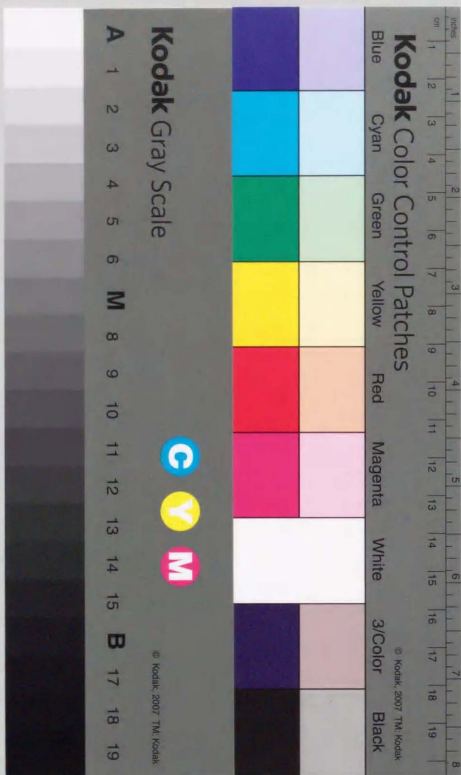


Precise Estimation for Large Deviations

大偏差原理の精密評価について

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論文の内容の要旨

論文題目 Precise Estimation for Large Deviations

大偏差原理の精密評価について

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この論文は大偏差理論の精密評価に関して研究した。

d 次元ユークリッド空間 \mathbb{R}^d を考える。 $C_b(\mathbb{R}^d)$ は \mathbb{R}^d 上の有界な連続関数全体であり、 $\|\cdot\|_\infty$ をノルムとして持つ。 $M(\mathbb{R}^d)$ を \mathbb{R}^d 上の有限変動を持つ符号付き測度の全体とし、ノルムは全変動ノルムである。また、 $M(\mathbb{R}^d)$ 上の弱*トポロジーも考える。すべての $f \in C_b(\mathbb{R}^d)$ と $R \in M(\mathbb{R}^d)$ に対して、ベアリングを $\langle f, R \rangle = \int_{\mathbb{R}^d} f dR$ とする。 \mathbb{R}^d 上の確率測度全体および全測度が 0 であるような有限変動の符号付き測度全体をそれぞれ $\mathcal{P}(\mathbb{R}^d)$ と $\mathcal{M}_0(\mathbb{R}^d)$ と書く。 $\mathcal{P}(\mathbb{R}^d)$ 上のプロホロフ距離を $\text{dist}(\cdot, \cdot)$ と書く。 $\mathcal{P}(\mathbb{R}^d)$ 上では、プロホロフ距離によって導かれるトポロジーは弱*トポロジーと一致する。パス空間は連続写像 $\omega: [0, \infty) \rightarrow \mathbb{R}^d$ の全体 $\Omega = C([0, \infty), \mathbb{R}^d)$ とする。 $\mathcal{F}_t = \sigma(\omega(s); s \leq t)$ とし、 $\mathcal{F} = \bigvee_t \mathcal{F}_t$ とする。

a_{ij} と $b_i, i, j = 1, \dots, d$ は \mathbb{R}^d 上の関数で、次を満たすとする。

(A1) $a_{ij} \in C_0^\infty(\mathbb{R}^d)$, $i, j = 1, \dots, d$, しかも $(a_{ij}(x))_{i,j=1}^d$ は $x \in \mathbb{R}^d$ に関して一様正定である。また、 $b_i \in C^\infty(\mathbb{R}^d)$, $i = 1, \dots, d$ であり、正定数 $C_1 > 0$ が存在して、行列の意味で $\nabla b + (\nabla b)^t \leq C_1 \cdot I_d$ が成り立つ。ただし、 $b = (b_1, \dots, b_d)$ であり、 I_d は $d \times d$ の単位行列である。すなわち、すべての $x = (x_1, \dots, x_d), \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ に対して、 $\sum_{i,j=1}^d (\frac{\partial b_i}{\partial x_j}(x) + \frac{\partial b_j}{\partial x_i}(x)) \xi_i \xi_j \leq C_1 \sum_{i=1}^d \xi_i^2$ 。

$$L_0 = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x^i}$$

とおく。このとき、任意の $x \in \mathbb{R}^d$ から出発して、 L_0 によって決められる拡散過程が存在する。その分布を P_x と書く。 $\{P_x\}_{x \in \mathbb{R}^d}$ に対応する $C_b(\mathbb{R}^d)$ 上の有界線形作用素の半群を $\{P_t\}_{t \geq 0}$ と書く。次を仮定する。

(A2) すべての $t > 0$ において、 $P_t: C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$ はコンパクトである。

このとき、 (P_t) -不変確率測度 $\pi \in \mathcal{P}(\mathbf{R}^d)$ が唯一存在する。 $\text{supp}(\pi) = \mathbf{R}^d$, $\pi(dx) \ll dx$ であり、 $(y \mapsto \frac{\pi(dy)}{dy}) \in C^\infty(\mathbf{R}^d)$ である。

$$b_i^* = \sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j} + \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j} (\log \frac{d\pi}{dx}) - b_i,$$

$i = 1, \dots, d$, とおくと、

$$L_0^{**} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i^*(x) \frac{\partial}{\partial x^i}$$

は L_0 の $L^2(d\pi)$ 中の双対作用素であり、 L_0 によって決められる拡散過程の時間逆過程の生成作用素である。次を仮定する。

(A3) $\nabla b^* + (\nabla b^*)^t \leq C_1 \cdot I_d$, すなわち、時間逆方向の拡散過程も仮定 A1 を満たす。

時間逆方向の拡散過程に対応する $C_b(\mathbf{R}^d)$ 上のすべての $t > 0$ について、 P_t^* は P_t の $L^2(d\pi)$ 中の双対作用素である。さらに次を仮定する。

(A4) すべての $t > 0$ において、 P_t^* も $C_b(\mathbf{R}^d)$ 上の作用素としてコンパクトである。

このとき、次のような大偏差原理を得た。

定理 0.1 $\mathcal{P}(\mathbf{R}^d)$ 上の関数 I を

$$I(\nu) = \sup \left\{ - \int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty, u \geq 1, u \text{ と } L_0 u \text{ は有界} \right\}, \quad \nu \in \mathcal{P}(\mathbf{R}^d)$$

のように定義する。(A1) ~ (A4) を仮定すれば、 I はエントロピー関数であり、すべての $x, y \in \mathbf{R}^d$ において、 I は $\frac{1}{2} \int_0^t \delta_{X_s} ds$ の $P_x(\cdot | X_t = y)$ の下での分布の大偏差を表す。すなわち、 $\mathcal{P}(\mathbf{R}^d)$ のすべてのボレル可測な部分集合 A に対して、

$$\begin{aligned} -\inf_{A^0} I &\leq \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A \mid X_t = y \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A \mid X_t = y \right) \leq -\inf_{A^2} I. \end{aligned}$$

但し、 A^0 と A^2 はそれぞれ A の内部と閉包を表す。

$\Phi: \mathcal{M}(\mathbf{R}^d) \rightarrow \mathbf{R}$ は有界な関数で、 $\Phi|_{\mathcal{P}(\mathbf{R}^d)}$ がプロボロフ距離に関して連続であるとする。定理 0.1 より、次の式を得た。

系 0.2 すべての $x, y \in \mathbf{R}^d$ において、

$$\frac{1}{T} \log E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \mid X_T = y \right) \right] \rightarrow \lambda,$$

但し、 $\lambda = \sup \{ \Phi(\nu) - I(\nu); \nu \in \mathcal{P}(\mathbf{R}^d) \}$ 。

すべての $x, y \in \mathbf{R}^d$ に対して、 $Z_T^{x,y} = E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \mid X_T = y \right) \right]$ と置き、 $T \rightarrow \infty$ の時の $Z_T^{x,y}$ の精密な漸近挙動を調べる。

$$K = \{ \nu \in \mathcal{P}(\mathbf{R}^d); \Phi(\nu) - I(\nu) = \lambda \}$$

と置くと、 K は $\mathcal{P}(\mathbf{R}^d)$ の空でないコンパクトな部分集合である。次を仮定する。

(A5) K は唯一つの元 ν_0 より成る、すなわち、 $K = \{ \nu_0 \}$ 。

Φ は次を満たすとする。

(A6) Φ は 3-回連続 Fréchet 微分可能な関数で、次を満たす。 $\Phi^{(1)} \in C_b(\mathcal{P}(\mathbf{R}^d) \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(2)} \in C_b(\mathcal{P}(\mathbf{R}^d) \times \mathbf{R}^d \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(3)} \in C_b(\mathcal{P}(\mathbf{R}^d) \times (\mathbf{R}^d)^3, \mathbf{R})$ が存在して、すべての $\nu \in \mathcal{P}(\mathbf{R}^d)$ と $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{R}^d)$ に対して、

$$D\Phi(\nu)(R_1) = \int_{\mathbf{R}^d} \Phi^{(1)}(\nu, x) R_1(dx),$$

$$D^2\Phi(\nu)(R_1, R_2) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy),$$

$$D^3\Phi(\nu)(R_1, R_2, R_3) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz).$$

また、 $\Phi^{(1)}(\nu_0, x) \in C_b^1(\mathbf{R}^d, \mathbf{R})$ であり、 $\nabla_x \Phi^{(2)}(\nu_0, x, y)$, $\nabla_y \Phi^{(2)}(\nu_0, x, y)$ および $\nabla_x \nabla_y \Phi^{(2)}(\nu_0, x, y)$ が存在し、 $C_b(\mathbf{R}^d \times \mathbf{R}^d)$ に属する。

$\phi^{v_0}(x) = \Phi^{(1)}(\nu_0, x) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0)$, $x \in \mathbf{R}^d$ と置くと、 $L_0 + \phi^{v_0}$ は唯一つ正な固有関数 h を持つ。 $L = L_0 + \frac{\nabla h}{h} \cdot \nabla$ に対応する拡散過程を $\{Q_s\}_{s \in \mathbf{R}^d}$ とする。これは ν_0 を不変測度として持つ。

すべての $f \in C_b(\mathbf{R}^d)$ に対して、 $\int_0^\infty (Q_t f - \int_{\mathbf{R}^d} f d\nu_0) dt$ は収束し、 $C_b(\mathbf{R}^d)$ に属する。このマップを G と書く。 G^* を G の $L^2(d\nu_0)$ 中の共役オペレータとし、 $\bar{G} = G + G^*$ とする。

$\Gamma(f_1, f_2) \equiv \int_{\mathbf{R}^d} f_1 \bar{G} f_2 d\nu_0$, $f_1, f_2 \in C_b(\mathbf{R}^d)$ とする。また、 $C_b(\mathbf{R}^d)$ 上の同値関係 \sim を $f \sim g \Leftrightarrow f - g \equiv \text{constant}$ のように定義し、 $\bar{C}_b(\mathbf{R}^d) \equiv C_b(\mathbf{R}^d) / \sim$ とする。このとき、 Γ は $\bar{C}_b(\mathbf{R}^d)$ 上の内積である。 $H \equiv \left(\bar{C}_b(\mathbf{R}^d) \right)^\Gamma$ と置くと、但し、 $\bar{C}_b(\mathbf{R}^d)$ は $\bar{C}_b(\mathbf{R}^d)$ の Γ に関する完備化を表す。 H はヒルベルト空間であり、ノルムは $\| \bar{G} f d\nu_0 \|_H^2 \equiv \int_{\mathbf{R}^d} f \bar{G} f d\nu_0$ である。 H は $\nu_0(\mathbf{R}^d)$ の稠密な部分集合と見なせる。

また、すべての $f \in C_b(\mathbf{R}^d)$ に対して、

$$(f, \bar{G} f)_{L^2(d\nu_0)} \geq D^2\Phi(\nu_0)(\bar{G} f d\nu_0, \bar{G} f d\nu_0)$$

である。すなわち、 $D^2\Phi(\nu_0)|_{H \times H}$ の固有値はすべて 1 以下である。次を仮定する。

(A7) $D^2\Phi(\nu_0)|_{H \times H}$ の固有値はすべて 1 より小さい。

(A8) すべての $\delta > 0$ に対して、定数 $\varepsilon > 0$ と対称な有界連続関数 $K_\delta: \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}$ が存在して、次を満たす: (1) $\nabla_x K_\delta, \nabla_y K_\delta, \nabla_x \nabla_y K_\delta$ がすべて存在し、 $C_b(\mathbf{R}^d \times \mathbf{R}^d)$ に属する。(2) $\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y) \bar{G}_x \bar{G}_y K_\delta(x, y) \nu_0(dx) \nu_0(dy) \leq \delta^2$ 。(3) すべての $R \in \mathcal{P}(\mathbf{R}^d)$ と $\nu \in \mathcal{P}(\mathbf{R}^d)$ に対して、 $\text{dist}(R, \nu_0) < \delta$, $\text{dist}(\nu, \nu_0) < \delta$ であれば、

$$D^3\Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0) \leq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y) (\nu - \nu_0)(dx) (\nu - \nu_0)(dy).$$

定理 0.3 (A1) ~ (A8) を仮定する。このとき、すべての $x, y \in \mathbb{R}^d$ に対して、

$$\lim_{T \rightarrow \infty} e^{-T\lambda} Z_T^{x,y} = \frac{h(x)}{h(y)} \cdot \exp \left\{ \frac{1}{2} \int_{\mathbb{R}^d} (\bar{G} \otimes I) \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u,u)} \nu_0(du) \right\} \\ \times \det_2(I_H - D^2 \Phi(\nu_0))^{-1/2}.$$

Kusuoka-Tamura は対称な場合、すなわち、すべての $t > 0$ において、 P_t は $L^2(dx)$ で自己共役作用素である場合に関してこの問題を考えた。

また、この論文は独立同分布な確率変数の場合およびトラス上の拡散過程の場合に関しても同じように大偏差理論の精密評価の問題を調べ、それぞれ結果を得た。

Introduction

大偏差理論は Donsker-Varadhan 等によって詳しく研究されているが、多くの問題で、もっと精密な評価が必要になることがある。バナッハ空間上 B で値をとる独立同分布な確率変数の和に関して、Bolthausen [1] と Chiyonobu [3] は B 上で中心極限定理が成り立つという条件を仮定し、この精密化問題を考えた。拡散過程に関しては、Bolthausen-Deuschel-Tamura [2] はコンパクト空間上の拡散過程場合について、また Kusuoka-Tamura [5] は対称なマルコフ過程について、それぞれやはり中心極限定理が成り立つような仮定の下で、この精密化問題を考えた。

この論文では中心極限定理の仮定なしに、独立同分布な確率変数の和の場合、トラス上の拡散過程、そしてユークリッド空間上の拡散過程に関して、それぞれ精密評価の問題を考えた。

第一章と第二章の説明

第一章と第二章は先述のように独立同分布の確率変数の和に関する大偏差理論の精密評価について研究した。

まず次の例を考える。 n 粒子の統計力学モデル (連続場モデル) を考える。各粒子の取りうる状態の全体がコンパクトな距離空間 M であるとし、(相互作用がない時の) 各粒子の状態の分布は M 上の確率測度 μ_0 に従うとする。さらに、状態 x の粒子と状態 y の粒子の間の相互作用は $\frac{1}{n} V(x, y)$ (平均場近似) であるとする。ここで、 $V: M \times M \rightarrow \mathbb{R}$ は連続関数。この時、 n 粒子が状態 $\underline{x} = (x_1, \dots, x_n)$ をとる確率は $\nu_n(d\underline{x}) \equiv Z_n^{-1} \exp \left(\frac{1}{n} \sum_{i,j=1}^n V(x_i, x_j) \right) \mu_0^{\otimes n}(d\underline{x})$ で与えられる。ただし、 Z_n は正規化定数。統計力学で興味のある磁化率や相関係数は $\int_{M^n} n^{-m} \sum_{i_1, \dots, i_m=1}^n f(x_{i_1}, \dots, x_{i_m}) \nu_n(dx)$ の極限に帰着される。これらの量が収束するか、収束すれば何に収束するかというのが考える問題である。

この問題は次のように一般化される。 μ は可分な実バナッハ空間 $(B, \|\cdot\|)$ 上の確率測度で、ある定数 $C > 0$ が存在して、 $\int_B \exp(C\|x\|^2) \mu(dx) < \infty$ を満たすとする。 Φ を B 上の連続関数で、 $\|x\|$ の一次式で抑えられるとする。 $Z_n \equiv E^{\mu^{\otimes n}}[\exp(n\Phi(\frac{1}{n}S_n))]$ と置く。ただし、 $S_n \equiv \sum_{i=1}^n x_i$ である。任意の有界連続関数 $F: B \rightarrow \mathbb{R}$ に対して、 $E^{\mu^{\otimes n}}[F(n^{-1}S_n) \exp(n\Phi(n^{-1}S_n))]$ の厳密な漸近挙動を求めるといふ問題を考える。上の例においては、 $B = C(M)^*$ と置き、 μ は δ_x の $\mu_0(dx)$ の下での分布とすれば、

磁化率等の計算はこの問題に帰着される。

Donsker-Varadhan [4] より, $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \sup_{x \in B} \{\Phi(x) - h(x)\} (= \lambda \text{ とおく})$ が分かる、但し、 h はエントロピー関数で、次のように与えられる。

$$h(x) = \sup_{\phi \in B^*} \{\phi(x) - \log M(\phi)\}, \quad x \in B,$$

B^* は B の双対空間で、 M は次のように定義される。 $M(\phi) = \int_B e^{\phi(x)} \mu(dx)$, $\phi \in B^*$ 。しかし、これでは先の問題に答えるには不十分であり、 $n \rightarrow \infty$ の時の Z_n の $1 + o(1)$ オーダーまでの精密評価が必要である。

$V = \{x \in B; \Phi(x) - h(x) = \lambda\}$ と置く。先の設定の下で、Bolthausen [1] より、 V は空でないコンパクト集合である。任意の V の元 $x \in V$ に対して、 B 上の確率測度 ν_x を次のように定義する：

$$\nu_x(dy) = \exp(D\Phi(x)(y)) \mu(dy) / M(D\Phi(x)). \quad (0.1)$$

x が $\Phi - h$ を最大化するので、次のことが分かる。

$$\begin{aligned} \int_B y \nu_x(dy) &= x, \\ h(x) &= D\Phi(x)(x) - \log M(D\Phi(x)). \end{aligned} \quad (0.2)$$

$\nu_{x,0}$ を ν_x を平均 0 のように平行移動したものとする、すなわち、

$$d\nu_{x,0}(y) = d\nu_x(y+x), \quad y \in B.$$

また、 B^* 上の covariance Γ_x を

$$\Gamma_x(\phi, \psi) = \int_B \phi(y) \psi(y) \nu_{x,0}(dy), \quad \phi, \psi \in B^*$$

と置き、 $H_x \equiv (\overline{B^*})^*$ と置く。但し、 $\overline{B^*}^{\perp x}$ は B^* の Γ_x に関する完備化を表す。 H_x の内積を $(\cdot, \cdot)_x$ と書き、ノルムを $\|\cdot\|_x$ と書く。 $S_x : B^* \rightarrow B$, $S_x \phi \equiv \int_B \phi(y) y \nu_{x,0}(dy)$, $\phi \in B^*$ と置くと、 $S_x(B^*) \subset H_x$ であり、すべての $\phi, \psi \in B^*$ に対して、 $\psi(S_x \phi) = \Gamma_x(\phi, \psi)$ である、しかも

$$\begin{aligned} \|S_x \phi\|_x^2 &= \sup \{\psi(S_x \phi)^2; \Gamma_x(\psi, \psi) \leq 1\} \\ &= \sup \{\Gamma_x(\phi, \psi)^2; \Gamma_x(\psi, \psi) \leq 1\} \\ &= \Gamma_x(\phi, \phi). \end{aligned}$$

すなわち、すべての $\phi, \psi \in B^*$ に対して、 $(S_x \phi, S_x \psi)_x = \Gamma_x(\phi, \psi)$ が成り立つ。また、任意の $x \in V$ において、 H_x は B の稠密な部分集合と見なせ、しかも、すべての連続な双線形関数 $A : B \times B \rightarrow \mathbf{R}$ に対して、 $A|_{H_x \times H_x}$ は Hilbert-Schmidt 型な関数である。

x が $\Phi - h$ を最大化するという条件より、

$$\Gamma_x(\phi, \phi) \geq D^2\Phi(x)(S_x \phi, S_x \phi),$$

はすべての $\phi \in B^*$ に対して成り立つ。

$$A_x \equiv \{\phi \in B^* : \Gamma_x(\phi, \phi) = D^2\Phi(x)(S_x \phi, S_x \phi)\}$$

と置く。

次を仮定する：

(A) 定数 $C_4, \delta_0 > 0$ と対称で連続な双線形関数 $K : B \times B \rightarrow \mathbf{R}$ が存在して、すべての $y \in B$ とすべての V の δ_0 -近傍 V_{δ_0} の元 $x \in V_{\delta_0}$ において、

$$|D^3\Phi(x)(y, y, y)| \leq C_4 \|y\| K(y, y).$$

第一章では非退化の場合、すなわち、 V が唯一の元 x^* を持ち、 $A_{x^*} = \{0\}$ である場合に関して研究し、次の結果を得た。

THEOREM 0.1 (A) を仮定する。 $V = \{x^*\}$ で、しかも $A_{x^*} = \{0\}$ であれば、

$$\begin{aligned} \lim_{n \rightarrow \infty} \exp(-n(\Phi(x^*) - h(x^*))) E^{n, \infty} \left[\exp(n\Phi(\frac{S_n}{n})) \right] \\ = \exp\left(\frac{1}{2} \int_B D^2\Phi(x^*)(x, x) \nu_0(dx)\right) \cdot \det_{l_2} (I_{H_{x^*}} - D^2\Phi(x^*))^{-\frac{1}{2}}. \end{aligned}$$

Bolthausen [1] は同じ問題を違う設定の下で考え、次の結果を得た。

THEOREM 0.2 定理 0.1 と同じ条件を仮定し、さらに次を仮定する。

(B) ν_0 は中心極限定理を満たす、すなわち、 $\nu_n = \nu_n(A) = \nu_0^n(\sqrt{n}A)$ のように置けば、ある B 上のガウス分布 γ が存在して、 $n \rightarrow \infty$ の時、 ν_n が γ に弱収束する。

このとき、

$$\lim_{n \rightarrow \infty} \exp(-n(\Phi(x^*) - h(x^*))) E^{n, \infty} \left[\exp(n\Phi(\frac{S_n}{n})) \right] = \int_B \exp(\frac{1}{2} D^2\Phi(x^*)[y, y]) \gamma(dy).$$

条件 (B) が成立すれば、 (H, B, γ) が抽象ウィナー空間になるので、 $D^2\Phi(x^*)$ は nuclear 型の関数となる。よって、Bolthausen の定理で出てきた定数 $\int_B \exp(\frac{1}{2}D^2\Phi(x^*)[y, y])$ が定義できる。このとき、この定数は我々の定理の中の定数と等しい。しかし、一般には、 $D^2\Phi(x^*)|_{H \times H}$ は nuclear とは限らないので、Bolthausen の方の定数は定義できない。一方、 $D^2\Phi(x^*)$ は常に Hilbert-Schmidt 型の関数なので、我々の定理で使った定数は (今の設定の下では) いつでも定義できる。また、条件 (B) は一般には成立せず、成立するかどうかのチェックも難しい。

第二章ではこの問題を退化を許す場合に一般化し、次の結果を得た。

THEOREM 0.3 条件 (A) を仮定する。このとき、自然数 $d \geq 1$ と B の d -次元部分多様体 M 及び連続な関数 $x: M \rightarrow B$, $a: M \rightarrow [0, \infty)$ と $b: M \rightarrow (0, \infty)$ が存在して、次を満たす。

(1) M はリーマン距離を持ち、 $V \subset M$ であり、しかもすべての $x \in V$ において、 $S_x(A_x) \subset T_x M$ である。

(2) $x(\cdot) \in C^2(M)$, しかもすべての $z \in V$ において $x(z) = z$,

(3) $a(z) = 0 \iff z \in V$,

(4) すべての有界な連続関数 $f: B \rightarrow \mathbf{R}$ に対して、 $n \rightarrow \infty$ の時、

$$E^{\mu^{\otimes n}} \left[f\left(\frac{S_n}{n}\right) \exp(n\Phi(\frac{S_n}{n})) \right] = e^{n\lambda} n^{\frac{d}{2}} \int_M f(x(z)) b(z) e^{-na(z)} v_M(dz) (1 + o(1))$$

である。但し、 v_M は M 上の体積要素である。

$a(z)$ および $b(z)$, $z \in M$, の具体的な数式に関しては、第二章 Lemma 5.2 を参照。

Chiyonobu [3] では中心極限定理が成り立つと仮定し、同じ問題考えた。

第三章の説明

第三章はトラス上の拡散過程における大偏差理論の精密評価に関して研究した。

d 次元トラス $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ 上の $L_0 \equiv \frac{1}{2}\Delta + b \cdot \nabla$ を生成作用素とする拡散過程を考える。ただし、 b はトラス上の滑らかな関数。対応するパス空間 $\Omega = C([0, \infty); \mathbf{T}^d)$ 上の確率測度の族を $\{P_x\}_{x \in \mathbf{T}^d}$ と書く。 $\mathcal{M}(\mathbf{T}^d)$ を $C(\mathbf{T}^d)$ の双対空間とする。これは \mathbf{T}^d 上の有限変動を持つ符号付き測度の全体であり、ノルムは全変動ノルムである。また、 $\mathcal{M}(\mathbf{T}^d)$ 上の弱*トポロジーも考える。 \mathbf{T}^d 上の確率測度全体および全測度が 0 であるような有限変動の符号付き測度全体をそれぞれ $\wp(\mathbf{T}^d)$ と $\mathcal{M}_0(\mathbf{T}^d)$ と書く。 $\wp(\mathbf{T})$ 上のプロホロフ距離を $\text{dist}(\cdot, \cdot)$ と書く。 $\wp(\mathbf{T}^d)$ 上では、プロホロフ距離によって導かれるトポロジーは弱*トポロジーと一致する。

$\{P_x\}$ は唯一つの不変確率測度 μ を持つ。しかも、 μ は \mathbf{T}^d 上のリーマン体積に関して絶対連続であり、正の密度関数を持つ。すべての $T > 0$ に対して、 $\{X_{T-t}(\omega)\}_{0 \leq t \leq T}$ の $P_\mu(d\omega)$ の下での分布はまた拡散過程であり、生成作用素は L_0 の $L^2(d\mu)$ の中の双対作用素 L_0^{**} である。 $b_0^* = \nabla(\log \frac{d\mu}{dx}) - b_0$ と置けば、 $b_0^* \in C^\infty(\mathbf{T}^d; \mathbf{R}^d)$ であり、 $L_0^{**} = \frac{1}{2}\Delta + b_0^* \cdot \nabla$ である。

すべての $t > 0$ において、 P_t は μ に関して滑らかで正の密度関数 $(p_t(x, y))_{x, y \in \mathbf{T}^d} \in C^\infty(\mathbf{T}^d \times \mathbf{T}^d)$ を持つ。

Φ を $\mathcal{M}(\mathbf{T}^d)$ 上の有界な実数値関数とする。

$$Z_T^{x, y} \equiv E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) \middle| X_T = y \right]$$

と置く。Donsker-Varadhan [4] の結果より、すべての $x, y \in \mathbf{T}^d$ に対して、 $\lim_{T \rightarrow \infty} \frac{1}{T} \log Z_T^{x, y} = \sup \{ \Phi(\nu) - I(\nu); \nu \text{ は } \mathbf{T}^d \text{ 上の確率測度} \} (= \lambda \text{ とおく})$ が成り立つ。但し、 $I: \wp(\mathbf{T}^d) \rightarrow \mathbf{R}$ はエントロピー関数で、次のように与えられる：

$$I(\nu) = \sup \left\{ - \int_{\mathbf{T}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty, u \geq 1 \right\}, \quad \nu \in \wp(\mathbf{T}^d).$$

第一章と同じように、 $Z_T^{x, y}$ の $T \rightarrow \infty$ の時の精密な漸近挙動を調べる。

次を仮定する。

(A1) $\Phi: \mathcal{M}(\mathbf{T}^d) \rightarrow \mathbf{R}$ は 3 回連続 Fréchet 微分可能である。また、連続な関数 $\Phi^{(1)} \in C(\wp(\mathbf{T}^d) \times \mathbf{T}^d, \mathbf{R})$, $\Phi^{(2)} \in C(\wp(\mathbf{T}^d) \times \mathbf{T}^d \times \mathbf{T}^d, \mathbf{R})$ および $\Phi^{(3)} \in C(\wp(\mathbf{T}^d) \times (\mathbf{T}^d)^3, \mathbf{R})$ が存在して、すべての $\nu \in \wp(\mathbf{T}^d)$ および $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{T}^d)$ に対して、次が成り立つ：

$$D\Phi(\nu)(R_1) = \int_{\mathbf{T}^d} \Phi^{(1)}(\nu, x) R_1(dx),$$

$$D^2\Phi(\nu)(R_1, R_2) = \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy),$$

$$D^3\Phi(\nu)(R_1, R_2, R_3) = \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz).$$

$K = \{\nu \in \wp(\mathbf{T}^d); \Phi(\nu) - I(\nu) = \lambda\}$ と置くと、独立同分布の場合と同じように、 K は $\wp(\mathbf{T}^d)$ の空でないコンパクトな部分集合である。次を仮定する。

(A2) K は唯一つの元 ν_0 より成る、すなわち、 $K = \{\nu_0\}$ 。

ν_0 を不変測度として持ち、生成作用素が $L = \frac{1}{2}\Delta + \tilde{b} \cdot \nabla$ の形であるような $C([0, \infty); \mathbf{T}^d)$ 上の確率測度の族 (Q_x) を構成することが出来る。実は、 $\phi^{x_0}(x) = \Phi^{(1)}(\nu_0, x) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0)$, $x \in \mathbf{T}^d$ と置くと、 $L_0 + \phi^{x_0}$ は唯一つ正な固有関数 h

を持つ。\$L = L_0 + \frac{\nabla^2}{h} \cdot \nabla\$ である。すべての \$f \in C(\mathbb{T}^d)\$ に対して、\$\int_0^\infty (Q_t f - \int_{\mathbb{T}^d} f d\nu_0) dt\$ は収束し、\$C(\mathbb{T}^d)\$ に属する。このマップを \$G\$ と書く。\$G^*\$ を \$G\$ の共役オペレータとし、\$\bar{G} = G + G^*\$ とする。

\$\Gamma(f_1, f_2) \equiv \int_{\mathbb{T}^d} f_1 \bar{G} f_2 d\nu_0\$、\$f_1, f_2 \in C(\mathbb{T}^d)\$ とする。また、\$C(\mathbb{T}^d)\$ 上の同値関係 \$\sim\$ を \$f \sim g \Leftrightarrow f - g \equiv \text{constant}\$ のように定義し、\$\bar{C}(\mathbb{T}^d) \equiv C(\mathbb{T}^d)/\sim\$ とする。このとき、\$\Gamma\$ は \$\bar{C}(\mathbb{T}^d)\$ 上の内積である。\$H \equiv (\bar{C}(\mathbb{T}^d))^{\perp}\$ と置く、但し、\$\bar{C}(\mathbb{T}^d)^{\perp}\$ は \$\bar{C}(\mathbb{T}^d)\$ の \$\Gamma\$ に関する完備化を表す。\$H\$ はヒルベルト空間であり、ノルムは \$||\bar{G} f d\nu_0||_H^2 \equiv \int_{\mathbb{T}^d} f \bar{G} f d\nu_0\$ である。\$H\$ は \$\varphi_0(\mathbb{T}^d)\$ の稠密な部分集合と見なせる。

また、すべての \$f \in C(\mathbb{T}^d)\$ に対して、

$$(f, \bar{G} f)_{L^2(d\nu_0)} \geq D^2 \Phi(\nu_0) (\bar{G} f d\nu_0, \bar{G} f d\nu_0)$$

である。すなわち、\$D^2 \Phi(\nu_0)|_{H \times H}\$ の固有値はすべて 1 以下である。次を仮定する：

(A3) \$D^2 \Phi(\nu_0)|_{H \times H}\$ の固有値はすべて 1 より小さい。

(A4) 任意の \$\delta > 0\$ に対して、定数 \$\varepsilon > 0\$ および対称な連続関数 \$K_\delta : \mathbb{T}^d \times \mathbb{T}^d \rightarrow \mathbb{R}\$ が存在して、次を満たす：\$\bar{K}_\delta(R_1, R_2) \equiv \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} K_\delta(x, y) R_1(dx) R_2(dy)\$、\$R_1, R_2 \in \mathcal{M}_0(\mathbb{T}^d)\$ のように定義される関数 \$\bar{K}_\delta\$ を定義すれば、

$$||\bar{K}_\delta||_{H \times H} ||_{H.S.} \leq \delta,$$

しかもすべての \$R, \nu \in \varphi(\mathbb{T}^d)\$、\$dist(R, \nu_0) < \varepsilon\$、\$dist(\nu, \nu_0) < \varepsilon\$ に対して、

$$D^2 \Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0, \nu - \nu_0) \leq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} K_\delta(x, y) (\nu - \nu_0)(dx) (\nu - \nu_0)(dy).$$

次の結果を得た。

THEOREM 0.4 (A1) \$\sim\$ (A4) を仮定する。この時、すべての \$x, y \in \mathbb{T}^d\$ に対して、

$$\lim_{T \rightarrow \infty} e^{-T\lambda} Z_T^{x,y} = \frac{h(x)}{h(y)} \cdot \exp \left\{ \frac{1}{2} \int_{\mathbb{T}^d} \bar{G}_x \Phi^{(2)}(\nu_0, \cdot, \cdot) \Big|_{(u,u)} \nu_0(du) \right\} \times det_2(I_H - D^2 \Phi(\nu_0))^{-1/2}.$$

Bolthausen-Deuschel-Tamura [2] は他の設定の下で、同じ問題を考えている。

第四章の説明

第四章はユークリッド空間上の拡散過程における大偏差理論の精密評価に関して研究した。

\$d\$ 次元ユークリッド空間 \$\mathbb{R}^d\$ を考える。\$C_b(\mathbb{R}^d)\$ は \$\mathbb{R}^d\$ 上の有界な連続関数全体であり、\$||\cdot||_\infty\$ をノルムとして持つ。\$\mathcal{M}(\mathbb{R}^d)\$ を \$\mathbb{R}^d\$ 上の有限変動を持つ符号付き測度の全体とし、ノルムは全変動ノルムである。また、\$\mathcal{M}(\mathbb{R}^d)\$ 上の弱*トポロジーも考える。すべての \$f \in C_b(\mathbb{R}^d)\$ と \$R \in \mathcal{M}(\mathbb{R}^d)\$ に対して、ベアリングを \$(f, R) = \int_{\mathbb{R}^d} f dR\$ とする。\$\mathbb{R}^d\$ 上の確率測度全体および全測度が 0 であるような有限変動の符号付き測度全体をそれぞれ \$\varphi(\mathbb{R}^d)\$ と \$\mathcal{M}_0(\mathbb{R}^d)\$ と書く。\$\varphi(\mathbb{R})\$ 上のプロホロフ距離を \$dist(\cdot, \cdot)\$ と書く。\$\varphi(\mathbb{R}^d)\$ 上では、プロホロフ距離によって導かれるトポロジーは弱*トポロジーと一致する。パス空間は連続写像 \$\omega : [0, \infty) \rightarrow \mathbb{R}^d\$ の全体 \$\Omega = C([0, \infty), \mathbb{R}^d)\$ とする。\$\mathcal{F}_t = \sigma\{\omega(s); s \leq t\}\$ とし、\$\mathcal{F} = \bigvee_t \mathcal{F}_t\$ とする。

\$a_{ij}\$ と \$b_i, i, j = 1, \dots, d\$ は \$\mathbb{R}^d\$ 上の関数で、次を満たすとする。

(A1) \$a_{ij} \in C_b^\infty(\mathbb{R}^d)\$、\$i, j = 1, \dots, d\$、しかも \$(a_{ij}(x))_{i,j=1}^d\$ は \$x \in \mathbb{R}^d\$ に関して一様だ円型である。また、\$b_i \in C^\infty(\mathbb{R}^d)\$、\$i = 1, \dots, d\$ であり、正定数 \$C_1 > 0\$ が存在して、行列の意味で \$\nabla b + (\nabla b)^t \leq C_1 \cdot I_d\$ が成り立つ。ただし、\$b = (b_1, \dots, b_d)\$ であり、\$I_d\$ は \$d \times d\$ の単位行列である。すなわち、すべての \$x = (x_1, \dots, x_d)\$、\$\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d\$ に対して、\$\sum_{i,j=1}^d \frac{\partial b_i}{\partial x_j}(x) + \frac{\partial b_j}{\partial x_i}(x) \xi_i \xi_j \leq C_1 \sum_{i=1}^d \xi_i^2\$。

$$L_0 = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x^i}$$

とおく。このとき、任意の \$x \in \mathbb{R}^d\$ から出発して、\$L_0\$ によって決められる拡散過程が存在する。その分布を \$P_x\$ と書く。\$\{P_x\}_{x \in \mathbb{R}^d}\$ に対応する \$C_b(\mathbb{R}^d)\$ 上の有界線形作用素の半群を \$\{P_t\}_{t \geq 0}\$ と書く。次を仮定する。

(A2) すべての \$t > 0\$ において、\$P_t : C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)\$ はコンパクトである。

このとき、\$(P_t)\$-不変確率測度 \$\pi \in \varphi(\mathbb{R}^d)\$ が唯一つ存在する。\$\text{supp}(\pi) = \mathbb{R}^d\$、\$\pi(dx) < dx\$ であり、\$(y \mapsto \frac{\pi(dy)}{dy}) \in C^\infty(\mathbb{R}^d)\$ である。

$$b_i^* = \sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j} + \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j} (\log \frac{d\pi}{dx}) - b_i,$$

\$i = 1, \dots, d\$、とおくと、

$$L_0^{**} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i^*(x) \frac{\partial}{\partial x^i}$$

は \$L_0\$ の \$L^2(d\pi)\$ の中の双対作用素であり、\$L_0\$ によって決められる拡散過程の時間逆過程の生成作用素である。次を仮定する。

(A3) $\nabla b^* + (\nabla b^*)^t \leq C_1 \cdot I_d$. すなわち、時間逆方向の拡散過程も仮定 A1 を満たす。

時間逆方向の拡散過程に対応する $C_b(\mathbf{R}^d)$ 上の有界線形作用素の半群 $\{P_t^*\}_{t \geq 0}$ と書く。すべての $t > 0$ について、 P_t^* は P_t の $L^2(d\pi)$ の中の双対作用素である。次を仮定する。

(A4) すべての $t > 0$ において、 P_t^* も $C_b(\mathbf{R}^d)$ 上の作用素としてコンパクトである。

このとき、トーラスの場合と同じように、次のような大偏差原理を得た。

THEOREM 0.5 $\varphi(\mathbf{R}^d)$ 上の関数 I を

$$I(\nu) = \sup \left\{ - \int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty, u \geq 1, u \text{ と } L_0 u \text{ は有界} \right\}, \quad \nu \in \varphi(\mathbf{R}^d)$$

のように定義する。(A1) ~ (A4) を仮定すれば、 I はエントロピー関数であり、すべての $x, y \in \mathbf{R}^d$ において、 I は $\frac{1}{t} \int_0^t \delta_{X_s} ds$ の $P_x(\cdot | X_t = y)$ の下での分布の大偏差を表す。すなわち、 $\varphi(\mathbf{R}^d)$ のすべてのボレル可測な部分集合 A に対して、

$$\begin{aligned} -\inf_{A^0} I &\leq \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A | X_t = y \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A | X_t = y \right) \leq -\inf_A I. \end{aligned}$$

但し、 A^0 と \bar{A} はそれぞれ A の内部と閉包を表す。

$\Phi: \mathcal{M}(\mathbf{R}^d) \rightarrow \mathbf{R}$ は有界な関数で、 $\Phi|_{\varphi(\mathbf{T}^d)}$ がプロホロフ距離に関して連続であるとする。定理 0.5 より、次の式を得た。

COROLLARY 0.6 すべての $x, y \in \mathbf{R}^d$ において、

$$\frac{1}{T} \log E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) | X_T = y \right] \rightarrow \lambda,$$

但し、 $\lambda = \sup \{ \Phi(\nu) - I(\nu); \nu \in \varphi(\mathbf{R}^d) \}$ 。

先と同じように、 $T \rightarrow \infty$ の時の $Z_T^{x,y} = E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) | X_T = y \right]$ の精密な漸近挙動を調べる。

$$K = \{ \nu \in \varphi(\mathbf{R}^d) : \Phi(\nu) - I(\nu) = \lambda \}$$

と置くと、 K は $\varphi(\mathbf{T}^d)$ の空でないコンパクトな部分集合である。次を仮定する。

(A5) K は唯一つの元 ν_0 より成る、すなわち、 $K = \{\nu_0\}$ 。

Φ は次を満たすとする。

(A6) Φ は 3-回連続 Fréchet 微分可能な関数で、次を満たす。 $\Phi^{(1)} \in C_b(\varphi(\mathbf{R}^d) \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(2)} \in C_b(\varphi(\mathbf{R}^d) \times \mathbf{R}^d \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(3)} \in C_b(\varphi(\mathbf{R}^d) \times (\mathbf{R}^d)^3, \mathbf{R})$ が存在して、すべての $\nu \in \varphi(\mathbf{R}^d)$ と $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{R}^d)$ に対して、

$$\begin{aligned} D\Phi(\nu)(R_1) &= \int_{\mathbf{R}^d} \Phi^{(1)}(\nu, x) R_1(dx), \\ D^2\Phi(\nu)(R_1, R_2) &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy), \\ D^3\Phi(\nu)(R_1, R_2, R_3) &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz). \end{aligned}$$

また、 $\Phi^{(1)}(\nu_0, x) \in C_b^1(\mathbf{R}^d; \mathbf{R})$ であり、 $\nabla_x \Phi^{(2)}(\nu_0, x, y)$, $\nabla_y \Phi^{(2)}(\nu_0, x, y)$ および $\nabla_x \nabla_y \Phi^{(2)}(\nu_0, x, y)$ が存在し、 $C_b(\mathbf{R}^d \times \mathbf{R}^d)$ に属する。

トーラスの時と同じように、 $\phi^{(0)}(x) = \Phi^{(1)}(\nu_0, x) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0)$, $x \in \mathbf{R}^d$ と置くと、 $L_0 + \phi^{(0)}$ は唯一つの正な固有関数 h を持つ。 $L = L_0 + \frac{\nabla h}{h} \cdot \nabla$ に対応する拡散過程 $\{Q_s\}_{s \in \mathbf{R}^d}$ は ν_0 を不変測度として持つ。

すべての $f \in C_b(\mathbf{R}^d)$ に対して、 $\int_0^\infty (Q_t f - \int_{\mathbf{R}^d} f d\nu_0) dt$ は収束し、 $C_b(\mathbf{R}^d)$ に属する。このマップを G と書く。 G^* を G の $L^2(d\nu_0)$ の中の共役オペレータとし、 $\bar{G} = G + G^*$ とする。

$\Gamma(f_1, f_2) \equiv \int_{\mathbf{R}^d} f_1 \bar{G} f_2 d\nu_0$, $f_1, f_2 \in C_b(\mathbf{R}^d)$ とする。また、 $C_b(\mathbf{R}^d)$ 上の同値関係 \sim を $f \sim g \Leftrightarrow f - g \equiv \text{constant}$ のように定義し、 $\bar{C}_b(\mathbf{R}^d) \equiv C_b(\mathbf{R}^d) / \sim$ とする。このとき、 Γ は $\bar{C}_b(\mathbf{R}^d)$ 上の内積である。 $H \equiv (\bar{C}_b(\mathbf{R}^d))^{\Gamma}$ と置くと、 H はヒルトベルト空間であり、ノルムは $\|\bar{G} f d\nu_0\|_H^2 \equiv \int_{\mathbf{R}^d} f \bar{G} f d\nu_0$ である。 H は $\varphi_0(\mathbf{R}^d)$ の稠密な部分集合と見なせる。

また、すべての $f \in C_b(\mathbf{R}^d)$ に対して、

$$(f, \bar{G} f)_{L^2(d\nu_0)} \geq D^2\Phi(\nu_0)(\bar{G} f d\nu_0, \bar{G} f d\nu_0)$$

である。すなわち、 $D^2\Phi(\nu_0)|_{H \times H}$ の固有値はすべて 1 以下である。次を仮定する。

(A7) $D^2\Phi(\nu_0)|_{H \times H}$ の固有値はすべて 1 より小さい。

(A8) すべての $\delta > 0$ に対して、定数 $\varepsilon > 0$ と対称な有界連続関数 $K_\delta: \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}$ が存在して、次を満たす: (1) $\nabla_x K_\delta, \nabla_y K_\delta, \nabla_x \nabla_y K_\delta$ がすべて存在し、 $C_b(\mathbf{R}^d \times \mathbf{R}^d)$ に属する。(2) $\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y) \bar{G}_x \bar{G}_y K_\delta(x, y) \nu_0(dx) \nu_0(dy) \leq \delta^2$ 。(3) すべての $R \in \varphi(\mathbf{R}^d)$ と $\nu \in \varphi(\mathbf{R}^d)$ について、 $\text{dist}(R, \nu_0) < \delta$, $\text{dist}(\nu, \nu_0) < \delta$ であ

れば、

$$D^2\Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0) \leq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y)(\nu - \nu_0)(dx)(\nu - \nu_0)(dy).$$

THEOREM 0.7 (A1) ~ (A8) を仮定する。このとき、すべての $x, y \in \mathbf{R}^d$ に対して、

$$\lim_{T \rightarrow \infty} e^{-T\lambda} Z_T^{-y} = \frac{h(x)}{h(y)} \cdot \exp \left\{ \frac{1}{2} \int_{\mathbf{R}^d} (\bar{G} \otimes I) \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u, u)} \nu_0(du) \right\} \\ \times \det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.$$

Kusuoka-Tamura [5] は対称な場合、すなわち、すべての $t > 0$ において、 P_t は $L^2(d\pi)$ で自己共役作用素である場合に関してこの問題を考えた。

また、第五章では、ユークリッド空間上の拡散過程におけるエルゴード性について議論した。

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Chapter 1

Laplace Approximations for Sums of Independent Random Vectors

Abstract

Let $X_i, i \in \mathbf{N}$, be i.i.d. B -valued random variables, where B is a real separable Banach space. Let Φ be a mapping $B \rightarrow \mathbf{R}$. Under a central limit theorem assumption, an asymptotic evaluation of $Z_n = E(\exp(n\Phi(\sum_{i=1}^n X_i/n)))$, up to a factor $(1+o(1))$, has been gotten in Bolthausen [1]. In this paper, we show that the same asymptotic evaluation can be gotten without the central limit theorem assumption.

1 Introduction

Let B be a real separable Banach space with norm $\|\cdot\|$, μ be a probability measure on B . We assume that the smallest closed affine space that contains $\text{supp}\mu$ is B . Moreover we assume

$$(A1) \quad \int_B \exp(t\|x\|)\mu(dx) < \infty, \quad \text{for all } t \in \mathbf{R}.$$

Let $\Phi : B \rightarrow \mathbf{R}$ be a three times continuously Fréchet differentiable function satisfying the following:

(A2) There exist constants $C_1, C_2 > 0$, such that

$$\Phi(x) \leq C_1 + C_2\|x\|, \quad \text{for any } x \in B.$$

Let X_n and $S_n, n \in \mathbf{N}$, be the random variables defined by $X_n(x) = x_n$ and $S_n(x) = \sum_{k=1}^n x_k$ for any $x = (x_1, x_2, x_3, \dots) \in B^{\mathbf{N}}$.

By Donsker-Varadhan [3], we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E^{n \otimes \infty} \left[\exp(n\Phi(\frac{S_n}{n})) \right] = \sup_{x \in B} \{\Phi(x) - h(x)\},$$

where h is the entropy function of μ :

$$h(x) = \sup_{\phi \in B^*} \{\phi(x) - \log M(\phi)\}, \quad x \in B,$$

B^* is the dual Banach space of B and $M(\phi) = \int_B e^{\phi(x)} \mu(dx)$ for any $\phi \in B^*$.

It has been shown by Bolthausen [1] that there is at least one $x^* \in B$ with $\Phi(x^*) - h(x^*) = \sup_{x \in B} \{\Phi(x) - h(x)\}$. Also, we assume the following as in Bolthausen [1]:

(A3) There is a unique $x^* \in B$ with $\Phi(x^*) - h(x^*) = \sup_{x \in B} \{\Phi(x) - h(x)\}$.

We will use x^* exclusively for this point.

Let ν be the probability measure on B given by

$$\nu(dx) = \frac{\exp(D\Phi(x^*)(x))\mu(dx)}{M(D\Phi(x^*))}.$$

As it has been shown by Bolthausen [1], the following proposition holds.

PROPOSITION 1.1 *Under the assumptions (A1), (A2), (A3),*

$$x^* = \int_B x \nu(dx), \quad (1.1)$$

$$h(x^*) = D\Phi(x^*)(x^*) - \log M(D\Phi(x^*)). \quad (1.2)$$

Let ν_0 be the 0-centered ν , i.e. $\nu_0 = \nu \theta_a^{-1}$, where $\theta_a : B \rightarrow B$ is defined by $\theta_a(x) = x - a$, $x \in B$.

Let $\Gamma(\varphi, \psi) = \int_B \varphi(x) \psi(x) \nu_0(dx)$ be the covariance of φ and ψ for any $\varphi, \psi \in B^*$. Then Γ becomes an inner product on B^* . Let $H \equiv (\overline{B^*})^*$, where $\overline{B^*}$ means the completion of B^* with respect to Γ . Then we can show that H can be regarded as a dense subset of B . (See Proposition 2.1.)

In this paper, we assume the following, which is a little stronger than (A1):

(A1') There exists a constant $C_3 > 0$, such that

$$\int_B \exp(C_3 \|x\|^2) \mu(dx) < \infty.$$

It has been shown by Bolthausen [1] that the following holds:

$$\Gamma(\phi, \phi) \geq D^2\Phi(x^*)(\iota(\phi), \iota(\phi)), \quad \text{for any } \phi \in B^*,$$

where $\iota(\phi) \equiv \int_B \phi(x) x \nu_0(dx)$, $\phi \in B^*$. From this, we see that all of the eigenvalues of the operator $D^2\Phi(x^*)|_{H \times H}$ are smaller than or equal to 1. Furthermore we assume the following

(A4) All of the eigenvalues of $D^2\Phi(x^*)|_{H \times H}$ are smaller than 1.

(A5) There exist constants $C_4 > 0$ and $\delta > 0$, and a continuous bilinear function $K : B \times B \rightarrow \mathbf{R}$, such that

$$|D^3\Phi(x)(y, y, y)| \leq C_4 \|y\| K(y, y)$$

for any $y \in B$ and any $x \in B$ with $\|x - x^*\| < \delta$.

The following is our main result:

THEOREM 1.2 *Under the assumptions (A1'), (A2) \sim (A5) above, we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \exp(-n(\Phi(x^*) - h(x^*))) E^{n\varphi_n} \left[\exp(n\Phi\left(\frac{S_n}{n}\right)) \right] \\ = \exp\left(\frac{1}{2} \int_B D^2\Phi(x^*)(x, x) \nu_0(dx)\right) \cdot \det_2(I_H - D^2\Phi(x^*))^{-\frac{1}{2}}. \end{aligned}$$

Remark: The fact that $D^2\Phi(x^*)|_{H \times H}$ is a Hilbert-Schmidt function, which ensures that the factor $\det_2(I_H - D^2\Phi(x^*))$ above is well-defined, can be gotten from Proposition 2.2 later.

Bolthausen [1] studied the same problem under the different assumption. He showed the following

THEOREM 1.3 *Assume the following*

(B) ν_0 satisfies central limit theorem, i.e., ν_n defined by $\nu_n(A) = \nu_0^n(\sqrt{n}A)$ converges weakly to a Gaussian measure γ on B .

Furthermore, assume (A1) \sim (A4), then

$$\lim_{n \rightarrow \infty} \exp(-n(\Phi(x^*) - h(x^*))) E^{n\varphi_n} \left[\exp(n\Phi\left(\frac{S_n}{n}\right)) \right] = \int_B \exp\left(\frac{1}{2} D^2\Phi(x^*)(y, y)\right) \gamma(dy).$$

If we assume that ν_0 satisfies central limit theorem as in Bolthausen [1], (H, B, γ) becomes an abstract Wiener Space, and so from Kuo [4] (Page 83, Theorem 4.6 (Goodman)), we can get that $D^2\Phi(x^*)|_{H \times H}$ is a nuclear function. In this situation, the integration $\int_B \exp(\frac{1}{2} D^2\Phi(x^*)(y, y)) \gamma(dy)$ appeared in Bolthausen's theorem is nothing but $\exp(\frac{1}{2} \int_B D^2\Phi(x^*)(x, x) \nu_0(dx)) \cdot \det_2(I - D^2\Phi(x^*))^{-\frac{1}{2}}$, which is just the limit appeared in our theorem. And when the operator $D^2\Phi(x^*)|_{H \times H}$ is not nuclear, but just a Hilbert-Schmidt function, Bolthausen's one is not defined, while our one is still well-defined. The point here is that the condition that a function is Hilbert-Schmidt can be easily checked by integration, while the condition nuclear is not. Moreover, if B is a Hilbert space, then (A5) is also satisfied.

As mentioned above, the central limit theorem assumption is actually a very strong assumption as we are dealing with infinite dimension space. Our theorem

claims that without the assumption that ν_0 satisfies central limit theorem, the result still holds under the assumptions (A1') and (A5).

Remark. In most of our proofs, (A1') can be substituted by (A1), but in the proof of Lemma 3.5, we use (A1') essentially to derive (3.18). We do not know whether one can weaken the assumption (A1').

2 Preparations

PROPOSITION 2.1 H can be regarded as a dense subset of B .

Proof. The fact that H can be regarded as a subset of B can be seen from the definition of H , the continuity of $\iota : (B^*, \|\cdot\|_{H^*}) \rightarrow (B, \|\cdot\|_B)$, and the completeness of B .

The denseness can be seen from the extension theory and the assumption that the closed affine space that contains $\text{supp } \nu_0$ is B , by using contradiction. ■

PROPOSITION 2.2 Under the assumption (A1') (or just (A1)), for any continuous bilinear function $A : B \times B \rightarrow \mathbf{R}$, $A|_{H \times H}$ is a Hilbert-Schmidt function.

Proof. Since A is continuous, there exists a constant $C_0 > 0$, such that

$$|A(y_1, y_2)| \leq C_0 \|y_1\| \cdot \|y_2\|, \quad \text{for any } y_1, y_2 \in B.$$

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal base of H^* with $\{e_n\}_{n=1}^\infty \subset B^*$. Then $\{\iota(e_n)\}_{n=1}^\infty$ is the corresponding base of H . Let $f_{n,m} : B \times B \rightarrow \mathbf{R}$ be defined as

$$f_{n,m}(y_1, y_2) := \langle e_n, y_1 \rangle \cdot \langle e_m, y_2 \rangle, \quad y_1, y_2 \in B,$$

then $(f_{n,m}, f_{n',m'})_{L^2(d\nu_0^{\otimes 2})} = \delta_{nn'} \cdot \delta_{mm'}$ for any $n, m, n', m' \in \mathbf{N}$. Therefore,

$$\begin{aligned} \|A\|_{H,S}^2 &= \sum_{n,m=1}^\infty A(\iota(e_n), \iota(e_m))^2 \\ &= \sum_{n,m=1}^\infty \left(\int_B \int_B A(y_1, y_2) f_{n,m}(y_1, y_2) \nu_0(dy_1) \nu_0(dy_2) \right)^2 \\ &\leq \int_B \int_B |A(y_1, y_2)|^2 \nu_0(dy_1) \nu_0(dy_2) \\ &\leq C_0^2 \left(\int_B \|y\|^2 \nu_0(dy) \right)^2, \end{aligned}$$

which is finite by assumption (A1') (or just (A1)). ■

3 Basic Lemmas

For any $R > 2$, let $\tilde{\nu}_R$ be the probability measure on \mathbf{R} given by

$$\tilde{\nu}_R(\{R\}) = \frac{3}{4R^2 - 1}, \quad \tilde{\nu}_R(\{\frac{1}{2}\}) = \frac{R-2}{2R-1}, \quad \tilde{\nu}_R(\{-\frac{1}{2}\}) = \frac{R+2}{2R+1}.$$

By a simple calculation, we have

$$E^{\tilde{\nu}_R}[Y] = 0, \quad E^{\tilde{\nu}_R}[Y^2] = 1.$$

For any $a > 0$, let ρ_a be the probability measures on \mathbf{R} given by

$$\rho_a(dR) = C_a \exp\left(-\frac{aR^2}{2}\right) dR, \quad R > 2,$$

where C_a is the normalizing constant, i.e. $C_a = (\int_2^\infty e^{-\frac{aR^2}{2}} dR)^{-1}$. Let γ_a be the probability measures on \mathbf{R} given by

$$\gamma_a(dy) = \int \tilde{\nu}_R(dy) \rho_a(dR),$$

and let Y_i be i.i.d. random variables s.t. $P(Y_i \in dy) = \gamma_a(dy)$.

Lemma 3.1 For any $a > 0$, there exists a constant D_a , depends only on a , such that

$$P\left(\left|\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i\right| \geq z\right) \leq 2 \exp\left(-\frac{1}{4D_a} z^2\right) \quad (3.1)$$

for any $z \geq 0$ and any $n \geq 1$.

