博士論文

論文題目　　Affinoids in the Lubin-Tate perfectoid space and special cases of the local Langlands correspondence in positive characteristic

（Lubin-Tate パーフェクトイド空間の アフィノイドと正標数の局所 Langlands 対応の 特別な場合について）

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Affinoids in the Lubin-Tate perfectoid space and special cases of the local Langlands correspondence in positive characteristic

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Introduction

Let K be a non-archimedean local field with residue field k of characteristic $p > 0$. Let W_K be the Weil group of K and D the central division algebra over K of invariant $1/n$. Denote by $\mathcal{O}_K \subset K$ the valuation ring and by $\mathfrak{p} \subset \mathcal{O}_K$ the maximal ideal. We fix an algebraic closure $\overline{k}$ of k. Let $n \geq 1$ be an integer. Then the Lubin-Tate spaces are defined to be certain deformation spaces of a one-dimensional formal $\mathcal{O}_K$-module over $\overline{k}$ with level structures. The Lubin-Tate spaces naturally form a projective system, called the Lubin-Tate tower, and the non-abelian Lubin-Tate theory asserts that the cohomology of the Lubin-Tate tower, which admits a natural action of a large subgroup of $\mathrm{GL}_n(K) \times D^\times \times W_K$, realizes the local Langlands correspondence for $\mathrm{GL}_n(K)$ and the local Jacquet-Langlands correspondence simultaneously. However, as the proofs of this fact ([Boye99], [HT01]) make heavy use of the theory of automorphic representations and the global geometry, the geometry of the Lubin-Tate spaces and its relation to representations are not yet fully understood.

Among the studies on the geometry of the Lubin-Tate spaces is a work [Yos10] of Yoshida. There he constructed a semistable model of the Lubin-Tate space of level $\mathfrak{p}$ and proved that an affine open subscheme of the reduction is isomorphic to a Deligne-Lusztig variety for $\mathrm{GL}_n(k)$. The appearance of the Deligne-Lusztig variety reflects the fact that some irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ can be constructed from irreducible cuspidal representations of $\mathrm{GL}_n(k)$. Note that this open subscheme can also be obtained as the reduction of an affinoid subspace in the Lubin-Tate space by considering its tube.

More recently, Weinstein showed in [Wei14] that a certain limit space of the Lubin-Tate tower makes sense as a perfectoid space. While it is no longer an ordinary finite-type analytic space, the Lubin-Tate perfectoid space has a simpler geometry; with coordinates not available on the individual Lubin-Tate spaces, the defining equation is simpler and the group actions can be made very explicit. Taking advantage of these properties, Weinstein[Wei14], Boyarchenko-Weinstein[BW16] and Imai-Tsushima[IT15a] constructed families of affinoid subspaces and their formal models in the Lubin-Tate perfectoid space such that the cohomology of the reduction of each formal model realizes the local Langlands and Jacquet-Langlands correspondences for some representations. The aim of this paper is to establish the existence of such a family of affinoids related to certain other representations, under a simplify-

ing assumption that K is of characteristic $p > 0$.

Let $\ell \neq p$ be a prime number. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Set $G = \mathrm{GL}_n(K) \times D^\times \times W_K$. Here is our main theorem:

Theorem. *Suppose that K is of equal-characteristic and that p does not divide n. Let $\nu > 0$ be an integer which is coprime to n. Let L/K be a totally ramified extension of degree n. Then there exist an affinoid $\mathcal{Z}_\nu$ and a formal model $\mathscr{Z}_\nu$ of $\mathcal{Z}_\nu$ in the Lubin-Tate perfectoid space such that the following hold.*

(1) The stabilizer Stab_ν of $\mathcal{Z}_\nu$ naturally acts on the reduction $\overline{\mathscr{Z}}_\nu$.

(2) For an irreducible smooth representation π of $\mathrm{GL}_n(K)$, we have

$$\mathrm{Hom}_{\mathrm{GL}_n(K)} \left(\text{c-Ind}_{\mathrm{Stab}_\nu}^G H_c^{n-1} \left(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell \right) ((1-n)/2), \pi \right) \neq 0$$

if and only if the image τ of π under the local Langlands correspondence is a character twist of an n-dimensional irreducible smooth representation of the form $\mathrm{Ind}_{L/K} \xi$ for a character ξ of $L^\times$ which is non-trivial on U_L^ν, but trivial on $U_L^{\nu+1}$. Moreover, if the above space is non-zero, it is isomorphic to $\rho \boxtimes \tau$ as a representation of $D^\times \times W_K$, where ρ is the image of π under the local Jacquet-Langlands correspondence.

Here, ξ is identified with a character of the Weil group W_L of L via the Artin reciprocity map and $\mathrm{Ind}_{L/K}$ denotes the smooth induction from W_L to W_K.

Let us compare Theorem with the preceding results. The affinoid $\mathcal{Z}_1$ and the formal model $\mathscr{Z}_1$ in Theorem are essentially identical to those constructed in [IT15a]. Also, in [Wei14], the affinoids and the formal models in Theorem are constructed when $n = 2$ and $p \neq 2$, along with those related to the unramified case in a suitable sense. Thus, Theorem generalizes [IT15a][1] and partially [Wei14], in the equal-characteristic setting. In the terminology of Definition 4.4, which is essentially taken from [BH05b], the above condition for π to occur in the compact induction is equivalent to being parametrized by a minimal admissible pair $(L/K, \xi)$ with the jump at ν. Let F/K be an unramified extension of degree n. The affinoids and the formal models constructed in [BW16] are related, in the same way as in Theorem, to irreducible supercuspidal representations π parametrized by minimal admissible

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[1]However, Imai and Tsushima announced that they also obtained a corresponding result for n divisible by p.

pairs $(F/K, \xi)$ with the jump at some ν (see Remark 4.5 (5) for more on the comparison with the preceding results). The author learned from Imai and Tsushima that they had previously constructed what should be $\mathcal{Z}_2$ and $\mathscr{Z}_2$ in our notation, computed the reduction and verified the non-triviality of the cohomology. Although this unannounced result preceded ours, our result was obtained independently. On a related note, in a recent article [IT15b], the corresponding affinoids in the Lubin-Tate space of level $\mathfrak{p}^2$ are studied when K is of equal-characteristic and $n = 3$.

We note some properties of the affinoids $\mathcal{Z}_\nu$ and the reductions $\overline{\mathscr{Z}}_\nu$. While only those with ν coprime to n are relevant to Theorem, the affinoids $\mathcal{Z}_\nu$ and the formal models $\mathscr{Z}_\nu$ are constructed for any $\nu > 0$ in a certain uniform way. The reductions $\overline{\mathscr{Z}}_\nu$ are related to the perfections of algebraic varieties Z_ν, which turn out to be periodic in ν with period $2n$. They are quite different, according to whether ν is odd or even. If ν is odd, Z_ν is the variety obtained by pulling back the Artin-Schreier covering $\mathbb{A}^1_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$ by a morphism $\mathbb{A}^{n-1}_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$ corresponding to a quadratic form depending on ν. If ν is even, the defining equation of Z_ν is more involved. However, it can be described in terms of the Lang torsor of an algebraic group $\mathcal{G}_\nu$ and a morphism related to a quadratic form (see 2.6 for more details). Here we imitated a similar description found in [BW16], but the analogy is not so straightforward; the relevant algebraic groups are not the same and no quadratic forms occur there.

In Section 1, we review some basic facts on the Lubin-Tate perfectoid space and a formal model, following [Wei14], [BW16] and [IT15a]. In particular, a power series δ is defined which essentially serves as the defining equation of the formal model of the Lubin-Tate perfectoid space. Also the actions of the relevant groups on the formal model are described explicitly.

In Section 2, a family of affinoids and formal models is constructed, and the reductions are studied along with the induced actions of the stabilizers. Building on the notion of CM points and related facts found in [BW16], [IT15a], which are recalled in Subsection 2.1, we construct affinoids $\mathcal{Z}_\nu$ in Subsection 2.2. The construction of formal models $\mathscr{Z}_\nu$ and the computation of the reductions $\overline{\mathscr{Z}}_\nu$ given in Subsection 2.4 are based on the behavior of the power series δ under a certain change of coordinates, which is the subject of Subsection 2.3. While motivated by that in [Wei14], our argument is more intricate. Thus we give a rather detailed account. In Subsection 2.5 we compute the stabilizers Stab_ν of the affinoids $\mathcal{Z}_\nu$ and the induced

actions on the reductions $\overline{\mathscr{Z}}_\nu$. The algebraic groups $\mathcal{G}_\nu$ appearing in the alternative description of Z_ν given in Subsection 2.6 are modeled on the actions of $\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)$.

In Section 3 we compute the cohomology of $\overline{\mathscr{Z}}_\nu$ together with the relevant group actions. This is reduced to the corresponding computation for Z_ν and is treated separately according to whether ν is odd or even. If ν is odd, we compute the cohomology for any ν. In particular, it turns out that the middle-degree cohomology is non-trivial if and only if ν is coprime to n. If ν is even, our understanding is not as complete and we restrict to the cases where ν is coprime to n; this suffices for the proof of the main theorem. Subsections 3.1, 3.2 (resp. Subsection 3.3) contain key ingredients for the computations for the odd (resp. even) cases.

In Section 4 we prove the main theorem described above. To this end, we apply the theory of essentially tame local Langlands and Jacquet-Langlands correspondences developed in [BH05a], [BH05b], [BH10], [BH11], as well as the results obtained in the previous sections. The review of the theory in the special cases that we need is given in Subsection 4.1. In Subsection 4.2 we finally achieve the main theorem.

Let us end this introduction by making a remark on the equal-characteristic assumption. Although this assumption is in force throughout the paper, it plays only a minor role in Sections 3 and 4. It seems reasonable to expect very similar varieties to appear as the reductions of suitable affinoids also in the mixed-characteristic setting. On the other hand, our computation of the reductions and the stabilizers in Section 2 heavily relies on this assumption, especially on the particularly simple expression of δ and the group actions. We hope to consider the problem of extending our results to the mixed-characteristic setting in a future work.

Notation For any non-archimedean valuation field F, we denote the valuation ring by $\mathcal{O}_F \subset F$ and its maximal ideal by $\mathfrak{p}_F \subset \mathcal{O}_F$. If R is a topological ring, we denote by $R^{\circ\circ}$ the set of topologically nilpotent elements.

For any non-archimedean local field F, we write v_F for the additive valuation normalized so that $v_F(\varpi_F) = 1$ for any uniformizer $\varpi_F \in F$. We denote by W_F the Weil group of F and the Artin reciprocity map $\mathrm{Art}_F \colon F^\times \to W_F^{\mathrm{ab}}$ is normalized so that the uniformizers are mapped to geometric Frobenius elements. A multiplicative character ξ of F is often identified with a character of W_F via Art_F.

