 t\right)^{-1/2},$$

and the assertion is concluded. □

Lemma 2.5 (finite time blow-up) *Suppose v is a solution of Problem 1. Under the same assumption of Theorem B, there exists a finite time $T_\star > 0$ such that the maximum of $\{\kappa_j = (v_j + b_j)/a_j\}$ blows up to infinity as $t \nearrow T_\star$:*

$$T_\star \leq \min\{T_1,\ T_2,\ T_3\}.$$

Here T_1, T_2, and T_3 have been defined in Theorem B.

Proof. The Assumption (A2) implies $v_j(0) \geq b_{\max}$. Then Lemma 2.3 provides $v_j(t) \geq b_{\max} \geq b_j$. Hence, we get T_1 by a similar proof of Lemma 2.4.

Integration of $-\dot{\mathcal{L}}(t) = 2\tan(\Delta\theta/2)\sum_j v_j \geq 2\tan(\Delta\theta/2)\sum_j b_j$ over $(0,t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) - 2\tan\frac{\Delta\theta}{2}\sum_j b_j t, \tag{2.9}$$

and then we obtain T_2.

Substitute the inequality (2.9) to (2.8), integrate it over $(0,t)$, and solve it. Then we get $t \leq T_3$. □

3 Point-extinction (proof of Theorems A and B)

Before we give the proof of Theorem A and B, we present the following theorem.

Theorem 3.1 *Assume* (A1), *or* (A1)', *or* (A2). *If the area $\mathcal{A}(t)$ is bounded away from zero, then a solution v of Problem 1 is uniformly bounded for $t \in [0, T_\star)$, where the blow-up time $T_\star$ attains $\mathcal{A}(T_\star) = 0$.*

Remark 3.2 This theorem does not claim that the polygon shrinks to a single point.

We use the analogue of several estimates by Gage–Hamilton [GH] for the curvature and by Girão [Gir] for the weighted curvature. For reader's convenience, we do not omit the proofs except completely the same one.

Lemma 3.3 *There exists a constant $C_1 = C_1(v(0), \Delta\theta) \geq 0$ such that*

$$2\tan\frac{\Delta\theta}{2}\sum_{0\leq j<n}(\mathrm{D}_+ v)_j^2 \leq 2\tan\frac{\Delta\theta}{2}\sum_{0\leq j<n} v_j^2 + C_1.$$

Proof. It can be shown that the next estimate:

$$2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j (v^2 - (\mathrm{D}_+ v)^2)_j = 4\tan\frac{\Delta\theta}{2}\sum_j a_j^{-1}(v_j + b_j)^2(\Delta_\theta v + v)_j^2 \geq 0.$$

By the integration of this inequality over $(0,t)$ and putting

$$C_1 \geq \max\left\{-2\tan\frac{\Delta\theta}{2}\sum_j (v(0)^2 - (\mathrm{D}_+ v(0))^2)_j,\ 0\right\},$$

we get the assertion. $\square$

One can easily get: $\sum_{j=1}^{[n/2]} \sin\theta_j \leq 2\cot(\Delta\theta/2)$, where $[n/2]$ is $n/2$ for n even and $(n-1)/2$ for n odd, since the left-hand side equals to $\cot(\Delta\theta/2)$ for n even, and $(1+\sec(\Delta\theta/2))\cot(\Delta\theta/2)/2$ for n odd.

We introduce the median normal velocity which is a similar to the median curvature in [GH] and the median discrete weighted curvature in [Gir].

Definition 3.4 (median normal velocity) $v_*(t) := \max_{0\leq j<n} \min_{j+1\leq i\leq j+[n/2]} v_i(t)$.

Lemma 3.5 *Assume* (A1), *or* (A1)', *or* (A2). *Fix* $t \in [0,T_*)$. *If* $\mathcal{A}(t)$ *is bounded away from zero, then* $v_*(t)$ *is bounded.*

Proof. Let j_0 be a value of j which attains the maximum of v. A polygon lies between parallel lines whose distance is less than

$$\begin{aligned}\sum_{j=j_0+1}^{j_0+[n/2]} \sin(\theta_j - \theta_{j_0})d_j &= 2\tan\frac{\Delta\theta}{2}\sum_{j=1}^{[n/2]}\frac{a_{j+j_0}\sin\theta_j}{v_{j+j_0}+b_{j+j_0}} \\ &\leq \frac{2\tan(\Delta\theta/2)a_{\max}}{v_*+b_{\min}}\sum_{j=1}^{[n/2]}\sin\theta_j \leq \frac{4a_{\max}}{v_*+b_{\min}}.\end{aligned}$$

If we assume (A1)', then we have $v_{\min}(0)+b_{\min} > 0$. We also have $v_*(t) \geq v_{\min}(0)$ by Lemma 2.3. Therefore the denominator $v_* + b_{\min}$ is positive for all $t \geq 0$.

The diameter is bounded by $\mathcal{L}/2$, and the area is bounded by the width times the diameter:

$$\mathcal{A}(t) \leq \frac{2a_{\max}\mathcal{L}(t)}{v_*(t)+b_{\min}}.$$

Hence $v_*(t) \leq 2a_{\max}\mathcal{L}(0)/\mathcal{A}(t) - b_{\min}$.

The assertion is proved in a similar way if we assume (A1) or (A2). $\square$

Definition 3.6 *Let the entropy be:*

$$\mathcal{E}(t) := 2\tan\frac{\Delta\theta}{2}\sum_{0\leq j<n}\left(a(\boldsymbol{n}_j)\log\kappa_j(t) + \frac{b(\boldsymbol{n}_j)}{\kappa_j(t)}\right).$$

Lemma 3.7 *Assume* (A1), *or* (A1)', *or* (A2). *Fix* $t \in [0,T_*)$. *It there exists a constant* $C_* > 0$ *such that* $v_*(\tau) \leq C_*$ *for* $0 \leq \tau \leq t$, *then* $\mathcal{E}(t)$ *is bounded.*

Proof. By using the summation by parts (2.4), one has

$$\dot{\mathcal{E}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j (v^2 - (\mathrm{D}_+ v)^2)_j.$$

We use the same estimates as in the proof of Girão[Gir] (section 2, *Fourth*), and have the next estimate (see also [Y2]):

$$2\tan\frac{\Delta\theta}{2}\sum_j (v^2 - (\mathrm{D}_+ v)^2)_j \le 2n\tan\frac{\Delta\theta}{2}v_*^2 - 2v_*\dot{\mathcal{L}}(t).$$

Hence, $\mathcal{E}(t) \le \mathcal{E}(0) + 2n\tan(\Delta\theta/2)C_*^2 T_\star + 2C_*\mathcal{L}(0)$ holds. □

Lemma 3.8 *Assume* (A1), *or* (A1)', *or* (A2). *If $\mathcal{E}(t)$ is bounded, then for any $\delta > 0$ there exists a constant $C_2 > \max\{1, a_{\max}\} - b_{\min}$ if $b \le 0$ and $C_2 > a_{\max}$ if $b > 0$ such that $v_j(t) \le C_2$ except for θ_j in intervals of length less than δ for $t \in [0, T_\star)$.*

Proof. If $v_j \ge C_2$ for m values of j and $m\Delta\theta \ge \delta$, then

$$\begin{aligned}\mathcal{E}(t) &\ge 2\tan\frac{\Delta\theta}{2}\left(ma_{\min}\log\frac{C_2 + b_{\min}}{a_{\max}} - a_{\max}(n-m)\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right) + B \\ &\ge \frac{2}{\Delta\theta}\tan\frac{\Delta\theta}{2}\left(\delta a_{\min}\log\frac{C_2 + b_{\min}}{a_{\max}} - a_{\max}(2\pi - \delta)\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right) + B\end{aligned}$$

where $B = b_{\min}\mathcal{L}(0)$ when $b \le 0$ and

$$\begin{aligned}\mathcal{E}(t) &\ge 2\tan\frac{\Delta\theta}{2}\left(ma_{\min}\log\frac{C_2}{a_{\min}} - (n-m)a_{\max}\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right) \\ &\ge \frac{2}{\Delta\theta}\tan\frac{\Delta\theta}{2}\left(\delta a_{\min}\log\frac{C_2}{a_{\min}} - (2\pi - \delta)a_{\max}\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right)\end{aligned}$$

when $b > 0$. This gives a contradiction when C_2 is large. □

Lemma 3.9 *Assume* (A1), *or* (A1)', *or* (A2). *For $t \in [0, T_\star)$, if $v_j(t) \le C_2$ for some constant $C_2 \gg 1$ except for θ_j in intervals of length less than δ and $\delta > 0$ is small enough, then $v_{\max}(t)$ is bounded.*

Proof. As in the proof of Girão[Gir] (section 2, *Sixth*), we have the next estimate:

$$\begin{aligned}v_j &= v_i + \sum_{i\le m<j}(v_{m+1} - v_m) \\ &\le C_2 + \left(\sum_{i\le m<j}\frac{2(1-\cos\Delta\theta)}{2\tan(\Delta\theta/2)}\right)^{1/2}\left(2\tan\frac{\Delta\theta}{2}\sum_{i\le m<j}(\mathrm{D}_+ v)_m^2\right)^{1/2} \\ &\le C_2 + \sqrt{(j-i)\sin\Delta\theta}\left(2\tan\frac{\Delta\theta}{2}\sum_{0\le m<n}v_m^2 + C_1\right)^{1/2} \\ &\le C_2 + \sqrt{\delta}\left(2n\tan\frac{\Delta\theta}{2}v_{\max}^2 + C_1\right)^{1/2} \\ &\le C_2 + \sqrt{\delta}\left(\sqrt{2\sqrt{2}\pi}v_{\max} + \sqrt{C_1}\right)\end{aligned}$$

since $v_j \le C_2$ and $\theta_i - \theta_j \le \delta$. Here we have used Lemma 3.3.

Hence $(1-\sqrt{2\sqrt{2}\pi\delta})v_{\max} \le C_2 + \sqrt{C_1\delta}$ holds, and we get $v_{\max} \le (C_2+\sqrt{C_1\delta})/(1-\sqrt{2\sqrt{2}\pi\delta})$ for small δ. □

Proof of Theorem 3.1. Suppose that a side of $\mathcal{P}_t$ disappears for $t < T_\star$ where $T_\star$ attains $\mathcal{A}(T_\star) = 0$. Put t_0 as the first time that happens (n.b. $t_0 > 0$ is clear). Then $\mathcal{A}(t) > 0$ for $0 \le t \le t_0$, and the estimates above imply that $\sup_{0\le t\le t_0} v_{\max}(t)$ is bounded, so $d_{\min}(t_0) > 0$. This is a contradiction. Hence the assertion holds. □

We are now ready to present of the proof of Theorem A and B.

Proof of Theorem A *and* B. By Theorem 3.1, we have $\mathcal{A}(T_\star) = 0$. If n is odd, then $\mathcal{L}(T_\star) = 0$ since the angle between two adjacent sides of polygon is always $\pi - \Delta\theta$ and we have no two sides which is parallel each other. Suppose that n is even. Then the jth side and the $(j+n/2)$th side are parallel. Let w_j be the distance between the jth and the $(j+n/2)$th side, and we have

$$w_m = \sum_{j=m+1}^{m+n/2} \sin(\theta_j - \theta_m)\, d_j = \sum_{j=1}^{n/2} \sin\theta_j\, d_{j+m}, \quad \text{or} \quad w_m = -\sum_{j=n/2+1}^{n} \sin\theta_j\, d_{j+m}.$$

Therefore,

$$2w_m = \sum_j |\sin\theta_j|\, d_{j+m} = 2\tan\frac{\Delta\theta}{2}\sum_j |\sin\theta_j| \frac{a_{j+m}}{v_{j+m}+b_{j+m}}.$$

Then we have

$$\begin{aligned}
\dot{w}_m &= -\tan\frac{\Delta\theta}{2}\sum_j |\sin\theta_j|\,(\Delta_\theta v + v)_{j+m} \\
&= -\tan\frac{\Delta\theta}{2}\sum_j v_j\,(\Delta_\theta|\sin\theta| + |\sin\theta|)_{j-m} \\
&= -(v_m + v_{m+n/2})
\end{aligned}$$

since

$$(\Delta_\theta|\sin\theta| + |\sin\theta|)_i = \begin{cases} \cot(\Delta\theta/2) & \text{if } \; i = 0,\; n/2; \\ 0 & \text{if } \; \textit{otherwise}. \end{cases}$$

Proof of Theorem A. Put $C > 0$ such as $\dot{\mathcal{A}}(t) \ge -2\tan(\Delta\theta/2)\sum_j a_j + b_{\min}\mathcal{L}(0) =: -C$. By Theorem 3.1, we have $\mathcal{A}(t) \le C(T_\star - t)$. Then $\dot{w}_m \le -v_m \le -2\tan(\Delta\theta/2)a_m d_m^{-1}$ and $\mathcal{A}(t) \ge w_m d_m/2$ yield

$$\frac{\dot{w}_m}{w_m} \le -a_m\frac{\tan(\Delta\theta/2)}{\mathcal{A}(t)} \le -a_m\frac{\tan(\Delta\theta/2)}{C(T_\star - t)}.$$

Hence, by integration over $(0,t)$, we have

$$w_m(t) \le w_m(0)\left(\frac{T_\star - t}{T_\star}\right)^{a_m\tan(\Delta\theta/2)/C}$$

and $w_m(T_\star) = 0$ for all m. Then $\mathcal{L}(T_\star) = 0$ is concluded. $\Box$

Proof of Theorem B. Since $b > 0$, it holds that $\mathcal{A}(t) \le C(T_\star - t)$ for a positive constant $C > 0$. By the condition (A2), we have $v_j(t) \ge b_{\max}$ and

$$-v_m = -a_m \frac{2\tan(\Delta\theta/2)}{d_m} + b_m \le -a_m \frac{2\tan(\Delta\theta/2)}{d_m} + v_{m+n/2}.$$

Then $\dot{w}_m \le -2\tan(\Delta\theta/2) a_m d_m^{-1}$ and $\mathcal{A}(t) \ge w_m d_m/2$ provide the point-extinction in a way similar to the proof of Theorem A. $\Box\Box$

4 Geometric expansion (proof of Theorem C)

We shall prove Theorem C by the super- and subsolution method, or the comparison principle.

Lemma 4.1 *Let v be a solution of Problem* 1. *Under the Assumption* (A3), *we have the followings.*

(1) Let v_u be a solution of

$$\frac{d}{dt} v_u(t) = a_{\max}^{-1} (v_u(t) + b_{\min})^2 v_u(t) \quad with \quad v_u(0) = v_{\max}(0). \tag{4.1}$$

Then we have $v_u(t) \ge v_j(t)$ for all $t \ge 0$. Moreover,

$$v_u(t) \le \frac{v_u(0) + b_{\min}}{1 - a_{\max}^{-1} v_u(0)(v_u(0) + b_{\min})t} - b_{\min} \tag{4.2}$$

holds for all $t \ge 0$.

(2) Let v_l be a solution of

$$\frac{d}{dt} v_l(t) = a_{\min}^{-1} (v_l(t) + b_{\max})^2 v_l(t) \quad with \quad v_l(0) = v_{\min}(0). \tag{4.3}$$

Then we have $v_j(t) \ge v_l(t)$ for all $t \ge 0$. Moreover,

$$v_l(t) \ge \frac{v_l(0) + b_{\max}}{1 + a_{\min}^{-1} b_{\max}(v_l(0) + b_{\max})t} - b_{\max} \tag{4.4}$$

holds for all $t \ge 0$.

Corollary 4.2 *Under the same assumption as in Lemma* 4.1, *we have*

$$-b_{\max} \le v_{ll}(t) \le v_j(t) \le v_{uu}(t) \le v_u(0) < 0$$

for all $t \ge 0$ where v_{uu} (resp., v_{ll}) is the right-hand side of inequality (4.2) *(resp.,* (4.4)*).*

Proof of Lemma 4.1. Assumption (A3) provides $v_u(0) < 0$, and so $v_u(t) \leq 0$ holds by Lemma 2.3. Remark 2.1 asserts that there exists a polygon corresponding a solution v_u of (4.1). This means that there is a crystalline curvature, say $\kappa_u(t)$ such that $v_u(t) = a_{\max}\kappa_u(t) - b_{\min}$ with $\kappa_u(0) = (v_{\max}(0) + b_{\min})/a_{\max}$. Therefore $v_u(t) + b_{\min} = a_{\max}\kappa_u(t) > 0$ holds.

Define $u_j(t) := v_u(t) - v_j(t)$. Then

$$\begin{aligned}
\dot{u}_j &= a_{\max}^{-1}(v_u + b_{\min})^2 v_u - a_j^{-1}(v_j + b_j)^2(\Delta_\theta v + v)_j \\
&\geq a_j^{-1}(v_j + b_j)^2(\Delta_\theta u)_j + a_j^{-1}(v_j + b_j)^2 u_j + a_j^{-1}\left((v_u + b_{\min})^2 - (v_j + b_j)^2\right) v_u \\
&= a_j^{-1}(v_j + b_j)^2(\Delta_\theta u)_j + a_j^{-1}(v_j + b_j)^2 u_j \\
&\qquad + a_j^{-1}(v_u + b_{\min} + v_j + b_j)(v_u - v_j + b_{\min} - b_j)v_u \\
&\geq a_j^{-1}(v_j + b_j)^2(\Delta_\theta u)_j + a_j^{-1}\left((v_j + b_j)^2 + (v_u + b_{\min} + v_j + b_j)v_u\right) u_j.
\end{aligned}$$

Here we have used the inequalities: $a_{\max}^{-1}v_u \geq a_j^{-1}v_u$, $v_u + b_{\min} > 0$, $v_u < 0$ and etc. Hence the initial estimate $u_j(0) = v_u(0) - v_j(0) \geq 0$ provides the first assertion by Lemma 2.2.

Inequality $v_u(t) \leq v_u(0)$ yields

$$\dot{v}_u(t) \leq a_{\max}^{-1}(v_u(t) + b_{\min})^2 v_u(0).$$

Solution of the above inequality satisfies (4.2). Then the proof of (1) is completed.

Proof of the first assertion in (2) is a similar to (1). To prove (4.4) use the relation $v_l(t) \geq -b_{\max}$, and solve $\dot{v}_l \geq -a_{\min}^{-1}(v_l + b_{\max})^2 b_{\max}$. □

We are now ready to present the proof of Theorem C.

Proof of Theorem C. By Corollary 4.2, we can estimate $\dot{\mathcal{L}}(t)$:

$$v_{ll}(t) \leq \frac{\dot{\mathcal{L}}(t)}{-2n\tan(\Delta\theta/2)} \leq v_{uu}(t).$$

Integration over $(0, t)$ yields

$$\begin{aligned}
\frac{\mathcal{L}(t)}{2n\tan(\Delta\theta/2)} &\leq \frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} - \frac{a_{\min}}{b_{\max}}\log\left(1 + \frac{b_{\max}}{a_{\min}}(v_l(0) + b_{\max})t\right) + b_{\max}t \\
&\leq \frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} + b_{\max}t
\end{aligned}$$

for all $t > 0$ and

$$\begin{aligned}
\frac{\mathcal{L}(t)}{2n\tan(\Delta\theta/2)} &\geq \frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} + \frac{a_{\max}}{v_u(0)}\log\left(1 - \frac{v_u(0)}{a_{\max}}(v_u(0) + b_{\min})t\right) + b_{\max}t \\
&\geq c_1 + c_2\log t + b_{\min}t
\end{aligned}$$

for all $t \geq 1$. Here constant c_1 depends on a, b, $v_u(0)$ and $\mathcal{P}_0$, and $c_2 = a_{\max}/v_u(0)$.

By Corollary 4.2 and (2.7), we have

$$-v_{uu}(t)\mathcal{L}(t) \le \dot{\mathcal{A}}(t) \le -v_{ll}(t)\mathcal{L}(t) \le b_{\max}\mathcal{L}(t).$$

Hence

$$\mathcal{A}(t) \le \mathcal{A}(0) + b_{\max}\mathcal{L}(0)t + n\tan\frac{\Delta\theta}{2}b_{\max}^2 t^2$$

holds for all $t > 0$ and

$$\mathcal{A}(t) \ge c_3 + c_4\log t + c_5 t + n\tan\frac{\Delta\theta}{2}b_{\min}^2 t^2$$

holds for all $t \ge t_0$ where t_0 is large enough. Here constants c_3, c_4 and c_5 depend on a, b, $v_u(0)$, $\mathcal{P}_0$ and t_0.

Therefore it holds that the limits: $\mathcal{L}(t)$, $\mathcal{A}(t) \to \infty$ as $t \to \infty$. Moreover, one can easily calculate the limit of isoperimetric ratio $\mathcal{I}(t) = \mathcal{L}(t)^2/(4n\tan(\Delta\theta/2)\mathcal{A}(t))$:

$$\limsup_{t\to\infty}\mathcal{I}(t) \le \left(\frac{b_{\max}}{b_{\min}}\right)^2.$$

In particular, if b is a constant, then $\limsup_{t\to\infty}\mathcal{I}(t) \le 1$ and the Bonnesen's type inequality (see [Eg]) provide that $\mathcal{P}_t$ converges to a regular polygon in the Hausdorff metric. This completes the proof of Theorem C. □

5 Lower bound of the blow-up time (proof of Theorems D and E)

We will use Schwarz inequality twice to obtain a lower bound of blow-up time. A similar idea was used in Giga–Yama-uchi [GY] to give a bound for the mean curvature flow in higher dimension.

Proof of Theorem D *and* E. By Schwarz inequality, we have

$$\begin{aligned}
-\dot{\mathcal{A}}(t) &= 2\tan\frac{\Delta\theta}{2}\sum_j a_j - 2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1} \\
&= 2\tan\frac{\Delta\theta}{2}\sum_j a_j^{1/2}\kappa_j^{1/2}a_j^{1/2}\kappa_j^{-1/2} - 2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1} \\
&\le \sqrt{a_{\max}}\left(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{1}{2}\frac{d}{dt}\mathcal{L}(t)^2\right)^{1/2} - 2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1}.
\end{aligned}$$

Integration the above inequality over $(0, T_\star)$, the point-extinction and Schwarz inequality yield

$$\begin{aligned}
\mathcal{A}(0) &\le \sqrt{a_{\max}} \int_0^{T_\star} \left(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{d}{dt}\frac{\mathcal{L}(t)^2}{2} \right)^{1/2} dt - B \\
&\le \sqrt{a_{\max}} \left(\int_0^{T_\star} dt \right)^{1/2} \left(\int_0^{T_\star} \left(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{d}{dt}\frac{\mathcal{L}(t)^2}{2} \right) dt \right)^{1/2} - B \\
&= \sqrt{a_{\max}}\sqrt{T_\star} \left(\frac{\mathcal{L}(0)^2}{2} + 2\tan\frac{\Delta\theta}{2}\sum_j b_j \int_0^{T_\star} \mathcal{L}(t)\,dt \right)^{1/2} - B
\end{aligned}$$

where $B = 2\tan(\Delta\theta/2)\int_0^{T_\star}\sum_j b_j/k_j\,dt$.

Proof of Theorem D. Since $b \le 0$, we get

$$\mathcal{A}(0) \le \sqrt{\frac{a_{\max}}{2}}\mathcal{L}(0)\sqrt{T_\star} - b_{\min}\mathcal{L}(0)T_\star.$$

Assumption $b \not\equiv 0$ means $b_{\min} < 0$. Hence the solution of this inequality provides the lower bound of $T_\star$. □

Proof of Theorem E. Since $b > 0$, we get

$$\mathcal{A}(0)^2 \le a_{\max}T_\star \left(2\tan\frac{\Delta\theta}{2}\sum_j b_j\mathcal{L}(0)T_\star + \frac{1}{2}\mathcal{L}(0)^2 \right).$$

The solution of this inequality provides the lower bound of $T_\star$. □□

Appendices

A Gradient flow of a total interfacial energy

If the interfacial energy on the curve Γ is distributed uniformly as constant 1, then the total interfacial energy of Γ is given by

$$E[\Gamma] = \int_\Gamma 1 ds = \int_{\boldsymbol{T}} |\boldsymbol{x}_\theta| d\theta, \quad (ds = |\boldsymbol{x}_\theta| d\theta : \text{ the arc-length parameter}),$$

where $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ is the flat torus. The first variation of E has the form:

$$\frac{\delta E[\Gamma^\varepsilon_{\boldsymbol{z}}]}{\delta \boldsymbol{z}} := \left.\frac{d}{d\varepsilon}E[\Gamma^\varepsilon_{\boldsymbol{z}}]\right|_{\varepsilon=0} = \int_\Gamma \langle -\boldsymbol{t}_s, \boldsymbol{z}\rangle ds$$

where $\Gamma^\varepsilon_{\boldsymbol{z}} = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{x} + \varepsilon\boldsymbol{z}(\theta),\ \boldsymbol{x} \in \Gamma_t,\ \theta \in \boldsymbol{T}\}$. Hence the gradient of E with L^2-metric is $\operatorname{grad} E[\Gamma] = -\boldsymbol{t}_s$. Then Frenet-Serret formula $\boldsymbol{t}_s = \kappa\boldsymbol{n}$ yields $\boldsymbol{x}_t = -\operatorname{grad} E[\Gamma] =$

$\kappa \boldsymbol{n}$, i.e. $v = \langle \boldsymbol{x}_t, \boldsymbol{n} \rangle = \kappa$. This equation is called the classical curve-shortening equation, and is investigated by many authors (see [GH, Gry, ?] and references therein).