Proof. Let $f(\xi) \equiv \int_{\mathbf{R}} e^{\xi y} \gamma_a(dy)$. Then it can be shown that

$$D_a \equiv \sup_{\xi \neq 0} \frac{1}{|\xi|^2} \log f(\xi) < \infty.$$

Therefore,

$$\begin{aligned} P\left(\left|\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i\right| \geq z\right) &\leq e^{-\xi \cdot \sqrt{n} z} E\left[e^{\xi \sum_{i=1}^n Y_i}\right] + e^{-\xi \cdot \sqrt{n} z} E\left[e^{-\xi \sum_{i=1}^n Y_i}\right] \\ &\leq 2e^{-\xi \cdot \sqrt{n} z} \cdot \exp(n D_a |\xi|^2) \end{aligned}$$

for any $\xi \neq 0$. Letting $\xi = \frac{z}{2D_a \sqrt{n}}$, we get (3.1). ■

Lemma 3.2 Under the assumption (A1') in section 1, for any $c > 0$, there exists a $a_0 > 0$ small enough, such that for any $n \geq 3$ and any $a \in (0, a_0]$,

$$c^n \left(\int_B ||x||^{2n} \nu_0(dx) \right)^{1/2} \leq \int_{\mathbf{R}} y^n \gamma_a(dy). \quad (3.2)$$

Proof. From assumption (A1') and the definition of ν_0 , there exists a constant $C'_3 > 0$, such that $C_5 \equiv \int_B e^{C'_3 ||x||^2} \nu_0(dx) < \infty$. So for any $t > 0$,

$$f(x) \equiv \nu_0(||X|| \geq t) \leq C_5 e^{-C'_3 t^2}.$$

Therefore, for any $n \geq 3$,

$$\int_B ||x||^n \nu_0(dx) \leq n \int_{(0,\infty)} y^{n-1} \cdot e^{-\frac{y^2}{2}} dy \cdot C_5 \cdot \left(\frac{1}{\sqrt{2C'_3}} \right)^n. \quad (3.3)$$

On the other hand, from the definition of $\tilde{\nu}_R$, we can get by a calculation that for any $R > 2$,

$$\int_{\mathbf{R}} y^n \tilde{\nu}_R(dy) \geq \frac{3}{4} R^{n-2}$$

for any $n \geq 3$. So let $\rho_{0,a}, a > 0$ be the probability measures given by

$$\rho_{0,a}(dR) = \frac{2\sqrt{a}}{\sqrt{2\pi}} e^{-\frac{aR^2}{2}} dR, \quad R > 0,$$

then we have that for any $a < a_0$ and any $n \geq 3$,

$$\begin{aligned} \int_{\mathbf{R}} y^n \gamma_a(dy) &\geq \frac{3}{4} \int_{(0,\infty)} R^{n-2} \rho_a(dR) \geq \frac{3}{4} \int_{(0,\infty)} R^{n-2} \rho_{0,a_0}(dR) \\ &= \frac{3}{4} \frac{2}{\sqrt{2\pi}} \int_{(0,\infty)} y^{n-2} e^{-\frac{y^2}{2}} dy \cdot \left(\frac{1}{\sqrt{a_0}} \right)^{n-2}. \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), to prove the lemma, we only need to show that

$$\begin{aligned} c^{2n} \cdot 2n \int_{(0,\infty)} y^{2n-1} e^{-\frac{y^2}{2}} dy \cdot C_5 \cdot \left(\frac{1}{\sqrt{2C'_3}} \right)^n \\ \leq \left(\frac{3}{2\sqrt{2\pi}} \right)^2 \cdot \left(\int_{(0,\infty)} y^{n-2} \cdot e^{-\frac{y^2}{2}} dy \right)^2 \cdot \left(\frac{1}{a_0} \right)^{n-2} \end{aligned} \quad (3.5)$$

holds for any $n \geq 3$ if $a_0 > 0$ is small enough. But this is easy to be seen by a simple calculation and Stirling's formula. ■

Lemma 3.3 Assume the assumption (A1') in section 1. Let Ψ be a symmetric, bilinear function that satisfies the following conditions:

1. There exists a constant $C_0 > 0$, such that $|\Psi(x, y)| \leq C_0 ||x|| \cdot ||y||$ for any $x, y \in B$,

2. $\int_B \Psi(x, y)^2 \nu_0(dx) \nu_0(dy) = 1$.

Then, there exists an $a_0 > 0$, depending only on C_0 and $\int_B ||x||^2 \nu_0(dx)$, such that

$$E^{\gamma_a^{\otimes \infty}} \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \right] \leq E^{\gamma_a^{\otimes \infty}} \left[\prod_{k=1}^m Y_{i_k} Y_{j_k} \right] \quad (3.6)$$

holds for any $m \in \mathbf{N}$, any $i_1, \dots, i_m, j_1, \dots, j_m \in \mathbf{N}$ with $1 \leq i_k < j_k \leq n, k = 1, \dots, m$, and any $a \in (0, a_0]$, where $\{X_i\}_{i=1}^{\infty}$ is the sequence of random variables defined in section 1, and $\{Y_i\}_{i=1}^{\infty}$ is defined by $Y_n(y) = y_n, y = (y_1, y_2, \dots) \in \mathbf{R}^{\mathbf{N}}$.

Note. As ν_0 has mean 0, we get from the bilinearity of Ψ that $\int_B \Psi(x, y) \nu_0(dy) = 0$ for any $x \in B$.

Proof. To simplify the notation, in the proof of this lemma, we will write just E , which means the expectation with respect to $\nu_0^{\otimes \infty}$ when deal with $\{X_i\}_{i=1}^{\infty}$, and $\gamma_a^{\otimes \infty}$ when deal with $\{Y_i\}_{i=1}^{\infty}$, when there is no risk of being confused.

Let us consider the graph that consists all the i_k, j_k 's as its nodes and all the $i_k j_k$'s as its lines. We may assume that the graph is connected, since if not, from the independent of the X_i 's and Y_i 's, we can consider each connected component, respectively.

Let

$$\alpha_\ell = \sharp\{k : i_k = \ell \text{ or } j_k = \ell\}, \quad 1 \leq \ell \leq n.$$

If there exists a ℓ such that $\alpha_\ell = 1$, then (3.6) obviously holds as $0 = 0$. So, we may assume that $\alpha_\ell = 0$ or $\alpha_\ell \geq 2$ for all ℓ . Let

$$L = \{\ell; \alpha_\ell \geq 2\}, \quad L_0 = \{\ell; \alpha_\ell \geq 3\}.$$

If $L = L \setminus L_0$, then all of the i_k 's appear exactly twice, so from Schwartz's inequality and the independence of the X_i 's and the assumptions, it could be seen that

$$\begin{aligned} E^{\nu_0^{\otimes \infty}} \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \right] &\leq \prod_{k=1}^m E \left[\Psi(X_{i_k}, X_{j_k})^2 \right]^{1/2} = 1 \\ &= E^{\gamma_a^{\otimes \infty}} \left[\prod_{k=1}^m Y_{i_k} Y_{j_k} \right]. \end{aligned} \quad (3.7)$$

To see the inequality in the first line, we only need to notice that when r is an odd number,

$$\begin{aligned}
 & E[\Psi(x, X_1)\Psi(X_1, X_2) \cdots \Psi(X_r, y)] \\
 &= E[(\Psi(x, X_1)\Psi(X_2, X_3) \cdots \Psi(X_{r-1}, X_r)) \\
 &\quad \cdot (\Psi(X_1, X_2)\Psi(X_3, X_4) \cdots \Psi(X_r, y))] \\
 &\leq E[(\Psi(x, X_1)\Psi(X_2, X_3) \cdots \Psi(X_{r-1}, X_r))^2]^{1/2} \\
 &\quad \cdot E[(\Psi(X_1, X_2)\Psi(X_3, X_4) \cdots \Psi(X_r, y))^2]^{1/2} \\
 &= E[\Psi(x, X_1)^2]^{1/2} E[\Psi(X_1, X_2)^2]^{1/2} \cdots E[\Psi(X_r, y)^2]^{1/2} \\
 &= E[\Psi(x, X_1)^2]^{1/2} \cdot E[\Psi(X_r, y)^2]^{1/2}
 \end{aligned}$$

for any x, y . The case when r is even is the same.

For the case when $L \neq L_0$, by using (3.7), we have that

$$\begin{aligned}
 & E^{v_0^{\otimes \infty}} \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \right] \\
 &= E \left[E \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \mid \sigma\{X_x, x \in L_0\} \right] \right] \\
 &\leq E \left[\prod_{k:i_k, j_k \in L_0} |\Psi(X_{i_k}, X_{j_k})| \left(\prod_{k \in A} E[\Psi(X_{i_k}, X_{j_k})^2 \mid \sigma\{X_x, x \in L_0\}] \right)^{1/2} \right], \quad (3.8)
 \end{aligned}$$

where A in the third production is defined as

$$\begin{aligned}
 A &= \{k : (i_k \in L_0 \quad \& \quad j_k \in L \setminus L_0), \\
 &\quad \text{or} \quad (j_k \in L_0 \quad \& \quad i_k \in L \setminus L_0)\}.
 \end{aligned}$$

So, let $g(x) = E[\Psi(x, X_1)^2]^{1/2}$ and

$$\beta_\ell = \sharp\{k : (i_k = \ell \& j_k \in L \setminus L_0), \text{ or } (i_k \in L \setminus L_0 \& j_k = \ell)\},$$

then we can get from (3.8) that

$$\begin{aligned}
 E \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \right] &\leq E \left[\prod_{k:i_k, j_k \in L_0} |\Psi(X_{i_k}, X_{j_k})|^2 \right]^{1/2} \\
 &\quad \cdot E \left[\prod_{k \in A} E[\Psi(X_{i_k}, X_{j_k})^2 \mid \sigma\{X_x, x \in L_0\}] \right]^{1/2} \\
 &= E \left[\prod_{k:i_k, j_k \in L_0} |\Psi(X_{i_k}, X_{j_k})|^2 \right]^{1/2} \cdot E \left[\prod_{\ell \in L_0} g(X_\ell)^{2\beta_\ell} \right]^{1/2} \quad (3.9)
 \end{aligned}$$

Since $|\Psi(x, y)| \leq C_0 \|x\| \cdot \|y\|$ for any $x, y \in B$ by the assumption,

$$\begin{aligned}
 E \left[\prod_{k:i_k, j_k \in L_0} |\Psi(X_{i_k}, X_{j_k})|^2 \right]^{1/2} &\leq E \left[\prod_{k:i_k, j_k \in L_0} C_0^2 \|X_{i_k}\|^2 \|X_{j_k}\|^2 \right]^{1/2} \\
 &= C_0^{\frac{1}{2} \sum_{\ell \in L_0} (\alpha_\ell - \beta_\ell)} \prod_{\ell \in L_0} E \left[\|X_\ell\|^{2(\alpha_\ell - \beta_\ell)} \right]^{1/2} \quad (3.10)
 \end{aligned}$$

Also, from the definition of g , we have

$$\begin{aligned}
 g(x) &= E \left[|\Psi(x, X_1)|^2 \right]^{1/2} \\
 &\leq E \left[C_0^2 \|x\|^2 \|X_1\|^2 \right]^{1/2} = C_0 \|x\| E^{v_0} \left[\|X_1\|^2 \right]^{1/2} \\
 &= C_0 \|x\|,
 \end{aligned}$$

where $C_6 \equiv C_0 E^{v_0} [\|X_1\|^2]^{1/2}$. So,

$$\begin{aligned}
 E \left[\prod_{\ell \in L_0} g(X_\ell)^{2\beta_\ell} \right]^{1/2} &= \prod_{\ell \in L_0} E \left[g(X_\ell)^{2\beta_\ell} \right]^{1/2} \\
 &\leq \prod_{\ell \in L_0} E \left[(C_6 \|X_\ell\|)^{2\beta_\ell} \right]^{1/2} = \prod_{\ell \in L_0} C_6^{\beta_\ell} E \left[\|X_\ell\|^{2\beta_\ell} \right]^{1/2} \\
 &= C_6^{\sum_{\ell \in L_0} \beta_\ell} \prod_{\ell \in L_0} E \left[\|X_\ell\|^{2\beta_\ell} \right]^{1/2}. \quad (3.11)
 \end{aligned}$$

Let $C_7 \equiv \max\{C_0, C_6, 1\}$, then from (3.9), (3.10), (3.11), we see that

$$\begin{aligned}
 & E \left[\prod_{k=1}^m \Psi(X_{i_k}, X_{j_k}) \right] \\
 &\leq C_0^{\frac{1}{2} \sum_{\ell \in L_0} (\alpha_\ell - \beta_\ell)} \prod_{\ell \in L_0} E \left[\|X_\ell\|^{2(\alpha_\ell - \beta_\ell)} \right]^{1/2} \cdot C_6^{\sum_{\ell \in L_0} \beta_\ell} \prod_{\ell \in L_0} E \left[\|X_\ell\|^{2\beta_\ell} \right]^{1/2} \\
 &\leq C_7^{\frac{1}{2} \sum_{\ell \in L_0} (\alpha_\ell + \beta_\ell)} \prod_{\ell \in L_0} (E \left[\|X_\ell\|^{2(\alpha_\ell - \beta_\ell)} \right] E \left[\|X_\ell\|^{2\beta_\ell} \right])^{1/2} \\
 &\leq C_7^{\sum_{\ell \in L_0} \alpha_\ell} \prod_{\ell \in L_0} E^{v_0^{\otimes \infty}} \left[\|X_\ell\|^{2\alpha_\ell} \right]^{1/2}.
 \end{aligned}$$

On the other hand,

$$E^{\gamma_0^{\otimes \infty}} \left[\prod_{k=1}^m (Y_{i_k} Y_{j_k}) \right] = \prod_{\ell \in L_0} E^{\gamma_0^{\otimes \infty}} [Y_\ell^{\alpha_\ell}].$$

So we only need to take a proper a_0 , such that for any $a \leq a_0$, the following holds:

$$C_7^{\alpha_\ell} E^{v_0^{\otimes \infty}} \left[\|X_\ell\|^{2\alpha_\ell} \right]^{1/2} \leq E^{\gamma_0^{\otimes \infty}} [Y_\ell^{\alpha_\ell}], \quad \text{for any } \ell \in L_0,$$

but this could be gotten from Lemma 3.2. \blacksquare

The following lemma has been proved in Kusuoka-Tamura [5] (Lemma 2.1 in [5]). We write it here as it will be used later.

Lemma 3.4 Let $Z_i, i \in \mathbf{N}$ be i.i.d. \mathbf{R}^d -valued random variables, with mean 0 and finite variance. Assume that there exist constants A_1, A_2, A_3 , such that

$$\begin{aligned} E[Z_1, {}^t Z_1] &\leq A_1 \cdot I_d, \\ E[\exp(A_2 |Z_1|)] &\leq A_3. \end{aligned}$$

Then for any $b < \frac{1}{2A_1}$, there exist constants $\delta > 0$ and $A_4 > 0$, such that

$$E \left[\exp \left(b \cdot \frac{1}{n} \left| \sum_{i=1}^n Z_i \right|^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Z_i \right| < \delta \right] \leq A_4, \quad \text{for any } n \in \mathbf{N},$$

where δ depends only on A_1, A_2, A_3 and b , and A_4 depends only on d, A_1, A_2, A_3 and b .

Lemma 3.5 Assume the same assumptions and use the same notations as in Lemma 3.3. Then for any $b < \frac{1}{2}$, there exists a $\varepsilon > 0$, depends only on a_0 and b , where a_0 is the one chosen in Lemma 3.3, such that

$$\sup_{n \in \mathbf{N}} E^{\nu_0^{\otimes \infty}} \left[\exp \left(b \cdot \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \Psi(X_i, X_j) \right), \left| \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} \Psi(X_i, X_j) \right| < \varepsilon \right] < \infty. \quad (3.12)$$

Proof. First, since $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, there exists a constant $C_8 > 0$, such that $n! \geq C_8^{-1} n^n e^{-2n}$. So, for $m = [n\varepsilon e^2]$,

$$\begin{aligned} \sum_{k=m+1}^{\infty} \frac{(n\varepsilon)^{2k}}{(2k)!} &\leq C_8 \sum_{k=m+1}^{\infty} \left(\frac{n\varepsilon e^2}{2k} \right)^{2k} \leq C_8 \sum_{k=0}^{\infty} \left(\frac{n\varepsilon e^2}{2m+2} \right)^k \\ &\leq C_8 \frac{1}{1 - \frac{n\varepsilon e^2}{2m+2}} \leq 2C_8. \end{aligned} \quad (3.13)$$

Also, in general, for any random variable Z ,

$$\begin{aligned} E[\exp(nZ), |Z| \leq \varepsilon] \\ \leq 2E \left[\sum_{k=0}^m \frac{(nZ)^{2k}}{(2k)!}, |Z| \leq \varepsilon \right] + 2E \left[\sum_{k=m+1}^{\infty} \frac{(nZ)^{2k}}{(2k)!}, |Z| \leq \varepsilon \right], \end{aligned} \quad (3.14)$$

and we can get from Lemma 3.3 that

$$E^{\nu_0^{\otimes \infty}} \left[\left(\sum_{1 \leq i \neq j \leq n} \Psi(X_i, X_j) \right)^m \right] \leq E^{\nu_0^{\otimes \infty}} \left[\left(\sum_{i=1}^n Y_i \right)^{2m} \right] \quad (3.15)$$

for any $m \in \mathbf{N}$ and any $a \leq a_0$, where a_0 is the one chosen in Lemma 3.3.

So, let $P_m(\xi) = \sum_{k=0}^m \frac{\xi^{2k}}{(2k)!}$, $m \in \mathbf{N}$, and we can get from (3.13), (3.14), (3.15) that for $m = [bn\varepsilon e^2]$,

$$\begin{aligned} &E \left[\exp \left(b \cdot \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \Psi(X_i, X_j) \right), \left| \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} \Psi(X_i, X_j) \right| \leq \varepsilon \right] \\ &\leq 4C_8 + 2E \left[\sum_{k=0}^m \frac{(b \frac{1}{n} \sum_{i \neq j} \Psi(X_i, X_j))^{2k}}{(2k)!} \right] \\ &\leq 4C_8 + 2E \left[P_m \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| < \delta \right] \\ &\quad + 2E \left[P_m \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right], \quad \text{for any } \delta > 0. \end{aligned} \quad (3.16)$$

For the second term in the last expression, from the definition of γ_a and the calculation in Lemma 3.1, we see that all of the conditions in Lemma 3.4 is satisfied. So, from Lemma 3.4, for any $b < \frac{1}{2}$, there exists a $\delta > 0$, such that

$$\begin{aligned} &E \left[P_m \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| < \delta \right] \\ &\leq E \left[\exp \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| < \delta \right] \\ &\leq 3C_9. \end{aligned} \quad (3.17)$$

Note that δ does not depend on ε here.

For the last term, since

$$P_m(\xi) \leq c^{-2m} \exp(c|\xi|)$$

for any $c \in (0, 1)$, we can get that

$$\begin{aligned} &E \left[P_m \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right] \\ &\leq c^{-2m} E \left[\exp \left(cb \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right] \\ &\leq c^{-2m} E \left[\exp \left(2cb \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right) \right]^{\frac{1}{2}} P \left(\left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right)^{\frac{1}{2}}. \end{aligned} \quad (3.18)$$

But here, from the definition of Y_i , we can get from Lemma 3.1 that if $A \equiv \frac{1}{4D_n} - 2cb > 0$, which can be done for any fixed a and b by taking c small enough, then

$$E \left[\exp \left(2cb \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right) \right] \leq \frac{4cb}{A} + 1 < \infty. \quad (3.19)$$

Also, by Cramér's Theorem (cf. [6] page 29, Theorem 1.3.13), we see that

$$\begin{aligned} \gamma_n^{\otimes \infty} \left(\left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right) \\ \leq \exp(-nI_{\gamma_n}(\delta)) + \exp(-nI_{\gamma_n}(-\delta)) \\ \leq 2e^{-n\alpha(\delta)}, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} I_{\gamma_n}(\delta) &= \sup\{\xi\delta - \log \int e^{\xi x} \gamma_n(dx), \xi \geq 0\} > 0, \\ I_{\gamma_n}(-\delta) &= \sup\{-\xi\delta - \log \int e^{\xi x} \gamma_n(dx), \xi \leq 0\} > 0, \\ \alpha(\delta) &\equiv I_{\gamma_n}(\delta) \wedge I_{\gamma_n}(-\delta). \end{aligned}$$

We have taken m to be $m = \lceil bnc\varepsilon^2 \rceil$, so if we take $\varepsilon > 0$ small enough, such that

$$\frac{\alpha(\delta)}{2} + 2b\varepsilon e^2 \log c > 0,$$

then from (3.18), (3.19), (3.20), we have

$$\begin{aligned} E \left[P_m \left(b \frac{1}{n} \left(\sum_{i=1}^n Y_i \right)^2 \right), \left| \frac{1}{n} \sum_{i=1}^n Y_i \right| \geq \delta \right] \\ \leq \left(\frac{4cb}{A} + 1 \right)^{\frac{1}{2}} e^{-2m \log c} (2e^{-n\alpha(\delta)})^{\frac{1}{2}} \\ \leq 2 \left(\frac{4cb}{A} + 1 \right)^{\frac{1}{2}} e^{2 \log c} \exp \left(-n \left(\frac{\alpha(\delta)}{2} + 2b\varepsilon e^2 \log c \right) \right) \\ < \exists C_{10}, \quad \text{for any } n \in \mathbf{N}, \end{aligned} \quad (3.21)$$

the c here is the one chosen before.

(3.16), (3.17) and (3.21) completes the proof of the lemma. ■

Lemma 3.6 Assume the same conditions as in Lemma 3.5. Then for any $b < \frac{1}{2}$, there exist constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that the following holds:

$$\begin{aligned} \sup_{n \in \mathbf{N}} E \nu_n^{\otimes \infty} \left[\exp \left(b \cdot n \Psi \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \right. \\ \left. \left\{ \left| \frac{1}{n^2} \sum_{i=1}^n \Psi(X_i, X_i) \right| < \varepsilon_1 \right\} \cap \left\{ \left| \frac{1}{n} \sum_{i=1}^n X_i \right| < \varepsilon_2 \right\} \right] < \infty. \end{aligned}$$

Proof. Let $N_0 \equiv \frac{2bC_0}{C_3}$. For $n = 1, \dots, N_0$, the item is obviously bounded. So we only need to do with $n > N_0$. Since $b < \frac{1}{2}$, there exists a $p > 1$ small enough

such that $p \cdot b < \frac{1}{2}$. Let q be the dual number of p , that is, $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality and Lemma 3.5, we only need to show that

$$\sup_{n > N_0} E \nu_n^{\otimes \infty} \left[\exp \left(qb \cdot \frac{1}{n} \sum_{i=1}^n \Psi(X_i, X_i) \right) \right] < \infty.$$

But by Hölder's inequality, for any $n > N_0 = \frac{2bC_0}{C_3}$,

$$E \nu_n^{\otimes \infty} \left[\exp \left(qb \cdot \frac{1}{n} \sum_{i=1}^n \Psi(X_i, X_i) \right) \right] \leq E \nu_0 \left[\exp(C'_3 \|X\|^2) \right]^{\frac{2bC_0}{C_3}} < \infty.$$

This finished the proof of the lemma. ■

Lemma 3.7 Assume the assumption (A1') in section 1. Assume that Ψ is a symmetric, bilinear function that satisfies the following conditions:

1. There exists a constant $C_0 > 0$, such that

$$|\Psi(x, y)| \leq C_0 \|x\| \cdot \|y\|, \quad \text{for any } x, y \in B,$$

2. $\int_B \Psi(x, y)^2 \nu_0(dx) \nu_0(dy) \equiv b < \frac{1}{2}$.

Then there exists a $\varepsilon > 0$, such that

$$\sup_{n \in \mathbf{N}} E \nu_n^{\otimes \infty} \left[\exp \left(\frac{1}{n} \sum_{i,j=1}^n \Psi(X_i, X_j) \right), \left| \frac{1}{n} \sum_{i=1}^n X_i \right| < \varepsilon \right] < \infty. \quad (3.22)$$

Proof. Since $\Psi(x, y) \leq C_0 \|x\| \cdot \|y\|$ for any $x, y \in B$, we have

$$\begin{aligned} \nu_0^{\otimes \infty} \left(\left| \frac{1}{n^2} \sum_{i=1}^n \Psi(X_i, X_i) \right| \geq \varepsilon_1 \right) &\leq \nu_0^{\otimes \infty} \left(\sum_{i=1}^n \|X_i\|^2 \geq \frac{\varepsilon_1}{C_0} \cdot n^2 \right) \\ &\leq e^{-\frac{\varepsilon_1}{C_0} \cdot n^2 \cdot C_3^2} \cdot (E \nu_0 [e^{C_3^2 \|X_1\|^2}])^n. \end{aligned}$$

Therefore,

$$\begin{aligned} E \nu_n^{\otimes \infty} \left[\exp \left(n \Psi \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\{ \left| \frac{1}{n^2} \sum_{i=1}^n \Psi(X_i, X_i) \right| > \varepsilon_1 \right\} \cap \left\{ \left| \frac{1}{n} \sum_{i=1}^n X_i \right| < \varepsilon_2 \right\} \right] \\ \leq E \nu_n^{\otimes \infty} \left[\exp \left(2n \Psi \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left| \frac{1}{n} \sum_{i=1}^n X_i \right| < \varepsilon_2 \right]^{1/2} \\ \cdot \nu_n^{\otimes \infty} \left(\left| \frac{1}{n^2} \sum_{i=1}^n \Psi(X_i, X_i) \right| > \varepsilon_1 \right)^{1/2} \\ \leq \exp(nC_0\varepsilon_2^2) \cdot \left(\exp \left(-\frac{\varepsilon_1}{C_0} \cdot n^2 \cdot C_3^2 \right) \cdot (E \nu_0 [e^{C_3^2 \|X_1\|^2}])^n \right)^{\frac{1}{2}}, \end{aligned}$$

which is obviously bounded for $n \in \mathbf{N}$.

This accompanied with Lemma 3.6 gives our assertion. ■

4 Proof of the Main Theorem

In this section, we will give the proof of the main theorem.

As in Bolthausen [1], by a easy calculation and Proposition 1.1, we can get that

$$\begin{aligned} & \exp(-n(\Phi(x^*) - h(x^*))) E^{\mu^{\otimes \infty}} \left[\exp(n\Phi(\frac{S_n}{n})) \right] \\ = & E^{\nu_0^{\otimes \infty}} \left[\exp\left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i\right) + nR(x^*, \frac{1}{n} \sum_{i=1}^n X_i)\right) \right], \end{aligned}$$

where $R(x^*, \frac{1}{n} \sum_{i=1}^n X_i)$ is the 3rd remainder of the Taylor's formula.

Therefore, to proof Theorem 1.2, we only need to show that the following two lemmas hold:

Lemma 4.1 *There exists a constant $\varepsilon > 0$, such that*

$$\begin{aligned} & \lim_{n \rightarrow \infty} E^{\nu_0^{\otimes \infty}} \left[\exp\left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i\right) + nR(x^*, \frac{1}{n} \sum_{i=1}^n X_i)\right) \right. \\ & \quad \left. \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon \right] \\ = & \exp\left(\frac{1}{2} \int_B D^2 \Phi(x^*)(x, x) \nu_0(dx)\right) \cdot \det_2(I - D^2 \Phi(x^*))^{-\frac{1}{2}} \equiv A. \end{aligned} \quad (4.1)$$

Lemma 4.2 *For any $\varepsilon > 0$,*

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log E^{\nu_0^{\otimes \infty}} \left[\exp\left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i\right) \right. \right. \\ & \quad \left. \left. + nR(x^*, \frac{1}{n} \sum_{i=1}^n X_i)\right), \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| \geq \varepsilon \right] < 0. \end{aligned} \quad (4.2)$$

Lemma 4.2 can be gotten from the following proposition, which has been shown by Donsker-Varadhan [3]:

PROPOSITION 4.3 *1. $h(x)$ is a lower semi-continuous function, and $\{x : h(x) \leq r\}$ is compact in B for $\forall r \in [0, \infty)$,*

2. For any closed set $K \subset B$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu^{\otimes \infty} \left(\left\{ \mathbf{x}; \frac{1}{n} \sum_{i=1}^n x_i \in K \right\} \right) \leq -\inf \{h(x); x \in K\},$$

3. For any open set $G \subset B$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mu^{\otimes \infty} \left(\left\{ \mathbf{x}; \frac{1}{n} \sum_{i=1}^n x_i \in G \right\} \right) \geq -\inf \{h(x); x \in G\}.$$

To prove Lemma 4.1, we will give the following proposition and lemma first:

PROPOSITION 4.4 *Let $\Psi : B \times B \rightarrow \mathbf{R}$ be a function such that $\Psi|_{H \times H}$ is a Hilbert-Schmidt function with eigenvalues $a_\ell, \ell = 1, 2, \dots$ and eigenvectors $e_\ell, \ell = 1, 2, \dots, i.e.$*

$$\Psi(x, y) = \sum_{k=1}^{\infty} a_k(e_k, x)(e_k, y), \quad \text{for all } x, y \in H.$$

Then e_k can be extended to the whole B for any k that satisfies $a_k \neq 0$, and $\sum_{k=1}^N a_k(e_k, x)(e_k, y)$ converges to $\Psi(x, y)$ in $L^2(d\nu_0^{\otimes 2}, B \times B)$ as $N \rightarrow \infty$.

Proof. $\{e_\ell\}_{\ell \in \mathbf{N}}$ is a complete orthonormal base of H^* . Let $f_\ell, \ell \in \mathbf{N}$ be the dual base of H . Since $\Psi(f_\ell, x) = a_\ell(e_\ell, x)$ for any $x \in H$ for each ℓ , and the left hand side is continuous with respect to $x \in B$, we can extend e_ℓ to the whole B in this way if $a_\ell \neq 0$. The others are easy. ■

Lemma 4.5 *Under the assumptions (A1'), (A2) \sim (A5) in section 1, there exist constants $p > 1$ and $\varepsilon > 0$, such that*

$$\sup_{n \in \mathbf{N}} E^{\nu_0^{\otimes \infty}} \left[\exp\left(p \cdot \frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i\right)\right), \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon \right] < \infty.$$

Proof. Let $a_\ell \in \mathbf{R}$ and $e_\ell \in H^*, \ell \in \mathbf{N}$ be the eigenvalues and the corresponding eigenvectors of $D^2 \Phi(x^*)|_{H \times H}$, then

$$D^2 \Phi(x^*)(x, y) = \sum_{\ell=1}^{\infty} a_\ell(e_\ell, x)(e_\ell, y), \quad \text{for any } x, y \in H.$$

$e_\ell, \ell = 1, 2, \dots$ becomes a orthonormal base of H^* . Let $f_\ell, \ell = 1, 2, \dots$ be the dual base of H , then as done in Proposition 4.4, for any ℓ with $a_\ell \neq 0$, we can assume that $e_\ell \in B^*$.

For any $N \in \mathbf{N}$, let

$$\begin{aligned} \Psi_1^{(N)}(x, y) &= \sum_{k=1}^N a_k(e_k, x)(e_k, y), \\ \Psi_2^{(N)}(x, y) &= D^2 \Phi(x^*)(x, y) - \Psi_1(x, y), \quad x, y \in B. \end{aligned}$$

Since $D^2 \Phi(x^*)$ is a Hilbert-Schmidt function from Proposition 2.2, we can see that $\Psi_2^{(N)}$ is also a Hilbert-Schmidt function. Also, from Proposition 4.4, for any $\delta > 0$, there exists a $N_0 \in \mathbf{N}$ large enough, such that $\int_{B \times B} \Psi_2^{(N_0)}(x, y)^2 \nu_0(dx) \nu_0(dy) < \delta$.

For the sake of simply, from now on, we will write Ψ_i for $\Psi_i^{(N_0)}$, $i = 1, 2$. From the definition of Ψ_1 and Ψ_2 , we see that they are bilinear and symmetric.

From Hölder's inequality, for any $r, s > 1 : \frac{1}{r} + \frac{1}{s} = 1$, we have

$$\begin{aligned} & E^{\otimes \infty} \left[\exp \left(p \left\{ \frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i \right) + n R(x^*, \frac{1}{n} \sum_{i=1}^n X_i) \right\} \right), \right. \\ & \quad \left. \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon \right] \\ & \leq E^{\otimes \infty} \left[\exp \left(p \cdot r \cdot \frac{n}{2} \Psi_1 \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right]^{\frac{1}{r}} \end{aligned} \quad (4.3)$$

$$\cdot E^{\otimes \infty} \left[\exp \left(p \cdot s \cdot \frac{n}{2} \Psi_2 \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right]^{\frac{1}{s}}. \quad (4.4)$$

For (4.3), since Ψ_1 is a finite type, we can consider X_i 's as finite dimensional valued random variables. Also, since $a_k < 1, k \in \mathbf{N}$ from the assumption (A4), and $a_n \rightarrow 0$ as $n \rightarrow \infty$ from the fact that $\sum_{n=1}^{\infty} a_n^2 < \infty$, there exists a constant $a < 1$, such that $a_n < a$ for any $n \in \mathbf{N}$. Take $p > 1$ such that $a \cdot p < 1$, and fix it. Then take $r > 1$ small enough, and we can get from Lemma 3.4 that this term is bounded for $n \in \mathbf{N}$, for $\varepsilon > 0$ small enough. Note that the $p > 1$ and $r > 1$ here depend only on $a_k, k \in \mathbf{N}$, and are independent to N .

For (4.4), as mentioned above, Ψ_2 satisfies all of the conditions in Lemma 3.7 except (3). But for any fixed s , we can take δ small enough such that (3) is being satisfied. So, from Lemma 3.7, (4.4) is bounded for $n \in \mathbf{N}$, for N_0 large enough such that $\delta > 0$ is small enough.

This completes the proof of the lemma. ■

Now, we will give the proof of Lemma 4.1, using the proposition and lemma above.

Proof of Lemma 4.1. Here, from Lemma 4.2, we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[E \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) + n R(x^*, \frac{S_n}{n}) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right] - A \right] \\ & = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left[E \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) + n R(x^*, \frac{S_n}{n}) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right] - A \right] \\ & \leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left[E \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) + n R(x^*, \frac{S_n}{n}) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right] \right. \\ & \quad \left. - E \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right] \right] \end{aligned} \quad (4.5)$$

$$+ \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} E \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right] - A, \quad (4.6)$$

so the lemma will be shown if we can show that (4.5) equals 0, and that there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon < \varepsilon_0$,

$$E^{\otimes \infty} \left[\exp \left(\frac{n}{2} D^2 \Phi(x^*) \left(\frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i \right) \right), \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon \right] \rightarrow A, \quad n \rightarrow \infty. \quad (4.7)$$

Let us show (4.7) first. Here, as in Kusuoka-Tamura [5], we can take a separable Hilbert space H_1 such that $L_0^2(B, d\nu_0)$ is a dense linear subspace of H_1 , and the inclusion map from $L_0^2(B, d\nu_0)$ to H_1 is a Hilbert-Schmidt operator. Then, let W be an H_1 -valued random variable such that

$$E \left[\exp(\sqrt{-1}(W, u)) \right] = \exp \left(-\frac{1}{2} \|u\|_{L_0^2(B, d\nu_0)}^2 \right), \quad \text{for all } u \in H_1^* \subset L_0^2(B, d\nu_0).$$

Since

$$E^{\otimes \infty} \left[u \left(\frac{S_n}{n} \right)^2 \right] = \|u\|_{L_0^2(B, d\nu_0)}^2,$$

from the central limit theorem for independently identically distributed Hilbert space valued random variables, we see that the law of $\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ under $\nu_0^{\otimes \infty}$ converges to W in distribution as $n \rightarrow \infty$.

So,

$$\begin{aligned} & \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \sum_{k=1}^N a_k(e_k, X_i)(e_k, X_j) \\ & = \sum_{k=1}^N a_k(e_k, \frac{1}{\sqrt{n}} S_n)(e_k, \frac{1}{\sqrt{n}} S_n) - \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^N a_k(e_k, X_i)^2 \\ & \rightarrow \sum_{k=1}^N a_k(e_k, W)^2 - \sum_{k=1}^N a_k = \sum_{k=1}^N a_k ((e_k, W)^2 - 1), \quad \text{for any } N \in \mathbf{N}, \end{aligned}$$

where the " \rightarrow " above means the convergence in distribution. Therefore, since

$$E \left[\left\{ \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \left(D^2 \Phi(x^*)(X_i, X_j) - \sum_{k=1}^N a_k(e_k, X_i)(e_k, X_j) \right) \right\}^2 \right] \rightarrow 0, \quad N \rightarrow \infty,$$

which is uniformly in n , we see that $\frac{1}{n} \sum_{1 \leq i \neq j \leq n} D^2 \Phi(x^*)(X_i, X_j)$ under $\nu_0^{\otimes \infty}$ converges to $D^2 \Phi(x^*)(W, W)$ in distribution as $n \rightarrow \infty$, where $D^2 \Phi(x^*)(x, x)$ is defined as the $L^2(d\bar{\mu})$ -limit of $\sum_{\ell=1}^N a_\ell((e_\ell, x)^2 - 1)$ as $N \rightarrow \infty$.

Therefore, (4.7) can be gotten from Lemma 4.5.

Now, let us show that (4.5) equals 0. Write it as $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \phi(n, \varepsilon)$. Let $p > 1$ be the one chosen in Lemma 4.5, and let q be determined by $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\phi(n, \varepsilon) \leq E \left[\exp \left(p \cdot \frac{n}{2} D^2 \Phi(x^*) \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\| \frac{S_n}{n} \right\| < \varepsilon \right]^{1/p} \quad (4.8)$$

$$\cdot E \left[\left| \exp(n R(x^*, \frac{S_n}{n})) - 1 \right|^q, \left\| \frac{S_n}{n} \right\| < \varepsilon \right]^{1/q}. \quad (4.9)$$

The boundness of (4.8) for $n \in \mathbb{N}$ has been established. As for (4.9), by Lemma 3.7,

$$\sup_{n \in \mathbb{N}} E \left[e^{p \cdot q C_4 \varepsilon n K(\frac{\tilde{S}_n}{n}, \frac{\tilde{S}_n}{n})} \cdot \left\| \frac{S_n}{n} \right\| < \varepsilon \right] < \infty$$

if $\varepsilon > 0$ is small enough, so from the fact that $|e^x - 1|^q \leq (e^{|x|} - 1)^q \leq e^{q|x|} - 1$, we have

$$\begin{aligned} (4.9)^q &\leq E \left[e^{q n R(\varepsilon^*, \frac{\tilde{S}_n}{n})} \cdot \left\| \frac{S_n}{n} \right\| < \varepsilon \right] - \nu_0 \left(\left\| \frac{S_n}{n} \right\| < \varepsilon \right) \\ &\leq E \left[e^{q C_4 \varepsilon K(\frac{\tilde{S}_n}{\sqrt{n}}, \frac{\tilde{S}_n}{\sqrt{n}})} \cdot \left\| \frac{S_n}{n} \right\| < \varepsilon \right] - \nu_0 \left(\left\| \frac{S_n}{n} \right\| < \varepsilon \right) \\ &\rightarrow E \left[\exp(q C_4 \varepsilon : K(W, W) :) \right] \cdot e^{C_4 \varepsilon \int_B K(y, y) \nu_0(dy)} - 1, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which converges to 0 as $\varepsilon \rightarrow 0$.

This completes the proof of the lemma. ■

5 Remark

Let $U \equiv \det_2(I - D^2\Phi(x^*))^{-\frac{1}{2}} < \infty$, and let $P_n, n \in \mathbb{N}$, be the probability measures given by

$$dP_n/d\mu^{\otimes \infty}(x) = \exp \left(n\Phi \left(\frac{S_n}{n} \right) \right) / E^{\mu^{\otimes \infty}} \left[\exp(n\Phi(\frac{S_n}{n})) \right], \quad x = (x_1, x_2, \dots).$$

Since we did not assume the existence of the Gaussian measure on B as in Bolthausen [1], we can not write in B the limit of the distribution of $\sqrt{n}(\frac{S_n}{n} - x^*)$ under P_n , but we can still get the following:

THEOREM 5.1 Assume the same conditions as in Theorem 1.2, then for any $n \in \mathbb{N}$, and any $u_k \in B^*, k = 1, 2, \dots, n$, the distribution of $\{B^*(u_k, \sqrt{n}(\frac{S_n}{n} - x^*))\}_B$, $u_k \in B^*, k = 1, 2, \dots, n$ under P_n converge weakly to the Normal distribution $N(0, (\sum_{i=1}^{\infty} u_i^k u_j^k \frac{1}{1-u_k})_{i,j})$, where $a_i, e_k, \ell \in \mathbb{N}$ are the ones defined in the proof of lemma 4.5, and $u_i = \sum_k u_i^k e_k$, $i = 1, 2, \dots$.

The proof is similar with the one above, and will be omitted.

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Chapter 2

Laplace Approximations for Sums of Independent Random Vectors – the degenerate case –

Abstract

Let $X_i, i \in \mathbf{N}$, be i.i.d. B -valued random variables, where B is a real separable Banach space. Let $\Phi : B \rightarrow \mathbf{R}$ be a mapping. The problem is to give an asymptotic evaluation of $Z_n = E(\exp(n\Phi(\sum_{i=1}^n X_i/n)))$, up to a factor $(1+o(1))$. Bolthausen [1] studied this problem in the case that there is a unique point maximizing $\Phi - h$, where h is the so-called entropy function, and the curvature at the maximum is nonvanishing, (these two will be called as *nondegenerate assumptions*), with some central limit theorem assumption. Kusuoka-Liang [5] studied the same problem, and succeeded in eliminating the central limit theorem assumption, but the nondegenerate assumptions are still left. In this paper, we study the same problem not assuming the central limit theorem assumption and the nondegenerate assumptions.

1 Introduction

Let B be a real separable Banach space with norm $\|\cdot\|$, μ be a probability measure on B . We assume that the smallest closed affine space that contains $\text{supp } \mu$ is B . Moreover we assume

(A1) There exists a constant $C_1 > 0$, such that

$$\int_B \exp(C_1 \|x\|^2) \mu(dx) < \infty.$$

Let $\Phi : B \rightarrow \mathbf{R}$ be a three times continuously Fréchet differentiable function satisfying the following:

(A2) There exist constants $C_2, C_3 > 0$, such that

$$\Phi(x) \leq C_2 + C_3 \|x\|, \quad \text{for any } x \in B.$$

Let X_n and S_n , $n \in \mathbf{N}$, be the random variables defined by $X_n(\underline{x}) = x_n$, $S_n(\underline{x}) = \sum_{k=1}^n x_k$, $\underline{x} = (x_1, x_2, x_3, \dots) \in B^{\mathbf{N}}$. Let $Z_n = E^{\mu^{\otimes n}} \left[\exp(n\Phi(\frac{S_n}{n})) \right]$.

By Donsker-Varadhan [3], we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \sup_{x \in B} \{\Phi(x) - h(x)\} \equiv \lambda,$$

where h is the entropy function of μ :

$$h(x) = \sup_{\phi \in B^*} \{\phi(x) - \log M(\phi)\},$$

B^* is the dual Banach space of B and $M(\phi) = \int_B e^{\phi(x)} \mu(dx)$, $\phi \in B^*$. Let

$$V = \{x \in B : \Phi(x) - h(x) = \lambda\}.$$

For each $x \in V$, let ν_x be the probability measure on B defined by

$$\nu_x(dy) = \exp(D\Phi(x)(y)) \mu(dy) / M(D\Phi(x)). \quad (1.1)$$

Then by the fact that $x \in V$ maximizes $\Phi - h$, we have

$$\begin{aligned} \int_B y \nu_x(dy) &= x, \\ h(x) &= D\Phi(x)(x) - \log M(D\Phi(x)). \end{aligned} \quad (1.2)$$

Let $\nu_{x,0}$ be the 0-centered ν_x , that is,

$$d\nu_{x,0}(y) = d\nu_x(y+x), \quad \forall y \in B,$$

and also, let Γ_x be the covariance on B^* defined by

$$\Gamma_x(\phi, \psi) = \int_B \phi(y)\psi(y) \nu_{x,0}(dy), \quad \forall \phi, \psi \in B^*.$$

Then by the fact that $x \in V$ maximizes $\Phi - h$, we have

$$\Gamma_x(\phi, \phi) \geq D^2\Phi(x)(S_x\phi, S_x\phi), \quad \forall \phi \in B^*,$$

where $S_x : B^* \rightarrow B$ is defined as $S_x\phi \equiv \int_B \phi(y) y \nu_{x,0}(dy)$, $\forall \phi \in B^*$. Define

$$A_x \equiv \{\phi \in B^* : \Gamma_x(\phi, \phi) = D^2\Phi(x)(S_x\phi, S_x\phi)\}.$$

Let $H_x \equiv (\overline{B^*}^{\Gamma_x})^*$, where $\overline{B^*}^{\Gamma_x}$ means the completion of B^* with respect to Γ_x , denote by $(\cdot, \cdot)_x$ the inner product in H_x , and $\|\cdot\|_x$ the norm of it. Then $S_x(B^*) \subset H_x$, $\psi(S_x\phi) = \Gamma_x(\phi, \psi)$ for any $\phi, \psi \in B^*$, and

$$\begin{aligned} \|S_x\phi\|_x^2 &= \sup\{\psi(S_x\phi)^2; \Gamma_x(\psi, \psi) \leq 1\} \\ &= \sup\{\Gamma_x(\phi, \psi)^2; \Gamma_x(\psi, \psi) \leq 1\} \\ &= \Gamma_x(\phi, \phi). \end{aligned}$$

So $(S_x\phi, S_x\psi)_x = \Gamma_x(\phi, \psi)$ for any $\phi, \psi \in B^*$.

Also, as it has been shown in Kusuoka-Liang [5, Proposition 2.1, Proposition 2.2], H_x can be regarded as a dense subset of B , and for any continuous bilinear function $A: B \times B \rightarrow \mathbf{R}$, $A|_{H_x \times H_x}$ is a Hilbert-Schmidt function for any $x \in V$.

Moreover, we assume the following:

(A3) There exist constants $C_4 > 0$ and $\delta_0 > 0$, and a continuous bilinear symmetric function $K: B \times B \rightarrow \mathbf{R}$, such that

$$|D^3\Phi(x)(y, y, y)| \leq C_4 \|y\| K(y, y), \quad \text{for any } y \in B \text{ and } x \in V_{\delta_0},$$

where V_{δ_0} denotes the δ_0 -neighborhood of V in B .

Our result in this paper is the following:

THEOREM 1.1 *Under the above assumptions, there exist an integer $d \geq 1$ and a d -dimensional manifold M embedded in B with Riemann metric such that $V \subset M$ and $S_x(A_x) \subset T_x M$ for any $x \in V$, and there exist continuous functions $x: M \rightarrow B$, $a: M \rightarrow [0, \infty)$ and $b: M \rightarrow (0, \infty)$ such that*

- (1) $x(\cdot) \in C^2(M)$, and $x(z) = z$ for any $z \in V$,
- (2) $a(z) = 0$ if and only if $z \in V$, and
- (3) for any bounded continuous function $f: B \rightarrow \mathbf{R}$,

$$E^{n \otimes \infty} \left[f\left(\frac{S_n}{n}\right) \exp\left(n\Phi\left(\frac{S_n}{n}\right)\right) \right] = e^{n\lambda} n^{\frac{d}{2}} \int_M f(x(z)) b(z) e^{-n\alpha(z)} v_M(dz) (1 + o(1))$$

as $n \rightarrow \infty$, where v_M is the volume element on M .

See Lemma 5.2 for the precise expression of $a(z)$ and $b(z)$, $z \in M$.

As a corollary, we get the following:

COROLLARY 1.2 *Under the above assumptions, there exist an integer $d \geq 1$ and a d -dimensional manifold M embedded in B with Riemann metric such that*

$V \subset M$ and $S_x(A_x) \subset T_x M$ for any $x \in V$, and there exist continuous functions $a: M \rightarrow [0, \infty)$ and $b: M \rightarrow (0, \infty)$ such that

- (1) $a(z) = 0$ if and only if $z \in V$, and
- (2)

$$Z_n = e^{n\lambda} n^{\frac{d}{2}} \int_M b(z) e^{-n\alpha(z)} v_M(dz) (1 + o(1))$$

as $n \rightarrow \infty$, where v_M is the volume element on M .

Chiyonobu [2] studied the same problem under a certain central limit theorem assumption.

2 Manifold reflecting singularities

In this section, we will show the existence of the manifold M with the properties described in our main theorem.

The following is well-known (c.f. Bolthausen [1]):

Lemma 2.1 (1) h is non-negative, lower semicontinuous and convex, and is strongly convex on $\{x \in B: h(x) < \infty\}$.

(2) $h(x) = 0$ if and only if $x = \int y \mu(dy)$.

(3) For all $r \in [0, \infty)$, $\{x: h(x) \leq r\}$ is compact in B .

(4) $\lim_{r \rightarrow \infty} \inf_{\{x\} \geq r} h(x)/r = \infty$.

As in Kusuoka-Tamura [6] and Chiyonobu [2], we can show that

Lemma 2.2 V is a non-void compact set.

Proof. First take $z_n \in B$, $n = 1, 2, \dots$, such that $\Phi(z_n) - h(z_n) \rightarrow \lambda$, where λ is the maximum of $\Phi - h$, as defined in section 1. Then z_n is bounded by assumption (A2) and Lemma 2.1 (4). So $h(z_n)$ is bounded. By Lemma 2.1 (3), this implies that there exists a subsequence n_k and a $z \in B$, such that $z_{n_k} \rightarrow z$, as $k \rightarrow \infty$. So

$$\Phi(z) - h(z) \geq \limsup_{k \rightarrow \infty} (\Phi(z_{n_k}) - h(z_{n_k})) = \lambda,$$

which implies that $z \in V$. Therefore, V is non-void.

The same argument implies that V is compact.

DEFINITION 2.3 We say that M is a manifold reflecting singularities if M is a submanifold embedded in B , $V \subset M$, and $S_x(A_x) \subset T_x M$ for each $x \in V$.

In the rest of this section, we will show that a manifold reflecting singularities exists.

Define $\tilde{\Gamma}(\phi, \psi) \equiv \int_B \phi(y) \psi(y) e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy) / \int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy)$, $\phi, \psi \in B^*$. Note that this is finite for all $\phi, \psi \in B^*$ by the assumption (A1).

PROPOSITION 2.4 *There exists a common constant $C > 0$ independent to $z \in V$, such that*

$$\Gamma_z(\phi, \phi) \leq C^2 \tilde{\Gamma}(\phi, \phi), \quad \text{for any } \phi \in B^* \text{ and any } z \in V.$$

Proof. From the compactness of V and the continuity of $D\Phi$, there exist constants $K_1 > 0$ and $K_2 > 0$ such that $\|z\| \leq K_1$ for all $z \in V$ and $|D\Phi(z)(y)| \leq K_2 \|y\|$ for all $z \in V$ and all $y \in B$. So from the fact that $e^{K_2 x - \frac{1}{2}C_1 x^2} \leq e^{\frac{K_2^2}{2C_1}}$ for all $x \in \mathbf{R}$, we have that for all $z \in V$,

$$\int_B \phi(y)^2 e^{D\Phi(z)(y)} \mu(dy) \leq e^{\frac{K_2^2}{2C_1}} \int_B \phi(y)^2 e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy).$$

Therefore

$$\begin{aligned} \Gamma_z(\phi, \phi) &= \frac{\int_B \phi(y)^2 e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} - \phi(z)^2 \\ &\leq \frac{\int_B \phi(y)^2 e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \\ &\leq e^{\frac{K_2^2}{2C_1}} \cdot \frac{\int_B \phi(y)^2 e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy)}{\int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy)} \cdot \frac{\int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \\ &\leq e^{\frac{K_2^2}{2C_1}} \cdot \frac{\int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy)}{\int_B e^{-K_2 \|y\|} \mu(dy)} \cdot \tilde{\Gamma}(\phi, \phi), \quad \text{for all } \phi \in B^*. \end{aligned}$$

This gives our assertion with $C \equiv \left(e^{\frac{K_2^2}{2C_1}} \cdot \int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy) / \int_B e^{-K_2 \|y\|} \mu(dy) \right)^{1/2}$. ■

Let $\tilde{H} \equiv (\tilde{B}^*)^*$, then \tilde{H} is a Hilbert space. Let $\tilde{S}: B^* \rightarrow B$ be given by

$$\tilde{S}\phi \equiv \int_B \phi(y) y e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy) / \int_B e^{\frac{1}{2}C_1 \|y\|^2} \mu(dy), \quad \phi \in B^*.$$

Then we have $(\tilde{S}\phi, \tilde{S}\psi)_{\tilde{H}} = \tilde{\Gamma}(\phi, \psi)$ for any $\phi, \psi \in B^*$. The norm of \tilde{H} will be denoted by $\|\cdot\|_{\tilde{H}}$ in this paper. \tilde{H} is separable, and by the same method as in Kusuoka-Liang [5, Proposition 2.1, Proposition 2.2], \tilde{H} can be considered as a dense subset of B , and for any continuous bilinear function $A: B \times B \rightarrow \mathbf{R}$, $A|_{\tilde{H} \times \tilde{H}}$ is a Hilbert-Schmidt function.

From Proposition 2.4 and the definition of $\|\cdot\|_{\tilde{H}}$,

$$\|\varphi\|_{H_z^*} \leq C \|\varphi\|_{\tilde{H}^*}, \quad \text{for any } \varphi \in \tilde{H}^* \text{ and any } z \in V. \quad (2.1)$$

Therefore

$$\|x\|_{\tilde{H}} \leq \frac{1}{C} \|x\|_{H_z}, \quad \text{for any } x \in H_z \text{ and any } z \in V. \quad (2.2)$$

That is, H_z can be embedded into \tilde{H} naturally for each $z \in V$.

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal base of \tilde{H}^* with $\{e_n\} \subset B^*$. Then $\{\tilde{S}e_n\}_{n=1}^\infty$ is the corresponding base of \tilde{H} . Let $Q_n: B \rightarrow B$ be defined by $Q_n(z) = \sum_{i=1}^n e_i(z) \tilde{S}e_i$, $\forall z \in B$, $\forall n \in \mathbf{N}$. Then $Q_n: B \rightarrow B$, $n \in \mathbf{N}$, is a sequence of bounded operators that satisfies the following: $\dim(\text{Image } Q_n) = n$, $Q_n^2 = Q_n$, $Q_{n+1}Q_n = Q_nQ_{n+1} = Q_n$ for any $n \in \mathbf{N}$ and $\cup_{n \in \mathbf{N}} Q_n(B)$ is dense in B .

Lemma 2.5 *There exists an integer d_1 such that for any $d \geq d_1$, $P_d D\Phi|_V$ is injective, where $P_d: B^* \rightarrow B^*$ denotes the adjoint operator of Q_d .*

Proof. It is obvious that the lemma can be seen if we can show that $P_{d_1} D\Phi|_V$ is injective for $d_1 \in \mathbf{N}$ large enough. Suppose not. Then for each $d \in \mathbf{N}$, there exist $z_d^1, z_d^2 \in V$ such that $P_d D\Phi(z_d^1) = P_d D\Phi(z_d^2)$, and $z_d^1 \neq z_d^2$. Combining with (1.1) and (1.2), we see that $D\Phi(z_d^1) \neq D\Phi(z_d^2)$.

Since V is compact, we may assume that $z_d^1 \rightarrow z^1$ and $z_d^2 \rightarrow z^2$ as $d \rightarrow \infty$. Then $D\Phi(z^1)(Q_d u) = D\Phi(z^2)(Q_d u)$ for any $u \in B$ and any $d \in \mathbf{N}$. So $D\Phi(z^1) = D\Phi(z^2)$, hence $z^1 = z^2$. We write this same point by $z \in V$.

Let $f: B^* \rightarrow B^*$ be defined by $f(\phi) = D\Phi(\int_B y e^{\phi(y)} \mu(dy) / M(\phi))$, $\phi \in B^*$. Then f is continuous, Fréchet differentiable, and $f(D\Phi(w)) = D\Phi(w)$ for any $w \in V$. Therefore,

$$\begin{aligned} D\Phi(z_d^1) - D\Phi(z_d^2) &= f(D\Phi(z_d^1)) - f(D\Phi(z_d^2)) \\ &= Df(D\Phi(z))(D\Phi(z_d^1) - D\Phi(z_d^2)) \\ &\quad + \int_0^1 [Df(D\Phi(z_d^1) + t(D\Phi(z_d^1) - D\Phi(z_d^2))) - Df(D\Phi(z))] \\ &\quad \quad \quad (D\Phi(z_d^1) - D\Phi(z_d^2)) dt. \end{aligned}$$

Let $\varphi_d = \frac{D\Phi(z_d^1) - D\Phi(z_d^2)}{\|D\Phi(z_d^1) - D\Phi(z_d^2)\|_{H_z^*}}$, which is well-defined since $D\Phi(z_d^1) \neq D\Phi(z_d^2)$. Then from the fact that $z_d^1 \rightarrow z$, $z_d^2 \rightarrow z$ as $d \rightarrow \infty$, we see from the equality above that $\varphi_d - Df(D\Phi(z))\varphi_d \rightarrow 0$ in H_z^* as $d \rightarrow \infty$. From the fact $z \in V$, we have that for

any $\psi \in B^*$,

$$\begin{aligned}
 Df(D\Phi(z))(\psi) &= \frac{d}{dt} \left(D\Phi \left(\frac{\int_B y e^{D\Phi(z)(y)+t\psi(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)+t\psi(y)} \mu(dy)} \right) \right) \Big|_{t=0} \\
 &= D^2\Phi \left(\frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right) \left(\frac{\int_B \psi(y) y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right. \\
 &\quad \left. - \frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \cdot \frac{\int_B \psi(y) e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right) \\
 &= D^2\Phi(z) \left(\int_B \psi(y) y \nu_z(dy) - \psi(z) z, \cdot \right) \\
 &= D^2\Phi(z) \left(\int_B \psi(y) y \nu_{z,0}(dy), \cdot \right) \\
 &= D^2\Phi(z)(S_z\psi, \cdot).
 \end{aligned}$$

Since $D^2\Phi(z)|_{H_z \times H_z}$ is a compact operator, we see from the above that there exists a $\varphi \in H_z^*$ such that $\varphi_d \rightarrow \varphi$ in H_z^* . Then $\|\varphi\|_{H_z^*} = 1$ since $\|\varphi_d\|_{H_z^*} = 1$. On the other hand, $\varphi_d(Qdy) = 0$ for all $y \in B$, which implies that $\varphi \equiv 0$. This makes a contradiction. ■

Lemma 2.6 *There exists an integer d_2 large enough such that for any $d \geq d_2$, any $z \in V$ and any $x \in B$ with $x \neq 0$,*

$$x - \int_B (D^2\Phi(z)(x, y) - D^2\Phi(z)(x, Q_dy)) y \nu_{z,0}(dy) \neq 0.$$

Proof. If not, then for any $n \in \mathbb{N}$, there exist $d_n \in \mathbb{N}$, $z_n \in V$ and $x_n \neq 0$, such that

$$\begin{aligned}
 x_n &= \int_B (D^2\Phi(z_n)(x_n, y) - D^2\Phi(z_n)(x_n, Q_{d_n}y)) y \nu_{z_n,0}(dy) \\
 &= S_{z_n} \left((I - P_{d_n}) D^2\Phi(z_n)(x_n, \cdot) \right). \tag{2.3}
 \end{aligned}$$

From the compactness of V , by taking subsequence if necessary, we may assume that z_n converge in V , i.e., $z_n \rightarrow z \in V$ as $n \rightarrow \infty$. Note that $x_n \in H_{z_n} \subset \tilde{H}$ for any $n \in \mathbb{N}$, so by dividing the both side by $\|x_n\|_{\tilde{H}}$ if necessary, we may assume that $\|x_n\|_{\tilde{H}} = 1$. Therefore, from the fact that $D^2\Phi(z)|_{\tilde{H} \times \tilde{H}}$ is a compact operator, by taking subsequence if necessary, we may assume that $D^2\Phi(z)(x_n, \cdot)$ converge in \tilde{H}^* . On the same time, from the assumption that $\|x_n\|_{\tilde{H}} = 1$, we have that $\|x_n\|$ is bounded for $n \in \mathbb{N}$. So from the continuity of $D^2\Phi$, $D^2\Phi(z_n)(x_n, \cdot)$ is also convergent in \tilde{H}^* . This implies that $(I - P_{d_n}) D^2\Phi(z_n)(x_n, \cdot) \rightarrow 0$ in \tilde{H}^* as $n \rightarrow \infty$.