We take a prime number $\ell \neq p$, and fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ of fields. Smooth representations over $\overline{\mathbb{Q}}_\ell$ are always identified with those over $\mathbb{C}$ by this isomorphism.

1 Preliminaries on Lubin-Tate perfectoid space

We summarize the relevant materials on the Lubin-Tate spaces, the Lubin-Tate perfectoid space and its formal model. Our basic references are [Wei14], [BW16] and [IT15a]. In many parts, we closely follow their expositions.

Let K be a non-archimedean local field of characteristic $p > 0$ and k its residue field. Denote by $\mathfrak{p} = \mathfrak{p}_K$ the maximal ideal of $\mathcal{O}_K$. We write q for the cardinality of k. We fix an algebraic closure $\overline{K}$ of K and denote by $\overline{k}$ the residue field of $\overline{K}$. Let $\mathbb{C}_p$ be the completion of $\overline{K}$.

Let n be a positive integer. Let Σ_0 be a one-dimensional formal $\mathcal{O}_K$-module over $\overline{k}$ of height n, which is unique up to isomorphism. Let K^{ur} be the maximal unramified extension of K and $\widehat{K}^{\mathrm{ur}}$ its completion. We denote by $\mathcal{C}$ the category of complete Noetherian local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field $\overline{k}$. Let $R \in \mathcal{C}$. A pair (Σ, ι) consisting of a formal $\mathcal{O}_K$-module Σ over R and an isomorphism $\iota \colon \Sigma_0 \to \Sigma \otimes_R \overline{k}$ is said to be a deformation of Σ_0 to R. For a formal $\mathcal{O}_K$-module Σ over R and an integer $m \geq 0$, we mean by "a Drinfeld level $\mathfrak{p}^m$-structure on Σ" what is called "a structure of level m on Σ" in [Dri74, p. 572 Definition].

We define a functor $\mathcal{C} \to \mathrm{Sets}$ by associating to $R \in \mathcal{C}$ the set of isomorphism classes of triples (Σ, ι, ϕ) in which (Σ, ι) is a deformation of Σ_0 over R and $\phi \colon (\mathfrak{p}^{-m}/\mathcal{O}_K)^n \to \Sigma[\mathfrak{p}^m](R)$ is a Drinfeld level $\mathfrak{p}^m$-structure on Σ.

This functor is representable by a regular local ring R_m of dimension n by [Dri74, Proposition 4.3]. These rings R_m naturally form an inductive system $\{R_m\}$. We denote by $R_\infty = (\varinjlim R_m)^\wedge$ the completion of the inductive limit with respect to the ideal generated by the maximal ideal of R_0. We put $\mathcal{M}_{\Sigma_0,\infty} = \operatorname{Spf} R_\infty$.

Let K^{ab} be the maximal abelian extension of K and $\widehat{K}^{\mathrm{ab}}$ its completion. We denote by $\wedge\Sigma_0$ the formal $\mathcal{O}_K$-module of height 1 over $\overline{k}$. Then from the above discussion a formal scheme $\mathcal{M}_{\wedge\Sigma_0,\infty}$ is defined. By the Lubin-Tate theory we have $\mathcal{M}_{\wedge\Sigma_0,\infty} \simeq \operatorname{Spf} \mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$.

We take a uniformizer ϖ of K. Let (Σ, ι) be a deformation of Σ_0 to $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ and A its coordinate ring. We set $\widetilde{\Sigma}_0 = \operatorname{Spf}(\varinjlim A)^\wedge$, where the transition maps are ring homomorphisms corresponding to the multiplication by ϖ of Σ and the completion is taken with respect to the ideal generated by a defining ideal of A. Then $\widetilde{\Sigma}_0$ is a K-vector space object in the category of complete, adic $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras. It is shown in [Wei14, Proposition 2.4.2] that $\widetilde{\Sigma}_0$, as a K-vector space object, does not depend on the choice of (Σ, ι) and is isomorphic to $\operatorname{Spf} \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[X^{q^{-\infty}}]]$ as a formal scheme. Here $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[X^{q^{-\infty}}]]$ is defined to be the (ϖ, X)-adic completion of $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[X^{q^{-\infty}}] = \varinjlim_{X \mapsto X^q} \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[X]$.

Theorem 1.1. *There is a canonical Cartesian diagram of formal schemes*:

$$
\begin{array}{ccc}
\mathcal{M}_{\Sigma_0,\infty} & \longrightarrow & \mathcal{M}_{\wedge\Sigma_0,\infty} \\
\downarrow & & \downarrow \\
\widetilde{\Sigma}_0^n & \longrightarrow & \widetilde{\wedge\Sigma}_0.
\end{array}
\tag{1.1}
$$

Proof. This is proved in [Wei14]. The constructions of morphisms are given in [Wei14, 2.5-2.7] and the fact that the diagram is Cartesian is proved in [Wei14, Theorem 2.7.3]. □

Let $\operatorname{Map}(U_K, \mathcal{O}_{\mathbb{C}_p})$ denote the $\mathcal{O}_{\mathbb{C}_p}$-algebra of continuous maps from U_K to $\mathcal{O}_{\mathbb{C}_p}$. We use similar notations for other topological rings as well. We set

$$
R_{\infty,\mathcal{O}_{\mathbb{C}_p}} = R_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}} \operatorname{Map}(U_K, \mathcal{O}_{\mathbb{C}_p}),
$$

where the right factor is considered as an $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$-algebra via

$$
\alpha \colon \mathcal{O}_{\widehat{K}^{\mathrm{ab}}} \to \operatorname{Map}(U_K, \mathcal{O}_{\mathbb{C}_p}); \ a \mapsto (\operatorname{Art}_K(u)(a))_{u \in U_K}.
$$

With this algebra, we define the Lubin-Tate perfectoid space as

$$
\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}} = \{|\cdot| \in \operatorname{Spa}(R_{\infty,\mathcal{O}_{\mathbb{C}_p}}, R_{\infty,\mathcal{O}_{\mathbb{C}_p}}) \mid |\varpi| \neq 0\}.
$$

We also set

$$\mathcal{M}^{\mathrm{ad}}_{\wedge\Sigma_0,\infty,\overline{\eta}} = \{|\cdot| \in \mathrm{Spa}(\mathrm{Map}(U_K, \mathcal{O}_{\mathbb{C}_p}), \mathrm{Map}(U_K, \mathcal{O}_{\mathbb{C}_p})) \mid |\varpi| \neq 0\}.$$

Setting $B_n = \mathcal{O}_{\mathbb{C}_p}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]]$ and $B_1 = \mathcal{O}_{\mathbb{C}_p}[[T^{q^{-\infty}}]]$, we similarly define

$$\widetilde{\Sigma}^{n,\mathrm{ad}}_{0,\infty,\overline{\eta}} = \{|\cdot| \in \mathrm{Spa}(B_n, B_n) \mid |\varpi| \neq 0\},$$
$$\widetilde{\wedge\Sigma}^{\mathrm{ad}}_{0,\infty,\overline{\eta}} = \{|\cdot| \in \mathrm{Spa}(B_1, B_1) \mid |\varpi| \neq 0\}.$$

In what follows, we give an explicit description of the following diagram

$$
\begin{array}{ccc}
R_{\infty,\mathcal{O}_{\mathbb{C}_p}} & \longleftarrow & \mathrm{Map}(U_K, \mathcal{O}_{\mathbb{C}_p}) \\
\uparrow & & \uparrow \\
B_n & \longleftarrow & B_1
\end{array}
$$

induced by the Cartesian diagram (1.1) and α, in terms of coordinates. Basically, we closely follow the general treatment in [IT15a, 1.1], but adopt and specialize it to our equal-characteristic setting (see also [Wei14, 2.10, 2.11], [Wei13, 5.1]).

For a formal $\mathcal{O}_K$-module Σ and $a \in K$, we denote by $[a]_\Sigma$ the multiplication by a of Σ. Take a model of Σ_0 so that

$$[\varpi]_{\Sigma_0}(X) = X^{q^n}, \quad [\zeta]_{\Sigma_0}(X) = \zeta X \text{ for } \zeta \in k.$$

We set $\mathcal{O}_D = \mathrm{End}\,\Sigma_0$ and $D = \mathcal{O}_D \otimes_{\mathcal{O}_K} K$. Then D is a central division algebra over K of invariant $1/n$. Denoting by $[a]$ the action of $a \in \mathcal{O}_D$, we define $\varphi_D \in \mathcal{O}_D$ to be the element such that $[\varphi_D](X) = X^q$. Let k_n be the field extension of k of degree n, so that $K_n = K \otimes_k k_n$ is the unramified extension of degree n of K. We define a K-algebra embedding $K_n \to D$ by $[\zeta](X) = \zeta X$ for $\zeta \in k_n$. Then D is generated over K_n by φ_D and we have $\varphi_D \zeta = \zeta^q \varphi_D$.

Let $\widehat{\wedge\Sigma_0}$ be the one-dimensional formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ defined by

$$[\varpi]_{\widehat{\wedge\Sigma_0}}(X) = \varpi X + (-1)^{n-1}X^q, \quad [\zeta]_{\widehat{\wedge\Sigma_0}}(X) = \zeta X \text{ for } \zeta \in k.$$

Let $\{t_m\}_{m \geq 1}$ be a system of elements of $\overline{K}^{\circ\circ}$ such that

$$t_m \in \overline{K}^{\circ\circ}, \quad [\varpi]_{\widehat{\wedge\Sigma_0}}(t_1) = 0, \quad t_1 \neq 0, \quad [\varpi]_{\widehat{\wedge\Sigma_0}}(t_m) = t_{m-1} \text{ for } m \geq 2$$

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and set $t = \lim_{m \to \infty} (-1)^{(n-1)(m-1)} t_m^{q^{m-1}} \in \mathcal{O}_{\mathbb{C}_p}$. We denote by $v = v_K$ the normalized valuation on K and extend it to $\mathbb{C}_p$ by continuity. Then $v(t) = 1/(q-1)$.