If the interfacial energy $f = f(\boldsymbol{n})$ is a positively homogeneous of degree one in $C^2(\boldsymbol{R}^2 \setminus \{\boldsymbol{0}\})$, then the gradient flow of total interfacial energy $E[\Gamma] = \int_\Gamma f(\boldsymbol{n})ds$ is computed as $\boldsymbol{x}_t = {}^t(\kappa \mathrm{Hess}\, f(\boldsymbol{n})\boldsymbol{t})^\perp$ (see Elliott[E]). Here ${}^t(x_1, x_2)^\perp = {}^t(-x_2, x_1)$. Then we obtain $v = \langle \boldsymbol{x}_t, \boldsymbol{n} \rangle = \langle (\kappa \mathrm{Hess}\, f(\boldsymbol{n})\boldsymbol{t})^\perp, \boldsymbol{n} \rangle = \kappa \langle \mathrm{Hess}\, f(\boldsymbol{n})\boldsymbol{t}, \boldsymbol{t} \rangle$. Moreover, if we put $f(\theta) = f(\boldsymbol{n}(\theta))$, then we get the weighted curvature flow $v = \omega = (f + f'')\kappa$ since $\langle \mathrm{Hess}\, f(\boldsymbol{n})\boldsymbol{t}, \boldsymbol{t} \rangle = f + f''$ holds. The function $f + f''$ is the inverse of curvature on the boundary $\partial \mathcal{W}_f$ of the Wulff shape $\mathcal{W}_f$. Indeed, the locus of the boundary of the Wulff shape $\partial \mathcal{W}_f$ is

$$\partial \mathcal{W}_f = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{y}(\theta) = -f(\theta)\boldsymbol{n}(\theta) + f'(\theta)\boldsymbol{t}(\theta), \quad \theta \in \boldsymbol{T}\},$$

and then its curvature is $\kappa_{\mathcal{W}} = -\langle \boldsymbol{y}_\theta, \boldsymbol{y}_{\theta\theta}^\perp \rangle |\boldsymbol{y}_\theta|^{-3} = (f + f'')^{-1}$.

Incidentally, if the interfacial energy is a spatially inhomogeneous $f = f(\boldsymbol{x}) > 0$, then the gradient flow of total interfacial energy provides the anisotropic and advected curvature flow: $v = f(\boldsymbol{x})\kappa - \langle \nabla f(\boldsymbol{x}), \boldsymbol{n} \rangle$. See [NMHS].

B Discrete curvature

Suppose a subarc of a curve Γ, say Γ_{sub}, is Gauss-parametrized and strictly convex as follows:

$$\Gamma_{\mathrm{sub}} = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{x}(\theta), \ \ \theta \in [\theta_{j-1}, \theta_{j+1}], \ \ \theta_{j-1} < \theta_j < \theta_{j+1}\}.$$

We define a part of circumscribed piecewise linear curve, say $\mathcal{P}_{\mathrm{sub}}$, of Γ_{sub} such as

$$\Gamma_{\mathrm{sub}} \cap \mathcal{P}_{\mathrm{sub}} = \{\boldsymbol{x}(\theta_{j-1}), \boldsymbol{x}(\theta_j), \boldsymbol{x}(\theta_{j+1})\}.$$

See Figure 1.

We call the side including $\boldsymbol{x}(\theta_j)$ of $\mathcal{P}_{\mathrm{sub}}$ the jth side. The length of the jth side is denoted by d_j. The jth side is a part of tangent line which has the orientation $\boldsymbol{t}(\theta_j) = {}^t(-\sin\theta_j, \cos\theta_j)$ since the inward normal at $\boldsymbol{x}(\theta_j)$ is $\boldsymbol{n}(\theta_j)$. We note that the transition number is $\chi_j = +1$.

Let $\kappa(\theta_j)$ be the curvature at $\boldsymbol{x}(\theta_j) \in \Gamma_{\mathrm{sub}}$ and $\kappa_j = \gamma_j / d_j$ the discrete curvature defined on the jth side of $\mathcal{P}_{\mathrm{sub}}$.

The relation between κ_j and $\kappa(\theta_j)$ is calculated as follows (cf. section 3 in [Gir]). First, we decompose the length of the jth side as $d_j = d_j^+ + d_j^-$ (see Figure 1). Next, we obtain

$$d_j^+ = \frac{1}{\kappa(\theta_j)} \left(\frac{\Delta\theta_{j+1}}{2} - \frac{(\Delta\theta_{j+1})^2}{6} \frac{\kappa'(\theta_j)}{\kappa(\theta_j)} + \mathrm{O}\left((\Delta\theta_{j+1})^3\right) \right)$$

by the Taylor expansion of

$$\boldsymbol{x}(\theta_{j+1}) - \boldsymbol{x}(\theta_j) = \int_{\theta_j}^{\theta_{j+1}} \frac{\boldsymbol{t}(\theta)}{\kappa(\theta)} d\theta = \int_0^{\Delta\theta_{j+1}} \frac{\boldsymbol{t}(\theta_j + \mu)}{\kappa(\theta_j + \mu)} d\mu$$

around θ_j and the decomposition:

$$d_j^+ = \langle \boldsymbol{x}(\theta_{j+1}) - \boldsymbol{x}(\theta_j), \boldsymbol{t}_j - \cot \Delta\theta_{j+1} \boldsymbol{n}_j \rangle.$$

In the same way, we obtain

$$d_j^- = \frac{1}{\kappa(\theta_j)} \left(\frac{\Delta\theta_j}{2} + \frac{(\Delta\theta_j)^2}{6} \frac{\kappa'(\theta_j)}{\kappa(\theta_j)} + \mathrm{O}\left((\Delta\theta_j)^3\right) \right).$$

Therefore we have

$$\kappa_j = \frac{\gamma_j}{d_j^+ + d_j^-} = \kappa(\theta_j) + \frac{\kappa'(\theta_j)}{3}(\Delta\theta_{j+1} - \Delta\theta_j) + \mathrm{O}\left((\Delta\theta_{\max})^2\right) \tag{B.1}$$

since

$$\begin{aligned} \gamma_j &= \tan\frac{\Delta\theta_{j+1}}{2} + \tan\frac{\Delta\theta_j}{2} \\ &= \frac{\Delta\theta_{j+1} + \Delta\theta_j}{2} + \frac{(\Delta\theta_{j+1})^3 + (\Delta\theta_j)^3}{24} + \mathrm{O}\left((\Delta\theta_{j+1})^5 + (\Delta\theta_j)^5\right) \end{aligned}$$

holds. Here $\Delta\theta_{\max} = \max\{\Delta\theta_{j+1}, \Delta\theta_j\}$ and $\mathrm{O}(\cdot)$ in equation (B.1) depends on

$$\sum_{1\le \ell \le 2} \max_{\theta\in[\theta_{j-1},\theta_{j+1}]} \left| \frac{d^\ell}{d\theta^\ell}\kappa(\theta) \right| \quad \text{and} \quad \min_{\theta\in[\theta_{j-1},\theta_{j+1}]} \kappa(\theta).$$

Hence, it is reasonable to use $\mathcal{N}_\star$, i.e. $\Delta\theta_j \equiv \Delta\theta$ as an admissible normal set from a numerical point of view.

Furthermore, if $\Delta\theta_j \equiv \Delta\theta$, then we have the relation $1/\kappa_j = \chi_j \times d_j/2\tan(\Delta\theta/2)$, that is,

$$\begin{aligned} & 1/\text{discrete curvature} \\ = \ & \chi_j \times \text{radius of the largest (inscribed circle of) inscribed regular polygon.} \end{aligned}$$

This relation is a discretized version of the inverse of usual curvature:

$$1/\text{curvature} = \text{sign} \times \text{radius of the largest inscribed circle.}$$

See Figure 2.

C Crystalline curvature

Let f be a crystalline energy. Then the Wulff shape $\mathcal{W}_f$ is a polygon and the distance between the origin and the jth side (which has the orientation $\boldsymbol{n}_j \in \mathcal{N}$) is f_j (see Figure 3).

If we decompose the length of the jth side such as $l(\boldsymbol{n}_j) = l_j^+ + l_j^-$ (see Figure 3 again), then we get $l(\boldsymbol{n}_j) = \gamma_j(f + \Delta_\theta f)_j$ since

$$\begin{aligned} l_j^+ &= f_{j+1}\sin\Delta\theta_{j+1} - \frac{f_j - f_{j+1}\cos\Delta\theta_{j+1}}{\tan\Delta\theta_{j+1}}, \quad \text{and} \\ l_j^- &= f_{j-1}\sin\Delta\theta_j - \frac{f_j - f_{j-1}\cos\Delta\theta_j}{\tan\Delta\theta_j} \end{aligned}$$

hold (awake around equation (1.2) again).

The discrete curvature of the polygon $\partial\mathcal{W}_f$ is given as $\gamma_j/l(\boldsymbol{n}_j) = (f + \Delta_\theta f)_j^{-1}$. Hence the crystalline curvature $\omega_j(t)$ is

$$\omega_j(t) = \frac{\text{discrete curvature of } \mathcal{P}_t}{\text{discrete curvature of polygon } \partial\mathcal{W}_f} = \frac{\kappa_j(t)}{(f + \Delta_\theta f)_j^{-1}} = (f + \Delta_\theta f)_j \kappa_j(t).$$

This is a discrete version of weighted curvature $\omega(\theta, t)$:

$$\omega(\theta, t) = \frac{\text{curvature of } \Gamma_t}{\text{curvature of } \partial\mathcal{W}_f} = \frac{\kappa(\theta, t)}{(f(\theta) + f''(\theta))^{-1}} = (f(\theta) + f''(\theta))\kappa(\theta, t)$$

at the point (θ_j, t) if f is smooth. Namely, the crystalline curvature is discrete weighted curvature.

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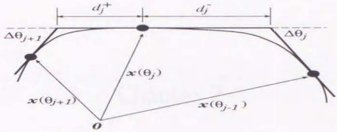

Figure 1: $\mathcal{P}_{\rm sub}$ (outside: the part of circumscribed piecewise linear curve of $\Gamma_{\rm sub}$), and $\Gamma_{\rm sub}$ (inside: the subarc of Γ).

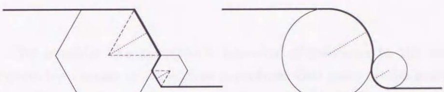

Figure 2: Symbolic figure to compare the discrete curvature and the usual curvature. Thick solid = piecewise linear curve $\mathcal{P}$ (left) and curve Γ (right), Solid = the largest inscribed polygon (left), and the largest inscribed circle (right), Dashed = radius (both), Long dashed = half of diagonal (left).

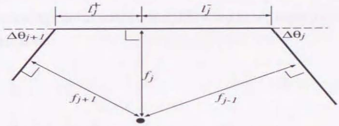

Figure 3: The jth side of the Wulff shape $\mathcal{W}_f$ if f is a crystalline energy.

Chapter 2

Existence of self-similar solutions to an anisotropic motion by a power of crystalline curvature

Outline: We consider the asymptotic behavior of solutions to the motion of polygonal curves by a power of crystalline curvature. Our main results state that if the power is sufficiently large, then any solution polygon shrinks to a single point in a finite time with its shape approaching a self-similar solution. On the other hand, if the power is very small, this point-extinction result does not hold. More precisely, we present examples in which the solution polygons collapses to a line segment of positive length in a finite time. We also give upper and lower bounds of the "extinction time" (when a solution polygon shrinks to a point) or the "collapse time" (when some of the sides of the polygon vanishes).
Key Words: crystalline motion, crystalline curvature, motion by a power of curvature, curve-shortening, point-extinction, blow-up rate, asymptotic self-similar, estimates of blow-up time.

1 Introduction and main results

This chapter is concerned with the asymptotic behavior of solutions to a motion of polygonal curves in the plane. A typical example of motion of smooth curves by its curvature is the classical curve-shortening equation $v = \kappa$: the normal velocity v of the curve is proportional to its curvature κ. One of the specific features of the classical curve-shortening motion is that any Jordan curve shrinks to a single point approaching a shrinking circle. See Gage–Hamilton [GH] and Grayson [Gry]. We call this property the asymptotic self-similarity since any shrinking circle is self-similar, i.e. homothetically shrinking solution. Gage–Li [GL] and Dohmen–Giga–Mizoguchi [DGM] show the asymptotic self-similarity for an anisotropic motion by curvature $v = g(\boldsymbol{n})\kappa$. Here $g(\cdot)$ is a smooth (in [GL]) or bounded (in [DGM]) function defined on S^1, assumed π-periodicity, and $\boldsymbol{n}$ is the normal vector of the curvature.

Such an equation arises in a model describing the motion of interface separating two materials in an anisotropic medium (see Gurtin [Gu1]). Actually, if the interfacial energy is smooth, and isotropic (resp., anisotropic), then a gradient flow of total interfacial energy (with a suitable metric) leads to the motion by curvature $v = \kappa$ (resp., the anisotropic motion $v = g(\boldsymbol{n})\kappa$). See, e.g., Elliott [E].

Besides the smooth interfacial energy, some materials have a nonsmooth interfacial energy, for instance, crystalline. In this case, we do not calculate the gradient flow of total interfacial energy in the usual sense. For such an energy, Angenent–Gurtin [AGu], and correspondingly Taylor [T1] introduced a weak formulation: the motion of admissible piecewise linear curves by crystalline curvature. See also Taylor [T2], [T3] and Gurtin [Gu1]. This motion is called the crystalline flow, or the crystalline motion. The crystalline flow and its application have been investigated by many authors. See, e.g., [TCH, GirK2, GMHG1, Gu2] for including a survey, and [AlT, GGu, GMHG2, GGuM, Ry, GP, Y] for an application and development. For a perspective application including numerical approximation of crystalline flow, we refer to [GirK1, FG, GMHG3, GMHG5, EGS, Gir, UY, IS, E].

The asymptotic behavior of solutions to a crystalline flow is investigated by Stancu [S1, S2, S3]. The author showed the asymptotic self-similarity in [S3], and the uniqueness of self-similar solution, under a symmetric assumption, in [S1].

The aim of this chapter is to extend the result of Stancu [S3] to the motion of polygonal curves by a power of crystalline curvature. This generalization is already introduced by Andrews [And] and Taniyama–Matano [TMa] for the motion of curves by a power of curvature. The above two papers show that there is the critical power $1/3$ such that if a power is less than $1/3$, then there exists the case where the isotropic ratio diverges to infinity in a finite time, or the so-called type II blow-up occurs (see Remark 4.3). We also refer to Sapiro–Tannenbaum [SaTa] for the motion of curves in the case where the power is $1/3$.

Our main results say that if the power is sufficiently large, then a solution polygon shrinks to a single point approaching a self-similar solution in a finite time. We present some examples which show that this point-extinction result does not necessarily hold in the case where the power is small enough. We also give upper and lower bounds of the extinction time.

We now introduce our problem setting. Let $\mathcal{P}_0$ be a convex closed polygon in the plane $\boldsymbol{R}^2$ with the angle between two adjacent sides of $\mathcal{P}_0$ is $\pi - \Delta\theta$ where

$$\Delta\theta := \frac{2\pi}{n}, \quad n \geq 4,$$

and n is a number of sides. We consider the evolution problem to find a family of polygons $\{\mathcal{P}(t)\}_{0\le t<T}$ satisfying

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_j(t) = v_j(t)\boldsymbol{n}_j, \quad 0 \le j < n, \quad 0 \le t < T, \\ \mathcal{P}(0) = \mathcal{P}_0 \end{cases} \tag{1.1a}$$

for some $T > 0$, where $\boldsymbol{n}_j := -{}^t(\cos\theta_j, \sin\theta_j)$ is the inward normal vector of the jth side, $\boldsymbol{x}_j$ is the position vector of the point of intersection between on the line containing the jth side and the line spanned by $\boldsymbol{n}_j$ and v_j is the inward normal velocity of the jth side. Here and hereafter we denote $\theta_j = j\Delta\theta$. We note that the angle between two adjacent sides of $\mathcal{P}(t)$ is always $\pi - \Delta\theta$ as long as solution polygons exist.

In this chapter we consider the case where the normal velocity is homogeneous of some degree $\alpha > 0$ in the crystalline curvature:

$$v_j(t) = g(\boldsymbol{n}_j)\kappa_j(t)^\alpha, \quad 0 \le j < n \tag{1.1b}$$

where g is a positive smooth function defined on S^1 and κ_j is the crystalline curvature:

$$\kappa_j(t) = \frac{2\tan(\Delta\theta/2)}{d_j(t)}, \quad 0 \le j < n. \tag{1.1c}$$

Here $d_j(t)$ is the length of the jth side of polygon $\mathcal{P}(t)$.

Our main results are the following. Let $\mathcal{L}(t)$ and $\mathcal{A}(t)$ be the length and the enclosed area of $\mathcal{P}(t)$, respectively.

Theorem A (finite time blow-up) *Let $\alpha > 0$ and $n \ge 4$. There exists a finite time, say $T_\star > 0$ such that the maximum solution v of Problem (1.1) blows up to infinity as $t \nearrow T_\star$. This number $T_\star$ satisfies $T_\star \le \min\{T_1, T_2\}$ where*

$$T_1 := \frac{1}{(\alpha+1)\min_{0\le j<n} g(\boldsymbol{n}_j)} \left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)}\right)^{\alpha+1}, \quad \textit{and}$$
$$T_2 := \frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)\min_{0\le j<n} v_j(0)}.$$

Remark 1.1 We call $T_\star$ the "extinction time," or the "blow-up time".

Theorem B (point-extinction) *Assume $\alpha \ge 1$ and $n \ge 4$. Then any solution polygon shrinks to a single point in finite time $T_\star$. No side of the polygon vanishes before t reaches $T_\star$.*

Remark 1.2 (non point-extinction) When $g \equiv 1$, regular polygons shrink to a single point in finite time for any $\alpha > 0$ (see Remark 3.3). However, if a polygon is not regular and α is sufficiently small, then the polygon does not necessarily shrink to a single point but may collapse to, say, a line segment (see section 4.3 and Figure 2).

Theorem C (lower bound of the blow-up time) *Let $\alpha > 0$ and $n \geq 4$. If the enclosed area $\mathcal{A}(t)$ converges to zero as t approaches $T_\star$, then the following lower bounds hold: $T_\star \geq \max\{T_\ell,\ T_r\}$, where*

$$T_\ell = \left(\frac{\alpha+1}{\alpha \max_{0\leq j<n} g(\boldsymbol{n}_j)}\right)^{\alpha} \left(\frac{\mathcal{A}(0)}{\mathcal{L}(0)}\right)^{\alpha+1} \quad \textit{and} \quad T_r = \frac{\min_{0\leq j<n} g(\boldsymbol{n}_j)^{1/\alpha}}{(\alpha+1)\max_{0\leq j<n} v_j(0)^{1+1/\alpha}}.$$

Remark 1.3 We remark that the order of T_1 and T_2 (resp., T_ℓ and T_r) depend on the initial condition $\mathcal{P}_0$ and the weight function g. We denote $\max_{0\leq j<n}(\cdot)_j$, $\min_{0\leq j<n}(\cdot)_j$ and $\sum_{0\leq j<n}(\cdot)_j/n$, by $(\cdot)_{\max}$, $(\cdot)_{\min}$ and $(\cdot)_{\mathrm{av}}$, respectively.

– If the initial polygon $\mathcal{P}_0$ is a regular polygon, then we put $\kappa_j(0) \equiv \kappa(0)$ and have

$$T_\star \leq T_1 = ((\alpha+1)g_{\min}\kappa(0)^{\alpha+1})^{-1} < T_2 = (\alpha+1)T_1,$$
$$T_\star \geq T_r = (g_{\min}/g_{\max})^{1+1/\alpha}T_1, \quad T_\ell = (g_{\min}/g_{\max}^{\alpha})((\alpha+1)/2)^{\alpha+1}\alpha^{-\alpha}T_1;$$

– if $g \equiv c > 0$, then $T_\star = ((\alpha+1)c\kappa(0)^{\alpha+1})^{-1}$ and

$$T_\ell \leq T_r = T_\star = T_1 < T_2;$$

Equality $T_\ell = T_r$ holds iff $c^{1-\alpha}((\alpha+1)/2)^{\alpha+1}\alpha^{-\alpha} = 1$ (e.g. $\alpha = c = 1$);

– if $g \not\equiv c$ and $\alpha = 1$, then $T_\star = (2g_{\mathrm{av}}\kappa(0)^2)^{-1}$ and

$$T_r < T_\ell = (2g_{\max}\kappa(0)^2)^{-1} < T_\star < T_1 = (2g_{\min}\kappa(0)^2)^{-1} < T_2.$$

– If $\mathcal{P}_0$ is not necessarily a regular polygon and if $g_j = c\kappa_j(0)^{-\alpha}$, then

$$T_1 = \kappa_{\max}(0)^{\alpha}(1/\kappa(0))_{\mathrm{av}}^{\alpha+1}/(\alpha+1)/c, \quad \text{and} \quad T_2 = (1/\kappa(0))_{\mathrm{av}}/c;$$

Inequality $T_1 \geq T_2$ holds iff $(\kappa_{\max}(0)(1/\kappa(0))_{\mathrm{av}})^{\alpha} \geq (\alpha+1)$.

Theorem D (type I blow-up) *Assume $\alpha \geq 1$ and $n \geq 4$. Then a solution v of Problem* (1.1) *diverges to infinity in finite time $T_\star$ with at most the self-similar rate. That is, there exist $t_0 \in [0, T_\star)$ and a positive constant C which depends only on α, g, $\Delta\theta$, $v(0)$ and $v(t_0)$ such that*

$$v_j(t) \leq C(T_\star - t)^{-\alpha/(\alpha+1)}, \quad 0 \leq j < n, \quad t_0 \leq t < T_\star.$$

Remark 1.4 (type II blow-up) When the above inequality holds, we call a solution v which undergoes the "type I blow-up." There is the case that the maximum of v diverges to infinity faster than the type I blow-up rate when α is small enough. We call this type of blow-up the "type II blow-up." See Definition 3.4 and section 4.3.

Theorem E (asymptotic self-similarity) *Assume $\alpha > 0$ and $n \geq 4$. If the solution polygon shrinks to a single point and a solution v undergoes the type* I *blow-up, then any solution polygon approaches a shrinking self-similar solution in the sense that the rescaled polygon approaches the set of stationary solutions of Problem* 2' *in section* 3 *with respect to the Hausdorff metric.*

See section 3 for the definition of the rescaling.

Corollary E.1 (existence of self-similar solutions) *Under the same assumption of Theorem* E, *there exist self-similar solutions to Problem* (1.1).

Remark 1.5 (non uniqueness of self-similar solutions) In the case where $\alpha < 1$ and $g \equiv 1$, we have self-similar solutions except the regular n-gon. For instance, an "almost regular" $(n/2)$-gon is a self-similar solution for n even. See section 4.2 and Figure 1.

The organization of this chapter is as follows: in section 2, we present fundamental properties of solutions, and give a proof of Theorem A. In section 3, the asymptotic behavior of solutions is given, and Theorems B, C, E and D are proved by using entropy estimates and a rescaling technique. In the last section 4, we discuss a stability of the regular polygon solutions, and give self-similar solutions except the regular polygons. We also present some examples of type II blow-up and non point-extinction.