So from (2.3) and (2.1), $\|x_n\|_{H_{z_n}} = \|(I - P_{d_n}) D^2\Phi(z_n)(x_n, \cdot)\|_{H_{z_n}^*} \rightarrow 0$ as $n \rightarrow \infty$. By (2.2), this implies that $\|x_n\|_{\tilde{H}} \rightarrow 0$ as $n \rightarrow \infty$. This contradicts with the assumption that $\|x_n\|_{\tilde{H}} = 1$ for all $n \in \mathbb{N}$. ■

Lemma 2.7 *There exists an integer $d_3 \in \mathbb{N}$ large enough such that for any $d \geq d_3$, any $z \in V$, $\varphi \in A_z$ and $\varphi \neq 0$ imply $Q_d S_z \varphi \neq 0$.*

Proof. As the way of proof is similar to that of Lemma 2.6, we only give the sketch here.

Same as in the proof of Lemma 2.5, we only need to show that for $d_3 \in \mathbb{N}$ large enough, for any $z \in V$, $\varphi \in A_z$ and $\varphi \neq 0$ imply $Q_{d_3} S_z \varphi \neq 0$.

If not, then for any $d \in \mathbb{N}$, there exist $z_d \in V$ and $\varphi_d \in A_{z_d}$ with $\varphi_d \neq 0$ but $Q_d S_{z_d} \varphi_d = 0$. Without loss of generality, we can assume, by the compactness of V , that there exists a $z \in V$, such that $z_d \rightarrow z$. Also, we can assume that $\|\varphi_d\|_{\tilde{H}^*} = 1$. Since $\varphi_d \in A_{z_d}$, it can be seen that $\varphi_d = D^2\Phi(z_d)(S_{z_d} \varphi_d, \cdot)$. By Proposition 2.4, $\|S_{z_d} \varphi\|_{\tilde{H}} \leq 1$. So from the fact that $D^2\Phi(z)|_{\tilde{H} \times \tilde{H}}$ is a compact operator and $D^2\Phi(\cdot)$ is continuous, $\varphi_d = D^2\Phi(z_d)(S_{z_d} \varphi_d, \cdot)$ converges in \tilde{H}^* . Write the limit as φ_0 . Then $\|\varphi_0\|_{\tilde{H}^*} = 1$. On the other hand, $Q_d \varphi_0 = 0$ for any $d \in \mathbb{N}$, so $\varphi_0 = 0$. This is a contradiction. ■

Let d be the maximum of d_1 , d_2 , and d_3 , the integers chosen in Lemma 2.5, Lemma 2.6, and Lemma 2.7, respectively. Let $W \equiv \text{Image } P_d$, let $\|\cdot\|_W$ and $\text{dist}_W(\cdot, \cdot)$ denote the norm and the distance on it, respectively.

THEOREM 2.8 *There exist a neighborhood U of $\{P_d D\Phi(z), z \in V\}$ in W small enough, and a map $X(\cdot) : U \rightarrow B$, which is a C^2 -diffeomorphism. In particular, there is a manifold reflecting singularities.*

Proof. The proof will be divided into several steps.

Step 1. Let $f : B \times W \rightarrow B$ be defined by

$$f(z, \varphi) = z - \frac{\int y e^{D\Phi(z)(y) - D\Phi(z)(Q_dy) + \varphi(y)} \mu(dy)}{\int e^{D\Phi(z)(y) - D\Phi(z)(Q_dy) + \varphi(y)} \mu(dy)}, \quad \forall z \in B, \quad \forall \varphi \in W.$$

Then f is twice continuously differentiable with respect to z , and $f(z, P_d D\Phi(z)) = 0$ for any $z \in V$. For any $z \in V$, let $\varphi_z \equiv P_d D\Phi(z)$. Then by Lemma 2.6, for any $x \in B$ not equal to 0,

$$\begin{aligned}
 D_z f(z, \varphi)(x) \Big|_{\varphi=\varphi_z} &= x - \left(\frac{\int_B (D^2\Phi(z)(x, y) - D^2\Phi(z)(x, Q_dy)) y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right. \\
 &\quad \left. - \frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \cdot \frac{\int_B (D^2\Phi(z)(x, y) - D^2\Phi(z)(x, Q_dy)) e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right) \\
 &= x - \int_B (D^2\Phi(z)(x, y) - D^2\Phi(z)(x, Q_dy)) y \nu_{z,0}(dy) \\
 &\neq 0.
 \end{aligned}$$

So by implicit function theorem, for any $z \in V$, there exists U_{φ_z} , a neighborhood of φ_z in W , and a unique twice continuously differentiable function $G_z(\varphi)$ defined on U_{φ_z} , such that $f(G_z(\varphi), \varphi) = 0$ on $\varphi \in U_{\varphi_z}$.

Step 2. In this step, we will show that the functions $\{G_z\}_{z \in V}$ are consistent if the neighborhoods $\{U_{\varphi_z}\}_{z \in V}$ are taken small enough.

For any $z \in V$, let $U_{z,n} \equiv U_{\varphi_z} \cap B_W(P_d D\Phi(z), \frac{1}{n})$, where U_{φ_z} is the one chosen before, and $B_W(\varphi, \varepsilon)$ means the neighborhood of φ in W with radius ε . Let $U_n = \bigcup_{z \in V} U_{z,n}$. We only need to show that there exists an integer $n \in \mathbb{N}$ large enough, such that for any $z, w \in V$, G_z and G_w are consistent on $U_{z,n} \cap U_{w,n}$. If not, then for any $n \in \mathbb{N}$, there exist $z_n, w_n \in V$ and $\varphi_n \in U_{z_n,n} \cap U_{w_n,n}$, such that $G_{z_n}(\varphi_n) \neq G_{w_n}(\varphi_n)$. From the compactness of V , by taking subsequence if necessary, we may suppose that there exist $z, w \in V$, such that $z_n \rightarrow z, w_n \rightarrow w$. If $z \neq w$, then $U_{z_n,n} \cap U_{w_n,n} = \emptyset$ for n large enough from Lemma 2.5 and the continuity of $D\Phi$. This is a contradiction. So we have $z = w$. From the definition of $U_{z_n,n}$, for any $\varepsilon > 0$, there exists a integer $N \in \mathbb{N}$, such that for any $n > N$, $\varphi_n \in B_W(P_d D\Phi(z), \varepsilon)$. As has been shown in step 1, if we take $\varepsilon > 0$ small enough, there exists only a unique $G_z(\varphi_n)$ that satisfies $f(G_z(\varphi_n), \varphi_n) = 0$. This makes a contradiction since $f(G_{z_n}(\varphi_n), \varphi_n) = 0$, $f(G_{w_n}(\varphi_n), \varphi_n) = 0$ and $G_{z_n}(\varphi_n) \neq G_{w_n}(\varphi_n)$ from the assumption.

Step 3. Now, we have shown that there exist U , a neighborhood of $\{P_d D\Phi(z); z \in V\}$ in W , and a twice continuously differentiable function $X(\varphi)$ on U , such that $f(X(\varphi), \varphi) = 0$ on $\varphi \in U$, i.e.

$$X(\varphi) = \frac{\int_B y e^{D\Phi(X(\varphi))(y) - D\Phi(X(\varphi))(Q_d y) + \varphi(y)} \mu(dy)}{\int_B e^{D\Phi(X(\varphi))(y) - D\Phi(X(\varphi))(Q_d y) + \varphi(y)} \mu(dy)},$$

and that $X(\varphi_z) = z$ for any $z \in V$. Differentiating the both side at φ_z , and we have for any $z \in V$ and any $\psi \in W$,

$$\begin{aligned} & DX(\varphi_z)(\psi) \\ &= \int_B \left(D^2\Phi(z)(DX(\varphi_z)(\psi), y) - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d y) + \psi(y) \right) y \nu_z(dy) \\ &\quad - \left(D^2\Phi(z)(DX(\varphi_z)(\psi), z) - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d z) + \psi(z) \right) z \\ &= \int_B \left(D^2\Phi(z)(DX(\varphi_z)(\psi), y) \right. \\ &\quad \left. - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d y) + \psi(y) \right) y \nu_{z,0}(dy). \end{aligned} \quad (2.4)$$

So if $DX(\varphi_z)(\psi) = 0$, then $\int_B \psi(y) y \nu_{z,0}(dy) = 0$, hence $\int_B \psi(y) y \nu_{z,0}(dy) = 0$. Therefore from the assumption that the smallest closed affine space that contains $\text{supp} \mu$ is B , we get $\psi = 0$. That is, $DX(\varphi_z)(\psi) \neq 0$ whenever $\psi \neq 0$. So we can take U

small enough one more time again if needed, such that $\varphi \mapsto X(\varphi), \varphi \in U$, is a local diffeomorphism.

Step 4. In this step, we will show that we can take the neighborhood U small enough such that $\varphi \mapsto X(\varphi), \varphi \in U$, is an injective, which accompanying with the step 3 implies that $\varphi \mapsto X(\varphi), \varphi \in U$, is not only a local diffeomorphism, but also a diffeomorphism.

If not, for any $m \in \mathbb{N}$, there exist $\varphi_m, \psi_m \in W$, $\varphi_m \neq \psi_m$, $X(\varphi_m) = X(\psi_m)$, and $\text{dist}_W(\varphi_m, \tilde{V}) + \text{dist}_W(\psi_m, \tilde{V}) \rightarrow 0$ as $m \rightarrow \infty$, where $\tilde{V} \equiv \{P_d D\Phi(z); z \in V\}$. \tilde{V} is compact in W , so by taking subsequence if necessary, we may assume that there exist $\varphi_\infty, \psi_\infty \in \tilde{V}$, such that $\varphi_m \rightarrow \varphi_\infty$ and $\psi_m \rightarrow \psi_\infty$ in W as $m \rightarrow \infty$. So $X(\varphi_\infty) = X(\psi_\infty)$ from the continuity of the map $\varphi \mapsto X(\varphi)$. Accompanying with the fact that $\varphi_\infty, \psi_\infty \in V$, this implies that $\varphi_\infty = \psi_\infty$. From the fact that $\varphi \mapsto X(\varphi), \varphi \in U$ is a local diffeomorphism, this implies that there exists a $M \in \mathbb{N}$ large enough, such that for all $m \geq M$, $X(\varphi_m) \neq X(\psi_m)$. This makes a contradiction.

Step 5. Let $M = \{X(\varphi); \varphi \in U\}$. In the following, we will check that M satisfies all of the conditions in Definition 2.3. The first is obvious from the fact that $\varphi \mapsto X(\varphi), \varphi \in U$ is a diffeomorphism. The second is true since $z = X(P_d D\Phi(z))$ for any $z \in V$. For the third one, for any $u \in A_z$, we have $u = D^2\Phi(z)(S_z u, \cdot)$, where the operator S_z is defined in section 1. So

$$S_z u = S_z(D^2\Phi(z)(S_z u, \cdot)) - S_z(P_d D^2\Phi(z)(S_z u, \cdot)) + S_z(P_d u).$$

Combining this with (2.4), we see that both $S_z u$ and $DX(\varphi_z)(P_d u)$ (where $\varphi_z \equiv P_d D\Phi(z)$ as before) are solutions of

$$X = S_z((I - P_d)D^2\Phi(z)(X, \cdot)) + S_z(P_d u). \quad (2.5)$$

So from the uniqueness of the solution of the equality (2.5), which comes from Lemma 2.6, $S_z u = DX(\varphi_z)(P_d u)$. Hence $S_z u \in T_z M$ for any $z \in V$ and any $u \in A_z$. This completes the proof of the fact that M is a manifold reflecting singularities.

This finishes the proof of the theorem. \blacksquare

3 Resolution of singularity

In this section, we construct a family of functions Φ_ε defined on B , $z \in M \cap V_\delta$, for $\delta > 0$ small enough, such that $\Phi_\varepsilon, z \in M \cap V_\delta$, are not degenerate.

First, we show the following

Lemma 3.1 *There exist an integer $k \in \mathbb{N}$ large enough and a $\delta \in (0, \delta_0)$ such that $Q_k|_{M \cap V_\delta}$ is injective, where δ_0 is the one in the assumption (A3), and for any $\varphi \in \{P_d D\Phi(z), z \in V\}$ and $\psi \in W$, $Q_k DX(\varphi)(\psi) = 0$ implies $\psi = 0$.*

Proof. If not, then for any $n \in \mathbb{N}$, there exist $x_n, y_n \in M \cap V_{1/n}$, such that $Q_n x_n = Q_n y_n$, but $x_n \neq y_n$. It is easy to see that by taking subsequence if necessary, we can assume that there exist $x, y \in M \cap V$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. $Q_n x_n = Q_n y_n$ for any $n \in \mathbb{N}$ implies that $x = y$. That is, x_n and y_n converge to a same limit as $n \rightarrow \infty$. As $x_n, y_n \in M$, by the definition of M , there exist $\varphi_n, \psi_n \in W$, such that $x_n = X(\varphi_n)$ and $y_n = X(\psi_n)$, $n \in \mathbb{N}$. $\frac{\varphi_n - \psi_n}{\|\varphi_n - \psi_n\|_W} \in W$ is bounded, so by taking subsequence if necessary, we can assume that it converges. Also, x_n and y_n converge as $n \rightarrow \infty$ implies that φ_n and ψ_n converge as $n \rightarrow \infty$, too. Note that $\|x_n - y_n\|_M = \|\varphi_n - \psi_n\|_W$ for any $n \in \mathbb{N}$. Therefore,

$$w_n \equiv \frac{x_n - y_n}{\|x_n - y_n\|} = \int_0^1 DX(\psi_n + t(\varphi_n - \psi_n)) \left(\frac{\varphi_n - \psi_n}{\|\varphi_n - \psi_n\|} \right) dt$$

converges as $n \rightarrow \infty$. Write the limit as $w \in M$. Then as done before, from the assumption, $\|w\|_M = 1$, but $Q_n w = 0$ for any $n \in \mathbb{N}$, hence $w = 0$. This is a contradiction.

For the second part, we use the contradiction, too. If not, then for any $n \in \mathbb{N}$, there exist a $\varphi_n \in \{P_d D\Phi(z), z \in V\}$, and a $\psi_n \in W (\equiv \text{Im } P_d)$ with $\psi_n \neq 0$, such that $Q_n DX(\varphi_n)(\psi_n) = 0$. Without loss of generality, we may assume that $\|\psi_n\|_W = 1$, $n \in \mathbb{N}$. So, by taking subsequence if necessary, we may assume that ψ_n converges to a $\psi \in W$ in W . Hence, $\|\psi\|_W = 1$. On the same time, from the compactness of V , by taking subsequence if necessary, we may assume that there exists a $\varphi \in \{P_d D\Phi(z), z \in V\}$, such that $\varphi_n \rightarrow \varphi$. So, by the continuity, we get that $Q_n DX(\varphi)(\psi) = 0$ for every $n \in \mathbb{N}$. Therefore, $DX(\varphi)(\psi) = 0$, which implies that $\psi = 0$. This makes a contradiction. ■

Obviously, we can assume that $k \geq d$. (Otherwise, just take $\max\{k, d\}$ as the new k .) For any $z \in M \cap V_\delta$, let

$$\Phi_z(y) = \Phi(y) - \frac{1}{2} \|Q_k y - Q_k z\|_{\text{Im } Q_k}^2, \quad \text{for any } y \in B,$$

where $\|\cdot\|_{\text{Im } Q_k}$ means the norm of $\text{Im } Q_k$ as considered as a subspace of \tilde{H} . Let λ_z denote the supremum of $\Phi_z - h$. (Note that $\lambda_z \leq \lambda$ for all $z \in M \cap V_\delta$.) Then we have the following

PROPOSITION 3.2 *The function $z \mapsto \lambda_z$, $z \in M \cap V_\delta$ is continuous.*

Proof. During and after the proof of this proposition, we will use, with a little abuse of the notation, $\|Q_k V - Q_k z\|_{\text{Im } Q_k}$ to denote the distance between $Q_k V$ and $Q_k z$ under $\|\cdot\|_{\text{Im } Q_k}$.

Take an arbitrary $\varepsilon > 0$ and fix it for a while. Now, note that for any $z \in M \cap V_\delta$, $\lambda_z \geq \lambda - \frac{1}{2} \|Q_k V - Q_k z\|_{\text{Im } Q_k}^2$. So for any $y \in B$ with $\|Q_k V - Q_k y\|_{\text{Im } Q_k} \geq 2\|Q_k V - Q_k z\|_{\text{Im } Q_k} + \varepsilon$, which implies $\|Q_k y - Q_k z\|_{\text{Im } Q_k} \geq \|Q_k V - Q_k z\|_{\text{Im } Q_k} + \varepsilon$, we have

$$\begin{aligned} & \Phi(y) - h(y) - \frac{1}{2} \|Q_k y - Q_k z\|_{\text{Im } Q_k}^2 \\ & \leq \lambda - \frac{1}{2} (\|Q_k V - Q_k z\|_{\text{Im } Q_k} + \varepsilon)^2 \\ & < \lambda - \frac{1}{2} \|Q_k V - Q_k z\|_{\text{Im } Q_k}^2 \leq \lambda_z. \end{aligned}$$

Write $a_z \equiv 2\|Q_k V - Q_k z\|_{\text{Im } Q_k} + \varepsilon$. Then we get

$$\lambda_z = \sup \{ \Phi(y) - h(y) - \frac{1}{2} \|Q_k y - Q_k z\|_{\text{Im } Q_k}^2; \|Q_k V - Q_k y\|_{\text{Im } Q_k} \leq a_z \}.$$

Therefore, for any $z_1, z_2 \in M \cap V_\delta$,

$$\begin{aligned} \lambda_{z_1} - \lambda_{z_2} &= \sup \left\{ \Phi(y) - h(y) - \frac{1}{2} \|Q_k y - Q_k z_1\|_{\text{Im } Q_k}^2; \right. \\ & \quad \left. \|Q_k V - Q_k y\|_{\text{Im } Q_k} \leq a_{z_1} \vee a_{z_2} \right\} \\ & \quad - \sup \left\{ \Phi(y) - h(y) - \frac{1}{2} \|Q_k y - Q_k z_2\|_{\text{Im } Q_k}^2; \right. \\ & \quad \left. \|Q_k V - Q_k y\|_{\text{Im } Q_k} \leq a_{z_1} \vee a_{z_2} \right\} \\ & \leq \sup \left\{ -\frac{1}{2} \|Q_k y - Q_k z_1\|_{\text{Im } Q_k}^2 + \frac{1}{2} \|Q_k y - Q_k z_2\|_{\text{Im } Q_k}^2; \right. \\ & \quad \left. \|Q_k V - Q_k y\|_{\text{Im } Q_k} \leq a_{z_1} \vee a_{z_2} \right\}. \end{aligned}$$

Now, our proposition can be easily seen from the definition of $\|\cdot\|_{\text{Im } Q_k}$. ■

For any $z \in M \cap V_\delta$, let $K_z \equiv \{x; \Phi_z(x) - h(x) = \lambda_z\}$. (Note that if $z \in V$, then $K_z = \{z\}$ by Lemma 3.1.) As in Bolthausen [1], K_z is compact and non-empty. For any $x_z \in K_z$, the probability measure $\nu_{z,0}^{x_z}$ defined by

$$\nu_{z,0}^{x_z}(dy) = e^{D\Phi_z(x_z)(y)} \mu(dy) / M(D\Phi_z(x_z))$$

has mean x_z . Let $\nu_{z,0}^{x_z}$ be the 0-centered $\nu_{z,0}^{x_z}$, and let $\Gamma_{z,0}^{x_z}$ be the inner product on B^* defined by

$$\Gamma_{z,0}^{x_z}(\phi, \psi) = \int_B \phi(y) \psi(y) \nu_{z,0}^{x_z}(dy), \quad \phi, \psi \in B^*.$$

Let $H_z^{x_z} \equiv (\overline{B}^{-1} \nu_z^{x_z})^*$, and $S_z^{x_z} \varphi \equiv \int \varphi(y) y \nu_z^{x_z}(dy)$, as done before. Then as in Kusuoka-Liang [5], we can show that $D^2 \Phi_z(x_z)|_{H_z^{x_z} \times H_z^{x_z}}$ is a Hilbert-Schmidt function for any $z \in M \cap V_\delta$ and any $x_z \in K_z$. Also, we can show the following

PROPOSITION 3.3 Choose $z_n \in M \cap V_{1/n}$ with $z_n \rightarrow z \in V$. Then for any $x_n \in K_{z_n}$, $x_n \rightarrow z$.

Proof. Choose any subsequence of the natural numbers \mathbf{N} , for the sake of simplicity, we write it as n , too. Since x_n maximizes $\Phi_{z_n} - h$,

$$\Phi(x_n) - h(x_n) - \frac{1}{2} \|Q_k x_n - Q_k z_n\|_{\text{Im} Q_k}^2 \geq \lambda - \frac{1}{2} \|Q_k z - Q_k z_n\|_{\text{Im} Q_k}^2.$$

So $\|Q_k x_n - Q_k z_n\|_{\tilde{H}} = \|Q_k x_n - Q_k z_n\|_{\text{Im} Q_k} \leq \|Q_k z - Q_k z_n\|_{\text{Im} Q_k} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $Q_k x_n \rightarrow Q_k z$ in B as $n \rightarrow \infty$, and $\liminf_{n \rightarrow \infty} (\Phi(x_n) - h(x_n)) \geq \lambda$. By doing in the same way as in the proof of Lemma 2.2, there exist a subsequence x_{n_j} and a $x \in V$ such that $x_{n_j} \rightarrow x$. $Q_k x_{n_j} \rightarrow Q_k x$, too, hence $Q_k x = Q_k z$, this and Lemma 3.1 imply that $x = z$, i.e., $x_{n_j} \rightarrow z$ as $j \rightarrow \infty$. This is true for any subsequence of \mathbf{N} .

This finishes the proof of our proposition. \blacksquare

PROPOSITION 3.4 All of the eigenvalues of $D^2 \Phi_z(x_z)|_{H_z^{x_z} \times H_z^{x_z}}$ are smaller than 1, if $\delta > 0$ is small enough.

Proof. From the continuity showed in Proposition 3.3, we only need to show that for any $z \in V$ and any $\varphi \in B^*$ with $\varphi \neq 0$,

$$D^2 \Phi(x_z)(S_z^{x_z} \varphi, S_z^{x_z} \varphi) - D^2 \left(\frac{1}{2} \|Q_k(\cdot) - Q_k z\|_{\text{Im} Q_k}^2 \right) (x_z)(S_z^{x_z} \varphi, S_z^{x_z} \varphi) < \varphi(S_z^{x_z} \varphi). \quad (3.1)$$

But here, from the definition of $\|\cdot\|_{\text{Im} Q_k}$,

$$\|Q_k y - Q_k z\|_{\text{Im} Q_k}^2 = \sum_{i=1}^k (e_i(y) - e_i(z))^2,$$

so

$$D \left(\|Q_k(\cdot) - Q_k z\|_{\text{Im} Q_k}^2 \right) (y)(u) = 2 \sum_{i=1}^k (e_i(y) - e_i(z)) e_i(u)$$

for any y and any u in B . For any $z \in V$, since $x_z = z$, the above implies that $D \Phi_z(x_z) = D \Phi(x_z) = D \Phi(z)$, so from the definition of $\nu_z^{x_z}$, $\nu_{z,0}^{x_z}$, $\Gamma_z^{x_z}$, $H_z^{x_z}$, and $S_z^{x_z}$

at the beginning of this section, we see that these quantities coincide with ν_z , $\nu_{z,0}$, Γ_z , H_z , and S_z , the ones defined in section 1, respectively. Moreover,

$$D^2 \left(\frac{1}{2} \|Q_k(\cdot) - Q_k z\|_{\text{Im} Q_k}^2 \right) (z)(u, u) = \sum_{i=1}^k e_i(u)^2 = \|Q_k u\|_{\text{Im} Q_k}^2 (\geq 0).$$

So (3.1) is equivalent to

$$D^2 \Phi(z)(S_z \varphi, S_z \varphi) - \|Q_k S_z \varphi\|_{\text{Im} Q_k}^2 < \varphi(S_z \varphi). \quad (3.2)$$

The inequality above is obviously if $\varphi \notin A_z$, from the definition of A_z . For $\varphi \in A_z$, by Lemma 2.7, $Q_k S_z \varphi \neq 0$, hence $Q_k S_z \varphi \neq 0$, too, so

$$\begin{aligned} & D^2 \Phi(z)(S_z \varphi, S_z \varphi) - \|Q_k S_z \varphi\|_{\text{Im} Q_k}^2 \\ &= \varphi(S_z \varphi) - \|Q_k S_z \varphi\|_{\text{Im} Q_k}^2 \\ &< \varphi(S_z \varphi). \end{aligned}$$

That is, (3.2) still holds.

This gives our assertion. \blacksquare

Lemma 3.5 $\delta > 0$ can be chosen small enough, such that for any $z \in M \cap V_\delta$, there is a unique x_z attains the maximum of $\Phi_z - h$. Moreover, the map $z \mapsto x_z, z \in M \cap V_\delta$ is in C^2 .

Proof. First, we show the uniqueness. If not, for any $n \in \mathbf{N}$, there exist $z_n \in M \cap V_{1/n}$ and $x_n^1, x_n^2 \in B$, such that $x_n^1 \neq x_n^2$ and both of them maximize $\Phi_{z_n} - h$. By taking subsequence if necessary, we can assume that there exists a $z \in V$, such that $z_n \rightarrow z$ as $n \rightarrow \infty$. Also, by Proposition 3.3, we can assume that x_n^1 and x_n^2 converge to z , too, by taking subsequence if necessary.

From the definition of x_n^1 and x_n^2 , we have that

$$\begin{aligned} \frac{\int_B y e^{D \Phi_{z_n}(x_n^1)(y)} \mu(dy)}{\int_B e^{D \Phi_{z_n}(x_n^1)(y)} \mu(dy)} &= x_n^1, \\ \frac{\int_B y e^{D \Phi_{z_n}(x_n^2)(y)} \mu(dy)}{\int_B e^{D \Phi_{z_n}(x_n^2)(y)} \mu(dy)} &= x_n^2. \end{aligned}$$

Let $f_n(x) \equiv \frac{\int_B y e^{D \Phi_{z_n}(x)(y)} \mu(dy)}{\int_B e^{D \Phi_{z_n}(x)(y)} \mu(dy)}$, then as before,

$$\begin{aligned} x_n^2 - x_n^1 &= f_n(x_n^2) - f_n(x_n^1) \\ &= D f_n(z)(x_n^2 - x_n^1) + \int_0^1 [D f_n(x_n^1 + t(x_n^2 - x_n^1)) - D f_n(z)] (x_n^2 - x_n^1) dt. \end{aligned}$$

But

$$Df_n(x)(u) = \frac{\int_B D^2\Phi_{z_n}(x)(u, y) e^{D\Phi_{z_n}(x)(y)} \mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)} \mu(dy)} - \frac{\int_B y e^{D\Phi_{z_n}(x)(y)} \mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)} \mu(dy)} \cdot \frac{\int_B D^2\Phi_{z_n}(x)(u, y) e^{D\Phi_{z_n}(x)(y)} \mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)} \mu(dy)}.$$

So by a simple calculation and the fact that both x_n^1 and x_n^2 converge to z as $n \rightarrow \infty$, we get

$$\psi\left(\frac{x_n^2 - x_n^1}{\|x_n^2 - x_n^1\|}\right) - D^2\Phi_{z_n}(z)\left(\frac{x_n^2 - x_n^1}{\|x_n^2 - x_n^1\|}, S_z\psi\right) \rightarrow 0, \quad \text{for any } \psi \in B^*,$$

therefore,

$$\psi\left(\frac{x_n^2 - x_n^1}{\|x_n^2 - x_n^1\|}\right) - D^2\Phi_z(z)\left(\frac{x_n^2 - x_n^1}{\|x_n^2 - x_n^1\|}, S_z\psi\right) \rightarrow 0, \quad \text{for any } \psi \in B^*.$$

As in the proof of Lemma 2.5, we can get from this that $\frac{x_n^2 - x_n^1}{\|x_n^2 - x_n^1\|}$ converges as $n \rightarrow \infty$. Write the limit as x , then $\psi(x) = D^2\Phi_z(z)(x, S_z\psi)$ for any $\psi \in B^*$. This contradicts with Proposition 3.4.

We just showed that for $z \in M \cap V_\delta$ with $\delta > 0$ small enough, x_z is the unique solution of the following equation with respect to x :

$$\frac{\int_B y e^{D\Phi_z(x)(y)} \mu(dy)}{\int_B e^{D\Phi_z(x)(y)} \mu(dy)} - x = 0.$$

Let the left hand side above be denoted by $f(x, z)$. Then f is twice continuously differentiable with respect to x , and by Proposition 3.4, $\frac{\partial f}{\partial z}(x_z, z)(u) \neq 0$ whenever $u \neq 0$. So by implicit function theorem, we get that $z \mapsto x_z$ is twice continuously differentiable.

This finishes the proof of our lemma. \blacksquare

From the uniqueness of x_z from Lemma 3.5, from now on, we will abbreviate $\nu_z^{x_z}, \nu_{z,0}^{x_z}, \Gamma_z^{x_z}$, and $H_z^{x_z}$ as $\nu_z, \nu_{z,0}, \Gamma_z$, and H_z , respectively.

4 Uniform estimate

As in Kusuoka-Liang [5], for any $R > 2$, let $\tilde{\nu}_R$ be the probability measure of \mathbf{R} given by

$$\tilde{\nu}_R(\{R\}) = \frac{3}{4R^2 - 1}, \quad \tilde{\nu}_R(\{\frac{1}{2}\}) = \frac{R-2}{2R-1}, \quad \tilde{\nu}_R(\{-\frac{1}{2}\}) = \frac{R+2}{2R+1}.$$

By a simple calculation, we have

$$E^{\tilde{\nu}_R}[Y] = 0, \quad E^{\tilde{\nu}_R}[Y^2] = 1.$$

Let $\rho_a, a > 0$, be the probability measures given by

$$\rho_a(dR) = C_a \exp\left(-\frac{aR^2}{2}\right) dR, \quad R > 2,$$

where C_a is the normalizing constant, i.e. $C_a = (\int_2^\infty e^{-\frac{aR^2}{2}} dR)^{-1}$. Let $\nu_a, a > 0$ be the probability measure given by

$$\nu_a(dy) = \int \tilde{\nu}_R(dy) \rho_a(dR).$$

Lemma 4.1 For any $a > 0$, there exists a constant D_a , depends only on a , such that for i.i.d. random variables $Y_i, i = 1, 2, \dots$ with law ν_a ,

$$P\left(\left|\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \geq z\right| \leq 2 \exp\left(-\frac{1}{4D_a} z^2\right), \quad \forall z \geq 0. \quad (4.1)$$

PROPOSITION 4.2 There exists a constant $C'_1 > 0$, independent to $z \in M \cap V_\delta$, such that

$$C'_5 \equiv \sup_{z \in M \cap V_\delta} \int_B e^{C'_1 \|x\|^2} \nu_{z,0}(dx) < \infty$$

Note: As mentioned above, the meaning of this proposition is that: there exist a $\delta > 0$ and a $C'_1 > 0$, where C'_1 is independent to δ , such that the expression holds. All of the lemmas that follows hold in the same meaning, and we will not emphasize it any more.

Proof. First, from the definition of $x_z, \|x_z\|, z \in M \cap V_\delta$ is bounded. Write C_6 as its upper bound.

Also, since V is compact and $z \mapsto x_z, z \in M \cap V_\delta$ is continuous at V from Lemma 3.5, we see that if $\delta > 0$ is small enough, $D\Phi_z(x_z), z \in M \cap V_\delta$ is bounded in B^* . So $\int_B e^{2D\Phi_z(x_z)(x)} \mu(dx)$ is bounded above and $\int_B e^{D\Phi_z(x_z)(x)} \mu(dx)$ is bounded from 0 for $z \in M \cap V_\delta$. Therefore,

$$\begin{aligned} & \int_B e^{C'_1 \|x\|^2} \nu_{z,0}(dx) \\ &= \int_B e^{C'_1 \|x-x_z\|^2} e^{D\Phi_z(x_z)(x)} \mu(dx) / M(D\Phi_z(x_z)) \\ &\leq \left(\int_B e^{2C'_1 \|x-x_z\|^2} \mu(dx)\right)^{1/2} \left(\int_B e^{2D\Phi_z(x_z)(x)} \mu(dx)\right)^{1/2} / \int_B e^{D\Phi_z(x_z)(x)} \mu(dx) \\ &\leq \left(\int_B e^{4C'_1 \|x\|^2} \mu(dx)\right)^{1/2} e^{2C'_1 C_6^2} \left(\int_B e^{2D\Phi_z(x_z)(x)} \mu(dx)\right)^{1/2} / \int_B e^{D\Phi_z(x_z)(x)} \mu(dx) \\ &< \infty \end{aligned}$$

for $C'_i \leq \frac{C_i}{k}$ by the assumption (A1). \blacksquare

The following can be gotten from proposition 4.2, by using the same method as in Kusuoka-Liang [5], Lemma 3.2, Lemma 3.3, Lemma 3.5, Lemma 3.6, Lemma 3.7. We omit the proofs here.

Lemma 4.3 Under the assumption (A1) in section 1, for any $c > 0$, there exists a $a_0 > 0$ small enough, such that for any $a < a_0$, the following holds:

$$e^n \left(\int_B \|x\|^{2n} \nu_{z,0}(dx) \right)^{1/2} \leq \int_{\mathbf{R}} y^n \nu_a(dy), \quad \forall n \geq 3, \forall z \in M \cap V_\delta. \quad (4.2)$$

Lemma 4.4 Let $\Psi_z, z \in M \cap V_\delta$ be a family of symmetric bilinear functions that satisfies:

1. $\int_B \Psi_z(x, y) \nu_0(dy) = 0, \quad \forall x \in B, \forall z \in M.$
2. There exists a constant $C_0 > 0$, independent to z , such that

$$|\Psi_z(x, y)| \leq C_0 \|x\| \cdot \|y\|, \quad \forall x, y \in B, \quad \forall z \in M \cap V_\delta,$$

3. $\int_B \Psi_z(x, y)^2 \nu_{z,0}(dx) \nu_{z,0}(dy) = 1.$

Then, there exists an $a_0 > 0$, depends only on C_0 and $\sup_{z \in M \cap V_\delta} \int_B \|y\|^2 \nu_{z,0}(dy)$, satisfying the following:

$$E^{\nu_{z,0}^\infty} \left[\prod_{k=1}^m \Psi_z(X_{i_k}, X_{j_k}) \right] \leq E^{\nu_a^\infty} \left[\prod_{k=1}^m Y_{i_k} Y_{j_k} \right], \quad (4.3)$$

$$\forall m \in \mathbf{N}, 1 \leq i_k < j_k \leq n, k = 1, \dots, m, 0 < a < a_0, \forall z \in M \cap V_\delta,$$

where $\{X_i\}_{i=1}^\infty$ is the sequence of random variables defined in section 1, and $\{Y_i\}_{i=1}^\infty$ is defined by $Y_n(\underline{y}) = y_n, \forall \underline{y} = (y_1, y_2, \dots) \in \mathbf{R}^{\mathbf{N}}$.

Lemma 4.5 Assume the same assumptions and use the same notations as in lemma 4.4. Then for $\forall b < \frac{1}{2}$, there exists $\varepsilon > 0$, such that

$$\sup_{z \in M \cap V_\delta} \sup_{n \in \mathbf{N}} E^{\nu_{z,0}^\infty} \left[\exp(b \cdot \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \Psi_z(X_i, X_j)), \left| \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} \Psi_z(X_i, X_j) \right| < \varepsilon \right] < \infty. \quad (4.4)$$

Lemma 4.6 Assume the same conditions as above. Then, for any $\forall b < \frac{1}{2}$, there exist constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that the following holds:

$$\sup_{z \in M \cap V_\delta} \sup_{n \in \mathbf{N}} E^{\nu_{z,0}^\infty} \left[\exp \left(b \cdot n \Psi_z \left(\frac{S_n}{n}, \frac{S_n}{n} \right) \right), \left\{ \left| \frac{1}{n^2} \sum_{i=1}^n \Psi_z(X_i, X_i) \right| < \varepsilon_1 \right\} \cap \left\{ \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon_2 \right\} \right] < \infty.$$

Lemma 4.7 Assume that $\Psi_z, z \in M \cap V_\delta$ is a family of symmetric, bilinear functions that satisfy the following conditions:

$$1. \int_B \Psi_z(x, y) \nu_{z,0}(dy) = 0, \quad \forall x \in B, \forall z \in M \cap V_\delta,$$

2. There exists a constant $C_0 > 0$, such that

$$|\Psi_z(x, y)| \leq C_0 \|x\| \cdot \|y\|, \quad \forall x, y \in B, \quad \forall z \in M \cap V_\delta,$$

$$3. \int_B \Psi_z(x, y)^2 \nu_{z,0}(dx) \nu_{z,0}(dy) \equiv b_z \leq \bar{b} < \frac{1}{2}.$$

Then there exists a $\varepsilon > 0$, such that

$$\sup_{z \in M \cap V_\delta} \sup_{n \in \mathbf{N}} E^{\nu_{z,0}^\infty} \left[\exp \left(\frac{1}{n} \sum_{i,j=1}^n \Psi_z(X_i, X_j) \right), \left\| \frac{1}{n} \sum_{i=1}^n X_i \right\| < \varepsilon \right] < \infty. \quad (4.5)$$

5 Proof of the Theorem

First, note that by Donsker-Varadhan [3],

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log E^{\nu_{z,0}^\infty} \left[e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \notin V_{\delta/2} \right] < \lambda. \quad (5.1)$$

So we only need to do with the integration on the set $\{\frac{S_n}{n} \in V_{\delta/2}\}$ from now on.

Now, let $M \cap V_\delta$ be the new M . It is obvious that the new M still has the same property as the old one. Equip with M the Riemann metric, written as d_M , and use $v_M(dx)$ to denote the volume element on M .

PROPOSITION 5.1 There exists a continuous $C : V_{\delta/2} \rightarrow \mathbf{R}$, such that

$$C(w)^{-1} n^{\frac{3}{2}} \int_{\{\|z-w\| \leq \delta/2\} \cap M} \exp \left(-\frac{1}{2} n \cdot \|Q_k z - Q_k w\|_{mq_k}^2 \right) v_M(dz) \rightarrow 1 \quad (5.2)$$

uniformly in $w \in V_{\delta/2}$ as $n \rightarrow \infty$.

Proof. From the definition of M , for any $z \in M$, there exists a $\varphi \in U$ such that $z = X(\varphi)$. Let $\varphi_0 \equiv X^{-1}(w)$, then

$$\begin{aligned} & n^{\frac{\delta}{2}} \int_{\{\|z-w\| \leq \delta/2\} \cap M} \exp\left(-\frac{1}{2}n \cdot \|Q_k z - Q_k w\|_{\text{Im}Q_k}^2\right) v_M(dz) \\ &= n^{\frac{\delta}{2}} \int_{\{\|\varphi_0 - \varphi\| \leq \delta/2\} \cap U} \exp\left(-\frac{1}{2}n \cdot \|Q_k X(\varphi) - Q_k X(\varphi_0)\|_{\text{Im}Q_k}^2\right) v_U(d\varphi). \end{aligned} \quad (5.3)$$

But $X(\varphi) = X(\varphi_0) + DX(\varphi_0)(\varphi - \varphi_0) + r(\varphi)$, where $r(\varphi)$ is the 2nd Taylor remainder, and by the continuity, there exists a constant $K > 0$, such that $r(\varphi) \leq K\|\varphi - \varphi_0\|^2$. Now, for any $w \in M \cap V_{\delta/2}$, let

$$B_w(\psi, \psi) = \sum_{i=1}^k \left[c_i \left(DX(X^{-1}(w))(\psi) \right) \right]^2, \quad \psi \in W = \text{Im}P_d,$$

it is bilinear on $W \times W$, and by Lemma 3.1, if $\delta > 0$ is small enough, there exists a constant $C > 0$, such that $B_w(\psi, \psi) \geq C\|\psi\|^2$ for all $\psi \in W$. So by the definition of Q_k , if $\delta' > 0$ is small enough,

$$\begin{aligned} & n^{\frac{\delta}{2}} \int_{\{\|\varphi_0 - \varphi\| \leq \delta'/2\} \cap U} \exp\left(-\frac{1}{2}n \cdot \|Q_k DX(\varphi_0)(\varphi - \varphi_0)\|_{\text{Im}Q_k}^2\right) v_U(d\varphi) \\ &= n^{\frac{\delta}{2}} \int_{\{\|\varphi_0 - \varphi\| \leq \delta'/2\} \cap \text{Im}P_d} \exp\left(-\frac{1}{2}n \cdot B_w(\varphi - \varphi_0, \varphi - \varphi_0)\right) v_{\text{Im}P_d}(dz), \end{aligned}$$

which, by the discussion above, converges to $(2\pi)^{\frac{\delta}{2}}(\det B_w)^{-\frac{1}{2}}$ as $n \rightarrow \infty$. As stated before, the $r(\varphi)$ is a high order of $\|\varphi - \varphi_0\|^2$, so (5.3) converges to the same limit. The uniformness can be gotten in the same way by the continuity. ■

Now, we are ready to proof the following proposition, which certainly gives our main theorem. Write $C(w)$ as C_w from now on.

Lemma 5.2 For any bounded continuous function $f: B \rightarrow \mathbf{R}$,

$$E^{n^{\otimes \infty}} \left[f\left(\frac{S_n}{n}\right) e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2} \right] = e^{n\lambda} n^{\frac{\delta}{2}} \int_M f(x(z)) b(z) e^{-n\alpha(z)} v_M(dz) (1 + o(1))$$

as $n \rightarrow \infty$, where $b(z) = C_x^{-1} \exp(\frac{1}{2} \int_B D^2 \Phi_z(x_z)(y, y) \nu_{z,0}(dy)) \times \det_2(I - D^2 \Phi_z(x_z))^{-1/2}$, $\alpha(z) = \lambda - \lambda_z$, and $x(z) = x_z$, $z \in M$.

Note. From the definitions of $x(z)$, $\alpha(z)$, and $b(z)$, and the discussions before, it is easy that they are continuous, and satisfy conditions (1), (2) of Theorem 1.1.

Proof. By Proposition 5.1,

$$C_w^{-1} n^{\frac{\delta}{2}} \int_M \exp\left(-\frac{1}{2}n \cdot \|Q_k z - Q_k w\|_{\text{Im}Q_k}^2\right) v_M(dz) \rightarrow 1$$

uniformly in $w \in V_{\delta/2}$, so

$$\begin{aligned} & E^{n^{\otimes \infty}} \left[f\left(\frac{S_n}{n}\right) e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2} \right] \\ & \sim E^{n^{\otimes \infty}} \left[f\left(\frac{S_n}{n}\right) e^{n\Phi(\frac{S_n}{n})} C_{\frac{S_n}{n}}^{-1} n^{\frac{\delta}{2}} \int_M \exp\left(-\frac{n}{2} \|Q_k z - Q_k \left(\frac{S_n}{n}\right)\|_{\text{Im}Q_k}^2\right) v_M(dz), \right. \\ & \quad \left. \frac{S_n}{n} \in V_{\delta/2} \right] \\ & = e^{n\lambda} n^{\frac{\delta}{2}} \int_M e^{-n(\lambda - \lambda_z)} \cdot e^{-n\lambda_z} E^{n^{\otimes \infty}} \left[C_{\frac{S_n}{n}}^{-1} f\left(\frac{S_n}{n}\right) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2} \right] v_M(dz), \end{aligned}$$

where λ_z is the one defined in section 3.

Therefore, if we can show that $e^{-n\lambda_z} E^{n^{\otimes \infty}} [C_{\frac{S_n}{n}}^{-1} f(\frac{S_n}{n}) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}]$ is bounded for $z \in M (= M \cap V_{\delta})$, and converges to $b(z)f(x(z))$ for each $z \in M$, it will complete the proof of the lemma.

The convergence to $b(z)f(x(z))$ for each $z \in M$ can be shown by using the same method as in Kusuoka-Liang [5]. In fact, as there, we have

$$\begin{aligned} & e^{-n\lambda_z} E^{n^{\otimes \infty}} [C_{\frac{S_n}{n}}^{-1} f\left(\frac{S_n}{n}\right) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}] \\ &= E^{\nu_{z,0}^{\otimes \infty}} [C_{(\frac{S_n}{n} + x_z)}^{-1} f\left(\frac{S_n}{n} + x_z\right) \exp\left(\frac{n}{2} D^2 \Phi_z(x_z)\left(\frac{S_n}{n}, \frac{S_n}{n}\right) + nR(x_z, \frac{S_n}{n})\right), \\ & \quad \frac{S_n}{n} + x_z \in V_{\delta/2}], \end{aligned}$$

where $R(x_z, \frac{S_n}{n})$ is the 3rd remainder of the Taylor's formula. But $\frac{S_n}{n} \rightarrow 0$ almost surely under $\nu_{z,0}^{\otimes \infty}$. So by Kusuoka-Liang [5], we get the convergence here for each $z \in M$.

For the boundedness, since C_w^{-1} , $w \in V_{\delta/2}$ is bounded from the continuity of $C: V_{\delta/2} \rightarrow \mathbf{R}$, and f is bounded, we only need to show that $e^{-n\lambda_z} E^{n^{\otimes \infty}} [e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}]$ is bounded for $z \in M$.

Here, for every $z \in M \cap V_{\delta}$, let a_l^z and f_l^z , $l \in \mathbf{N}$, be the eigenvalues and the corresponding eigenvectors of $D^2 \Phi_z(x_z)|_{H_z}$, where $|a_l^z|^2$ is decreasing with respect to l for each z . Let

$$\begin{aligned} \Psi_{z,1}^{(N)}(x, y) &= \sum_{l=1}^N a_l^z(f_l, x)_{H_z}(f_l, y)_{H_z}, \\ \Psi_{z,2}^{(N)}(x, y) &= D^2 \Phi_z(x_z)(x, y) - \Psi_{z,1}^{(N)}(x, y). \end{aligned}$$

Since $D^2 \Phi_z(x_z)|_{H_z}$ is a Hilbert-Schmidt function, for any $\eta > 0$, there exists an $n_z \in \mathbf{N}$ large enough, such that

$$\int_B \int_B \Psi_{z,2}^{(n_z)}(x, y)^2 \nu_{z,0}(dx) \nu_{z,0}(dy) = \sum_{l=n_z+1}^{\infty} |a_l^z|^2 < \eta/2.$$

Also, since $D^2\Phi_z(x_z)$ is continuous with respect to z , we have that a_i^z is upper semi-continuous with respect to z , therefore, for each $z_0 \in M \cap V_\delta$, we can find a neighborhood U_{z_0} of z_0 in $M \cap V_\delta$, such that for every $z \in U_{z_0}$, we have

$$\sum_{l=n_{z_0}+1}^{\infty} |a_l^z|^2 < \eta.$$

Therefore, for any $\eta > 0$, we can find a N (independent to $z \in M \cap V_\delta$), such that

$$\int_B \int_B \Psi_{z,2}^{(N)}(x,y)^2 \nu_{z,0}(dx) \nu_{z,0}(dy) = \sum_{l=N+1}^{\infty} |a_l^z|^2 < \eta, \quad \text{for all } z \in M \cap V_\delta.$$

Therefore, the boundness can be gotten from Lemma 4.7 and the Lemma 2.1 in Kusuoka-Tamura [6], since the boundness of $D^2\Phi_z, z \in M \cap V_\delta$ in $B^* \times B^*$ is easy from the fact that M is a manifold embedded in B .

This gives our assertion. ■

6 Example

In this section, we will give an example, in which our conditions are satisfied, but the central limit theorem is not.

Example. Let B be the space $\mathcal{M}(\mathbf{T})$ of all signed measures on the torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, which is equal to the dual space of $C(\mathbf{T})$, with the norm induced by it. (B is not separable now, but our argument still works.) Let

$$U(z) = 2 \sum_{k=1}^{\infty} \left(\frac{\cos((4k+1)z)}{(4k+1)\log(4k+1)} - \frac{\cos((4k+3)z)}{(4k+3)\log(4k+3)} \right).$$

$U(z)$ is well-defined, i.e., the series above converges for any $z \in [0, 2\pi]$, and $U(z)$ is continuous with respect to z . Actually, $F(z) = \sum_{n=4}^{\infty} \frac{\sin n z}{n \log n}$ is well-defined and absolutely continuous with respect to z by Edwards [4], and $U(z) = U(x + \frac{z}{2}) + U(-x + \frac{z}{2})$. Let $V(x, y) = CU(x - y)$, where the constant C is chosen so that $\int_0^{2\pi} \int_0^{2\pi} V(x, y)^2 dx dy \leq \pi^2$. V is symmetric and continuous. Let

$$\Phi(R) = \int_0^{2\pi} \int_0^{2\pi} V(x, y) R(dx) R(dy)$$

for $R \in B$. Let $\mu(dx) = \frac{1}{2\pi} dx$, and consider

$$Z_n = E^{w \otimes \infty} \left[\exp(n \Phi(n^{-1} \sum_{i=1}^n \delta_{X_i})) \right].$$

The entropy function now is

$$h(\nu) = \int_{\mathbf{T}} \left(\log \frac{2\pi d\nu(x)}{dx} \right) 2\pi \nu(dx)$$

if $\nu(dx) \ll dx$ and $\log \frac{d\nu(x)}{dx}$ is integrable, and $h(\nu) = 0$ otherwise. So by the conditions above, $\nu_0(dx) = \frac{1}{2\pi} dx$ maximize $\Phi - h$. Therefore, the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are constant times $\frac{1}{(4k+1)\log(4k+1)} - \frac{1}{(4k+3)\log(4k+3)}$, $k = 1, 2, \dots$. So, although $D^2\Phi(\nu_0)$ is a Hilbert-Schmidt function, it is not a nuclear function, hence the central limit theorem does not hold.

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Chapter 3

Laplace Approximations for Diffusion Processes on Torus: nondegenerate case.

Abstract

Let $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, and consider the family of probability measures $\{P_x\}_{x \in \mathbf{T}^d}$ on $C([0, \infty); \mathbf{T}^d)$ given by the infinitesimal generator $L_0 \equiv \frac{1}{2} \Delta + b \cdot \nabla$, where $b: \mathbf{T}^d \rightarrow \mathbf{R}^d$ is a continuous function. Let Φ be a mapping $\mathcal{M}(\mathbf{T}^d) \rightarrow \mathbf{R}$. Under a nuclearity assumption on the second Fréchet differential of Φ , an asymptotic evaluation of $Z_T^{-y} \equiv E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta X_t dt \right) \right) \middle| X_T = y \right]$, up to a factor $(1 + o(1))$, has been gotten in Bolthausen-Deuschel-Tamura [2]. In this paper, we show that the same asymptotic evaluation holds without the nuclearity assumption.

1 Introduction

We consider the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, which is a compact manifold. The tangent space $T(\mathbf{T}^d)$ can be identified with \mathbf{R}^d . Let $\mathcal{B}(\mathbf{T}^d)$ be the set of all Borel sets in \mathbf{T}^d .

Let $\mathcal{M}(\mathbf{T}^d)$ be the dual space of $C(\mathbf{T}^d)$. $\mathcal{M}(\mathbf{T}^d)$ is the set of all signed measures on \mathbf{T}^d with finite total variation, and denote the norm derived by it, the total variation, by $\|\cdot\|$. We also think of the weak*-topology in $\mathcal{M}(\mathbf{T}^d)$. Let $\wp(\mathbf{T}^d)$ and $\mathcal{M}_0(\mathbf{T}^d)$ be the set of all probability measures on \mathbf{T}^d and the set of all signed measures on \mathbf{T}^d with total measure 0, respectively. Let $\text{dist}(\cdot, \cdot)$ denote the Prohorov metric on $\wp(\mathbf{T})$. Note that the topology induced by the Prohorov metric and the weak*-topology coincide.

The path space $\Omega = C([0, \infty), \mathbf{T}^d)$ is the set of continuous functions $\omega: [0, \infty) \rightarrow \mathbf{T}^d$. Let $X_t(\omega) = \omega(t)$, $t \geq 0$, let $\mathcal{F}_t = \sigma\{\omega(s); s \leq t\}$, and let $\mathcal{F} = \vee_t \mathcal{F}_t$.

Let $L_0 = \frac{1}{2} \Delta + b_0 \cdot \nabla$, where $b_0: \mathbf{T}^d \rightarrow \mathbf{R}^d$ is a C^∞ function. Let $\{P_x\}_{x \in \mathbf{T}^d}$ be the family of probability measures on Ω of the martingale problem L_0 , i.e., for any $f \in C^\infty(\mathbf{T}^d; \mathbf{R})$,

- (1) $f(\omega_t) - f(\omega_0) - \int_0^t L_0 f(\omega_s) ds$ is a $(\Omega, \{\mathcal{F}_t\}, P_x)$ martingale,
- (2) $P_x(\omega_0 = x) = 1$.

Denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$ by $\{P_t\}_{t \geq 0}$. $\{P_x\}$ has a unique invariant probability measure μ , which is absolutely continuous with respect to the Riemann volume on \mathbf{T}^d , and $\frac{d\mu}{dx}$ is a strictly positive smooth function. For any $T > 0$, the distribution law of $\{X_{T-t}(\omega)\}_{0 \leq t \leq T}$ under $P_\mu(d\omega)$ is also a diffusion process. The infinitesimal generator of it is the adjoint operator of L_0 in $L^2(d\mu)$, and can be written as $L_0^* \mu = \frac{1}{2} \Delta + b_0^* \cdot \nabla$ for some $b_0^* \in C^\infty(\mathbf{T}^d; \mathbf{R}^d)$. Actually, $b_0^* = \nabla(\log \frac{d\mu}{dx}) - b_0$.

Also, for each $t > 0$, there exist transition probability densities $(p_t(x, y))_{x, y \in \mathbf{T}^d}$ of P_t with respect to μ , which satisfy $p_t \in C^\infty(\mathbf{T}^d \times \mathbf{T}^d)$ and p_t is strictly positive.

Let $\Phi: \mathcal{M}(\mathbf{T}^d) \rightarrow \mathbf{R}$ be a bounded and three times continuously Fréchet differentiable function satisfying the following:

A 1 There exist functions $\Phi^{(1)} \in C(\wp(\mathbf{T}^d) \times \mathbf{T}^d, \mathbf{R})$, $\Phi^{(2)} \in C(\wp(\mathbf{T}^d) \times \mathbf{T}^d \times \mathbf{T}^d, \mathbf{R})$, and $\Phi^{(3)} \in C(\wp(\mathbf{T}^d) \times (\mathbf{T}^d)^3, \mathbf{R})$, such that for any $\nu \in \wp(\mathbf{T}^d)$ and any $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{T}^d)$,

$$\begin{aligned} D\Phi(\nu)(R_1) &= \int_{\mathbf{T}^d} \Phi^{(1)}(\nu, x) R_1(dx), \\ D^2\Phi(\nu)(R_1, R_2) &= \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy), \\ D^3\Phi(\nu)(R_1, R_2, R_3) &= \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz). \end{aligned}$$

Then by Donsker-Varadhan [4], we have (c.f. Lemma 4.4)

$$\frac{1}{T} \log E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta X_t dt \right) \right) \middle| X_T = y \right] \rightarrow \lambda$$

for every $x, y \in \mathbf{T}^d$, where $\lambda = \sup\{\Phi(\nu) - I(\nu); \nu \in \wp(\mathbf{T}^d)\}$ and I is given by

$$I(\nu) = \sup \left\{ - \int_{\mathbf{T}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty, u \geq 1 \right\}, \quad \nu \in \wp(\mathbf{T}^d).$$

The aim of this paper is to give a more precise evaluation of

$$Z_T^{x,y} \equiv E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) \middle| X_T = y \right]$$

up to order $1 + o(1)$ under some assumptions given below.

Define

$$K = \{\nu \in \varphi(\mathbf{T}^d) : \Phi(\nu) - I(\nu) = \lambda\}.$$

We can easily see that K is not empty and is compact in $\varphi(\mathbf{T}^d)$. In this paper, we assume that

A 2 *There exists only one element in K , say ν_0 , that is, $K = \{\nu_0\}$.*

Now, let us construct a diffusion which has ν_0 as its invariant measure following Bolthausen-Deuschel-Tamura [2] and Bolthausen-Deuschel-Schmock [1]. For any $\varphi \in C(\mathbf{T}^d)$, let

$$P_t^\varphi(x, A) = E^{P_x}[\exp(\int_0^t \varphi(X_s) ds), X_t \in A], \quad A \in \mathcal{B}(\mathbf{T}^d),$$

and

$$\Lambda(\varphi) = \sup\{\int_{\mathbf{T}^d} \varphi d\nu - I(\nu), \nu \in \varphi(\mathbf{T}^d)\}.$$

Then P_t^φ has strictly positive right- and left-hand principal eigenfunctions h^φ and $l^\varphi \in C(\mathbf{T}^d)$, i.e.,

$$P_t^\varphi h^\varphi = \exp(\Lambda(\varphi)t) h^\varphi, \quad t \geq 0, \\ \int_{\mathbf{T}^d} \mu(dy) l^\varphi(y) P_t^\varphi(y, dz) = \exp(\Lambda(\varphi)t) l^\varphi(z) \mu(dz).$$

They are unique if they are appropriately normalized by

$$\int_{\mathbf{T}^d} (h^\varphi)^2 d\mu = 1, \quad d\pi^\varphi \equiv l^\varphi h^\varphi d\mu \in \varphi(\mathbf{T}^d).$$

PROPOSITION 1.1 *π^φ is the stationary measure of the diffusion process whose transition probability $Q_t^\varphi(x, dy)$ is given by*

$$Q_t^\varphi(x, dy) \equiv e^{-\Lambda(\varphi)t} \frac{1}{h^\varphi(x)} P_t^\varphi(x, dy) h^\varphi(y).$$

Let

$$\begin{aligned} \phi^{\nu_0}(x) &= D\Phi(\nu_0)(\delta_x - \nu_0) + \Phi(\nu_0) \\ &= \Phi^{(1)}(\nu_0, x) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0), \quad x \in \mathbf{T}^d. \end{aligned}$$

Then we have $\lambda = \Lambda(\phi^{\nu_0})$. Denote h^{ν_0} by h , and l^{ν_0} by l .