We put $\varpi' = (-1)^{n-1}\varpi$ and $t'_m = (-1)^{(n-1)(m-1)}t_m$ for $m \geq 1$. Then we have

$$[\varpi']_{\widehat{\wedge \Sigma_0}}(X) = \varpi' X + X^q, \quad [\zeta]_{\widehat{\wedge \Sigma_0}} = \zeta X \text{ for } \zeta \in k,$$

$$[\varpi']_{\widehat{\wedge \Sigma_0}}(t'_1) = 0, \quad [\varpi']_{\widehat{\wedge \Sigma_0}}(t'_m) = t'_{m-1} \text{ for } m \geq 2, \quad t = \lim_{m \to \infty} t'^{q^{m-1}}_m.$$

By [Wei14, Proposition 2.3.3, Corollary 2.8.14] we have $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}} = \overline{k}[[t^{q^{-\infty}}]]$ and the continuous $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-homomorphism $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[T^{q^{-\infty}}]] \to \mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$ induced by the right vertical morphism of (1.1) sends T to t. Thus the morphism $B_1 \to \mathrm{Map}(U_K, \mathcal{O}_{\mathbb{C}_p})$ induced by α is described as

$$B_1 \to \mathrm{Map}(U_K, \mathcal{O}_{\mathbb{C}_p}); \ f(T) \mapsto \left(f\left(\mathrm{Art}_K(u)(t)\right)\right)_{u \in U_K}.$$

We denote by $\Delta(X_1, \ldots, X_n)$ the Moore determinant

$$\Delta(X_1, \ldots, X_n) = \det(X_i^{q^{j-1}})_{1 \leq i,j \leq n} \in \mathbb{Z}[X_1, \ldots, X_n]$$

and put

$$\delta(X_1, \ldots, X_n) = \sum_{m_1 + \cdots + m_n = 0} \Delta(X_1^{q^{m_1 n}}, \ldots, X_n^{q^{m_n n}}) \in B_n.$$

For $m \in \mathbb{Z}$, we also put

$$\delta_m(X_1, \ldots, X_n) = \delta(X_1, \ldots, X_n)^{q^{-m}} = \sum_{m_1 + \cdots + m_n = 0} \Delta(X_1^{q^{m_1 n - m}}, \ldots, X_n^{q^{m_n n - m}})$$

for later use. Then the continuous $\mathcal{O}_{\mathbb{C}_p}$-algebra homomorphism $B_1 \to B_n$ induced by the lower horizontal morphism in (1.1) is

$$B_1 \to B_n; \ f(T) \mapsto f(\delta(X_1, \ldots, X_n)).$$

This can be found in [IT15a, 1.1] (see also [BW16, Theorem 2.15] and [Wei13, Lemma 5.1.2]). We call δ the determinant map.

Remark 1.2. *Let $x_1, \ldots, x_n \in B_n^{\circ\circ}$. Then, since B_n is perfect, we can substitute X_i in $\delta(X_1, \ldots, X_n)$ with x_i for $1 \leq i \leq n$ in a natural way. The following properties of δ_0 are easy to check, but useful;*

$$\delta_0(x_1^{q^n}, \ldots, x_n) = (-1)^{n-1}\delta_{-1}(x_1, \ldots, x_n),$$
$$\delta_0(x_1^{q^{-n}}, \ldots, x_n) = (-1)^{n-1}\delta_1(x_1, \ldots, x_n),$$
$$\delta_0(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \operatorname{sgn}(\sigma)\delta_0(x_1, \ldots, x_n).$$

In particular, we have

$$\delta_m(x_1, \ldots, x_n) = \delta_{m+1}(x_n^{q^n}, x_1, \ldots, x_{n-1}),$$
$$\delta_m(x_1, \ldots, x_n) = \delta_{m-1}(x_2, \ldots, x_n, x_1^{q^{-n}}).$$

Put $G = \operatorname{GL}_n(K) \times D^\times \times W_K$. We define N_G by

$$N_G\colon \operatorname{GL}_n(K) \times D^\times \times W_K \to K^\times; \ (g, d, \sigma) \mapsto \det g^{-1}\operatorname{Nrd} d \operatorname{Art}_K^{-1}\sigma,$$

where $\operatorname{Nrd}\colon D^\times \to K^\times$ is the reduced norm, and set $G^0 = \operatorname{Ker}(v \circ N_G)$. The Lubin-Tate perfectoid space $\mathcal{M}_{\Sigma_0, \infty, \overline{\eta}}^{\mathrm{ad}}$ and its formal model $R_{\infty, \mathcal{O}_{\mathbb{C}_p}}$ carry natural actions of G^0 induced by that on the Lubin-Tate tower, studied in the non-abelian Lubin-Tate theory. Following [IT15a, 1.2], we explicitly write down the action of G^0 on $R_{\infty, \mathcal{O}_{\mathbb{C}_p}}$ in the equal-characteristic setting (see also [BW16, 2.11]). We describe the left action of G on B_n which induces that of G^0 on $R_{\infty, \mathcal{O}_{\mathbb{C}_p}}$.

Let $g = (a_{i,j})_{1 \leq i,j \leq n} \in \operatorname{GL}_n(K)$ and write $a_{i,j} = \sum_l a_{i,j}^{(l)}\varpi^l \in K$ with $a_{i,j}^{(l)} \in k$. Then g acts on the ring B_n as a continuous $\mathcal{O}_{\mathbb{C}_p}$-homomorphism defined by

$$g^*\colon B_n \to B_n; \ X_i \mapsto \sum_{1 \leq j \leq n} \sum_l a_{j,i}^{(l)} X_j^{q^{ln}} \quad \text{for } 1 \leq i \leq n. \tag{1.2}$$

Let $d \in D^\times$ and write $d^{-1} = \sum_l d_l \varphi_D^l$ with $d_l \in k_n$. Then d acts on the ring B_n as a continuous $\mathcal{O}_{\mathbb{C}_p}$-homomorphism defined by

$$d^*\colon B_n \to B_n; \ X_i \mapsto \sum_l d_l X_i^{q^l} \quad \text{for } 1 \leq i \leq n. \tag{1.3}$$

Let $\sigma \in W_K$ and set $n_\sigma = v(\operatorname{Art}_K^{-1}(\sigma))$. Then $\sigma \in W_K$ acts on the ring B_n as a continuous ring homomorphism defined by

$$\sigma^*\colon B_n \to B_n; \ X_i \mapsto X_i^{q^{-n_\sigma}}, \quad x \mapsto \sigma(x), \quad \text{for } 1 \leq i \leq n \text{ and } x \in \mathcal{O}_{\mathbb{C}_p}. \tag{1.4}$$

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We remark that a natural action of $K^\times$ (resp. of U_K) on B_1 (resp. on $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$) is deduced as a special case of the above. With respect to these actions the upper horizontal morphism (resp. the lower horizontal morphism) of the base change to $\mathcal{O}_{\mathbb{C}_p}$ of the Cartesian diagram (1.1) is G^0-equivariant (resp. G-equivariant) via N_G.

2 Affinoids and the reductions of formal models

2.1 CM points

We briefly review some facts on CM points, following [BW16] and [IT15a].

For a deformation Σ of Σ_0 over $\mathcal{O}_{\mathbb{C}_p}$, we set

$$T_{\mathfrak{p}}\Sigma = \varprojlim \Sigma[\mathfrak{p}^m](\mathcal{O}_{\mathbb{C}_p}),$$
$$V_{\mathfrak{p}}\Sigma = T_{\mathfrak{p}}\Sigma \otimes_{\mathcal{O}_K} K,$$

where each transition map is the multiplication by ϖ. By [BW16, Definition 2.10.1], a point $\xi \in \mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}(\mathbb{C}_p)$ defines a corresponding triple (Σ, ι, ϕ), where Σ is a formal $\mathcal{O}_K$-module over $\mathcal{O}_{\mathbb{C}_p}$, $\iota \colon \Sigma_0 \to \Sigma \otimes_{\mathbb{C}_p} \overline{k}$ is an isomorphism and $\phi \colon \mathcal{O}_K^n \to T_{\mathfrak{p}}\Sigma$ is an isomorphism of $\mathcal{O}_K$-modules.

Definition 2.1. Let $L \subset \mathbb{C}_p$ be an extension of K of degree n and let Σ be a deformation of Σ_0 to $\mathcal{O}_{\mathbb{C}_p}$.

We say that Σ has CM by L if there exists a K-isomorphism $L \xrightarrow{\sim} (\operatorname{End}\Sigma) \otimes_{\mathcal{O}_K} K$ such that the induced homomorphism $L \to \operatorname{End}(\operatorname{Lie}\Sigma) \simeq \mathbb{C}_p$ agrees with the inclusion $L \subset \mathbb{C}_p$. We also say that $\xi \in \mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}(\mathbb{C}_p)$ has CM by L if the corresponding deformation has CM by L.

Note that the K-isomorphism in the definition is determined uniquely by the compatibility with the induced homomorphism, if it exists.

A point $\xi \in \mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}(\mathbb{C}_p)$ with CM by L defines a K-embedding $i_\xi \colon L \to M_n(K)$; $x \mapsto i_\xi(x)$ by the commutativity of the following diagram

$$
\begin{array}{ccc}
K^n & \xrightarrow{\phi \otimes \mathrm{id}} & V_{\mathfrak{p}}\Sigma \\
{\scriptstyle i_\xi(x)}\downarrow & & \downarrow{\scriptstyle V_{\mathfrak{p}}(x)} \\
K^n & \xrightarrow[\phi \otimes \mathrm{id}]{} & V_{\mathfrak{p}}\Sigma
\end{array}
$$

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and similarly defines $i_\xi^D \colon L \to D$; $x \mapsto i_\xi^D(x)$ by

$$
\begin{array}{ccc}
\Sigma_0 & \xrightarrow{\ \iota\ } & \Sigma \otimes_{\mathcal{O}_{\mathbb{C}_p}} \overline{k} \\
{\scriptstyle i_\xi^D(x)} \downarrow & & \downarrow {\scriptstyle x \otimes \mathrm{id}} \\
\Sigma_0 & \xrightarrow[\ \iota\ ]{} & \Sigma \otimes_{\mathcal{O}_{\mathbb{C}_p}} \overline{k},
\end{array}
$$

where morphisms are considered up to isogeny in both diagrams.

We set $\Delta_\xi = (i_\xi, i_\xi^D) \colon L \to M_n(K) \times D$.

The following are consequences of the Lubin-Tate theory as proved in [BW16, Lemmas 3.1.2, 3.1.3] (see also [IT15a, Lemmas 1.9, 3.4]).