2 Fundamental properties of solutions

Throughout this chapter we use the notation $\sum_j u_j$, $u_{\max}$, $u_{\min}$ and $\dot{u}(t)$ for $\sum_{0\leq j<n} u_j$, $\max_{0\leq j<n} u_j$, $\min_{0\leq j<n} u_j$ and $du(t)/dt$, respectively. Hereafter we denote $g_j := g(\boldsymbol{n}_j)$ for simplicity, and assume $n \geq 4$. We note again $\theta_j = j\Delta\theta$.

2.1 Equivalent formulation

Let $\mathcal{P}(t)$ be a solution of Problem (1.1). The jth vertex $\boldsymbol{B}_j(t)$ of $\mathcal{P}(t)$ is given as the following:

$$\begin{aligned} \boldsymbol{B}_j(t) &= \langle \boldsymbol{x}_{j-1}(t) - \boldsymbol{x}_j(t), \boldsymbol{t}_j + \boldsymbol{n}_j \cot\Delta\theta \rangle \, \boldsymbol{t}_j + \boldsymbol{x}_j(t), \\ &= \boldsymbol{B}_0(t) + \sum_{0\leq m<j} d_m(t)\boldsymbol{t}_m, \quad 1 \leq j \leq n, \quad 0 \leq t < T \end{aligned} \tag{2.1}$$

with $\boldsymbol{B}_0(t) \equiv \boldsymbol{B}_n(t)$, where $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j)$ is the tangent vector, since the position vector $\boldsymbol{x}_j$ is on the line containing the jth side (n.b. $\boldsymbol{x}_j$ is not necessarily on the jth side), and $\langle \cdot, \cdot \rangle$ is the usual inner product. Then the time evolution of the length of the jth side $d_j(t)$ is given as the following (cf. Figure 10C in [AGu]):

$$\frac{d}{dt}d_j(t) = \frac{d}{dt}|\boldsymbol{B}_{j+1}(t) - \boldsymbol{B}_j(t)| = -2\tan\frac{\Delta\theta}{2}(\Delta_\theta\, v + v)_j. \tag{2.2}$$

Here the operator Δ_θ is defined as

$$(\Delta_\theta\,(\cdot))_j := \frac{(\cdot)_{j+1} - 2(\cdot)_j + (\cdot)_{j-1}}{2(1-\cos\Delta\theta)}$$

which is a kind of central difference operator. Then we obtain a discretized version of the equation (2.20) in the book [Gu1]:

$$\frac{d}{dt}\kappa_j(t) = \kappa_j^2(\Delta_\theta\, v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T.$$

Therefore we can restate Problem (1.1) as follows.

Problem 1 *Let $n \ge 4$. Find a function $v(t) = (v_0, v_1, ..., v_{n-1}) \in [C^1(0,T)]^n$ for some $T > 0$ satisfying*

$$\frac{d}{dt}v_j(t) = \alpha g_j^{-1/\alpha} v_j^{1+1/\alpha}(\Delta_\theta\, v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T, \tag{2.3a}$$

$$v_j(0) = g(\boldsymbol{n}_j)\kappa_j(0)^\alpha, \quad 0 \le j < n, \tag{2.3b}$$

$$v_{-1}(t) = v_{n-1}(t), \quad v_n(t) = v_0(t), \quad 0 \le t < T, \tag{2.3c}$$

where $\kappa_j(0)$ is the initial crystalline curvature of $\mathcal{P}_0$.

Remark 2.1 (eigenvalue and eigenvector) The kth eigenvalue and the kth eigenvector for the difference operator $-\Delta_\theta$ (under the periodic boundary condition (2.3c)) are

$$\lambda_k = \frac{1-\cos(k\Delta\theta)}{1-\cos\Delta\theta} = \frac{\sin^2(k\Delta\theta/2)}{\sin^2(\Delta\theta/2)}, \quad 0 \le k < n, \quad \text{and}$$

$$\psi_j^k = C_1\cos(k\theta_j) + C_2\sin(k\theta_j), \quad 0 \le j < n, \quad 0 \le k < n$$

for any constants $(C_1, C_2) \ne (0,0)$, respectively. In particular, $(\Delta_\theta\, \boldsymbol{t})_j = -\boldsymbol{t}_j$ holds since $\lambda_1 = 1$.

Remark 2.2 (equivalence) Problem (1.1) and Problem 1 are equivalent except the indefiniteness of position of the polygon. Indeed if we suppose $v(t)$ is a solution of Problem 1, then we have

$$\frac{1}{2\tan(\Delta\theta/2)}\frac{d}{dt}\sum_j \frac{g_j^{1/\alpha}}{v_j(t)^{1/\alpha}}\boldsymbol{t}_j = -\sum_j(\Delta_\theta\, v + v)_j\boldsymbol{t}_j = -\sum_j(\Delta_\theta\, \boldsymbol{t} + \boldsymbol{t})_j v_j = \boldsymbol{0}.$$

Here we have used the relation of summation by parts:

$$\sum_j f_j(\Delta_\theta\, g)_j = -\sum_j(\mathrm{D}_+\, f)_j(\mathrm{D}_+\, g)_j = \sum_j g_j(\Delta_\theta\, f)_j, \tag{2.4}$$

and the relation $(\Delta_\theta \boldsymbol{t})_j = -\boldsymbol{t}_j$. The operator D_+ is defined as

$$(\mathrm{D}_+ f)_j := \frac{f_{j+1} - f_j}{2\sin(\Delta\theta/2)}. \tag{2.5}$$

Hence by equation (2.1) we can construct a closed convex n-gon, whose length of the jth side is $2g_j^{1/\alpha}\tan(\Delta\theta/2)v_j(t)^{-1/\alpha} =: d_j(t)$ and the jth normal vector is $\boldsymbol{n}_j$, as long as $v(t)$ is a solution of Problem 1. This n-gon is the very solution polygon of Problem (1.1).

2.2 Comparison principle

We present the comparison principle and its application in this subsection.

Lemma 2.3 *Fix $T > 0$. Let $(p_j(t))_{0\le j<n} > 0$ and $(q_j(t))_{0\le j<n}$ be defined on $t \in [0,T]$. If $u = (u_j(t))_{0\le j<n} \in [C^1(0,T)\cap C[0,T]]^n$ is a solution of*

$$\begin{cases} \dfrac{d}{dt}u_j \ge p_j(\Delta_\theta u)_j + q_j u_j, \quad 0 \le j < n, \quad 0 < t < T, \\ u_{-1}(t) = u_{n-1}(t), \quad u_n(t) = u_0(t), \quad 0 \le t \le T, \\ u_j(0) \ge 0, \quad 0 \le j < n, \end{cases}$$

then $u_j(t) \ge 0$ holds for $0 \le j < n$ and $0 \le t \le T$.

The proof is done by a similar argument to the one for the maximum principle. See, e.g., [Y].

As an application of the above lemma, we obtain the next:

Lemma 2.4 *Let $\alpha > 0$. For the solution v of Problem 1, we have the followings.*

(1) If $v_j(0) \ge \psi_j^k$, then $v_j(t) \ge \psi_j^k$ for $k = 1,2$. In particular, $v_{\min}(0)$ is the lower bound of $v(t)$. Here ψ_j^k is the kth eigenvector of $-\Delta_\theta$.

(2) If $v_j(0) \le v_r(0)$, then $v_j(t) \le v_r(t)$ where $v_r(t)$ is the solution of

$$\frac{d}{dt}v_r(t) = \frac{\alpha v_r(t)^{2+1/\alpha}}{g_{\min}^{1/\alpha}}, \quad v_r(0) = v_{\max}(0),$$

that is, $v_r(t) = ((\alpha+1)g_{\min}^{-1/\alpha}(T_r - t))^{-\alpha/(\alpha+1)}$. Here the blow-up time T_r has been defined in Theorem C.

Proof. For each proposition, put (1) $u_j = v_j - \psi_j^k$; (2) $u_j = v_r - v_j$; and apply Lemma 2.3. □

2.3 The length and the area

The (total) length of solution polygon $\mathcal{P}(t)$ is

$$\mathcal{L}(t) := \sum_j d_j = 2\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha} v_j^{-1/\alpha}, \tag{2.6}$$

and the rate of change of $\mathcal{L}(t)$ can be computed by

$$\dot{\mathcal{L}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j v_j(t). \tag{2.7}$$

Since $\dot{\mathcal{L}}(t) < 0$, motion of solution polygons is discretized curve-shortening.

The enclosed area of solution polygon $P(t)$ is

$$\mathcal{A}(t) := -\frac{1}{2}\sum_j \langle \boldsymbol{x}_j(t), \boldsymbol{n}_j\rangle d_j(t) = -\tan\frac{\Delta\theta}{2}\sum_j \langle \boldsymbol{x}_j(t), \boldsymbol{n}_j\rangle g_j^{1/\alpha} v_j(t)^{-1/\alpha},$$

and the rate of change of $\mathcal{A}(t)$ can be computed by

$$\dot{\mathcal{A}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha} v_j^{1-1/\alpha}. \tag{2.8}$$

Here we use equations (2.2) and (2.4), definition $\langle \dot{\boldsymbol{x}}_j, \boldsymbol{n}_j\rangle = v_j$ and geometric relation $d_j = -2\tan(\Delta\theta/2)(\Delta_\theta \langle \boldsymbol{x}, \boldsymbol{n}\rangle + \langle \boldsymbol{x}, \boldsymbol{n}\rangle)_j$.

2.4 Finite time blow-up (proof of Theorem A)

Proof of Theorem A. Since $n = \sum_j 1 = \sum_j (g_j^{-1}v_j)^{-1/(\alpha+1)}(g_j^{-1}v_j)^{1/(\alpha+1)}$, Hölder's inequality, equations (2.6) and (2.7) yield

$$\begin{aligned} 2n\tan\frac{\Delta\theta}{2} &\le 2\tan\frac{\Delta\theta}{2}\left(\sum_j (g_j v_j^{-1})^{1/\alpha}\right)^{\alpha/(\alpha+1)}\left(\sum_j g_j^{-1}v_j\right)^{1/(\alpha+1)} \\ &\le g_{\min}^{-1/(\alpha+1)}\mathcal{L}(t)^{\alpha/(\alpha+1)}(-\dot{\mathcal{L}}(t))^{1/(\alpha+1)}, \end{aligned}$$

and then we have

$$\frac{d}{dt}\mathcal{L}(t)^{\alpha+1} \le -(\alpha+1)g_{\min}\left(2n\tan\frac{\Delta\theta}{2}\right)^{\alpha+1}.$$

By the general argument for ordinary differential equation, solution v of Problem (1.1) or Problem 1 exists uniquely, and locally in time. Put $T_\star > 0$ such as maximal existing time. Take $0 < t < T_\star$. Then we obtain

$$\mathcal{L}(t) \le 2n\tan\frac{\Delta\theta}{2}\left(\left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)}\right)^{\alpha+1} - (\alpha+1)g_{\min}t\right)^{1/(\alpha+1)}.$$

Since $\mathcal{L}(t) \geq 2n\tan(\Delta\theta/2) g_{\min}^{1/\alpha} v_{\max}^{-1/\alpha}$, we have

$$v_{\max} \geq (\alpha+1)^{-\alpha/(\alpha+1)} g_{\min}^{1/(\alpha+1)} \left(\frac{1}{(\alpha+1) g_{\min}} \left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} \right)^{\alpha+1} - t \right)^{-\alpha/(\alpha+1)},$$

and obtain T_1. By Lemma 2.4 (1) and equation (2.7), we have

$$\mathcal{L}(t) \leq \mathcal{L}(0) - 2n\tan(\Delta\theta/2) v_{\min}(0) t,$$

and obtain T_2. $\square$

Remark 2.5 *Let $\alpha > 0$ and $g \equiv 1$. For any k such as the eigenvalue $\lambda_k < 1$, we have*

$$\frac{d}{dt} \sum_j \frac{\psi_j^k}{v_j^{1/\alpha}} = -\sum_j \psi_j^k (\Delta_\theta v + v)_j = -(1-\lambda_k) \sum_j \psi_j^k v_j \leq -(1-\lambda_k) \sum_j (\psi_j^k)^2 < 0.$$

Integration over $(0,t)$ yields $t \leq \sum_j \psi_j^k v_j(0)^{-1/\alpha} \left((1-\lambda_k) \sum_j (\psi_j^k)^2 \right)^{-1} < \infty$. This gives another proof of the finite time blow-up when $g \equiv 1$.

3 Asymptotic behavior of solutions when $\alpha \geq 1$

Here we will prove the rest theorems by using entropy estimates and a rescaling technique.

3.1 Self-similar rescaling and the blow-up rate

We magnify $\boldsymbol{R}^2$ by

$$h(t) = ((\alpha+1)(T_\star - t))^{-1/(\alpha+1)}, \tag{3.1}$$

and replace the time t by

$$\tau = \tau(t) = -\frac{1}{\alpha+1} \log\left(\frac{T_\star - t}{T_\star} \right) : \ [0, T_\star) \ni t \mapsto \tau \in [0, \infty).$$

Then the rescaled velocity is given by

$$\bar{v}_j(\tau) = h(t)^{-\alpha} v_j(t), \quad 0 \leq j < n,$$

and Problem 1 is transformed into the following rescaled problem:

Problem 2 *Let $\alpha > 0$. Find a function $\bar{v}(\tau) = (\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1}) \in [C^1(\boldsymbol{R}_+)]^n$ satisfying*

$$\frac{d}{d\tau} \bar{v}_j(\tau) = \alpha g_j^{-1/\alpha} \bar{v}_j^{1+1/\alpha} (\Delta_\theta \bar{v} + \bar{v} - g^{1/\alpha} \bar{v}^{-1/\alpha})_j, \quad 0 \leq j < n, \quad \tau \geq 0, \tag{3.2a}$$

$$\bar{v}_j(0) = ((\alpha+1)T_\star)^{\alpha/(\alpha+1)} v_j(0), \quad 0 \leq j < n, \tag{3.2b}$$

$$\bar{v}_{-1}(\tau) = \bar{v}_{n-1}(\tau), \quad \bar{v}_n(\tau) = \bar{v}_0(\tau), \quad \tau \geq 0. \tag{3.2c}$$

The stationary problem of Problem 2 is stated as follows:

Problem 2' *Let $\alpha > 0$. Find a positive vector $\bar{v} = (\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1}) \in \boldsymbol{R}^n_+$ satisfying*

$$(\Delta_\theta \bar{v} + \bar{v} - g^{1/\alpha}\bar{v}^{-1/\alpha})_j = 0, \quad 0 \le j < n, \tag{3.3a}$$

$$\bar{v}_{-1} = \bar{v}_{n-1}, \quad \bar{v}_n = \bar{v}_0. \tag{3.3b}$$

Now we define the term "self-similar":

Definition 3.1 (self-similar) *We call a solution of Problem 2' a "self-similar solution." Moreover, we call a solution polygon $\mathcal{P}(t)$ a "self-similar" if and only if there exists a function $l(t)$ and a fixed polygon $\bar{\mathcal{P}}$ such that $\mathcal{P}(t) = l(t)\bar{\mathcal{P}} + \bar{\boldsymbol{x}}$, where $\bar{\boldsymbol{x}}$ is a fixed point in $\boldsymbol{R}^2$. In our case, $l(t) = h(t)^{-1}$.*

Remark 3.2 If a solution polygon $\mathcal{P}(t)$ is a self-similar, then a fixed polygon $\bar{\mathcal{P}}$ is a solution of Problem 2' in the sense of Remark 2.2.

Remark 3.3 (regular polygon) When $g \equiv 1$ and $\bar{v}_j \equiv 1$, i.e. $v_j(t) \equiv v_r(t)$ is a self-similar solution for any $\alpha > 0$:

$$v_r(t) = h(t)^\alpha, \quad \mathcal{A}(t) = n \tan\frac{\Delta\theta}{2} h(t)^{-2}, \quad \mathcal{L}(t) = 2n\tan\frac{\Delta\theta}{2} h(t)^{-1}, \quad 0 \le t < T_\star$$

where $T_\star = \left((\alpha+1)v_r(0)^{1+1/\alpha}\right)^{-1}$. In other words, regular polygons are self-similar when $g \equiv 1$.

Definition 3.4 (blow-up rate) *Let $\alpha > 0$. We call that the solution undergoes the "type I blow-up" when the blow-up rate of the maximum of solution v is at most the self-similar rate, that is,*

$$\lim_{t\to T_\star} \max_{0\le j<n} v_j(t)(T_\star - t)^{\alpha/(\alpha+1)} < \infty,$$

and we call that the solution undergoes the "type II blow-up" when the blow-up rate of the maximum of solution v is faster than the self-similar rate, that is,

$$\lim_{t\to T_\star} \max_{0\le j<n} v_j(t)(T_\star - t)^{\alpha/(\alpha+1)} = \infty.$$

Remark 3.5 If the "point-extinction," and the "type I blow-up" hold, then the solution v_j blows up with the same order of the self-similar rate for all j (see Lemma 3.22).

In order to prove Theorem E, we will use the self-similar rescaling technique (see section 3.6). Remark 3.3 will suggest to consider this strategy. Another reason to do so is that the self-similar rate lies between $v_{\min}$ and $v_{\max}$: the following lemma is the most we can say about the blow-up rate at this stage. It will be clear that the self-similar rate is a dominant rate in seciton 3.5.

Lemma 3.6 *Let $\alpha > 0$. Then it holds that*

$$\min_{0\le j<n} v_j(t) \le \left(\max_{0\le j<n} g_j\right)^{1/(\alpha+1)} ((\alpha+1)(T_\star - t))^{-\alpha/(\alpha+1)},$$

and that

$$\max_{0\le j<n} v_j(t) \ge \left(\min_{0\le j<n} g_j\right)^{1/(\alpha+1)} ((\alpha+1)(T_\star - t))^{-\alpha/(\alpha+1)}.$$

Proof. The following proof is based on the proof of Lemma 2.2 in Stancu [S3].

If $v_{\min}(t) = v_{j_t}(t)$ for some $0 \le j_t < n$, then

$$\frac{d}{dt} v_{j_t}(t) = \alpha g_{j_t}^{-1/\alpha} v_{j_t}^{1+1/\alpha} (\Delta_\theta v + v)_{j_t} \ge \alpha g_{\max}^{-1/\alpha} v_{j_t}^{2+1/\alpha}$$

for any $t \in (0, T_\star)$. The function $v_{\min}(t)$ is a continuous function, but it may not be differentiable. It is, however, Lipschitz. Therefore we have

$$\frac{d^-}{dt} v_{\min}(t) := \liminf_{\varepsilon\uparrow 0} \frac{v_{\min}(t+\varepsilon) - v_{\min}(t)}{\varepsilon} \ge \alpha g_{\max}^{-1/\alpha} v_{\min}^{2+1/\alpha}.$$

Since comparison principle (similar result to Lemma 2.3 for the operator d^-/dt) holds, once $v_{\min}$ is bigger than $w(t)$, the solution of $\dot{w} = \alpha g_{\max}^{-1/\alpha} w^{2+1/\alpha}$, it must stay bigger. This comparison holds when $w(t) = g_{\max}^{1/(\alpha+1)}((\alpha+1)(T_\star - \delta - t))^{-\alpha/(\alpha+1)}$ for any $\delta > 0$ and any t. Then $v_{\min}$ must blow up at $T_\star - \delta$, earlier time than the blow-up time $T_\star$. This is a contradiction. Hence $v_{\min}(t) < g_{\max}^{1/(\alpha+1)}((\alpha+1)(T_\star - \delta - t))^{-\alpha/(\alpha+1)}$ holds and this proves the first assertion.

The second assertion will be proved in a similar way. $\square$

3.2 Entropy estimates

Definition 3.7 *Let the entropy be:*

$$\mathcal{E}(t) := \begin{cases} \dfrac{2\tan(\Delta\theta/2)}{\alpha - 1} \displaystyle\sum_{0\le j<n} g_j^{1/\alpha} v_j(t)^{1-1/\alpha} & \text{if } \alpha > 1; \\ 2\tan\dfrac{\Delta\theta}{2} \displaystyle\sum_{0\le j<n} g_j \log v_j(t) & \text{if } \alpha = 1. \end{cases}$$

The next lemma is a discrete analogue of estimates due to Tsutsumi [Tsu] for the curve-shortening equation.

Lemma 3.8 *Let $\alpha > 1$. Then there exists $t_1 \in [0, T_\star)$ such that*

$$\mathcal{E}(t) \le C(T_\star - t)^{(1-\alpha)/(1+\alpha)}, \quad t_1 \le t < T_\star$$

for some constant $C = C(\alpha, \Delta\theta, g, v(t_1)) > 0$.

Proof. Differentiating of $\mathcal{E}(t)$ and making use of the relation (2.4), we have

$$\dot{\mathcal{E}}(t) = \frac{2}{\alpha}\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha} v_j^{-1/\alpha}\dot{v}_j, \quad \text{and} \quad \ddot{\mathcal{E}}(t) = \frac{4}{\alpha}\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha} v_j^{-(1+1/\alpha)}(\dot{v}_j)^2.$$

Claim: There exists $t_1 \in [0, T_\star)$ such that $\dot{\mathcal{E}}(t_1) > 0$. If not, $\dot{\mathcal{E}}(t) \le 0$ for all $t \in [0, T_\star)$. Then we have $\mathcal{E}(t) \le \mathcal{E}(0)$ for all $t \in [0, T_\star)$ which contradicts that $v_{\max}$ and so $\mathcal{E}(t)$ blows up at time $T_\star$.

From this claim and $\ddot{\mathcal{E}}(t) > 0$ for all $t \in [0, T_\star)$, we see that $\dot{\mathcal{E}}(t) > 0$ for all $t \in [t_1, T_\star)$. Schwarz inequality gives

$$(\dot{\mathcal{E}}(t))^2 \le \frac{\alpha - 1}{2\alpha}\ddot{\mathcal{E}}(t)\mathcal{E}(t), \quad t_1 \le t < T_\star$$

from which it follows that

$$\dot{\mathcal{E}}(t)\mathcal{E}(t)^{2\alpha/(1-\alpha)} \ge \dot{\mathcal{E}}(t_1)\mathcal{E}(t_1)^{2\alpha/(1-\alpha)} > 0, \quad t_1 \le t < T_\star.$$

Hence

$$\frac{d}{dt}\mathcal{E}(t)^{(1+\alpha)/(1-\alpha)} \le \frac{1+\alpha}{1-\alpha}\dot{\mathcal{E}}(t_1)\mathcal{E}(t_1)^{2\alpha/(1-\alpha)}, \quad t_1 \le t < T_\star.$$

Integration over $(t, T_\star)$ yields $\mathcal{E}(t) \le C(T_\star - t)^{(1-\alpha)/(1+\alpha)}$ for a constant $C = C(\alpha, \Delta\theta, g, v(t_1)) > 0$. □

Since $\mathcal{E}(t) \ge 2\tan(\Delta\theta/2) g_j^{1/\alpha} v_j^{1-1/\alpha}/(\alpha - 1)$ for any $0 \le j < n$, we have the next:

Corollary 3.9 *Let $\alpha > 1$. Suppose that t_1 is the same as in Lemma 3.8. Then we have*

$$v_j(t) \le C(T_\star - t)^{-\alpha/(1+\alpha)}, \quad 0 \le j < n, \quad t_1 \le t < T_\star.$$

Lemma 3.10 *Let $\alpha = 1$. Then there exists $t_2 \in [0, T_\star)$ such that*

$$\frac{d}{dt}\mathcal{E}(t) \le \tan\frac{\Delta\theta}{2}\sum_{0 \le j < n} g_j (T_\star - t)^{-1}, \quad t_2 \le t < T_\star.$$

Proof. As in the proof of Lemma 3.8, we have the following estimates:

$$\dot{\mathcal{E}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j g_j v_j^{-1}\dot{v}_j = 2\tan\frac{\Delta\theta}{2}\sum_j v_j(\Delta_\theta v + v)_j,$$

$$\ddot{\mathcal{E}}(t) = 4\tan\frac{\Delta\theta}{2}\sum_j g_j v_j^{-2}(\dot{v}_j)^2 \ge \frac{\dot{\mathcal{E}}^2}{\tan(\Delta\theta/2)\sum_j g_j}.$$

As in the proof of Lemma 3.8, there exists $t_2 \in [0, T_\star)$ such that $\dot{\mathcal{E}}(t_2) > 0$. Then we obtain

$$\frac{\ddot{\mathcal{E}}(t)}{\dot{\mathcal{E}}(t)^2} \ge \frac{1}{\tan(\Delta\theta/2)\sum_j g_j}, \quad t_2 \le t < T_\star.$$

Hence, integration of

$$-\frac{d}{dt}\dot{\mathcal{E}}(t)^{-1} \ge \frac{1}{\tan(\Delta\theta/2)\sum_j g_j}$$

over $(t, T_\star)$ concludes the assertion since $\dot{\mathcal{E}}(t)$ diverges to infinity when t tends to $T_\star$. □

3.3 Point-extinction (proof of Theorem B)

Before presenting the proof of Theorem B, we show the following theorem.