Let $(Q_x)_{x \in \mathbf{T}^d}$ be the probability measures given by

$$\frac{dQ_x(\omega)}{dP_x(\omega)} \Big|_{\mathcal{F}_t} = e^{-\lambda t} \frac{h(X_t(\omega))}{h(x)} \exp\left(\int_0^t \phi^{\nu_0}(X_s(\omega)) ds\right).$$

$\{Q_x\}$ is a diffusion process. Denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$ by (Q_t) , and the infinitesimal generator of (Q_t) by L . Actually, $h \in C^1(\mathbf{T}^d)$, and $L = L_0 + \frac{\nabla h}{h} \cdot \nabla$. (c.f. Proposition 2.3). As has been shown in Bolthausen-Deuschel-Tamura [2], $\pi^{\nu_0} = \nu_0$. So by proposition 1.1, we have

Lemma 1.2 *(Q_x) has ν_0 as its invariant measure.*

As a result, ν_0 is absolutely continuous with respect to μ , and $\frac{d\nu_0}{d\mu} > 0$ is continuous, also, $\text{supp} \nu_0 = \mathbf{T}^d$.

Now, for any $t > 0$ and any $x \in \mathbf{T}^d$, let $q_t(x, \cdot)$ be the density function of $Q_t(x, \cdot)$ with respect to ν_0 with $q_t \in C^+(\mathbf{T}^d \times \mathbf{T}^d)$. We will write it as $q(t, x, y)$ sometimes, too. By Bolthausen-Deuschel-Tamura [2] and Bolthausen-Deuschel-Schmock [1], $\sup_{x, y \in \mathbf{T}^d} |q_t(x, y) - 1| \rightarrow 0$ exponentially fast as $t \rightarrow \infty$. So we can define

$$g(x, y) = \int_0^\infty (q_t(x, y) - 1) dt. \quad (1.1)$$

Define $G : L^2(d\nu_0) \rightarrow L^2(d\nu_0)$ by

$$Gf(x) = \int_{\mathbf{T}^d} g(x, y) f(y) \nu_0(dy) = \int_0^\infty (Q_t f(x) - \int_{\mathbf{T}^d} f d\nu_0) dt.$$

Let G^* be the adjoint operator of it in $L^2(d\nu_0)$, i.e., $G^* f(x) = \int_{\mathbf{T}^d} g(y, x) f(y) \nu_0(dy)$, and let $\bar{G} = G + G^*$.

In this paper, we will need the following operators: For $f_1, f_2 \in L^2(d\nu_0)$, let $(\bar{G} \otimes \bar{G})(f_1 \otimes f_2)(x, y) = (\bar{G}f_1)(x)(\bar{G}f_2)(y)$, and denote the continuous linear expansion of it on $L^2(d\nu_0) \otimes L^2(d\nu_0)$ as $\bar{G} \otimes \bar{G}$, too. Define $\bar{G}_x \equiv \bar{G} \otimes I$ and $\bar{G}_y \equiv I \otimes \bar{G}$ in the same way, where I means the identity operator on $L^2(d\nu_0)$. (So $\bar{G}_x \bar{G}_y = \bar{G} \otimes \bar{G}$.) $G_x, G_y, \bar{G}_x, \bar{G}_y$ are defined similarly.

Let $\Gamma(f_1, f_2) \equiv \int_{\mathbf{T}^d} f_1 \bar{G} f_2 d\nu_0$, $f_1, f_2 \in C(\mathbf{T}^d)$. Then it is easy to see (c.f. Proposition 2.5 below) that $\Gamma(f, f) = \int_{\mathbf{T}^d} \|\nabla(Gf)(x)\|^2 \nu_0(dx) \geq 0$, so $\Gamma(f, f) = 0$ if and only if $f \equiv \text{constant}$. Let us define an equivalent relation \sim by $f \sim g \Leftrightarrow f - g \equiv \text{constant}$, and let $\tilde{C}(\mathbf{T}^d) \equiv C(\mathbf{T}^d)/\sim$. Then Γ is an inner product on $\tilde{C}(\mathbf{T}^d)$. Let $H \equiv (\tilde{C}(\mathbf{T}^d))^\Gamma$, where $(\tilde{C}(\mathbf{T}^d))^\Gamma$ means the completion of $\tilde{C}(\mathbf{T}^d)$ with respect to Γ . Since $\tilde{C}(\mathbf{T}^d)^*$ is identified with $\mathcal{M}_0(\mathbf{T}^d)$, H can be regarded as a dense subset of $(\nu_0(\mathbf{T}^d))^\Gamma$, (see Proposition 2.6). H is a Hilbert space with norm $\|\bar{G}f d\nu_0\|_H^2 \equiv \int_{\mathbf{T}^d} f \bar{G} f d\nu_0$.

Also, as has been shown in Bolthausen-Deuschel-Tamura [2], for any $f \in C(\mathbf{T}^d)$,

$$(f, \overline{G}f)_{L^2(d\nu_0)} \geq D^2\Phi(\nu_0)(\overline{G}f, d\nu_0, \overline{G}f, d\nu_0),$$

which means that all of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are less than or equal to 1. In addition, we assume the following

A 3 All of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are smaller than 1.

A 4 For any $\delta > 0$, there exist a constant $\varepsilon > 0$ and a symmetric continuous function $K_\delta : \mathbf{T}^d \times \mathbf{T}^d \rightarrow \mathbf{R}$, such that the function \tilde{K}_δ given by $\tilde{K}_\delta(R_1, R_2) \equiv \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} K_\delta(x, y) R_1(dx) R_2(dy)$, $R_1, R_2 \in \mathcal{M}_0(\mathbf{T}^d)$, satisfies

$$\|\tilde{K}_\delta\|_{H \times H} \|H.S.\| \leq \delta,$$

and

$$D^2\Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0) \leq \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} K_\delta(x, y)(\nu - \nu_0)(dx)(\nu - \nu_0)(dy)$$

for any $R \in \mathcal{P}(\mathbf{T}^d)$ with $\text{dist}(R, \nu_0) < \varepsilon$ and any $\nu \in \mathcal{P}(\mathbf{T}^d)$ with $\text{dist}(\nu, \nu_0) < \varepsilon$.

Our main result is the following

THEOREM 1.3 Under the assumptions above, for any $x, y \in \mathbf{T}^d$,

$$\lim_{T \rightarrow \infty} e^{-TX} Z_T^{x,y} = \frac{h(x)}{h(y)} \cdot \exp \left\{ \frac{1}{2} \int_{\mathbf{T}^d} \overline{G}_x \Phi^{(2)}(\nu_0, \cdot, \cdot) \Big|_{(u,u)} \nu_0(du) \right\} \\ \times \det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.$$

Remark 1 The fact that $D^2\Phi(\nu_0)|_{H \times H}$ is a Hilbert-Schmidt type function, which ensures that the factor $\det_2(I_H - D^2\Phi(\nu_0))$ above is well-defined, can be seen from the Proposition 2.8.

2 Preparations

In this section, we will show in the first half an extended Ito's formula for Gf , where f is a continuous function. Also, we will give the proofs of the several facts claimed in section 1.

In general, consider a operator L given by $L \equiv \frac{1}{2} \Delta + b \cdot \nabla$, where $b \in C(\mathbf{T}^d; \mathbf{R}^d)$. For each $x \in \mathbf{T}^d$, let P_x^L denote the probability law of the diffusion process generated

by L starting at x . Write the invariant measure of $\{P_x^L\}$ as μ_L . Let $\{P_t^L\}_{t \geq 0}$ denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$. Also, let G_L be the corresponding Green operator, i.e., $G_L f \equiv \int_0^\infty (P_t^L f - \int_{\mathbf{T}^d} f d\mu_L) dt$, $f \in C(\mathbf{T}^d)$. Let $\|\cdot\|_{op}$ denote the operator norm in $C(\mathbf{T}^d) \rightarrow C(\mathbf{T}^d)$. Then we have the following

PROPOSITION 2.1 P_t^L is a compact operator on $C(\mathbf{T}^d)$ for any $t > 0$.

Proof. Let $L_B \equiv \frac{1}{2} \Delta$ and let P_t^0 be the semigroup of linear operators on $C(\mathbf{T}^d)$ corresponding to it. Then P_t^0 maps $C(\mathbf{T}^d)$ to $C^2(\mathbf{T}^d)$, and $\|\nabla P_t^0\|_{op} \leq \frac{2\sqrt{d}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{t}}$ for any $t > 0$. So P_t^0 is a compact operator for any $t > 0$. Also,

$$P_t^L = P_t^0 + \int_0^t P_s^L b \cdot \nabla P_{t-s}^0 ds,$$

where $b \cdot \nabla P_s^0$ is compact for any $s > 0$. Thus, P_t^L is compact for any $t > 0$. ■

By Proposition 2.1, every number in the spectrum of P_t^L except 0 is an eigenvalue of it. Let $W_p^2(\mathbf{T}^d)$ denote the Sobolev space, i.e., $W_p^2(\mathbf{T}^d) = \{(1 - \Delta)^{-1} f; f \in L^p\}$. Then we have the following

Lemma 2.2 G_L maps $C(\mathbf{T}^d)$ into $W_p^2(\mathbf{T}^d)$ for any $p \in [1, \infty)$, and it is a bounded linear map. Also, for any $f \in C(\mathbf{T}^d)$ with $\int f d\mu_L = 0$, $u \equiv -G_L f$ is a solution of the equation $Lu = f$ in the sense of generalized functions. Also, if $v \in W_p^2(\mathbf{T}^d)$, $\int_{\mathbf{T}^d} v d\mu_L = 0$, and $Lv = f$ in L^p for some $p > 1$, then $u = v$ in $W_p^2(\mathbf{T}^d)$. Moreover, let $\{X_t\}$ be the diffusion process generated by L , and let $B_t = X_t - X_0 - \int_0^t b(X_s) ds$. Then $\{B_t\}_{t \geq 0}$ is a Brownian motion, and

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) dB_s + \int_0^t f(X_s) ds. \quad (2.1)$$

Proof. The fact that $\{B_t\}_{t \geq 0}$ is a Brownian motion is trivial since by the definition of $\{X_t\}$, $g(X_t) - g(X_0) - \int_0^t (Lg)(X_s) ds$ is a \mathcal{F}_t -martingale for any $g \in C^2(\mathbf{T}^d)$.

Since $b \in C(\mathbf{T}^d; \mathbf{R}^d)$ and $f \in C(\mathbf{T}^d)$, we can find $b_n \in C^\infty(\mathbf{T}^d; \mathbf{R}^d)$ and $f_n \in C^\infty(\mathbf{T}^d)$, such that $b_n \rightarrow b$ in $C(\mathbf{T}^d; \mathbf{R}^d)$ and $f_n \rightarrow f$ in $C(\mathbf{T}^d)$ as $n \rightarrow \infty$, and $\int f_n d\mu_L = 0$. Let $L_n \equiv \frac{1}{2} \Delta + b_n \cdot \nabla$, and write the invariant probability measure, the semigroup of linear operators on $C(\mathbf{T}^d)$ and the Green operator corresponding to it as μ_n , P_n^L and G_n , respectively. Also, let $u_n \equiv -G_n f_n$. Then $u_n \in C^\infty(\mathbf{T}^d)$, and $L_n u_n = f_n - \int_{\mathbf{T}^d} f_n d\mu_n$. We show that $u_n \rightarrow u$ in $C(\mathbf{T}^d)$, and $\lim_{n, m \rightarrow \infty} \|u_n - u_m\|_{W_2^2} = 0$.

P_t^L is ergodic since P_t^L has strictly positive density with respect to μ_L . Therefore, by Perron-Frobenius argument, we see that 1 is the only eigenvalue of P_t^L

that has a positive eigenfunction, 1 is a simple eigenvalue, and the absolute value of any other eigenvalue is smaller than 1. Let E_t be the projection to the eigenspace of P_t^L with respect to 1, and let $K_t = P_t^L - E_t$. Then the spectral radius of K_t is smaller than 1. Let $E = E_1$. By Dunford-Schwartz [3, Theorem VII.3.19], $E_t = E$. Therefore, we have that P_t^L can be written as $P_t^L = E + K_t$, where E is a spectral projection, $\{K_t\}$ is a semi-group of linear operators on $C(\mathbf{T}^d)$ with $\|K_t\|_{op} \leq C_1 e^{-\lambda_1 t}$ for some constants $C_1, \lambda_1 > 0$, and $E K_t = K_t E = 0$.

In the same way, P_t^n can be written as $P_t^n = E_n + K_t^n$.

By using Cameron-Martin-Maruyama-Girsanov formula, we get from the definition of P_t^n and P_t^L that $P_t^n \rightarrow P_t^L$ in the operator norm as $n \rightarrow \infty$ for any $t > 0$. So by Dunford-Schwartz [3, Lemma VII.6.5], $\|E_n - E\|_{op} \rightarrow 0$. Therefore, $\|K_t^n - K_t\|_{op} \rightarrow 0$ for any $t > 0$. This implies that $\|K_t^n\|_{op} \leq C_2 e^{-\lambda_2 t}$ for some $C_2, \lambda_2 > 0$ for all $t > 0$. On the other hand, $u_n = -\int_0^\infty K_t^n f_n dt$ and $u = -\int_0^\infty K_t f dt$. This completes the proof that $u_n \rightarrow u$ in $C(\mathbf{T}^d)$ as $n \rightarrow \infty$.

For the second one, we first show that $u_n, n \in \mathbf{N}$, is bounded in W_p^2 . From the definition of u_n , $\Delta u_n = 2(f_n - b_n \cdot \nabla u_n)$. So, from the boundedness of b_n in $C(\mathbf{T}^d)$ for $n \in \mathbf{N}$, there exists a constant $C_3 > 0$, such that

$$\|u_n\|_{W_p^2} \leq C_3 (\|f_n\|_{L_p} + \|u_n\|_{L_p} + \|\nabla u_n\|_{L_p}).$$

By Friedman [5, Theorem 1.8.1], for any $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$, such that

$$\|g\|_{W_p^1} \leq \varepsilon \|g\|_{W_p^2} + C(\varepsilon) \|g\|_{L_p}, \quad \text{for all } g \in W_p^2.$$

So we get

$$(1 - \varepsilon C_3) \|u_n\|_{W_p^2} \leq C_3 (1 + C(\varepsilon)) (\|f_n\|_{L_p} + \|u_n\|_{L_p}), \quad n \geq 1, \quad (2.2)$$

for any $\varepsilon > 0$. Take $\varepsilon > 0$ small enough such that $1 - \varepsilon C_3 > 0$, and we see that $\sup_{n \in \mathbf{N}} \|u_n\|_{W_p^2} < \infty$.

Next, we use the boundedness of u_n in W_p^2 to show that it is a Cauchy sequence in W_p^2 , by using the same method above. Here, for any $n, m \in \mathbf{N}$, from the definition of u_n, u_m and the boundedness mentioned above, we see that there exists a constant $C_4 > 0$, such that

$$\|\Delta u_n - \Delta u_m\|_{L_p} \leq C_4 (\|f_n - f_m\|_{L_p} + \|\nabla u_n - \nabla u_m\|_{L_p} + \|b_n - b_m\|_\infty).$$

So as before, we see that for any $\varepsilon > 0$, there exists a constant $C'_\varepsilon > 0$ such that

$$(1 - \varepsilon C_4) \|u_n - u_m\|_{W_p^2} \leq C'_\varepsilon (\|f_n - f_m\|_{L_p} + \|u_n - u_m\|_{L_p} + \|b_n - b_m\|_\infty).$$

Therefore, $u_n, n \in \mathbf{N}$, is a Cauchy sequence in W_p^2 .

From the conclusion above and the completeness of W_p^2 , we see that $u \in W_p^2$ for any $p > 1$, and $u_n \rightarrow u$ in W_p^2 as $n \rightarrow \infty$.

Since $u_n \in C^\infty(\mathbf{T}^d)$, Ito's formula certainly holds for each u_n , i.e.,

$$u_n(X_t) = u_n(X_0) + \int_0^t \nabla u_n(X_s) dB_s + \int_0^t Lu_n(X_s) ds.$$

But $Lu_n = f_n - f_n d\mu_0 + (b - b_n) \cdot \nabla u_n = f_n - E_n f_n + (b - b_n) \cdot \nabla u_n \rightarrow f$ in $C(\mathbf{T}^d)$ as $n \rightarrow \infty$. Therefore, taking $n \rightarrow \infty$ in the equation above, we have (2.1).

Now, we only need to show the linearity, the boundedness of $G_L : C(\mathbf{T}^d) \rightarrow W_p^2(\mathbf{T}^d)$, and the uniqueness of the solution. The linearity is trivial. Also, from (2.2), there exists a constant $C > 0$ independent to f , such that

$$\|u\|_{W_p^2} \leq C_5 (\|f\|_{L_p} + \|u\|_{L_p}). \quad (2.3)$$

So the boundedness of $G_L : C(\mathbf{T}^d) \rightarrow W_p^2(\mathbf{T}^d)$ follows from that of $G_L : C(\mathbf{T}^d) \rightarrow C(\mathbf{T}^d)$.

For the uniqueness of the solution of the equation $Lu = f$ in $W_p^2(\mathbf{T}^d)$, let $p > 1$ be large enough, and let $v \in W_p^2(\mathbf{T}^d)$ satisfies $Lv = f$, we show that $v = u (= -G_L f)$ in $W_p^2(\mathbf{T}^d)$. Since $v \in W_p^2(\mathbf{T}^d)$, there exist $v_n \in C^\infty(\mathbf{T}^d)$ with $\int_{\mathbf{T}^d} v_n d\mu_L = 0$, such that $v_n \rightarrow v$ in $W_p^2(\mathbf{T}^d)$. Let $g_n = Lv_n$, then $v_n = \int_{\mathbf{T}^d} v_n d\mu_L - G_L g_n = -G_L g_n$. Therefore, from the completeness of W_p^2 , we only need to show that $G_L g_n \rightarrow G_L f$ in L^p . But $g_n \rightarrow f$ in L^p from the definition, so this is easy to see from the definition of G and the fact that $\sup_{x,y \in \mathbf{T}^d} |\frac{P_t^L(x,y)}{\mu_L(dy)}| \rightarrow 0$ exponentially as $t \rightarrow \infty$. ■

Now, let us come back to our situation described in section 1, i.e., let L be the infinitesimal generator corresponding to $\{Q_x\}$. Let L^{*v_0} denote the adjoint operator of L in $L^2(d\nu_0)$. L^{*v_0} is the infinitesimal generator of the diffusion process $\{X_{T-t}(\omega)\}_{0 \leq t \leq T}$ under $Q_{v_0}(d\omega)$ for any $T > 0$. Note that the G^* defined in section 1 is nothing but the Green operator with respect to L^{*v_0} . We have the following

PROPOSITION 2.3 $h \in C^1(\mathbf{T}^d)$, and $L = L_0 + \frac{\nabla h}{h} \cdot \nabla$. Also, $\ell \in C^1(\mathbf{T}^d)$, and $L^{*v_0} = L_0^{*v_0} + \frac{\nabla \ell}{\ell} \cdot \nabla$.

Proof. As the proof is the same, we only give the proof of the first assertion. By the definition of h , $h = h^{\phi^{v_0}}$, (and $\Lambda(\phi^{v_0}) = \lambda$), for any $x \in \mathbf{T}^d$,

$$E^{P_x} \left[\exp \left(\int_0^t \phi^{v_0}(X_s) ds \right) h(X_t) \right] = e^{\lambda t} h(x).$$

So we have $\lim_{t \rightarrow 0} \frac{1}{t}(P_t h - h) = \lambda h - \phi^{v_0}(x)h$ in $C(\mathbf{T}^d)$. Acting G_0 on the both side, we have $G_0(P_t h - h) = t \int h d\mu - \int_0^t P_s h ds$, and from the continuity of G_0 we get that

$$h - \int_{\mathbf{T}^d} h d\mu = G_0(\phi^{v_0} h - \lambda h).$$

Therefore, by Lemma 2.2 applied to L_0 , $h \in W_p^2$ for any $p > 1$, which implies $h \in C^1(\mathbf{T}^d)$, and

$$h(X_t) = h(X_0) + \int_0^t \nabla h(X_s) dB_s + \int_0^t (\lambda h - \phi^{v_0} h)(X_s) ds.$$

Therefore, by Ito's formula, we have

$$\begin{aligned} \log h(X_t) &= \log h(X_0) + \int_0^t \frac{\nabla h}{h}(X_s) dB_s \\ &\quad + \lambda t - \int_0^t \phi^{v_0}(X_s) ds - \frac{1}{2} \int_0^t \left| \frac{\nabla h}{h}(X_s) \right|^2 ds, \end{aligned}$$

which implies that

$$\begin{aligned} e^{-\lambda t} \frac{h(X_t)}{h(X_0)} \exp\left(\int_0^t \phi^{v_0}(X_s) ds\right) \\ = \exp\left(\int_0^t \frac{\nabla h}{h}(X_s) dB_s - \frac{1}{2} \int_0^t \left| \frac{\nabla h}{h}(X_s) \right|^2 ds\right). \end{aligned}$$

The left hand side above is nothing but $\frac{dQ_{X_0}}{dP_{X_0}}(\omega) \Big|_{\mathcal{F}_t}$. This gives our assertion. ■

From Lemma 2.2 and Proposition 2.3, we have the following

COROLLARY 2.4 G maps $C(\mathbf{T}^d)$ into $W_p^2(\mathbf{T}^d)$ for any $p > 1$, and it is a bounded linear map. Also, for any $f \in C(\mathbf{T}^d)$, $u \equiv -Gf$ is the unique solution of the equation $Lu = f$ in the sense of generalized functions. Moreover, let $\{X_t\}$ be the diffusion process generated by L , and let $B_t \equiv X_t - X_0 - \int_0^t (b_0 + \frac{\nabla h}{h})(X_s) ds$, $t \geq 0$, then $\{B_t\}_{t \geq 0}$ is a Brownian motion, and

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) dB_s + \int_0^t f(X_s) ds, \quad \text{a.s.} \quad (2.4)$$

PROPOSITION 2.5 For any $f \in C(\mathbf{T}^d)$,

$$\Gamma(f, f) = \int_{\mathbf{T}^d} \|\nabla(Gf)\|^2 d\nu_0 = \int_{\mathbf{T}^d} \|\nabla(G^* f)\|^2 d\nu_0.$$

Proof. We only give the proof of the first equality. The second is the same.

First, since ν_0 is $\{Q_t\}$ invariant, and L is the infinitesimal generator of it, we have that $\int_{\mathbf{T}^d} Lg d\nu_0 = 0$ for any $g \in C^2(\mathbf{T}^d)$. Also, by Proposition 2.3, for any $g \in C^2(\mathbf{T}^d)$,

$$gLg = \frac{1}{2}L(g^2) - \frac{1}{2}\|\nabla g\|^2,$$

so

$$-2 \int_{\mathbf{T}^d} Lg \cdot g d\nu_0 = \int_{\mathbf{T}^d} \|\nabla g\|^2 d\nu_0$$

for any $g \in C^2(\mathbf{T}^d)$. So the same is true for any $g \in \cap_{p>1} W_p^2(\mathbf{T}^d)$ (actually, with some $p > 1$ large enough, for any $g \in W_p^2(\mathbf{T}^d)$).

Now, for any $f \in C(\mathbf{T}^d)$, let $g \equiv Gf$. Then by Corollary 2.4, $g \in W_p^2(\mathbf{T}^d)$ for any $p > 1$. Also, $f = -Lg + a$ as generalized functions, where $a = \int_{\mathbf{T}^d} f d\nu_0$, and $\int_{\mathbf{T}^d} g d\nu_0 = 0$. Therefore,

$$\begin{aligned} \int_{\mathbf{T}^d} f \bar{G}f d\nu_0 &= 2 \int_{\mathbf{T}^d} f Gf d\nu_0 \\ &= -2 \int_{\mathbf{T}^d} Lg \cdot g d\nu_0 + 2a \int_{\mathbf{T}^d} g d\nu_0 \\ &= \int_{\mathbf{T}^d} \|\nabla g\|^2 d\nu_0 = \int_{\mathbf{T}^d} \|\nabla Gf\|^2 d\nu_0. \end{aligned}$$

PROPOSITION 2.6 H can be regarded as a subset of $\mathcal{M}_0(\mathbf{T}^d)$, which is dense in $\mathcal{M}_0(\mathbf{T}^d)$ with respect to the weak*-topology.

Proof. The fact that $H \subset \mathcal{M}_0(\mathbf{T}^d)$ is trivial since $H = (\bar{C}(\mathbf{T}^d))^*{}^\perp$ and $\mathcal{M}_0(\mathbf{T}^d) = \bar{C}(\mathbf{T}^d)^*$. So to finish the proof, we only need to show that $\mathcal{M}_0(\mathbf{T}^d) \subset \bar{H}^{weak*}$. If not, there will exists a $\nu \in \mathcal{M}_0(\mathbf{T}^d)$ satisfying $\nu \notin \bar{H}^{weak*}$. For the sake of simplicity, we denote the equivalent class which contains f by f , too. So there exists a function $f \in C(\mathbf{T}^d)$, such that $(f, \nu) = 1$, and $(f, h) = 0$ for any $h \in H$. So $f = 0 \in H^*$, which means $\int_{\mathbf{T}^d} f \bar{G}f d\nu_0 = 0$. Therefore $f \equiv \text{constant}$, and so $\int_{\mathbf{T}^d} f d\nu = 0$, which makes a contradiction. ■

PROPOSITION 2.7 For any symmetric continuous function $V : \mathbf{T}^d \times \mathbf{T}^d \rightarrow \mathbf{R}$, let $U_1(x, y) \equiv -G_x V(x, y)$, and let $U \equiv -G_y^* U_1$, then $U \in C^1(\mathbf{T}^d \times \mathbf{T}^d)$, $\nabla_x U(x, y)$ is continuously differentiable with respect to y , and $\nabla_y \nabla_x U(x, y) \in C(\mathbf{T}^d \times \mathbf{T}^d)$. Also,

$$\int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) \bar{G}_x \bar{G}_y V(x, y) \nu_0(dx) \nu_0(dy) = \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \|\nabla_x \nabla_y U(x, y)\|^2 \nu_0(dx) \nu_0(dy). \quad (2.5)$$

Proof. From the compactness of \mathbf{T}^d and the continuity of V , V is uniformly continuous, and the map $\mathbf{T}^d \rightarrow C(\mathbf{T}^d), y \mapsto V(\cdot, y)$, is continuous.

$U_1(x, y) = -G_x V(x, y)$, so by Corollary 2.4, $U_1(\cdot, y) \in C^1(\mathbf{T}^d)$ for any $y \in \mathbf{T}^d$, and $y \mapsto \nabla_x U_1(\cdot, y) \in C(\mathbf{T}^d)$ is continuous.

Now, from the definition of G^* , we see that G^* is continuous in $C(\mathbf{T}^d)$. So $\nabla_x U(x, y) = -\nabla_x(G_y^* U_1(x, y)) = -G_y^*(\nabla_x U_1(x, y))$. Therefore, $\nabla_x U(x, \cdot) \in C^1(\mathbf{T}^d)$ for any $x \in \mathbf{T}^d$, and the function $x \mapsto \nabla_y \nabla_x U(x, \cdot) \in C(\mathbf{T}^d)$ is continuous. i.e., $\nabla_y \nabla_x U(x, y) \in C(\mathbf{T}^d \times \mathbf{T}^d)$.

We show (2.5) now. First, if V can be expressed as $V(x, y) = \sum_{k=1}^n \varphi_k(x) \psi_k(y)$ for some $n \in \mathbf{N}$ and some $\varphi_k, \psi_k \in C(\mathbf{T}^d), k = 1, \dots, n$, then (2.5) is obvious by Proposition 2.5. For general V , by Weierstrass-Stone Theorem, there exist V_n , such that V_n has the expression above, and $V_n \rightarrow V$ in $C(\mathbf{T}^d \times \mathbf{T}^d)$. From the boundedness of $\bar{G}_x \bar{G}_y$ in $C(\mathbf{T}^d \times \mathbf{T}^d)$, the left hand side of (2.5) for V_n converges to that for V . For the right hand side, we have from Sobolev's inequality and (2.3) that for any $f \in C(\mathbf{T}^d)$, $u = -Gf$ is in $C^1(\mathbf{T}^d)$, and for $p > 1$ large enough, we have

$$\|\nabla u\|_\infty \leq C_6 \|\nabla u\|_{W_2^1} \leq C_7 \|u\|_{W_2^1} \leq C_8 (\|f\|_{L^p} + \|u\|_{L^p}) \leq C_9 \|f\|_\infty$$

for some proper constants C_6, C_7, C_8, C_9 . So the right hand converges, too. Therefore, (2.5) is true for general V . ■

PROPOSITION 2.8 Given any continuous symmetric function $V : \mathbf{T}^d \times \mathbf{T}^d \rightarrow \mathbf{R}$, define a bilinear and continuous function $A_V : \mathcal{M}_0(\mathbf{T}^d) \times \mathcal{M}_0(\mathbf{T}^d) \rightarrow \mathbf{R}$ by $A_V(R_1, R_2) \equiv \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) R_1(dx) R_2(dy)$. Then $A_V|_{H \times H}$ is a Hilbert-Schmidt type function.

Proof. Let $\{f_n\}_{n=1}^\infty$ be a complete orthonormal base of H^* with $\{f_n\}_{n=1}^\infty \in \tilde{C}(\mathbf{T}^d)$. Then by Proposition 2.5 and Proposition 2.7,

$$\begin{aligned} \|A_V\|_{H.S.}^2 &= \sum_{n,m=1}^\infty A(\bar{G} f_n d\nu_0, \bar{G} f_m d\nu_0)^2 \\ &= \sum_{m=1}^\infty \sum_{n=1}^\infty \left(\int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) \bar{G} f_n(x) \bar{G} f_m(y) \nu_0(dx) \nu_0(dy) \right)^2 \\ &= \sum_{k=1}^d \int_{\mathbf{T}^d} \sum_{m=1}^\infty \left(\frac{\partial}{\partial x_k} G_x V(x, \cdot), f_m \right)_{H^*}^2 \nu_0(dx) \\ &= \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \|\nabla_x \nabla_y G_x G_y V(x, y)\|^2 \nu_0(dx) \nu_0(dy) \\ &= \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) \bar{G}_x \bar{G}_y V(x, y) \nu_0(dx) \nu_0(dy) \\ &< \infty, \end{aligned}$$

since V and $\bar{G}_x \bar{G}_y V$ are bounded. ■

3 Lemmas

The following lemma is easy to see, from the definition of multiple integral.

Lemma 3.1 Let $\{W_t\}_{t \geq 0}$ be a Brownian motion. Then for any $T > 0$, and any symmetric function $h(\cdot, \cdot) : [0, T] \times [0, T] \rightarrow \mathbf{R}$ that satisfies

$$\int_0^T \int_0^T h(t_1, t_2)^2 dt_1 dt_2 < \frac{1}{4},$$

we have

$$E^W \left[\exp \left(\int_0^T \int_0^T h(t_1, t_2) dW_{t_1} dW_{t_2} \right) \right] \leq \exp \left(\int_0^T \int_0^T h(t_1, t_2)^2 dt_1 dt_2 \right).$$

Proof. Let A be the symmetric operator on $L^2[0, T]$ given by $A : L^2[0, T] \rightarrow L^2[0, T]$,

$$Af(t) = \int_0^T h(t, s) f(s) ds.$$

A is a Hilbert-Schmidt operator. Therefore, it has discrete spectrum (except 0). So all of its spectrums except 0 are its eigenvalues. Write them as $\{\lambda_k\}_{k=1}^\infty$. By the assumption,

$$\sum_{k=1}^\infty \lambda_k^2 = \|A\|_{H.S.}^2 = \int_0^T \int_0^T h(s, t)^2 ds dt < \frac{1}{4},$$

so $|\lambda_k| < 1/2$ for any $k \in \mathbf{N}$. Write the corresponding orthonormal eigenvectors as $e_k, k = 1, 2, \dots$, so $h(s, t) = \sum_{k=1}^\infty \lambda_k e_k(s) e_k(t)$ in $L^2([0, T] \times [0, T])$. $\int_0^T e_k(s) dW_s, k = 1, 2, \dots$, are i. i. d. normal distributed random variables. Note that $\frac{1}{\sqrt{1-2x}} e^{-x} \leq e^{x^2}$ for any $x < 1/2$, so we get

$$\begin{aligned} &E^W \left[\exp \left(\int_0^T \int_0^T h(t_1, t_2) dW_{t_1} dW_{t_2} \right) \right] \\ &= E^W \left[\exp \left(\sum_{k=1}^\infty \lambda_k \left[\left(\int_0^T e_k(s) dW_s \right)^2 - 1 \right] \right) \right] \\ &= \prod_{i=1}^\infty \frac{1}{\sqrt{1-2\lambda_i}} e^{-\lambda_i} \leq \exp \left(\sum_{i=1}^\infty \lambda_i^2 \right) \\ &= \exp \left(\int_0^T \int_0^T h(t_1, t_2)^2 dt_1 dt_2 \right). \end{aligned}$$

Lemma 3.2 For any probability measure ν on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0})$, any continuous ν -local-martingale (M_t) with $M_0 = 0$, any pair of dual numbers p_1, q_1 , i.e., $\frac{1}{p_1} + \frac{1}{q_1} = 1$, any $T > 0$, and any $A \in \mathcal{F}_T$,

$$E^\nu [e^{M_T}, A] \leq E^\nu \left[\exp\left(\frac{p_1 q_1}{2} \langle M \rangle_T\right), A \right]^{1/q_1}.$$

proof. Since (M_t) is a continuous ν -local-martingale, $(p_1 M_t)$ is a continuous ν -local-martingale, too, so $\exp(p_1 M_t - \frac{p_1^2}{2} \langle M \rangle_t)$ is also a continuous ν -local-martingale, and so a ν -super-martingale. Therefore,

$$\begin{aligned} E^\nu [e^{M_T}, A] &\leq E^\nu \left[\exp(p_1 \cdot (M_T - \frac{p_1}{2} \langle M \rangle_T)) \right]^{1/p_1} \\ &\quad \times E^\nu \left[\exp(q_1 \cdot (\frac{p_1}{2} \langle M \rangle_T)) \right]^{1/q_1} \\ &\leq E^\nu \left[\exp(\frac{p_1 q_1}{2} \langle M \rangle_T), A \right]^{1/q_1}. \end{aligned}$$

Now, we are ready to prove the following:

Lemma 3.3 Let $V : \mathbf{T}^d \times \mathbf{T}^d \rightarrow \mathbf{R}$ be a symmetric, continuous function that satisfies the following:

1. $\int_{\mathbf{T}^d} V(x, y) \nu_0(dy) = 0$ for any $x \in \mathbf{T}^d$,
2. $\int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) \overline{G_x G_y} V(x, y) \nu_0(dx) \nu_0(dy) < \frac{1}{128}$.

Then there exists a constant $\varepsilon_0 > 0$, such that for any $x, y \in \mathbf{T}^d$, and any $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \sup_{T > 0} E^{Q_x} \left[\exp\left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt\right), \right. \\ \left. \text{dist}\left(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0\right) < \varepsilon | X_T = y \right] < \infty. \end{aligned}$$

Proof. First, we have that for any $T > 1$,

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T \int_0^T V(X_s, X_t) ds dt - \frac{1}{T} \int_1^{T-1} \int_1^{T-1} V(X_s, X_t) ds dt \right| \\ &\leq \frac{4T-4}{T} \|V\|_\infty \leq 4\|V\|_\infty. \end{aligned}$$

Let $C_{10} = \sup_{x, y \in \mathbf{T}^d} \{q^*(1, x, y), q^*(1, x, y)\} < \infty$, where $q^*(1, x, y) \equiv \frac{Q_1^*(x, dy)}{\nu_0(dy)} \in C(\mathbf{T}^d \times \mathbf{T}^d)$ and $q^*(1, x, y) > 0$. Then for any $A \in \mathcal{F}_T$,

$$\begin{aligned} &E^{Q_x} \left[\exp\left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt\right), A | X_T = y \right] \\ &\leq E^{Q_{\nu_0}} [q(1, x, X_1) q^*(1, y, X_{T-1}) \cdot \exp\left(\frac{1}{T} \int_1^{T-1} \int_1^{T-1} V(X_t, X_s) ds dt + 4\|V\|_\infty\right), A] \\ &\leq C_{10}^2 e^{8\|V\|_\infty} E^{Q_{\nu_0}} \left[\exp\left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt\right), A \right]. \end{aligned}$$

Therefore, it is sufficient to prove that

$$\sup_{T > 0} E^{Q_{\nu_0}} \left[\exp\left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt\right), \text{dist}\left(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0\right) < \varepsilon \right] < \infty.$$

Since ν_0 is the invariant measure of (Q_x) as mentioned before, $(X_{T-t})_{t=0}^T$ under (Q_{ν_0}) is still a diffusion process for any $T > 0$, with the infinitesimal generator $L^{*\nu_0} = L_0^{*\nu_0} + \frac{\nabla_x^2}{T} \cdot \nabla$. Let $U_1(x, y) \equiv -(G_x V)(x, y)$ and $U(x, y) \equiv -(G_y^* U_1)(x, y)$ as in Proposition 2.7. By condition, $\int_{\mathbf{T}^d} V(x, y) \nu_0(dy) = 0$ for any $x \in \mathbf{T}^d$, so

$$L_x L_y^{*\nu_0} U(x, y) = L_y^{*\nu_0} L_x U(x, y) = V(x, y), \quad \text{for any } x, y \in \mathbf{T}^d$$

in the sense of generalized functions. From the condition (2) and Proposition 2.7, we have that $\nabla_x \nabla_y U$ exists, is continuous, and

$$\int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \|\nabla_x \nabla_y U(x, y)\|^2 \nu_0(dx) \nu_0(dy) < \frac{1}{128}.$$

Let $\rho_T \equiv \frac{1}{T} \int_0^T \delta_{X_t} dt$ and $A_\varepsilon \equiv \{\text{dist}(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0) < \varepsilon\}$. Then from the boundedness of $\|\nabla_x \nabla_y U(x, y)\|^2$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$,

$$\int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \|\nabla_x \nabla_y U(x, y)\|^2 \rho_T(dx) \rho_T(dy) < \frac{1}{128} \quad \text{on } A_\varepsilon.$$

From the definition of U_1 and Corollary 2.4,

$$U_1(X_T, X_t) = U_1(X_t, X_t) + \int_t^T \nabla_x U_1(X_s, X_t) dB_s + \int_t^T V(X_s, X_t) ds,$$

where $(B_t)_{t \geq 0}$ is the Brownian motion defined in Corollary 2.4. Therefore,

$$\begin{aligned} &\frac{1}{T} \int_0^T \int_0^T V(X_s, X_t) ds dt = \frac{2}{T} \int_0^T \int_t^T V(X_s, X_t) ds dt \\ &= \frac{2}{T} \left(\int_0^T (U_1(X_T, X_t) - U_1(X_t, X_t)) dt \right) - \frac{2}{T} \int_0^T dt \left(\int_t^T \nabla_x U_1(X_s, X_t) dB_s \right). \end{aligned}$$

Here, $\|U_1\|_\infty < \infty$ from the continuity of U_1 and the compactness of \mathbf{T}^d , and the second term is equal to $-\frac{2}{T} \int_0^T (\int_0^s \nabla_x U_1(X_s, X_t) dt) dB_s$ by stochastic Fubini's theorem (c.f. Ikeda-Watanabe [6, Lemma 3.4.1]), hence a continuous Q_{v_0} -martingale. So by Lemma 3.2 (with $p_1 = 2$ and $\nu = Q_{v_0}$),

$$\begin{aligned} & E^{Q_{v_0}} \left[\exp \left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt \right), A_\varepsilon \right] \\ & \leq \exp(4\|U_1\|_\infty) \cdot E^{Q_{v_0}} \left[\exp \left(-\frac{2}{T} \int_0^T \int_0^s \nabla_x U_1(X_s, X_t) dt \right), A_\varepsilon \right] \\ & \leq \exp(4\|U_1\|_\infty) \cdot E^{Q_{v_0}} \left[\exp \left(2 \int_0^T \int_0^s \left| \frac{2}{T} \int_0^s \nabla_x U_1(X_s, X_t) dt \right|^2 ds \right), A_\varepsilon \right]^{1/2}. \end{aligned}$$

So, the problem now turns to show that

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{8}{T^2} \int_0^T ds \int_0^s \nabla_x U_1(X_s, X_t) dt \right)^2, A_\varepsilon \right] < \infty$$

for some $\varepsilon > 0$. Since $(X_{T-t})_{t=0}^T$ under Q_{v_0} is a diffusion process for any $T > 0$, we have by Lemma 2.2 and the definition of U that $\tilde{B}_t^T \equiv X_{T-t} - \int_0^t (b_0^* + \frac{\nabla_x}{T}(X_{T-s})) ds$, $t \in [0, T]$, is a Brownian motion, and for any $s' \in (0, T)$,

$$\begin{aligned} \nabla_x U(X_{T-s'}, X_0) &= \nabla_x U(X_{T-s'}, X_{T-s'}) + \int_{s'}^T \nabla_y \nabla_x U(X_{T-s'}, X_{T-t'}) d\tilde{B}_{t'}^T \\ &\quad + \int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'}) dt'. \end{aligned}$$

So we have

$$\begin{aligned} & \frac{1}{T^2} \int_0^T ds \int_0^s \nabla_x U_1(X_s, X_t) dt^2 \\ &= \frac{1}{T^2} \int_0^T ds' \int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'}) dt'^2 \\ &\leq \frac{2}{T^2} \int_0^T |\nabla_x U(X_{T-s'}, X_0) - \nabla_x U(X_{T-s'}, X_{T-s'})|^2 ds' \\ &\quad + \frac{2}{T^2} \int_0^T \int_{s'}^T |\nabla_y \nabla_x U(X_{T-s'}, X_{T-t'}) d\tilde{B}_{t'}^T|^2 ds'. \end{aligned}$$

Here the first term is bounded by the compactness of \mathbf{T}^d and the continuity of $\nabla_x U$. So it is sufficient to show that for some $\varepsilon > 0$ small enough,

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{16}{T^2} \int_0^T \int_{s'}^T \nabla_y \nabla_x U(X_{T-s'}, X_{T-t'}) d\tilde{B}_{t'}^T \right)^2, A_\varepsilon \right] < \infty.$$

Let W_t be another d-dimension Brownian motion which is independent to $\{X_t\}_{t \in [0, \infty)}$. Write $g(t, s) \equiv \nabla_y \nabla_x U(X_{T-t}, X_{T-s})$, then by Lemma 3.2,

$$\begin{aligned} & E^{Q_{v_0}} \left[\exp \left(\frac{16}{T^2} \int_0^T \int_{t_1}^T \nabla_y \nabla_x U(X_{T-t_1}, X_{T-s}) d\tilde{B}_{s'}^T dt \right), A_\varepsilon \right] \\ &= E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{4\sqrt{2}}{T} \int_0^T \left(\int_{t_1}^T g(t, s) d\tilde{B}_{s'}^T \right) dW_t \right), A_\varepsilon \right] \right] \\ &= E^W \left[E^{Q_{v_0}} \left[\exp \left(\frac{4\sqrt{2}}{T} \int_0^T \left(\int_0^s g(t, s) dW_t \right) d\tilde{B}_{s'}^T \right), A_\varepsilon \right] \right] \\ &\leq E^W \left[E^{Q_{v_0}} \left[\exp \left(\frac{64}{T^2} \int_0^T \int_0^s g(t, s) dW_t^2 ds \right), A_\varepsilon \right] \right]^{1/2} \\ &= E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64}{T^2} \int_0^T \int_0^s g(t, s) dW_t^2 ds \right), A_\varepsilon \right] \right]^{1/2}. \end{aligned}$$

Here,

$$\begin{aligned} & \frac{1}{T^2} \int_0^T \int_0^s g(t, s) dW_t^2 ds \\ &= \frac{1}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \\ &\quad + \frac{1}{T^2} \int_0^T \left(\int_{t_1}^T |g(t, s)|^2 ds \right) dt. \end{aligned}$$

The second term is bounded from the compactness of \mathbf{T}^d and Proposition 2.7. So we only need to show that

$$\sup_{T>0} E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \right), A_\varepsilon \right] \right] < \infty.$$

On the other hand, as shown before, $\int_{\mathbf{T}^d} \mathbf{T}^d \|\nabla_x \nabla_y U(x, y)\|^2 \rho_T(dy) < \frac{1}{128}$ on A_ε , so

$$\begin{aligned} & \frac{64^2}{T^4} \int_0^T \int_0^T dt_1 dt_2 \left\| \int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right\|^2 \\ &\leq \frac{64^2}{T^4} \int_0^T dt_1 \int_0^T dt_2 \left(\int_{t_1}^T \|g(t_1, s)\|^2 ds \right) \left(\int_{t_2}^T \|g(t_2, s)\|^2 ds \right) \\ &= (64)^2 \left\{ \frac{1}{T^2} \int_0^T dt \left(\int_t^T \|g(t, s)\|^2 ds \right) \right\}^2 \\ &\leq \left\{ \frac{64}{T^2} \int_0^T \int_0^T \|g(t, s)\|^2 dt ds \right\}^2 \end{aligned}$$

$$\begin{aligned}
&= \left\{ 64 \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \|\nabla_x \nabla_y U(x, y)\|^2 \rho_T(dx) \rho_T(dy) \right\}^2 \\
&< 64^2 \cdot \left(\frac{1}{128}\right)^2 = \frac{1}{4} \quad \text{on } A_\varepsilon.
\end{aligned} \tag{3.1}$$

So from Lemma 3.1, we have

$$\begin{aligned}
&E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \right) \right], A_\varepsilon \right] \\
&\leq E^{Q_{v_0}} \left[\exp \left(\frac{64^2}{T^4} \int_0^T \int_0^T dt_1 dt_2 \left| \int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right|^2 \right) \right], A_\varepsilon < e^{\frac{1}{4}}.
\end{aligned}$$

This completes the proof of the lemma. \blacksquare

Lemma 3.4 For any $e \in C(\mathbf{T}^d)$ with $\int_{\mathbf{T}^d} e(y) \nu_0(dy) = 0$ and $\|e\|_{H^*} = 1$, and any $a < 1$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$,

$$\sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\frac{a}{2T} \left(\int_0^T e(X_t) dt \right)^2 \right) \right], A_\varepsilon | X_T = y < \infty,$$

where $A_\varepsilon = \{ \text{dist}(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0) < \varepsilon \}$ as in Lemma 3.3.

Proof. As in the proof of Lemma 3.3, we only need to show the assertion without the condition that $X_0 = x$ and $X_T = y$, i.e., it is sufficient if we prove

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{a}{2T} \left(\int_0^T e(X_t) dt \right)^2 \right) \right], A_\varepsilon < \infty.$$

Also, as there, since $\int_{\mathbf{T}^d} e(x) \nu_0(dx) = 0$, by Corollary 2.4, the function u defined by $u \equiv -Ge$ is in $W_p^2(\mathbf{T}^d)$ for any $p > 1$, hence in $C^1(\mathbf{T}^d)$, and

$$u(X_T) - u(X_0) = \int_0^T \nabla u(X_t) dB_t + \int_0^T e(X_t) dt.$$

So from the boundedness of u , it is sufficient if

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \left(\int_0^T \nabla u(X_t) dB_t \right)^2 \right) \right], A_\varepsilon < \infty$$

for $\varepsilon > 0$ small enough. Choose and fix a constant $\delta \in (0, \frac{1}{a} - 1)$ first. Since

$$\int_{\mathbf{T}^d} \|\nabla u(x)\|^2 \nu_0(dx) = \|e\|_{H^*}^2 = 1,$$

and $\|\nabla u(x)\|^2$ is bounded on \mathbf{T}^d , there exists an $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$, $\int_{\mathbf{T}^d} \|\nabla u(x)\|^2 \rho_T(dx) \leq 1 + \delta$ on A_ε . So, by Ikeda-Watanabe [6, Theorem II.7.2], there exists a standard Brownian motion \tilde{B} , such that

$$\begin{aligned}
\left(\int_0^T \nabla u(X_t) dB_t \right)^2 &= \left(\tilde{B} \left(\left[\int_0^T \nabla u(X_t) dB_t, \int_0^T \nabla u(X_t) dB_t \right]_T \right) \right)^2 \\
&= \tilde{B} \left(\int_0^T \|\nabla u(X_t)\|^2 dt \right) \\
&= \tilde{B} \left(T \cdot \int_{\mathbf{T}^d} \|\nabla u(x)\|^2 \rho_T(dx) \right)^2 \\
&\leq \sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)|^2 \quad \text{on } A_\varepsilon.
\end{aligned}$$

By the reflection principle, for any $T_0 > 0$ and any x ,

$$P \left(\sup_{0 \leq t \leq T_0} |\tilde{B}(t)| \geq x \right) \leq 2P \left(\sup_{0 \leq t \leq T_0} \tilde{B}(t) \geq x \right) = 2P(|\tilde{B}(T_0)| \geq x).$$

Therefore, since $\delta \in (0, \frac{1}{a} - 1)$, we have

$$\begin{aligned}
&\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \left(\int_0^T \nabla u(X_t) dB_t \right)^2 \right) \right], A_\varepsilon \\
&\leq \sup_{T>0} E \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)|^2 \right) \right] \\
&= \int_0^\infty P \left(\sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)| \geq x \right) d(e^{\frac{ax}{2}} + 1) \\
&\leq 2 \sup_{T>0} E \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} |\tilde{B}((1+\delta)T)|^2 \right) \right] - 1 \\
&= \frac{2}{\sqrt{1-a(1+\delta)}} - 1 < \infty.
\end{aligned}$$

This completes the proof of the lemma. \blacksquare

Using the two lemmas above, we get the following:

Lemma 3.5 For any continuous symmetric function $V : \mathbf{T}^d \times \mathbf{T}^d \rightarrow \mathbf{R}$, which satisfies $\int_{\mathbf{T}^d} V(x, y) \nu_0(dy) = 0$ for any $x \in \mathbf{T}^d$, define a symmetric, bilinear, and continuous function $A_V : \mathcal{M}_0(\mathbf{T}^d) \times \mathcal{M}_0(\mathbf{T}^d) \rightarrow \mathbf{R}$ by $A_V(R_1, R_2) = \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} V(x, y) R_1(dx) R_2(dy)$. Suppose that all of the eigenvalues of $A_V|_{H \times H}$ are smaller than 1. Then there exists a constant $\varepsilon > 0$ small enough, such that for any $x, y \in \mathbf{T}^d$,

$$\begin{aligned}
&\sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\frac{1}{2T} \int_0^T \int_0^T V(X_t, X_s) dtds \right) \right] \\
&\text{dist} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0 \right) < \varepsilon | X_T = y < \infty.
\end{aligned}$$

Proof. By Proposition 2.8, $Av|_{H \times H}$ is a Hilbert-Schmidt type function. Combining this with the condition, we see that the maximum of its eigenvalues, say a_0 , is also smaller than 1. Choose and fix a $p > 1$ such that $a_0 p < 1$.

Write the eigenvalues of $Av|_{H \times H}$ as $\{a_n\}_{n \in \mathbb{N}}$ with $|a_1| \geq |a_2| \geq |a_3| \geq \dots$, and the corresponding eigenvectors as $\{\bar{G}e_m d\nu_0\}_{m=1}^\infty$ with $\int_{\mathbb{T}^d} e_m(x) \bar{G}e_n(x) \nu_0(dx) = \delta_{mn}$. Then $Av(\bar{G}e_m d\nu_0, R) = a_m \int_{\mathbb{T}^d} e_m(x) R(dx)$ for any $R \in \mathcal{M}_0(\mathbb{T}^d)$. So for any $m \in \mathbb{N}$ with $a_m \neq 0$, from the continuity of $V(x, y)$, we can assume that $e_m \in \tilde{C}(\mathbb{T}^d)$.

Let q be the dual number of $p > 1$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Since $Av|_{H \times H}$ is a Hilbert-Schmidt function as claimed, there exists a $N \in \mathbb{N}$ large enough such that $\sum_{i=N+1}^\infty q^2 a_i^2 < \frac{1}{128}$. Apply lemma 3.3 to

$$V_1(x, y) := q \left(V(x, y) - \sum_{i=1}^N a_i e_i(x) \cdot e_i(y) \right), \quad x, y \in \mathbb{T}^d,$$

and use Hölder's inequality, so it is sufficient if

$$\sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\sum_{i=1}^N \frac{p}{2T} \int_0^T \int_0^T a_i e_i(X_t) e_i(X_s) ds dt \right), A_\varepsilon | X_T = y \right] < \infty$$

for $\varepsilon > 0$ small enough, where A_ε is as before.

Obviously, we can assume that $a_1, \dots, a_N \geq 0$, as if not, we can just omit the term corresponding to it. As in Kusuoka-Tamura [9], in general, we have that for any $\varepsilon_1 > 0$, there exists an integer $m > 0$ and $\xi_i = (\xi_i^1, \dots, \xi_i^N) \in \mathbb{R}^N$, $i = 1, \dots, m$, such that $\|\xi_i\|_{\mathbb{R}^d} = 1$, $i = 1, \dots, m$, and

$$\bigcup_{i=1}^m \left\{ x \in \mathbb{R}^N : (x, \xi_i) \leq \frac{1}{(1 + \varepsilon_1)^{1/2}} \right\} \subset \{x \in \mathbb{R}^N : \|x\| < 1\},$$

so

$$\|x\|^2 \leq (1 + \varepsilon_1) \max_{i=1, \dots, m} (x, \xi_i)^2, \quad x \in \mathbb{R}^N.$$

Replace ε_1 by $1 - pa_0$ in the above. Let $\tilde{e}_i = \sum_{j=1}^N \xi_i^j e_j$, $i = 1, \dots, m$. Then $(\bar{G}\tilde{e}_i, \tilde{e}_i)_{L^2(d\nu_0)} = 1$, $\int_{\mathbb{T}^d} \tilde{e}_i(x) \nu_0(dx) = 0$, $i = 1, \dots, m$, and

$$\begin{aligned} \sum_{j=1}^m \left(\int_0^T e_j(X_t) dt \right)^2 &\leq (1 + \varepsilon_1) \max_{i=1, \dots, m} \sum_{j=1}^N \left(\int_0^T e_j(X_t) dt \cdot \xi_i^j \right)^2 \\ &= (1 + \varepsilon_1) \max_{i=1, \dots, m} \left(\int_0^T \tilde{e}_i(X_t) dt \right)^2. \end{aligned}$$

Therefore,

$$\sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\sum_{i=1}^N \frac{p}{2T} \int_0^T \int_0^T a_i e_i(X_t) e_i(X_s) ds dt \right), A_\varepsilon | X_T = y \right]$$

$$\leq \sup_{T>0} \sum_{i=1}^m E^{Q_\varepsilon} \left[\exp \left(\frac{1 - \varepsilon_i^2}{2} \cdot \frac{1}{T} \left(\int_0^T \tilde{e}_i(X_t) dt \right)^2 \right), A_\varepsilon | X_T = y \right],$$

which is finite for $\varepsilon > 0$ small enough by Lemma 3.4.

This completes the proof of the lemma. \square

4 Proof of the Theorem

In this section, we will give the proof of the main theorem. Let

$$\begin{aligned} \tilde{\Phi}(\nu) &\equiv \Phi(\nu) - \int_{\mathbb{T}^d} \phi^{(0)}(\nu)(dy), \\ &= \Phi(\nu) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu - \nu_0), \quad \nu \in \mathcal{M}(\mathbb{T}^d). \end{aligned}$$

Also, let $A_\varepsilon = \{dist(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0) < \varepsilon\}$ as before. Since for any $A \in \mathcal{F}_T$,

$$\begin{aligned} &e^{-\lambda T} E^{P_\varepsilon} \left[\exp \left(T\Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A | X_T = y \right) \right] \\ &= \frac{h(x)}{h(y)} E^{Q_\varepsilon} \left[\exp \left(T\tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A | X_T = y \right) \right], \end{aligned}$$

the theorem will be shown if we can show the following two lemmas.

Lemma 4.1

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{Q_\varepsilon} \left[\exp \left(T\tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A_\varepsilon^c | X_T = y \right) \right] < 0$$

for any $\varepsilon > 0$.

Lemma 4.2

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} E^{Q_\varepsilon} \left[\exp \left(T\tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A_\varepsilon | X_T = y \right) \right] \\ &= \exp \left\{ \frac{1}{2} \int_{\mathbb{T}^d} \bar{G}_x \Phi^{(2)}(\nu_0, \cdot, \cdot) \Big|_{(y, y)} \nu_0(du) \right\} \times \det_2(I_H - D^2\Phi(\nu_0))^{-1/2}. \end{aligned}$$

We prove Lemma 4.1 in the first. By Donsker-Varadhan [4], we have the following

PROPOSITION 4.3 (1) For any $x \in \mathbb{T}^d$ and any closed set $C \subset \varphi(\mathbb{T}^d)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left[\frac{1}{t} \int_0^t \delta_{X_s} ds \in C \right] \leq -\inf \{I(\nu); \nu \in C\},$$

(2) for any $x \in \mathbf{T}^d$ and any open set $G \subset \wp(\mathbf{T}^d)$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} ds \in G \right] \geq -\inf\{I(\nu); \nu \in G\}.$$

From this, we get the following

Lemma 4.4 1. For any $x, y \in \mathbf{T}^d$ and any closed set $C \subset \wp(\mathbf{T}^d)$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} ds \in C \mid X_T = y \right] \leq -\inf\{I(\nu); \nu \in C\}.$$

2. For any $x, y \in \mathbf{T}^d$ and any open set $G \subset \wp(\mathbf{T}^d)$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} ds \in G \mid X_T = y \right] \geq -\inf\{I(\nu); \nu \in G\}.$$

Proof. We only give the proof of the first assertion, the second one can be proved in the same way.

First, for any path $\{X_t\}_{t \geq 0}$, $\|\frac{1}{T} \int_0^T \delta_{X_t} dt - \frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt\| \leq \frac{1}{T}$, therefore, for any $\varepsilon > 0$, there exists a $t_\varepsilon > 0$, such that for any $T > t_\varepsilon$ and any path $\{X_t\}_T$, $\text{dist}(\frac{1}{T} \int_0^T \delta_{X_t} dt, \frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt) \leq \varepsilon$. Now, let C_ε be the ε -neighborhood of C in $\wp(\mathbf{T}^d)$, and let C_{10} be the constant defined in the proof of Lemma 3.3, i.e., $q^*(1, x_1, x_2) \leq C_{10}$ for any $x_1, x_2 \in \mathbf{T}^d$, then for any $T > t_\varepsilon$,

$$\begin{aligned} & P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} dt \in C \mid X_T = y \right] \\ & \leq P_x \left[\frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt \in C_\varepsilon \mid X_T = y \right] \\ & = E^{P_x} \left[1_{\left\{ \frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt \in C_\varepsilon \right\}} q^*(1, y, X_{T-1}) \right] \\ & \leq C_{10} P_x \left[\frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt \in C_\varepsilon \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} dt \in C \mid X_T = y \right] \\ & \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log P_x \left[\frac{1}{T-1} \int_0^{T-1} \delta_{X_t} dt \in C_\varepsilon \right] \\ & \leq -\inf\{I(\nu); \nu \in C_\varepsilon\} \end{aligned}$$

for any $\varepsilon > 0$. The right hand side above converges to $-\inf\{I(\nu); \nu \in C\}$ as ε goes to 0. \square

Lemma 4.1 can now be seen by the same method as used for the not pinned one.