Proposition 2.2. *Let L and $\xi \in \mathcal{M}^{ad}_{\Sigma_0,\infty,\overline{\eta}}(\mathbb{C}_p)$ be as above.*

(1) *The group $G^0 \cap (\mathrm{GL}_n(K) \times D^\times)$ acts transitively on the set of points on $\mathcal{M}^{ad}_{\Sigma_0,\infty,\overline{\eta}}(\mathbb{C}_p)$ with CM by L. The stabilizer of ξ in this group is $\Delta_\xi(L^\times)$.*

(2) *Put $W_{L'} = \{\sigma \in W_K \mid \sigma(L) = L\}$. Then, for an element $\sigma \in W_K$, the translation[2] $(1, \varphi_D^{-n_\sigma}, \sigma)^*\xi$ has CM by L if and only if $\sigma \in W_{L'}$.*

(3) *If $\sigma \in W_L$, then $(1, (\mathrm{Art}_L^{-1}\sigma)^{-1}, \sigma)^*\xi = \xi$.*

By (1) and (2), for any $\sigma \in W_{L'}$, there exists an element $(g, d) \in \mathrm{GL}_n(K) \times D^\times$, uniquely up to multiplication by $\Delta_\xi(L^\times)$, such that $(g, d, \sigma) \in G^0$ and $(g, d, \sigma)^*\xi = \xi$. We define a map $j_\xi \colon W_{L'} \to L^\times \backslash (\mathrm{GL}_n(K) \times D^\times)$ by $j_\xi(\sigma) = L^\times(d, \sigma)$. Then the stabilizer $\mathcal{S}$ of ξ in G^0 is

$$
\mathcal{S} = \{(g, d, \sigma) \in \mathrm{GL}_n(K) \times D^\times \times W_{L'} \mid j_\xi(\sigma) = L^\times(g, d)\}.
$$

The assertion (3) says that $j_\xi(\sigma) = L^\times(1, (\mathrm{Art}_L^{-1}\sigma)^{-1})$ if $\sigma \in W_L$.

Now let $n \geq 2$ and assume $p \nmid n$. We put $n_q = \gcd(n, q-1)$.

For any uniformizer $\varpi \in K$, we set $L_\varpi = K[X]/(X^n - \varpi)$.

Lemma 2.3. *[IT15a, Lemma 2.1] Let $T(K, n)$ be the set of isomorphism classes of totally ramified extensions of K of degree n. Then the following map is a bijection:*

$$
\mu_{(q-1)/n_q}(K) \backslash (\mathfrak{p} - \mathfrak{p}^2)/\mathfrak{p}^2 \to T(K, n); \ \varpi \mapsto L_\varpi.
$$

---

[2]Somewhat awkwardly, we use the notation $g^*(\cdot)$ for the actions of $g \in G^0$ on both $R_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ and $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$.

Let L/K be a totally ramified extension of degree n in $\mathbb{C}_p$. From this point on, we work with this fixed field L. Although we do not indicate in the notation the constructions to follow depend on the choice of L.

By Lemma 2.3, there exists a uniformizer $\varphi_L \in L$ such that $\varpi = \varphi_L^n \in K$. We apply the arguments of Section 1 with respect to this uniformizer $\varpi \in K$. In particular, a model of Σ_0 is chosen, and $\varphi_D \in D$ and $\widehat{\wedge \Sigma_0}$ are defined. We set

$$\varphi = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ \varpi & \mathbf{0} \end{pmatrix} \in M_n(K).$$

Note that $\sigma \in W_K$ lies in $W_{L'}$ if and only if $\varphi_L^{-1}\sigma(\varphi_L) \in \mu_{n_q}(K)$.

For a point $\xi \in \mathcal{M}_{\Sigma_0,\infty,\overline{\eta}}^{\mathrm{ad}}(\mathbb{C}_p)$, we write $(\xi_1,\dots,\xi_n) \in \mathbb{C}_p^n$ for the coordinate with respect to $X_1,\dots,X_n \in B_n$.

Proposition 2.4. *Let $L \subset \mathbb{C}_p$, $\varphi_L \in L$, $\varpi \in K$ be as above. There exists a point $\xi \in \mathcal{M}_{\Sigma_0,\infty,\overline{\eta}}^{ad}(\mathbb{C}_p)$ with CM by L satisfying the following conditions:*

(1) $\xi_i = \xi_{i+1}^q$ for $1 \leq i \leq n-1$.

(2) $v(\xi_i) = 1/(nq^{i-1}(q-1))$ for $1 \leq i \leq n$.

(3) $i_\xi(\varphi_L) = \varphi$, $i_\xi^D(\varphi_L) = \varphi_D$.

(4) For any $\sigma \in W_{L'}$, there exists an element $\zeta \in \overline{K}$ such that $\zeta^{q-1} = \varphi_L^{-1}\sigma(\varphi_L)$ and $\xi_1^{-1}\sigma(\xi_1) \equiv \zeta \mod \mathfrak{p}_{\overline{K}}$.

Proof. This is essentially [IT15a, Lemma 2.2], where an explicit construction of $\xi \in \mathcal{M}_{\Sigma_0,\infty,\overline{\eta}}^{\mathrm{ad}}(\mathbb{C}_p)$ is given. The assertion (4) is not stated there but follows from the construction. $\square$

2.2   Construction of affinoids

Put

$$Y_i = X_i - \xi_i \in B_n \text{ for all } 1 \leq i \leq n,$$
$$Z = \sum_{1 \leq i \leq n} Y_i^{q^{i-1}} \in B_n.$$

For each integer $\nu > 0$, we define an affinoid $\mathcal{X}_\nu \subset \widetilde{\Sigma}_{0,\infty,\overline{\eta}}^{n,\mathrm{ad}}$ by

$$|Z| \leq |\xi_1|^{q^\nu}, \quad |Y_i| \leq \begin{cases} |\xi_i|^{q^\mu(q+1)/2} & \text{if } \nu = 2\mu+1 \text{ is odd} \\ |\xi_i|^{q^\mu} & \text{if } \nu = 2\mu \text{ is even} \end{cases}$$

and an affinoid $\mathcal{Z}_\nu \subset \mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ by the pull-back of $\mathcal{X}_\nu$ in $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$. Take a square root $\xi_n^{1/2}$ of ξ_n and put $\xi_i^{1/2} = (\xi_n^{1/2})^{q^{n-i}}$. We also define a formal model $\mathscr{X}_\nu$ of $\mathcal{X}_\nu$ by

$$\mathscr{X}_\nu = \mathrm{Spf}\,\mathcal{O}_{\mathbb{C}_p}\langle z'^{q^{-\infty}}, y_2'^{\,q^{-\infty}}, \ldots, y_n'^{\,q^{-\infty}}\rangle,$$

$$z' = \frac{Z}{\xi_1^{q^\nu}}, \;\; y_i' = \begin{cases} \frac{Y_i}{\xi_i^{q^\mu(q+1)/2}} & \text{if } \nu = 2\mu+1 \text{ is odd} \\ \frac{Y_i}{\xi_i^{q^\mu}} & \text{if } \nu = 2\mu \text{ is even.} \end{cases} \tag{2.1}$$

To construct a formal model of $\mathcal{Z}_\nu$ and to study its special fiber, we prove several lemmas on the determinant map δ.

2.3   Lemmas on the determinant map

Lemma 2.5. *Let* $1 \le i \le n$ *be an integer. Let* $x_n, T_i \in R^{\circ\circ}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$. *Put* $x_{n+m} = x_n^{q^{-m}}$ *for all* $m \in \mathbb{Z}$. *Then*

$$\delta_0(T_i^{q^{i-1}}, x_2, \ldots, x_n) = \delta_0(x_1, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_n).$$

Proof. To prove the equality we may assume $i > 1$. By definition of δ_0 we have

$$\delta_0(T_i^{q^{i-1}}, x_2, \ldots, x_n) = \sum_{m=(m_j)\in S} \mathrm{sgn}(\sigma_m) T_i^{q^{i-1+m_1}} x_2^{q^{m_2}} \cdots x_n^{q^{m_n}},$$

$$\delta_0(x_1, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_n) = \sum_{m=(m_j)\in S} \mathrm{sgn}(\sigma_m) x_1^{q^{m_1}} \cdots x_{i-1}^{q^{m_{i-1}}} T_i^{q^{m_i}} x_{i+1}^{q^{m_{i+1}}} \cdots x_n^{q^{m_n}},$$

where

$$S = \left\{ (m_j) \in \mathbb{Z}^n \,\middle|\, \sum_{j=1}^n m_j = \sum_{j=1}^n (j-1), \; m_j \not\equiv m_{j'} \mod n \; (\text{if } j \ne j') \right\}$$

and

$$\sigma_m = \begin{pmatrix} \overline{0} & \overline{1} & \ldots & \overline{n-1} \\ \overline{m_1} & \overline{m_2} & \ldots & \overline{m_n} \end{pmatrix}$$

is a permutation of $\mathbb{Z}/n\mathbb{Z}$. (Here and in the rest of the proof $\overline{j}$ denotes the image of $j \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$.) As the two series converge absolutely, it suffices to

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construct a bijection $f\colon S \to S$ such that, for $m = (m_j) \in S$,

$$f(m)_i = m_1 + i - 1, \tag{2.2}$$

$$[m_j + n - j \mid 2 \le j \le n] = [f(m)_j + n - j \mid 1 \le j \le n,\ j \ne i] \quad \text{(as multisets)}, \tag{2.3}$$

$$\operatorname{sgn}\left(\begin{array}{cccc} \overline{m_1} & \overline{m_2} & \cdots & \overline{m_n} \\ f(m)_1 & f(m)_2 & \cdots & f(m)_n \end{array}\right) = 1. \tag{2.4}$$

Given $m = (m_j) \in S$, we are to define $f(m) \in S$. This is done based on the following inductive steps. Put $j_0 = i$ and $j_1 = 1$.

Given $1 \le j_a, j_{a+1} \le n$, define

$$m'_{j_a} = m_{j_{a+1}} - j_{a+1} + j_a \tag{2.5}$$

and j_{a+2} to be the unique integer $1 \le j_{a+2} \le n$ such that

$$m_{j_{a+2}} \equiv m'_{j_a} \mod n. \tag{2.6}$$

We repeat this procedure for $a = 0, 1, \ldots, b - 2$, where $b \ge 2$ is the smallest integer such that $j_b = j_c$ for some $0 \le c < b$. We claim that $c = 0$ here. To see this, note that

$$m_{j_a} \equiv m_{j_1} - j_{a-1} + j_0 \mod n \tag{2.7}$$

for all $a \ge 1$ by construction. Therefore, if $c > 0$, then

$$m_{j_1} - j_{c-1} + j_0 \equiv m_{j_c} = m_{j_b} \equiv m_1 - j_{b-1} + j_0 \mod n,$$

which implies $j_{c-1} = j_{b-1}$, contradicting the minimality of b.