Theorem 3.11 *Let $\alpha \geq 1$. If the area $\mathcal{A}(t)$ is bounded away from zero, then a solution v is uniformly bounded for $t \in [0, T_*)$, where the blow-up time T_* attains $\mathcal{A}(T_*) = 0$.*

Remark 3.12 This theorem does not claim that a solution polygon shrinks to a single point.

We use analogue of several estimates by Gage–Hamilton [GH] for the curvature and by Girão [Gir] for the weighted curvature. For reader's convenience, we do not omit the proofs except the completely the same one. Recall the forward difference (2.5).

Lemma 3.13 *Let $\alpha > 0$. There exists a constant $C_1 = C_1(v(0), \Delta\theta) \geq 0$ such that*

$$2\tan\frac{\Delta\theta}{2}\sum_{0\leq j<n}(\mathrm{D}_+ v)_j^2 \leq 2\tan\frac{\Delta\theta}{2}\sum_{0\leq j<n} v_j^2 + C_1.$$

Proof. It can be shown that the next estimate:

$$2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j(v^2 - (\mathrm{D}_+ v)^2)_j = 4\alpha\tan\frac{\Delta\theta}{2}\sum_j v_j^{1+1/\alpha}(\Delta_\theta v + v)_j^2 \geq 0.$$

By integration of this inequality over $(0, t)$ and putting

$$C_1 \geq \max\left\{-2\tan\frac{\Delta\theta}{2}\sum_j(v(0)^2 - (\mathrm{D}_+ v(0))^2)_j,\ 0\right\},$$

we obtain the assertion. □

One can easily obtain: $\sum_{j=1}^{[n/2]}\sin\theta_j \leq 2\cot(\Delta\theta/2)$, where $[n/2]$ is $n/2$ for n even and $(n-1)/2$ for n odd, since the left hand side equals to $\cot(\Delta\theta/2)$ for n even, and $(1 + \sec(\Delta\theta/2))\cot(\Delta\theta/2)/2$ for n odd.

We introduce the median normal velocity which is a similar to the median curvature in [GH].

Definition 3.14 (median normal velocity) $v_*(t) := \max\limits_{0\leq j<n}\ \min\limits_{j+1\leq i\leq j+[n/2]} v_i(t)$.

Lemma 3.15 *Let $\alpha > 0$ and fix $t \in [0, T_*)$. If $\mathcal{A}(t)$ is bounded away from zero, then $v_*(t)$ is bounded.*

Proof. We assume that j_0 is a value of j which attains the maximum of v_j. A polygon lies between parallel lines whose distance is less than

$$\begin{aligned}\sum_{j=j_0+1}^{j_0+[n/2]} \langle d_j \boldsymbol{t}_j, \boldsymbol{n}_{j_0}\rangle &= \sum_{j=j_0+1}^{j_0+[n/2]} \sin(\theta_j - \theta_{j_0}) d_j = 2\tan\frac{\Delta\theta}{2} \sum_{j=1}^{[n/2]} g_{j_0+j}^{1/\alpha} \sin\theta_j v_{j_0+j}^{-1/\alpha} \\ &\leq \frac{2\tan(\Delta\theta/2) g_{\max}^{1/\alpha}}{v_*^{1/\alpha}} \sum_{j=1}^{[n/2]} \sin\theta_j \leq \frac{4 g_{\max}^{1/\alpha}}{v_*^{1/\alpha}}\end{aligned}$$

The diameter is bounded by $\mathcal{L}/2$ and the area is bounded by the width times the diameter:

$$\mathcal{A}(t) \leq \frac{2\mathcal{L}(t) g_{\max}^{1/\alpha}}{v_*(t)^{1/\alpha}}.$$

Hence $v_*(t) \leq \left(\dfrac{2\mathcal{L}(0) g_{\max}^{1/\alpha}}{\mathcal{A}(t)}\right)^{\alpha}$. □

The next lemma, due to Girão [Gir], is the core of strategy for the proof of Theorem 3.11. One can find the original idea in Gage-Hamilton [GH] for the motion by curvature. Although the proof of the next lemma is almost the same as in [Gir], for reader's convenience we do not omit the proof.

Lemma 3.16 (Girão [Gir], section 2, especially *Fourth***)** *Let $v_0, \ldots, v_{n-1}$ be n positive numbers with $v_n = v_0$ and $v_{-1} = v_{n-1}$. Then it holds that*

$$2\tan\frac{\Delta\theta}{2} \sum_{0\leq j<n} (v^2 - (\mathrm{D}_+ v)^2)_j \leq 2n\tan\frac{\Delta\theta}{2} v_*^2 + 4\tan\frac{\Delta\theta}{2} v_* \sum_{0\leq j<n} v_j$$

where v_* is the median of v defined in Definition 3.14.

To prove this lemma, we will use the following discrete version of Wirtinger's inequality:

Proposition 3.17 (discrete version of Wirtinger's inequality) *Let $f_0, \ldots, f_m$ be $m+1$ real numbers with $f_0 = f_m = 0$. Assume $m \geq 2$. Then we have $\sum_{j=0}^{m-1} (f^2 - (\mathrm{D}_+ f)^2)_j \leq 0$.*

Proof. Let $\Lambda = (\Lambda_{ij})$ be a $(m-1)\times(m-1)$ symmetric matrix such that Λ_{ij} takes 2, -1 and 0 for $j = i$, $j = i \pm 1$ and *otherwise*, respectively. Put $(\Delta_\Lambda f)_j := f_{j+1} - 2f_j + f_{j-1}$. Since the eigenvalue of Λ is $\mu_k = 2(1 - \cos(k\pi/m))$ for $k = 1, \ldots, m-1$ and $\mu_1 \leq \mu_k$, we have $-f_j(\Delta_\Lambda f)_j \geq \mu_1 f_j^2$ and obtain the following inequality ([Gir], equation (9)):

$$\sum_{i=0}^{m-1} f_i^2 \leq \frac{1}{2(1-\cos(\pi/m))} \sum_{i=0}^{m-1} (f_{i+1} - f_i)^2.$$

Note that $2\sin^2(\Delta\theta/2) = 1 - \cos\Delta\theta$ and $2(1-\cos(\pi/m)) \geq 2(1-\cos\Delta\theta)$ if $2 \leq m \leq n/2$. Therefore we obtain the assertion. □

Proof of Lemma 3.16. The set $U := \{i \in \boldsymbol{N} \mid v_i > v_*\}$ can be divided uniquely as the union of a maximal subsets of the form $U_j := \{i_j, i_j+1, \ldots, i_j + m_j - 2\}$. Let

$I_j := \{i_j - 1\} \cup U_j = \{i_j - 1, i_j, \ldots, i_j + m_j - 2\}$, $I := \bigcup_j I_j$ and $J := \boldsymbol{N} \setminus I$. Note that I_j has m_j elements ($m_j \geq 2$), and note also that $m_j - 1 \leq n/2 - 1$, or $m_j - 1 \leq n/2$, because by the definition of v_* there are at most $[n/2] - 1$ v_i's corresponding to adjacent sides and with $v_i > v_*$. We have

$$\sum_{j \in J \cap \{0,1,\ldots,n-1\}} (v^2 - (\mathrm{D}_+ v)^2)_j \leq \sum_{j \in J \cap \{0,1,\ldots,n-1\}} v_*^2 \leq n v_*^2.$$

By Proposition 3.17 with $m = m_j \geq 2$, $f_\ell = v_{i_j+\ell-1} - v_*$ for $1 \leq \ell \leq m_j - 1$, and the fact that $v_{i_j-1} \leq v_* < v_{i_j}$ and $v_{i_j+m_j-2} > v_* \geq v_{i_j+m_j-1}$, we have

$$\begin{aligned}\sum_{\ell=1}^{m_j-2} (v^2 - (\mathrm{D}_+ v)^2)_{i_j+\ell-1} \;&\leq\; 2v_* \sum_{\ell=1}^{m_j-2} v_{i_j+\ell-1} - v_*^2(m_j - 2) + \left(\frac{v_{i_j} - v_*}{2\sin(\Delta\theta/2)}\right)^2 \\ &\quad - (v_{i_j+m_j-2} - v_*)^2 + \left(\frac{v_{i_j+m_j-2} - v_*}{2\sin(\Delta\theta/2)}\right)^2,\end{aligned}$$

and so

$$\begin{aligned}&\sum_{i \in I_j} (v^2 - (\mathrm{D}_+ v)^2)_i \\ &= (v^2 - (\mathrm{D}_+ v)^2)_{i_j-1} + \sum_{i=i_j}^{i_j+m_j-3} (v^2 - (\mathrm{D}_+ v)^2)_i + (v^2 - (\mathrm{D}_+ v)^2)_{i_j+m_j-2} \\ &= (v^2 - (\mathrm{D}_+ v)^2)_{i_j-1} + \sum_{\ell=1}^{m_j-2} (v^2 - (\mathrm{D}_+ v)^2)_{i_j+\ell-1} + (v^2 - (\mathrm{D}_+ v)^2)_{i_j+m_j-2} \\ &\leq 2v_* \sum_{\ell=1}^{m_j-1} v_{i_j+\ell-1}.\end{aligned}$$

Combining the above two estimates lead the assertion. □

Lemma 3.18 *Let $\alpha \geq 1$ and fix $t \in [0, T_*)$. It there exists a constant $C_* > 0$ such that $v_*(\tau) \leq C_*$ for $0 \leq \tau \leq t$, then $\mathcal{E}(t)$ is bounded.*

Proof. By using the summation by parts (2.4), one has

$$\dot{\mathcal{E}}(t) = 2\tan\frac{\Delta\theta}{2} \sum_j (v^2 - (\mathrm{D}_+ v)^2)_j$$

for all $\alpha \geq 1$.

By the Lemma 3.16, we have the next estimate:

$$2\tan\frac{\Delta\theta}{2} \sum_j (v^2 - (\mathrm{D}_+ v)^2)_j \leq 2n\tan\frac{\Delta\theta}{2} v_*^2 - 2v_* \dot{\mathcal{L}}(t).$$

Hence $\mathcal{E}(t) \leq \mathcal{E}(0) + 2n\tan\frac{\Delta\theta}{2} C_*^2 T_* + 2C_* \mathcal{L}(0)$ holds. □

Lemma 3.19 *Let $\alpha \geq 1$. If $\mathcal{E}(t)$ is bounded, then for any $\delta > 0$ there exists a constant $C_2 > 1$ such that $v_j(t) \leq C_2$ except for θ_j in intervals of length less than δ for $t \in [0, T_\star)$.*

Proof. If $v_j \geq C_2$ for m values of j and $m\Delta\theta \geq \delta$, then

$$\mathcal{E}(t) \geq \frac{2g_{\min}^{1/\alpha}}{\alpha - 1} \tan\frac{\Delta\theta}{2}(mC_2^{1-1/\alpha} + (n-m)v_{\min}(0)^{1-1/\alpha}) \geq \frac{2g_{\min}^{1/\alpha}}{(\alpha-1)\Delta\theta} \tan\frac{\Delta\theta}{2}\delta C_2^{1-1/\alpha}$$

when $\alpha > 1$ and

$$\begin{aligned} \mathcal{E}(t) &\geq 2g_{\min} \tan\frac{\Delta\theta}{2}(m \log C_2 + (n-m)\log v_{\min}(0)) \\ &\geq \frac{2g_{\min}}{\Delta\theta} \tan\frac{\Delta\theta}{2}(\delta \log C_2 + (2\pi - \delta)\log v_{\min}(0)) \end{aligned}$$

when $\alpha = 1$ (we have assumed $v_{\min}(0) < 1$). This gives a contradiction when C_2 is large. □

Lemma 3.20 *Let $\alpha > 0$. For $t \in [0, T_\star)$, if $v_j(t) \leq C_2$ for some constant $C_2 > 1$ except for θ_j in intervals of length less than δ and $\delta > 0$ is small enough, then $\max_{0 \leq j < n} v_j(t)$ is bounded.*

Proof. As in the proof of Girão[Gir] (section 2, *Sixth*), we have the next estimate:

$$\begin{aligned} v_j &= v_i + \sum_{i \leq m < j} (v_{m+1} - v_m) \\ &\leq C_2 + \left(\sum_{i \leq m < j} \frac{2(1-\cos\Delta\theta)}{2\tan(\Delta\theta/2)}\right)^{1/2} \left(2\tan\frac{\Delta\theta}{2} \sum_{i \leq m < j} (\mathrm{D}_+ v)_m^2\right)^{1/2} \\ &\leq C_2 + \sqrt{(j-i)\sin\Delta\theta}\left(2\tan\frac{\Delta\theta}{2} \sum_{0 \leq m < n} v_m^2 + C_1\right)^{1/2} \\ &\leq C_2 + \sqrt{\delta}\left(2n\tan\frac{\Delta\theta}{2} v_{\max}^2 + C_1\right)^{1/2} \\ &\leq C_2 + \sqrt{\delta}\left(\sqrt{2\sqrt{2}\pi}v_{\max} + \sqrt{C_1}\right) \end{aligned}$$

since $v_j \leq C_2$ and $\theta_i - \theta_j \leq \delta$. Here we have used Lemma 3.13.

Hence $(1 - \sqrt{2\sqrt{2}\pi\delta})v_{\max} \leq C_2 + \sqrt{C_1\delta}$ holds, and we obtain $v_{\max} \leq (C_2 + \sqrt{C_1\delta})/(1 - \sqrt{2\sqrt{2}\pi\delta})$ for small δ. □

Proof of Theorem 3.11. Suppose a side of the polygon disappears for $t < T_\star$ where $T_\star$ attains $\mathcal{A}(T_\star) = 0$. Put t_0 as the first time that happens (n.b. $t_0 > 0$ is clear). Then $\mathcal{A}(t) > 0$ for $0 \leq t \leq t_0$ and the above estimates imply that $\sup_{0 \leq t \leq t_0} v_{\max}(t)$ is bounded, so $d_{\min}(t_0) > 0$. This is a contradiction. Hence the assertion holds. □

We are now ready to present of the proof of Theorem B.

Proof of Theorem B. By Theorem 3.11, we have $\mathcal{A}(T_\star) = 0$. If n is odd, then $\mathcal{L}(T_\star) = 0$ for $\alpha \geq 1$ since the angle between two adjacent sides of polygon is always $\pi - \Delta\theta$ and we have no two sides which is parallel each other. Suppose that n is even. Then jth side and $(j + n/2)$th side are parallel. Let w_j be the distance between j and $(j + n/2)$th side, and we have

$$w_m = \sum_{j=m+1}^{m+n/2} \sin(\theta_j - \theta_m)\, d_j = \sum_{j=1}^{n/2} \sin\theta_j\, d_{j+m}, \quad \text{or} \quad w_m = -\sum_{j=n/2+1}^{n} \sin\theta_j\, d_{j+m}.$$

Therefore,

$$2w_m = \sum_j |\sin\theta_j|\, d_{j+m} = 2\tan\frac{\Delta\theta}{2} \sum_j |\sin\theta_j|\, g_{j+m}^{1/\alpha} v_{j+m}^{-1/\alpha}.$$

Then by (2.4) we have

$$\dot{w}_m = -(v_m + v_{m+n/2})$$

since

$$(\Delta_\theta |\sin\theta| + |\sin\theta|)_i = \begin{cases} \cot(\Delta\theta/2) & \text{if } i = 0,\ n/2; \\ 0 & \text{if } \textit{otherwise}, \end{cases}$$

holds.

Case $\alpha > 1$. Since $\dot{\mathcal{A}}(t) = -(\alpha - 1)\mathcal{E}(t)$, by Lemma 3.8, there exists $t_1 \in [0, T_\star)$ such that

$$\dot{\mathcal{A}}(t) = -(\alpha - 1)\mathcal{E}(t) \geq -C(T_\star - t)^{(1-\alpha)/(1+\alpha)}, \quad t_1 \leq t < T_\star$$

for some constant $C = C(\alpha, \Delta\theta, g, v(t_1)) > 0$. Integration over $(t, T_\star)$ and $\mathcal{A}(T_\star) = 0$ yield

$$\mathcal{A}(t) \leq C(T_\star - t)^{2/(1+\alpha)}, \quad t_1 \leq t < T_\star. \tag{3.4}$$

Therefore by this inequality, $\mathcal{A}(t) \geq w_m(t) d_m(t)$ and Corollary 3.9, we have

$$w_m(t) \leq \frac{\mathcal{A}(t)}{2\tan(\Delta\theta/2)} \left(\frac{v_m}{g_m}\right)^{1/\alpha} \leq C(T_\star - t)^{1/(1+\alpha)}, \quad t_1 \leq t < T_\star$$

for any m and some positive constant $C = C(\alpha, \Delta\theta, g, v(t_1))$. Hence $w_m(T_\star) = 0$ for all m and $\mathcal{L}(T_\star) = 0$.

Case $\alpha = 1$. By $\dot{w}_m \leq -v_m$ and $\mathcal{A}(t) = 2\tan(\Delta\theta/2) \sum_j g_j (T_\star - t) \geq w_m d_m$, we have

$$\frac{\dot{w}_m}{w_m} \leq -\frac{2\tan(\Delta\theta/2)}{\mathcal{A}(t)} g_m = -\frac{g_m}{\sum_j g_j (T_\star - t)}.$$

Hence, by integration over $(0, t)$, we have

$$w_m(t) \leq w_m(0) \left(\frac{T_\star - t}{T_\star}\right)^{g_m / \sum_j g_j},$$

and $w_m(T_\star) = 0$ for all m. Then $\mathcal{L}(T_\star) = 0$ is concluded. $\square$

3.4 Lower bound of the blow-up time (proof of Theorem C)

Giga–Yama-uchi [GY] remarks that by using the Schwarz inequality twice, it is easy to obtain a lower bound of the blow-up time for the motion of smooth surfaces by mean curvature in higher dimension. Here we will use the Hölder's inequality twice to obtain T_ℓ, and apply Lemma 2.4 (2) to calculate T_r.

Proof of Theorem C. We apply the Hölder's inequality to the equation (2.8):

$$\begin{aligned} -\dot{\mathcal{A}}(t) &= 2\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha} v_j^{-1/(\alpha(\alpha+1))} v_j^{\alpha/(\alpha+1)} \\ &\leq 2\tan\frac{\Delta\theta}{2}\left(\sum_j g_j^{1+1/\alpha} v_j^{-1/\alpha}\right)^{1/(\alpha+1)}\left(\sum_j v_j\right)^{\alpha/(\alpha+1)} \\ &\leq g_{\max}^{1/(\alpha+1)}\left(\frac{\mathcal{L}(t)}{2\tan(\Delta\theta/2)}\right)^{1/(\alpha+1)}\left(-\frac{\dot{\mathcal{L}}(t)}{2\tan(\Delta\theta/2)}\right)^{\alpha/(\alpha+1)}. \end{aligned}$$

Here we have used the equations (2.6) and (2.7). Integration over $(0, T_\star)$ of this inequality, assumption $\mathcal{A}(T_\star) = 0$ and the Hölder's inequality yield

$$\begin{aligned} \mathcal{A}(0) &\leq g_{\max}^{1/(\alpha+1)}\left(\int_0^{T_\star} dt\right)^{1/(\alpha+1)}\left(\int_0^{T_\star}\frac{-\alpha}{\alpha+1}\frac{d}{dt}\mathcal{L}(t)^{1+1/\alpha}dt\right)^{\alpha/(\alpha+1)} \\ &\leq g_{\max}^{\alpha/(\alpha+1)} T_\star^{1/(\alpha+1)}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha/(\alpha+1)}\mathcal{L}(0). \end{aligned}$$

Hence we obtain T_ℓ. By Lemma 2.4 (2), we have $v_j(t) \leq v_r(t)$. Therefore v_r blows up to infinity faster than solution v. In other words $T_r \leq T_\star$ holds. □

3.5 Type I blow-up (proof of Theorem D)

By using Theorem B, we show the next lemma:

Lemma 3.21 *Let $\alpha = 1$. Suppose t_2 is the same as in Lemma* 3.10. *Then there exists $t_3 \in [t_2, T_\star)$ such that*

$$v_j(t) \leq C_j(T_\star - t)^{-1/2}, \quad 0 \leq j < n, \quad t_3 \leq t < T_\star$$

where $C_j = v_j(t_2)\sqrt{T_\star - t_2} > 0$, and so $C_j \leq C = C(g, \Delta\theta, v(0), v(t_2))$.

Proof. Integration of $\dot{\mathcal{E}}(t) \leq \tan(\Delta\theta/2)\sum_j g_j(T_\star - t)^{-1}$ over (t_2, t) is

$$\mathcal{E}(t) \leq \mathcal{E}(t_2) - \tan\frac{\Delta\theta}{2}\sum_j g_j \log\frac{T_\star - t}{T_\star - t_2}, \quad t_2 \leq t < T_\star. \tag{3.5}$$

By Theorem B, $v_{\min}$ blows up at $T_\star$, and equation (2.3a) shows that $v_{\min}$ dose not decrease in time. Then there exists $t_3 \in [t_2, T_\star)$ such that $v_{\min}(t) \geq v_{\max}(t_2)$ for all $t \in [t_3, T_\star)$. Hence the next inequality holds.

$$\mathcal{E}(t) - \mathcal{E}(t_2) = 2\tan\frac{\Delta\theta}{2}\sum_i g_i \log\frac{v_i(t)}{v_i(t_2)} \geq 2\tan\frac{\Delta\theta}{2}\log\frac{v_j(t)}{v_j(t_2)}\sum_i g_i$$

for $0 \leq j < n$ and $t_3 \leq t < T_\star$. Combination of equation (3.5) and the above inequality leads the assertion. □

Corollary 3.9 and Lemma 3.21 conclude the proof of Theorem D.

3.6 Asymptotic self-similarity (proof of Theorem E)

If the "point-extinction," and the "type I blow-up" hold, then we have the following:

Lemma 3.22 *Let $\alpha > 0$. If a solution polygon shrinks to a single point and the solution v undergoes the type* I *blow-up, then the rescaled length $\bar{\mathcal{L}}(\tau) = h(t)\mathcal{L}(t)$ is bounded. Moreover, a solution $\bar{v}(\tau)$ of Problem* 2 *is bounded away from zero and is bounded for all time $\tau \geq 0$.*

Proof. By equation (2.7), and the assumption of the "type I blow-up," there exist $t_0 \in [0, T_\star)$ and a positive constant C which depends only on α, g, $\Delta\theta$, $v(0)$ and $v(t_0)$ such that

$$\dot{\mathcal{L}}(t) \geq -C(T_\star - t)^{-\alpha/(\alpha+1)}, \quad t_0 \leq t < T_\star.$$

Integration over $(t, T_\star)$ yields

$$\mathcal{L}(t) \leq C(T_\star - t)^{1/(\alpha+1)}, \quad t_0 \leq t < T_\star$$

since the "point-extinction" holds.

Hence we have

$$\bar{\mathcal{L}}(\tau) \leq C, \quad \tau \geq \tau(t_0).$$

For $t < t_0$ the length $\mathcal{L}(t)$ is bounded, and so the rescaled length $\bar{\mathcal{L}}(\tau)$ is also bounded for $\tau < \tau(t_0)$. Therefore the first assertion holds.

Next, since the rescaled length is given by

$$\bar{\mathcal{L}}(\tau) = 2\tan\frac{\Delta\theta}{2}\sum_i g_i^{1/\alpha}\bar{v}_i^{-1/\alpha} \geq 2\tan\frac{\Delta\theta}{2} g_{\min}^{1/\alpha}\bar{v}_j^{-1/\alpha}, \quad 0 \leq j < n,$$

we have $\bar{v}_j \geq (2\tan(\Delta\theta/2))^\alpha g_{\min}\bar{\mathcal{L}}(\tau)^{-\alpha}$. Then $\bar{v}(\tau)$ is bounded away from zero for all time $\tau \geq 0$ by the first assertion. And the "type I blow-up" property leads the upper bound of $\bar{v}$. Therefore the second assertion holds. □

Now we prove Theorem E. In the sense of the dynamical systems, Theorem E implies that there exists a Lyapunov function, and so the ω-limit set consists of the equilibrium solution only, if the "point-extinction," and the "type I blow-up" hold.