For Lemma 4.2, we follow the way as used in Kusuoka-Tamura [9] and Kusuoka-Liang [8].

Lemma 4.5 There exist constants $p > 1$ and $\varepsilon > 0$, such that

$$\sup_{T > 0} E^{Q_x} \left[e^{\varepsilon T \tilde{\Phi}(\frac{1}{T} \int_0^T \delta_{X_t} dt)}, A_\varepsilon \mid X_T = y \right] < \infty.$$

Proof. The proof is similar with the one in Kusuoka-Liang [8]. Let $R(\nu_0, \cdot)$ be the 3rd remainder of the Taylor expansion around ν_0 , i.e., $R(\nu_0, \nu - \nu_0) = \tilde{\Phi}(\nu) - D^2\Phi(\nu_0)(\nu - \nu_0, \nu - \nu_0)$. Then for any $p > 1$ and any $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$, by Hölder's inequality,

$$\begin{aligned} & E^{Q_x} \left[e^{\varepsilon T \tilde{\Phi}(\frac{1}{T} \int_0^T \delta_{X_t} dt)}, A_\varepsilon \mid X_T = y \right] \\ & = E^{Q_x} \left[\exp \left\{ p \cdot \frac{T}{2} D^2\Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) \right. \right. \\ & \quad \left. \left. + p \cdot T R(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0) \right\}, A_\varepsilon \mid X_T = y \right] \\ & \leq E^{Q_x} \left[\exp \left\{ p \cdot \frac{T}{2} \cdot r D^2\Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) \right\}, \right. \\ & \quad \left. A_\varepsilon \mid X_T = y \right]^{1/r} \end{aligned} \quad (4.1)$$

$$\times E^{Q_x} \left[\exp \left\{ p \cdot T \cdot s R(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0) \right\}, A_\varepsilon \mid X_T = y \right]^{1/s}. \quad (4.2)$$

Now, for any function $U(\cdot, \cdot)$, define

$$\bar{U}(x, y) \equiv U(x, y) - \int_{\mathbf{T}^d} U(x, y) \nu_0(dx) - \int_{\mathbf{T}^d} U(x, y) \nu_0(dy) + \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} U(x, y) \nu_0(dx) \nu_0(dy),$$

and

$$\tilde{U}(R_1, R_2) = \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} U(x, y) R_1(dx) R_2(dy),$$

then $\int \bar{U}(x, y) \nu_0(dx) = 0$ for any $x \in \mathbf{T}^d$, and $\tilde{U}(R_1, R_2) = \bar{U}(R_1, R_2)$ for any $R_1, R_2 \in \mathcal{M}_0(\mathbf{T}^d)$.

Since the maximum a_0 of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ is smaller than 1 by the assumption 4, we can find a $p > 1$ such that $a_0 \cdot p < 1$. For this p , there exists a $r > 1$ such that $a_0 \cdot p \cdot r < 1$. So since

$$\begin{aligned} & T \cdot D^2\Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0 \right) \\ & = \frac{1}{T} \int_0^T \int_0^T \Phi^{(2)}(\nu_0, \cdot, \cdot) \Big|_{(X_s, X_t)} dt ds, \end{aligned}$$

we get by Lemma 3.5 that (4.1) is bounded for $T > 0$ if $\varepsilon > 0$ is small enough.

For (4.2), let s be the dual number of $r > 1$, choose a $\delta \in (0, \frac{1}{2pr})$ and fix it. By the assumption 4, for this $\delta > 0$, there exist a constant $\varepsilon' > 0$ and a K_δ , such that $\|\overline{K}_\delta\|_{H \times H} \|H, s\| \leq \delta$, and

$$\begin{aligned} & |TR(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0)| \\ & \leq T \cdot \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} K_\delta(x, y) (\frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0)^{\otimes 2} (dx \otimes dy) \\ & = \frac{1}{T} \cdot \int_0^T \int_0^T \overline{K}_\delta(X_t, X_s) ds dt \quad \text{on } A_{\varepsilon'}. \end{aligned}$$

So by using Lemma 3.5 again, we get that (4.2) is bounded for $T > 0$ if $\varepsilon' > 0$ is small enough.

This completes the proof of the lemma. \blacksquare

Proof of Lemma 4.2. As in Kusuoka-Tamura [9], Q_x has the strong mixing property, so X_T and $\sqrt{T}(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0)$ are asymptotically independent as $T \rightarrow \infty$ under Q_x for any $x \in \mathbf{T}^d$, also,

$$\begin{aligned} & E^{Q_x} \left[\exp \left(\sqrt{-1} \sqrt{T} \int_{\mathbf{T}^d} u(x) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) (dx) \right) \right] \\ & \rightarrow \exp \left(-\frac{1}{2} \int_{\mathbf{T}^d} u(y) \overline{G} u(y) \nu_0(dy) \right), \quad \text{as } T \rightarrow \infty \end{aligned}$$

for any $u \in L^2(\mathbf{T}^d, d\nu_0)$.

Take a separable Hilbert space H_1 such that the set $\{\overline{G} u d\nu_0 \mid \int_{\mathbf{T}^d} u \overline{G} u d\nu_0 < \infty\}$ is a dense linear subspace of H_1 , and the inclusion map is a Hilbert-Schmidt operator. Let W be an H_1 -valued random variable with distribution γ such that

$$E \left[\exp(\sqrt{-1}(u, W)) \right] = \exp \left(-\frac{1}{2} \int_{\mathbf{T}^d} u(y) \overline{G} u(y) \nu_0(dy) \right)$$

for any $u \in H_1$.

So from the central limit theorem for Hilbert space valued random variables, the distribution of $(X_T, \sqrt{T}(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0))$ under Q_x converges weakly to $\nu_0 \otimes \gamma$ as $T \rightarrow \infty$ on $\mathbf{T}^d \times H_1$.

As before, $D^2\Phi(\nu_0)(\cdot, \cdot)|_{H \times H}$ is a Hilbert-Schmidt function. Write the eigenvectors and the corresponding eigenvalues as a_m and $\overline{G} e_m d\nu_0$, $m = 1, 2, \dots$. Then $\sum_{m=1}^N a_m ((e_m, W)^2 - 1)$ converges in $L^2(d\gamma)$. Let $D^2\Phi(\nu_0)(W, W)$ be the $L^2(d\gamma)$ -limit of $\sum_{m=1}^N a_m ((e_m, W)^2 - 1)$.

It is easy that

$$\begin{aligned} & \frac{1}{T} \int_0^T \int_0^T \sum_{m=1}^N a_m e_m(X_s) e_m(X_t) ds dt - \frac{1}{T} \int_0^T \sum_{m=1}^N a_m e_m(X_s) \overline{G} e_m(X_s) ds \\ & \rightarrow \sum_{m=1}^N a_m ((e_m, W)^2 - 1) \end{aligned}$$

under Q_x in distribution for any $N \in \mathbf{N}$ and any $x \in \mathbf{T}^d$. Also,

$$\begin{aligned} & \sup_{T>0} E^{Q_x} \left[\left\{ \left(\frac{1}{T} \int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s) ds dt - \frac{1}{T} \int_0^T \overline{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(X_s, X_s)} ds \right) \right. \right. \\ & \left. \left. - \left(\frac{1}{T} \int_0^T \int_0^T \sum_{m=1}^N a_m e_m(X_s) e_m(X_t) ds dt - \frac{1}{T} \int_0^T \sum_{m=1}^N a_m e_m(X_s) \overline{G} e_m(X_s) ds \right) \right\}^2 \right] \\ & \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$. Therefore,

$$\frac{1}{T} \int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s) ds dt - \frac{1}{T} \int_0^T \overline{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(X_s, X_s)} ds \rightarrow D^2\Phi(\nu_0)(W, W) :$$

in distribution as $T \rightarrow \infty$. Also,

$$\frac{1}{T} \int_0^T \overline{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(X_s, X_s)} ds \rightarrow \int_{\mathbf{T}^d} \overline{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u, u)} \nu_0(du)$$

Q_x -almost surely as $T \rightarrow \infty$, and

$$TR(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt) \rightarrow 0$$

under Q_x in distribution as $T \rightarrow \infty$. Therefore, we have that

$$T \tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \rightarrow D^2\Phi(\nu_0)(W, W) : + \int_{\mathbf{T}^d} \overline{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u, u)} \nu_0(du)$$

in distribution as $T \rightarrow \infty$. This together with Lemma 4.5 gives our assertion. \blacksquare

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Chapter 4

Laplace Approximations for Diffusion Process on Euclidean Space

1 Introduction

We consider the Euclid space \mathbf{R}^d and let $\mathcal{B}(\mathbf{R}^d)$ be its Borel field. $C_b(\mathbf{R}^d)$ denotes the Banach space consisting of bounded continuous mappings $\mathbf{R}^d \rightarrow \mathbf{R}$ with the supremum norm $\|\cdot\|_\infty$. Also, we will use C_b^∞ to denote the set of bounded smooth functions defined in \mathbf{R}^d whose derivatives of any order are bounded, and $C_b^1(\mathbf{R}^d)$ the set of bounded 1-time differentiable functions with bounded derivatives. Let $\mathcal{M}(\mathbf{R}^d)$ be the set of all finite signed measures on E , with the total variation norm $\|\cdot\|$. For any $f \in C_b(\mathbf{R}^d)$ and any $R \in \mathcal{M}(\mathbf{R}^d)$, define $\langle f, R \rangle = \int_E f dR$. We also think of the weak*-topology in $\mathcal{M}(\mathbf{R}^d)$. Let $\wp(\mathbf{R}^d)$ and $\mathcal{M}_0(\mathbf{R}^d)$ be the set of all probability measures on \mathbf{R}^d and the set of all finite signed measures on \mathbf{R}^d with total measure 0, respectively. Let $\text{dist}(\cdot, \cdot)$ denote the Prohorov metric on $\wp(\mathbf{R}^d)$. Note that the topology induced by the Prohorov metric and the weak*-topology coincide. The path space $\Omega = C([0, \infty), \mathbf{R}^d)$ is the set of continuous functions $\omega : [0, \infty) \rightarrow \mathbf{R}^d$. Let $\mathcal{F}_t = \sigma\{\omega(s); s \leq t\}$, and let $\mathcal{F} = \vee_t \mathcal{F}_t$.

Let a_{ij} and b_i , $i, j = 1, \dots, d$, be a set of functions satisfying

A 1 $a_{ij} \in C_b^\infty(\mathbf{R}^d)$, $i, j = 1, \dots, d$, and $(a_{ij}(x))_{i,j=1}^d$ is uniformly positive-definite with respect to $x \in \mathbf{R}^d$. $b_i \in C^\infty(\mathbf{R}^d)$, $i = 1, \dots, d$, and there exists a positive constant $C_1 > 0$, such that the matrix $\nabla b + (\nabla b)^t \leq C_1 \cdot I_d$, where $b = (b_1, \dots, b_d)$, I_d means the identity matrix, and t means the transpose of the matrix. i.e., for any $x = (x_1, \dots, x_d)$, $\xi = (\xi_1, \dots, \xi_d) \in \mathbf{R}^d$, $\sum_{i,j=1}^d (\frac{\partial b_i}{\partial x_j}(x) + \frac{\partial b_j}{\partial x_i}(x)) \xi_i \xi_j \leq C_1 \sum_{i=1}^d \xi_i^2$.

Under the assumption above, there exists a set of probability measures $\{P_x\}$ on

(Ω, \mathcal{F}) with

$$L_0 = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x^i}$$

as its infinitesimal generator, i.e., $\{P_x\}$ is a strong Markovian system satisfying the following: $P_x(\omega_0 = x) = 1$ for any $x \in \mathbf{R}^d$, and for any $f \in C_b^2(\mathbf{R}^d)$,

$$f(\omega_t) - f(\omega_0) - \int_0^t (L_0 f)(\omega_s) ds \text{ is a } (\Omega, \mathcal{F}_t, P_x)\text{-martingale for any } x \in \mathbf{R}^d.$$

One can construct such a Markovian system in the following way. Let $\beta_i = b_i$, and let $(\alpha_{ij})_{i,j=1}^d \in (C_b^\infty(\mathbf{R}^d))^{d \times d}$ be a symmetric matrix satisfying

$$a_{ij}(x) = \sum_{k=1}^d \alpha_{ik}(x) \alpha_{jk}(x).$$

Let $\{B_1^t, \dots, B_d^t\}_{t \geq 0}$ be a standard Brownian motion. Then for any $x \in \mathbf{R}^d$, there exists a solution $\{X(t, x)\}_{t \geq 0}$ to the stochastic differential equation (c.f. Stroock-Varadhan [14])

$$dX^i(t, x) = \sum_{j=1}^d \alpha_{ij}(X(t, x)) dB^j(t) + \beta_i(X(t, x)) dt, \quad i = 1, \dots, d, \quad (1.1)$$

with initial condition $(X^1(0, x), \dots, X^d(0, x)) = x$. P_x is the law of this solution $\{X(t, x)\}_{t \geq 0}$.

Let $\{P_t\}_{t \geq 0}$ denote the corresponding semigroup of bounded linear operators in $C_b(\mathbf{R}^d)$, i.e., $P_t f(x) = E^{P_x}[f(\omega_t)]$.

We assume that

A 2 For any $t > 0$, $P_t : C_b(\mathbf{R}^d) \rightarrow C_b(\mathbf{R}^d)$ is a compact operator.

By Kusuoka-Liang [9], under the assumptions A1 and A2, there exists a unique (P_t) -invariant probability measure $\pi \in \wp(\mathbf{R}^d)$ satisfying $\text{supp}(\pi) = \mathbf{R}^d$ and $\pi(dx) \ll dx$. $(y \mapsto \pi(\frac{dy}{dy})) \in C^\infty(\mathbf{R}^d)$. Also, the density function $p_t(x, y)$ of $P_t(x, dy)$ with respect to $\pi(dy)$ is in $C^\infty(\mathbf{R}^d)$, and $p_t(x, y) > 0$ for any $t > 0$ and any $x, y \in \mathbf{R}^d$.

Let $b_i^* = \sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j} + \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j} (\log \frac{dy}{dx}) - b_i$, $i = 1, \dots, d$, and let $b^* = (b_1^*, \dots, b_d^*)$. We assume the following

A 3 There exists a constant $C_1' > 0$, such that $\nabla b^* + (\nabla b^*)^t \leq C_1' \cdot I_d$.

We can certainly assume without loss of generality that $C_1 = C_1'$.

For any $T > 0$, $\{X_{T-t}(\omega)\}_{t \in [0, T]}$ under $P_x(d\omega)$ is also a diffusion process. The infinitesimal generator of it is

$$L_0^{*x} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b_i^*(x) \frac{\partial}{\partial x^i},$$

the adjoint operator of L_0 in $L^2(d\pi)$.

Moreover, we assume the following

A 4 For any $t > 0$, $P_t^* : C_b(\mathbf{R}^d) \rightarrow C_b(\mathbf{R}^d)$ is a compact operator, too.

We will show the following in section 2.

THEOREM 1.1 Let I be an entropy function given by

$$I(\nu) = \sup \left\{ - \int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty(\mathbf{R}^d), u \geq 1, u \text{ and } L_0 u \text{ are bounded} \right\}, \quad \nu \in \wp(\mathbf{R}^d).$$

Then under Assumptions A1 ~ A4, I is a good rate function, and governs the large deviation principle of the distribution of $\int_0^t \delta_{X_s} ds$ under $P_x(\cdot | X_t = y)$ for any $x, y \in \mathbf{R}^d$, i.e., for any $A \in \mathcal{B}(\mathbf{R}^d)$,

$$\begin{aligned} -\inf_{A^0} I &\leq \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A \mid X_t = y \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left(\frac{1}{t} \int_0^t \delta_{X_s} ds \in A \mid X_t = y \right) \leq -\inf_{\bar{A}} I, \end{aligned}$$

where A^0 and \bar{A} means the interior and the closure of A , respectively.

Let $\Phi : \mathcal{M}(\mathbf{R}^d) \rightarrow \mathbf{R}$ be a bounded function such that $\Phi|_{\wp(\mathbf{R}^d)}$ is continuous with respect to Prohorov Metric.

From Theorem 1.1, we get the following (c.f. section 6).

COROLLARY 1.2

$$\frac{1}{T} \log E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) \mid X_T = y \right] \rightarrow \lambda$$

for every $x, y \in \mathbf{R}^d$, where $\lambda = \sup \{ \Phi(\nu) - I(\nu); \nu \in \wp(\mathbf{R}^d) \}$.

The aim of this paper is to give a more precise evaluation of

$$Z_T^{x,y} \equiv E^{P_x} \left[\exp \left(T \Phi \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right) \right) \mid X_T = y \right]$$

up to order $1 + o(1)$ under some assumptions given below.

Define

$$K = \{\nu \in \wp(\mathbf{R}^d) : \Phi(\nu) - I(\nu) = \lambda\}.$$

K is not empty and is compact in $\wp(\mathbf{R}^d)$. In this paper, we assume the following as in Kusuoka-Liang [8].

A 5 There exists only one element in K , say ν_0 , i.e., $K = \{\nu_0\}$.

A 6 $\Phi : \mathcal{M}(\mathbf{R}^d) \rightarrow \mathbf{R}$ is a three times continuously Fréchet differentiable function, satisfies the following: there exist functions $\Phi^{(1)} \in C(\wp(\mathbf{R}^d) \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(2)} \in C(\wp(\mathbf{R}^d) \times \mathbf{R}^d \times \mathbf{R}^d, \mathbf{R})$, $\Phi^{(3)} \in C(\wp(\mathbf{R}^d) \times (\mathbf{R}^d)^3, \mathbf{R})$, such that for any $\nu \in \wp(\mathbf{R}^d)$ and any $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{R}^d)$,

$$\begin{aligned} D\Phi(\nu)(R_1) &= \int_{\mathbf{R}^d} \Phi^{(1)}(\nu, x) R_1(dx), \\ D^2\Phi(\nu)(R_1, R_2) &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy), \\ D^3\Phi(\nu)(R_1, R_2, R_3) &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz). \end{aligned}$$

Also, we assume that $\Phi^{(1)}(\nu_0, x) \in C_b^1(\mathbf{R}^d; \mathbf{R})$, $\nabla_x \Phi^{(2)}(\nu_0, x, y)$, $\nabla_y \Phi^{(2)}(\nu_0, x, y)$, and $\nabla_x \nabla_y \Phi^{(2)}(\nu_0, x, y)$ are well-defined, and all of them and $\Phi^{(2)}(\nu_0, x, y)$ are in $C_b^1(\mathbf{R}^d \times \mathbf{R}^d)$.

In section 3 and section 4, we will construct, as in Bolthausen-Deuschel-Schmock [1, Lemma 2.7], a family of diffusion laws $\{Q_x\}_{x \in \mathbf{R}^d}$ which has ν_0 as its invariant measure. Actually, let ϕ^{ν_0} be the function given by

$$\begin{aligned} \phi^{\nu_0}(x) &= D\Phi(\nu_0)(\delta_x - \nu_0) + \Phi(\nu_0) \\ &= \Phi^{(1)}(\nu_0, x) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0), \quad x \in \mathbf{R}^d. \end{aligned}$$

Then λ is the principle eigenvalue of the operator $L_0 + \phi^{\nu_0}$, and has a unique $L^2(\pi)$ -normalized eigenfunction h corresponding to it. We will show that $h \in C_b^1(\mathbf{R}^d)$ and $(Q_x)_{x \in E}$ is the diffusion process with the infinitesimal generator $L = L_0 + (a_{ij})_{i,j=1}^d \frac{\nabla}{h} \cdot \nabla$.

As in Bolthausen-Deuschel-Tamura [2], We also assume the following condition, which is not a mathematically precise statement:

A 7 $I - D^2\Phi(\nu_0)$ is strictly positive definite.

To state it in a precise form needs some preparation, and will be done in section 4.

Finally, we assume that

A 8 For any $\delta > 0$, there exist a constant $\varepsilon > 0$ and a symmetric continuous bounded function $K_\delta : \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}$ with $\nabla_x K_\delta, \nabla_y K_\delta, \nabla_x \nabla_y K_\delta$ well-defined and in $C_b(\mathbf{R}^d \times \mathbf{R}^d)$, such that the function \bar{K}_δ given by $\bar{K}_\delta(R_1, R_2) \equiv \int_E \int_E K_\delta(x, y) R_1(dx) R_2(dy)$, $R_1, R_2 \in \mathcal{M}_0(\mathbf{R}^d)$ satisfies

$$\|\bar{K}_\delta\|_{H \times H} \leq \delta, \quad (1.2)$$

and

$$D^3\Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0) \leq \int_E \int_E K_\delta(x, y)(\nu - \nu_0)(dx)(\nu - \nu_0)(dy)$$

for any $R \in \wp(\mathbf{R}^d)$ with $\text{dist}(R, \nu_0) < \delta$ and any $\nu \in \wp(\mathbf{R}^d)$ with $\text{dist}(\nu, \nu_0) < \delta$.

Here, H is the Hilbert space used in the precise stating of the assumption 6, the norm of it is essentially the second Fréchet differential of the entropy function I . As I is not smooth, this description is not mathematically precise. We will give the precise definition of it in section 4. H can be regarded as a dense subset of $\mathcal{M}_0(\mathbf{R}^d)$.

Remark 1 Let G be the Green operator corresponding to L , i.e.,

$$Gf = \int_0^\infty (Q_t f - \int f d\nu_0) dt$$

for any $f \in C_b(\mathbf{R}^d)$. Let G^* be the dual of G in $L^2(d\nu_0)$, and $\bar{G} = G + G^*$. (c.f. section 4.) Then (1.2) is equivalent to

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y) \bar{G}_x \bar{G}_y K_\delta(x, y) \nu_0(dx) \nu_0(dy) \leq \delta^2.$$

THEOREM 1.3 Under Assumptions A1 ~ A8, for any $x, y \in \mathbf{R}^d$,

$$\lim_{T \rightarrow \infty} e^{-T\lambda} Z_T^{x,y} = \frac{h(x)}{h(y)} \cdot \exp \left\{ \frac{1}{2} \int (\bar{G} \otimes I) \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u,u)} \nu_0(du) \right\} \times \det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.$$

Remark 2 $D^2\Phi(\nu_0)|_{H \times H}$ is an Hilbert-Schmidt type function. Also, all of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are smaller than or equal to 1, and are smaller than 1 by the assumption 7, so the \det_2 above is well-defined.

The same kind of problem was studied by Kusuoka-Tamura [13] for reversible case, i.e., when P_t is self-adjoint in $L^2(\pi)$ for any $t > 0$, by Bolthausen-Deuschel-Tamura [2] for the case that the state space is compact under some conditions which imply the central limit condition, and by Kusuoka-Liang [8] for the general compact case, but for the special type of infinitesimal generator $-\frac{1}{2}\Delta + b \cdot \nabla$.

2 Large Deviation Principle

Let $\{P_x\}_{x \in E}$ be a family of probability measures on Ω associated with the Markov process given by an infinitesimal generator L_0 with invariant probability measure π . (We are dealing with the general ones for a while, and the meanings of the notations are different from the ones in section 1). Let $\{P_t\}_{t \geq 0}$ be the corresponding semigroup of bounded linear operators in $C_b(E)$. Let P_t^* be the dual operator of P_t in $L^2(d\pi)$, i.e., P_t^* is the one corresponding to the time inverse Markov process.

In this section, we show the large deviation principle of the pinned process $P_x(\cdot | X_t = y)$ for some $x, y \in \mathbf{R}^d$ under the following conditions.

H 1 $P_t(x, dy)$ has smooth density function $p_t(x, y) \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ with respect to π , and $p_t(x, y) > 0$ for any $x, y \in \mathbf{R}^d$.

H 2 Both P_t and P_t^* are compact operators in $C_b(\mathbf{R}^d)$ for any $t > 0$.

H 3 For any $r > 0$, $\sup_{x \in \mathbf{R}^d, |y| \leq r} p(t, x, y) < \infty$, and $\sup_{|x| \leq r, y \in \mathbf{R}^d} p(t, x, y) < \infty$.

Remark 3 As stated in section 1, the existence of the $\{P_t\}_{t \geq 0}$ -invariant probability measure π can be gotten if we assume that $P_t : C_b(\mathbf{R}^d) \rightarrow C_b(\mathbf{R}^d)$ is a compact operator for any $t > 0$.

For any $\varphi \in C_b(E)$, define the semigroup of transition kernels $\{P_t^\varphi\}_{t \geq 0}$ by

$$P_t^\varphi(x, A) = E^{P_x} \left[\exp \left(\int_0^t \varphi(X_u) du \right) \mathbf{1}_A(X_t) \right], \quad x \in E, A \in \mathcal{E}(\mathbf{R}^d), t \geq 0.$$

The corresponding semigroup of bounded linear operators in $C_b(E)$ will be denoted by $\{P_t^\varphi\}_{t \geq 0}$, too. Let $\|\cdot\|_{op}$ denote the operator norm of the bounded linear operator in $C_b(E)$. The logarithmic spectral radius of P_t^φ given by

$$\Lambda^\varphi = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|P_t^\varphi\|_{op}$$

satisfies $|\Lambda^\varphi| \leq \|\varphi\|_\infty$. We will write it as $\Lambda(\varphi)$ sometimes, too.

Lemma 2.1 For any $\varphi \in C_b(\mathbf{R}^d)$ and any $t > 0$, P_t^φ is a compact operator in $C_b(\mathbf{R}^d)$.

Proof. From the definition of P_t^φ , for any $f \in C_b(E)$, $(P_t^\varphi f - P_t f)(x) = E^{P_x} [f(X_t) (e^{\int_0^t \varphi(X_s) ds} - 1)]$, so $\|P_t^\varphi - P_t\|_{op} \leq e^{t\|\varphi\|_\infty} - 1$, which converges to 0 as $t \rightarrow 0$. Therefore, for any $t > 0$, $\|(P_t^\varphi - P_t)^n\|_{op} \rightarrow 0$ as $n \rightarrow \infty$. Also, from H2, P_s is a compact operator for any $s > 0$, so $P_t^\varphi - (P_t^\varphi - P_t)^n = (P_t^\varphi - P_t)^n + (P_t^\varphi - P_t)^{n-1} - (P_t^\varphi - P_t)^{n-1} + (P_t^\varphi - P_t)^n$ is a compact operator. Therefore, P_t^φ is a compact operator in $C_b(E)$, too. ■

Lemma 2.2 For any $t > 0$, $\sup_{x \in \mathbf{R}^d} P_x(|X_t| \geq n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $\varphi : \mathbf{R}^d \rightarrow \mathbf{R}$ be a smooth function satisfying the following: $\varphi \in C^\infty(\mathbf{R}^d)$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $|x| \leq 1$, and $\varphi(x) = 0$ if $|x| \geq 2$. Let $\varphi_n(x) = \varphi(\frac{x}{n})$, $n \in \mathbf{N}$. $\{\varphi_n\}_{n \in \mathbf{N}}$ is bounded in $C_b(\mathbf{R}^d)$. So since P_t is a compact operator, by taking subsequence if necessary, we can assume that $P_t \varphi_n$ converges in $C_b(\mathbf{R}^d)$ as $n \rightarrow \infty$. $\varphi_n \rightarrow 1$ pointwisely as $n \rightarrow \infty$, so from dominated convergence theorem, we have that $P_t \varphi_n \rightarrow 1$ pointwisely as $n \rightarrow \infty$. Therefore, $P_t \varphi_n \rightarrow 1$ in $C_b(\mathbf{R}^d)$ as $n \rightarrow \infty$. So

$$\sup_{x \in \mathbf{R}^d} P_x(|X_t| \geq 2n) \leq \sup_{x \in \mathbf{R}^d} \int_{\mathbf{R}^d} p_t(x, y) (1 - \varphi_n(y)) \pi(dy) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This completes the proof of our assertion. ■

Lemma 2.3 Under the assumption H2, we have that $\inf_{x \in \mathbf{R}^d, |y| \leq r} p(t, x, y) > 0$, and $\inf_{|x| \leq r, y \in \mathbf{R}^d} p(t, x, y) > 0$.

Proof. We only give the proof of the first assertion, the second one is the same.

First, for any $t > 0$, $p_t(x, y) > 0$ for any $x, y \in \mathbf{R}^d$, so let B_r denote the set $\{x \in \mathbf{R}^d; |x| < r$ for any $r > 0$, then $\inf_{u, v \in B_r} p_t(u, v) > 0$ for any $r \in (0, \infty)$. Also, from the last lemma, we see that there exists a $r_0 > 0$, such that for any $r \geq r_0$, $\inf_{x \in \mathbf{R}^d} P_x(|X_{t/2}| \leq r) \geq \frac{1}{2}$. Therefore,

$$\begin{aligned} \inf_{x \in \mathbf{R}^d, |y| \leq r} p_t(x, y) &\geq \inf_{x \in \mathbf{R}^d, |y| \leq r} \int_{\mathbf{R}^d} p_{t/2}(x, z) p_{t/2}(z, y) \pi(dz) \\ &\geq \inf_{|u|, |v| \leq r} p_{t/2}(u, v) \inf_{x \in \mathbf{R}^d} P_x(|X_{t/2}| \leq r) \\ &\geq \frac{1}{2} \inf_{|u|, |v| \leq r} p_{t/2}(u, v). \end{aligned}$$

This gives our assertion. ■

Lemma 2.4 For any $\varphi \in C_b(\mathbf{R}^d)$ and any $t > 0$, P_t^φ has continuous density function $p_t^\varphi(x, y)$ with respect to π , and $|\log p_t^\varphi(x, y)| \leq \|\varphi\|_\infty t + |\log p_t(x, y)|$.

Proof. The proof is similar to the one for ultrabounded one, only that we use the locally boundedness of $\|p_t(x, \cdot)\|_\infty, x \in \mathbf{R}^d$, instead of the boundedness.

Actually, the fact that $P(x, dy) < \infty$ for any $x \in \mathbf{R}^d$ is trivial from the definition. Also, $\text{supp } \pi = \mathbf{R}^d$. So we can define $p_t^\varphi(x, \cdot) \equiv dP_t^\varphi(x, \cdot)/d\pi$. The inequality $|\log p_t^\varphi(x, y)| \leq \|\varphi\|_\infty t + |\log p_t(x, y)|$ is easy. We show its continuity.

For any $s, t > 0$ with $2s < t$, define

$$p_{s,t}^{\varphi}(x, y) = \int_{\mathbf{R}^d} p_s(x, \tilde{x}) \int_{\mathbf{R}^d} p_{t-2s}^{\varphi}(\tilde{x}, \tilde{y}) p_s(\tilde{y}, y) \pi(d\tilde{y}) \pi(d\tilde{x}), \quad x, y \in E.$$

$p_s(x, \tilde{x})$ and $p_s(\tilde{y}, y)$, $x, y \in B_r, \tilde{x}, \tilde{y} \in \mathbf{R}^d$ is bounded for any $r > 0$, $p_{t-2s}^{\varphi}(\tilde{x}, \tilde{y}) \leq e^{\|\varphi\|_{\infty}(t-2s)} p_t(\tilde{x}, \tilde{y})$, which is intergable, and $p_s(x, \tilde{x})$ and $p_s(\tilde{y}, y)$ are continuous with respect to $x, y \in \mathbf{R}^d$, so $p_{s,t}^{\varphi}(x, y) \in C(\mathbf{R}^d \times \mathbf{R}^d)$.

Also, for any $x, y \in \mathbf{R}^d$,

$$|p_t^{\varphi}(x, y) - p_{s,t}^{\varphi}(x, y)| \leq e^{t\|\varphi\|} (e^{2s\|\varphi\|} - 1) p_t(x, y)$$

whenever $0 < s < t/2$. And $p_t(x, y)$ is locally bounded with respect to $x, y \in \mathbf{R}^d$. This shows that $p_t^{\varphi} \in C_b((\mathbf{R}^d)^2, (0, \infty))$ for all $t > 0$. ■

Remark 4 The lemma above says that $p_t^{\varphi}(x, y)$ satisfies the locally boundedness stated in H3.

Lemma 2.5 If $f \in C(\mathbf{R}^d)$, $f \geq 0$, and $\inf_{x \in \mathbf{R}^d} P_t^{\varphi} f(x) = 0$ for some $\varphi \in C_b(\mathbf{R}^d)$ and some $t > 0$, then $f \equiv 0$.

Proof. Suppose there exists a $x_0 \in \mathbf{R}^d$ such that $f(x_0) > 0$. From the continuity of f , there exists a $\delta > 0$, such that $f(y) \geq \frac{f(x_0)}{2} > 0$ for any $y \in \mathbf{R}^d$ with $|y - x_0| \leq \delta$. So for any $x \in \mathbf{R}^d$,

$$\begin{aligned} P_t^{\varphi} f(x) &\geq \int_{|y-x_0| \leq \delta} p_t^{\varphi}(x, y) f(y) \pi(dy) \\ &\geq \frac{f(x_0)}{2} \pi(B(x_0, \delta)) \inf_{\tilde{x} \in \mathbf{R}^d, |y-x_0| \leq \delta} p_t^{\varphi}(\tilde{x}, y), \end{aligned}$$

which is greater than 0. This contradicts with the assumption. ■

Lemma 2.6 For any $t > 0$, 1 is a simple eigenvalue of $e^{-\Lambda^{\varphi} t} P_t^{\varphi} : C_b(\mathbf{R}^d) \rightarrow C_b(\mathbf{R}^d)$, and is the only eigenvalue of it with a positive eigenfunction. Also, the absolute value of any other eigenvalue is smaller than 1.

Proof. This comes from Lemma 2.1 and Lemma 2.5 by using Perron-Frobenius argument.

Actually, for any $f_0 > 0$, which is strictly positive, and $0 < \inf f_0 \leq \sup f_0 = 1$, let

$$\alpha_0 = \inf_{x \in \mathbf{R}^d} \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0(x)}{f_0(x)}, \quad \beta_0 = \sup_{x \in \mathbf{R}^d} \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0(x)}{f_0(x)}.$$

Also, let $f_1 = \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{\|P_t^{\varphi} f_0\|_{\infty}}$. Then by Lemma 2.5, $\alpha_0 > 0$, and f_1 satisfies the same condition as f_0 . So we can define α_1 and β_1 in the same way. By a simple calculation, we get that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0.$$

Also, for any $g \in C_b(\mathbf{R}^d)$, $e^{-\Lambda(\varphi)t} P_t^{\varphi} g(x) \geq \inf e^{-\Lambda(\varphi)t} P_t^{\varphi} g$ for any $x \in \mathbf{R}^d$, apply this to $g = \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}} - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{f_0} \right) \frac{f_0}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}}$, and we get that

$$\begin{aligned} &\left(\frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}} \right)^{-1} \cdot \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} (e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0)}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}} \\ &\geq \inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{f_0} \\ &\quad + \inf e^{-\Lambda(\varphi)t} P_t^{\varphi} \left(e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0 - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{f_0} \right) f_0 \right) \cdot \frac{1}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}}. \end{aligned}$$

Take infimum with respect to $x \in \mathbf{R}^d$, and we get that

$$\alpha_1 \geq \alpha_0 + \inf e^{-\Lambda(\varphi)t} P_t^{\varphi} \left(e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0 - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0}{f_0} \right) f_0 \right) \cdot \frac{1}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}}.$$

Define $\alpha_n, \beta_n, f_n, n \in \mathbf{N}$ in the same way, and we get that

$$0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \beta_2 \leq \beta_1 \leq \beta_0, \quad (2.1)$$

and

$$\alpha_{n+1} \geq \alpha_n + \inf e^{-\Lambda(\varphi)t} P_t^{\varphi} \left(e^{-\Lambda(\varphi)t} P_t^{\varphi} f_n - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_n}{f_n} \right) f_n \right) \cdot \frac{1}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_0\|_{\infty}}. \quad (2.2)$$

From (2.1), by taking subsequence if necessary, we can assume that there exists a $\alpha > 0$, such that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Also, $\{f_n\}_{n \in \mathbf{N}}$ is bounded in $C_b(\mathbf{R}^d)$, so from the compactness of P_t^{φ} , by taking subsequence if necessary, we can assume that there exists a $f_{\infty} \in C_b(\mathbf{R}^d)$ with $f_{\infty} \neq 0$, such that $e^{-\Lambda(\varphi)t} P_t^{\varphi} f_n \rightarrow f_{\infty}$ as $n \rightarrow \infty$, so $f_{n+1} = \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} f_n}{\|e^{-\Lambda(\varphi)t} P_t^{\varphi} f_n\|_{\infty}} \rightarrow \frac{f_{\infty}}{\|f_{\infty}\|_{\infty}}$. Define $h = \frac{f_{\infty}}{\|f_{\infty}\|_{\infty}}$. From (2.2), this means that

$$\inf e^{-\Lambda(\varphi)t} P_t^{\varphi} \left(e^{-\Lambda(\varphi)t} P_t^{\varphi} h - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} h}{h} \right) h \right) = 0.$$

Apply Lemma 2.5, and we get that

$$e^{-\Lambda(\varphi)t} P_t^{\varphi} h - \left(\inf \frac{e^{-\Lambda(\varphi)t} P_t^{\varphi} h}{h} \right) h = 0. \quad (2.3)$$

$h \in C_b(\mathbf{R}^d)$, $f_n \geq 0$, and $\|f_n\|_\infty = 1$, $n \in \mathbf{N}$, so (2.3) shows that h is strictly positive, i.e., we showed that $e^{-\Lambda^\varphi t} P_t^\varphi$ has a positive eigenvalue $\alpha > 0$ with strictly positive eigenfunction h , i.e., $e^{-\Lambda^\varphi t} P_t^\varphi h = \alpha h$.

Same argument can be applied to P_t^* , so there exist a $\alpha' > 0$ and a strictly positive $l \in C_b(\mathbf{R}^d)$, such that $e^{-\Lambda^\varphi t} P_t^{*\varphi} l = \alpha' l$. So

$$\alpha \int_{\mathbf{R}^d} h l d\pi = \int_{\mathbf{R}^d} e^{-\Lambda^\varphi t} P_t^\varphi h l d\pi = \int_{\mathbf{R}^d} e^{-\Lambda^\varphi t} P_t^{*\varphi} l h d\pi = \alpha' \int_{\mathbf{R}^d} h l d\pi.$$

$\int_{\mathbf{R}^d} h l d\pi > 0$, so $\alpha = \alpha'$.

By using l , we get in the same way as above that if x is a strictly positive eigenfunction of $e^{-\Lambda^\varphi t} P_t^\varphi$ with eigenvalue a , i.e. $e^{-\Lambda^\varphi t} P_t^\varphi x = ax$, then $a = \alpha$ and $x = h$.

Next, we show that α is a simple eigenvalue of $e^{-\Lambda^\varphi t} P_t^\varphi$. If not, then there exists a $y \in C_b(\mathbf{R}^d)$, s.t., $e^{-\Lambda^\varphi t} P_t^\varphi y = \alpha y + h$. Take integral of the both side above with respect to $l d\pi$, and we get that $\int h l d\pi = 0$, which makes a contradiction. Therefore, α is a simple eigenvalue.

Finally, we show that the absolute value of any other eigenvalues is smaller than α . If we can show this, then it is obvious that $\alpha = 1$. Let λ be an eigenvalue of it different from α . Let g be an eigenfunction with respect to λ (which is a function that may take complex values), i.e., $e^{-\Lambda^\varphi t} P_t^\varphi g = \lambda g$. So $|\lambda| |g| \leq e^{-\Lambda^\varphi t} P_t^\varphi |g|$, and the equality holds if and only if g is equal to some complex constant times a real valued function.

$$|\lambda| \int_{\mathbf{R}^d} |g| l d\pi \leq \int_{\mathbf{R}^d} e^{-\Lambda^\varphi t} P_t^\varphi |g| l d\pi = \int_{\mathbf{R}^d} \alpha |g| l d\pi.$$

So $|\lambda| \leq \alpha$. Also, if $|\lambda| = \alpha$, then as mentioned before, ignoring the constant scalar, we must have that g is a positive eigenfunction of $e^{-\Lambda^\varphi t} P_t^\varphi$. Therefore, by the results of the proof of the first part, $g = h$ and $\lambda = \alpha$, which contradicts with our assumption. This completes the proof of the lemma. ■

From Lemma 2.6, for any $t > 0$, there exists a unique $h_t^\varphi \in C_b(\mathbf{R}^d)$, such that $P_t^\varphi h_t^\varphi = e^{\Lambda^\varphi t} h_t^\varphi$. As in the ultrabounded case, since P_t^φ is a semigroup, we get from the uniqueness that $h_t^\varphi = h_1^\varphi$ for any $t > 0$. Write it as h^φ .

Remark 5 By Lemma 2.5, we know that h^φ is strictly positive, i.e., $\|\log h^\varphi\|_\infty < \infty$.

Now, we can define a set of probability measures Q_x^φ , $x \in \mathbf{R}^d$, on (Ω, \mathcal{F}) , such that

$$Q_x^\varphi(A) = \frac{e^{-\Lambda^\varphi t}}{h^\varphi(x)} E^{P_x} \left[1_A(X_t) \exp \left(\int_0^t \varphi(X_u) du \right) h^\varphi(X_t) \right]$$

for all $x \in E$, $t \geq 0$, and $A \in \mathcal{F}_t$. Let $\{Q_t^\varphi\}$ be the corresponding semigroup of bounded linear operators in $C_b(\mathbf{R}^d)$. Q_t^φ has strictly positive continuous density function

$$q_t^\varphi(x, y) = \frac{e^{-\Lambda^\varphi t}}{h^\varphi(x)} p_t^\varphi(x, y) h^\varphi(y), \quad x, y \in E, t > 0.$$

Doing in the same way as in Lemma 2.6, we get that there exists a unique $l^\varphi \in C_b(\mathbf{R}^d)$, such that $P_t^{*\varphi} l^\varphi = e^{\Lambda^\varphi t} l^\varphi$, i.e., $\int p_t^\varphi(x, y) l^\varphi(x) \pi(dx) = e^{\Lambda^\varphi t} l^\varphi(y)$. Also, l^φ is strictly positive, i.e., $\|\log l^\varphi\|_\infty < \infty$. Let $\pi^\varphi = h^\varphi l^\varphi d\pi$. Obviously, $\{Q_t^\varphi\}$ is π^φ -invariant. Also, 1 is a simple eigenvalue of Q_t^φ with eigenfunction 1, is the only eigenvalue with a positive eigenfunction, and the absolute value of any other eigenvalue is smaller than 1. Therefore, π^φ is the only invariant measure of Q_t^φ , and there exist constants $C, \varepsilon > 0$ depend on φ , such that $\|Q_t^\varphi f - \int f d\pi^\varphi\|_\infty \leq C e^{-\varepsilon t} \|f\|_\infty$ for any $t \geq 1$ and any $f \in C_b(\mathbf{R}^d)$.

Also, we have the following one for $e^{-\Lambda^\varphi t} P_t^\varphi$.

Lemma 2.7 For any $\varphi \in C_b(\mathbf{R}^d)$, there exist constants $C_\varphi, \alpha_\varphi > 0$, such that $\|e^{-\Lambda^\varphi t} P_t^\varphi f - (\int f l^\varphi d\pi) h^\varphi\|_\infty \leq C_\varphi e^{-\alpha_\varphi t} \|f\|_\infty$ for any $f \in C_b(\mathbf{R}^d)$ and any $t > 1$.

Let L_t denote $\frac{1}{t} \int_0^t \delta_{X_s} ds$, where δ denote the delta measure. For any $t > 0$ and any $x, y \in \mathbf{R}^d$, let $\mu_t^{x,y}$ be the probability measure given by $\mu_t^{x,y}(A) = P_x(L_t \in A | X_t = y)$. We are going to give the large deviation principle of $\{\mu_t^{x,y}\}_{t>0}$.

For any $\varphi \in C_b(\mathbf{R}^d)$, $p_t^\varphi(x, y) = P_{t-1}^\varphi(p_1^\varphi(\cdot, y))(x)$, and $p_1^\varphi(\cdot, y)$ is bounded and strictly positive, so by Lemma 2.7,

$$\frac{1}{t} \log p_t^\varphi(x, y) = \frac{t-1}{t} \Lambda^\varphi + \frac{1}{t} \log \left(e^{-\Lambda^\varphi(t-1)} P_{t-1}^\varphi(p_1^\varphi(\cdot, y))(x) \right) \rightarrow \Lambda^\varphi$$

as $t \rightarrow \infty$. In the same way, $p_t(x, y) = P_{t-1}(p_1(\cdot, y))(x)$, so $\frac{1}{t} \log p_t(x, y) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$\frac{1}{t} \log \int e^{t(\varphi, \nu)} \mu_t^{x,y}(d\nu) = \frac{1}{t} \log E^{P_x} \left[e^{\int_0^t \varphi(X_s) ds} | X_t = y \right] = \frac{1}{t} \log \frac{p_t^\varphi(x, y)}{p_t(x, y)} \rightarrow \Lambda^*$$

as $t \rightarrow \infty$. This is true for any $\varphi \in C_b(\mathbf{R}^d)$. Let $\Lambda^*(\nu) = \sup\{\int_{\mathbf{R}^d} \phi d\nu - \Lambda^\phi; \phi \in C_b(\mathbf{R}^d)\}$, $\nu \in \mathcal{P}(\mathbf{R}^d)$. Then by Deuschel-Stroock [3, Theorem 2.2.4], we have the following

Lemma 2.8 Λ^* is a non-negative, lower semi-continuous, convex function, and for any $G \subset \subset \mathcal{P}(\mathbf{R}^d)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in G | X_t = y) \leq - \inf_{\nu \in G} \Lambda^*(\nu).$$

Lemma 2.9 $\{\mu_t\}$ is exponentially tight.

Proof. As showed before, for any $t > 0$, $\sup_{x \in \mathbf{R}^d} P_x(|X_t| \geq n) \rightarrow 0$ as $n \rightarrow \infty$. Let $V: \mathbf{R} \rightarrow \mathbf{R}$ be the function given by $V(u) = \log((\sup_{x \in \mathbf{R}^d} P_x(|X_1| \geq u))^{-1/2})$, $u \in \mathbf{R}$. Then $V(u) \rightarrow \infty$ as $u \rightarrow \infty$.

$$\sup_{x \in \mathbf{R}^d} E^{P_x}[e^{V(|X_1|)}] \leq \sup_{x \in \mathbf{R}^d} \left(- \int_{\mathbf{R}} (P_x(|X_1| \geq y))^{-1/2} dP_x(|X_1| \geq y) \right) = 2.$$

So for any $s > 1$,

$$\sup_{x \in \mathbf{R}^d} E^{P_x}[e^{V(X_s)}] = \sup_{x \in \mathbf{R}^d} P_{s-1}(E^P[e^{V(|X_1|)}])(x) \leq 2.$$

Therefore,

$$\sup_{x \in \mathbf{R}^d} E^{P_x} \left[\exp \left(\int_1^2 V(X_s) ds \right) \right] \leq \sup_{x \in \mathbf{R}^d} \int_1^2 E^{P_x} [e^{V(X_s)}] ds \leq 2.$$

Therefore, by Holder's inequality and Markov property,

$$\begin{aligned} & E^{P_x} \left[\exp \left(\frac{1}{2} \int_2^{2^n} V(X_s) ds \right) \right] \\ & \leq E^{P_x} \left[\exp \left(\sum_{k=1}^{n-1} \int_{2k}^{2k+1} V(X_s) ds \right) \right]^{1/2} E^{P_x} \left[\exp \left(\sum_{k=2}^n \int_{2k-1}^{2k} V(X_s) ds \right) \right]^{1/2} \\ & \leq \left(\sup_{x \in \mathbf{R}^d} E^{P_x} \left[\exp \left(\int_1^2 V(X_s) ds \right) \right] \right)^{n-1} \leq \sup_{x \in \mathbf{R}^d} \int_1^2 E^{P_x} [e^{V(X_s)}] ds \leq 2^{n-1}. \end{aligned}$$

So there exists constants $A_1, A_2 > 0$, such that for any $T > 1$,

$$E^{P_x} \left[\exp \left(\int_1^T V(X_s) ds \right) \right] \leq A_1 e^{TA_2}.$$

For any $l > 0$, there exists a $k_l \in \mathbf{N}$, s.t. for any $k \geq k_l$, $\inf_{B_k^c} V \geq l^2$. Also, for any $a > 0$,

$$P_x(L_T(B_k^c) > a) \leq e^{-T(\inf_{B_k^c} V)^2} E^{P_x} \left[e^{\int_0^T V(X_s) ds} \right] \leq e^{-T(\inf_{B_k^c} V)^2} A_1 e^{TA_2}.$$

Therefore,

$$P_x(L_T(B_k^c) > \frac{1}{l}) \leq A_1 e^{-T(l-A_2)}.$$

Now, for any $L \in \mathbf{N}$, let

$$C_L \equiv \bigcap_{l \geq L} \left\{ \nu; \nu(B_{k_l}) \geq 1 - \frac{1}{l} \right\}.$$

C_L is a compact set, and for any $T \geq 1$,

$$P_x(L_T \in C_L^c) \leq \sum_{l=L}^{\infty} P_x(L_T(B_{k_l}^c) \geq \frac{1}{l}) \leq \sum_{l=L}^{\infty} A_1 e^{-T(l-A_2)} \leq \frac{A_1 e^{-T(L-A_2)}}{1 - e^{-1}}.$$

We get our assertion from this.

Therefore, as in the proof of Deuschel-Stroock [3, Lemma 2.1.5], we have that

Lemma 2.10 For any closed $C \subset \mathfrak{P}(\mathbf{R}^d)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in C | X_t = y) \leq - \inf_{\nu \in K} \Lambda^*(\nu).$$

Next, we show the lower bound. First, we have the following

Lemma 2.11 For any $\phi \in C_b(\mathbf{R}^d)$, any $K \subset \subset \mathbf{R}^d$, and any $\varepsilon > 0$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in B(\pi^\phi, \varepsilon) | X_t = y) \geq -\Lambda^*(\pi^\phi)$$

uniformly for $x, y \in K$, where $B(\nu, \varepsilon)$ means the set $\{\eta \in \mathfrak{P}(\mathbf{R}^d) | \text{dist}(\nu, \eta) < \varepsilon\}$ for any $\nu \in \mathfrak{P}(\mathbf{R}^d)$ and any $\varepsilon > 0$.

Proof. Since $\phi \in C_b(\mathbf{R}^d)$, for any $\delta > 0$, there exists a $\varepsilon_1 > 0$, such that $|\int \phi d\nu - \int \phi d\pi^\phi| \leq \delta$ for any $\nu \in \mathfrak{P}(\mathbf{R}^d)$ with $\text{dist}(\nu, \pi^\phi) \leq \varepsilon_1$. We can certainly assume that $\varepsilon_1 \leq \varepsilon$. π^ϕ is the invariant measure of $\{Q_t^\phi\}_{t \in \mathbf{R}^d}$, so by the large number theorem of diffusion process, we see that $Q_t^\phi(L_t \in B(\pi^\phi, \varepsilon_1)) \rightarrow 1$ as $t \rightarrow \infty$. Also, L_t and X_t are asymptotically independent under Q_t^ϕ . Therefore,

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in B(\pi^\phi, \varepsilon) | X_t = y) \\ & \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \log \left(e^{\Lambda(\phi)t} E^{Q_t^\phi} \left[e^{-t \int_{\mathbf{R}^d} \phi dL_1} 1_{L_t \in B(\pi^\phi, \varepsilon_1)} | X_t = y \right] \right) \\ & \geq \Lambda(\phi) - \int_{\mathbf{R}^d} \phi d\pi^\phi - \delta \\ & \geq -\Lambda^*(\pi^\phi) - \delta. \end{aligned}$$

This is true for any $\delta > 0$, so our assertion follows by letting $\delta \rightarrow 0$. The uniformity can be gotten in the same time.

Remark 6 From the proof of the last lemma, we see that $\Lambda(\psi) - \int_{\mathbf{R}^d} \psi d\pi^\phi \geq \Lambda(\phi) - \int_{\mathbf{R}^d} \phi d\pi^\phi$ for any $\psi \in C_b(\mathbf{R}^d)$. Therefore, $\Lambda^*(\pi^\phi) = \int_{\mathbf{R}^d} \phi d\pi^\phi - \Lambda(\phi)$.

Now, for any $\nu \in \varphi(\mathbf{R}^d)$, we define

$$J_\varepsilon(\nu) = \inf \left\{ \sum_{i=1}^n \lambda_i \Lambda^*(\pi^{\phi_i}); n \in \mathbf{N}, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, \right. \\ \left. \phi_i \in C_b(\mathbf{R}^d), \text{dist}(\nu, \sum_{i=1}^n \lambda_i \pi^{\phi_i}) \leq \varepsilon \right\}, \quad \varepsilon > 0,$$

and define

$$J(\nu) = \lim_{\varepsilon \rightarrow 0} J_\varepsilon(\nu) = \sup_{\varepsilon > 0} J_\varepsilon(\nu).$$

Lemma 2.12 For any $\nu_1, \nu_2, \mu_1, \mu_2 \in \varphi(\mathbf{R}^d)$ and any $\lambda \in [0, 1]$,

$$\text{dist}(\lambda \nu_1 + (1 - \lambda) \nu_2, \lambda \mu_1 + (1 - \lambda) \mu_2) \leq \max\{\text{dist}(\nu_1, \mu_1), \text{dist}(\nu_2, \mu_2)\}.$$

Proof. Without loss of generality, we may assume that $\text{dist}(\nu_1, \mu_1) \geq \text{dist}(\nu_2, \mu_2)$. For any $A \in \mathcal{B}(\mathbf{R}^d)$, let A_ε denote the ε -neighborhood of A , then from the definition of Prohorov metric, for any $R_1, R_2 \in \varphi(\mathbf{R}^d)$,

$$\text{dist}(R_1, R_2) = \inf\{\varepsilon > 0; R_1(A) \leq R_2(A_\varepsilon) + \varepsilon, R_2(A) \leq R_1(A_\varepsilon) + \varepsilon, \text{ for any } A \in \mathcal{B}(\mathbf{R}^d)\}.$$

Therefore, for any $\varepsilon_1 > \text{dist}(\nu_1, \mu_1) \geq \text{dist}(\nu_2, \mu_2)$, we have that

$$\begin{aligned} (\lambda \nu_1 + (1 - \lambda) \nu_2)(A) &\leq \lambda(\mu_1(A_{\varepsilon_1}) + \varepsilon_1) + (1 - \lambda)(\mu_2(A_{\varepsilon_1}) + \varepsilon_1) \\ &\leq (\lambda \mu_1 + (1 - \lambda) \mu_2)(A_{\varepsilon_1}) + \varepsilon_1 \end{aligned}$$

The opposite inequality is the same. So $\text{dist}(\lambda \nu_1 + (1 - \lambda) \nu_2, \lambda \mu_1 + (1 - \lambda) \mu_2) \leq \varepsilon_1$. Take infimum with respect to ε_1 , and we get our assertion. ■

Lemma 2.13 J is convex and lower semi-continuous.

Proof. For the lower semicontinuity, choose any ν and $\nu_n \in \varphi(\mathbf{R}^d)$, $n \in \mathbf{N}$, with $\text{dist}(\nu_n, \nu) \rightarrow 0$. Then for any $\varepsilon > 0$, there exists a $N \in \mathbf{N}$, for any $n > N$, $\text{dist}(\nu_n, \nu) \leq \varepsilon$. Therefore, from the definition of J_ε , we have that $J_{2\varepsilon}(\nu) \leq J_\varepsilon(\nu_n) \leq J(\nu_n)$. Take $\varepsilon \rightarrow 0$, and we get that $J(\nu) \leq \liminf_{n \rightarrow \infty} J(\nu_n)$. i.e., J is lower semi-continuous.

For the convexity, from the definition of Prohorov metric and the fact that Λ^* is always positive, we get that for any $\varepsilon > 0$, any $\lambda \in (0, 1)$ and any $\nu_1, \nu_2 \in \varphi(\mathbf{R}^d)$, $J_\varepsilon(\lambda \nu_1 + (1 - \lambda) \nu_2) \leq \lambda J_\varepsilon(\nu_1) + (1 - \lambda) J_\varepsilon(\nu_2)$. Actually, for any $n_1, n_2 \in \mathbf{N}$, any $\lambda_1, \dots, \lambda_{n_1}, \lambda_{n_1+1}, \dots, \lambda_{n_1+n_2} \in [0, 1]$ with $\sum_{i=1}^{n_1} \lambda_i = 1$ and $\sum_{j=1}^{n_2} \lambda_{n_1+j} = 1$, any

$\phi_1, \dots, \phi_{n_1} \in C_b(\mathbf{R}^d)$, and any $\phi_{n_1+1}, \dots, \phi_{n_1+n_2} \in C_b(\mathbf{R}^d)$, if $\text{dist}(\nu_1, \sum_{i=1}^{n_1} \lambda_i \pi^{\phi_i}) \leq \varepsilon$ and $\text{dist}(\nu_2, \sum_{j=1}^{n_2} \lambda_{n_1+j} \pi^{\phi_{n_1+j}}) \leq \varepsilon$, then by Lemma 2.12,

$$\text{dist}(\lambda \nu_1 + (1 - \lambda) \nu_2, \sum_{i=1}^{n_1} \lambda \lambda_i \pi^{\phi_i} + \sum_{j=1}^{n_2} (1 - \lambda) \lambda_{n_1+j} \pi^{\phi_{n_1+j}}) \leq \varepsilon.$$

So for any $\varepsilon > 0$,

$$J_\varepsilon(\lambda \nu_1 + (1 - \lambda) \nu_2) \leq \lambda J_\varepsilon(\nu_1) + (1 - \lambda) J_\varepsilon(\nu_2).$$

Take $\varepsilon \rightarrow 0$, and we get our assertion. ■

Lemma 2.14 $J(\nu) = \Lambda^*(\nu)$ for any $\nu \in \varphi(\mathbf{R}^d)$. Also, $\Lambda(\phi) = J^*(\phi) = \sup\{\int \phi d\nu - J(\nu); \nu \in \varphi(\mathbf{R}^d)\}$.

Proof. First, we show that $J(\nu) \geq \Lambda^*(\nu)$ for any $\nu \in \varphi(\mathbf{R}^d)$. Since Λ^* is convex and lower semi-continuous, we have that for any $\delta > 0$, there exists a $\varepsilon > 0$, such that $\Lambda^*(\nu') \geq \Lambda^*(\nu) - \delta$ for any $\nu' \in \varphi(\mathbf{R}^d)$ with $\text{dist}(\nu, \nu') \leq \varepsilon$. Therefore,

$$\begin{aligned} J(\nu) &\geq J_\varepsilon(\nu) \\ &\geq \inf \left\{ \Lambda^* \left(\sum_{i=1}^n \lambda_i \pi^{\phi_i} \right); n \in \mathbf{N}, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, \right. \\ &\quad \left. \phi_i \in C_b(\mathbf{R}^d), \text{dist}(\nu, \sum_{i=1}^n \lambda_i \pi^{\phi_i}) \leq \varepsilon \right\} \\ &\geq \Lambda^*(\nu) - \delta. \end{aligned}$$

This is true for any $\delta > 0$. So $J(\nu) \geq \Lambda^*(\nu) \geq \int \phi d\nu - \Lambda(\phi)$ for any $\phi \in C_b(\mathbf{R}^d)$. Therefore, $\Lambda(\phi) \geq \int \phi d\nu - J(\nu)$ for any $\nu \in \varphi(\mathbf{R}^d)$ and any $\phi \in C_b(\mathbf{R}^d)$.