Thus we have defined $j_0, \ldots, j_{b-1}, j_b = i$ and $m_{j_0}, \ldots, m_{j_{b-2}}$. Define $m'_{j_{b-1}}$ by (2.5) for $a = b - 1$. Finally define $f(m) \in \mathbb{Z}^n$ by

$$f(m)_j = \begin{cases} m'_j & \text{if } j = j_a \text{ for some } 0 \le a \le b - 1, \\ m_j & \text{otherwise.} \end{cases}$$

By (2.5) we clearly have

$$\sum_{0 \le a \le b-1} m_{j_a} = \sum_{0 \le a \le b-1} m'_{j_a}$$

and hence $\sum_{1 \le j \le n} m_j = \sum_{1 \le j \le n} f(m)_j$. By (2.6), the definition of $m'_{j_{b-1}}$ and (2.7), the map $\overline{m}_j \mapsto \overline{f(m)}_j$ defines a permutation on $\mathbb{Z}/n\mathbb{Z}$. Therefore, we see that $f(m) \in S$.

The properties (2.2) and (2.3) follow from (2.5). The property (2.4) can be checked as

$$\mathrm{sgn} \begin{pmatrix} \overline{m}_1 & \overline{m}_2 & \cdots & \overline{m}_n \\ \overline{f(m)}_1 & \overline{f(m)}_2 & \cdots & \overline{f(m)}_n \end{pmatrix} = \mathrm{sgn} \begin{pmatrix} 0 & 1 & \cdots & b-2 & b-1 \\ 2 & 3 & \cdots & 0 & 1 \end{pmatrix} = 1.$$

Also, we can readily construct the inverse map in a similar way. This completes the proof. $\qquad\square$

In the rest of this subsection, let $|\cdot| \in \mathcal{M}^{\mathrm{ad}}_{\Sigma_0, \infty, \bar{\eta}}$.

Lemma 2.6. *Let $r_1 \ge r_2 \ge \cdots \ge r_n$ be rational numbers. Let $x \in \mathcal{O}^{\circ\circ}_{\mathbb{C}_p}$ and $x_i \in R^{\circ\circ}_{\infty, \mathcal{O}_{\mathbb{C}_p}}$ $(1 \le i \le n)$. Suppose that $|x_i| \le |x|^{r_i}$ for all $1 \le i \le n$ and that $r_1 < q^n r_n$. Then*

$$|\delta_0(x_1, \ldots, x_n)| \le |x|^{\sum q^{i-1} r_i},$$

$$\left| \delta_0(x_1, \ldots, x_n) - \sum (\mathrm{sgn}\, \sigma) x_{\sigma(1)} x^q_{\sigma(2)} \cdots x^{q^{n-1}}_{\sigma(n)} \right| < |x|^{\sum q^{i-1} r_i},$$

where the summation in the left-hand side of the second inequality is taken over the permutations $\sigma \in \mathfrak{S}_n$ such that $r_{\sigma(1)} \ge \cdots \ge r_{\sigma(n)}$.

Proof. Put $d_i = \log_q r_i$ for all $1 \le i \le n$, so that $d_1 - d_n < n$. The monomials occurring in $\delta_0(x_1, \ldots, x_n)$ are all of the form $x^{q^{m_1}}_1 \cdots x^{q^{m_n}}_n$ for some $(m_1, \ldots, m_n) \in S$, where S is as in the proof of the preceding lemma, and we have

$$|x^{q^{m_1}}_1 \cdots x^{q^{m_n}}_n| = \prod_i |x_i|^{q^{m_i}} \le |x|^{\sum q^{m_i} r_i} = |x|^{\sum q^{m_i + d_i}}.$$

To facilitate our argument, we introduce a total order structure $\ge$ on the set of all the multisets of n real numbers by deeming $[m_1, \ldots, m_n] \ge [m'_1, \ldots, m'_n]$ if and only if, when altering the indexing so that $m_1 \ge \cdots \ge m_n$ and $m'_1 \ge \cdots \ge m'_n$, we have $(m_1, \ldots, m_n) \ge (m'_1, \ldots, m'_n)$ with respect to the lexicographic order on $\mathbb{R}^n$. Then it is easily verified that, assuming $\sum_{1 \le i \le n} m_i = \sum_{1 \le i \le n} m'_i$, we have $\sum_{1 \le i \le n} q^{m_i} \ge \sum_{1 \le i \le n} q^{m'_i}$ if and only

if $[m_1, \ldots, m_n] \geq [m'_1, \ldots, m'_n]$. Thus, putting $f(m_1, \ldots, m_n) = [m_1 + d_1, \ldots, m_n + d_n]$, we are to show that the set

$$O = \{f(m_1, \ldots, m_n) \mid (m_1, \ldots, m_n) \in S\}$$

admits the smallest element $f(0, \ldots, n-1)$ (with respect to the induced order structure) and that it is attained only by those $(m_1, \ldots, m_n) \in S$ such that

$$\{m_1, \ldots, m_n\} = \{0, \ldots, n-1\} \text{ and } d_{\sigma(1)} \geq \cdots \geq d_{\sigma(n)}, \qquad (2.8)$$

where

$$\sigma = \begin{pmatrix} m_1 + 1 & \cdots & m_n + 1 \\ 1 & \cdots & n \end{pmatrix} \in \mathfrak{S}_n.$$

Let us show that if $(m_1, \ldots, m_n) \in S$ does not satisfy (2.8) then $f(m_1, \ldots, m_n)$ admits a strictly smaller element in O.

First assume that $\{m_1, \ldots, m_n\} \neq \{0, \ldots, n-1\}$. Then there exist $1 \leq i, j \leq n$ such that $m_i \geq n$ and $m_j \leq -1$. Now replacing m_i with $m_j + n$ and m_j with $m_i - n$ yields a strictly smaller element:[3]

$$f(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_n) > f(m_1, \ldots, m_j + n, \ldots, m_i - n, \ldots, m_n)$$

because $m_i + d_i > m_j + d_j, \ m_j + n + d_i, \ m_i - n + d_j$.

Next assume that $\{m_1, \ldots, m_n\} = \{0, \ldots, n-1\}$ but (2.8) does not hold. Then there exists $1 \leq i \leq n-1$ such that $d_{\sigma(i)} < d_{\sigma(i+1)}$ with σ as before (so that $\sigma(i) > \sigma(i+1)$). Now interchanging $m_{\sigma(i)} = i - 1$ and $m_{\sigma(i+1)} = i$ yields a strictly smaller element:

$$f(m_1, \ldots, \overset{\sigma(i+1)}{\overbrace{i}}, \ldots, i-1, \ldots, m_n) > f(m_1, \ldots, i-1, \ldots, \overset{\sigma(i)}{\overbrace{i}}, \ldots, m_n).$$

Finally, given an element $(m_1, \ldots, m_n) \in S$ such that $f(m_1, \ldots, m_n) \neq f(0, \ldots, n-1)$, we may apply the above procedures finitely many times to obtain strict inequalities $f(m_1, \ldots, m_n) > f(m'_1, \ldots, m'_n) > \cdots$ until we eventually find some element $(\tilde{m}_1, \ldots, \tilde{m}_n) \in S$ such that

$$f(m_1, \ldots, m_n) > \cdots > f(\tilde{m}_1, \ldots, \tilde{m}_n) = f(0, \ldots, n-1).$$

Therefore, $f(0, \ldots, n-1)$ is indeed the smallest in O. The same argument shows that $f(m_1, \ldots, m_n) = f(0, \ldots, n-1)$ only if $(m_1, \ldots, m_n)$ satisfies (2.8). Now the proof is complete. $\qquad \square$

---

[3]Here the inequality is written as if $i < j$, but this is only for a notational convenience. We do not assume $i < j$ and the argument clearly works without this assumption.

Although the following two lemmas are in principle simple applications of the preceding lemma, they involve many cases. To state them concisely we define, for an integer $0 \le \mu \le n-1$ and a rational number $1 \le c < q$,

$$M_1(\mu) = (n + \mu(q-1))q^{n-1},$$

$$M_2(\mu, c) = \begin{cases} (n + 2(c-1) + 2\mu(q-1))q^{n-1} & \text{if } 0 \le \mu < n/2 \\ (n + 2(c-1) + (2\mu - n)(q-1))q^n & \text{if } n/2 \le \mu < n. \end{cases}$$

Lemma 2.7. *Let $0 \le \mu \le n-1$ be an integer. Let $x_n \in \mathcal{O}_{\mathbb{C}_p}^{\circ\circ}$, $T \in R_{\infty, \mathcal{O}_{\mathbb{C}_p}}^{\circ\circ}$. Put $x_{n+m} = x_n^{q^{-m}}$ for all $m \in \mathbb{Z}$. Suppose that $|T| \le |x_1|^{q^\mu}$. Put*

$$\delta = \delta_0(T, x_2, \ldots, x_n)$$

and $M_1 = M_1(\mu)$.
   Then the following assertions hold.

(1) Suppose that $\mu = 0$. Then

$$|\delta| \le |x_n|^{M_1},$$
$$\left| \delta - x_n^{(n-1)q^{n-1}} T \right| < |x_n|^{M_1}.$$

(2) Suppose that $\mu > 0$. Then

$$|\delta| \le |x_n|^{M_1},$$

$$\left| \delta - (-1)^\mu x_n^{(n-\mu-1+(\mu-1)q)q^{n-1}} (x_n^{q^n} T^{q^{-\mu}} - x_n^{q^{n-1}} T^{q^{-\mu+1}}) \right| < |x_n|^{M_1}.$$

Proof. The case (1) follows immediately from Lemma 2.6.
   Suppose that $\mu > 0$. By Remark 1.2 we have

$$\delta = \delta_{-1}(x_2, \ldots, x_n, T^{q^{-n}})$$
$$= (-1)^\mu \delta_{-1}(x_2, \ldots, x_{n-\mu}, T^{q^{-n}}, x_{n-\mu+1}, \ldots, x_n).$$

Now Lemma 2.6 shows

$$|\delta| \le |x_n|^M,$$
$$\left| \delta - (-1)^\mu x_2^q \cdots x_{n-\mu}^{q^{n-\mu-1}} ((T^{q^{-n}})^{q^{n-\mu}} x_{n-\mu+1}^{q^{n-\mu+1}} - (T^{q^{-n}})^{q^{n-\mu+1}} x_{n-\mu+1}^{q^{n-\mu}}) x_{n-\mu+2}^{q^{n-\mu+2}} \cdots x_n^{q^n} \right| < |x_n|^M,$$

where $M = ((n - \mu - 1)q^{n-1} + q^{n-1} + \mu q^n) = M_1$. The left-hand side of the second inequality is easily seen to agree with the statement of the lemma and the case (2) follows. $\qquad\square$

Lemma 2.8. *Let* $1 \leq i < j \leq n$ *be integers,* $0 \leq \mu \leq n - 1$ *an integer and* $1 \leq c < q$ *a rational number. Let* $x_n \in \mathcal{O}_{\mathbb{C}_p}^{\infty}$ *and* $T_i, T_j \in R_{\infty, \mathcal{O}_{\mathbb{C}_p}}^{\infty}$*. Put* $x_{n+m} = x_n^{q^{-m}}$ *for all* $m \in \mathbb{Z}$*. Suppose that* $|T_i| \leq |x_i|^{q^{\mu}c}$ *and* $|T_j| \leq |x_j|^{q^{\mu}c}$*. Put*
$$\delta = \delta_0(x_1, \ldots, T_i, \ldots, T_j, \ldots, x_n)$$
and $M_2 = M_2(\mu, c)$*.*

Then the following assertions hold.