Definition 3.23 *Let the functional be:*

$$\mathcal{J}[\bar{v}(\tau)] := \begin{cases} -2\tan(\Delta\theta/2) \sum_{0\le j<n} \bar{v}_j \left(\Delta_\theta \bar{v} + \bar{v} - \frac{2\alpha}{\alpha-1}\left(\frac{g}{\bar{v}}\right)^{1/\alpha}\right)_j & \text{if } \alpha \neq 1; \\ -2\tan(\Delta\theta/2) \sum_{0\le j<n} \bar{v}_j \left(\Delta_\theta \bar{v} + \bar{v} - 2\frac{g}{\bar{v}}\log \bar{v}\right)_j & \text{if } \alpha = 1. \end{cases}$$

In fact, we see that $\mathcal{J}$ is the Lyapunov function:

Lemma 3.24 *Under the same assumption of Lemma* 3.22, $\mathcal{J}$ *does not increase in time* τ *and is bounded from below.*

Proof. By the direct calculation, one can obtain:

$$\frac{d}{d\tau}\mathcal{J}[\bar{v}(\tau)] = -\frac{4}{\alpha}\tan\frac{\Delta\theta}{2}\sum_j g_j^{1/\alpha}\bar{v}_j^{-(1+1/\alpha)}\left(\frac{d}{d\tau}\bar{v}_j\right)^2 \le 0$$

for all $\alpha > 0$.

By Lemma 3.22, for $\alpha > 0$ there exist positive constants C_1, C_2 such that $C_1 \le \bar{v}(\tau) \le C_2$ for all $\tau \ge 0$. Hence the summation by parts (2.4) yields

$$\mathcal{J}[\bar{v}(\tau)] \ge \begin{cases} -2n\tan(\Delta\theta/2)C_2^2 & \text{if } \alpha > 1; \\ -2n\tan(\Delta\theta/2)C_2^2 + 4n\tan\frac{\Delta\theta}{2} g_{\max} C_1^{-1}\log C_1 & \text{if } \alpha = 1; \\ -2n\tan(\Delta\theta/2)C_2^2 + \frac{4\alpha}{\alpha-1}\tan\frac{\Delta\theta}{2} g_{\max}^{1/\alpha} C_1^{1-1/\alpha} & \text{if } \alpha < 1, \end{cases}$$

here we have assumed $C_1 < 1$ if $\alpha = 1$. □

Proof of Theorem E. The previous lemma implies $\lim_{\tau\to\infty} d\mathcal{J}[\bar{v}(\tau)]/d\tau = 0$, and so we have the limit $\lim_{\tau\to\infty} d\bar{v}_j(\tau)/d\tau = 0$ for all $0 \le j < n$.

By Lemma 3.22, $\bar{v}$ is bounded and bounded away from zero, and so there exists a sequence τ_i of time diverging to infinity such that $\bar{v}(\tau_i)$ has a converging subsequence and converges positive value. Therefore the converging subsequence must converge to a solution of Problem 2', that is, a self-similar solution. □

4 The case where $\alpha < 1$

In this section, we will discuss about a stability of the regular polygon solution, Remark 1.2 and Remark 1.4. Throughout this section, we assume the isotropic motion $g \equiv 1$.

4.1 Stability of the regular polygon solution

In this subsection, we discuss the stability of the regular polygon solution from the linearization. By the assumption $g \equiv 1$, we see that the constant $\bar{v}_j \equiv 1$ is a stationary solution to (3.2a). We linearize (3.2a) around 1. Substitute $\bar{v}_j = 1 + e^{\nu\tau}\varphi_j$ and neglect the second order terms $\varphi_i\varphi_j$ and also the higher order terms. Then we would obtain

$$(\Delta_\alpha \varphi)_j := \alpha(\Delta_\theta\,\varphi)_j + (\alpha+1)\varphi_j = \alpha\left(\Delta_\theta\,\varphi + \frac{\alpha+1}{\alpha}\varphi\right)_j = \nu\varphi_j$$

The eigenvalues of Δ_α are $\nu_k = 1 - \alpha(\lambda_k - 1)$ and the eigenvectors are $\varphi_j = \psi_j^k$ where λ_k and ψ_j^k are defined in Remark 2.1. For $k = 0, 1, 2, \ldots$

$$\nu_k = 1+\alpha,\ \ 1,\ \ 1-\alpha(2\cos\Delta\theta + 1), \cdots.$$

Remark 4.1 (see [Ma]) Because of the positiveness $\nu_0 = 1 + \alpha > 0$, it seems that the regular polygons are linearly instable solutions for any $\alpha > 0$. This instability, however, comes from the difference of suitable rescaling between the stationary solution and its perturbed solution, in other words, the difference of the blow-up time before rescaling. Hence we can remove this instability if we take a suitable rescaling. Although we have another instability $\nu_1 = 1$, this instability comes from the non-closeness of the perturbed solution. Hence we can also remove this instability if we set initial perturbed solution polygon is closed.

This remark implies the next lemma.

Lemma 4.2 *Assume $g \equiv 1$. Then a shrinking regular solution polygon to Problem 1 are linearly stable if $\alpha > \alpha_n := 1/(2\cos\Delta\theta + 1)$.*

Remark 4.3 For $n = 4, 6, 8, \ldots$ We have

$$\alpha_n = 1,\ \frac{1}{2},\ \frac{1}{1+\sqrt{2}}, \cdots \to \frac{1}{3} \quad \text{as} \quad n \to \infty.$$

The power $\alpha = 1/3$ is the critical power (of the blow-up type) for the motion of closed convex curves by power α of curvature: if $\alpha \geq 1/3$ there is the type I blow-up only, and if $\alpha < 1/3$ there exists a type II blow-up. See Andrews [And], Taniyama-Matano [TMa] and references therein.

4.2 Self-similar solutions other than the regular polygons

For $k = 2, 3, 4, \ldots$ the k-peaked (in regard to the crystalline curvature) self-similar solution may bifurcate from the regular polygons solution since the eigenvectors of Δ_α are ψ_j^k. We provide these special solutions from a example under the symmetric assumption:

$$d_{2j}(t) \equiv d_0(t), \quad d_{2j+1}(t) \equiv d_1(t), \quad n\text{: even}, \quad 0 \leq j < n/2, \quad 0 \leq t < T_\star. \qquad (4.1)$$

If we put

$$d_1(t) = \mu d_0(t), \quad \mu > 0,$$

and substitute it in equation (3.2a) through the relation $v_j(t) = (2\tan(\Delta\theta/2)/d_j(t))^\alpha$, then we would obtain the equation:

$$F(\mu) := \mu^{\alpha+1} + \frac{\mu^\alpha - \mu}{\cos\Delta\theta} - 1 = 0 \quad \text{if} \quad n \geq 6$$

for $\alpha > 0$. When $n = 4$, we have the equation $\mu^\alpha - \mu = 0$, then any rectangles are the self-similar solutions if $\alpha = 1$. Hereafter we consider the case $n \geq 6$.

We can easily check the properties of F:

$$F(1) = 0, \quad F(0) = -1, \quad F(1/\mu) = -F(\mu)/\mu^{\alpha+1} \tag{4.2}$$

for $\mu > 0$, $\alpha > 0$ and $n \geq 6$. Hence we suffice to consider F in the interval $0 < \mu < 1$.

In the case there $\alpha \geq 1$, we have $F''(\mu) > 0$ for $0 < \mu < 1$, $F'(0) \leq 0$ and $F'(1) > 0$, then the solution of $F(\mu) = 0$ does not exist in the interval $0 < \mu < 1$. In other words, the self-similar solution are the only regular polygons ($\mu = 1$) under the symmetric assumption (4.1).

Now let us consider the case where $\alpha < 1$. One can easily check

$$F''(\mu) \leq \alpha\mu^{\alpha-2}F'(1), \quad F'(1) = \alpha + 1 - \frac{1-\alpha}{\cos\Delta\theta}.$$

Hence if α is sufficiently small such as

$$\alpha < \tilde{\alpha}_n := \frac{1 - \cos\Delta\theta}{1 + \cos\Delta\theta},$$

then $F'(1) < 0$ and so $F''(\mu) < 0$. Therefore by the properties in (4.2), we see there exists unique solution of $F(\mu) = 0$ in $0 < \mu < 1$ if $\alpha < \tilde{\alpha}_n$. Here we have $\tilde{\alpha}_n \leq \alpha_n$ and the equality $\tilde{\alpha}_n = \alpha_n$ holds iff $n = 4$. Figure 1 indicates some examples of this kind of self-similar solutions except regular polygons.

We note that $\partial F/\partial\alpha < 0$ if $0 < \mu < 1$ and so the zero-point μ_α of $F = 0$, i.e. the ratio $\mu_\alpha \equiv d_1/d_0$ is decreasing as α ($< \tilde{\alpha}_n$) approaches to 0. This means that this self-similar n-gon is an "almost regular" $(n/2)$-gon if α is small enough.

4.3 Examples of type II blow-up and non point-extinction

In this subsection, we shall present a type II blow-up, and non point-extinction by three exaples when $g \equiv 1$ and $n = 4$, 6, 8. Consequently, the order of type II blow-up $\mathrm{O}((T_\star - t)^{-\alpha})$ as $t \nearrow T_\star$, will be given. See also Figure 2.

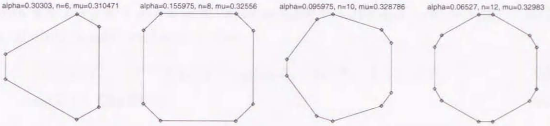

Figure 1: Self-similar solutions except regular polygons in the case where $\alpha = \tilde{\alpha}_n/(1+\varepsilon)$ with $\varepsilon = 0.1$. From left to right $n = 6, 8, 10, 12$.

Lemma 4.4 *Let $g \equiv 1$ and $n = 4$. Assume $v_2(0) = v_0(0)$ and $v_3(0) = v_1(0)$. If $v_1(0) > v_0(0)$ and $\alpha < 1$, then*

$$(v_0(T_\star)(T_\star - t))^{-\alpha} \le v_1(t) \le (v_0(0)(T_\star - t))^{-\alpha}, \quad 0 \le t < T_\star, \tag{4.3a}$$

$$v_0(T_\star) = (v_0(0)^{1-1/\alpha} - v_1(0)^{1-1/\alpha})^{\alpha/(\alpha-1)}, \tag{4.3b}$$

$$T_\star = \int_0^{v_1(0)^{-1/\alpha}} (\xi^{1-\alpha} - v_1(0)^{1-1/\alpha} + v_0(0)^{1-1/\alpha})^{\alpha/(1-\alpha)}\, d\xi. \tag{4.3c}$$

Proof. Under the assumption, evolution equations are

$$\dot{v}_0 = \alpha v_0^{1+1/\alpha} v_1 \quad \text{and} \quad \dot{v}_1 = \alpha v_1^{1+1/\alpha} v_0. \tag{4.4}$$

By Theorem A, v_0 or v_1 blow up at $T_\star$. If v_0 blows up, then the assumption $v_1(0) > v_0(0)$ leads that there exists t_0 such that $v_0(t_0) = v_1(t_0)$, and so the axi-symmetry leads $v_0(t) = v_1(t)$ for $t \ge t_0$. Hence anyhow v_1 blows up.

Direct calculation yields

$$\frac{d}{dt}\left(v_0(t)^{1-1/\alpha} - v_1(t)^{1-1/\alpha}\right) = 0. \tag{4.5}$$

Integration over $(0, T_\star)$ gives (4.3b) if $\alpha < 1$ and $v_1(0) > v_0(0)$.

We have $v_0(0) \le v_0(t) \le v_0(T_\star)$ since $\dot{v}_0 \ge 0$. Thus by Lemma 2.4 (1), inequality

$$v_0(0) \le -\frac{d}{dt} v_1(t)^{-1/\alpha} \le v_0(T_\star)$$

holds from (4.4, right). Hence integration over $(0, T_\star)$ gives (4.3a).

From (4.5) and (4.4, right), we have

$$-\frac{d}{dt} v_1^{-1/\alpha} = v_0(t) = \left(v_1(t)^{1-1/\alpha} - v_1(0)^{1-1/\alpha} + v_0(0)^{1-1/\alpha}\right)^{\alpha/(1-\alpha)},$$

and this leads (4.3c). □

Lemma 4.5 *Let $g \equiv 1$ and $n = 6$. Assume $v_3(0) = v_0(0)$ and $v_2(0) = v_4(0) = v_5(0) = v_1(0)$. If $v_1(0) \gg v_0(0)$ and $\alpha \ll 1$, then*

$$(v_0(T_\star)(T_\star - t))^{-\alpha} \le v_1(t) \le (v_0(0)(T_\star - t))^{-\alpha}, \quad 0 \le t < T_\star, \tag{4.6a}$$

$$v_0(T_\star) \le C(v(0), \alpha). \tag{4.6b}$$

Proof. Under the assumption, evolution equations are

$$\dot{v}_0 = \alpha v_0^{1+1/\alpha}(2v_1 - v_0) \quad \text{and} \quad \dot{v}_1 = \alpha v_1^{1+1/\alpha} v_0. \tag{4.7}$$

By Theorem A, v_0 or v_1 blow up at $T_\star$. If v_0 blows up and v_1 remains bounded at $T_\star$, then for sufficiently large t, $\dot{v}_0$ is negative by (4.7, left). This contradicts that v_0 blows up. Hence anyhow v_1 must blow up. From (4.7, right) and Lemma 2.4 (1), we have $-dv_1^{-1/\alpha}/dt = v_0(t) \ge v_0(0)$ and so $v_1(t) \le (v_0(0)(T_\star - t))^{-\alpha}$. Then by (4.7, left), it holds that

$$-\frac{d}{dt} v_0^{-1/\alpha} \le 2(v_0(0)(T_\star - t))^{-\alpha} - v_0(0),$$

and that its integration over $(0, T_\star)$:

$$v_0(T_\star)^{-1/\alpha} \ge v_0(0)^{-1/\alpha} + v_0(0)T_\star - \frac{2T_\star^{1-\alpha}}{(1-\alpha)v_0(0)^\alpha}.$$

We put $d_1(0) = \mu d_0(0)$ and $v_0(0) = 1$ without loss of generality, then we would obtain the upper bounds of the blow-up time $T_1 = T_2^{\alpha+1}/(\alpha+1)$, $T_2 = (1 + 2\mu)/3$ and so

$$v_0(T_\star)^{-1/\alpha} \ge 1 + T_\star - \frac{2T_\star^{1-\alpha}}{1-\alpha} \ge 1 - \frac{2T_1^{1-\alpha}}{1-\alpha} =: c(\mu, \alpha).$$

Hence $c(\mu, \alpha)$ is bounded away from zero if $T_\star \le T_1 \ll 1$ i.e. $\mu \ll 1$ and $\alpha \ll 1$. For example, if $\mu = 1/8$ and $\alpha = 1/4$, then we have $c(1/8, 1/4) = 0.00725391\ldots > 0$, and so $v_0(T_\star) \le 3.42655\ldots < \infty$.

The lower estimate of (4.6a) follows easily. □

Lemma 4.6 *Let $g \equiv 1$ and $n = 8$. Assume $v_4(0) = v_0(0)$, $v_3(0) = v_5(0) = v_7(0) = v_1(0)$, and $v_6(0) = v_2(0)$. If $v_2(0) > v_1(0) > v_0(0)$, $\sqrt{2}v_1(0) - v_2(0) > a_0$ where $a_0 := (1 + 2^{(\alpha+1)/(2\alpha)})^{-1} v_0(0)$, and $\alpha \ll 1$, then*

$$(2 - \sqrt{2})^\alpha((v_0(T_\star) - a_0)(T_\star - t))^{-\alpha} \le v_1(t) \le v_2(t) \le (b_0(T_\star - t))^{-\alpha} \tag{4.8a}$$

holds for $0 \le t < T_\star$ and

$$v_0(T_\star) \le C(v(0), \alpha) \tag{4.8b}$$

holds. Here $b_0 := a_0/(\sqrt{2} - 1)$.

Proof. Under the assumption, evolution equations are

$$\begin{aligned}&\dot{v}_0 = \alpha' v_0^{1+1/\alpha}(2v_1 - \sqrt{2}v_0), \quad \dot{v}_1 = \alpha' v_1^{1+1/\alpha}(v_2 - \sqrt{2}v_1 + v_0) \quad \text{and} \\ &\dot{v}_2 = \alpha' v_2^{1+1/\alpha}(2v_1 - \sqrt{2}v_2)\end{aligned} \tag{4.9}$$

where $\alpha' = \alpha/(2-\sqrt{2})$.

Claim 1: *At least v_1 blows up.* By Theorem A, some of v_0, v_1 and v_2 blow up at T_*. We use the same argument as in the proof of Lemma 4.5: if v_0 blows up, v_1 must blow up by (4.9, left); if v_2 blows up, v_1 must blow up by (4.9, right); anyhow v_1 blows up and so v_0 or v_2 must blow up by (4.9, middle).

Claim 2: *v_2 blows up and $v_2 \geq v_0$ holds.* If v_0 blows up and v_2 remains bounded at T_*, then the assumption $v_2(0) > v_0(0)$ leads there exists $t_0 > 0$ such that $v_2(t_0) = v_0(t_0)$ and $v_2(t) > v_0(t)$ for $t < t_0$. Thus the axi-symmetric assumption leads that $v_2(t) \equiv v_0(t)$ for all $t \geq t_0$. This is a contradiction. Hence anyhow v_2 blows up and $v_2 \geq v_0$ holds.

Claim 3: *$v_2 \geq v_1 \geq v_0$ holds.* If there is a time $t_0 > 0$ such as the first time that $v_1(t_0) = v_2(t_0)$ happens, and so $v_1(t) < v_2(t)$ for $t \in [0, t_0)$, then $\dot{v}_1(t_0) \leq \dot{v}_2(t_0)$ holds at $t = t_0$ by Claim 2. This yields $v_2 \geq v_1$. The right inequality $v_1 \geq v_0$ will be proved in a similar way.

Claim 4: *$\sqrt{2}v_1 - v_2 \geq a_0$ holds.* If there is a time $t_0 > 0$ such as the first time that $\sqrt{2}v_1(t_0) - v_2(t_0) = a_0$ happens, and so $\sqrt{2}v_1(t) - v_2(t) > a_0$ for $t \in [0, t_0)$, then $\sqrt{2}\dot{v}_1(t_0) > \dot{v}_2(t_0)$ holds. Here we have used $v_0(t_0) \geq v_0(0)$ and $a_0 > 0$. This yields $\sqrt{2}v_1 - v_2 \geq a_0$.

By Claim 4, we have $\dot{v}_2 \geq \sqrt{2}\alpha' v_2^{1+1/\alpha} a_0$, and so

$$v_2(t) \leq (b_0(T_* - t))^{-\alpha}.$$

Substitution the above estimate into (4.9, left), and integration over $(0, T_*)$ yield

$$v_0(T_*)^{-1/\alpha} \geq v_0(0)^{-1/\alpha} + \frac{v_0(0)T_*}{\sqrt{2}-1} - \frac{2T_*^{1-\alpha}}{(2-\sqrt{2})(1-\alpha)b_0^\alpha}.$$

Now we put $d_2(0) = \mu_2 d_0(0)$, $d_1(0) = \mu_1 d_0(0)$, and $v_0(0) = 1$ without loss of generality. The assumption leads the relations $\mu_2 < \mu_1 < 1$, $\sqrt{2}\mu_1^{-\alpha} - \mu_2^{-\alpha} > a_0$, and the lower and upper bounds of the blow-up time T_*: $T_r = \mu_2^{\alpha+1}/(\alpha+1)$, $T_1 = T_2^{\alpha+1}/(\alpha+1)$ and $T_2 = (1 + 2\mu_1 + \mu_2)/4 < 1$. Thus we have

$$v_0(T_*)^{-1/\alpha} \geq 1 + \frac{T_r}{\sqrt{2}-1} - \frac{\sqrt{2}(1 + 2^{(\alpha+1)/(2\alpha)})^\alpha T_1}{(\sqrt{2}-1)^{(1-\alpha)}(1-\alpha)} =: c(\mu_1, \mu_2, \alpha).$$

Hence $c(\mu_1, \mu_2, \alpha)$ is bounded away from zero if μ_1, μ_2 and α are small enough. For example, if $\mu_1 = 1/15$, $\mu_2 = 1/20$ and $\alpha = 1/4$, then the assumptions are satisfied and $c(1/15, 1/20, 1/4) = 0.0217784\ldots > 0$ holds, and so we have $v(T_*) \leq 2.60312\ldots < \infty$.

The lower estimate of (4.8a) follows easily. □

Numerical examples. In Figure 2, we show the comparison to the asymptotic behavior of solution polygons between the type I blow-up case (upper figure), and the type II blow-up case (lower figure). The initial figure is the outmost polygon. We can observe the point-extinction in Figure 2 (a)(b)(c), and the non point-extinction in Figure 2 (a')(b')(c'). The parameters are $\mu_1 = 9/10$, $\mu_1 = 1/8$, and $(\mu_1, \mu_2) = (1/15, 1/20)$ from left to right. Here $d_1(0) = \mu_1 d_0(0)$ and $d_2(0) = \mu_2 d_0(0)$ in each figure. Note that the initial polygon in Figure 2 (a)(a') is rectangle. All figures are performed by using the scheme developed in Ushijima-Yazaki [UY] at Graduate School of Mathematical Sciences, The University of Tokyo, Japan.

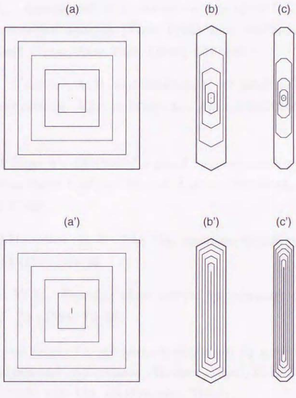

Figure 2: Numerical expamples of Lemma 4.4, 4.5 and 4.6. The upper figure (resp., the lower figure) is in the case where $\alpha = 1$ (resp., $\alpha = 1/4$). From left to right, the number of sides is $n = 4$, 6 and 8.

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Chapter 3

Asymptotic behavior of solutions to an expanding motion by a negative power of crystalline curvature

Outline: This chapter is concerned with an expanding motion of polygonal curves by a negative power of crystalline curvature. Main results say that solution polygonal curves expand to infinity approaching a regular polygon as time tends to infinity, or in a finite time, depending on the power. We prove these results by using a comparison principle, a discrete version of Aleksandrov reflection method, estimates of an isoperimetric ratio and some fundamental inequalities. We also prove an isoperimetric inequality for polygonal curves.
Key words: crystalline motion, crystalline curvature, motion by a negative curvature, geometric expansion, blow-up, asymptotic self-similar, isoperimetric ratio, isoperimetric inequality.

1 Introduction and main results

In this chapter we study the asymptotic behavior of solutions to a motion of polygonal curves in the plane. In the last several years, many authors have investigated in the asymptotic behavior, especially the asymptotic self-similarity of solutions to a motion of plane curves by a function of their curvature. The typical example for contracting flows is the classical curve-shortening equation, or the motion by curvature. A property of the motion is that any given Jordan curve shrinks to a single point, and its asymptotic shape just before disappear is a circle. This result was given by Gage–Hamilton [GH], and Grayson [Gry]. Subsequently, Gage–Li [GL] and Dohmen–Giga–Mizoguchi [DGM] extended the result for an anisotropic motion by curvature which is derived in Gurtin [Gu]. More recently, Andrews [And] and Taniyama–Matano [TMa] study a motion of closed convex curves by a power of curvature. Besides contracting flows, there has been considerable interest in expanding flows: motion of plane curves by their principal radius, or the inverse of their curvature. See Urbas [Ur], Chow–Tsai [CTs], Tsai [Ts], Andrews [And] and references therein. A part of the main results in this chapter is proved by using a discrete version of proposition in [CTs]. We will touch upon this later.

The classical curve-shortening equation comes from a physical context: the equation is derived from a gradient flow of total smooth interfacial energy defined on the curves which are boundaries separating two materials. Anisotropic interfacial energy leads the weighted curvature flow, or an anisotropic motion by curvature. See Angenent–Gurtin [AGu].