Also, it is easy from the definition of J that $J(\pi^\phi) \leq \Lambda^*(\pi^\phi)$, which is equal to $\int \phi d\pi^\phi - \Lambda(\phi)$ as mentioned in Remark 6. So we have that $\Lambda(\phi) \leq \int \phi d\pi^\phi - J(\pi^\phi)$.

Therefore, $\Lambda(\phi) = \sup\{\int \phi d\nu - J(\nu); \nu \in \varphi(\mathbf{R}^d)\}$ for any $\phi \in C_b(\mathbf{R}^d)$. Now, we can get our assertion from Lemma 2.13 by Deuschel-Stroock [3, Theorem 2.2.15]. ■

Lemma 2.15 For any $\nu \in \varphi(\mathbf{R}^d)$, any $\varepsilon > 0$, and any $x, y \in \mathbf{R}^d$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in B(\nu, \varepsilon) | X_t = y) \geq -\Lambda^*(\nu).$$

Proof. For the simplicity of the notations, we write $L_{t_1, t_2} \equiv \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \delta_{X(s)} ds$ for $0 \leq t_1 < t_2 < \infty$.

For any $n \in \mathbf{N}$ and any $\lambda_i \in [0, 1]$, $\phi_i \in C_b(\mathbf{R}^d)$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, and $\text{dist}(\nu, \sum_{i=1}^n \lambda_i \pi^{\phi_i}) < \frac{\varepsilon}{2}$, we have that

$$\begin{aligned} & \bigcap_{i=1}^n \left\{ L_{\sum_{j=0}^{i-1} \lambda_j t, \sum_{j=0}^i \lambda_j t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} \\ & \subset \left\{ L_t \in B(\sum_{i=1}^n \lambda_i \pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} \\ & \subset \{L_t \in B(\nu, \varepsilon)\}. \end{aligned}$$

Therefore, for any $C \subset \subset \mathbf{R}^d$ with $\pi(C) > 0$ and $x, y \in C$, write $x_0 = x$ and $x_n = y$, then we have from the Markov property that

$$\begin{aligned} & P_x(L_t \in B(\nu, \varepsilon) | X_t = y) \\ & \geq P_x \left(\bigcap_{i=1}^n \left\{ L_{\sum_{j=0}^{i-1} \lambda_j t, \sum_{j=0}^i \lambda_j t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} | X_t = y \right) \\ & \geq \int_{x_1, \dots, x_{n-1} \in C} \prod_{i=1}^n P_{x_{i-1}} \left(L_{\lambda_i t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) | X_{\lambda_i t} = x_i \right) \\ & \quad \prod_{i=1}^n p_{\lambda_i t}(x_{i-1}, x_i) \pi(dx_1) \cdots \pi(dx_{n-1}) \\ & \geq \prod_{i=1}^n \left(\inf_{x_{i-1}, x_i \in C} P_{x_{i-1}} \left(L_{\lambda_i t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) | X_{\lambda_i t} = x_i \right) \right. \\ & \quad \times \left. \prod_{i=1}^n \left(\inf_{x_{i-1}, x_i \in K} p_{\lambda_i t}(x_{i-1}, x_i) \right) \pi(K)^{n-1} \right). \end{aligned}$$

Since C is compact, the second term in the last expression is uniformly bounded and separated from 0 for $T > 0$. Therefore, we get from Lemma 2.11 and the definition of J that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in B(\nu, \varepsilon) | X_t = y) \\ & \geq \liminf_{t \rightarrow \infty} \sum_{i=1}^n \frac{1}{t} \log \left(\inf_{x_{i-1}, x_i \in C} P_{x_{i-1}} \left(L_{\lambda_i t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) | X_{\lambda_i t} = x_i \right) \right) \\ & \geq - \sum_{i=1}^n \lambda_i \Lambda^*(\pi^{\phi_i}). \end{aligned}$$

Takes infimum with respect to $n \in \mathbf{N}$ and $\lambda_i, \phi_i, i = 1, \dots, n$, and we get that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in B(\nu, \varepsilon) | X_t = y) \geq -J_{\varepsilon/2}(\nu) \geq -J(\nu).$$

This accompanied with Lemma 2.14 gives our assertion. \square

Lemma 2.16 For any open subset $G \subset \wp(\mathbf{R}^d)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x(L_t \in G | X_t = y) \geq - \inf_{\nu \in G} \Lambda^*(\nu).$$

Proof. This is easy from Lemma 2.15. \square

Up to now, we have showed that Λ^* is a good rate function and governs the large deviation principle of $\mu_t^{\varepsilon, \varphi}$ for any $x, y \in \mathbf{R}^d$. Next, we give another expression of Λ^* . Let $D(L_0)$ be the domain of L_0 , and $D_b(L_0) = \{u \in D(L_0); L_0 u \text{ is bounded}\}$. Let

$$I(\nu) = \sup \left\{ \int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty(\mathbf{R}^d), u \in D_b(L_0) \cap C_b(\mathbf{R}^d), u \geq 1 \right\}, \quad \nu \in \wp(\mathbf{R}^d).$$

Note that the condition $u \geq 1$ can certainly be substituted by the condition that u is strictly positive.

Lemma 2.17 I is a convex and lower semi-continuous function.

Proof. This is easy from the definition of I . \square

Lemma 2.18 $I = \Lambda^* = J$.

Proof. The second equality is from Lemma 2.14. We show the first one.

First, for any $\varphi \in C^\infty(\mathbf{R}^d) \cap C_b(\mathbf{R}^d)$, as discussed before, there exists a unique $h^\varphi \in C_b(\mathbf{R}^d)$ which is strictly positive and $P_t^\varphi h^\varphi = e^{\Lambda(\varphi)t} h^\varphi$. So $h^\varphi \in C^\infty(\mathbf{R}^d) \cap D_b(L_0)$, and $L_0 h^\varphi + \varphi h^\varphi = \Lambda(\varphi) h^\varphi$. Let $L^\varphi f = (h^\varphi)^{-1}(L_0 + \varphi - \Lambda(\varphi))(h^\varphi f) = (h^\varphi)^{-1} L_0(h^\varphi f) + (\varphi - \Lambda(\varphi))f$. L^φ is nothing but the infinitesimal generator corresponding to the $\{Q_s^\varphi\}$ defined before. So

$$\begin{aligned} I(\nu) &= \sup \left\{ - \int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty(\mathbf{R}^d) \cap D_b(L_0), u \text{ is bounded and strictly positive} \right\} \\ &= \int_{\mathbf{R}^d} \varphi d\nu - \Lambda(\varphi) \\ &\quad + \sup \left\{ - \int_{\mathbf{R}^d} \frac{L^\varphi u}{u} d\nu; u \in D_b(L_0), u \text{ is bounded and strictly positive} \right\} \\ &\geq \int_{\mathbf{R}^d} \varphi d\nu - \Lambda(\varphi). \end{aligned}$$

This is true for any $\varphi \in C^\infty(\mathbf{R}^d) \cap C_b(\mathbf{R}^d)$. Therefore, $I(\nu) \geq \Lambda^*(\nu)$.

For the opposite one, for any bounded and strictly positive $u \in C^\infty(\mathbf{R}^d) \cap D(L_0)$ with $L_0 u$ bounded, since $(L_0 - \frac{L_0 u}{u})u = 0$, we have $\Lambda(-\frac{L_0 u}{u}) = 0$, so $-\int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu = -\int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu - \Lambda(-\frac{L_0 u}{u}) \leq \Lambda^*(\nu)$. Therefore, $I(\nu) \leq \Lambda^*(\nu)$. This completes the proof of our assertion. \square

Put things together, and we get Theorem 1.1.

3 Perturbations

Use the same notation as in section 2. Also, assume the same conditions as there. Let G_0 be the Green operator corresponding to $\{P_t\}$, i.e., $G_0 f = f(P_t f - f f d\pi) dt$, the integral on the right hand side converges by Lemma 2.7 with $\varphi = 0$. We will need some information about the behavior of $\Lambda^\varphi, h^\varphi, l^\varphi, \pi^\varphi$, and so on, as $\varphi \rightarrow 0$.

First, we have the following continuity.

Lemma 3.1 1. The map $C_b(\mathbf{R}^d) \ni \varphi \mapsto \Lambda^\varphi$ is continuous,

2. The map $C_b(\mathbf{R}^d) \ni \varphi \mapsto h^\varphi \in C_b(\mathbf{R}^d)$ is continuous,

3. The map $C_b(\mathbf{R}^d) \ni \varphi \mapsto l^\varphi \in C_b(\mathbf{R}^d)$ is continuous.

Proof. The continuity of $\varphi \mapsto \Lambda^\varphi$ is easy, since $|\Lambda^\varphi - \Lambda^\psi| \leq \|\varphi - \psi\|_\infty$ for any $\varphi, \psi \in C_b(\mathbf{R}^d)$ by Lemma 2.14. The proofs of the second and the third assertion are the same, and we will give only the proof of the second one.

From the definition of P_t^φ , we see that if $\varphi_n \rightarrow \varphi$ in $C_b(\mathbf{R}^d)$, then $\|P_t^{\varphi_n} - P_t^\varphi\|_{op} \leq e^{\|\varphi\|_\infty} (e^{\|\varphi_n - \varphi\|_\infty} - 1) \rightarrow 0$. Also, as claimed in (1), $\Lambda^{\varphi_n} \rightarrow \Lambda^\varphi$, so $e^{-\Lambda^{\varphi_n}} P_t^{\varphi_n} \rightarrow e^{-\Lambda^\varphi} P_t^\varphi$ as operators in $C_b(\mathbf{R}^d)$. They are all compact, and 1 is the simple eigenvalues of them, with projection operators $f \mapsto h^{\varphi_n} \int_{\mathbf{R}^d} f l^{\varphi_n} d\pi$ and $f \mapsto h^\varphi \int_{\mathbf{R}^d} f l^\varphi d\pi$, respectively. Therefore, by Dunford-Schwartz [4, Lemma VII.6.5], we see that $\|h^{\varphi_n} \langle \cdot, l^{\varphi_n} \rangle_\pi - h^\varphi \langle \cdot, l^\varphi \rangle_\pi\|_{op} \rightarrow 0$ as $n \rightarrow \infty$. This accompanied with the fact that $\int (h^{\varphi_n})^2 d\pi = \int (h^\varphi)^2 d\pi = 1$ gives us that $\|h^{\varphi_n} - h^\varphi\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. ■

Now, let $f \in C_b(\mathbf{R}^d)$ be a bounded continuous function with $\int_{\mathbf{R}^d} f d\pi = 0$ and fix it for a while. Denote $\Lambda^{\varepsilon f}$ by $\Lambda(\varepsilon)$, $h^{\varepsilon f}$ by h^ε , and so forth. Set

$$\Lambda_{T,x,y}(\varepsilon) = \frac{1}{T} \log E^{P_x} \left[\exp \left(\varepsilon \int_0^T f(X_s) ds \right) \middle| X_T = y \right]$$

for any $x, y \in \mathbf{R}^d$ and any $T > 0$.

Let $Q_t^{\varepsilon,*}$ be the adjoint operator of Q_t^ε in $L^2(d\pi^\varepsilon)$. $\{X_{T-t}\}_{t=0}^T$ under $Q_t^{\varepsilon,*}$ is a Markov process, and the semigroup of continuous linear operators in $C_b(\mathbf{R}^d)$ generated by it is $\{Q_t^{\varepsilon,*}\}$.

Lemma 3.2 There exists a $\varepsilon_0 > 0$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $t > 0$,

$$\|Q_t^\varepsilon - \langle \cdot \rangle_{d\pi^\varepsilon}\|_{op} \leq 2C_0 e^{-\frac{\alpha_0}{2}t}.$$

The same is true for $\{Q_t^{\varepsilon,*}\}$.

Proof. P_t^ε is continuous with respect to ε , also, h^ε and Λ^ε are continuous with respect to ε by Lemma 3.1, with h^ε strictly positive for every ε , $Q_t^\varepsilon f = \frac{e^{-\Lambda(\varepsilon)t}}{h^\varepsilon} P_t^\varepsilon(h^\varepsilon f)$, so Q_t^ε is continuous with respect to ε , too. The continuity of h^ε and l^ε with respect to ε gives the continuity of $\langle \cdot \rangle_{\pi^\varepsilon}$ with respect to ε . Also,

$$\|Q_t^\varepsilon - P_t\|_{op} \leq \max\{e^{2\|f\|_\infty t} \frac{\sup h^\varepsilon}{\inf h^\varepsilon} - 1, 1 - e^{-2\|f\|_\infty t} \frac{\inf h^\varepsilon}{\sup h^\varepsilon}\} \quad (3.1)$$

for any $t > 0$.

Now, note that $\frac{1}{t} \log \|P_t - \langle \cdot \rangle_\pi\|_{op}$ is monotone non-increasing with respect to t , and converges as $t \rightarrow \infty$ to the logarithm spectral radius of $P_t - \langle \cdot \rangle_\pi$, which is $\leq -\alpha_0 < -\frac{\alpha_0}{2}$. So we can find a $T > 0$, such that $\frac{1}{T} \log \|P_T - \langle \cdot \rangle_\pi\|_{op} < -\frac{\alpha_0}{2}$. For this T , we see from (3.1) and the continuity of h^ε that $\frac{1}{T} \log \|Q_T^\varepsilon - \langle \cdot \rangle_{\pi^\varepsilon}\|_{op} < -\frac{\alpha_0}{2}$ if $|\varepsilon|$ is small enough. Therefore, by the monotonicity again, we see that there exists a ε_0 , such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $t \geq T$,

$$\|Q_t^\varepsilon - \langle \cdot \rangle_{\pi^\varepsilon}\|_{op} \leq e^{-\frac{\alpha_0}{2}t}.$$

The part for $t \in [0, T]$ is easy from (3.1) and the continuity of h^ε . ■

Lemma 3.3 For any $x \in \mathbf{R}^d$, there exist constants $0 < C_1 < \infty$ and $\varepsilon_0 > 0$ (which may depend on x), such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $t > 1$,

$$\|q_t^\varepsilon(x, \cdot) - 1\|_\infty \leq 2C_0 C_1 e^{-\frac{\alpha_0}{2}(t-1)}.$$

The same is true for the time inverse process.

Proof. For any $y \in \mathbf{R}^d$, $q_t^\varepsilon(x, y) = Q_{t-1}^\varepsilon(q_1^\varepsilon(x, \cdot))(y)$, and $\int_{\mathbf{R}^d} q_1^\varepsilon(x, \cdot) d\pi^\varepsilon = 1$. So by Lemma 3.2,

$$\|q_t^\varepsilon(x, \cdot) - 1\|_\infty \leq 2C_0 e^{-\frac{\alpha_0}{2}(t-1)} \|q_1^\varepsilon(x, \cdot)\|_\infty.$$

But

$$\|q_1^\varepsilon(x, \cdot)\|_\infty \leq \frac{\sup h^\varepsilon}{\inf h^\varepsilon} e^{2\|x\|_\infty} \|p_1(x, \cdot)\|_\infty,$$

and $\frac{\sup h^\varepsilon}{\inf h^\varepsilon}$ is bounded for $\varepsilon \in [0, 1]$ from the continuity. This gives our assertion. ■

Lemma 3.4 For any $x \in \mathbf{R}^d$, there exist constants $K_x > 0$, $K_{x,f} > 0$, and $\varepsilon_0 > 0$, (where $K_{x,f}$ and ε_0 depend on f), such that for any $y \in \mathbf{R}^d$, any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and any $T \geq 2$,

$$\left| \Lambda_{T,x,y}(\varepsilon) - \frac{\varepsilon^2}{2} \int_{\mathbf{R}^d} f G_0 f d\pi \right| \leq K_x \left(\frac{\varepsilon \|f\|_\infty}{T} + \frac{\varepsilon^2 \|f\|_\infty^2}{\sqrt{T}} + \varepsilon^3 K_{x,f} \right).$$

Proof. The proof is similar to the one in Bolthausen-Deuschel-Tamura [2]. Define Q_t^{ef} , Q_x^{ef} , q_t^{ef} , and so on as before. The proof will be divided into several steps.

Let $A_{t_1, t_2} = \int_{t_1}^{t_2} f(X_s) ds$ and write $A_{0, t}$ as A_t . Let $E_{x, y}^{\varepsilon, T}[\cdot] = E^{P_{x, y}^{\varepsilon, T}}[\cdot | X_T = y]$. Note that

$$E_{x, y}^{\varepsilon, T}[F] = \frac{E_{x, y}^{0, T}[F e^{\varepsilon A_T}]}{E_{x, y}^{0, T}[e^{\varepsilon A_T}]}$$

if F is bounded and \mathcal{F}_T -measurable.

Lemma 3.5 Set $\phi_{T, x, y}(\varepsilon) = E_{x, y}^{0, T}[e^{\varepsilon A_T}]$. Then

$$\Lambda_{T, x, y}(\varepsilon) = \frac{1}{T} \log \phi_{T, x, y}(\varepsilon).$$

Also,

$$\begin{aligned} \Lambda_{T, x, y}'(\varepsilon) &= \frac{d}{d\varepsilon} \Lambda_{T, x, y}(\varepsilon) = \frac{1}{T} \frac{\phi_{T, x, y}'(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} = \frac{1}{T} E_{x, y}^{\varepsilon, T}[A_T], \\ \Lambda_{T, x, y}''(\varepsilon) &= \frac{1}{T} \left\{ \frac{\phi_{T, x, y}''(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} - \left(\frac{\phi_{T, x, y}'(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} \right)^2 \right\} = \frac{1}{T} E_{x, y}^{\varepsilon, T}[(A_T - E_{x, y}^{\varepsilon, T}[A_T])^2], \\ \Lambda_{T, x, y}'''(\varepsilon) &= \frac{1}{T} \left\{ \frac{\phi_{T, x, y}'''(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} - 3 \frac{\phi_{T, x, y}''(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} \frac{\phi_{T, x, y}'(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} + 2 \left(\frac{\phi_{T, x, y}'(\varepsilon)}{\phi_{T, x, y}(\varepsilon)} \right)^3 \right\} \\ &= \frac{1}{T} E_{x, y}^{\varepsilon, T}[(A_T - E_{x, y}^{\varepsilon, T}[A_T])^3]. \end{aligned}$$

Lemma 3.6 There exists a constant $K_x > 0$, such that for any $y \in \mathbf{R}^d$ and any $T \geq 2$,

$$|\Lambda_{T, x, y}'(0)| \leq \frac{K_x}{T} \|f\|_{\infty} \quad (3.2)$$

$$|\Lambda_{T, x, y}''(0) - \int_{\mathbf{R}^d} f \bar{G}_0 f d\pi| \leq \frac{K_x}{\sqrt{T}} \|f\|_{\infty}^2. \quad (3.3)$$

Proof. First, we have that for any $s \geq 1$, $p_s(x, y) = P_{s-1}^*(p_1(x, \cdot))(y)$, so $\|p_1(x, \cdot)\|_{\infty} < \infty$ accompanied with Lemma 2.7 applied to $\{P_t\}$ gives that

$$\sup_{s \geq 1} \|p_s(x, \cdot)\|_{\infty} < \infty.$$

Therefore, since $\int_{\mathbf{R}^d} p_s(x, \cdot) f d\pi = P_s f(x)$ for any $t > 0$, by applying Lemma 2.7 to $\{P_t\} = \{P_t^0\}$ and $\{P_t^*\}$, we have that for any $y \in \mathbf{R}^d$ and any $T \geq 1$,

$$\left| p_T(x, y) E^{P_x} \left[\int_1^{T-1} f(X_s) ds \mid X_T = y \right] \right|$$

$$\begin{aligned} &= \int_1^{T-1} P_{T-s}^*(p_s(x, \cdot))(y) ds \\ &\leq \int_1^{T-1} \|P_{T-s}^*(f p_s(x, \cdot))\|_{\infty} ds \\ &\leq \int_1^{T-1} |P_s f(x)| ds + C_0 \int_1^{T-1} e^{-\alpha_0(T-s)} ds \|f\|_{\infty} \sup_{1 \leq s \leq T-1} \|p_s(x, \cdot)\|_{\infty} \\ &\leq K_x^1 \|f\|_{\infty} \end{aligned}$$

for some constant $K_x^1 > 0$. Since $p_T(x, y)$, $y \in \mathbf{R}^d$, is strictly positive for any $x \in \mathbf{R}^d$, this gives that $E_{x, y}^{0, T}[A_{1, T-1}]$, $y \in \mathbf{R}^d$, is bounded. Also, by Lemma 3.5,

$$|\Lambda_{T, x, y}'(0)| = \left| \frac{1}{T} E_{x, y}^{0, T}[A_T] \right| \leq \frac{2\|f\|_{\infty}}{T} + \frac{1}{T} |E_{x, y}^{0, T}[A_{1, T-1}]|.$$

Therefore, (3.1) holds.

The proof for (3.2) is similar. By Lemma 3.5 and (3.2), we only need to show that

$$\left| \frac{1}{T} E_{x, y}^{0, T}[A_T^2] - \int_{\mathbf{R}^d} f \bar{G}_0 f d\pi \right| \leq \frac{K_x^2}{\sqrt{T}} \|f\|_{\infty} \quad (3.4)$$

for some constant $K_x^2 > 0$.

We have that

$$p_T(x, y) E_{x, y}^{0, T}[(\int_1^{T-1} f(X_s) ds)^2] = 2 \int_1^{T-1} dt \int_1^t ds P_{T-t}^*(f P_{t-s}^*(f p_s(x, \cdot)))(y).$$

By the first part of the proof, we have that

$$\sup_{1 \leq t \leq T-1} \int_1^t \|P_{t-s}^*(f p_s(x, \cdot))\|_{\infty} ds \leq K_x^1 \|f\|_{\infty}.$$

So

$$\begin{aligned} &\left| p_T(x, y) E_{x, y}^{0, T}[(\int_1^{T-1} f(X_s) ds)^2] - 2 \int_1^{T-1} dt \int_1^t ds \int_{\mathbf{R}^d} f P_{t-s}^*(f p_s(x, \cdot)) d\pi \right| \\ &\leq 2C_0 K_x^1 \int_1^{T-1} e^{-\alpha_0(T-t)} dt \|f\|_{\infty}^2 \leq K_x^2 \|f\|_{\infty}^2 \end{aligned} \quad (3.5)$$

for some constant $K_x^3 > 0$. Also, $\int_{\mathbf{R}^d} f P_{t-s}^*(f p_s(x, \cdot)) d\pi = P_s(f P_{t-s} f)(x)$, and $\int_{\mathbf{R}^d} f d\pi = 0$, so we have

$$\begin{aligned} &\left| 2 \int_1^{T-1} dt \int_1^t ds \left(\int_{\mathbf{R}^d} f P_{t-s}^*(f p_s(x, \cdot)) d\pi - \int_{\mathbf{R}^d} f P_{t-s} f d\pi \right) \right| \\ &\leq 2C_0^2 \|f\|_{\infty}^2 \int_1^{T-1} dt \int_1^t ds e^{-\alpha_0 s} e^{-\alpha_0(t-s)} \\ &\leq K_x^4 \|f\|_{\infty}^2. \end{aligned} \quad (3.6)$$

Moreover, since $G_0 f = \int_0^\infty P_t f dt$, and $|P_t f| \leq \|f\|_\infty C_0 e^{-\alpha_0 t}$ for any $t > 0$, we have that

$$\begin{aligned} & \left| \int_1^{T-1} dt \int_1^t ds \int_{\mathbf{R}^d} f P_{t-s} f d\pi - (T-2) \int_{\mathbf{R}^d} f G_0 f d\pi \right| \\ & \leq \int_1^{T-1} dt \int_{t-1}^\infty ds \left| \int_{\mathbf{R}^d} f P_s f d\pi \right| \\ & \leq C_0 \|f\|_\infty^2 \int_1^{T-1} dt \int_{t-1}^\infty e^{-\alpha_0 s} ds \leq K_x^5 \|f\|_\infty^2. \end{aligned} \quad (3.7)$$

But $\|p_T(x, \cdot) - 1\|_\infty \leq C_0 e^{-\alpha_0(T-1)} \|p_1(x, \cdot)\|_\infty$, so there exists a constant $K_x^6 > 0$ such that

$$\left| \frac{1}{T} E_{x,y}^{0,T} [(A_{1,T-1})^2] - \int_{\mathbf{R}^d} f \overline{G_0} f d\pi \right| \leq \frac{K_x^6}{T} \|f\|_\infty^2,$$

in particular, $\sup_{T>0} \frac{1}{T} E_{x,y}^{0,T} [(A_{1,T-1})^2] \leq K_x^6$. So

$$\begin{aligned} & \left| E_{x,y}^{0,T} [A_T^2] - E_{x,y}^{0,T} [(A_{1,T-1})^2] \right| \\ & \leq 2 \|f\|_\infty (2 E_{x,y}^{0,T} [A_{1,T-1}] + 2 \|f\|_\infty) \\ & \leq 4 \|f\|_\infty^2 + 4 \|f\|_\infty E_{x,y}^{0,T} [(A_{1,T-1})^2]^{1/2} \\ & \leq 4 \|f\|_\infty^2 + 4 \|f\|_\infty (T K_x^6 \|f\|_\infty^2)^{1/2} \\ & \leq K_x^7 \|f\|_\infty^2 \sqrt{T}. \end{aligned}$$

Therefore, (3.4) is true. This gives our assertion. ■

Lemma 3.7 For any $x \in \mathbf{R}^d$, there exist constants $K_x > 0$ and $\varepsilon_0 > 0$, such that for any $T \geq 2$, any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and any $y \in \mathbf{R}^d$,

$$|\Lambda_{T,x,y}'''(\varepsilon)| \leq K_x \|f\|_\infty^3.$$

Proof. Let $f^\varepsilon = f - f_{\mathbf{R}^d} f d\pi^\varepsilon$, and let $A_T^\varepsilon = \int_0^T f^\varepsilon(X_s) ds$. $\|f^\varepsilon\|_\infty \leq 2 \|f\|_\infty$, and

$$\Lambda_{T,x,y}'''(\varepsilon) = \frac{1}{T} E_{x,y}^{0,T} [(A_T^\varepsilon)^3] - 3 \Lambda_{T,x,y}''(\varepsilon) \cdot T \Lambda_{T,x,y}'(\varepsilon) - T^2 \Lambda_{T,x,y}'(\varepsilon).$$

By Lemma 3.2 and Lemma 3.3, we get in the same way as in the proof of Lemma 3.6 that there exists a K_x , such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $T > 2$,

$$|\Lambda_{T,x,y}'(\varepsilon)| \leq K_x \|f\|_\infty, \quad |\Lambda_{T,x,y}''(\varepsilon)| \leq K_x \|f\|_\infty^2.$$

So it is enough if we can show that there exists a $K_x > 0$, such that for any $T > 0$, any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $y \in \mathbf{R}^d$,

$$\left| \frac{1}{T} E_{x,y}^{0,T} [(A_T^\varepsilon)^3] \right| \leq K_x \|f\|_\infty^3.$$

This is true if

$$\left| \frac{1}{T} E_{x,y}^{0,T} \left[\left(\int_1^{T-1} f^\varepsilon(X_s) ds \right)^3 \right] \right| \leq K_x^8 \|f\|_\infty^3 \quad (3.8)$$

for some $K_x^8 > 0$.

By Lemma 3.2 and Lemma 3.3, doing in the same way as in the proof of Lemma 3.6, we have that for any $1 \leq s < t < u < T-1$,

$$\begin{aligned} & \|f^\varepsilon Q_{u-t}^\varepsilon (f^\varepsilon Q_{t-s}^{\varepsilon,*} (f^\varepsilon q_s^\varepsilon(x, \cdot)))\|_\infty \\ & \leq 4C_0^2 \|f^\varepsilon\|^3 e^{-\frac{3\alpha_0}{2}(u-s)} + 4C_0^2 \|f^\varepsilon\|^3 e^{-\frac{3\alpha_0}{2}t} \\ & \quad + 4C_0^2 C_x \|f^\varepsilon\|^3 e^{-\frac{3\alpha_0}{2}(u-1)} + 4C_0^2 \|f^\varepsilon\|^2 e^{-\frac{\alpha_0}{2}s} e^{-\frac{\alpha_0}{2}(u-t)}. \end{aligned}$$

Note that

$$\begin{aligned} & q_T^\varepsilon(x, y) E_{x,y}^{0,T} \left[\left(\int_1^{T-1} f^\varepsilon(X_s) ds \right)^3 \right] \\ & = 6 \int_1^{T-1} ds \int_s^{T-1} dt \int_t^{T-1} du Q_{T-u}^{\varepsilon,*} (f^\varepsilon Q_{u-t}^{\varepsilon,*} (f^\varepsilon Q_{t-s}^{\varepsilon,*} (f^\varepsilon q_s^\varepsilon(x, \cdot))))(y), \end{aligned}$$

and

$$\int f^\varepsilon Q_{u-t}^{\varepsilon,*} (f^\varepsilon Q_{t-s}^{\varepsilon,*} (f^\varepsilon q_s^\varepsilon(x, \cdot))) d\pi^\varepsilon = Q_s^\varepsilon (f^\varepsilon Q_{t-s}^\varepsilon (f^\varepsilon Q_{u-t}^\varepsilon (f^\varepsilon)))(x).$$

Therefore,

$$\begin{aligned} & \left| \frac{1}{T} q_T^\varepsilon(x, y) E_{x,y}^{0,T} \left[\left(\int_1^{T-1} f^\varepsilon(X_s) ds \right)^3 \right] \right| \\ & \quad - \frac{6}{T} \int_1^{T-1} ds \int_s^{T-1} dt \int_t^{T-1} du Q_s^\varepsilon (f^\varepsilon Q_{t-s}^\varepsilon (f^\varepsilon Q_{u-t}^\varepsilon (f^\varepsilon)))(x) \\ & \leq \frac{6}{T} \int_1^{T-1} ds \int_s^{T-1} dt \int_t^{T-1} du 2C_0 e^{-\frac{3\alpha_0}{2}(T-u)} \|f^\varepsilon Q_{u-t}^{\varepsilon,*} (f^\varepsilon Q_{t-s}^{\varepsilon,*} (f^\varepsilon q_s^\varepsilon(x, \cdot)))\|_\infty \\ & \leq K_x^9 \|f^\varepsilon\|_\infty^3 \end{aligned}$$

for some $K_x^9 > 0$.

Also, by Lemma 3.2, for any $1 \leq s < t < u < T-1$, by using the fact that for any $r > 0$, $Q_r^{\varepsilon,*}$ is the dual operator of $Q_r^\varepsilon \in L^2(d\pi^\varepsilon)$, we have that

$$\begin{aligned} & \|Q_s^\varepsilon (f^\varepsilon Q_{t-s}^\varepsilon (f^\varepsilon Q_{u-t}^\varepsilon (f^\varepsilon)))\|_\infty \\ & \leq (2C_0)^2 \|f^\varepsilon\|^3 e^{-\frac{3\alpha_0}{2}(u-s)} + 2(2C_0)^3 \|f^\varepsilon\|^3 e^{-\frac{3\alpha_0}{2}u} + (2C_0)^2 \|f^\varepsilon\|^3 e^{-\frac{\alpha_0}{2}s} e^{-\frac{\alpha_0}{2}(u-t)}, \end{aligned}$$

hence there exists a K_x^{10} , such that

$$\frac{6}{T} \int_1^{T-1} ds \int_s^{T-1} dt \int_t^{T-1} du Q_s^\varepsilon (f^\varepsilon Q_{t-s}^\varepsilon (f^\varepsilon Q_{u-t}^\varepsilon (f^\varepsilon)))(x) \leq K_x^{10} \|f\|_\infty^3.$$

These accompanied with Lemma 3.3 gives (3.8). ■

Proof of Lemma 3.4 By the mean value theorem, for any ε , there exists a $\tilde{\varepsilon} \in [-\varepsilon, \varepsilon]$, such that

$$\Lambda_{T,x,y}(\varepsilon) = \varepsilon \Lambda'_{T,x,y}(0) + \frac{\varepsilon^2}{2} \Lambda''_{T,x,y}(0) + \frac{\varepsilon^3}{3!} \Lambda'''_{T,x,y}(\tilde{\varepsilon}).$$

This with Lemma 3.5, Lemma 3.6, and Lemma 3.7 gives our assertion. ■

From this, we can get the following

COROLLARY 3.8

$$\Lambda(\varepsilon) = \frac{\varepsilon^2}{2} \int_{\mathbf{R}^d} f \overline{G_0} f d\pi + O(\varepsilon^3).$$

Proof. We see from Lemma 3.4 that

$$\Lambda_{T,x}(\varepsilon) \equiv \frac{1}{T} \log P_T^{\varepsilon f} 1(x) = \frac{1}{T} \log \int_{\mathbf{R}^d} e^{T \Lambda_{T,x,y}(\varepsilon)} p_T(x, y) \pi(dy)$$

satisfies the same inequality as $\Lambda_{T,x,y}(\varepsilon)$ there, i.e.,

$$\left| \Lambda_{T,x}(\varepsilon) - \frac{\varepsilon^2}{2} \int f \overline{G_0} f d\pi \right| \leq K_x \left(\frac{\varepsilon \|f\|_\infty}{T} + \frac{\varepsilon^2 \|f\|_\infty^2}{\sqrt{T}} + \varepsilon^3 \|f\|_\infty^3 K_x \right).$$

By Remark 5, $0 < \inf h^\varepsilon \leq \sup h^\varepsilon < \infty$. Also, for any $x \in \mathbf{R}^d$ and any $t > 0$,

$$P_t^{\varepsilon f} 1(x) \geq P_t^{\varepsilon f} \left(\frac{h^\varepsilon}{\sup h^\varepsilon} \right) (x) = \frac{1}{\sup h^\varepsilon} e^{\Lambda(\varepsilon)t} h^\varepsilon(x) \geq \inf h^\varepsilon e^{\Lambda(\varepsilon)t}.$$

Therefore, $\lim_{T \rightarrow \infty} \Lambda_{T,x}(\varepsilon) \geq \Lambda(\varepsilon)$. The opposite inequality can be shown in the same way. Therefore, $\lim_{T \rightarrow \infty} \Lambda_{T,x}(\varepsilon) = \Lambda(\varepsilon)$ for any $x \in \mathbf{R}^d$. This gives our assertion. ■

Lemma 3.9 (a) $h^\varepsilon = 1 + \varepsilon G_0 f + r_1(\varepsilon)$,

(b) $l^\varepsilon = 1 + \varepsilon \overline{G_0} f + r_2(\varepsilon)$,

(c) $\pi^\varepsilon = (1 + \varepsilon \overline{G_0} f + r_3(\varepsilon))\pi$, where $\|r_i(\varepsilon)\|_\infty = o(\varepsilon)$, $i = 1, 2, 3$,

(d) $I(\pi^\varepsilon) = \frac{\varepsilon^2}{2} (f, \overline{G_0} f)_\pi + o(\varepsilon^2)$.

Proof. From the definition of h^ε , we have that $P_t^{\varepsilon f} h^\varepsilon = e^{\Lambda(\varepsilon)t} h^\varepsilon$ for any $t > 0$, so

$$\lim_{t \rightarrow 0} \frac{1}{t} (P_t h^\varepsilon - h^\varepsilon) = \Lambda(\varepsilon) h^\varepsilon - \varepsilon f h^\varepsilon,$$

which implies that $h^\varepsilon - \int h^\varepsilon d\pi = G_0(\varepsilon f h^\varepsilon - \Lambda(\varepsilon) h^\varepsilon)$, hence

$$h^\varepsilon - \int_{\mathbf{R}^d} h^\varepsilon d\pi - \varepsilon G_0 f = -\varepsilon G_0(f(1 - h^\varepsilon)) - \Lambda(\varepsilon) G_0 h^\varepsilon, \quad (3.9)$$

denote the right hand side above by $q(\varepsilon)$.

From the continuity earned in Lemma 3.1,

$$h^\varepsilon = 1 + o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

So the continuity of G_0 in $C_b(\mathbf{R}^d)$ and Corollary 3.8 implies that $\|q(\varepsilon)\|_\infty = o(\varepsilon)$. Also, $\int_{\mathbf{R}^d} q(\varepsilon) d\pi = 0$, so we have from (3.9) that

$$1 = \int_{\mathbf{R}^d} (h^\varepsilon)^2 d\pi = \left(\int_{\mathbf{R}^d} h^\varepsilon d\pi \right)^2 + \varepsilon^2 \int_{\mathbf{R}^d} (G_0 f)^2 d\pi + o(\varepsilon^2).$$

Hence

$$\int_{\mathbf{R}^d} h^\varepsilon d\pi = 1 + o(\varepsilon),$$

and therefore, by (3.9) again,

$$h^\varepsilon = \int h^\varepsilon d\pi + \varepsilon G_0 f + q(\varepsilon) = 1 + \varepsilon G_0 f + o(\varepsilon).$$

The assertion with respect to l^ε can be gotten in the same way. The assertion for π^ε is obvious now since $d\pi^\varepsilon = h^\varepsilon l^\varepsilon d\pi$.

Finally, by Remark 6 and Lemma 2.18, $I(\pi^\varepsilon) = \int_{\mathbf{R}^d} \varphi d\pi^\varepsilon - \Lambda(\varphi)$ for any $\varphi \in C_b(\mathbf{R}^d)$, also, $\int_{\mathbf{R}^d} f d\pi = 0$, so we get from (c) and Corollary 3.8 that

$$\begin{aligned} I(\pi^\varepsilon) &= \varepsilon \int_{\mathbf{R}^d} f d\pi^\varepsilon - \Lambda(\varepsilon) \\ &= \varepsilon \int_{\mathbf{R}^d} f (1 + \varepsilon \overline{G_0} f + o(\varepsilon)) d\pi - \frac{\varepsilon^2}{2} \int_{\mathbf{R}^d} f \overline{G_0} f d\pi + o(\varepsilon^2) \\ &= \frac{\varepsilon^2}{2} (f, \overline{G_0} f)_\pi + o(\varepsilon^2). \end{aligned}$$

4 Preparations

In section 2 and section 3, we dealt with general Markov processes. we have constructed a family of probability measures $\{Q_\varphi\}$ for any $\varphi \in C_b(\mathbf{R}^d)$, showed that h^φ is bounded and strictly positive. Also, we showed that there exists a unique probability measure π^φ which is Q_φ^ε -invariant, and showed that $\|Q_\varphi^\varepsilon - (\cdot)_{\pi^\varphi}\|_{op} < C_\varphi e^{-\alpha_\varphi t}$ for some $C_\varphi, \alpha_\varphi > 0$, where $\|\cdot\|_{op}$ means the operator norm in $C_b(\mathbf{R}^d)$.

From this section on, let us go back to the situation described in section 1. By Lemma 2.2, we have that under A6,

$$\sup_{x \in \mathbf{R}^d} P_x(|X_t| \geq n) \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

for any $t > 0$. Also, the same is true for the time inverse diffusion process.

Notice that the assumptions H1, H2, H3 are satisfied in our situation. (c.f. Kusuoka-Stroock [9]).

Let ν_0 be the unique element of K as assumed in section 1, i.e., ν_0 maximizes $\Phi - I$. Let $\phi^{\nu_0} = \Phi^{(1)}(\nu_0; \cdot) - D\Phi(\nu_0)(\nu_0) + \Phi(\nu_0)$. $\phi^{\nu_0} \in C_b(\mathbf{R}^d)$ by the assumption A6, so by the discussion before, we can construct $\{Q_x^{\phi^{\nu_0}}\}_{x \in \mathbf{R}^d}$.

Lemma 4.1 $\pi^{\phi^{\nu_0}} = \nu_0$, i.e., ν_0 is the invariant probability measure of $\{Q_x^{\phi^{\nu_0}}\}$.

Proof. In the same way as in the proof of Lemma 2.11, we have that

$$I^{\phi^{\nu_0}}(\nu) = I(\nu) - \int_{\mathbf{R}^d} \phi^{\nu_0} d\nu + \Lambda(\phi^{\nu_0})$$

for any $\nu \in \varphi(\mathbf{R}^d)$. Since $\pi^{\phi^{\nu_0}}$ is the only invariant probability measure of $\{Q_x^{\phi^{\nu_0}}\}$, $I^{\phi^{\nu_0}}(\nu) = 0$, or equivalently ν minimize $I^{\phi^{\nu_0}}$, if and only if $\nu = \pi^{\phi^{\nu_0}}$.

But since ν_0 maximize $\Phi - I$ and I is convex, we have that for any $t \in (0, 1)$ and any $\nu \in \varphi(\mathbf{R}^d)$,

$$\begin{aligned} & \Phi(\nu_0) - I(\nu_0) \\ & \geq \Phi(t\nu + (1-t)\nu_0) - I(t\nu + (1-t)\nu_0) \\ & \geq \Phi(t\nu + (1-t)\nu_0) - tI(\nu) - (1-t)I(\nu_0), \end{aligned}$$

therefore,

$$\frac{\Phi(t\nu + (1-t)\nu_0) - \Phi(\nu_0)}{t} \leq I(\nu) - I(\nu_0).$$

Let $t \rightarrow 0$, the left hand side converges to $D\Phi(\nu_0)(\nu - \nu_0) = \int_{\mathbf{R}^d} \phi^{\nu_0} d\nu - \int_{\mathbf{R}^d} \phi^{\nu_0} d\nu_0$. So

$$I(\nu_0) - \int \phi^{\nu_0} d\nu_0 \leq I(\nu) - \int \phi^{\nu_0} d\nu$$

for any $\nu \in \varphi(\mathbf{R}^d)$. Therefore, ν_0 minimize $I^{\phi^{\nu_0}}$, and hence $I^{\phi^{\nu_0}}(\nu_0) = 0$, which implies that $\nu_0 = \pi^{\phi^{\nu_0}}$. ■

To simplify the notations, from now on, we will omit the superscript ϕ^{ν_0} , and write $Q_x^{\phi^{\nu_0}}$ as Q_x , $Q_t^{\phi^{\nu_0}}$ as Q_t , and so on, when there is no . Also, $h^{\phi^{\nu_0}}$ as h , $l^{\phi^{\nu_0}}$ as l , and so on.

Now, we can define the Green operator G corresponding to $\{Q_t\}$ by $Gf = \int_0^\infty (Q_t f - \int f d\nu_0) dt$. G is a bounded operator in $C_b(\mathbf{R}^d)$. Let G^* be the adjoint operator of G in $L^2(d\nu_0)$, and let $\bar{G} = G + G^*$. Let $\Gamma(f, g) = \int_{\mathbf{R}^d} f \bar{G} g d\nu_0$, $f, g \in C_b(\mathbf{R}^d)$. Then $\Gamma(f, f) \geq 0$, and $\Gamma(f, f) = 0$ if and only if $f = \text{constant}$. (c.f. Lemma 4.10). Define the equivalent relation \sim in $C_b(\mathbf{R}^d)$ by $f \sim g \Leftrightarrow f - g = \text{constant}$, and let $\bar{C}_b(\mathbf{R}^d) = C_b(\mathbf{R}^d)/\sim$, then Γ is a inner product in $\bar{C}_b(\mathbf{R}^d)$. Let $H = (\bar{C}_b(\mathbf{R}^d))^*$. H is a Hilbert space, and H can be regarded as a dense subspace of $\mathcal{M}_0(\mathbf{R}^d)$.

Since ν_0 maximize $\Phi - I = \Phi - I^{\phi^{\nu_0}} + \int_{\mathbf{R}^d} \phi^{\nu_0} d\nu - \Lambda(\phi^{\nu_0})$, by apply Lemma 3.9 to $\{Q_t\}$, we get the following

PROPOSITION 4.2 For any $f \in C_b(\mathbf{R}^d)$,

$$D^2\Phi(\nu_0)(\bar{G}f d\nu_0, \bar{G}f d\nu_0) \leq (f, \bar{G}f)_{L^2(d\nu_0)}.$$

Proposition 4.2 means that all of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are not greater than 1. Now, we are ready to give a precise formulation of the assumption A7.

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$$D^2\Phi(\nu_0)(\bar{G}f d\nu_0, \bar{G}f d\nu_0) < (f, \bar{G}f)_{L^2(d\nu_0)}$$

for any $f \in C_b(\mathbf{R}^d)$, i.e., all of the eigenvalues of $D^2\Phi(\nu_0)|_{H \times H}$ are less than 1.

Before show that Γ is an inner product in $\bar{C}_b(\mathbf{R}^d)$, we will first show some kind to Ito's formula for Gf with $f \in C_b(\mathbf{R}^d)$. We will need some preparations.

Let $B_n = \{x \in \mathbf{R}^d; |x| \leq n\}$. For any $\varphi \in C_b(\mathbf{R}^d)$, let G^φ be the Green operator with respect to $\{P_t^\varphi\}$, i.e.,

$$G^\varphi f = \int_0^\infty \left(e^{-\Lambda(\varphi)t} P_t^\varphi f - h^\varphi \int_{\mathbf{R}^d} \frac{f}{h^\varphi} d\pi^\varphi \right) dt, \quad f \in C(\mathbf{R}^d).$$

The integral in the right hand side converges by the virtue of Lemma 2.7.

Lemma 4.3 If $\varphi, \{\varphi_n\}_{n \in \mathbf{N}} \subset C_b(\mathbf{R}^d)$ is bounded, and $\varphi_n \rightarrow \varphi$ uniformly on B_m as $n \rightarrow \infty$ for any $m \in \mathbf{N}$, then $\|P_t^{\varphi_n} - P_t^\varphi\|_{op} \rightarrow 0$ as $n \rightarrow \infty$ for any $t > 0$.

Proof. Let K be a bound of φ_n , $n \in \mathbf{N}$ and φ . From Martin-Cameron formula,

$$\begin{aligned} & \|P_t^{\varphi_n} - P_t^\varphi\|_{op} \\ & \leq \sup_{x \in \mathbf{R}^d} E^{P_x} \left[e^{\int_0^t (\varphi_n - \varphi)(X_s) ds} - 1 \right] e^{\|\varphi\|_\infty} \\ & \leq \left(\frac{1}{t} \int_0^t \sup_{x \in \mathbf{R}^d} E^{P_x} \left[e^{\|\varphi_n - \varphi\|(X_s)} \right] ds - 1 \right) e^{\|\varphi\|_\infty}. \end{aligned}$$

But for any $s \geq 0$,

$$\begin{aligned} & \sup_{x \in \mathbf{R}^d} E^{P_s} \left[e^{t|\varphi_n - \varphi|(X_s)} \right] \\ & \leq e^{t\|\varphi_n - \varphi\|_{C_b(B_m)}} + e^{2Kt} \sup_{x \in \mathbf{R}^d} P_x(|X_s| \geq m) \end{aligned}$$

for any $m \in \mathbf{N}$. So from the conditions, we have that for any $s > 0$,

$$\sup_{x \in \mathbf{R}^d} E^{P_s} \left[e^{t|\varphi_n - \varphi|(X_s)} \right] \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$

Also, $\sup_{x \in \mathbf{R}^d} E^{P_s} \left[e^{t|\varphi_n - \varphi|(X_s)} \right]$ is bounded for $s \in [0, t]$, so by dominated convergence theorem, we get our assertion. ■

From the last lemma, we get in the same way as for Q_t^φ that

Lemma 4.4 Assume the same as in the last lemma. Then $\Lambda(\varphi_n) \rightarrow \Lambda(\varphi)$, $h^{\varphi_n} \rightarrow h^\varphi$, and $\|G^{\varphi_n} - G^\varphi\|_{op} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. By Peron-Frobenius argument, 1 is a simple eigenvalue of $e^{\Lambda(\varphi)t} P_t^\varphi$ with eigenvector h^φ , the only eigenvalue of it with positive eigenvector, and the absolute value of any other eigenvalue is smaller than 1. Use the last lemma, and we get our assertion in the same way as we did for $\{Q_t\}$. ■

Lemma 4.5 For any $\varphi \in C_b(\mathbf{R}^d)$, $G^\varphi f$ is differentiable for any $f \in C_b(\mathbf{R}^d)$. Also, if $\{\varphi_n\}_{n \in \mathbf{N}} \subset C^\infty(\mathbf{R}^d)$, is bounded in $C_b(\mathbf{R}^d)$, and converges uniformly on any B_m to a bounded continuous function $\varphi \in C_b(\mathbf{R}^d)$, then for any $m \in \mathbf{N}$ and any $u_m \in C_b(\mathbf{R}^d)$ with $\text{supp}(u_m) \subset B_m$ and $0 \leq u_m \leq 1$, we have that

$$\sup_{n \in \mathbf{N}} \|u_m \nabla G^{\varphi_n}\|_{op} < \infty.$$

Therefore, $\|(\nabla G^{\varphi_n})|_{B_m}\|_{op}$, $n \in \mathbf{N}$, is bounded for any $m \in \mathbf{N}$.

Proof. For any $m \in \mathbf{N}$, let u_m be given and fix it. From Kusuoka-Stroock [11] and Kusuoka-Stroock [12], we have that there exist constants $C, \gamma > 0$, such that for any $t \in (0, 1]$,

$$\left| \nabla_y \frac{P_t^*(x, dy)}{\pi(dy)} \right| \leq \frac{C}{t^{(d+1)/2}} e^{-\frac{\gamma(|x-y|^2 \wedge 1)}{2t}}$$

for any $x, y \in \mathbf{R}^d$ with $|y| \leq m$. So $P_t f$ is differentiable for any $f \in C(\mathbf{R}^d)$, and there exists a constant $C_m > 0$, such that

$$\|u_m \nabla P_t\|_{op} \leq \frac{C_m}{\sqrt{t}}, \quad \text{for any } t \in (0, 1].$$

Now, from the definition of P_t^φ ,

$$P_t^\varphi = P_t + \int_0^t P_s \varphi P_{t-s}^\varphi ds$$

for any $t > 0$. Actually, both of the two hand sides above have infinitesimal generator $L_0 + \varphi$. Therefore, P_t^φ is differentiable and

$$\nabla P_t^\varphi = \nabla P_t + \int_0^t \nabla P_s \varphi P_{t-s}^\varphi ds.$$

Also, $e^{-\Lambda(\varphi)t} \|P_t^\varphi\|_{op} \leq 1$, so

$$\|u_m \nabla P_t^\varphi\|_{op} \leq \frac{C_m}{\sqrt{t}} + \int_0^t \frac{C_m}{\sqrt{s}} \|\varphi\|_\infty e^{\|\varphi\|_\infty s} ds \leq \frac{C_m}{\sqrt{t}} + 2C_m \sqrt{t} \|\varphi\|_\infty e^{\|\varphi\|_\infty}$$

for any $t \in [0, 1]$. We get from

$$G_0 f = \int_0^1 (P_t f - \int_{\mathbf{R}^d} f d\pi) dt + P_1 G_0 f$$

that G_0 is differentiable and

$$\|u_m \nabla G_0\|_{op} \leq \int_0^1 \left(\frac{C_m}{\sqrt{t}} \right) dt + C_m \|G_0\|_{op} < \infty.$$

Also, $h^\varphi = G_0(\varphi h^\varphi - \Lambda(\varphi)h^\varphi)$. Therefore,

$$\|u_m \nabla h^\varphi\|_\infty = \|u_m \nabla G_0(\varphi h^\varphi - \Lambda(\varphi)h^\varphi)\|_\infty \leq \|u_m \nabla G_0\|_{op} 2\|\varphi\|_\infty \|h^\varphi\|_\infty.$$

Now,

$$G^\varphi f = \int_0^1 (P_t^\varphi f - h^\varphi \int_{\mathbf{R}^d} f d\pi) dt + P_1^\varphi G^\varphi f,$$

so we get that G^φ is differentiable and

$$\begin{aligned} & \|u_m \nabla G^\varphi\|_{op} \\ & \leq \int_0^1 \left(\frac{C_m}{\sqrt{t}} + 2C_m \sqrt{t} \|\varphi\|_\infty e^{\|\varphi\|_\infty} \right) dt + \|u_m \nabla G_0\|_{op} 2\|\varphi\|_\infty \|h^\varphi\|_\infty \\ & \quad + (C_m + 2C_m \|\varphi\|_\infty e^{\|\varphi\|_\infty}) \|G^\varphi\|_{op}. \end{aligned}$$

These accompanied with the last lemma gives our last assertion. ■

To make the mean of the notations easy to be understood, in this paper, we will sometimes use $\mathbf{u} \cdot ((a)\mathbf{v})$ to denote $\sum_{i,j=1}^d u_i a_{ij} v_j$ for any $\mathbf{u} = (u_1, \dots, u_d)$, $\mathbf{v} = (v_1, \dots, v_d) \in \mathbf{R}^d$. Therefore, $\nabla f \cdot ((a)\nabla g) = \sum_{i,j=1}^d \frac{\partial f}{\partial x_i} a_{ij} \frac{\partial g}{\partial x_j}$ for any $f, g \in C^1(\mathbf{R}^d)$. The same is used for $(\alpha_{ij})_{i,j=1}^d$, i.e., $\mathbf{u} \cdot ((\alpha)\mathbf{v}) = \sum_{i,j=1}^d u_i \alpha_{ij} v_j$.

Lemma 4.6 If $\{\varphi_n\}_{n \in \mathbf{N}}, \{f_n\}_{n \in \mathbf{N}} \subset C^\infty(\mathbf{R}^d)$ are bounded, and converges to $\varphi \in C_b(\mathbf{R}^d)$ and $f \in C_b(\mathbf{R}^d)$, respectively, uniformly on B_m as $n \rightarrow \infty$ for any $m \in \mathbf{N}$, then $\|G^{\varphi_n} f_n\|_{B_m} \|w_p(B_m)\|_{B_m}$, $n \in \mathbf{N}$, is bounded for any $p > 1$ and any $m \in \mathbf{N}$.

Proof. For any $m \in \mathbf{N}$, choose $u_m \in C^\infty(\mathbf{R}^d)$, such that $u_m \in [0, 1]$, $u = 1$ on $B_{m/2}$, and $\text{supp}(u_m) \subset B_m$.

$\varphi_n, f_n \in C^\infty(\mathbf{R}^d)$, so from Gilbarg-Trudinger [6], $G^{\varphi_n} f_n \in C^\infty(\mathbf{R}^d)$. For the sake of simplicity, denote h^{φ_n} by h_n . From the definition of G^{φ_n} ,

$$(L_0 + \varphi_n)G^{\varphi_n} f_n = -f_n + h_n \int_{\mathbf{R}^d} \frac{f_n}{h_n} d\pi^{\varphi_n} + \Lambda(\varphi)G^{\varphi_n} f_n.$$

So

$$\begin{aligned} (L_0 + \varphi_n)(u_m G^{\varphi_n} f_n) &= u_m(-f_n + h_n \int_{\mathbf{R}^d} \frac{f_n}{h_n} d\pi^{\varphi_n} + \Lambda(\varphi)G^{\varphi_n} f_n) \\ &\quad + (L_0 u_m)G^{\varphi_n} f_n + \nabla u_m \cdot ((a) \nabla G^{\varphi_n} f_n). \end{aligned}$$

$u_m G^{\varphi_n} f_n = 0$ on the boundary of B_m , and b is bounded in B_m , $\varphi_n, n \in \mathbf{N}$, is bounded, so by Friedman [5], there exists a constant $C_m > 0$, such that

$$\begin{aligned} &\|u_m G^{\varphi_n} f_n\|_{W_p^2(B_m)} \\ &\leq C_m (\|u_m G^{\varphi_n} f_n\|_{L^p(B_m)} + \|\nabla(u_m G^{\varphi_n} f_n)\|_{L^p(B_m)} \\ &\quad + \|(L_0 - b \cdot \nabla)u_m G^{\varphi_n} f_n\|_{L^p(B_m)}) \\ &\leq C_m (\|u_m G^{\varphi_n} f_n\|_{L^p(B_m)} + \|\nabla u_m\|_{L^p(B_m)} \|G^{\varphi_n} f_n\|_{L^p(B_m)} \\ &\quad + \|u_m\|_{L^p(B_m)} \|\nabla G^{\varphi_n} f_n\|_{L^p(B_m)} \\ &\quad + \|u_m(-f_n + h_n \int_{\mathbf{R}^d} \frac{f_n}{h_n} d\pi^{\varphi_n} + \Lambda(\varphi)G^{\varphi_n} f_n) \\ &\quad + (L_0 u_m)G^{\varphi_n} f_n + \nabla u_m \cdot ((a) \nabla G^{\varphi_n} f_n)\|_{L^p(B_m)}), \end{aligned}$$

which is bounded for $n \in \mathbf{N}$ by the last lemma. This shows that for any $m \in \mathbf{N}$, $\|G^{\varphi_n} f_n\|_{B_{m/2}} \|w_p(B_{m/2})\|_{B_{m/2}}$, $n \in \mathbf{N}$, is bounded. ■

Lemma 4.7 Let $\varphi \in C_b(\mathbf{R}^d)$. For any $f \in C(\mathbf{R}^d)$, let $g = -G^\varphi f$. Then $g \in C^1(\mathbf{R}^d)$. Also, for any $\psi \in C_b(\mathbf{R}^d)$, let $\{X_t\}$ be the diffusion process corresponding to $L_0 + \psi \cdot \nabla$, and let $B_t = \int_0^t \alpha^{-1}(X_s) dX_s - \int_0^t \alpha(X_s)^{-1} (b + \psi)(X_s) ds$, then $\{B_t\}_{t \geq 0}$ is a Brownian motion. Moreover,

$$\begin{aligned} g(X_t) &= g(X_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial g}{\partial x_i} (X_s) \alpha_{ij}(X_s) dB_s^j \\ &\quad + \int_0^t (f + (\varphi - \Lambda(\varphi))G^\varphi f - \psi \nabla G^\varphi f - h^\varphi \int_{\mathbf{R}^d} f l^\varphi d\pi)(X_s) ds. \end{aligned}$$

Proof. The fact that $\{B_t\}$ is a Brownian motion is easy (cf. Ikeda-Watanabe [7]).

Since $\varphi, f \in C_b(\mathbf{R}^d)$, we can find sequences of functions $\varphi_n, f_n \in C^\infty(\mathbf{R}^d)$, such that $\{\varphi_n\}_{n \in \mathbf{N}}$ and $\{f_n\}_{n \in \mathbf{N}}$ are bounded in $C_b(\mathbf{R}^d)$, and $\varphi_n \rightarrow \varphi, f_n \rightarrow f$ in $C_b(\mathbf{R}^d)$ for any $m \in \mathbf{N}$. So by the lemma before, for any $p > 1$, and any $m \in \mathbf{N}$, $\|G^{\varphi_n} f_n\|_{B_m} \|w_p(B_m)\|_{B_m}$, $n \in \mathbf{N}$, is bounded. Let $p > 1$ be large enough, then from Sobolev's Imbedding Theorem, $G^{\varphi_n} f_n$, $n \in \mathbf{N}$, is bounded in $C^1(B_m)$ for any $m \in \mathbf{N}$. Also, by the lemma before, $G^{\varphi_n} f_n \rightarrow G^\varphi f$ in $C_b(\mathbf{R}^d)$. Therefore, $G^\varphi f \in C^1(\mathbf{R}^d)$, and $G^{\varphi_n} f_n \rightarrow G^\varphi f$ in $C_b^1(B_m)$ as $n \rightarrow \infty$ for any $m \in \mathbf{N}$.