(1) $|\delta| \leq |x_n|^{M_2}$*.*

(2) Suppose that $c > 1$ *and* $\mu < n/2$*. Then*
$$|x_n|^{M_2} > \begin{cases} \left| \delta - x_n^{(n-2\mu-2+2\mu q)q^{n-1}} T_i^{q^{i-\mu-1}} T_j^{q^{j-\mu-1}} \right| & \text{if } \mu < j - i < n - \mu \\ |\delta| & \text{otherwise.} \end{cases}$$

(3) Suppose that $c > 1$ *and* $\mu \geq n/2$*. Then*
$$|x_n|^{M_2} > \begin{cases} \left| \delta - x_n^{(2(n-\mu-1)+(2\mu-n)q)q^n} T_i^{q^{i-\mu}} T_j^{q^{j-\mu}} \right| & \text{if } n - \mu \leq j - i \leq \mu \\ |\delta| & \text{otherwise.} \end{cases}$$

(4) Suppose that $c = 1$ *and* $\mu = 0$*. Then*
$$|x_n|^{M_2} > \left| \delta - x_n^{(n-2)q^{n-1}} T_i^{q^{i-\mu}} T_j^{q^{j-\mu}} \right|.$$

(5) Suppose that $c = 1$ *and* $0 < \mu < n/2$*.*
   Then $|x_n|^{M_2} > |\delta - x_n^{(n-2\mu-2+(2\mu-2)q)q^{n-1}} d_1|$*, where*
$$d_1 = \begin{cases} (x_n^{q^n} T_i^{q^{i-\mu-1}} - x_n^{q^{n-1}} T_i^{q^{i-\mu}})(x_n^{q^n} T_j^{q^{j-\mu-1}} - x_n^{q^{n-1}} T_j^{q^{j-\mu}}) & \text{if } \mu < j - i < n - \mu \\ -(x_n^{q^n} T_i^{q^{i-\mu-1}} - x_n^{q^{n-1}} T_i^{q^{i-\mu}}) T_j^{q^{j-\mu}} & \text{if } \mu = j - i < n - \mu \\ -T_i^{q^{i-\mu}} (x_n^{q^n} T_j^{q^{j-\mu-1}} - x_n^{q^{n-1}} T_j^{q^{j-\mu}}) & \text{if } \mu < j - i = n - \mu \\ 0 & \text{otherwise.} \end{cases}$$

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(6) *Suppose that $c = 1$ and $\mu = n/2$ (hence n is even). Then*

$$|x_n|^{M_2} > \begin{cases} \left| \delta - x_n^{(n-2)q^n} T_i^{q^{i-\mu}} T_j^{q^{j-\mu}} \right| & \text{if } j - i = n/2 \\ |\delta| & \text{otherwise.} \end{cases}$$

(7) *Suppose that $c = 1$ and $\mu > n/2$.*

Then $|x_n|^{M_2} > |\delta - x_n^{(2(n-\mu-1)+(2\mu-n-2)q)q^n} d_2|$, where

$$d_2 = \begin{cases} (x_n^{q^{n+1}} T_i^{q^{i-\mu}} - x_n^{q^n} T_i^{q^{i-\mu+1}})(x_n^{q^{n+1}} T_j^{q^{j-\mu}} - x_n^{q^n} T_j^{q^{j-\mu+1}}) & \text{if } n - \mu < j - i < \mu \\ T_i^{q^{i-\mu}}(x_n^{q^{n+1}} T_j^{q^{j-\mu}} - x_n^{q^n} T_j^{q^{j-\mu+1}}) & \text{if } n - \mu = j - i < \mu \\ (x_n^{q^{n+1}} T_i^{q^{i-\mu}} - x_n^{q^n} T_i^{q^{i-\mu+1}})T_j^{q^{j-\mu}} & \text{if } n - \mu < j - i = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us first prove (1).

Suppose that $j - i \leq \mu$. Then

$$\delta = \delta_{n-j}(x_{j+1-n}, \ldots, x_0, x_1, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_{j-1}, T_j)$$
$$= (-1)^\mu \delta_{n-j}(x_{j+1-n}, \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_{j-1})$$
$$= (-1)^\mu \delta_{n-i-1}(x_{i+1-n}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{i-1}, T_i)$$
$$= \begin{cases} (-1)^\mu (-1)^{\mu+1} \delta_{n-i-1}(x_{i+1-n}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots, x_{i-\mu-1}, T_i, x_{i-\mu}, \ldots \\ \qquad \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{i-1}) \\ \qquad\qquad\qquad\qquad \text{if } j - i \leq \mu \text{ and } j - i < n - \mu \\ (-1)^\mu (-1)^\mu \delta_{n-i-1}(x_{i+1-n}, \ldots, x_{i-\mu-1}, T_i, x_{i-\mu}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots \\ \qquad \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{i-1}) \\ \qquad\qquad\qquad\qquad \text{if } n - \mu \leq j - i \leq \mu. \end{cases}$$

$$\tag{2.9}$$

Thus, if $j - i \leq \mu$ and $j - i < n - \mu$, then Lemma 2.6 shows that $|\delta| \leq |x_n|^M$, where

$$M = ((j - i - 1) + (n - \mu - (j - i) - 1)q^{-1} + cq^{-1} + (j - i) + c + (\mu - (j - i))q)q^n$$
$$= (n - \mu - (j - i) + c - 1 + (2(j - i) + c - 1)q + (\mu - (j - i))q^2)q^{n-1}.$$

Now

$$\frac{1}{q^{n-1}}(M - (n + 2(c-1) + 2\mu(q-1))q^{n-1})$$
$$= \mu - (j-i) - (c-1) + (2(j-i-\mu) + c - 1)q + (\mu - (j-i))q^2$$
$$= (\mu - (j-i))(q-1)^2 + (c-1)(q-1)$$
$$\geq 0. \tag{2.10}$$

Here, the equality holds if and only if $\mu = j - i$ and $c = 1$ (in which case $\mu < n/2$). Similarly, noting $q > c$ we have

$$\frac{1}{q^{n-1}}(M - (n + 2(c-1) + (2\mu - n)(q-1))q^n)$$
$$= n - \mu - (j-i) + c - 1 + (2(j - i - (n-\mu)) - (c-1))q + (n - \mu - (j-i))q^2$$
$$= (n - \mu - (j-i))(q-1)^2 - (c-1)(q-1)$$
$$> (n - \mu - (j-i) - 1)(q-1)^2$$
$$\geq 0. \tag{2.11}$$

Applying similarly Lemma 2.6, we see that

$$\text{if } n - \mu \leq j - i \leq \mu, \text{ then } |\delta| \leq |x_n|^{M_2}. \tag{2.12}$$

Suppose that $j - i > \mu$. We proceed in a way similar to that in the above case.

$$\delta = \delta_{n-j}(x_{j+1-n}, \ldots, x_0, x_1, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_{j-1}, T_j)$$
$$= (-1)^\mu \delta_{n-j}(x_{j+1-n}, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{j-1})$$
$$= (-1)^\mu \delta_{n-j+\mu+1}(T_j^{q^n}, x_{j-\mu-n}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_{j-\mu-1},)$$
$$= \begin{cases} (-1)^\mu(-1)^{\mu-1}\delta_{n-j+\mu+1}(T_j^{q^n}, x_{j-\mu-n}, \ldots, x_{i-\mu-1}, T_i, x_{i-\mu}, \ldots \\ \qquad \ldots, x_{j-1-n}, x_{j+1-n}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-\mu-1},) \\ \qquad\qquad\qquad \text{if } j - i > \mu \text{ and } j - i \geq n - \mu \\ (-1)^\mu(-1)^\mu \delta_{n-j+\mu+1}(T_j^{q^n}, x_{j-\mu-n}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots \\ \qquad \ldots, x_{i-\mu-1}, T_i, x_{i-\mu}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-\mu-1},) \\ \qquad\qquad\qquad \text{if } \mu < j - i < n - \mu. \end{cases}$$
$$\tag{2.13}$$

If $j - i > \mu$ and $j - i \geq n - \mu$, then Lemma 2.6 shows that $|\delta| \leq |x_n|^{M'}$, where $M' = (j - i - \mu + c - 1 + (2n - 2(j - i) + c - 1)q + (j - i - (n - \mu))q^2)q^{n-1}$. We have

$$\frac{1}{q^{n-1}}(M' - (n + 2(c-1) + 2\mu(q-1))q^{n-1})$$

$$= (j - i - (n - \mu))(q - 1)^2 + (c - 1)(q - 1) \geq 0, \tag{2.14}$$

$$\frac{1}{q^{n-1}}(M' - (n + 2(c-1) + (2\mu - n)(q-1))q^n)$$

$$= (j - i - \mu)(q - 1)^2 - (c - 1)(q - 1) > (j - i - \mu - 1)(q - 1)^2 \geq 0. \tag{2.15}$$

In the first inequality, the equality holds if and only if $j - i = \mu$ and $c = 1$ (in which case $\mu < n/2$). Similarly, it follows from Lemma 2.6 that

$$\text{if } \mu < j - i < n - \mu, \text{ then } |\delta| \leq |x_n|^{M_2}. \tag{2.16}$$

The assertion (1) follows from (2.10), (2.11), (2.12), (2.14), (2.15), (2.16).