Some materials have nonsmooth interfacial energies, for instance, crystalline. In this case the problem is reduced to the motion of polygonal curves by its crystalline curvature (defined below) with a driving force term. This motion is called crystalline motion, or crystalline flow, which is proposed by [AGu], [Gu] and correspondingly Taylor [T3]. On the recent development of crystalline motion and its application, see, e.g., Taylor–Cahn–Handwerker [TCH], Giga–Giga [GMHG4], Giga–Gurtin–Matias [GGuM], Roosen–Taylor [RT], Rybka [Ry].

The asymptotic self-similarity of solutions to a crystalline motion, except a driving force term, is studied by Stancu [S2]. Uniqueness of self-similar solutions is proved in [S1] which Taylor [T2] conjectured. Recently Yazaki [Y1] studies the asymptotic self-similarity of solutions to a motion by a power of crystalline curvature. See also [Y2] for a generalized crystalline motion.

Our aim in this chapter is to study the asymptotic behavior of solutions to an expanding crystalline flow, or motion by a negative power of crystalline curvature, and to show the asymptotic self-similarity of solutions.

Many authors have recently studied an approximation of curvature-dependent motions by using crystalline motions in their articles. See Remark 1.1 below.

Now we state our problem setting, the motion of polygons by a negative power of crystalline curvature as follows. Let $\mathcal{P}_0$ be a convex closed polygon in the plane $\boldsymbol{R}^2$ with the angle between two adjacent sides is $\pi - \Delta\theta$ where

$$\Delta\theta := \frac{2\pi}{n},$$

and n is a number of sides. We consider the evolution problem to find a family of polygons $\{\mathcal{P}(t)\}_{0\le t<T}$ satisfying

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_j(t) = v_j(t)\boldsymbol{n}_j, \quad 0 \le j < n, \quad 0 \le t < T, \\ \mathcal{P}(0) = \mathcal{P}_0 \end{cases} \tag{1.1a}$$

for some $T > 0$, where $\boldsymbol{n}_j := -{}^t(\cos\theta_j, \sin\theta_j)$ is the inward normal vector of the jth side, $\boldsymbol{x}_j$ is the position vector of the point of intersection between on the line containing the

jth side and the line spanned by $\boldsymbol{n}_j$ and v_j is the inward normal velocity of the jth side. Here and hereafter we denote $\theta_j = j\Delta\theta$. We note that the angle between two adjacent sides of $\mathcal{P}(t)$ is always $\pi - \Delta\theta$ as long as solution polygons exist.

In this chapter, we treat of the case that the normal velocity is homogeneous of some degree $-\beta$ in the crystalline curvature:

$$v_j(t) = \kappa_j(t)^{-\beta}, \quad 0 \le j < n, \quad 0 < t < T \tag{1.1b}$$

where β is a positive parameter, and κ_j is the crystalline curvature:

$$\kappa_j(t) = \frac{2\tan(\Delta\theta/2)}{d_j(t)}, \quad 0 \le j < n, \quad 0 \le t < T. \tag{1.1c}$$

Here $d_j(t)$ is the length of the jth side of polygon $\mathcal{P}(t)$.

The main result of this chapter is the following.

Theorem A *Let $n \ge 4$ and $\mathcal{P}_0$ a convex closed polygon in the plane $\boldsymbol{R}^2$ with the angle between two adjacent sides is $\pi - \Delta\theta$. Then there exists a solution polygon $\mathcal{P}(t)$ of Problem* (1.1) *which expands to infinity as t tends to T_*, the maximal existence time of solution polygon. If $0 < \beta \le 1$, then $T_* = \infty$ and if $\beta > 1$, then $T_* < \infty$. For any $\beta > 0$, a rescaled solution polygon $\tilde{\mathcal{P}}(t) := R(t)^{-1}\mathcal{P}(t)$ converges to a regular polygon in the Hausdorff metric as t tends to $T_* \le \infty$. Here the rescaling rate $R(t)$ is given as the following:*

$$R(t) = \begin{cases} C(C_{\mathrm{av}} + (1-\beta)t)^{1/(1-\beta)} & \text{if } 0 < \beta < 1, \\ Ce^t & \text{if } \beta = 1, \\ C(T_* - t)^{-1/(\beta-1)} & \text{if } \beta > 1 \end{cases}$$

for some constant $C = C(\mathcal{P}_0, \beta, \Delta\theta) > 0$, where $C_{\mathrm{av}} := (\sum_{0\le j<n} v_j(0)^{1/\beta}/n)^{1-\beta}$.

Main tools of the proof of Theorem A include the comparison principle, a discrete version of Aleksandrov reflection method which is used in [CG] and [CTs], an estimate of isoperimetric ratio, and some fundamental inequalities. We will define an isoperimetric ratio for polygons, and show the isoperimetric inequalities in section 3. In section 2, we will discuss the fundamental properties of solutions, and see solution polygons expand to infinity in a finite time or infinite time dependeing on the power β. In section 4 and 5, an asymptotic behavior of solutions will be given, and a proof of Theorem A will be completed.

At the end of this section, we mention an approximation by using crystalline motion.

Remark 1.1 (approximation) One of the view points of approximation is to present the relation between a motion of smooth curves, and polygonal curves through the convergence. If curves are graph-like, convergence results are shown in both papers Girão–Kohn [GirK] and Fukui–Giga [FG]. In Elliott–Gardiner–Schätzle [EGS], the properties of a solution in the sense of [FG] are investigated, and several numerical examples are presented for which visualize the results. The new notion of solutions to a fully non-linear equation including crystalline motion is introduced and analyzed in Giga–Giga [GMHG1, GMHG2, GMHG4]. Its notion is in the realm of viscosity solution theory, and so is based on comparison principle which is an extension of Giga–Gurtin [GGu]. Convergence results are discussed in [GMHG3, GMHG5] for the solutions in its notion.

Girão [Gir] showed that a crystalline motion approximates a weighted curvature flow if the curves are closed and convex. This result was extended by Ushijima–Yazaki [UY1] for the motion by a positive power of curvature. Moreover, they constructed a crystalline algorithm to the motion of nonconvex curves by a power of curvature in [UY2]. Implicit crystalline algorithm is considerable interest in [UY3] for an area-preserving motion by curvature. In Ishii–Soner [IS], they show the crystalline motion approximates the curve-shortening equation through the level set method. See the survey Elliott [E] for more general information about an approximation of curvature-dependent motions.

2 Preliminaries

Throughout this chapter we use the notation $\sum_j u_j$, $u_{\max}$, $u_{\min}$ and $\dot{u}(t)$ for $\sum_{0\le j<n} u_j$, $\max_{0\le j<n} u_j$, $\min_{0\le j<n} u_j$ and $du(t)/dt$, respectively. Hereafter we assume $n \geq 4$. We note again $\theta_j = j\Delta\theta$.

2.1 Restatement of Problem (1.1)

Let $\mathcal{P}(t)$ be a solution of Problem (1.1). The jth vertex $\boldsymbol{B}_j(t)$ of $\mathcal{P}(t)$ is given as the following:

$$\begin{aligned}\boldsymbol{B}_j(t) &= \langle \boldsymbol{x}_{j-1}(t) - \boldsymbol{x}_j(t), \boldsymbol{t}_j + \boldsymbol{n}_j \cot\Delta\theta\rangle\, \boldsymbol{t}_j + \boldsymbol{x}_j(t), \\ &= \boldsymbol{B}_0(t) + \sum_{0\le m<j} d_m(t)\boldsymbol{t}_m, \quad 1 \le j \le n, \quad 0 \le t < T \qquad (2.1)\end{aligned}$$

with $\boldsymbol{B}_0(t) \equiv \boldsymbol{B}_n(t)$, since the position vector $\boldsymbol{x}_j$ is on the line containing the jth side (n.b. $\boldsymbol{x}_j$ is not necessarily on the jth side). Here $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j)$ is the tangent vector, and $\langle\cdot,\cdot\rangle$ is the usual inner product. Then the time evolution of the length of the jth side $d_j(t)$ is given as the following (cf. Figure 10C in [AGu]):

$$\frac{d}{dt}d_j(t) = \frac{d}{dt}|\boldsymbol{B}_{j+1}(t) - \boldsymbol{B}_j(t)| = 2\tan\frac{\Delta\theta}{2}(\Delta_\theta v + v)_j. \qquad (2.2)$$

Here the operator Δ_θ is defined as

$$(\Delta_\theta v)_j := \frac{v_{j+1} - 2v_j + v_{j-1}}{2(1-\cos\Delta\theta)}$$

which is a kind of central difference operator. Then we obtain the following evolution equation:

$$\frac{d}{dt}\kappa_j(t) = -\kappa_j^2(\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T.$$

Therefore we can restate Problem (1.1) as follows.

Problem 1 *Assume $\beta > 0$ and $n \ge 4$. Find a function $v(t) = (v_0, v_1, \ldots, v_{n-1}) \in [C^1(0,T)]^n$ for some $T \in (0, \infty]$ satisfying*

$$\frac{d}{dt}v_j(t) = \beta {v_j}^{1-1/\beta}(\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 < t < T, \tag{2.3a}$$

$$v_j(0) = \kappa_j(0)^{-\beta}, \quad 0 \le j < n, \tag{2.3b}$$

$$v_{-1}(t) = v_{n-1}(t), \quad v_n(t) = v_0(t), \quad 0 \le t < T \tag{2.3c}$$

where $\kappa_j(0)$ is the initial crystalline curvature of $\mathcal{P}_0$.

Remark 2.1 (equivalence) Problem (1.1) and Problem 1 are equivalent except the indefiniteness of position of the polygon. Indeed, suppose v is a solution of Problem 1, then we have

$$\frac{1}{2\tan(\Delta\theta/2)}\frac{d}{dt}\sum_j 2\tan\frac{\Delta\theta}{2}v_j(t)^{1/\beta}\boldsymbol{t}_j = \sum_j(\Delta_\theta v + v)_j\boldsymbol{t}_j = \sum_j(\Delta_\theta \boldsymbol{t} + \boldsymbol{t})_j v_j = \boldsymbol{0}.$$

Here we have used the relation of summation by parts:

$$\sum_j f_j(\Delta_\theta g)_j = -\sum_j(\mathrm{D}_+ f)_j(\mathrm{D}_+ g)_j = \sum_j g_j(\Delta_\theta f)_j, \quad (\mathrm{D}_+ f)_j := \frac{f_{j+1} - f_j}{2\sin(\Delta\theta/2)}; \tag{2.4}$$

and the relation $(\Delta_\theta \boldsymbol{t})_j = -\boldsymbol{t}_j$. Hence by equation (2.1), we can construct a closed convex n-gon, whose length of the jth side is $2\tan(\Delta\theta/2)v_j(t)^{1/\beta} =: d_j(t)$ and the jth normal vector is $\boldsymbol{n}_j$, as long as v is a solution of Problem 1. This n-gon is the very solution polygon of Problem (1.1).

2.2 Support function

The support function $h_j(t)$ of $\mathcal{P}(t)$ is defined by

$$h_j(t) = \langle \boldsymbol{x}_j(t), \boldsymbol{n}_j \rangle.$$

The length of the jth side is given as

$$d_j(t) = 2\tan\frac{\Delta\theta}{2}(\Delta_\theta h(t) + h(t))_j \tag{2.5}$$

by geometry. Then we have

$$v_j(t) = \kappa_j(t)^{-\beta} = (\Delta_\theta h(t) + h(t))_j^\beta. \tag{2.6}$$

Since $\dot{h}_j(t) = \langle \dot{\boldsymbol{x}}_j(t), \boldsymbol{n}_j \rangle = v_j(t)$ holds, we obtain the following problem equivalent to Problem 1:

Problem 1' *Assume $\beta > 0$ and $n \geq 4$. Find a function $h(t) = (h_0, h_1, \dots, h_{n-1}) \in [C^1(0,T)]^n$ for some $T \in (0, \infty]$ satisfying*

$$\frac{d}{dt}h_j(t) = (\Delta_\theta h + h)_j^\beta, \quad 0 \leq j < n, \quad 0 < t < T, \tag{2.7a}$$

$$h_j(0) = \langle \boldsymbol{x}_j(0), \boldsymbol{n}_j \rangle, \quad 0 \leq j < n, \tag{2.7b}$$

$$h_{-1}(t) = h_{n-1}(t), \quad h_n(t) = h_0(t), \quad 0 \leq t < T \tag{2.7c}$$

where $\boldsymbol{x}_j(0)$ is the position vector on the line containing the jth side of the initial $\mathcal{P}_0$.

2.3 The length and the area

The (total) length of polygon is

$$\mathcal{L}(t) := \sum_j d_j = 2\tan\frac{\Delta\theta}{2}\sum_j v_j^{1/\beta},$$

and the rate of change of $\mathcal{L}(t)$ can be computed by

$$\dot{\mathcal{L}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j v_j(t). \tag{2.8}$$

Since $\dot{\mathcal{L}}(t) > 0$, the motion of solution polygons is a discretized curve-lengthening, or an expanding flow.

The area enclosed by polygon is

$$\mathcal{A}(t) := \frac{1}{2}\sum_j h_j(t)d_j(t) = \tan\frac{\Delta\theta}{2}\sum_j h_j(t)v_j(t)^{1/\beta},$$

and the rate of change of $\mathcal{A}(t)$ can be computed by

$$\dot{\mathcal{A}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j v_j(t)^{1+1/\beta}.$$

Here we use equations (2.2), (2.4), (2.5) and (2.6).

2.4 Comparison principle and its application

The following comparison principle and its application play an important roll of this chapter.

Lemma 2.2 *Fix $T > 0$. Let $(p_j(t))_{0\leq j<n} > 0$ and $(q_j(t))_{0\leq j<n}$ be defined on $t \in [0,T]$. If $u = (u_j(t))_{0\leq j<n} \in [C^1(0,T) \cap C[0,T]]^n$ is a solution of*

$$\begin{cases} \dfrac{d}{dt}u_j \geq p_j(\Delta_\theta u)_j + q_j u_j, & 0 \leq j < n, \quad 0 < t < T, \\ u_{-1}(t) = u_{n-1}(t), \quad u_n(t) = u_0(t), & 0 \leq t \leq T, \\ u_j(0) \geq 0, \quad 0 \leq j < n, & \end{cases}$$

then $u_j(t) \geq 0$ holds for $0 \leq j < n$ and $0 \leq t \leq T$.

Proof. Put $\mu > 0$ large enough such that $\tilde{q}_j(t) := q_j(t) - \mu < 0$ for any $t \in [0,T]$. We set $\tilde{u}_j(t) := e^{-\mu t}u_j(t)$ on $0 \le t \le T$. Then $\tilde{u}_j$ satisfy $d\tilde{u}_j(t)/dt \ge p_j(\Delta_\theta \tilde{u})_j + \tilde{q}_j\tilde{u}_j$ for $0 \le j < n$ and $0 < t < T$. Suppose that $\min_{0\le t\le T, 0\le j<n} \tilde{u}_j(t)$ attains negative value at $j = j_0$ and $t = t_0$. At this point, however,

$$\frac{d}{dt}\tilde{u}_{j_0}(t_0) \le 0, \quad (\Delta_\theta \tilde{u}(t_0))_{j_0} \ge 0, \quad \text{and} \quad \tilde{q}_{j_0}(t_0)\tilde{u}_{j_0}(t_0) > 0.$$

This is a contradiction and hence $\tilde{u}_j(t) \ge 0$ holds for $0 \le t \le T$. □

As an application of the above lemma, we obtain the following:

Lemma 2.3 *Let $\beta \le 1$ and v a solution of Problem* 1. *The following comparisons hold.*

(1) *Let v_u be a solution of $\dot{v}_u = \beta v_u^{2-1/\beta}$ with $v_u(0) = v_{\max}(0)$. Then $v_u(t) \ge v_j(t)$ holds for all $0 < t < T_\star = \infty$.*

(2) *Let v_l be the solution of $\dot{v}_l = \beta v_l^{2-1/\beta}$ with $v_l(0) = v_{\min}(0)$. Then $v_j(t) \ge v_l(t)$ holds for all $0 < t < T_\star = \infty$.*

Proof. For each proposition, put (1) $u_j = v_u - v_j$; (2) $u_j = v_j - v_l$; and apply Lemma 2.2. □

Remark 2.4 Solutions v_u (resp., v_l) corresponds to a motion of a large (resp., small) regular polygon compare with the solution polygon $\mathcal{P}(t)$.

Lemma 2.3 implies that the solution v blows up to infinity as t tends to infinity when $\beta \le 1$. Moreover, the following corollaries hold.

Corollary 2.5 *It holds that*

$$v_{\min}(0)e^t \le v_j(t) \le v_{\max}(0)e^t \tag{2.9}$$

for $t \ge 0$ if $\beta = 1$ and that

$$c_1\left(C_{\mathrm{av}} + (1-\beta)t\right)^{\beta/(1-\beta)} \le \left(v_{\min}(0)^{-1+1/\beta} + (1-\beta)t\right)^{\beta/(1-\beta)} \le v_j(t), \tag{2.10}$$

and

$$v_j(t) \le \left(v_{\max}(0)^{-1+1/\beta} + (1-\beta)t\right)^{\beta/(1-\beta)} \le c_2\left(C_{\mathrm{av}} + (1-\beta)t\right)^{\beta/(1-\beta)}$$

for $t \ge t_0$ if $\beta < 1$. Here, for a given $t_0 > 0$, constants c_1 and c_2 depend on $v(0)$, β and t_0, and C_{av} is defined in Theorem A.

This corollary imply the following:

Corollary 2.6 *Let $\beta \leq 1$. The solution polygon $\mathcal{P}(t)$ of Problem* (1.1) *expands to infinity as $t \to \infty$, that is, $\lim_{t\to\infty} \min_{0\leq j<n} |\boldsymbol{x}_j(t)| = \infty$. Moreover, for any sequence $\{t_i\}$ of time diverging infinity, there is a convergent subsequence of polygons $\{\tilde{\mathcal{P}}_l\} := \{\mathcal{P}(t_{i_l})\}$ such that $\tilde{\mathcal{P}}_l := R(t_l)^{-1}\mathcal{P}(t_l)$ converges to a polygon $\tilde{\mathcal{P}}_\infty$ as l tends to infinity. Here $R(t)$ is the rate defined in Theorem* A.

In section 4, we will see the polygon $\tilde{\mathcal{P}}_\infty$ is a regular polygon, and a proof of Theorem A will be completed in the case where $\beta \leq 1$. See also Remark 3.3.

2.5 Finite time blow-up

In the case where $\beta > 1$, the maximum of solution v of Problem 1 diverges to infinity in a finite time:

Lemma 2.7 (Finite time blow-up) *Let $\beta > 1$. Suppose v is a solution of Problem 1. There exists a finite time $T_\star > 0$ such that the maximum of $\{v_j\}$ blows up to infinity as $t \nearrow T_\star$:*

$$T_\star \leq \frac{1}{\beta - 1}\left(\frac{2n\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right)^{\beta-1}.$$

Proof. Jensen's inequality yields

$$\mathcal{L}(t)^\beta = \left(2\tan\frac{\Delta\theta}{2}\right)^\beta \left(\sum_j v_j^{1/\beta}\right)^\beta \leq \left(2n\tan\frac{\Delta\theta}{2}\right)^{\beta-1} \dot{\mathcal{L}}(t)$$

By the general argument for ordinary differential equation, a solution v of Problem 1 exists uniquely and locally in time. Put $T_\star > 0$ such as maximal existing time and take $t \in (0, T_\star)$. We obtain

$$\mathcal{L}(t) \geq \left(\mathcal{L}(0)^{1-\beta} - (\beta - 1)\left(2n\tan\frac{\Delta\theta}{2}\right)^{1-\beta} t\right)^{-1/(\beta-1)}.$$

Since $\mathcal{L}(t) \leq 2n\tan(\Delta\theta/2)v_{\max}^{1/\beta}$, we have

$$v_{\max} \geq \left(2\tan\frac{\Delta\theta}{2}\right)^{-\beta}\left(\mathcal{L}(0)^{1-\beta} - (\beta-1)\left(2n\tan\frac{\Delta\theta}{2}\right)^{1-\beta} t\right)^{-\beta/(\beta-1)},$$

and the assertion is concluded. $\square$

3 Isoperimetric ratio and isoperimetric inequality

The isoperimetric ratio for a closed embedded curve Γ is usually defined as

$$\mathrm{iso}\,(\Gamma) = \frac{L^2}{4\pi A}$$

where L and A are the length and the enclosed area of Γ, respectively. It is well known that the isoperimetric inequality $\mathrm{iso}\,(\Gamma) \geq 1$ holds. Equality $\mathrm{iso}\,(\Gamma) = 1$ holds if and only if the curve Γ is a circle. In this sense, if Γ is a regular n-gon, say $\mathcal{P}_n$, then it holds that

$$\mathrm{iso}\,(\mathcal{P}_n) = \frac{n\tan(\Delta\theta/2)}{\pi} > 1, \quad \text{and that} \quad \lim_{n\to\infty} \mathrm{iso}\,(\mathcal{P}_n) = 1.$$

In this chapter, we define the isoperimetric ratio such as

$$\mathcal{I}(t) := \frac{\mathcal{L}(t)^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)} \tag{3.1}$$

since we consider the polygons $\mathcal{P}(t)$ which the angle between two adjacent side is $\pi - \Delta\theta$. For this isoperimetric ratio, the next isoperimetric inequality

$$\mathcal{I}(t) \geq 1 \tag{3.2}$$

holds. The equality $\mathcal{I}(t) = 1$ holds if and only if the polygon $\mathcal{P}(t)$ is a regular polygon.

We shall give a proof of this isoperimetric inequality and equality. We use the next result in which Stancu [S2] and Yazaki [Y1, Y2] show.

Proposition 3.1 (Point-extinction) *Let $n \geq 4$. Then any solution polygon of the problem:*

$$\text{(1.1a) and } v_j(t) = -\kappa_j(t) \text{ (instead of (1.1b)) with (1.1c)} \tag{3.3}$$

shrinks to a single point in finite time, say t_*. No side of the polygon vanishes before t reaches t_*.

Remark 3.2 We note that the each side of the solution polygon of Problem (3.3) moves toward the inward normal $-\boldsymbol{n}_j$, and Proposition 3.1 asserts $\mathcal{L}(t_*) = \mathcal{A}(t_*) = 0$.

The time gradient of the enclosed area $\mathcal{A}(t)$ is given as, from $v_j = -\kappa_j$,

$$\dot{\mathcal{A}}(t) = \sum_j v_j d_j = -\sum_j \kappa_j d_j = -2n\tan\frac{\Delta\theta}{2},$$

and then we have

$$4n\tan\frac{\Delta\theta}{2}\dot{\mathcal{A}}(t) = -2\left(\sum_j \kappa_j d_j\right)^2. \tag{3.4}$$

The length is $\mathcal{L}(t) = \sum_j d_j$ and its time gradient is given as

$$\dot{\mathcal{L}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j v_j = \sum_j v_j\kappa_j d_j = -\sum_j \kappa_j^2 d_j.$$

Here we have used $v_j = -\kappa_j$.

Now we apply Schwarz inequality to (3.4), and we obtain

$$4n\tan\frac{\Delta\theta}{2}\dot{A}(t) \geq -2\sum_j \kappa_j^2 d_j \sum_j d_j = \frac{d}{dt}\mathcal{L}(t)^2.$$

Integration of this inequality over (t, t_*) and the point-extinction result yield

$$-4n\tan\frac{\Delta\theta}{2}\mathcal{A}(t) \geq -\mathcal{L}(t)^2.$$

Hence the isoperimetric inequality (3.2) holds.

The equality of (3.2) holds if and only if the equality in Schwarz inequality holds, in other words, the polygon is a regular polygon.

We note that the next more accurate geometric inequalities - Bonnesen's inequalities hold for polygons (see, e.g., [Egg]):

$$h_{\max}\mathcal{L} - \mathcal{A} - n\tan\frac{\Delta\theta}{2}h_{\max}^2 \geq 0, \quad \text{and} \quad h_{\min}\mathcal{L} - \mathcal{A} - n\tan\frac{\Delta\theta}{2}h_{\min}^2 \geq 0.$$

By virtue of this inequalities, we can easily check that

$$\mathcal{I}(t) \geq 1 + \frac{n\tan(\Delta\theta/2)}{4\mathcal{A}(t)}(h_{\max}(t) - h_{\min}(t))^2 \geq 1. \tag{3.5}$$

Remark 3.3 For any solution polygon $\mathcal{P}(t)$ of Problem (1.1), if we show $\limsup_{t\to T_*}\mathcal{I}(t) \leq 1$, then we would obtain the assertion that the solution polygon $\mathcal{P}(t)$ expands to infinity approaching a regular polygon in the Hausdorff metric as $t \nearrow T_*$ since (3.5) holds for any $\mathcal{P}(t)$. In the following two sections, we will see the limit supremum of $\mathcal{I}(t)$ is less than or equals to 1.