Now, $G^{\varphi_n} f_n \in C^\infty$ for any $n \in \mathbf{N}$. Let τ_m be the first exiting time of B_m , i.e., $\tau_m = \inf\{t; X_t \notin B_m\}$. We know by Ito's formula that for any $m, n \in \mathbf{N}$,

$$\begin{aligned} G^{\varphi_n}(f_n)(X_{t \wedge \tau_m}) &= G^{\varphi_n}(f_n)(X_0) + \sum_{i,j=1}^d \int_0^{t \wedge \tau_m} \frac{\partial G^{\varphi_n}(f_n)}{\partial x_i} (X_s) \alpha_{ij}(X_s) dB_s^j \\ &\quad + \int_0^{t \wedge \tau_m} (L_0 + \psi \nabla)(G^{\varphi_n}(f_n))(X_s) ds. \end{aligned} \quad (4.1)$$

$G^{\varphi_n}(f_n) \rightarrow G^\varphi f$ in $C_b^1(B_m)$ as $n \rightarrow \infty$, and

$$\begin{aligned} L_0 G^{\varphi_n}(f_n) &= -f_n - (\varphi_n - \Lambda(\varphi))G^{\varphi_n}(f_n) + h_n \int_{\mathbf{R}^d} f_n l_n d\pi \\ &\rightarrow -f - (\varphi - \Lambda(\varphi))G^\varphi(f) + h \int_{\mathbf{R}^d} f l d\pi. \end{aligned}$$

Therefore, take $n \rightarrow \infty$ in (4.1), and we get that

$$\begin{aligned} G^\varphi f(X_{t \wedge \tau_m}) &= G^\varphi f(X_0) + \sum_{i,j=1}^d \int_0^{t \wedge \tau_m} \frac{\partial G^\varphi f}{\partial x_i} (X_s) \alpha_{ij}(X_s) dB_s^j \\ &\quad - \int_0^{t \wedge \tau_m} (f + (\varphi - \Lambda(\varphi))G^\varphi f + h \int_{\mathbf{R}^d} f l d\pi - \psi \nabla G^\varphi f)(X_s) ds. \end{aligned}$$

This is true for any $m \in \mathbf{N}$. $\tau_m \rightarrow \infty$ as $m \rightarrow \infty$ almost surely. So we get our assertion by taking $m \rightarrow \infty$. ■

PROPOSITION 4.8 $L^\varphi = L_0 + \sum_{i=1}^d \left(\sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j} (\log h^\varphi) \right) \frac{\partial}{\partial x_i} = L_0 + (a_{ij})_{i,j=1}^d \frac{\nabla h^\varphi}{h^\varphi} \cdot \nabla$. In the same way, $L^{\varphi, \varphi^* \varphi} = L_0^{\varphi^*} + (a_{ij})_{i,j=1}^d \frac{\nabla \varphi^*}{\varphi^*} \cdot \nabla$, where $L_0^{\varphi^*}$ is the adjoint operator of L_0 in $L^2(d\pi)$.

Proof. We have from the definition of h^φ that

$$\lim_{t \rightarrow 0} \frac{P_t h^\varphi - h^\varphi}{t} = (\Lambda(\varphi) - \varphi) h^\varphi,$$

act G_0 on the both sides, and we get from the continuity of G_0 that $h^\varphi = G_0(\varphi h^\varphi - \Lambda(\varphi)h^\varphi)$, so from Lemma lemma.preparation.7, $h^\varphi \in C^1(\mathbf{R}^d)$, and let $\{X_t\}$ be the diffusion process corresponding to L_0 , then since $\int_{\mathbf{R}^d} h^\varphi d\pi = 0$,

$$h^\varphi(X_t) = h^\varphi(X_0) + \int_0^t \nabla h^\varphi(X_s) \cdot (\alpha)(X_s) dB_s + \int_0^t (\Lambda(\varphi)h^\varphi - \varphi h^\varphi)(X_s) ds.$$

Therefore, from Ito's formula,

$$\begin{aligned} & \log h^\varphi(X_t) - \log h^\varphi(X_0) \\ &= \int_0^t \frac{1}{h^\varphi(X_s)} dh^\varphi(X_s) - \frac{1}{2} \int_0^t \frac{1}{(h^\varphi)^2(X_s)} d[h^\varphi(X_s), h^\varphi(X_s)]_s \\ &= \sum_{i,j=1}^d \int_0^t h^\varphi(X_s)^{-1} \frac{\partial h^\varphi}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j + \int_0^t (\Lambda(\varphi) - \varphi(X_s)) ds \\ & \quad - \frac{1}{2} \int_0^t \frac{1}{(h^\varphi)^2(X_s)} \sum_{i,j=1}^d \frac{\partial h^\varphi}{\partial x_i}(X_s) a_{ij}(X_s) \frac{\partial h^\varphi}{\partial x_j}(X_s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & e^{-\Lambda(\varphi)t} \frac{h^\varphi(X_t)}{h^\varphi(X_0)} \exp\left(\int_0^t \varphi(X_s) ds\right) \\ &= \exp\left(\sum_{i,j=1}^d \int_0^t h^\varphi(X_s)^{-1} \frac{\partial h^\varphi}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j \right. \\ & \quad \left. - \frac{1}{2} \int_0^t \frac{1}{(h^\varphi)^2(X_s)} \sum_{i,j=1}^d \frac{\partial h^\varphi}{\partial x_i}(X_s) a_{ij}(X_s) \frac{\partial h^\varphi}{\partial x_j}(X_s) ds\right). \end{aligned}$$

The left hand side above is nothing but $\frac{dQ_{X_0}^\varphi(\omega)}{dP_{X_0}}|_{\mathcal{F}_t}$. This gives our first assertion. The second assertion can be shown in the same way, and will be omitted here. \blacksquare

Note that by Kusuoka-Liang [8], if $\varphi \in C_b^1(\mathbf{R}^d)$, then $h^\varphi \in C_b^1(\mathbf{R}^d)$, so the $(a_{ij})_{i,j=1}^d \frac{\nabla h^\varphi}{h^\varphi}$ appeared in the last proposition is in $C_b(E)$.

For $\varphi \in C_b(\mathbf{R}^d)$, let \tilde{G}^φ denote the Green operator corresponding to $\{Q_t^\varphi\}$, which has infinitesimal generator $L_0 + (a_{ij})_{i,j=1}^d \frac{\nabla h^\varphi}{h^\varphi} \cdot \nabla$ by Proposition 4.8. i.e.,

$$\tilde{G}^\varphi f = \int_0^\infty (Q_t^\varphi f - \int_{\mathbf{R}^d} f d\pi^\varphi) dt.$$

Then for any $f \in C_b(\mathbf{R}^d)$,

$$\tilde{G}^\varphi f = \frac{1}{h^\varphi} G^\varphi(h^\varphi f).$$

Lemma 4.9 Let $\varphi \in C_b(\mathbf{R}^d)$. For any $f \in C(\mathbf{R}^d)$ with $\int_{\mathbf{R}^d} f d\pi^\varphi = 0$, let $g = -\tilde{G}^\varphi f$. Then $g \in C^1(\mathbf{R}^d)$. Also, let $\{X_t\}$ be the diffusion process corresponding

to $\{Q_t^\varphi\}$, and let $B_t = \int_0^t \alpha^{-1}(X_s) dX_s - \int_0^t \alpha(X_s)^{-1} (b + (a_{ij})_{i,j=1}^d \frac{\nabla h^\varphi}{h^\varphi})(X_s) ds$, then $\{B_t\}_{t \geq 0}$ is a Brownian motion. Moreover,

$$g(X_t) = g(X_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial g}{\partial x_i}(X_s) \alpha_{ij} dB_s^j + \int_0^t f(X_s) ds.$$

Proof. The fact that $\{B_t\}$ is a Brownian is easy (c.f. Ikeda-Watanabe [7]).

We have from Lemma 4.7 and Proposition 4.8 and the assumption $\int_{\mathbf{R}^d} f h^\varphi d\pi = 0$ that

$$\begin{aligned} G^\varphi(h^\varphi f)(X_t) &= G^\varphi(h^\varphi f)(X_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial G^\varphi(h^\varphi f)}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j \\ & \quad - \int_0^t \left(h^\varphi f + (\varphi - \Lambda(\varphi)) G^\varphi(h^\varphi f) \right. \\ & \quad \left. - h^\varphi(X_s)^{-1} \sum_{i,j=1}^d \frac{\partial h^\varphi}{\partial x_i} a_{ij} \frac{\partial G^\varphi(h^\varphi f)}{\partial x_j} \right) (X_s) ds. \end{aligned}$$

Also, in the same reason, since $\int_{\mathbf{R}^d} h^\varphi d\pi = 0$, we have

$$\begin{aligned} h^\varphi(X_t) &= h^\varphi(X_0) + \sum_{i,j=1}^d \int_0^t \frac{\partial h^\varphi}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j \\ & \quad - \int_0^t \sum_{i,j=1}^d \left((\varphi - \Lambda(\varphi)) h^\varphi - (h^\varphi)^{-1} \frac{\partial h^\varphi}{\partial x_i} a_{ij} \frac{\partial h^\varphi}{\partial x_j} \right) (X_s) ds, \end{aligned}$$

so by Ito's formula applied to $Y_t = h^\varphi(X_t)$ with the function $\frac{1}{x}$,

$$\begin{aligned} & h^\varphi(X_t)^{-1} \\ &= h^\varphi(X_0)^{-1} - \int_0^t \frac{1}{(h^\varphi)^2(X_s)} d h^\varphi(X_s) + \int_0^t \frac{1}{(h^\varphi)^3(X_s)} d[h^\varphi(X_s), h^\varphi(X_s)]_s \\ &= h^\varphi(X_0)^{-1} - \sum_{i,j=1}^d \int_0^t \frac{1}{(h^\varphi(X_s))^2} \frac{\partial h^\varphi}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j \\ & \quad + \sum_{i,j=1}^d \int_0^t \left(\frac{\varphi - \Lambda(\varphi)}{h^\varphi} - \frac{1}{(h^\varphi)^3} \frac{\partial h^\varphi}{\partial x_i} a_{ij} \frac{\partial h^\varphi}{\partial x_j} + \frac{1}{(h^\varphi)^3} \frac{\partial h^\varphi}{\partial x_i} \frac{\partial h^\varphi}{\partial x_j} \right) (X_s) ds \\ &= h^\varphi(X_0)^{-1} - \sum_{i,j=1}^d \int_0^t \frac{1}{(h^\varphi)^2(X_s)} \frac{\partial h^\varphi}{\partial x_i}(X_s) \alpha_{ij}(X_s) dB_s^j + \int_0^t \left(\frac{\varphi - \Lambda(\varphi)}{h^\varphi} \right) (X_s) ds. \end{aligned}$$

Therefore, by Ito's formula again, we have that

$$\begin{aligned} & (h^\varphi)^{-1} G^\varphi(h^\varphi f)(X_t) \\ &= (h^\varphi)^{-1} G^\varphi(h^\varphi f)(X_0) + \int_0^t (h^\varphi(X_s))^{-1} dG^\varphi(h^\varphi f)(X_s) \end{aligned}$$

$$\begin{aligned}
& + \int_0^t G^\varphi(h^\varphi f)(X_s) d\left((h^\varphi(X_s))^{-1}\right) + \int_0^t d\left[(h^\varphi(X_s))^{-1}, G^\varphi(h^\varphi f)(X_s)\right]_s \\
& = \frac{1}{h^\varphi} G^\varphi(h^\varphi f)(X_0) + \sum_{i,j=1}^d \int_0^t \left(\frac{1}{h^\varphi} \frac{\partial G^\varphi(h^\varphi f)}{\partial x_i} - \frac{1}{(h^\varphi)^2} \frac{\partial h^\varphi}{\partial x_i} G^\varphi(h^\varphi f) \right) \alpha_{ij}(X_s) dB_s^j \\
& \quad - \int_0^t \frac{1}{h^\varphi} \left(h^\varphi f + (\varphi - \Lambda(\varphi)) G^\varphi(h^\varphi f) - \frac{1}{h^\varphi} \sum_{i,j=1}^d \frac{\partial h^\varphi}{\partial x_i} \frac{\partial G^\varphi(h^\varphi f)}{\partial x_j} \right) (X_s) ds \\
& \quad + \int_0^t G^\varphi(h^\varphi f) \frac{(\varphi - \Lambda(\varphi)) h^\varphi}{(h^\varphi)^2} (X_s) ds - \int_0^t \frac{1}{(h^\varphi)^2} \sum_{i,j=1}^d \frac{\partial G^\varphi(h^\varphi f)}{\partial x_i} a_{ij} \frac{\partial h^\varphi}{\partial x_j} (X_s) ds \\
& = (h^\varphi)^{-1} G^\varphi(h^\varphi f)(X_0) + \int_0^t \frac{\partial}{\partial x_i} ((h^\varphi)^{-1} G^\varphi(h^\varphi f)) (X_s) \alpha_{ij}(X_s) dB_s^j - \int_0^t f(X_s) ds.
\end{aligned}$$

Write $(h^\varphi)^{-1} G^\varphi(h^\varphi f)$ back to $\overline{G}f = -g$, and we get our assertion. ■

As before, let G be the Green operator corresponding to $\{Q_x^{\varphi^\nu}\}_{x \in \mathbf{R}^d}$, which is written as $\{Q_x\}_{x \in \mathbf{R}^d}$.

Lemma 4.10 $\Gamma(f, f) = \int_E \frac{\partial Gf}{\partial x_i} a_{ij} \frac{\partial Gf}{\partial x_j} d\nu_0$ for any $f \in C_b(\mathbf{R}^d)$.

Proof. Let $\{X_t\}$ be the diffusion process corresponding to L , the infinitesimal generator corresponding to $\{Q_x\}_{x \in \mathbf{R}^d}$. Then for any $f \in C_b(\mathbf{R}^d)$ with $\int_{\mathbf{R}^d} f d\nu_0 = 0$,

$$\begin{aligned}
\Gamma(f, f) &= \int_{\mathbf{R}^d} f \overline{G}f d\nu_0 = \lim_{T \rightarrow \infty} \text{var}^{Q_{\nu_0}} \left(\frac{1}{\sqrt{T}} \int_0^T f(X_s) ds \right) \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} E^{Q_{\nu_0}} \left[\left(\int_0^T f(X_s) ds \right)^2 \right].
\end{aligned}$$

But as showed before,

$$\int_0^T f(X_s) ds = 2Gf(X_0) - 2Gf(X_T) + 2 \sum_{i,j=1}^d \int_0^T \frac{\partial Gf}{\partial x_i} (X_s) \alpha_{ij}(X_s) dB_s^j.$$

Let $M \equiv \sum_{i,j=1}^d \int_0^t \frac{\partial Gf}{\partial x_i} (X_s) \alpha_{ij}(X_s) dB_s^j = \int_0^t (\nabla Gf)^t(\alpha)(X_s) dB_s$, where t means the transpose, and (α) stands for $(\alpha_{ij})_{i,j=1}^d$. $\{M_t\}_t$ is a locally bounded local martingale, hence $E^{Q_{\nu_0}}\{[M, M]_T\} < \infty$ for all $T \geq 0$, so M is a martingale. Also, $\frac{1}{T} \int_0^T \delta X_s ds \rightarrow \nu_0$ almost surely as $T \rightarrow \infty$, so

$$\begin{aligned}
\frac{1}{T} E^{Q_{\nu_0}}\{M_T^2\} &= \frac{1}{T} E^{Q_{\nu_0}}\{[M, M]_T\} = \frac{1}{T} E^{Q_{\nu_0}} \left[\int_0^T (\nabla Gf(X_s))^t(a)(X_s) \nabla Gf(X_s) ds \right] \\
&= \int_{\mathbf{R}^d} (\nabla Gf)^t \cdot (a) \nabla Gf d\nu_0.
\end{aligned}$$

Gf is bounded, so

$$\begin{aligned}
& \frac{1}{T} E^{Q_{\nu_0}} \left[(Gf(X_0) - Gf(X_T))^2 \right] \rightarrow 0, \\
& \frac{2}{T} E^{Q_{\nu_0}} \left[(Gf(X_0) - Gf(X_T)) \int_0^T (\nabla Gf)^t(\alpha)(X_s) dB_s \right] \rightarrow 0, \quad T \rightarrow \infty.
\end{aligned}$$

Therefore,

$$\frac{1}{T} E^{Q_{\nu_0}} \left[\left(\int_0^T f(X_s) ds \right)^2 \right] \rightarrow \int_{\mathbf{R}^d} (\nabla Gf)^t(a) \nabla Gf d\nu_0$$

as $T \rightarrow \infty$. This completes the proof of our assertion. ■

COROLLARY 4.11 $\Gamma(f, f) = 0$ if and only if f is a constant.

Proof. This is obvious by Lemma 4.10 and the assumption that $(a_{ij})_{i,j=1}^d$ is strictly positive definite. ■

PROPOSITION 4.12 For any bounded continuous symmetric function $V : \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}$, $\nabla_x \nabla_y G_x G_y^* V(x, y)$ is well-defined and is in $C(\mathbf{R}^d \times \mathbf{R}^d)$. Also, define $A_V : \mathcal{M}_0(\mathbf{R}^d) \times \mathcal{M}_0(\mathbf{R}^d) \rightarrow \mathbf{R}$, $A_V(R_1, R_2) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) R_1(dx) R_2(dy)$. Then $A_V|_{H \times H}$ is a Hilbert-Schmidt function, and

$$\begin{aligned}
\|A_V\|_{H \times H}^2 &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G_x G_y^* V}(x, y) \nu_0(dx) \nu_0(dy) \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \sum_{i,j=1}^d \sum_{k,l=1}^d \left(\frac{\partial}{\partial x_k} \frac{\partial}{\partial y_i} G_x G_y^* V(x, y) \right) a_{ij}(y) a_{kl}(x) \\
&\quad \left(\frac{\partial}{\partial x_l} \frac{\partial}{\partial y_j} G_x G_y^* V(x, y) \right) \nu_0(dx) \nu_0(dy) \\
&\equiv \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V(x, y))^t ((a_{kl})_{k,l=1}^d \otimes (a_{ij})_{i,j=1}^d)(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V(x, y)) \nu_0(dx) \nu_0(dy). \quad (4.2)
\end{aligned}$$

Proof. The fact that $\nabla_x \nabla_y G_x G_y^* V(x, y)$ is well-defined and is in $C(\mathbf{R}^d \times \mathbf{R}^d)$ is same as in Kusuoka-Liang [8]. We will give a proof of the equality.

The first one is easy. Let $\overline{G}e_m d\nu_0$, $m \in \mathbf{N}$, be an orthogonal normalized base set of H , then from the continuity of the operator \overline{G}_y , we have from Lemma 4.10 that

$$\begin{aligned}
& \|A_V\|_{H \times H}^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}e_m(x) \overline{G}e_n(y) \nu_0(dx) \nu_0(dy) \right)^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \left\| \int_{\mathbf{R}^d} V(x, \cdot) \overline{G} e_m(x) \nu_0(dx) \right\|_{H^*}^2 \\
&= \sum_{m=1}^{\infty} \left(\int_{\mathbf{R}^d} V(x, y) \overline{G} e_m(x) \nu_0(dx) \right) \left(\int_{\mathbf{R}^d} \overline{G}_y V(x, y) \overline{G} e_m(x) \nu_0(dx) \right) \nu_0(dy) \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy) \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy).
\end{aligned}$$

Now, let us show the second equality. As just showed, $A_V|_{H \times H}$ is a Hilbert-Schmidt operator, so there exist $a_n \in \mathbf{R}$ and $e_n \in \overline{C_b(\mathbf{R}^d)}$, $n \in \mathbf{N}$, such that $\Gamma(e_n, e_m) = \delta_{n,m}$, $n, m \in \mathbf{N}$, and

$$A_V(R_1, R_2) = \sum_{n=1}^{\infty} a_n \int_{\mathbf{R}^d} e_n dR_1 \int_{\mathbf{R}^d} e_n dR_2$$

for any $R_1, R_2 \in H$. For any $n \in \mathbf{N}$ with $a_n \neq 0$, we may assume that $e_n \in C_b(\mathbf{R}^d)$, with $\int_{\mathbf{R}^d} e_n d\nu_0 = 0$, $n \in \mathbf{N}$. Let $V_N(x, y) = \sum_{n=1}^N a_n e_n(x) e_n(y)$, $x, y \in \mathbf{R}^d$, $N \in \mathbf{N}$. From Lemma 4.7, the equality holds for any $N \in \mathbf{N}$, i.e.,

$$\begin{aligned}
&\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V_N(x, y) \overline{G}_x \overline{G}_y V_N(x, y) \nu_0(dx) \nu_0(dy) \\
&= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V_N(x, y))^t ((a_{kl})_{k,l=1}^d (a_{ij})_{i,j=1}^d)(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V_N(x, y)) \nu_0(dx) \nu_0(dy). \tag{4.3}
\end{aligned}$$

The left hand side above is $\sum_{n=1}^N a_n^2$, which certainly converges to

$$\sum_{n=1}^{\infty} a_n^2 = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy)$$

as $N \rightarrow \infty$. Also, since $(a_{ij})_{i,j=1}^d$ is strictly positive definite, $(a_{ij})_{i,j=1}^d \otimes (a_{kl})_{k,l=1}^d$ is also strictly positive definite, (write it as $(a) \otimes (a)$ for sake of simplicity). Moreover,

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V_N)^t (a \otimes a) (\nabla_x \nabla_y G_x G_y^* (V - V_N)) d\nu_0 dx d\nu_0 = 0$$

for any $N \in \mathbf{N}$, therefore, we see that

$$\begin{aligned}
&\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V(x, y))^t ((a) \otimes (a))(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V(x, y)) \nu_0(dx) \nu_0(dy) \\
&\geq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V_N(x, y))^t ((a) \otimes (a))(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V_N(x, y)) \nu_0(dx) \nu_0(dy)
\end{aligned}$$

for any $N \in \mathbf{N}$. Therefore, take the limit inf as $N \rightarrow \infty$ in (4.3), and we get that

$$\begin{aligned}
&\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy) \\
&\leq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V(x, y))^t ((a) \otimes (a))(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V(x, y)) \nu_0(dx) \nu_0(dy). \tag{4.4}
\end{aligned}$$

We show that the opposite inequality holds too. Now, if $V \in C_0(\mathbf{R}^d \times \mathbf{R}^d)$, i.e., V is a continuous function with compact support, then by Weierstrass-Stone Theorem, we can find a sequence of functions U_n , $n \in \mathbf{N}$, such that U_n has the form $\sum_{k=1}^{N_n} f_{k,n}(x) g_{k,n}(y)$, and $U_n \rightarrow V$ in $C_b(\mathbf{R}^d \times \mathbf{R}^d)$. So (4.3) is true for any U_n . Now, from the local boundedness of the operator $\nabla_x \nabla_y \overline{G}_x \overline{G}_y$ in $C_b(\mathbf{R}^d \times \mathbf{R}^d)$, which comes from the boundedness of $G : C_b(\mathbf{R}^d) \rightarrow W_p^2(B_n)$ for any $p > 1$ and any $n \in \mathbf{N}$, we have that

$$\nabla_x \nabla_y \overline{G}_x \overline{G}_y U_n(x, y) \rightarrow \nabla_x \nabla_y \overline{G}_x \overline{G}_y V(x, y)$$

at every $x, y \in \mathbf{R}^d$. Also, as mentioned before, $(a_{ij})_{i,j=1}^d \otimes (a_{kl})_{k,l=1}^d$ is positively defined, so from Fatou's Lemma, we see that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* U_n(x, y))^t ((a) \otimes (a))(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* U_n(x, y)) \nu_0(dx) \nu_0(dy) \\
&\geq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V(x, y))^t (a \otimes a)(x, y) \\
&\quad (\nabla_x \nabla_y G_x G_y^* V(x, y)) \nu_0(dx) \nu_0(dy).
\end{aligned}$$

Also, the left hand side is equal to $\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} U_n(x, y) \overline{G}_x \overline{G}_y U_n(x, y) \nu_0(dx) \nu_0(dy) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy)$. Combining these with (4.4), and we see that (4.2) is true if $V \in C_0(\mathbf{R}^d \times \mathbf{R}^d)$.

Now, we show the equality for general $V \in C_b(\mathbf{R}^d \times \mathbf{R}^d)$. Let ϕ_n be a function in $C_0(\mathbf{R}^d)$ with $\phi_n(x) = 1$ for $x \in B_n$, $0 \leq \phi_n \leq 1$, and $\text{supp}(\phi_n) \subset B_{n+1}$, $n \in \mathbf{N}$. Let $V_n(x, y) = V(x, y) \phi_n(x) \phi_n(y)$. Then $V_n \in C_0(\mathbf{R}^d \times \mathbf{R}^d)$, so (4.2) holds for any V_n , $n \in \mathbf{N}$. $V = V_n$ on $B_n \times B_n$, hence $V_n \rightarrow V$ uniformly on any compact sets. If we can show that

$$\nabla_x \nabla_y G_x G_y^* V_n(x, y) \rightarrow \nabla_x \nabla_y G_x G_y^* V(x, y) \tag{4.5}$$

at every $x, y \in \mathbf{R}^d$, then by apply dominated convergence theorem to the left hand side and Fatou's Lemma to the right hand side of (4.2) for V_n , we get that

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G}_x \overline{G}_y V(x, y) \nu_0(dx) \nu_0(dy)$$

$$\geq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y G_x G_y^* V(x, y))^t ((a_{kl})_{k,l=1}^d \otimes (a_{ij})_{i,j=1}^d)(x, y) \\ (\nabla_x \nabla_y G_x G_y^* V(x, y)) \nu_0(dx) \nu_0(dy).$$

This accompanied with (4.4) gives our assertion.

Now, only (4.5) is left. It is enough if we can show the following: if $f_n \rightarrow f$ in $C_b(B_m)$ as $n \rightarrow \infty$ for any $m \in \mathbf{N}$, and $\sup_{n \in \mathbf{N}} \|f_n\|_\infty \leq \|f\|_\infty < \infty$, then $\nabla G f_n \rightarrow \nabla G f$ in $C_b(B_m)$ as $n \rightarrow \infty$ for any $m \in \mathbf{N}$. Actually, for any $\varphi \in C_b(\mathbf{R}^d)$, from the proof of Lemma 4.7, we see that $G^\varepsilon f_n$, $n \in \mathbf{N}$, are in $W_p^2(B_m)$, and $\sup_{n \in \mathbf{N}} \|G^\varepsilon f_n\|_{W_p^2(B_m)} < \infty$. In particular, in our case, $\sup_{n \in \mathbf{N}} \|\nabla G f_n\|_{C_b(B_m)} < \infty$. On the other hand, as showed before, for $q_t(x, y) = \frac{Q_t(x, dy)}{\nu_0(dy)}$, there exist constants $C_m, \alpha_m > 0$, such that $|q_t(x, y) - 1| \leq C_m e^{-\alpha_m t}$ for any $x \in B_m$, any $y \in \mathbf{R}^d$ and any $t > 1$. For any $\varepsilon > 0$, we can find $t > 0$ small enough and $T > 0$ large enough, such that $t \|f\|_\infty + C_m \int_T^\infty e^{-\alpha_m s} ds \|f\|_\infty < \varepsilon/2$. Then, since $\sup_{x \in B_m} \int_t^T |q_s(x, \cdot) - 1| ds$ is bounded, hence integrable with respect to ν_0 , we can find a $N \in \mathbf{N}$ large enough, such that for any $n \geq N$,

$$\sup_{x \in B_m} |G f_n(x) - G f(x)| \leq \varepsilon/2 + \|f\|_\infty \int_{B_m^c} \left(\sup_{x \in B_m} \int_t^T |q_s(x, \cdot) - 1| ds \right) \nu_0(dy) \leq \varepsilon.$$

i.e., $G f_n \rightarrow G f$ in $C_b(B_m)$ as $n \rightarrow \infty$. Therefore, $G f_n \rightarrow G f$ in $C_b^1(B_m)$ as $n \rightarrow \infty$.

5 Lemmas

It is showed in Kusuoka-Liang [8] that $G : C_b^1(\mathbf{R}^d) \rightarrow C_b^1(\mathbf{R}^d)$ is bounded. So by doing in the same way as in Kusuoka-Liang [8], by Lemma 4.9 and Proposition 4.8, we get the following

Lemma 5.1 For any continuous symmetric function $V : \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}$, which satisfies $\int_{\mathbf{R}^d} V(x, y) \nu_0(dy) = 0$ for any $x \in \mathbf{R}^d$, and $\nabla_x V(x, y), \nabla_y V(x, y), \nabla_x \nabla_y V(x, y) \in C_b(\mathbf{R}^d \times \mathbf{R}^d)$, define a symmetric, bilinear, and continuous function $A_V : \varphi_0(\mathbf{R}^d) \times \varphi_0(\mathbf{R}^d) \rightarrow \mathbf{R}$ by $A_V(R_1, R_2) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) R_1(dx) R_2(dy)$. Suppose that all of the eigenvalues of $A_V|_{H \times H}$ are smaller than 1. Then there exists a constant $\varepsilon > 0$ small enough, such that

$$\sup_{T > 0} E^{Q_\varepsilon} \left[\exp \left(\frac{1}{2T} \int_0^T \int_0^T V(X_t, X_s) dt ds \right), \right. \\ \left. dist \left(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0 \right) < \varepsilon | X_T = y \right] < \infty. \quad (5.1)$$

The proof of it will be divided into several lemmas.

First we have the following about multiple integral, which has been showed in Kusuoka-Liang [8].

Lemma 5.2 Let $\{W_t\}_{t \geq 0}$ be a Brownian motion. Then for any $T > 0$, and any symmetric function $h(\cdot, \cdot) : [0, T] \times [0, T] \rightarrow \mathbf{R}$ that satisfies

$$\int_0^T \int_0^T h(t_1, t_2)^2 dt_1 dt_2 < \frac{1}{4},$$

we have

$$E^W \left[\exp \left(\int_0^T \int_0^T h(t_1, t_2) dW_{t_1} dW_{t_2} \right) \right] \leq \exp \left(\int_0^T \int_0^T h(t_1, t_2)^2 dt_1 dt_2 \right).$$

For the sake of simplicity, let $\rho_T \equiv \frac{1}{T} \int_0^T \delta_{X_t} dt$ and $A_\varepsilon \equiv \{dist(\frac{1}{T} \int_0^T \delta_{X_t} dt, \nu_0) < \varepsilon\}$.

Lemma 5.3 Let V be a function as in Lemma 5.1. Suppose that

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} V(x, y) \overline{G_x G_y} V(x, y) \nu_0(dx) \nu_0(dy) < \frac{1}{128}, \quad (5.2)$$

where $C_1, C_2 > 0$ are constants such that $C_1 I_d \leq (a_{ij})_{i,j=1}^d \leq C_2 I_d$. Then (5.1) holds.

Proof. We follow the same way as in Kusuoka-Liang [8].

First, for any $T > 1$,

$$\left| \frac{1}{T} \int_0^T \int_0^T V(X_s, X_t) ds dt - \frac{1}{T} \int_1^{T-1} \int_1^{T-1} V(X_s, X_t) ds dt \right| \\ \leq \frac{4T-4}{T} \|V\|_\infty \leq 4 \|V\|_\infty.$$

Also, $C \equiv \sup_{z \in \mathbf{R}^d} \{q(1, x, z), q^*(1, y, z)\} < \infty$, where $q^*(1, u, v) \equiv \frac{Q_1^*(u, dv)}{\nu_0(dv)} \in C(\mathbf{R}^d \times \mathbf{R}^d)$ and $q^*(1, u, v) > 0$. Actually,

$$q_1(x, z) = \frac{1}{h(x)l(z)} p_1^{\phi^{v_0}}(x, z) e^{-\lambda},$$

so by Lemma 2.4, $\sup_{z \in \mathbf{R}^d} \{q(1, x, z)\} < \infty$. The one with respect to the time reserve one is the same.

So we have that for any $A \in \mathcal{F}_T$,

$$E^{Q_\varepsilon} \left[\exp \left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt \right), A | X_T = y \right] \\ \leq E^{Q_{v_0}} \left[q(1, x, X_1) q^*(1, y, X_{T-1}) \cdot \exp \left(\frac{1}{T} \int_1^{T-1} \int_1^{T-1} V(X_t, X_s) ds dt + 4 \|V\|_\infty \right), A \right] \\ \leq C^2 e^{8 \|V\|_\infty} E^{Q_{v_0}} \left[\exp \left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt \right), A \right].$$

Since ν_0 is the invariant measure of (Q_x) as mentioned before, $(X_{T-t})_{t=0}^T$ under (Q_{ν_0}) is still a diffusion process for any $T > 0$, with the infinitesimal generator $L^{\nu_0} = L_0^{\nu_0} + (a_{ij})_{i,j=1}^d \frac{\nabla_i \nabla_j}{T}$. Let $U_1(x, y) \equiv -(G_x V)(x, y)$ and $U(x, y) \equiv -(G_x^* U_1)(x, y)$. By condition, $\int_{\mathbf{R}^d} V(x, y) \nu_0(dy) = 0$ for any $x \in \mathbf{R}^d$, so from the conditions and the continuity of G , we get from Proposition 4.12 that $\nabla_x \nabla_y U$ exists, is continuous, and

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \nabla_x \nabla_y U(x, y) \cdot ((a) \otimes (a))(x, y) \nabla_x \nabla_y U(x, y) \nu_0(dx) \nu_0(dy) < \frac{1}{4} \frac{C_1}{32C_2}.$$

Also, from the conditions, we get by Kusuoka-Liang [9] that there exists a constant $\delta > 0$, such that for any $y \in \mathbf{R}^d$,

$$\|\nabla_x U_1(\cdot, y)\|_{\infty} = \|-\nabla_x G_x V(\cdot, y)\|_{\infty} \leq \delta (\|V(\cdot, y)\|_{\infty} + \|\nabla_x V(\cdot, y)\|_{\infty}),$$

which is bounded for $y \in \mathbf{R}^d$. In the same way, from the continuity of G ,

$$\begin{aligned} \|\nabla_x \nabla_y U(\cdot, \cdot)\|_{\infty} &= \|\nabla_y G_y \nabla_x G_x V(\cdot, \cdot)\|_{\infty} \\ &\leq \delta \sup_{x \in \mathbf{R}^d} (\|\nabla_x G_x V(x, \cdot)\|_{\infty} + \|\nabla_y G_y \nabla_x G_x V(x, \cdot)\|_{\infty}) \\ &\leq \delta^2 (\|V(\cdot, \cdot)\|_{\infty} + \|\nabla_x V(\cdot, \cdot)\|_{\infty} + \|\nabla_y V(x, \cdot)\|_{\infty} + \|\nabla_y \nabla_x V(x, \cdot)\|_{\infty}) \\ &< \infty. \end{aligned}$$

i.e., both $\nabla_x U_1$ and $\nabla_x \nabla_y U$ are continuous and bounded in $\mathbf{R}^d \times \mathbf{R}^d$. The same can be said about $\nabla_x U$.

From the boundedness of $\nabla_x \nabla_y U(x, y)$ and $(a_{ij})_{i,j=1}^d$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$,

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \nabla_x \nabla_y U(x, y) \cdot ((a) \otimes (a))(x, y) \nabla_x \nabla_y U(x, y) p_T(dx) p_T(dy) < \frac{1}{4} \frac{C_1}{32C_2} \quad \text{on } A_{\varepsilon}.$$

From the definition of U_1 and Lemma 4.9, for any $T, t > 0$ with $T > t$,

$$U_1(X_T, X_t) = U_1(X_t, X_t) + \int_t^T (\nabla_x U_1(X_s, X_t))^t (\alpha)(X_s) dB_s + \int_t^T V(X_s, X_t) ds,$$

where $(B_t)_{t \geq 0}$ is the Brownian motion defined there. Therefore, from the symmetry of V ,

$$\begin{aligned} &\frac{1}{T} \int_0^T \int_0^T V(X_s, X_t) ds dt = \frac{2}{T} \int_0^T \int_t^T V(X_s, X_t) ds dt \\ &= \frac{2}{T} \left(\int_0^T (U_1(X_T, X_t) - U_1(X_t, X_t)) dt \right) \\ &\quad - \frac{2}{T} \int_0^T dt \int_t^T (\nabla_x U_1(X_s, X_t))^t (\alpha)(X_s) dB_s. \end{aligned}$$

Here, $\|U_1\|_{\infty} < \infty$ from the boundedness of the operator G , and the second term is equal to $-\frac{2}{T} \int_0^T \left(\int_0^s \nabla_x U_1(X_s, X_t) \cdot (\alpha)(X_s) dt \right) dB_s$ by stochastic Fubini's theorem (c.f. Ikeda-Watanabe [7, Lemma 3.4.1]), which is bounded for any $T > 0$, hence a continuous Q_{ν_0} -martingale. So

$$\begin{aligned} &E^{Q_{\nu_0}} \left[\exp \left(\frac{1}{T} \int_0^T \int_0^T V(X_t, X_s) ds dt \right), A_{\varepsilon} \right] \\ &\leq \exp(4\|U_1\|_{\infty}) \cdot E^{Q_{\nu_0}} \left[\exp \left(-\frac{2}{T} \int_0^T \left(\int_0^s (\nabla_x U_1(X_s, X_t))^t (\alpha)(X_s) dt \right) dB_s \right), A_{\varepsilon} \right] \\ &\leq \exp(4\|U_1\|_{\infty}) \cdot E^{Q_{\nu_0}} \left[\exp \left(2 \int_0^T \left| \frac{2}{T} \int_0^s (\nabla_x U_1(X_s, X_t))^t (\alpha)(X_s) dt \right|^2 ds \right), A_{\varepsilon} \right]^{1/2} \\ &= \exp(4\|U_1\|_{\infty}) \cdot E^{Q_{\nu_0}} \left[\exp \left(2 \int_0^T \left(\frac{2}{T} \int_0^s \nabla_x U_1(X_s, X_t) dt \right)^t (a)(X_s) \left(\frac{2}{T} \int_0^s \nabla_x U_1(X_s, X_t) dt \right) ds, A_{\varepsilon} \right) \right]^{1/2}. \end{aligned}$$

$0 < C_1 I_d \leq (a_{ij})_{i,j=1}^d \leq C_2 I_d$, so it is sufficient to show that

$$\sup_{T>0} E^{Q_{\nu_0}} \left[\exp \left(\left(\frac{8C_2}{T^2} \int_0^T ds \int_0^s \nabla_x U_1(X_s, X_t) dt \right)^2 \right), A_{\varepsilon} \right] < \infty$$

for some $\varepsilon > 0$. Since $(X_{T-t})_{t=0}^T$ under Q_{ν_0} is a diffusion process for any $T > 0$, we have by Lemma 4.9 again and the definition of U that $\tilde{B}_t^T \equiv X_{T-t} - X_T - \int_0^t (b^* + (a_{ij})_{i,j=1}^d \frac{\nabla_i \nabla_j}{T} (X_{T-s})) ds$, $t \in [0, T]$, is a Brownian motion, and for any $s' \in (0, T)$,

$$\begin{aligned} \nabla_x U(X_{T-s'}, X_0) &= \nabla_x U(X_{T-s'}, X_{T-s'}) + \int_{s'}^T (\nabla_y \nabla_x U(X_{T-s'}, X_{T-t}))^t (\alpha)(X_{T-t}) d\tilde{B}_t^T \\ &\quad + \int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'}) dt'. \end{aligned}$$

So we have

$$\begin{aligned} &\frac{1}{T^2} \int_0^T ds \int_0^s \nabla_x U_1(X_s, X_t) dt \Big|^2 \\ &= \frac{1}{T^2} \int_0^T ds \int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'}) dt' \Big|^2 \\ &\leq \frac{2}{T^2} \int_0^T |\nabla_x U(X_{T-s'}, X_0) - \nabla_x U(X_{T-s'}, X_{T-t'})|^2 ds' \\ &\quad + \frac{2}{T^2} \int_0^T \int_{s'}^T |(\nabla_y \nabla_x U(X_{T-s'}, X_{T-t'}))^t (\alpha)(X_{T-t'}) d\tilde{B}_t^T|^2 ds'. \end{aligned}$$

Here, the first term is bounded as claimed before. So it is sufficient to show that for some $\varepsilon > 0$ small enough,

$$\sup_{T>0} E^{Q_{\nu_0}} \left[\exp \left(\frac{16C_2}{T^2} \int_0^T \int_{s'}^T |(\nabla_y \nabla_x U(X_{T-s'}, X_{T-t'}))^t (\alpha)(X_{T-t'}) d\tilde{B}_t^T|^2 ds' \right), A_{\varepsilon} \right] < \infty.$$

Let W_t be another d -dimension Brownian motion which is independent to $\{X_t\}_{t \in [0, \infty)}$. Let $g(t, s) \equiv \nabla_y \nabla_x U(X_{T-t}, X_{T-s})$, then

$$\begin{aligned} & E^{Q_{v_0}} \left[\exp \left(\frac{16C_2}{T^2} \int_0^T \int_t^T (\nabla_y \nabla_x U(X_{T-t}, X_{T-s}))^t (\alpha)(X_{T-s}) d\tilde{B}_s^T)^2 dt \right), A_\varepsilon \right] \\ &= E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{4\sqrt{2}C_2}{T} \int_0^T \left(\int_t^T g(t, s) \right)^t (\alpha)(X_{T-s}) d\tilde{B}_s^T \right), A_\varepsilon \right] \right] \\ &= E^W \left[E^{Q_{v_0}} \left[\exp \left(\frac{4\sqrt{2}C_2}{T} \int_0^T \left(\int_0^s g(t, s)^t (\alpha)(X_{T-s})^t dW_t \right) d\tilde{B}_s^T \right), A_\varepsilon \right] \right] \\ &\leq E^W \left[E^{Q_{v_0}} \left[\exp \left(\frac{64C_2}{T^2} \int_0^T \int_0^s \left(g(t, s)^t (\alpha)(X_{T-s})^t dW_t \right)^2 ds \right), A_\varepsilon \right] \right]^{1/2} \\ &= E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64C_2}{T^2} \int_0^T \left(\int_0^s g(t, s) dW_t \right)^t (a)(X_{T-s}) \left(\int_0^s g(t, s) dW_t \right) ds \right), A_\varepsilon \right] \right]^{1/2} \\ &\leq E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64C_2^2}{T^2} \int_0^T \int_0^s g(t, s) dW_t \right)^2 ds \right), A_\varepsilon \right]^{1/2}. \end{aligned}$$

Here,

$$\begin{aligned} & \frac{1}{T^2} \int_0^T \int_0^s g(t, s) dW_t \Big|^2 ds \\ &= \frac{1}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \\ & \quad + \frac{1}{T^2} \int_0^T \left(\int_t^T |g(t, s)|^2 ds \right) dt. \end{aligned}$$

The second term is bounded. So we only need to show that

$$\sup_{T>0} E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64C_2^2}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \right), A_\varepsilon \right] \right] < \infty.$$

On the other hand, as shown before,

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y U(x, y))^t (a)(x, y) \nabla_x \nabla_y U(x, y) \rho_T(dx) \rho_T(dy) < \frac{1}{4} \cdot \frac{C_1}{32C_2}$$

on A_ε , so

$$\begin{aligned} & \frac{64^2 C_2^2}{T^4} \int_0^T \int_0^T dt_1 dt_2 \left\| \int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right\|^2 \\ & \leq \frac{64^2 C_2^2}{T^4} \int_0^T \int_0^T dt_1 \int_0^T dt_2 \left(\int_{t_1}^T \|g(t_1, s)\|^2 ds \right) \left(\int_{t_2}^T \|g(t_2, s)\|^2 ds \right) \end{aligned}$$

$$\begin{aligned} &= (64C_2^2)^2 \left\{ \frac{1}{T^2} \int_0^T dt \left(\int_t^T \|g(t, s)\|^2 ds \right) \right\}^2 \\ &\leq \left\{ \frac{64C_2}{T^2} \int_0^T \int_0^T \|g(t, s)\|^2 dt ds \right\}^2 \\ &= \left\{ 64C_2^2 \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \|\nabla_x \nabla_y U(x, y)\|^2 \rho_T(dx) \rho_T(dy) \right\}^2 \\ &\leq \left\{ \frac{64C_2^2}{C_1^2} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla_x \nabla_y U(x, y))^t (a)(x, y) \nabla_x \nabla_y U(x, y) \rho_T(dx) \rho_T(dy) \right\}^2 \\ &< \frac{64^2 C_2^2}{C_1^2} \cdot \left(\frac{1}{4} \cdot \frac{C_1}{32C_2} \right)^2 = \frac{1}{4} \quad \text{on } A_\varepsilon. \end{aligned} \quad (5.3)$$

So from Lemma 5.2, we have

$$\begin{aligned} & E^{Q_{v_0}} \left[E^W \left[\exp \left(\frac{64C_2^2}{T^2} \int_0^T \int_0^T \left(\int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right) dW_{t_1} dW_{t_2} \right), A_\varepsilon \right] \right] \\ &\leq E^{Q_{v_0}} \left[\exp \left(\frac{64^2 C_2^2}{T^4} \int_0^T \int_0^T dt_1 dt_2 \left\| \int_{t_1 \vee t_2}^T g(t_1, s) \otimes g(t_2, s) ds \right\|^2 \right), A_\varepsilon \right] < e^{\frac{1}{4}}. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 5.4 For any $e \in C_b^1(\mathbf{R}^d)$ with $\int_{\mathbf{R}^d} e dv_0 = 0$ and $\Gamma(e, e) = 1$, and any $a < 1$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon < \varepsilon_0$,

$$\sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\frac{a}{2T} \left(\int_0^T e(X_t) dt \right)^2 \right), A_\varepsilon \mid X_T = y \right] < \infty. \quad (5.4)$$

Proof. As in the proof of last Lemma, we only need to show the assertion without the condition $X_0 = x$ and $X_T = y$, i.e., it is sufficient if we prove

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{a}{2T} \left(\int_0^T e(X_t) dt \right)^2 \right), A_\varepsilon \right] < \infty.$$

Also, as there, since $\int_{\mathbf{R}^d} e(x) \nu_0(dx) = 0$, by Lemma 4.9, the function u defined by $u \equiv -Ge$ is in $C^1(\mathbf{R}^d)$, and

$$u(X_T) - u(X_0) = \sum_{i,j=1}^d \int_0^T \frac{\partial u}{\partial x_i}(X_t) \alpha_{ij}(X_t) dB_t^j + \int_0^T e(X_t) dt.$$

So from the boundedness of u , it is sufficient if

$$\sup_{T>0} E^{Q_{v_0}} \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \left(\int_0^T (\nabla u(X_t))^t (\alpha)(X_t) dB_t \right)^2 \right), A_\varepsilon \right] < \infty$$

for $\varepsilon > 0$ small enough. Choose and fix a constant $\delta \in (0, \frac{1}{a} - 1)$ first. Since

$$\int_{\mathbf{R}^d} (\nabla u(x))^t (a)(x) \nabla u(x) \nu_0(dx) = \|e\|_{H^*}^2 = 1,$$

and $\nabla u(x)$ is bounded on \mathbf{R}^d , there exists an $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$,

$$\int_{\mathbf{R}^d} (\nabla u(x))^t (a)(x) \nabla u(x) \rho_T(dx) \leq 1 + \delta, \quad \text{on } A_\varepsilon.$$

So by Ikeda-Watanabe [7, Theorem II.7.2], there exists a standard Brownian motion \tilde{B} , such that

$$\begin{aligned} & \left(\int_0^T (\nabla u(X_t))^t (\alpha)(X_t) dB_t \right)^2 \\ &= \left(\tilde{B} \left(\int_0^T (\nabla u(X_t))^t (\alpha)(X_t) dB_t, \int_0^T (\nabla u(X_t))^t (\alpha)(X_t) dB_t \right)_T \right)^2 \\ &= \tilde{B} \left(\int_0^T (\nabla u(X_t))^t (a)(X_t) \nabla u(X_t) dt \right)^2 \\ &= \tilde{B} \left(T \cdot \int_{\mathbf{R}^d} (\nabla u(x))^t (a)(x) \nabla u(x) \rho_T(dx) \right)^2 \\ &\leq \sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)|^2 \quad \text{on } A_\varepsilon. \end{aligned}$$

By the reflection principle, for any $T_0 > 0$ and any x ,

$$P \left(\sup_{0 \leq t \leq T_0} |\tilde{B}(t)| \geq x \right) \leq 2P \left(\sup_{0 \leq t \leq T_0} \tilde{B}(t) \geq x \right) = 2P(|\tilde{B}(T_0)| \geq x).$$

Therefore, since $\delta \in (0, \frac{1}{a} - 1)$, we have

$$\begin{aligned} & \sup_{T>0} E^{Q_0} \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \left(\int_0^T (\nabla u(X_t))^t (\alpha)(X_t) dB_t \right)^2 \right), A_\varepsilon \right] \\ &\leq \sup_{T>0} E \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} \sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)|^2 \right) \right] \\ &= \int_0^\infty P \left(\sup_{0 \leq t \leq (1+\delta)T} |\tilde{B}(t)| \geq x \right) d(e^{\frac{ax^2}{2T}} + 1) \\ &\leq 2 \sup_{T>0} E \left[\exp \left(\frac{a}{2} \cdot \frac{1}{T} |\tilde{B}((1+\delta)T)|^2 \right) \right] - 1 \\ &= \frac{2}{\sqrt{1-a(1+\delta)}} - 1 < \infty. \end{aligned}$$

This completes the proof of the lemma.

Now, we are ready to prove Lemma 5.1

Proof of Lemma 5.1 By Proposition 4.12, $A_V|_{H \times H}$ is a Hilbert-Schmidt type function. Combining this with the condition, we see that the maximum of its eigenvalues, say a_0 , is also smaller than 1. Choose and fix a $p > 1$ such that $a_0 p < 1$.

Write the eigenvalues of $A_V|_{H \times H}$ as $\{a_n\}_{n \in \mathbf{N}}$ with $|a_1| \geq |a_2| \geq |a_3| \geq \dots$, and the corresponding eigenvectors as $\{\bar{G}e_m d\nu_0\}_{m=1}^\infty$ with $\int_{\mathbf{R}^d} e_m(x) \bar{G}e_n(x) \nu_0(dx) = \delta_{mn}$. Then $A_V(\bar{G}e_m d\nu_0, R) = a_m \int_{\mathbf{R}^d} e_m(x) R(dx)$ for any $R \in \mathcal{M}_0(\mathbf{R}^d)$. So for any $m \in \mathbf{N}$ with $a_m \neq 0$, from the conditions with respect to V , we can assume that $e_m \in \bar{C}_b^1(\mathbf{R}^d)$.

Let q be the dual number of $p > 1$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Since $A_V|_{H \times H}$ is a Hilbert-Schmidt function as claimed, there exists a $N \in \mathbf{N}$ large enough such that $\sum_{i=N+1}^\infty q^2 a_i^2 < \frac{1}{128}$. Apply Lemma 5.3 to

$$V_1(x, y) := q \left(V(x, y) - \sum_{i=1}^N a_i e_i(x) \cdot e_i(y) \right), \quad x, y \in \mathbf{R}^d,$$

and use Hölder's inequality, so it is sufficient if

$$\sup_{T>0} E^{Q_2} \left[\exp \left(\sum_{i=1}^N \frac{p}{2T} \int_0^T \int_0^T a_i e_i(X_t) e_i(X_s) ds dt \right), A_\varepsilon | X_T = y \right] < \infty$$

for $\varepsilon > 0$ small enough.

Obviously, we can assume that $a_1, \dots, a_N \geq 0$, as if not, we can just omit the term corresponding to it. As in Kusuoka-Tamura [13], in general, we have that for any $\varepsilon_1 > 0$, there exist an integer $m > 0$ and $\xi_i = (\xi_i^1, \dots, \xi_i^N) \in \mathbf{R}^N$, $i = 1, \dots, m$, such that $\|\xi_i\|_{\mathbf{R}^d} = 1$, $i = 1, \dots, m$, and

$$\bigcap_{i=1}^m \left\{ x \in \mathbf{R}^N : (x, \xi_i) \leq \frac{1}{(1+\varepsilon_1)^{1/2}} \right\} \subset \{x \in \mathbf{R}^N : \|x\| < 1\},$$

so

$$\|x\|^2 \leq (1+\varepsilon_1) \max_{i=1, \dots, m} (x, \xi_i)^2, \quad x \in \mathbf{R}^N.$$

Substitute ε_1 by $1 - pa_0$ in the inequality above. Let $\bar{e}_i = \sum_{j=1}^N \xi_i^j e_j$, $i = 1, \dots, m$. Then $(\bar{G}\bar{e}_i, \bar{e}_i)_{L^2(d\nu_0)} = 1$, $\int_{\mathbf{R}^d} \bar{e}_i(x) \nu_0(dx) = 0$, $i = 1, \dots, m$, and

$$\begin{aligned} \sum_{j=1}^N \left(\int_0^T e_j(X_t) dt \right)^2 &\leq (1+\varepsilon_1) \max_{i=1, \dots, m} \sum_{j=1}^N \left(\int_0^T e_j(X_t) dt \cdot \xi_i^j \right)^2 \\ &= (1+\varepsilon_1) \max_{i=1, \dots, m} \left(\int_0^T \bar{e}_i(X_t) dt \right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{T>0} E^{Q_\varepsilon} \left[\exp \left(\sum_{i=1}^N \frac{p}{2T} \int_0^T \int_0^T a_i \varepsilon_i(X_t) \varepsilon_i(X_s) ds dt \right), A_\varepsilon | X_T = y \right] \\ & \leq \sup_{T>0} \sum_{i=1}^m E^{Q_\varepsilon} \left[\exp \left(\frac{1-\varepsilon_i^2}{2} \cdot \frac{1}{T} \left(\int_0^T \tilde{c}_i(X_t) dt \right)^2 \right), A_\varepsilon | X_T = y \right], \end{aligned}$$

which is finite for $\varepsilon > 0$ small enough by the Lemma 5.4.

This completes the proof of the lemma. \square

6 Proof of the main theorem.

In this section, we give the proof of our main theorem. Let

$$\begin{aligned} \tilde{\Phi}(\nu) & \equiv \Phi(\nu) - \int_{\mathbf{R}^d} \phi^{(0)}(y) \nu(dy), \\ & = \Phi(\nu) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu - \nu_0), \quad \nu \in \mathcal{M}(\mathbf{R}^d). \end{aligned}$$

Since for any $A \in \mathcal{F}_T$,

$$\begin{aligned} & e^{-\lambda T} E^{P_\varepsilon} \left[\exp \left(T \tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A \right) | X_T = y \right] \\ & = \frac{h(x)}{h(y)} E^{Q_\varepsilon} \left[\exp \left(T \tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A \right) | X_T = y \right], \end{aligned} \quad (6.1)$$

the theorem will be shown if we can show the following two lemmas.

Lemma 6.1

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{Q_\varepsilon} \left[\exp \left(T \tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A_\varepsilon^C \right) | X_T = y \right] < 0$$

for any $\varepsilon > 0$, and any $x, y \in \mathbf{R}^d$.

Lemma 6.2 For any $x, y \in \mathbf{R}^d$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} E^{Q_\varepsilon} \left[\exp \left(T \tilde{\Phi} \left(\frac{1}{T} \int_0^T \delta_{X_t} dt \right), A_\varepsilon \right) | X_T = y \right] \\ & = \exp \left\{ \frac{1}{2} \int_{\mathbf{R}^d} \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) \Big|_{(u,u)} \nu_0(du) \right\} \times \det_2(I_H - D^2\Phi(\nu_0))^{-1/2}. \end{aligned}$$

Lemma 6.1 can be gotten from the result in section 2. First, we have the following

Lemma 6.3 Let Ψ be a function of $\wp(\mathbf{R}^d)$ which is upper semi-continuous and bounded from above. Then for any closed $A \subset \wp(\mathbf{R}^d)$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[\exp (T\Psi(\rho_T)), \rho_T \in A | X_T = y \right] \leq \sup_{\nu \in A} \{ \Psi(\nu) - I(\nu) \}.$$

Proof. We first show the assertion for $A = \wp(\mathbf{R}^d)$. Let $\sup_{\nu \in \wp(\mathbf{R}^d)} \{ \Psi(\nu) - I(\nu) \} = \Lambda$. Choose any $\varepsilon > 0$ and fix it for a while. From the upper semi-continuity of Ψ and the lower semi-continuity of I , for any $\nu \in \wp(\mathbf{R}^d)$ with $I(\nu) < \infty$, there exists a $r > 0$, such that for any $\tilde{\nu} \in B(\nu, r) = \{ R \in \wp(\mathbf{R}^d), \text{dist}(\nu, R) < r \}$, $\Psi(\tilde{\nu}) \leq \Psi(\nu) + \varepsilon$, and $I(\tilde{\nu}) \geq I(\nu) - \varepsilon$. Let $G_\nu = B(\nu, \frac{r}{2})$, and let $C_\nu = \bar{G}_\nu$.

$$\begin{aligned} & E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in G_\nu | X_T = y \right] \\ & \leq \exp \left(\sup_{\tilde{\nu} \in G_\nu} T\Psi(\tilde{\nu}) \right) P_\varepsilon \left[\rho_T \in C_\nu | X_T = y \right]. \end{aligned}$$

C_ν is a closed set, so from Theorem 1.1, we have that

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in G_\nu | X_T = y \right] \\ & \leq \Psi(\nu) + \varepsilon - \inf_{\tilde{\nu} \in C_\nu} I(\tilde{\nu}) \\ & \leq \Psi(\nu) - I(\nu) + 2\varepsilon \leq \Lambda + 2\varepsilon. \end{aligned}$$

Choose an arbitrary constant M and fix it for a while. Let $K_M \equiv \{ \nu \in \wp(\mathbf{R}^d); I(\nu) \leq M \}$. K_M is a compact set since I is a good rate function. So there exist a $N \in \mathbf{N}$ and N elements $\nu_1, \dots, \nu_N \in K_M$, such that $G_{\nu_1}, \dots, G_{\nu_N}$ covers K_M . Let $G \equiv \bigcup_{i=1}^N G_{\nu_i}$. G is an open set, and

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in G | X_T = y \right] \\ & \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(N \max_{i=1, \dots, N} E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in G_{\nu_i} | X_T = y \right] \right) \\ & \leq \max_{i=1, \dots, N} \limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in G_{\nu_i} | X_T = y \right] \\ & \leq \Lambda + 2\varepsilon. \end{aligned}$$

Let $C \equiv \wp(\mathbf{R}^d) \setminus G$. Ψ is upper bounded, let L be an upper bound of it. C is a closed set, so by Theorem 1.1,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[e^{T\Psi(\rho_T)}, \rho_t \in C | X_T = y \right] \\ & \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(e^{TL} P_\varepsilon(\rho_T \in C | X_T = y) \right) \\ & \leq L - \inf \{ I(\tilde{\nu}), \tilde{\nu} \in C \} \\ & \leq L - M. \end{aligned}$$

The above are true for any constant M . Let M be large enough such that $L - M \leq A$. Then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log E^{P_\varepsilon} \left[\exp(T\Psi(\rho_T)) | X_T = y \right] \leq \max\{\Lambda + 2\varepsilon, L - M\} = \Lambda + 2\varepsilon.$$

This is true for any $\varepsilon > 0$. Let $\varepsilon \rightarrow 0$, and we get our assertion with $A = \varphi(\mathbf{R}^d)$.