In proving the rest of the lemma, we first assume $c = 1$. If $\mu \geq n/2$ we have, by the second case of (2.9) and Lemma 2.6,

$$\delta = \delta_{n-i-1}(x_{i+1-n}, \ldots, x_{i-\mu-1}, T_i, x_{i-\mu}, \ldots, x_{j-1-n}, x_{j+1-n}, \ldots, x_{j-\mu-1}, T_j, x_{j-\mu}, \ldots, x_{i-1})$$

and hence $|\delta - d| < |x_n|^{M_2}$, where

$$d = \begin{cases} x_{i+1-n}^{q^{i+1-n}} \cdots x_{i-\mu-1}^{q^{i-\mu-1}}(T_i^{q^{i-\mu}} x_{i-\mu}^{q^{i-\mu+1}} - T_i^{q^{i-\mu+1}} x_{i-\mu}^{q^{i-\mu}})x_{i-\mu+1}^{q^{i-\mu+2}} \cdots x_{j-1-n}^{q^{j-n}} \\ \quad \cdot x_{j+1-n}^{q^{j+1-n}} \cdots x_{j-\mu-1}^{q^{j-\mu-1}}(T_j^{q^{j-\mu}} x_{j-\mu}^{q^{j-\mu+1}} - T_j^{q^{j-\mu+1}} x_{j-\mu}^{q^{j-\mu+1}})x_{j-\mu+1}^{q^{j-\mu+2}} \cdots x_{i-1}^{q^i} \\ \qquad \text{if } n - \mu < j - i < \mu \\ x_{i+1-n}^{q^{i+1-n}} \cdots x_{i-\mu-1}^{q^{i-\mu-1}} T_i^{q^{i-\mu}} \\ \quad \cdot x_{j+1-n}^{q^{j+1-n}} \cdots x_{j-\mu-1}^{q^{j-\mu-1}}(T_j^{q^{j-\mu}} x_{j-\mu}^{q^{j-\mu+1}} - T_j^{q^{j-\mu+1}} x_{j-\mu}^{q^{j-\mu+1}})x_{j-\mu+1}^{q^{j-\mu+2}} \cdots x_{i-1}^{q^i} \\ \qquad \text{if } n - \mu = j - i < \mu \\ x_{i+1-n}^{q^{i+1-n}} \cdots x_{i-\mu-1}^{q^{i-\mu-1}}(T_i^{q^{i-\mu}} x_{i-\mu}^{q^{i-\mu+1}} - T_i^{q^{i-\mu+1}} x_{i-\mu}^{q^{i-\mu}})x_{i-\mu+1}^{q^{i-\mu+2}} \cdots x_{j-1-n}^{q^{j-n}} \\ \quad \cdot x_{j+1-n}^{q^{j+1-n}} \cdots x_{j-\mu-1}^{q^{j-\mu-1}} T_j^{q^{j-\mu}} \\ \qquad \text{if } n - \mu < j - i = \mu \\ x_{i+1-n}^{q^{i+1-n}} \cdots x_{i-\mu-1}^{q^{i-\mu-1}} T_i^{q^{i-\mu}} \\ \quad \cdot x_{j+1-n}^{q^{j+1-n}} \cdots x_{j-\mu-1}^{q^{j-\mu-1}} T_j^{q^{j-\mu}} \\ \qquad \text{if } n - \mu = j - i = \mu \end{cases}$$

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From this, (2.11) and (2.15), the assertions (6) and (7) follow.

Except for minor complications the assertions (4) and (5) are proved in a way similar to the above; we apply Lemma 2.6 to the second case of (2.13) (resp. the first case of (2.9) with $j - i = \mu$ and the first case of (2.13) with $j - i = n - \mu$) to obtain the desired inequalities for those (i, j) such that $\mu < j - i < n - \mu$ (resp. $\mu = j - i < n - \mu$ and $\mu < j - i = n - \mu$). Note that in the second case of (2.13) we have to treat separately the cases where $\mu = 0$ and $\mu > 0$.

The assertions (2) and (3) are similarly and more easily proved by applying Lemma 2.6 to the second case of (2.13) and (2.9) respectively.

$\square$

2.4 Reductions of formal models

Let $\nu > 0$ be an integer and set $M_3(\nu) = (1 - s/n)q^r + (s/n)q^{r+1}$, where $\nu = rn + s$ with $r, s \in \mathbb{Z}$ and $0 \leq s \leq n - 1$. We put $U = T - t \in B_1$.

Proposition 2.9. *(1) The set-theoretic image of $\mathcal{X}_\nu$ in $\widetilde{\wedge\Sigma}_{0,\infty,\overline{\eta}}^{ad}$ under $\delta \colon \widetilde{\Sigma}_{0,\infty,\overline{\eta}}^{n,ad} \to \widetilde{\wedge\Sigma}_{0,\infty,\overline{\eta}}^{ad}$ is contained in a rational subset $\mathcal{Y}_\nu \subset \widetilde{\wedge\Sigma}_{0,\infty,\overline{\eta}}^{ad}$ defined by*

$$|U| \leq |t|^{M_3(\nu)}.$$

(2) The pull-back of $\mathcal{Y}_\nu$ in $\mathcal{M}_{\wedge\Sigma_0,\infty,\overline{\eta}}^{ad} \simeq U_K$, which we simply denote by $\mathcal{Y}_\nu \cap \mathcal{M}_{\wedge\Sigma_0,\infty,\overline{\eta}}^{ad}$, is identified with $U_K^{\lceil \nu/n \rceil}$.

Proof. Let us prove (1). In $\mathcal{O}_{\mathbb{C}_p}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]]$, we expand as follows:

$$
\begin{aligned}
\delta(X_1, \ldots, X_n) - t &= \delta(\xi_1 - Y_2^q - \cdots - Y_n^{q^{n-1}} + Z, \xi_2 + Y_2, \ldots, \xi_n + Y_n) - t \\
&= \delta(\xi_1, \ldots, \xi_n) - t \\
&\quad + \sum_{1 \le i \le n} \left(\delta(-Y_i^{q^{i-1}}, \ldots, \xi_n) + \delta(\xi_1, \ldots, Y_i, \ldots, \xi_n) \right) \\
&\quad + \delta(Z, \xi_2, \ldots, \xi_n) + \sum_{2 \le i,j \le n} \delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n) \\
&\quad + \sum_{2 \le i < j \le n} \delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n) + \cdots \\
&= \delta(Z, \xi_2, \ldots, \xi_n) + \sum_{2 \le i,j \le n} \delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n) \\
&\quad + \sum_{2 \le i < j \le n} \delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n) + \cdots,
\end{aligned}
$$

where we use Lemma 2.5 in the last equality. Note that terms not indicated here are negligible because $|Y_i| < |\xi_i|$.

Let $|\cdot| \in \mathcal{X}_\nu$. It suffices to prove that the valuation of each term explicitly appearing in the last two lines of the above is bounded by $|t|^{M_3(\nu)}$, for then $|\delta(X_1, \ldots, X_n) - t| \le |t|^{M_3(\nu)}$, which is to say, the image of $|\cdot|$ by δ lies in $\mathcal{Y}_\nu$. The required estimates are obtained by applying Remark 1.2 and Lemmas 2.7 and 2.8. For instance, expressing $\nu = rn + s$ with $r, s \in \mathbb{Z}$ and $0 \le s < n$, we have

$$
\begin{aligned}
|\delta(Z, \xi_2, \ldots, \xi_n)| &= |(-1)^{r(n-1)} \delta_{-r}(Z^{q^{-rn}}, \xi_2, \ldots, \xi_n)| \\
&\le |\xi_n|^{M_1(s)q^r} \\
&= |t|^{n^{-1} q^{-(n-1)} M_1(s) q^r} \\
&= |t|^{M_3(\nu)}.
\end{aligned}
$$

Similarly, if $c = (q+1)/2$ or $c = 1$ according to the parity of ν, and if $\mu = r'n + s'$ with $r', s' \in \mathbb{Z}$ and $0 \le s' < n$, we have

$$
\begin{aligned}
|\delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n)| &= |(-1)^{2r'(n-1)} \delta_{-2r'}(-(Y_i^{q^{-r'n}})^{q^{i-1}}, \xi_2, \ldots, Y_j^{q^{-r'n}}, \ldots, \xi_n)| \\
&\le |\xi_n|^{M_2(s',c)q^{2r'}} \\
&= |t|^{n^{-1} q^{-(n-1)} M_2(s',c) q^{2r'}}, \\
|\delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n)| &\le |t|^{n^{-1} q^{-(n-1)} M_2(s',c) q^{2r'}}
\end{aligned}
$$

and one can check the equality $n^{-1}q^{-(n-1)}M_2(s',c)q^{2r'} = M_3(\nu)$ by case-by-case calculations.

Now we prove (2). For $x = \sum_{m \geq 0} x_m \varpi'^m \in U_K$, we set

$$[x](t) = \mathrm{Art}(x)(t) = \lim_{m \to \infty} [x]_{\widehat{\wedge \Sigma_0}} (t'_m)^{q^{m-1}} = \sum_{m \geq 0} x_m t^{q^m}.$$

Then, under the canonical identifications $\mathcal{M}^{\mathrm{ad}}_{\wedge \Sigma_0, \infty, \bar{\eta}}(\mathbb{C}_p) = U_K$, $\widetilde{\wedge \Sigma}^{\mathrm{ad}}_{0, \infty, \bar{\eta}}(\mathbb{C}_p) = \mathbb{C}_p^{\circ\circ}$, the maps induced by the morphisms $\mathcal{M}^{\mathrm{ad}}_{\wedge \Sigma_0, \infty, \bar{\eta}} \to \widetilde{\wedge \Sigma}^{\mathrm{ad}}_{0, \infty, \bar{\eta}}$, $\mathcal{Y}_\nu \to \widetilde{\wedge \Sigma}^{\mathrm{ad}}_{0, \infty, \bar{\eta}}$ correspond to

$$U_K \to \mathbb{C}_p^{\circ\circ}; \ x \mapsto [x](t),$$
$$\left\{ y \in \mathbb{C}_p \mid |y - t| \leq |t|^{M_3(\nu)} \right\} \hookrightarrow \mathbb{C}_p^{\circ\circ}$$

respectively. The pull-back of these two maps are clearly $U_K^{[\nu/n]}$, as desired.
□

Take an n-th root $t^{1/n} \in \mathcal{O}_{\mathbb{C}_p}$. We define a formal model $\mathscr{Y}_\nu$ of $\mathcal{Y}_\nu$ by

$$\mathscr{Y}_\nu = \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p} \langle u^{q^{-\infty}} \rangle, \ u = \frac{T - t}{\xi_n^{nq^{n-1}M_3(\nu)}}.$$