4 Asymptotic behavior of solutions when $\beta \leq 1$

Corollary 2.6 asserts that there is a limit shape of solution polygon $\mathcal{P}(t)$ as subsequence of t tends to infinity and the convergence rate is given as $R(t)$ defined in Theorem A when $\beta \leq 1$. In this section, we will see the limit shape is a regular polygon, and the following two lemmas will complete a part of Theorem A. Strategy of the proofs is to use the comparison principle, and an estimate of isoperimetric ratio (3.1).

Lemma 4.1 *Let $\beta = 1$ and v a solution of Problem* 1. *Then* $\limsup_{t\to\infty}\mathcal{I}(t) \leq 1$.

Proof. Since $\dot{\mathcal{L}}(t) = \mathcal{L}(t)$, we have $\mathcal{L}(t) = \mathcal{L}(0)e^t$. Schwarz inequality leads

$$\dot{\mathcal{A}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j v_j^2 \geq \frac{2}{n}\tan\frac{\Delta\theta}{2}\left(\sum_j v_j\right)^2 = \frac{\mathcal{L}(t)^2}{2n\tan(\Delta\theta/2)} = \frac{\mathcal{L}(0)^2}{2n\tan(\Delta\theta/2)}e^{2t}.$$

Integration over $(0,t)$ yields

$$\mathcal{A}(t) \geq \mathcal{A}(0) + \frac{\mathcal{L}(0)^2}{4n\tan(\Delta\theta/2)}(e^{2t}-1).$$

Hence we have

$$\mathcal{I}(t) = \frac{\mathcal{L}(t)^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)} \leq \frac{\mathcal{L}(0)^2e^{2t}}{4n\tan(\Delta\theta/2)\mathcal{A}(0) + \mathcal{L}(0)^2(e^{2t}-1)},$$

and this implies the assertion. $\square$

Lemma 4.2 *Let $\beta < 1$ and v a solution of Problem* 1. *Then* $\limsup_{t\to\infty}\mathcal{I}(t) \leq 1$.

Proof. Let $r(t) := (C_{\mathrm{av}} + (1-\beta)t)^{-1/(1-\beta)}$ where

$$C_{\mathrm{av}} = \left(\frac{1}{n}\sum_j v_j(0)^{1/\beta}\right)^{1-\beta} = \left(2n\tan\frac{\Delta\theta}{2}\right)^{\beta-1}\mathcal{L}(0)^{1-\beta}.$$

Hölder's inequality leads

$$\dot{\mathcal{L}}(t) \leq 2\tan\frac{\Delta\theta}{2}n^{1-\beta}\left(\sum_j v_j^{1/\beta}\right)^{\beta} = \left(2n\tan\frac{\Delta\theta}{2}\right)^{1-\beta}\mathcal{L}(t)^{\beta}.$$

Hence we have

$$\frac{d}{dt}\mathcal{L}(t)^{1-\beta} \leq (1-\beta)\left(2n\tan\frac{\Delta\theta}{2}\right)^{1-\beta},$$

and so $\mathcal{L}(t) \leq 2n\tan(\Delta\theta/2)/r(t)$.

By equation (2.10) in Corollary 2.5, we have $\dot{\mathcal{A}}(t) \geq 2n\tan(\Delta\theta/2)/\tilde{r}(t)^{1+\beta}$ where

$$\tilde{r}(t) := \left(v_{\min}(0)^{-1+1/\beta} + (1-\beta)t\right)^{-1/(1-\beta)}.$$

Integration over $(0,t)$ leads $\mathcal{A}(t) \geq \mathcal{A}(0) + n\tan(\Delta\theta/2)(\tilde{r}(t)^{-2} - \tilde{r}(0)^{-2})$. Hence

$$\begin{aligned}
\mathcal{I}(t) &= \frac{(\mathcal{L}(t)r(t))^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)r(t)^2} \\
&\leq \frac{(2n\tan(\Delta\theta/2))^2}{4n\tan(\Delta\theta/2)(\mathcal{A}(0)r(t)^2 + n\tan(\Delta\theta/2)((r(t)/\tilde{r}(t))^2 - (r(t)/\tilde{r}(0))^2))},
\end{aligned}$$

and then the assertion holds since $\lim_{t\to\infty} r(t)/\tilde{r}(t) = 1$, and $\lim_{t\to\infty} r(t) = 0$. $\square$

5 The case where $\beta > 1$

The following lemma is a discrete version of Proposition 1 in Chow–Tsai [CTs], which is based on a version of Aleksandrov reflection method (see Chow–Gulliver [CG], especially Theorem 2.1). Although the following proof will be a parallel story to the proof in [CTs], for the reader's convenience, a proof will be given.

Lemma 5.1 *Let $\beta > 0$ and h a solution of Problem 1'. Then there exists a positive constant λ depending only on the initial data $h(0)$ such that*

$$|h_{j_1}(t) - h_{j_2}(t)| \le \lambda \left|\sin\left(\frac{j_1 - j_2}{2}\Delta\theta\right)\right|, \quad 0 \le j_1,\, j_2 < n, \quad 0 \le t < T_*.$$

Proof. Given integer k define

$$w_j^k := h_j(0) - h_{2k-j}(0).$$

For $j = k$ and $j = k - [n/2]$ we obtain

$$w_k^k = h_k(0) - h_{2k-k}(0) = 0,$$

and

$$w_{k-[n/2]}^k = h_{k-[n/2]}(0) - h_{k+[n/2]}(0) = h_{k+n-(n+1)/2}(0) - h_{k+(n-1)/2}(0) = 0.$$

Here and hereafter we set $[n/2] := (n-1)/2$ and $-[n/2] := -(n+1)/2$ for n odd.

Then there exists $\lambda_k \ge 0$ such that $\lambda_k \sin\theta_{k-j} \ge w_j^k$ for $k - [n/2] \le j \le k$ since $\sin\theta_{[n/2]} > 0$ holds if n is odd.

Let $h_j^\lambda(t) := h_{2k-j}(t) + \lambda_k \sin\theta_{k-j}$. We obtain the inequality

$$h_j^\lambda(0) = h_{2k-j}(0) + \lambda_k \sin\theta_{k-j} \ge h_{2k-j}(0) + w_j^k = h_j(0),$$

and the relation

$$(\Delta_\theta\, h^\lambda + h^\lambda)_j = (\Delta_\theta\, h + h)_{2k-j}.$$

Now we put

$$u_j(t) := h_j^\lambda(t) - h_j(t).$$

Note that $u_k(t) = 0$ and $u_{k-[n/2]}(t) \ge 0$. By mean value theorem, we obtain $\dot u_j = \beta\xi_{k,j}^{\beta-1}(\Delta_\theta\, u + u)_j$. Here $\xi_{k,j}$ is in the between κ_j^{-1} and κ_{2k-j}^{-1} since

$$\kappa_j^{-1} = (\Delta_\theta\, h + h)_j, \quad \text{and} \quad \kappa_{2k-j}^{-1} = (\Delta_\theta\, h + h)_{2k-j} = (\Delta_\theta\, h^\lambda + h^\lambda)_j$$

hold and so ξ_j is non zero function for any $t > 0$.

Fix $T \in (0, T_*)$. Recall $T_* = \infty$ if $\beta \leq 1$, and $T_* < \infty$ if $\beta > 1$. Let $a_j(t) > 0$ and $b_j(t)$ be defined on $t \in [0, T]$. If $u_j(t) \in C^1(0, T) \cap C[0, T]$ is a solution to

$$\dot{u}_j \geq a_j(\Delta_\theta u)_j + b_j u_j, \quad k - [n/2] < j < k, \quad 0 < t < T$$

with the initial condition $u_j(0) \geq 0$, and the boundary condition $u_{k-[n/2]}(t) \geq 0$, $u_k(t) = 0$, then $u_j(t) \geq 0$ holds for all $k - [n/2] \leq j \leq k$ and $0 \leq t \leq T$ by the comparison principle. The proof is a similar to Lemma 2.2.

Hence if we put $\lambda = \lambda(h(0)) > 0$ such as $\lambda \geq \max_k \lambda_k$, then we obtain

$$h_{2k-j}(t) + \lambda \sin \theta_{k-j} \geq h_j(t), \quad k - [n/2] \leq j \leq k, \quad 0 \leq t < T_*.$$

Setting $k = [(j_1 + j_2)/2]$ and $j = j_1$. We conclude that

$$h_{j_2}(t) + \lambda \sin\left(\frac{j_2 - j_1}{2}\Delta\theta\right) \geq h_{j_1}(t). \tag{5.1}$$

Here we have set $j_2 := j_2 - 1$ if $j_1 + j_2$ is odd. Switching j_1 and j_2 in (5.1) implies the assertion. $\square$

Corollary 5.2 *Let $\beta > 1$ and h a solution of Problem* 1' *with the blow-up time T_*. Then*

(1) $h_{\max}(t) - h_{\min}(t) \leq C$.

(2) $h_{\max}(t) \nearrow \infty$ *as* $t \nearrow T_*$.

(3) $h_{\min}(t) \nearrow \infty$ *and so* $\min_{0 \leq j < n} |\boldsymbol{x}_j(t)| \nearrow \infty$ *as* $t \nearrow T_*$.

Proof. First, Lemma 5.1 implies (1) where C is a positive constant depending only $h(0)$. Secondly, If $h_{\max}(t)$ is bounded, then h_j and so κ_j^{-1} is also bounded. Hence v_j is bounded. This is a contradiction to Lemma 2.7. Then (2) holds. Finally, $h_{\min} \geq h_{\max} - C$ holds by (1), and therefore the assertion (2) leads (3). $\square$

The following lemma asserts that any solution polygon expands to infinity approaching an expanding regular polygon.

Lemma 5.3 *Let $\beta > 1$ and h a solution of Problem* 1'. *Then* $\limsup_{t \nearrow T_*} \mathcal{I}(t) \leq 1$.

Proof. Inequality

$$h_j(t) \geq h_{\min}(t) \geq h_{\max}(t) - C \geq h_{\min}(t) - C > 0$$

holds for sufficiently large t since $h_{\min}$ blows up by Corollary 5.2 (3). Therefore we have

$$\mathcal{A}(t) \geq \mathcal{L}(t)(h_{\max}(t) - C)/2 > 0,$$

and equality $\sum_j v_j^{1/\beta} = \sum_j h_j$ yields

$$\mathcal{I}(t) \le \frac{\mathcal{L}(t)^2}{2n\tan(\Delta\theta/2)\mathcal{L}(t)(h_{\max}(t) - C)} \le \frac{h_{\max}(t)}{h_{\max}(t) - C}.$$

This leads the assertion. □

By the following lemma it will be clear that the specific order of the blow-up rate, and this will complete a proof of Theorem A.

Lemma 5.4 *Let $\beta > 1$. For sufficiently large $t_0 \in (0, T_*)$ there exist constants c_1 and c_2 depending on β, t_0 and $h(0)$ such that*

$$c_1(T_* - t)^{-1/(\beta-1)} \le h_j(t) \le c_2(T_* - t)^{-1/(\beta-1)}, \quad 0 \le j < n, \quad t_0 \le t < T_*.$$

Proof. We will prove the estimate $h_{\min}(t) \le ((\beta-1)(T_* - t))^{-1/(\beta-1)}$. The following proof is based on the proof of Lemma 2.2 in Stancu [S2].

If $h_{\min}(t) = h_{j_t}(t)$ for some $0 \le j_t < n$, then

$$\frac{d}{dt}h_{j_t}(t) = (\Delta_\theta h + h)_{j_t}^\beta \ge h_{j_t}^\beta$$

for any $t \in (0, T_*)$. The function $h_{\min}(t)$ is a continuous function, but it may not be differentiable. It is, however, Lipschitz. Therefore we have

$$\frac{d^-}{dt}h_{\min}(t) := \liminf_{\varepsilon \uparrow 0} \frac{h_{\min}(t+\varepsilon) - h_{\min}(t)}{\varepsilon} \ge h_{\min}^\beta.$$

Since comparison principle (similar result to Lemma 2.2 for the operator d^-/dt) holds, once $h_{\min}$ is bigger than $R(t)$, the solution of $\dot{R}(t) = R(t)^\beta$, it must stay bigger. This comparison holds when $R(t) = ((\beta-1)(T_* - \delta - t)^{-1/(\beta-1)}$ for any $\delta > 0$ and any t. Then $h_{\min}$ must blow up at $T_* - \delta$, earlier time than the blow-up time T_*. This is a contradiction. Hence $h_{\min}(t) < ((\beta-1)(T_* - \delta - t))^{-1/(\beta-1)}$ holds, and this proves the assertion.

One can prove the estimate $h_{\max}(t) \ge ((\beta-1)(T_* - t))^{-1/(\beta-1)}$ in a similar way.

The assertion will be proved by the combination of Corollary 5.2 and the above estimates. □

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Chapter 4

On an area-perserving crystalline motion

Outline: The asymptotic behavior of solutions to an area-preserving crystalline motion is investigated in this chapter. In this equation, the area enclosed by the solution polygon is preserved, while its circumference keeps on shrinking. By establishing several isoperimetric inequalities, we show that the asymptotic shape of every solution polygon is a regular polygon.
Key Words: crystalline motion, crystalline curvature, curve-shortening, area-preserving, entropy estimate, isoperimetric inequality, Bonnesen's inequality, Gage's inequality.

1 Introduction and main results

The isoperimetric inequality for closed embedded curves (in the plane) represents the variational problem: what is the shape which has the least total length of the curve for the fixed enclosed area? The answer is a circle. Gage [G2] considers the gradient flow of the length functional keeping the area enclosed by the curve constant; and shows that any convex curve in the plane which evolves by this gradient flow remains convex and converges to a circle in the C^∞-metric.

Our aim in this chapter is to answer the problem: what is the asymptotic shape of a solution polygon which evolved by the gradient flow of the length functional keeping the area enclosed by the polygon constant? For simplycity, we restrict the polygon is a convex closed polygon, say $\mathcal{P}$, in the plane with the angle between two adjacent sides of $\mathcal{P}$ is $\pi - \Delta\theta$, where $\Delta\theta = 2\pi/n$, and n is a number of sides of the polygon $\mathcal{P}$. We call this polygon the admissible polygon. From the admissiblity, without loss of generality, the inward normal vector of the j-th side is given as $\boldsymbol{n}_j := -{}^t(\cos\theta_j, \sin\theta_j)$, and the j-th tangent vector is given as $\boldsymbol{t}_j := {}^t(-\sin\theta_j, \cos\theta_j)$. Here and hereafter we use the notation

$\theta_j = j\Delta\theta$. Let $\boldsymbol{x}_j$ be the positon vector on (the line containing) the j-th side of polygon $\mathcal{P}$. Then the length of the j-th side, say d_j, is given as

$$d_j[\boldsymbol{x}] = \langle \boldsymbol{x}_{j+1} - \boldsymbol{x}_{j-1}, \boldsymbol{t}_j \rangle - \cot \Delta\theta \langle \boldsymbol{x}_{j+1} - 2\boldsymbol{x}_j + \boldsymbol{x}_{j-1}, \boldsymbol{n}_j \rangle,$$

and so the total length of $\mathcal{P}$ is $\mathcal{L}[\boldsymbol{x}] = \sum_{0 \le j < n} d_j[\boldsymbol{x}]$.

Now we shall derive the evolution equation from the gradient flow of $\mathcal{L}[\boldsymbol{x}]$ keeping the area enclosed by the polygon $\mathcal{P}$ constant. For a small perturbation $\varepsilon \boldsymbol{z}_j$, the change of the j-th length is $d_j[\boldsymbol{x} + \varepsilon \boldsymbol{z}] = d_j[\boldsymbol{x}] + \varepsilon d_j[\boldsymbol{z}]$. Then the first variation of $\mathcal{L}$ is

$$\left. \frac{d}{d\varepsilon} \mathcal{L}[\boldsymbol{x} + \varepsilon \boldsymbol{z}] \right|_{\varepsilon=0} = \sum_{0 \le j < n} d_j[\boldsymbol{z}] = \sum_{0 \le j < n} \langle -\kappa_j \boldsymbol{n}_j, \boldsymbol{z}_j \rangle d_j[\boldsymbol{x}],$$

where κ_j is called crystalline curvature (see below in details) defined as

$$\kappa_j = \frac{2\tan(\Delta\theta/2)}{d_j}, \quad 0 \le j < n. \tag{1.1a}$$

The enclosed area of $\mathcal{P}$ is $\mathcal{A}[\boldsymbol{x}] = \sum_{0 \le j < n} \langle \boldsymbol{x}_j, -\boldsymbol{n}_j \rangle d_j[\boldsymbol{x}]/2$, and so we have the first variation of $\mathcal{A}$:

$$\left. \frac{d}{d\varepsilon} \mathcal{A}[\boldsymbol{x} + \varepsilon \boldsymbol{z}] \right|_{\varepsilon=0} = \sum_{0 \le j < n} \langle -\boldsymbol{n}_j, \boldsymbol{z}_j \rangle d_j[\boldsymbol{x}].$$

Since $\sum_{0 \le j < n} \langle -\boldsymbol{n}_j, \kappa_j \boldsymbol{n}_j \rangle d_j = -2n \tan(\Delta\theta/2)$ holds, the gradient flow of $\mathcal{L}$ along polygons which enclose a fixed area $\mathcal{A} \equiv const.$ is

$$\frac{d}{dt} \boldsymbol{x}_j(t) = v_j \boldsymbol{n}_j, \quad 0 \le j < n, \tag{1.1b}$$

with the normal velocity:

$$v_j = \kappa_j - \frac{2n \tan(\Delta\theta/2)}{\mathcal{L}}, \quad 0 \le j < n. \tag{1.1c}$$

The problem of this chapter is "for a given admissible polygon $\mathcal{P}_0$, find a family of polygons $\{\mathcal{P}(t)\}_{0 \le t < T}$ satisfying (1.1) with $\mathcal{P}(0) = \mathcal{P}_0$ and a duration $T > 0$." We note that the angle between two adjacent sides of $\mathcal{P}(t)$ is always $\pi - \Delta\theta$ as long as solution polygons exist, that is a polygon $\mathcal{P}(t)$ is the admissible polygon.

Our main result in this chapter is the following.

Theorem A *Let $n \ge 4$, and $\mathcal{P}_0$ an admissible polygon. A solution polygon $\mathcal{P}(t)$ of Problem* (1.1) *exists globally in time, and the solution converges to a regular polygon in the Hausdorff metric as t tends to infinity, keeping the area enclosed by the polygon $\mathcal{P}(t)$ constant.*

Now we will refer to a background and related topics, especially for terminologies the "admissibility" and the "crystalline." The so-called Wulff's problem is a development form of the beginning variational problem: what is the shape which has the least total interfacial energy of the curve for the fixed enclosed area? The answer is the Wulff shape. The gradient flow of the total interfacial energy of the curve without the constraint fixed enclosed area provieds the weighted curvature flow: the normal velocity is propotional to the weighted curvature, i.e. the curvature times a weighted function. Actually, this weighted function is the inverce of curvature of the Wulff shape's boundary. Such an evolution equation arises in a model describing the motion of interface separating materials in an anisotropic medium. See Gurtin [Gu1]. Besides the smooth interfacial energy, some materials have a nonsmooth interfacial energy, for instance, called crystalline. In this case, we do not calculate the gradient flow of total interfacial energy in the usual sense. For such an energy, Angenent–Gurtin [AGu], and correspondingly Taylor [T1] (see also [T2], [T3], [Gu1]) introduced a weak formulation: the motion of admissible piecewise linear curves by crystalline curvature. This motion is called the crystalline flow, or the crystalline motion. Recently, many authors investigate the crystalline flow and its application. See, e.g., [TCH, GirK2, GMHG1, Gu2] for including a survey, and [AlT, GGu, GMHG2, GMHG4, GGuM, Ry, GP] for an application and development. The asymptotic behavior, especially the asymptotic self-similarity, of solutions to a crystalline flow is investigated by Stancu [S1, S2]. The author showed the asymptotic self-similarity in [S2], and the uniqueness of self-similar solution, under a symmetric assumption, in [S1]. For a perspective application including numerical approximation of crystalline flow, we refer to [GirK1, FG, GMHG3, GMHG5, EGS, Gir, UY1, IS, E]. In particular, an implicit crystalline algorithm for Problem (1.1) is studied in [UY2].

The organization of this chapter is as follows: in section 2, we give several fundamental properties of solutions to Problem (1.1). In section 3, some isoperimetric inequalities for admissible polygons, including a discrete version of Gage's inequality, will be shown. In section 4, we will give the proof of Theorem A, especially the time global existence of solutions via some estimates, e.g., entropy estimate.

2 Some properties

Throughout this chapter we use the notation $\sum_j u_j$, $u_{\max}$, $u_{\min}$ and $\dot{u}(t)$ for $\sum_{0\le j<n} u_j$, $\max_{0\le j<n} u_j$, $\min_{0\le j<n} u_j$ and $du(t)/dt$, respectively. Hereafter we assume $n \ge 4$.

2.1 Discrete curve-shortening

By virtue of the argument in section 1, the time gradient of the j-th side is

$$\dot{d}_j(t) = \frac{d}{dt} d_j[\boldsymbol{x}(t)] = d_j[\dot{\boldsymbol{x}}(t)] = -2\tan\frac{\Delta\theta}{2}(\Delta_\theta v + v)_j, \tag{2.1}$$

and so the time gradient of the length $\mathcal{L}(t) = \mathcal{L}[\boldsymbol{x}(t)]$ is

$$\dot{\mathcal{L}}(t) = \sum_j \dot{d}_j(t) = -2\tan\frac{\Delta\theta}{2}\sum_j v_j = -2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j + \frac{(2n\tan(\Delta\theta/2))^2}{\mathcal{L}(t)}.$$

Here the operator Δ_θ is a kind of central difference operator such as

$$(\Delta_\theta u)_j := \frac{u_{j+1} - 2u_j + u_{j-1}}{2(1-\cos\Delta\theta)} = \frac{(\mathrm{D}_+ u)_j - (\mathrm{D}_+ u)_{j-1}}{2\sin(\Delta\theta/2)}, \quad (\mathrm{D}_+ u)_j := \frac{u_{j+1} - u_j}{2\sin(\Delta\theta/2)}.$$

Using the Schwarz inequality we have

$$\left(2n\tan\frac{\Delta\theta}{2}\right)^2 = 2\tan\frac{\Delta\theta}{2}\Big(\sum_j \sqrt{\kappa_j}\sqrt{d_j}\Big)^2 \le 2\tan\frac{\Delta\theta}{2}\mathcal{L}\sum_j \kappa_j,$$

and then $\dot{\mathcal{L}}(t) \le 0$ holds even though the area is fixed: $\mathcal{A}(t) \equiv \mathcal{A}(0)$. Equality occurs when the polygon $\mathcal{P}$ is a regular polygon, otherwise the evolution of polygons is a discrete curve-shortening motion.

2.2 Equivalent formulation

From (1.1c), (2.1) and the length $\mathcal{L} = 2\tan(\Delta\theta/2)\sum_j \kappa_j^{-1}$, we can restate Problem (1.1) as follows:

Problem 1 *Let $n \ge 4$. Find a function $\kappa(t) = (\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) \in [C^1(0,T)]^n$, and a duration $T \in (0,\infty]$ satisfying*

$$\dot{\kappa}_j = \kappa_j^2(\Delta_\theta \kappa)_j + \kappa_j^3 - \frac{n\kappa_j^2}{\sum_{0\le i<n}\kappa_i^{-1}}, \quad 0 \le j < n, \quad 0 < t < T, \tag{2.2a}$$

$$\kappa_j(0) = \kappa_j^0, \quad 0 \le j < n, \tag{2.2b}$$

$$\kappa_{-1}(t) = \kappa_{n-1}(t), \quad \kappa_n(t) = \kappa_0(t), \quad 0 \le t < T \tag{2.2c}$$

where κ_j^0 is the initial crystalline curvature of $\mathcal{P}_0$.