For the general A , we only need to let

$$\Psi'(\nu) = \begin{cases} \Psi(\nu), & \text{if } \nu \in A, \\ -\infty, & \text{if } \nu \notin A. \end{cases}$$

and apply the first part of the proof to Ψ' . ■

Proof of Lemma 6.1. This is obvious from (6.1), the definition of A_ε , and Lemma 6.3. ■

Remark 7 Theorem 1.2 is now obvious, since the opposite inequality is trivial when Ψ is continuous.

For Lemma 6.2, we follow the way as used in Kusuoka-Tamura [13] and Kusuoka-Liang [8]. First, we prove the following

Lemma 6.4 There exist constants $p > 1$ and $\varepsilon > 0$, such that

$$\sup_{T > 0} E^{Q_\varepsilon} \left[e^{pT\tilde{\Phi}(\frac{1}{T} \int_0^T \delta_{X_t} dt)}, A_\varepsilon | X_T = y \right] < \infty.$$

Proof. The proof is similar with the one in Kusuoka-Liang [8]. Let $R(\nu_0, \cdot)$ be the 3rd remainder of the Taylor expansion of Φ around ν_0 , i.e., $R(\nu_0, \nu - \nu_0) = \tilde{\Phi}(\nu) - \frac{1}{2} D^2 \Phi(\nu_0)(\nu - \nu_0, \nu - \nu_0)$. Then for any $p > 1$ and any $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$, by Hölder's inequality,

$$\begin{aligned} & E^{Q_\varepsilon} \left[e^{pT\tilde{\Phi}(\frac{1}{T} \int_0^T \delta_{X_t} dt)}, A_\varepsilon | X_T = y \right] \\ &= E^{Q_\varepsilon} \left[\exp \left\{ p \cdot \frac{T}{2} D^2 \Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) \right. \right. \\ &\quad \left. \left. + p \cdot TR(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0) \right\}, A_\varepsilon | X_T = y \right] \\ &\leq E^{Q_\varepsilon} \left[\exp \left\{ p \cdot \frac{T}{2} \cdot r D^2 \Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) \right\}, \right. \\ &\quad \left. A_\varepsilon | X_T = y \right]^{1/r} \quad (6.2) \end{aligned}$$

$$\times E^{Q_\varepsilon} \left[\exp \left\{ p \cdot T \cdot s R(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0) \right\}, A_\varepsilon | X_T = y \right]^{1/s}. \quad (6.3)$$

Now, for any function $U(\cdot, \cdot)$, define

$$\bar{U}(x, y) \equiv U(x, y) - \int_{\mathbf{R}^d} U(x, y) \nu_0(dx) - \int_{\mathbf{R}^d} U(x, y) \nu_0(dy) + \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} U(x, y) \nu_0(dx) \nu_0(dy),$$

and

$$\bar{U}(R_1, R_2) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} U(x, y) R_1(dx) R_2(dy),$$

then $\int_{\mathbf{R}^d} \bar{U}(x, y) \nu_0(dx) = 0$ for any $x \in \mathbf{R}^d$, and $\bar{U}(R_1, R_2) = \bar{U}(R_1, R_2)$ for any $R_1, R_2 \in \mathcal{M}_0(\mathbf{R}^d)$.

Since the maximum a_0 of the eigenvalues of $D^2 \Phi(\nu_0)|_{H \times H}$ is smaller than 1 by the assumption A7, we can find a $p > 1$ such that $a_0 \cdot p < 1$. For this p , there exists a $r > 1$ such that $a_0 \cdot p \cdot r < 1$. So since

$$\begin{aligned} & T \cdot D^2 \Phi(\nu_0) \left(\frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0 \right) \\ &= \frac{1}{T} \int_0^T \int_0^T \bar{\Phi}^{(2)}(\nu_0, \cdot, \cdot)_{(X_t, X_s)} dt ds, \end{aligned}$$

and the other conditions of Lemma 5.1 are all satisfied, we get by Lemma 5.1 that (6.2) is bounded for $T > 0$ if $\varepsilon > 0$ is small enough.

For (6.3), let s be the dual number of $r > 1$, choose a $\delta \in (0, \frac{1}{2ps})$ and fix it. By the assumption A8, for this $\delta > 0$, there exist a constant $\varepsilon' > 0$ and a K_δ , such that $\|\bar{K}_\delta\|_{H \times H} \|h, s\| \leq \delta$, K_δ satisfies all of the conditions of Lemma 5.1, and

$$\begin{aligned} & |TR(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0)| \\ &\leq T \cdot \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} K_\delta(x, y) \left(\frac{1}{T} \int_0^T \delta_{X_t} dy - \nu_0 \right)^{\otimes 2} (dx \otimes dy) \\ &= \frac{1}{T} \cdot \int_0^T \int_0^T \bar{K}_\delta(X_t, X_s) ds dt \quad \text{on } A_{\varepsilon'}. \end{aligned}$$

So by using Lemma 5.1 again, we get that (6.3) is bounded for $T > 0$ if $\varepsilon' > 0$ is small enough.

This completes the proof of the lemma. ■

Now, we are ready to prove Lemma 6.2.

Proof of Lemma 6.2. As in Kusuoka-Tamura [13], Q_x has the strong mixing property, so X_T and $\sqrt{T}(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0)$ are asymptotically independent as $T \rightarrow \infty$ under Q_x for any $x \in \mathbf{R}^d$, also,

$$\begin{aligned} & E^{Q_x} \left[\exp \left(\sqrt{-1} \sqrt{T} \int_{\mathbf{R}^d} u(x) \left(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0 \right) (dx) \right) \right] \\ &\rightarrow \exp \left(-\frac{1}{2} \int_{\mathbf{R}^d} u(y) \bar{C} u(y) \nu_0(dy) \right), \quad \text{as } T \rightarrow \infty \end{aligned}$$

for any $u \in L^2(\mathbf{R}^d, d\nu_0)$.

Take a separable Hilbert space H_1 such that the set $\{\bar{G}u d\nu_0 \mid \int_{\mathbf{R}^d} u \bar{G}u d\nu_0 < \infty\}$ is a dense linear subspace of H_1 , and the inclusion map is a Hilbert-Schmidt operator. Let W be an H_1 -valued random variable with distribution γ such that

$$E \left[\exp(\sqrt{-1}(u, W)) \right] = \exp \left(-\frac{1}{2} \int_{\mathbf{R}^d} u(y) \bar{G}u(y) \nu_0(dy) \right)$$

for any $u \in H_1$.

So from the central limit theorem for Hilbert space valued random variables, the distribution of $(X_T, \sqrt{T}(\frac{1}{T} \int_0^T \delta_{X_t} dt - \nu_0))$ under Q_x converges weakly to $\nu_0 \otimes \gamma$ as $T \rightarrow \infty$ on $\mathbf{R}^d \times H_1$.

As claimed before, $D^2\Phi(\nu_0)(\cdot, \cdot)|_{H \times H}$ is a Hilbert-Schmidt function. Write the eigenvalues and the corresponding eigenvectors as a_m and $\bar{G}e_m d\nu_0$, $m = 1, 2, \dots$. Then $\sum_{m=1}^N a_m ((e_m, W)^2 - 1)$ converges in $L^2(d\gamma)$ as $N \rightarrow \infty$. Let $D^2\Phi(\nu_0)(W, W)$ denote the $L^2(d\gamma)$ -limit of $\sum_{m=1}^N a_m ((e_m, W)^2 - 1)$.

It is easy that

$$\begin{aligned} & \frac{1}{T} \int_0^T \int_0^T \sum_{m=1}^N a_m e_m(X_s) e_m(X_t) ds dt - \frac{1}{T} \int_0^T \sum_{m=1}^N a_m e_m(X_s) \bar{G}e_m(X_s) ds \\ & \rightarrow \sum_{m=1}^N a_m ((e_m, W)^2 - 1) \end{aligned}$$

under Q_x in distribution for any $N \in \mathbf{N}$ and any $x \in \mathbf{R}^d$. Also,

$$\begin{aligned} & \sup_{T>0} E^{Q_x} \left[\left\{ \left(\frac{1}{T} \int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s) ds dt - \frac{1}{T} \int_0^T \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) |_{(X_s, X_s)} ds \right) \right. \right. \\ & \left. \left. - \left(\frac{1}{T} \int_0^T \int_0^T \sum_{m=1}^N a_m e_m(X_s) e_m(X_t) ds dt - \frac{1}{T} \int_0^T \sum_{m=1}^N a_m e_m(X_s) \bar{G}e_m(X_s) ds \right) \right\}^2 \right] \\ & \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$. Therefore,

$$\frac{1}{T} \int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s) ds dt - \frac{1}{T} \int_0^T \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) |_{(X_s, X_s)} ds \rightarrow D^2\Phi(\nu_0)(W, W) :$$

in distribution as $T \rightarrow \infty$. Also,

$$\frac{1}{T} \int_0^T \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) |_{(X_s, X_s)} ds \rightarrow \int_{\mathbf{R}^d} \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) |_{(u, u)} \nu_0(du)$$

Q_x -almost surely as $T \rightarrow \infty$, and

$$T R(\nu_0, \frac{1}{T} \int_0^T \delta_{X_t} dt) \rightarrow 0$$

under Q_x in distribution as $T \rightarrow \infty$. Therefore, we have that

$$T \bar{\Phi}(\frac{1}{T} \int_0^T \delta_{X_t} dt) \rightarrow D^2\Phi(\nu_0)(W, W) : + \int_{\mathbf{R}^d} \bar{G}_x \Phi^{(2)}(\nu_0; \cdot, \cdot) |_{(u, u)} \nu_0(du)$$

in distribution as $T \rightarrow \infty$. This together with Lemma 6.4 gives our assertion. ■

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Chapter 5

On an Ergodic Property of Diffusion Semigroups on Euclidean Space

1 Introduction

Let N be an integer, W be the set of continuous \mathbf{R}^N -valued functions w defined in $[0, \infty)$ with $w(0) = 0$, and μ be the standard Wiener measure on W . Let $\mathcal{F}_t = \sigma\{w(s); s \leq t\}$, $t \geq 0$.

Let $\alpha_{ij} \in C_b^\infty(\mathbf{R}^N)$, $i, j = 1, \dots, N$, and assume that there are $c_0, c_1 > 0$ such that

$$c_0 \sum_{i=1}^N \xi_i^2 \leq \sum_{j=1}^N \left(\sum_{i=1}^N \alpha_{ij}(x) \xi_i \right)^2 \leq c_1 \sum_{i=1}^N \xi_i^2, \quad \text{for any } x, \xi \in \mathbf{R}^N.$$

Here $C_b^\infty(\mathbf{R}^N)$ denotes the set of bounded smooth functions defined in \mathbf{R}^N whose derivatives of any order are bounded. Let β_i , $i = 1, \dots, N$, be smooth functions defined in \mathbf{R}^N . We assume the following through the paper.

(A-1) There is a positive constant C_0 such that

$$\sum_{i,j=1}^N \xi_i \xi_j \nabla_i \nabla_j \beta_j(x) \leq C_0 \sum_{i=1}^N \xi_i^2, \quad x, \xi \in \mathbf{R}^N,$$

where $\nabla_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, N$.

Now let us think of the following SDE.

$$\begin{aligned} dX_i(t, x) &= \sum_{j=1}^N \alpha_{ij}(X(t, x)) dw_j(t) + \beta_i(X(t, x)) dt, & i = 1, \dots, N, \\ X(0, x) &= (X_1(0, x), \dots, X_N(0, x)) = x \in \mathbf{R}^N. \end{aligned} \quad (1.1)$$

Our first main result is the following.

THEOREM 1.1 *There is a modification $X : [0, \infty) \times \mathbf{R}^N \times W \rightarrow \mathbf{R}^N$ of the solution of the SDE (1.1) satisfying the following.*

- (1) $X(t, \cdot, w) : \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a smooth function for any $t \in [0, \infty)$ and $w \in W$.
- (2) $\frac{\partial}{\partial x^i} X(\cdot, \cdot, w) : [0, \infty) \times \mathbf{R}^N$ is continuous for any $w \in W$ and multi index γ .
- (3) For any $p > 1$ and $T > 0$,

$$\sup_{x \in \mathbf{R}^N} \sum_{i,j=1}^N E^\mu \left[\sup_{t \in [0, T]} |\nabla_i X^j(t, x)|^p \right] < \infty.$$

Let $C_b(\mathbf{R}^N)$ denotes the set of bounded continuous functions defined in \mathbf{R}^N . We may regard $C_b(\mathbf{R}^N)$ as a Banach space with a norm $\|f\|_\infty = \sup_{x \in \mathbf{R}^N} |f(x)|$, $f \in C_b(\mathbf{R}^N)$. For any $c \in C_b(\mathbf{R}^N)$, we define a semi-group of bounded linear operators P_t^c , $t \in [0, \infty)$ in $C_b(\mathbf{R}^N)$ by

$$(P_t^c f)(x) = E^\mu \left[\exp \left(\int_0^t c(X(s, x)) ds \right) f(X(t, x)) \right], \quad f \in C_b(\mathbf{R}^N).$$

In the case that $c = 0$, we denote P_t^c by P_t , $t \in [0, \infty)$. Then we have the following essentially due to Kusuoka-Stroock [3].

THEOREM 1.2 *There is a strictly positive smooth function p defined in $(0, \infty) \times \mathbf{R}^N \times \mathbf{R}^N$ such that*

$$(P_t f)(x) = \int_{\mathbf{R}^N} p(t, x, y) f(y) dy, \quad f \in C_b(\mathbf{R}^N),$$

$$\sup \left\{ \left| \frac{\partial}{\partial y^i} p(t, x, y) \right|; x, y \in \mathbf{R}^N, |y| \leq r \right\} < \infty \quad (1.2)$$

$$\sup \left\{ \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial y^j} p(t, x, y) \right|; x, y \in \mathbf{R}^N, |y| \leq r \right\} < \infty, \quad i = 1, \dots, N \quad (1.3)$$

for any $t > 0$, any multi index γ , and any $r > 0$.

Now we introduce the following assumption.

(A-2) For any $t > 0$,

$$\sup \{ \mu(|X(t, x)| \geq n); x \in \mathbf{R}^N \} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

THEOREM 1.3 *If there is an increasing convex function $\varphi : [0, \infty) \rightarrow \mathbf{R}$ such that*

$$\varphi(s) \rightarrow \infty, \quad \text{as } s \uparrow \infty, \quad \int_0^\infty \frac{ds}{\varphi(s)} < \infty,$$

and

$$x \cdot \beta(x) \leq -\varphi(|x|^2), \quad x \in \mathbf{R}^N,$$

then Assumption (A-2) holds.

Now let $C_b^1(\mathbf{R}^N)$ denote the set of continuously differentiable functions f such that f itself and its derivatives $\nabla_i f$, $i = 1, \dots, N$, are bounded.

THEOREM 1.4 *Assume that Assumptions (A-1) and (A-2) hold. Then we have the following.*

- (1) For any $c \in C_b(\mathbf{R}^N)$ and $t > 0$, the linear operator P_t^c defined in $C_b(\mathbf{R}^N)$ is compact. Moreover, there exist an $h^c \in C_b(\mathbf{R}^N)$, a probability measure ν^c on \mathbf{R}^N , constants $\lambda^c \in \mathbf{R}$ and $\varepsilon > 0$, such that

$$\inf \{ h(x); x \in \mathbf{R}^N \} > 0,$$

and

$$\| \exp(-\lambda^c t) P_t^c f - \left(\int_{\mathbf{R}^N} \frac{f}{h^c} d\nu^c \right) h^c \|_\infty \leq \varepsilon^{-1} \exp(-\varepsilon t) \|f\|_\infty, \quad t \geq 0, f \in C_b(\mathbf{R}^N).$$

- (2) If $c \in C_b^1(\mathbf{R}^N)$, then $h^c \in C_b^1(\mathbf{R}^N)$. Moreover, there exists a constant $\delta > 0$ such that

$$\begin{aligned} & \sum_{i=1}^N \| \exp(-\lambda^c t) \nabla_i (P_t^c f) - \left(\int_{\mathbf{R}^N} \frac{f}{h^c} d\nu^c \right) \nabla_i h^c \|_\infty \\ & \leq \delta^{-1} \exp(-\delta t) (\|f\|_\infty + \sum_{i=1}^N \|\nabla_i f\|_\infty), \quad t \geq 0, f \in C_b^\infty(\mathbf{R}^N). \end{aligned}$$

2 Proof of Theorem 1

Note that from the assumption (A-1) we have

$$(x-y) \cdot (\beta(x) - \beta(y)) = (x_j - y_j)(x_i - y_i) \int_0^1 dt \nabla_i \beta^j(y + t(x-y)) \leq C_0 |x-y|^2. \quad (2.1)$$

PROPOSITION 2.1 *There exists a version $X(t, x)$ of the solution of the SDE (1.1) such that $X(\cdot, \cdot, w) : [0, \infty) \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ is continuous for all $w \in W$.*

Proof. Note that

$$\begin{aligned} & |X(t, x) - X(t, y)|^2 \\ &= |x - y|^2 + 2 \int_0^t (X(s, x) - X(s, y)) \cdot (\beta(X(s, x)) - \beta(X(s, y))) ds \\ & \quad + \sum_{i,j=1}^N \int_0^t |\alpha_{ij}(X(s, x)) - \alpha_{ij}(X(s, y))|^2 ds \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{i,j=1}^N \int_0^t (X_i(s, x) - X_i(s, y)) (\alpha_{ij}(X(s, x)) - \alpha_{ij}(X(s, y))) dw_j(s) \\
\leq & |x - y|^2 + C \int_0^t |X(s, x) - X(s, y)|^2 ds \\
& + 2 \sum_{i,j=1}^N \int_0^t (X_i(s, x) - X_i(s, y)) (\alpha_{ij}(X(s, x)) - \alpha_{ij}(X(s, y))) dw_j(s).
\end{aligned}$$

Here $C = C_0 + \sum_{i,j,k=1}^N \|\nabla_k \alpha_{ij}\|_\infty$.

Let $\tau_n = \inf\{t > 0; |X(t, x) - X(t, y)| > n\}$, $n \geq 1$. Then we have by Doob's inequality and Hölder's inequality that

$$\begin{aligned}
& E^\mu [\sup\{|X(s, x) - X(s, y)|; s \in [0, t \wedge \tau_n]\}^{2p}] \\
\leq & 3^p (|x - y|^{2p} + C^p t^{p-1} \int_0^{t \wedge \tau_n} E^\mu |X(s, x) - X(s, y)|^{2p} ds \\
& + (\frac{p}{p-1})^p E^\mu [|\sum_{i,j=1}^N \int_0^{t \wedge \tau_n} (X_i(s, x) - X_i(s, y)) (\alpha_{ij}(X(s, x)) - \alpha_{ij}(X(s, y))) dw_j(s)|^p]).
\end{aligned}$$

Therefore by Burkholder's inequality, we see that there exists a $C' > 0$ depending only on C, T, p such that

$$\begin{aligned}
& E^\mu [\sup\{|X(s, x) - X(s, y)|; s \in [0, t \wedge \tau_n]\}^{2p}] \\
\leq & C' (|x - y|^{2p} + \int_0^t E^\mu [\sup\{|X(s, x) - X(s, y)|; s \in [0, u]\}^{2p}] du
\end{aligned}$$

for any $t \in [0, T]$, $x, y \in \mathbf{R}^N$, $n \geq 1$. Letting $n \rightarrow \infty$ and using Gronwall's inequality, we see that for any $p \in (1, \infty)$ and $T > 0$ there is a constant $C'' > 0$ such that

$$E^\mu [\sup\{|X(s, x) - X(s, y)|; s \in [0, t]\}^{2p}] \leq C'' |x - y|^{2p}, \quad x, y \in \mathbf{R}^N.$$

This and Kolomogorov's theorem (c.f. Stroock-Varadhan [4]) imply our assertion. ■

Now let us prove Theorem 1. By Proposition 2.1, we may assume that the solution $X(t, x)$ is continuous in (t, x) for $\mu - a.s.w$. Let

$$\tau_{r,n}(w) = \inf\{t > 0; \max_{|s| \leq t} |X(t, x, w)| \geq n\}, \quad r, n \geq 1.$$

Then we see $\tau_{r,n}(w) \uparrow \infty$, $n \rightarrow \infty$, for $\mu - a.s.w$. Let $\beta_{n,i} \in C_b^\infty(\mathbf{R}^N)$, $i = 1, \dots, N$, such that $\beta_{n,i}(x) = \beta_i(x)$, $|x| \leq n+1$, $i = 1, \dots, N$. Also, let $X_n(t, x)$ be a solution to the SDE

$$\begin{aligned}
dX_{n,i}(t, x) &= \sum_{j=1}^N \alpha_{ij}(X_n(t, x)) dw_j(t) + \beta_{n,i}(X(t, x)) dt, \quad i = 1, \dots, N, \\
X_n(0, x) &= x \in \mathbf{R}^N.
\end{aligned}$$

Then we see that $X_n(t, x)$ has a modification $X_n : [0, \infty) \times \mathbf{R}^N \times W \rightarrow \mathbf{R}^N$ such that $X(t, \cdot, w) : \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a smooth function for any $t \in [0, \infty)$ and $w \in W$, and that $\frac{\partial}{\partial x^\gamma} X(\cdot, w) : [0, \infty) \times \mathbf{R}^N$ is continuous for any $w \in W$ and multi index γ (c.f. Ikeda-Watanabe [2]).

Then the uniqueness of the solution of the SDE implies that

$$\begin{aligned}
& \mu(\{w \in W; X(t, x) = X_n(t, x) \text{ for any } (t, x) \in [0, T] \times \mathbf{R}^N \text{ with } |x| \leq r\}) \\
&= \mu(\tau_{r,n} \geq T) \rightarrow 1, \quad n \rightarrow \infty
\end{aligned}$$

for any $T > 0$ and $r \geq 1$. This implies Assertions (1) and (2) in Theorem 1.1.

Also, we see that

$$\begin{aligned}
d\nabla_i X_j(t, x) &= \sum_{k,\ell=1}^N \nabla_i X_k(t, x) \nabla_k \alpha_{j\ell}(X(t, x)) dw_\ell(t) + \sum_{k=1}^N \nabla_i X_k(t, x) \nabla_k \beta_j(X(t, x)) dt \\
\nabla_i X_j(0, x) &= \delta_{ij}, \quad i, j = 1, \dots, N.
\end{aligned}$$

for $\mu - a.s.w$. So we have

$$\begin{aligned}
& \sum_{i,j=1}^N |\nabla_i X_j(t, x)|^2 \\
\leq & N + 2 \sum_{i,j,k,\ell=1}^N \int_0^t \nabla_i X_j(s, x) \nabla_i X_k(s, x) \nabla_k \alpha_{j\ell}(X(s, x)) dw_\ell(s) \\
& + \sum_{i,j,\ell=1}^N \int_0^t (\sum_{k=1}^N \nabla_i X_k(s, x) \nabla_k \alpha_{j\ell}(X(s, x)))^2 ds + C_0 \int_0^t \sum_{i,j=1}^N |\nabla_i X_j(s, x)|^2 ds.
\end{aligned}$$

Then the same argument as in Proposition 2.1 implies Assertion (3) in Theorem 1.1. This completes the proof of Theorem 1.1. ■

3 Semigroup $\{Q_t\}$ and Proof of Theorem 2

In this section, we introduce another semigroup and prove Theorem 1.2.

Let $c \in C_b^1(\mathbf{R}^N)$. Let $E = C_b(\mathbf{R})^{1+N}$. Then E is a Banach space with a norm $\|\cdot\|_E$ given by $\|g\|_E = \sum_{i=0}^N \|g_i\|_\infty$, $g = (g_0, g_1, \dots, g_N) \in E$. For each $c \in C_b^1(\mathbf{R}^N)$, let

$$M^c(t, x) = \exp(\int_0^t c(X(s, x)) ds), \quad (t, x) \in [0, \infty) \times \mathbf{R}^N,$$

and let us define $Y_{ij}^c : [0, \infty) \times \mathbf{R}^N \times W \rightarrow \mathbf{R}$, $i, j = 0, 1, \dots, N$, by

$$Y_{00}^c(t, x) = M^c(t, x), \quad Y_{0i}^c(t, x) = 0,$$

$$Y_{i0}^c(t, x) = \nabla_i M^c(t, x), \quad Y_{ij}^c(t, x) = \nabla_i X_j(t, x) M^c(t, x), \quad i, j = 1, \dots, N.$$

Now let us define operators Q_t^c , $t \geq 0$, by

$$(Q_t^c g)_i(x) = \sum_{j=0}^N E^\mu[Y_{ij}^c(t, x) g_j(X(t, x))], \quad i = 0, 1, \dots, N, \quad x \in \mathbf{R}^N$$

for $g = (g_0, g_1, \dots, g_N) \in E$. In the case that $c = 0$, we denote Q_t^c by Q_t , and Y_{ij}^c by Y_{ij} . Since we have $\sup_{x \in \mathbf{R}^N} E^\mu[|Y_{ij}(t, x)|^2] < \infty$, $i, j = 0, 1, \dots, N$, and $X(t, x)$, $Y_{ij}(t, x)$, $i, j = 0, 1, \dots, N$ are continuous in x for μ -a.s., we see that Q_t^c is a bounded linear operator in E for each $t \geq 0$.

Let $a_{ij}^c \in C(\mathbf{R}^N)$, $i, j = 0, 1, \dots, N$ be given by

$$a_{00}^c(x) = c(x), \quad a_{i0}^c(x) = (\nabla_i c)(x),$$

$$a_{0j}^c(x) = 0, \quad a_{ij}^c(x) = c(x) \delta_{ij}, \quad i, j = 1, \dots, N.$$

Let A^c denotes the bounded linear operator given by

$$(A^c g)_i(x) = \sum_{j=0}^N a_{ij}^c(x) g_j(x), \quad i = 0, 1, \dots, N, \quad x \in \mathbf{R}^N, \quad g = (g_0, g_1, \dots, g_N) \in E.$$

PROPOSITION 3.1 Let $c \in C_b^1(\mathbf{R}^N)$.

- (1) $\{Q_t^c; t \geq 0\}$ is a semigroup of operators, i.e., $Q_{t+s}^c = Q_t^c Q_s^c$, $t, s \geq 0$.
- (2) For any $f \in C_b^1(\mathbf{R}^N)$ and $t \geq 0$, $P_t^c f \in C_b^1(\mathbf{R}^N)$ and

$$(P_t f, \nabla_1(P_t f), \dots, \nabla_N(P_t f)) = Q_t((f, \nabla_1 f, \dots, \nabla_N f))$$

- (3) For any $t > 0$ and $g \in E$,

$$Q_t^c g = Q_t g + \int_0^t Q_{t-s}^c A^c Q_s g \, ds.$$

Proof. Note that

$$dM^c(t, x) = c(X(t, x)) M^c(t, x) dt$$

and

$$d\nabla_i M^c(t, x) = \left(c(X(t, x)) \nabla_i M^c(t, x) + \sum_{k=1}^N \nabla_i X_k(t, x) (\nabla_k c)(X(t, x)) M^c(t, x) \right) dt.$$

Let $a_{ijk} \in C(\mathbf{R}^N)$, $i, j, k = 0, 1, \dots, N$, be given by

$$a_{000}(x) = a_{00k}(x) = a_{i00}(x) = a_{i0k}(x) = a_{0jk}(x) = a_{0jk}(x) = 0,$$

$$a_{ij0}(x) = \nabla_i \beta_j(x), \quad a_{ijk}(t, x) = \nabla_i a_{jk}(x), \quad i, j, k = 1, \dots, N.$$

Then we see that

$$\begin{aligned} dY_{ij}^c(t, x) &= \sum_{k=0}^N \sum_{\ell=1}^N Y_{ik}^c(t, x) a_{k\ell}^c(X(t, x)) dw_\ell(t) \\ &\quad + \sum_{k=0}^N Y_{ik}^c(t, x) (a_{k0}^c(X(t, x)) + a_{kj}^c(X(t, x))) dt, \\ Y_{ij}^c(0, x) &= \delta_{ij}, \quad i, j = 0, 1, \dots, N. \end{aligned}$$

Let $\theta : [0, \infty) \times W \rightarrow W$ be defined by $\theta(s, w)(t) = w(t+s) - w(s)$, $s, t \in [0, \infty)$, $w \in W$. Then we have from the uniqueness of the solution of the SDE that

$$Y_{ij}(t+s, x, w) = \sum_{k=0}^N Y_{ik}(t, x, w) Y_{kj}(s, X(t, x, w), \theta(t, w))$$

This implies Assertion (1).

For any $x, v \in \mathbf{R}^N$ and $f \in C_b^1(\mathbf{R}^N)$, we have

$$\begin{aligned} &(P_t^c f)(x+v) - (P_t^c f)(x) \\ &= \sum_{i=1}^N E^\mu \left[\int_0^t Y_{i0}^c(t, x+sv) f(X(t, x+sv)) ds \right] + \sum_{i,j=1}^N E^\mu \left[\int_0^t Y_{ij}^c(t, x+sv) \nabla_j f(X(t, x+sv)) ds \right]. \end{aligned}$$

Since $\sum_{i,j} \sup_{x \in \mathbf{R}^N} E[|Y_{ij}^c(t, x)|^2] < \infty$, we have Assertion (2).

Let $(Y_{ij}^{-1}(t, x))_{i,j=0}^N$ be the inverse matrix of $(Y_{ij}^0(t, x))_{i,j=0}^N$. Then by Ito's lemma, we have

$$\begin{aligned} &d \left(\sum_{k=0}^N Y_{ik}^c(t, x) Y_{kj}^{-1}(t, x) \right) \\ &= \sum_{k,\ell=0}^N Y_{ik}^c(t, x) a_{k\ell}^c(X(t, x)) Y_{\ell j}^{-1}(t, x) dt. \end{aligned}$$

Then we have

$$\begin{aligned} &Y_{ij}^c(t, x, w) - Y_{ij}^0(t, x, w) \\ &= \sum_{k,\ell=0}^N \int_0^t Y_{ik}^c(s, x, w) a_{k\ell}^c(X(s, x, w)) Y_{\ell j}^0(t-s, X(s, x, w), \theta(s, w)) ds. \end{aligned}$$

This implies Assertion (3). This completes the proof. \square

PROPOSITION 3.2 There are $q_{ij} \in C^\infty((0, \infty) \times \mathbf{R}^N \times \mathbf{R}^N)$, $i, j = 0, 1, \dots, N$, such that

$$q_{0j} = 0, \quad j = 1, \dots, N,$$

$$\sup\left\{\left|\frac{\partial^\gamma}{\partial y^\gamma}q_{ij}(t, x, y)\right|; x, y \in \mathbf{R}^N \text{ with } |y| \leq r\right\} < \infty$$

for any $t > 0$, $r \geq 1$, and any multi-index γ , and

$$(Q_t g)_i(x) = \sum_{j=0}^N \int_{\mathbf{R}^N} q_{ij}(t, x, y) g_j(y) dy, \quad t > 0, \quad g = (g_0, g_1, \dots, g_N) \in E.$$

Proof. The proof is quite similar to the proof of Kusuoka-Stroock [3, Theorem 4.5]. So we give only a sketch. Let us take an arbitrary $r \geq 1$ and fix it. Let us take a $\beta' \in C_0^\infty(\mathbf{R}^N; \mathbf{R}^N)$ such that $\beta'(x) = \beta(x)$, $x \in \mathbf{R}^N$ with $|x| \leq r+3$.

Let $X'(t, x)$ be the solution of SDE

$$\begin{aligned} dX'_i(t, x) &= \sum_{j=1}^N \alpha_{ij}(X'(t, x)) dw_j(t) + \beta'_i(X'(t, x)) dt \\ X'(0, x) &= x \in \mathbf{R}^N \end{aligned}$$

Then by Theorem 1.1, we may assume that $X'(t, x)$ is smooth in x . Let $\tau'(w, x) = \inf\{t \geq 0; |X'(t, x)| = r+3\}$. Then we see, by using the uniqueness of the SDE's for $X(t, x)$ and $\nabla_i X_j(t, x)$, $i, j = 1, \dots, N$, that $X(t, x, w) = X'(t, x, w)$, and $\nabla_i X_j(t, x) = \nabla_i X'_j(t, x)$, $t \leq \tau'(w, x)$, for $x \in \mathbf{R}^N$ with $|x| \leq r+1$. Define $Y'_{ij}: [0, \infty) \times \mathbf{R}^N \times W \rightarrow \mathbf{R}$, $i, j = 0, 1, \dots, N$, by $Y'_{ij}(t, x) = \delta_{ij}$, if $i = 0$ or $j = 0$, and $Y'_{ij}(t, x) = \nabla_i X'_j(t, x)$, if $i \neq 0$ and $j \neq 0$. Let Q'_t , $t > 0$ be the bounded linear operator in E given by

$$(Q'_t g)_i(x) = \sum_{j=0}^N E^\mu[Y'_{ij}(t, x) g_j(X'(t, x)), t < \tau'(x)], \quad g = (g_0, g_1, \dots, g_N) \in E$$

Let $B_R = \{x \in \mathbf{R}^N; |x| < R\}$, $R > 0$. Then by the same argument as in Kusuoka-Stroock [3, Section 4], we see that there is a $q'_{ij}(t, x, \cdot) \in L^1(\mathbf{R}^N, dx)$, $i, j = 0, 1, \dots, N$, such that

$$(Q'_t g)_i(x) = \sum_{j=0}^N \int_{\mathbf{R}^N} q'_{ij}(t, x, y) g_j(y) dy, \quad g \in E,$$

$$q'_{ij}(t, x, \cdot)|_{B_{r+2}} \in C^\infty(B_{r+2}), \quad (t, x) \in (0, \infty) \times \mathbf{R}^N \text{ with } |x| \leq r+1,$$

$$\sup\left\{\left|\frac{\partial^\gamma}{\partial y^\gamma}q'_{ij}(t, x, y)\right|; x, y \in B_{r+1}\right\} < \infty, \quad t > 0,$$

and

$$\sup\left\{\left|\frac{\partial^\gamma}{\partial y^\gamma}q'_{ij}(t, x, y)\right|; (t, x, y) \in (0, T] \times \mathbf{R}^N \times B_r \text{ with } |x| = r+1\right\} < \infty, \quad T > 0,$$

for any multi-index γ . Moreover, from the definition, we have $q'_{ij}(t, x, \cdot) = 0$, if $i = 0, j \neq 0$ or $i \neq 0, j = 0$. Now let us define $\tau_n(x, w)$, $n \geq 0$, and $\sigma_n(x, w)$, $n \geq 1$, $x \in \mathbf{R}^N$, inductively by

$$\tau_0(x, w) = 0,$$

$$\sigma_n(x, w) = \inf\{t \geq \tau_{n-1}(x, w); |X(t, x, w)| \leq r+1\},$$

$$\tau_n(x, w) = \inf\{t \geq \sigma_n(x, w); |X(t, x, w)| \geq r+3\},$$

for $n \geq 1$. Then we see that for any $g \in C_0(B_r)^{1+N} \subset E$,

$$\begin{aligned} (Q_t g)_i(x) &= \sum_{n=1}^{\infty} \sum_{j=0}^N E^\mu[Y_{ij}(t, x) g_j(X(t, x)), \sigma_n(x) \leq t < \tau_n(x)] \\ &= \sum_{n=1}^{\infty} \sum_{j=0}^N E^\mu[Y_{ij}(\sigma_n(x), x)(Q'_{t-\sigma_n(x)} g)_j(X(\sigma_n(x), x)), \sigma_n(x) \leq t] \\ &= \sum_{k=0}^N \int_{B_r} dy g_k(y) \left(\sum_{n=1}^{\infty} \sum_{j=0}^N E^\mu[Y_{ij}(\sigma_n(x), x)(q'_{jk}(t - \sigma_n(x), X(\sigma_n(x), x), y), \sigma_n(x) \leq t)] \right). \end{aligned}$$

So we have our assertion. \blacksquare

Now let us prove Theorem 1.2. For any $f \in C_0^\infty(\mathbf{R}^N)$, we have by Propositions 3.1 and Proposition 3.2 that

$$P_t f(x) = \int_{\mathbf{R}^N} q_{0,0}(t, x, y) f(y) dy,$$

$$(\nabla_i P_t f)(x) = \int_{\mathbf{R}^N} q_{i,0}(t, x, y) f(y) dy - \sum_{j=1}^N \int_{\mathbf{R}^N} \frac{\partial}{\partial y_j} q_{i,j}(t, x, y) f(y) dy.$$

So we have Theorem 1.2. \blacksquare

4 Proof of Theorem 3

We prove Theorem 3. Let $c = \sum_{i,j=1}^N \sup_{x \in \mathbf{R}^N} |a_{ij}(x)|^2$. Then there is an $s_0 > 0$ such that $\varphi(s_0) \geq 2(c+1)$. Then we see that

$$t_0 = \int_{s_0}^{\infty} \frac{ds}{\varphi(s) - c} \leq \int_{s_0}^{\infty} \frac{2}{\varphi(s)} ds < \infty.$$

Let $a(t) \geq s_0$, $t \in (0, t_0]$, be given by

$$\int_{a(t)}^{\infty} \frac{ds}{\varphi(s) - c} = t.$$

Note that

$$\begin{aligned} & E^\mu[|X(t, x)|^2] \\ &= |x|^2 + \sum_{i,j=1}^N \int_0^t E^\mu[|\alpha_{ij}(X(s, x))|^2] ds + \sum_{i=1}^N \int_0^t E[X_i(s, x)b_i(X(s, x))] ds. \end{aligned}$$

Let $v(t, x) = E^\mu[|X(t, x)|^2]$, $(t, x) \in [0, \infty) \times \mathbf{R}^N$. Then we have

$$\frac{d}{dt} v(t, x) \leq c - E^\mu[\varphi(|X(t, x)|^2)] \leq c - \varphi(v(t, x)).$$

Let $\tau(x) = \inf\{t \geq 0; v(t, x) \leq s_0\}$. Then we have

$$-\frac{1}{\varphi(v(t, x)) - c} \frac{d}{dt} v(t, x) \geq 1, \quad 0 < t < \tau(x),$$

and

$$v(t, x) \leq s_0, \quad t > \tau(x).$$

This implies that

$$\int_{v(t,x)}^{v(0,x)} \frac{ds}{\varphi(s) - c} \leq t, \quad 0 < t \leq \tau(x) \wedge t_0.$$

Thus we have $v(t, x) \leq a(t)$, $0 < t \leq \tau(x) \wedge t_0$. So we have

$$v(t, x) \leq a(t), \quad t \in (0, t_0).$$

Thus we see that

$$\sup_{x \in \mathbf{R}^N} \mu(|X(t, x)| > r) \leq \frac{a(t)}{r^2}, \quad r > 0, t \in (0, t_0].$$

This completes the proof of Theorem 3.

5 Proof of Theorem 4

In this section, we assume Assumptions (A-1) and (A-2) through out.

PROPOSITION 5.1 (1) For any $c \in C_b(\mathbf{R}^N)$ and $t > 0$, the linear operator P_t^c is a compact operator in $C_b(\mathbf{R}^N)$.

(2) For any $c \in C_b^1(\mathbf{R}^N)$ and $t > 0$, the linear operator Q_t^c is a compact operator in E .

Proof. Since proofs are similar, we prove Assertion (2). By Proposition 3.1 (3), we see that it is suffice if we can prove that Q_t , $t > 0$, is compact in E .

Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ be such that $\varphi(x) = 1, |x| \leq 1$, and $0 \leq \varphi \leq 1$. Also, let $\varphi_n(x) = \varphi(nx)$, $x \in \mathbf{R}^N$, $n \geq 1$. Let $Q_{t,n}$ be the linear operator given by $Q_{t,n}g = Q_t(\varphi_n^2 g)$, $g \in E$. First, we prove that $Q_{t,n}$, $t > 0$, $n \geq 1$, is compact in E .

Let $\{g^{(m)}\}_{m=1}^\infty$ be a bounded sequence in E . Then the sequence $\{(1-\Delta)^{-1}(\varphi_n g^{(m)})\}_{m=1}^\infty$ is relatively compact in E . So taking a subsequence if necessary, we may assume that $\{(1-\Delta)^{-1}(\varphi_n g^{(m)})\}_{m=1}^\infty$ is convergent in E . Since we have

$$(Q_{t,n}g^{(m)})_i(x)$$

$$= \sum_{j=0}^N \int_{\mathbf{R}^N} (1-\Delta_y)(\varphi_n(y)q_{ij}(t, x, y))(1-\Delta)^{-1}(\varphi_n g^{(m)})(y) dy,$$

we see that $\|Q_{t,n}g^{(m)} - Q_{t,n'}g^{(m')}\|_E \rightarrow 0$, as $m, m' \rightarrow \infty$ by Proposition 3.2. So we see that $Q_{t,n}$, $t > 0$, $n \geq 1$, is compact in E .

Note that for any $g \in E$

$$\begin{aligned} & \|Q_t g - Q_{t,n} g\|_E \\ & \leq \sum_{i,j=0}^N \sup_{x \in \mathbf{R}^N} |E^\mu[Y_{ij}^0(t, x)(1-\varphi_n(X(t, x))^2)g_j(X(t, x))]| \\ & \leq \left(\sum_{i,j=0}^N \sup_{x \in \mathbf{R}^N} E^\mu[|Y_{ij}^0(t, x)|^2]^{1/2} \right) \sup_{x \in \mathbf{R}^N} \mu(|X(t, x)| \geq n)^{1/2} \|g\|_E. \end{aligned}$$

So we see that the operator norm of $Q_t - Q_{t,n}$ in E converges to zero as $n \rightarrow \infty$. So we see Q_t is compact in E . This completes the proof. ■

PROPOSITION 5.2 Let $c \in C_b(\mathbf{R}^N)$ and $t > 0$. Then there are $h \in C_b(\mathbf{R}^N)$, $\lambda_0 > 0$, a probability measure ν in \mathbf{R}^N , $C > 0$ and $\varepsilon > 0$, such that

$$P_t^c h = \lambda_0 h, \quad \int_{\mathbf{R}^N} h d\nu = 1, \quad \inf\{h(x); x \in \mathbf{R}^N\} > 0.$$

and

$$\|\lambda_0^{-n}(P_t^c)^n f - \left(\int_{\mathbf{R}^N} \frac{f}{h} d\nu\right) h\|_\infty \leq C(1-\varepsilon)^n \|f\|_\infty, \quad f \in C_b(\mathbf{R}^N).$$

Proof. First, we prove the following.

Claim 1. If $f \in C_b(\mathbf{R}^N)$ satisfies $f \geq 0$ and $f \neq 0$, then $\inf\{(P_t^c f)(x); x \in \mathbf{R}^N\} > 0$.

Proof of Claim 1. By virtue of support theorem (c.f. Stroock-Varadhan [4]), we see that $\mu(X(t/2, x) \in U) > 0$, $x \in \mathbf{R}^N$, for any non-void open set U in \mathbf{R}^N . So we have

$(P_{t/2}^c f)(x) > 0$, $x \in \mathbf{R}^N$. By Assumption (A-2), we see that there is an $r > 0$ such that $\mu(|X(t/2, x)| \leq r) \geq 1/2$ for all $x \in \mathbf{R}^N$. Then we have

$$\inf\{(P_t^c f)(x); x \in \mathbf{R}^N\} = \inf\{P_{t/2}^c(P_{t/2}^c f)(x); x \in \mathbf{R}^N\} > 0.$$

This implies Claim 1.

Now let B be a complex Banach space given by $B = C_b^c(\mathbf{R}^N; \mathbf{C})$ with a norm $\|f\|_B = \sup_{x \in \mathbf{R}^N} |f(x)|$, $f \in B$. Then B is the complex extension of $C_b(\mathbf{R}^N)$. So the bounded linear operator P_t^c is extended to a bounded linear operator in B . We denote this operator by the same symbol P_t^c . Then P_t^c , $t > 0$, is a compact linear operator in B , the spectrum of P_t^c , $\sigma(P_t^c)$, has no cluster point except 0. Let $\lambda_0 = \max\{|\lambda|; \lambda \in \sigma(P_t^c)\}$.

Claim 2. Suppose that $\lambda \in \sigma(P_t^c)$ with $|\lambda| = \lambda_0$ and that $f \in B$ satisfies $f \neq 0$ and $P_t^c f = \lambda f$. Then $\lambda = \lambda_0$ and there is a $a \in \mathbf{C}$ such that $f = a|f|$.

Proof of Claim 2.

It is obvious that $P_t^c(|f|) - |\lambda||f| \geq 0$. It is sufficient to prove that $P_t^c(|f|) = |\lambda||f|$.

Let $h = P_t^c(|f|) \in C_b(\mathbf{R}^N)$. Then we see that $P_t^c h - \lambda_0 h \geq 0$, and that $\inf\{h(x); x \in \mathbf{R}^N\} > 0$ by Claim 1. Suppose that $P_t^c h - \lambda_0 h \neq 0$. Then by Claim 1, we see that there is a $\delta > 0$ such that $P_t^c(P_t^c h) \geq (\lambda_0 + \delta)P_t^c h$. Therefore we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|(P_t^c)^n h\|_B \geq \log(\lambda_0 + \delta).$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(P_t^c)^n\|_{\text{operator}} = \log \lambda_0.$$

This is contradiction. So we have $P_t^c h = \lambda_0 h$. This and Claim 1 imply that $P_t^c(|f|) = |\lambda||f|$. This completes the proof of Claim 2.

By Claims 1 and 2, we see that $\lambda_0 \in \sigma(P_t^c)$ and there is an $h \in C_b(\mathbf{R}^N)$ such that $P_t^c h = \lambda_0 h$, $\inf_x h(x) > 0$. Let $E(\lambda_0) = E(\lambda_0, P_t^c)$ be the projection operator as in Dunford-Schwarz [1, Chapter VII].

Claim 3. The dimension of the image of $E(\lambda_0)$ is one.

Proof of Claim 3. If not, by Claim 2 we see that there is an $f \in B$ such that $\lambda_0^{-n} \|(P_t^c)^n f\|_B \rightarrow \infty$, as $n \rightarrow \infty$. But there is a constant $c > 0$ such that $|f| \leq ch$. Then we have $\limsup_{n \rightarrow \infty} \lambda_0^{-n} \|(P_t^c)^n f\|_B \leq c \|h\|_\infty$. This is a contradiction.

By Claim 2, we see that $\sup\{|\lambda|; \lambda \in \sigma(P_t^c) \setminus \{\lambda_0\}\} < \lambda_0$. Combining this with Claim 3, we see that there is a bounded linear operator $S: B \rightarrow \mathbf{C}$ and $C, \varepsilon > 0$ such that

$$\|\lambda_0^{-n}(P_t^c)^n f - S(f)h\|_B \leq C(1 - \varepsilon)^n \|f\|_B, \quad f \in B.$$

We can easily see that $S(h) = 1$, and that $S(f) \geq 0$, for $f \in C_b(\mathbf{R}^N)$ with $f \geq 0$. Moreover $S(\lambda_0^{-1} P_t^c f) = S(f)$, $f \in B$. For each $n \geq 1$, let $\varphi_n(x) = ((n - |x|) \wedge 1) \vee 0$, $x \in \mathbf{R}^N$. Then there is a finite measure ν_n in \mathbf{R}^N such that $S(\varphi_n f) = \int f d\nu_n$, $f \in C_b(\mathbf{R}^N)$, by Rietz's theorem. By Assumption (A-2), we see that $(P_t^c(\varphi_n f))(x) \uparrow (P_t^c f)(x)$, $n \rightarrow \infty$, for $x \in \mathbf{R}^N$ and $f \in C_b(\mathbf{R}^N)$ with $f \geq 0$. Since P_t^c is compact, we see that $S(\varphi_n f) \rightarrow S(f)$ in B , $n \rightarrow \infty$, for $f \in C_b(\mathbf{R}^N)$. So there is a finite measure ν on \mathbf{R}^N such that $S(f) = \int f d\nu$, $f \in C_b(\mathbf{R}^N)$. Replacing ν and h by $\nu(\mathbf{R}^N)^{-1}\nu$ and $\nu(\mathbf{R}^N)h$, if necessary, we have our Proposition 5.2. ■

PROPOSITION 5.3 Let $c \in C_b(\mathbf{R}^N)$. Then there are $h \in C_b(\mathbf{R}^N)$, $\eta \in \mathbf{R}$, a probability measure ν in \mathbf{R}^N , $C > 0$ and $\varepsilon > 0$, such that

$$P_t^c h = \exp(\eta t)h, \quad t > 0, \quad \int_{\mathbf{R}^N} h d\nu = 1, \quad \inf\{h(x); x \in \mathbf{R}^N\} > 0.$$

and

$$\|\exp(-\eta t)(P_t^c)^n f - (\int_{\mathbf{R}^N} \frac{f}{h} d\nu)h\|_\infty \leq C \exp(-\varepsilon t) \|f\|_\infty, \quad t > 0, \quad f \in C_b(\mathbf{R}^N).$$

Proof. By Proposition 5.2, for each $n \geq 0$ there are $h_n \in C_b(\mathbf{R}^N)$, $\lambda_n > 0$, and a probability measure ν_n in \mathbf{R}^N , such that

$$P_{2^{-n}t}^c h_n = \lambda_n h_n, \quad \int_{\mathbf{R}^N} h_n d\nu_n = 1, \quad \inf\{h_n(x); x \in \mathbf{R}^N\} > 0.$$

and $\lambda_n^{-k} P_{2^{-n}t}^c f \rightarrow (\int_{\mathbf{R}^N} \frac{f}{h_n} d\nu_n)h_n$ in $C_b(\mathbf{R}^N)$, as $k \rightarrow \infty$, for $f \in C_b(\mathbf{R}^N)$. Then we see that $\nu_n = \nu_0$, $h_n = h_0$, and $\lambda_n = \lambda_0^{2^{-n}}$ for $n \geq 1$. Let $\eta = \log \lambda_0$. Since $(P_s^c h_0)(x) \rightarrow (P_t^c h_0)(x)$, as $s \rightarrow t$, for each $x \in \mathbf{R}^N$, we see that $P_t^c h_0 = \exp(\eta t)h_0$, $t > 0$. Also, we see that

$$\begin{aligned} & \|\exp(-\eta t)P_t^c f - (\int_{\mathbf{R}^N} \frac{f}{h_0} d\nu_0)h_0\|_\infty \\ & \leq \exp(\|c\|_\infty) \|\lambda_0^{-n}(P_t^c)^n f - (\int_{\mathbf{R}^N} \frac{f}{h_0} d\nu_0)h_0\|_\infty \end{aligned}$$

for $t \in [n, n+1]$, $f \in C_b(\mathbf{R}^N)$. These imply our Proposition. ■

Proposition 5.3 implies Assertion (1) of Theorem 1.4. Let us prove Assertion (2). For this purpose we make some more preparations.

Let $E^{\mathbf{C}} = C_b(\mathbf{R}^N; \mathbf{C}^{1+N})$. Then $E^{\mathbf{C}}$ is a complex extension of a real Banach space E . Let $c \in C_b^1(\mathbf{R}^N)$ and fix it in the rest of this section. Then for every $t > 0$, Q_t^c can be extended to a compact linear operator in $E^{\mathbf{C}}$. We use the same symbol Q_t^c for this operator. Let R_n denotes a linear operator $Q_{2^{-n}}^c$ in $E^{\mathbf{C}}$, $n \geq 0$. The spectrum $\sigma(R_n)$ has no cluster points except zero. Let $E(\lambda; R_n)$, $\lambda \neq 0$, denotes the projection, i.e.,

$E(\lambda; R_n) = F(R_n)$, where F is a function such that $F = 1$ in a neighborhood of λ and $F = 0$ in a neighborhood of $\sigma(R_n) \setminus \{\lambda\}$ (c.f. Dunford-Schwarz [1, Chapter VII]).

PROPOSITION 5.4 (1) $\sigma(R_n) = \{\lambda^2; \lambda \in \sigma(R_{n+1})\}$, $n \geq 0$. (2) For any $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \geq 0$,

$$E(\lambda^2, R_n) = E(\lambda, R_{n+1}) + E(-\lambda, R_{n+1}).$$

Proof. Since $R_n = R_{n+1}^2$, Assertion (1) follows from Dunford-Schwarz [1, Theorem VII 3.11].

Let F be a function such that $F = 1$ in a neighborhood of λ^2 and $F = 0$ in a neighborhood of $\sigma(R_n) \setminus \{\lambda^2\}$. Then the function $G(z) = F(z^2)$ satisfies that $G = 1$ in a neighborhood of $\{\lambda, -\lambda\}$, and $G = 0$ in a neighborhood of $\sigma(R_{n+1}) \setminus \{\lambda, -\lambda\}$. So we have

$$E(\lambda^2, R_n) = F(R_n) = G(R_{n+1}) = E(\lambda, R_{n+1}) + E(-\lambda, R_{n+1}).$$

This proves Assertion (2).

Now let h, ν , and η be as in Proposition 5.3. $S_n = \{\lambda \in \sigma(R_n); |\lambda| \geq \exp(2^{-n}(\eta - 2))\}$. Then by Proposition 5.4, we see that $S_n = \{\lambda^2; \lambda \in S_{n+1}\}$, $n \geq 0$. Then we see that the number of elements $\#(S_n)$ is increasing in n . Also, we have

$$\sum_{\lambda \in S_0} E(\lambda; R_0) = \sum_{\lambda \in S_n} E(\lambda; R_n).$$

So we see that $\#(S_n)$ is dominated by the dimension of image of $\sum_{\lambda \in S_0} E(\lambda; R_0)$. Thus there are $n_0 \geq 1$ and $M \geq 1$ such that $\#(S_n) = M$, $n \geq n_0$. So there are $\lambda_{n,i}$, $n \geq n_0$, $i = 1, 2, \dots, M$, such that $S_n = \{\lambda_{n,i}; i = 1, \dots, M\}$, $\lambda_{n,i} = \lambda_{n+1,i}$, $i = 1, \dots, M$, $n \geq n_0$. Then we see that $E_i = E(\lambda_{n,i}; R_n)$, $i = 1, 2, \dots, M$, is independent of $n \geq n_0$. By the same argument as in the proof of Proposition 5.3, we see that

$$Q_i^c E_i = E_i Q_i^c, \quad i = 1, \dots, M,$$

and that there is a $C > 0$ such that

$$\|Q_i^c - \sum_{i=1}^M Q_i^c E_i\|_{\text{operator}} \leq C \exp((\eta - 1)t), \quad t > 0.$$

Let R_i^t , $t \geq 0$, $i = 1, \dots, M$, be the restriction of Q_i^c on $\text{Image}(E_i)$. Then $\{R_i^t; t \geq 0\}$ be a continuous semigroup of linear operators in $\text{Image}(E_i)$. Moreover $\lambda_{n,i}$ is the

unique eigenvalue of $R_{i,n}^t$. So there is a $\eta_i \in \mathbb{C}$ such that $\exp(\eta_i t)$ is the unique eigenvalue of R_i^t , $i = 1, \dots, M$.

Now let $f \in C_b^1(\mathbb{R}^N)$. Then by Proposition 3.1 (2) we have for any $\psi_i \in C_0^\infty(\mathbb{R}^N)$, $i = 0, 1, \dots, N$,

$$\begin{aligned} & \int_{\mathbb{R}^N} dx (\psi_0(x) - \sum_{i=1}^N \nabla_i \psi_i(x)) (P_t^c f)(x) dx \\ &= \int_{\mathbb{R}^N} dx (\psi_0(x), \dots, \psi_N(x)) \cdot (Q_t^c(f, \nabla_1 f, \dots, \nabla_N f))(x). \end{aligned}$$

So we have

$$\begin{aligned} & \exp(-\eta t) \sum_{j=1}^M \int_{\mathbb{R}^N} dx (\psi_0(x), \dots, \psi_N(x)) \cdot (R_i^t(E_j(f, \nabla_1 f, \dots, \nabla_N f)))(x) \\ & \rightarrow \left(\int_{\mathbb{R}^N} \frac{f}{h} d\nu \right) \left(\int_{\mathbb{R}^N} dx (\psi_0(x) - \sum_{i=0}^N \nabla_i \psi_i(x)) h(x) \right). \end{aligned}$$

So we see that

$$\exp(-\eta t) \sum_{i=1}^M R_i^t(E_i(f, \nabla_1 f, \dots, \nabla_N f)) \rightarrow \left(\int_{\mathbb{R}^N} \frac{f}{h} d\nu \right) (h, \nabla_1 h, \dots, \nabla_N h)$$

in the sense of Schwartz' distribution. Since $\text{Image}(E_i)$, $i = 1, \dots, M$, are of finite dimensions and are linearly independent, we see that if $\eta_i \neq \eta$,

$$\exp(-\eta t) R_i^t(E_i(f, \nabla_1 f, \dots, \nabla_N f)) \rightarrow 0, \quad f \in C_b^1(\mathbb{R}^N),$$

and that if $\eta_i = \eta$,

$$\exp(-\eta t) R_i^t(E_i(f, \nabla_1 f, \dots, \nabla_N f)) \rightarrow \left(\int_{\mathbb{R}^N} \frac{f}{h} d\nu \right) (h, \nabla_1 h, \dots, \nabla_N h), \quad f \in C_b^1(\mathbb{R}^N).$$

These imply that if $\text{Re}(\eta_i) \geq \eta$ and $\eta_i \neq \eta$, then

$$E_i(f, \nabla_1 f, \dots, \nabla_N f) = 0, \quad f \in C_b^1(\mathbb{R}^N),$$

and that if $\eta_i = \eta$, then

$$E_i(f, \nabla_1 f, \dots, \nabla_N f) = \left(\int_{\mathbb{R}^N} \frac{f}{h} d\nu \right) (h, \nabla_1 h, \dots, \nabla_N h), \quad f \in C_b^1(\mathbb{R}^N).$$

So we see that $h \in C_b^1(\mathbb{R}^N)$ and that there are $C > 0$ and $\delta > 0$ such that

$$\|\exp(-\eta t) Q_i^c(f, \nabla_1 f, \dots, \nabla_N f) - \left(\int_{\mathbb{R}^N} \frac{f}{h} d\nu \right) (h, \nabla_1 h, \dots, \nabla_N h)\|_E \leq C \exp(-\delta t),$$

for any $t > 0$, $f \in C_b^1(\mathbb{R}^N)$. This completes the proof of Theorem 1.4. ■

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