Remark 2.1 (equivalence) Problem (1.1) and Problem 1 are equivalent except the indefiniteness of position of the polygon. Indeed, suppose κ is a solution of Problem 1, then we have

$$\begin{aligned}\frac{1}{2\tan(\Delta\theta/2)}\frac{d}{dt}\sum_j 2\tan\frac{\Delta\theta}{2}\kappa_j(t)^{-1}\boldsymbol{t}_j &= -\sum_j\left((\Delta_\theta \kappa + \kappa)_j - \frac{n}{\sum_i \kappa_i^{-1}}\right)\boldsymbol{t}_j \\ &= -\sum_j(\Delta_\theta \boldsymbol{t} + \boldsymbol{t})_j\kappa_j + \frac{n}{\sum_i \kappa_i^{-1}}\sum_j \boldsymbol{t}_j = \boldsymbol{0}.\end{aligned}$$

Here we have used the summation by parts:

$$\sum_j f_j(\Delta_\theta g)_j = -\sum_j (\mathrm{D}_+ f)_j(\mathrm{D}_+ g)_j = \sum_j g_j(\Delta_\theta f)_j, \tag{2.3}$$

and $(\Delta_\theta \boldsymbol{t})_j = -\boldsymbol{t}_j$.

Since the j-th vertex $\boldsymbol{B}_j(t)$ of $\mathcal{P}(t)$ is given as the following:

$$\begin{aligned} \boldsymbol{B}_j(t) &= \langle \boldsymbol{x}_{j-1}(t) - \boldsymbol{x}_j(t), \boldsymbol{t}_j + \boldsymbol{n}_j \cot\Delta\theta \rangle\, \boldsymbol{t}_j + \boldsymbol{x}_j(t), \\ &= \boldsymbol{B}_0(t) + \sum_{0\le m<j} d_m(t)\boldsymbol{t}_m, \quad 1 \le j \le n, \quad 0 \le t < T \end{aligned}$$

with $\boldsymbol{B}_0(t) \equiv \boldsymbol{B}_n(t)$, we can construct a closed convex n-gon, whose length of the j-th side is $2\tan(\Delta\theta/2)\kappa_j(t)^{-1} =: d_j(t)$ and the j-th normal vector is $\boldsymbol{n}_j$, as long as κ is a solution of Problem 1. This n-gon is the very solution polygon of Problem (1.1).

3 Isoperimetric inequalities

Let the j-th support function be:

$$h_j(t) = \langle \boldsymbol{x}_j(t), -\boldsymbol{n}_j \rangle, \quad 0 \le j < n.$$

It is easy to check that the next relation between d_j and h_j:

$$d_j(t) = 2\tan\frac{\Delta\theta}{2}(\Delta_\theta h + h)_j, \quad 0 \le j < n$$

by geometry.

3.1 Discrete version of isoperimetric inequality

The isoperimetric ratio for a closed embedded curve Γ is usually defined as

$$\mathrm{iso}\,(\Gamma) = \frac{L^2}{4\pi A},$$

where L and A are the length and the enclosed area of Γ, respectively. It is well known that the isoperimetric inequality $\mathrm{iso}\,(\Gamma) \ge 1$ holds. Equality $\mathrm{iso}\,(\Gamma) = 1$ holds if and only if the curve Γ is a circle. In this sense, if Γ is a regular n-gon, say $\mathcal{P}_n$, then it holds that

$$\mathrm{iso}\,(\mathcal{P}_n) = \frac{n\tan(\Delta\theta/2)}{\pi} > 1, \quad \text{and that} \quad \lim_{n\to\infty} \mathrm{iso}\,(\mathcal{P}_n) = 1.$$

In this chapter, we define the isoperimetric ratio such as

$$\mathcal{I}(t) := \frac{\mathcal{L}(t)^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)} \tag{3.1}$$

since we consider the polygons $\mathcal{P}(t)$ which the angle between two adjacent side is $\pi - \Delta\theta$. For this isoperimetric ratio, the next isoperimetric inequality

$$\mathcal{I}(t) \geq 1 \tag{3.2}$$

holds. The equality $\mathcal{I}(t) = 1$ holds if and only if the polygon $\mathcal{P}(t)$ is a regular polygon. The isoperimetric ineqality iso$(\Gamma) \geq 1$ for smooth curves is proved by using the motion by curvature. In a similar way, one can prove the isopermetric inequality (3.2). See, e.g., Yazaki [Y1].

3.2 Bonnesen's inequalities

The next Bonnesen's inequalities hold for any admissible polygons (see Eggleston [Eg]):

$$h_{\max}\mathcal{L} - \mathcal{A} - n\tan\frac{\Delta\theta}{2}h_{\max}^2 \geq 0, \quad h_{\min}\mathcal{L} - \mathcal{A} - n\tan\frac{\Delta\theta}{2}h_{\min}^2 \geq 0. \tag{3.3}$$

Fix any j. Then there exists $\mu \in [0,1]$ such that $h_j = \mu h_{\max} + (1-\mu)h_{\min}$. Therefore

$$\begin{aligned} h_j^2 &= \mu^2 h_{\max}^2 + (1-\mu)^2 h_{\min}^2 + 2\mu(1-\mu)h_{\min}h_{\max} \\ &\leq \mu h_{\max}^2 + (1-\mu)h_{\min}^2 + \mu(1-\mu)(h_{\min}^2 + h_{\max}^2) = \mu h_{\max}^2 + (1-\mu)h_{\min}^2. \end{aligned}$$

From (3.3) and the above inequality, we obtain:

$$h_j\mathcal{L} - \mathcal{A} - n\tan\frac{\Delta\theta}{2}h_j^2 \geq 0, \quad 0 \leq j < n. \tag{3.4}$$

Bonnesen's inequalities (3.3) is more accurate than the isoperimetric inequality (3.2) in the next sence:

$$\mathcal{I}(t) \geq 1 + \frac{n\tan(\Delta\theta/2)}{4\mathcal{A}}(h_{\max} - h_{\min})^2 \geq 1. \tag{3.5}$$

3.3 Discrete version of Gage's inequality

We shall present a discrete vertion of Gage's inequality [G1].

Multiply (3.4) by d_j, and sum them over $0 \leq j < n$:

$$\mathcal{L}\sum_j h_j d_j - \mathcal{A}\sum_j d_j - n\tan\frac{\Delta\theta}{2}\sum_j h_j^2 d_j \geq 0.$$

Then we have

$$\begin{aligned} \mathcal{L} &= \sum_j d_j = 2\tan\frac{\Delta\theta}{2}\sum_j h_j \leq 2\tan\frac{\Delta\theta}{2}\sqrt{\sum_j h_j^2 d_j \sum_j \frac{1}{d_j}} \\ &= \sqrt{2\tan\frac{\Delta\theta}{2}\sum_j h_j^2 d_j \sum_j \kappa_j} \leq \sqrt{\frac{2}{n}\mathcal{L}\mathcal{A}\sum_j \kappa_j}. \end{aligned}$$

Hence we obtain a discrete vertion of Gage's inequality:

$$\frac{n\tan(\Delta\theta/2)\mathcal{L}}{\mathcal{A}} \leq 2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j = \sum_j \kappa_j^2 d_j. \tag{3.6}$$

3.4 Estimate of $\mathcal{I}(t)$

By (3.6), we have

$$\dot{\mathcal{I}}(t) \leq -\frac{2n\tan(\Delta\theta/2)}{\mathcal{A}}(\mathcal{I}(t)-1).$$

Then

$$\mathcal{I}(t) \leq 1 + Ce^{-2n\tan(\Delta\theta/2)t/\mathcal{A}} \tag{3.7}$$

where $C = C(\mathcal{P}_0, \Delta\theta) \geq 0$.

Therefore if an evolving polygon does not develop singularilies for all time, then it converges to a regular polygon. By (3.5), the convergence sence is the one in the Hausdorff metric (see [G2], especially the proof of COROLLARY 2.5). Consequently,

Lemma 3.1 *If a solution polygon does not develop singularilies in a finite time, then it converges to a regular polygon in the Hausdorff metric as $t \nearrow \infty$.*

4 Proof of Theorem A

In this section we will prove the assumption in Lemma 3.1, that is, the time global existence of solution polygons, and the main theorem. The following strategy is a discrete version of Gage [G2], section 3 in particular.

As in the proof of LEMMA 3.1 in [G2], we can show the next lower bound of κ_j by using the maximum principle and the isoperimetric inequality (3.2).

Lemma 4.1 *$\kappa_{\min}(t)$ is a non-decreasing in time, and for a parameter satisfying $\mu > n\tan(\Delta\theta/2)/(4\mathcal{A})$, $\kappa_{\min}(t) \geq \kappa_{\min}(0)e^{-\mu t}$ holds uniformly in $\Delta\theta$.*

Proof. Let $u_j(t) = \kappa_j(t)e^{\mu t}$ for a constant μ. Then u_j satisfies the evolution equation

$$\dot{u}_j = \kappa_j^2(\Delta_\theta u)_j + f(\kappa_j)u_j \tag{4.1}$$

where f is a quadratic polynomial: $f(x) = x^2 - 2n\tan(\Delta\theta/2)x/\mathcal{L} + \mu$ whose discriminant is $\mathcal{D}_f := (2n\tan(\Delta\theta/2)/\mathcal{L})^2 - 4\mu$. The isoperimetric inequality (3.2) provides $\mathcal{D}_f \leq n\tan(\Delta\theta/2)/\mathcal{A} - 4\mu$. Hence for sufficiently large μ, $\mathcal{D}_f < 0$ holds, and so we have $f > 0$.

Let $u_\varepsilon := u_{\min}(0)/(1+\varepsilon)$ for any fixed $\varepsilon > 0$. Fix any $T > 0$. Suppose that $\min_{0\leq t<T, 0\leq j<n} u_j(t)$ attains u_ε at $j = j_0$ and $t = t_0$ (n.b. $t_0 > 0$ is clear). At this point, however, $\dot{u}_{j_0}(t_0) \leq 0$, $(\Delta_\theta u(t_0))_{j_0} \geq 0$ and $f(\kappa_{j_0}(t_0))u_{j_0}(t_0) > 0$. This is a contradiction to (4.1), and proves that $u_{\min}(t)$ is a non-decreasing function. Therefore

$$\kappa_{\min}(t) = u_{\min}(t)e^{-\mu t} \geq u_{\min}(0)e^{-\mu t} = \kappa_{\min}(0)e^{-\mu t} > 0$$

holds. $\square$

We introduce now the median crystalline curvature which is a similar to the median curvature in [GH], and the median discrete weighted curvature in [Gir].

Definition 4.2 (median crystalline curvature) $\kappa_*(t) := \max_{0\le j<n} \min_{j+1\le i\le j+[n/2]} \kappa_i(t)$.

One can easily obtain: $\sum_{j=1}^{[n/2]} \sin\theta_j \le 2\cot(\Delta\theta/2)$, where $[n/2]$ is $n/2$ for n even and $(n-1)/2$ for n odd, since the left hand side equals to $\cot(\Delta\theta/2)$ for n even, and $(1+\sec(\Delta\theta/2))\cot(\Delta\theta/2)/2$ for n odd.

In order to obtain the maximum crystalline curvature, we first prove the next geometric estimate.

Lemma 4.3 (geometric estimate) *$\kappa_*(t) \le 2\mathcal{L}(0)/\mathcal{A}$ holds for $t \ge 0$.*

Proof. We assume that j_0 is a value of j which attains the maximum of κ_j. A polygon lies between parallel lines whose distance is less than

$$\begin{aligned}\sum_{j=j_0+1}^{j_0+[n/2]} \langle d_j \boldsymbol{t}_j, \boldsymbol{n}_{j_0}\rangle &= \sum_{j=j_0+1}^{j_0+[n/2]} \sin(\theta_j-\theta_{j_0})d_j = 2\tan\frac{\Delta\theta}{2}\sum_{j=1}^{[n/2]} \sin\theta_j \kappa_{j_0+j}^{-1} \\ &\le \frac{2\tan(\Delta\theta/2)}{\kappa_*}\sum_{j=1}^{[n/2]}\sin\theta_j \le \frac{4}{\kappa_*}\end{aligned}$$

The diameter is bounded by $\mathcal{L}/2$ and the area is bounded by the width times the diameter:

$$\mathcal{A} \le \frac{\mathcal{L}(t)}{2}\frac{4}{\kappa_*(t)} = \frac{2\mathcal{L}(t)}{\kappa_*(t)}.$$

Hence $\kappa_*(t) \le 2\mathcal{L}(0)/\mathcal{A}$ holds. $\square$

Next, we will estimate the following entropy:

$$\mathcal{E}(t) := 2\tan\frac{\Delta\theta}{2}\sum_{0\le j<n}\log\kappa_j(t) \tag{4.2}$$

Lemma 4.4 (entropy estimate) *For any fixed $T>0$, $\mathcal{E}(t)$ is bounded on $[0,T)$.*

Proof. By using the summation by parts (2.3), one has

$$\dot{\mathcal{E}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j(\kappa^2-(\mathrm{D}_+\kappa)^2)_j - \frac{n(2\tan(\Delta\theta/2))^2}{\mathcal{L}}\sum_j\kappa_j.$$

We use the same estimates as in the proof of Girão[Gir] (section 2, *Fourth*), and have the next estimate (see also [Y2]):

$$2\tan\frac{\Delta\theta}{2}\sum_j(\kappa^2-(\mathrm{D}_+\kappa)^2)_j \le 2n\tan\frac{\Delta\theta}{2}\kappa_*^2 + 4\tan\frac{\Delta\theta}{2}\kappa_*\sum_j\kappa_j.$$

Let $C_* := 2\mathcal{L}(0)/\mathcal{A}$, then

$$\begin{aligned}\dot{\mathcal{E}}(t) &\le 2n\tan\frac{\Delta\theta}{2}\kappa_*^2 + \left(4\tan\frac{\Delta\theta}{2}\kappa_* - \frac{n(2\tan(\Delta\theta/2))^2}{\mathcal{L}}\right)\sum_j \kappa_j \\ &= 2n\tan\frac{\Delta\theta}{2}\kappa_*^2 + \left(2\kappa_* - \frac{2n\tan(\Delta\theta/2)}{\mathcal{L}}\right)\left(\frac{(2n\tan(\Delta\theta/2))^2}{\mathcal{L}} - \dot{\mathcal{L}}\right) \\ &\le 2n\tan\frac{\Delta\theta}{2}C_*^2 - 2C_*\dot{\mathcal{L}} + 2C_*\frac{(2n\tan(\Delta\theta/2))^2}{\sqrt{4n\tan(\Delta\theta/2)\mathcal{A}}}.\end{aligned}$$

Here we have used the equation

$$2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j = \frac{(2n\tan(\Delta\theta/2))^2}{\mathcal{L}} - \dot{\mathcal{L}},$$

and the isoperimetric inequality (3.2). Therefore

$$\mathcal{E}(t) \le \mathcal{E}(0) + 2n\tan\frac{\Delta\theta}{2}C_*^2 t + 2C_*\mathcal{L}(0) + 4C_*\sqrt{\frac{(n\tan(\Delta\theta/2))^3}{\mathcal{A}}}\,t \le C_0 + C_1 T \quad (4.3)$$

holds, where C_0 and C_1 depend only on $\mathcal{P}_0$ and $\Delta\theta$. □

Lemma 4.5 *For a constant $C_2 = C_2(\mathcal{P}_0, \Delta\theta)$ we have*

$$\begin{aligned}&2\tan\frac{\Delta\theta}{2}\sum_j (\mathrm{D}_+\kappa)_j^2 \\ &\le 2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j^2 + 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2\int_0^t \frac{d}{d\tau}\left(\frac{1}{\mathcal{L}(\tau)}\right)\sum_j \kappa_j(\tau)\,d\tau + C_2.\end{aligned}$$

Proof. We calculate, from (2.2a),

$$\begin{aligned}&2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j\left(\kappa^2 - (\mathrm{D}_+\kappa)^2 - \frac{2n\tan(\Delta\theta/2)}{\mathcal{L}}\kappa\right)_j \\ &= 2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j \kappa_j\left(\Delta_\theta\kappa + \kappa - \frac{2n\tan(\Delta\theta/2)}{\mathcal{L}}\right)_j \\ &= 4\tan\frac{\Delta\theta}{2}\sum_j\left(\frac{\dot\kappa}{\kappa}\right)_j^2 + \frac{n(2\tan(\Delta\theta/2))^2}{\mathcal{L}}\sum_j\dot\kappa_j + \frac{n(2\tan(\Delta\theta/2))^2}{\mathcal{L}^2}\dot{\mathcal{L}}\sum_j\kappa_j \\ &\ge n\left(2\tan\frac{\Delta\theta}{2}\right)^2\frac{d}{dt}\left(\frac{1}{\mathcal{L}}\sum_j\kappa_j\right) - 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2\frac{d}{dt}\left(\frac{1}{\mathcal{L}}\right)\sum_j\kappa_j.\end{aligned}$$

Then

$$\begin{aligned}2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j(\mathrm{D}_+\kappa)_j^2 &\le 2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j\kappa_j^2 - 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2\frac{d}{dt}\left(\frac{1}{\mathcal{L}}\sum_j\kappa_j\right) \\ &\quad + 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2\frac{d}{dt}\left(\frac{1}{\mathcal{L}}\right)\sum_j\kappa_j.\end{aligned}$$

Hence

$$
\begin{aligned}
2\tan\frac{\Delta\theta}{2}\sum_j (\mathrm{D}_+\kappa)_j^2 \;\le\; & 2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j^2 \\
& + 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2 \int_0^t \frac{d}{d\tau}\left(\frac{1}{\mathcal{L}(\tau)}\right)\sum_j \kappa_j(\tau)\,d\tau \\
& + 2\tan\frac{\Delta\theta}{2}\sum_j \left((\mathrm{D}_+\kappa(0))^2-\kappa(0)^2\right)_j \;+\; 2n\left(2\tan\frac{\Delta\theta}{2}\right)^2 \frac{1}{\mathcal{L}(0)}\sum_j \kappa_j(0).
\end{aligned}
$$

holds, and this leads the assertion. □

Corollary 4.6 *If $\kappa_{\max}(t) \le M$ on $[0,T)$, then for a constant $C_3 = C_3(\mathcal{P}_0, \Delta\theta)$*

$$
2\tan\frac{\Delta\theta}{2}\sum_{0\le j<n} (\mathrm{D}_+\kappa)_j^2 \le (MC_3)^2.
$$

Proof. By using the isoperimetric inequality (3.2), we have

$$
\begin{aligned}
2n\left(2\tan\frac{\Delta\theta}{2}\right)^2 \int_0^t \frac{d}{d\tau}\left(\frac{1}{\mathcal{L}(\tau)}\right)\sum_j \kappa_j(\tau)\,d\tau \;\le\; & 2\left(2n\tan\frac{\Delta\theta}{2}\right)^2 M\left(\frac{1}{\mathcal{L}(t)}-\frac{1}{\mathcal{L}(0)}\right) \\
\le\; & \frac{4(n\tan(\Delta\theta/2))^{3/2}}{\sqrt{\mathcal{A}}}M.
\end{aligned}
$$

Then

$$
\begin{aligned}
2\tan\frac{\Delta\theta}{2}\sum_{0\le j<n} (\mathrm{D}_+\kappa)_j^2 \;\le\; & 2n\tan\frac{\Delta\theta}{2}M^2 + \frac{4(n\tan(\Delta\theta/2))^{3/2}}{\sqrt{\mathcal{A}}}M + C_2 \\
=\; & M^2\left(2n\tan\frac{\Delta\theta}{2} + \frac{4(n\tan(\Delta\theta/2))^{3/2}}{\sqrt{\mathcal{A}}M} + \frac{C_2}{M^2}\right) \\
\le\; & (MC_3)^2.
\end{aligned}
$$

Here we have used $2n\tan(\Delta\theta/2) = \sum_j \kappa_j d_j$, $\dot{\mathcal{L}}(t) \le 0$ and so the fact that $M \ge 2n\tan(\Delta\theta/2)/\mathcal{L}(0)$. □

Lemma 4.7 *Let*

$$
M := \sup_{0\le t<T} \kappa_{\max}(t)
$$

then

$$
C_4 \log M \le C_5 + C_6 T,
$$

where C_4, C_5 and C_6 depend only on $\mathcal{P}_0$ and $\Delta\theta$.

Proof. Let t_1 be a value where $\kappa_{\max}(t_1) = 3M/4$, and j_1 a value of j which attains the maximum of $\kappa_j(t_1)$, i.e. $\kappa_{j_1}(t_1) = 3M/4$. By using the Schwarz inequality, $\sin\Delta\theta \le \Delta\theta$ and Corollary 4.6, we have

$$\begin{aligned}
\kappa_{j_1}(t_1) - \kappa_j(t_1) &= 2\sin\frac{\Delta\theta}{2} \sum_{j\le i<j_1} (\mathrm{D}_+ \kappa)_i \\
&\le 2\sin\frac{\Delta\theta}{2} \sqrt{\sum_{j\le i<j_1} \left(2\tan\frac{\Delta\theta}{2}\right)^{-1}} \sqrt{\sum_{j\le i<j_1} 2\tan\frac{\Delta\theta}{2}(\mathrm{D}_+ \kappa)_i^2} \\
&\le MC_3\sqrt{\theta_{j_1} - \theta_j}.
\end{aligned}$$

Therefore

$$\kappa_j(t_1) \ge \kappa_{j_1}(t_1) - MC_3\sqrt{\theta_{j_1} - \theta_j} = \left(\frac{3}{4} - C_3\sqrt{\theta_{j_1} - \theta_j}\right) M$$

holds.

If $C_3\sqrt{\theta_{j_1} - \theta_j} \le 1/4$ for m values of j (there is at least one value of j, $j = j_1$, then $m \ge 1$), we can estimate

$$\begin{aligned}
\mathcal{E}(t_1) &\ge 2\tan\frac{\Delta\theta}{2} \sum_{|\theta_{j_1}-\theta_j|\le(4C_1)^{-2}} \log\kappa_j(t_1) + 2\tan\frac{\Delta\theta}{2} \sum_{|\theta_{j_1}-\theta_j|>(4C_1)^{-2}} \log\kappa_j(t_1) \\
&\ge 2m\tan\frac{\Delta\theta}{2}\log\left(\frac{M}{2}\right) + 2(n-m)\tan\frac{\Delta\theta}{2}\log\left(\kappa_{\min}(0)e^{-\mu t_1}\right)
\end{aligned}$$

Hence by Lemma 4.1 and (4.3), the assertion is proved. □

Proof of Theorem A. First, by the general argument for ordinary differential equation, a solution polygon $\mathcal{P}(t)$ of Problem (1.1), that is a solution κ of Problem 1 exists uniquely and locally in time, say T; then Lemma 4.3 (the geometric esitimate), Lemma 4.4 (the entropy esitimate) and Lemma 4.7 imply that $\kappa_j(t)$ is bounded on $[0, T)$. Secondly, we use the local solutions to extend the time global on which the solution is defined. This follows that the solution can be defined globally in time. Finally, Lemma 3.1 provides the asymptotic behavior of solutions. □

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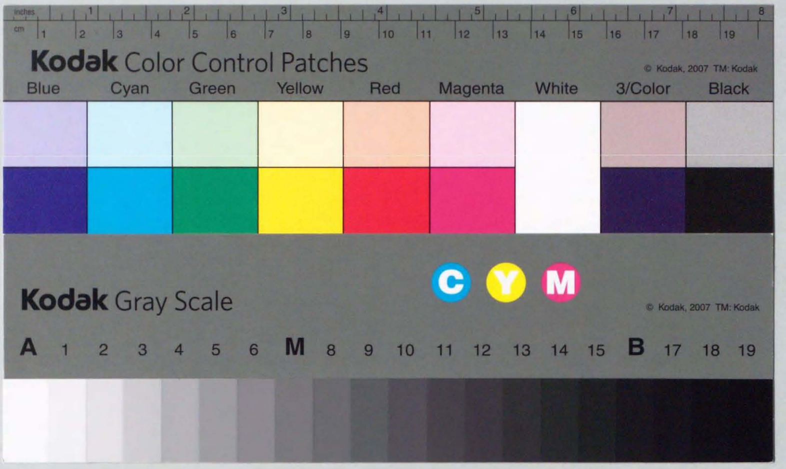
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