Then the morphism $U_K^{[\nu/n]} \simeq \mathcal{Y}_\nu \cap \mathcal{M}^{\mathrm{ad}}_{\wedge \Sigma_0, \infty, \bar{\eta}} \to \mathcal{Y}_\nu$ is induced by a morphism $\mathrm{Spf}\, \mathrm{Map}(U_K^{[\nu/n]}, \mathcal{O}_{\mathbb{C}_p}) \to \mathscr{Y}_\nu = \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p} \langle u^{q^{-\infty}} \rangle$ of formal models given by

$$f(u) \mapsto \left(f\left(\frac{[x](t) - t}{\xi_n^{nq^{n-1}M_3(\nu)}} \right) \right)_{x \in U_K^{[\nu/n]}}. \tag{2.17}$$

We finally define a formal model $\mathscr{Z}_\nu$ of $\mathcal{Z}_\nu$ by

$$\mathscr{Z}_\nu = \mathscr{X}_\nu \underset{\mathscr{Y}_\nu}{\times} \mathrm{Spf}\, \mathrm{Map}(U_K^{[\nu/n]}, \mathcal{O}_{\mathbb{C}_p}). \tag{2.18}$$

Theorem 2.10. *Let $\nu > 0$ be an integer. Let $\mathcal{Z}_\nu$ be the affinoid defined in 2.2 and $\mathscr{Z}_\nu$ its formal model defined by (2.18). For each integer $0 \leq m \leq n - 1$, define a set $T(m)$ by*

$$T(m) = \begin{cases} \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq n, \ m < j - i < n - m\} & \text{if } m < n/2 \\ \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq n, \ n - m \leq j - i \leq m\} & \text{if } m \geq n/2. \end{cases}$$

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Then the special fiber $\overline{\mathscr{Z}}_\nu$ of $\mathscr{Z}_\nu$ fits into the following Cartesian diagrams[4]

$$
\begin{array}{ccc}
\overline{\mathscr{Z}}_\nu & \longrightarrow & U_K^{[\nu/n]} = N_{L/K}U_L^\nu \\
\downarrow & & \downarrow \\
(Z_\nu)^{\mathrm{perf}} & \longrightarrow & N_{L/K}U_L^\nu / N_{L/K}U_L^{\nu+1} \\
\downarrow & & \downarrow \\
\overline{\mathscr{X}}_\nu = (\mathbb{A}_k^n)^{\mathrm{perf}} & \longrightarrow & \overline{\mathscr{Y}}_\nu = (\mathbb{A}_k^1)^{\mathrm{perf}},
\end{array}
\qquad (2.19)
$$

where Z_ν is a smooth affine variety defined below, $(\cdot)^{\mathrm{perf}}$ denotes the inverse perfection of an affine scheme in characteristic p and we simply write

$$
N_{L/K}U_L^\nu = \operatorname{Spec}\operatorname{Map}(N_{L/K}U_L^\nu, \overline{k}),
$$
$$
N_{L/K}U_L^\nu / N_{L/K}U_L^{\nu+1} = \operatorname{Spec}\operatorname{Map}(N_{L/K}U_L^\nu / N_{L/K}U_L^{\nu+1}, \overline{k}).
$$

(1) *Suppose that n divides ν (so that $N_{L/K}U_L^\nu / N_{L/K}U_L^{\nu+1}$ is identified with k via ϖ). Then Z_ν is the trivial affine space bundle $\amalg_k \mathbb{A}_k^{n-1}$ over $N_{L/K}U_L^\nu / N_{L/K}U_L^{\nu+1}$.*

(2) *Suppose[5] that ν is odd and $\nu \equiv \nu' \mod 2n$ with $0 < \nu' < n$. Define $\mu' < n/2$ by $\nu' = 2\mu' + 1$. Then Z_ν is an affine variety defined by*

$$
\begin{cases}
y_1 + \cdots + y_n = 0 \\
z^q - z = - \displaystyle\sum_{(i,j)\in T(\mu')} y_i y_j
\end{cases}
$$

in $\mathbb{A}_{\overline{k}}^{n+1}$.

(3) *Suppose that ν is odd and $\nu \equiv \nu' \mod 2n$ with $n < \nu' < 2n$. Define $\mu' \geq n/2$ by $\nu' = 2\mu' + 1$. Then Z_ν is an affine variety defined by*

$$
\begin{cases}
y_1 + \cdots + y_n = 0 \\
z^q - z = \displaystyle\sum_{(i,j)\in T(\mu')} y_i y_j
\end{cases}
$$

in $\mathbb{A}_{\overline{k}}^{n+1}$.

---

[4]Analogous Cartesian diagrams occur in [BW16] as well. We drew inspirations from their result.

[5]In the assertions to follow, the relations between various y_i (resp. z) and various y_i' (resp. z') are given in the proof.

(4) Suppose that ν is even and $\nu \equiv \nu'$ mod $2n$ with $0 < \nu' < n$. Define $\mu' < n/2$ by $\nu' = 2\mu'$. Then Z_ν is an affine variety defined by

$$\begin{cases} y_1 + \cdots + y_n = 0 \\ z^q - z = \displaystyle\sum_{(i,j) \in T(\mu')} (y_i^q - y_i)(y_j^q - y_j) + \sum_{j-i=\mu'} (y_i^q - y_i)y_j^q + \sum_{j-i=n-\mu'} y_i^q(y_j^q - y_j) \end{cases}$$

in $\mathbb{A}_{\overline{k}}^{n+1}$.

(5) Suppose that ν is even and $\nu \equiv \nu'$ mod $2n$ with $n < \nu' < 2n$. Define $\mu' > n/2$ by $\nu' = 2\mu'$. Then Z_ν is an affine variety defined by

$$\begin{cases} y_1 + \cdots + y_n = 0 \\ z^q - z = -\displaystyle\sum_{(i,j) \in T(n-\mu')} (y_i^q - y_i)(y_j^q - y_j) + \sum_{j-i=\mu'} (y_i^q - y_i)y_j + \sum_{j-i=n-\mu'} y_i(y_j^q - y_j) \end{cases}$$

in $\mathbb{A}_{\overline{k}}^{n+1}$.

Proof. It follows from (2.17) that $\operatorname{Spec}\operatorname{Map}(U_K^{[\nu/n]}, \overline{k}) = N_{L/K}U_L^\nu \to \overline{\mathscr{Y}}_\nu = (\mathbb{A}_{\overline{k}}^1)^{\mathrm{perf}} = \operatorname{Spec}\overline{k}[u^{q^{-\infty}}]$ is given by

$$\overline{k}[u^{q^{-\infty}}] \to \operatorname{Map}(U_K^{[\nu/n]}, \overline{k}); \quad u \mapsto \begin{cases} (0)_{x \in U_K^{[\nu/n]}} & \text{if } n \text{ does not divide } \nu \\ (\overline{y})_{1+y\varpi^{\nu/n} \in U_K^{\nu/n}} & \text{if } n \text{ divides } \nu, \end{cases}$$

where $\overline{y}$ is the image of $y \in \mathcal{O}_K$ in $k = N_{L/K}U_L^\nu/N_{L/K}U_L^{\nu+1}$. This shows the desired factorization of $N_{L/K}U_L^\nu \to \overline{\mathscr{Y}}_\nu$.

Thus, it now suffices for us to study (the perfection of) the variety $\overline{\mathscr{X}}_\nu \underset{\overline{\mathscr{Y}}_\nu}{\times} N_{L/K}U_L^\nu/N_{L/K}U_L^{\nu+1}$ defined by the lower Cartesian diagram. Note that the morphism $\overline{\mathscr{X}}_\nu \to \overline{\mathscr{Y}}_\nu$ is induced by the reduction of

$$\mathcal{O}_{\mathbb{C}_p}\langle u^{q^{-\infty}}\rangle \to \mathcal{O}_{\mathbb{C}_p}\langle z'^{q^{-\infty}}, y_2'^{q^{-\infty}}, \ldots, y_n'^{q^{-\infty}}\rangle; \quad u \mapsto \xi_n^{-nq^{n-1}M_3(\nu)}(\delta - t),$$

where $\delta = \delta(X_1, \ldots, X_n)$. We claim that (in (2) to (5)) an isomorphism is given by[6]

$$z' \mapsto z^{q^{-(r-\nu)}}, \quad y_i' \mapsto y_i^{q^{-(r-\mu+i-1)}} \ (2 \le i \le n), \tag{2.20}$$

___
[6]Since we are working with perfect rings, there are many other obvious possibilities; for instance, we may leave out all r from the definition of the above map.

where we write $\nu = rn + s$ with $r, s \in \mathbb{Z}$ and $0 \leq s < n$, and $\mu = \lfloor \nu/2 \rfloor$. As in the proof of Proposition 2.9 (1) we use Remark 1.2 to reduce to the case where $0 \leq \nu < n$ (resp. where $0 \leq \mu < n$), so that Lemma 2.7 (resp. Lemma 2.8) is applicable. Since the computation is rather complicated, we only indicate several typical cases.

Since $\delta(Z, \xi_2, \ldots, \xi_n) = (-1)^{r(n-1)} \delta_{-r}(Z^{q^{-rn}}, \xi_2, \ldots, \xi_n)$, we have, by Lemma 2.7, $|\delta(Z, \xi_2, \ldots, \xi_n) - f| < |\xi_n|^{nq^{n-1}M_3(\nu)}$, where, if $0 < s$,

$$f = (-1)^{r(n-1)} \left((-1)^s \xi_n^{(n-s-1+(s-1)q)q^{n-1}} \left(\xi_n^{q^n}(Z^{q^{-rn}})^{q^{-s}} - \xi_n^{q^{n-1}}(Z^{q^{-rn}})^{q^{-s+1}} \right) \right)^{q^r}$$

$$= (-1)^{r(n-1)+s} \left(\xi_n^{(n-s-1+(s-1)q)q^{n-1}} \left(\xi_n^{q^n}(\xi_n^{q^{n-1+\nu}}z')^{q^{-\nu}} - \xi_n^{q^{n-1}}(\xi_n^{q^{n-1+\nu}}z')^{q^{-\nu+1}} \right) \right)^{q^r}$$

$$= (-1)^{\nu-r} \left(\xi_n^{(n-s-1+(s-1)q)q^{n-1}} \left(\xi_n^{q^n+q^{n-1}}z'^{q^{-\nu}} - \xi_n^{q^{n-1}+q^n}z'^{q^{-\nu+1}} \right) \right)^{q^r}$$

$$= (-1)^{\nu-r} \xi_n^{(n-s+sq)q^{n-1} \cdot q^r} \left(z'^{q^{-\nu+r}} - z'^{q^{-\nu+r+1}} \right)$$

$$= (-1)^{\nu-r} \xi_n^{nq^{n-1}M_3(\nu)}(z - z^q)$$

and, if $s = 0$,

$$f = (-1)^{r(n-1)} \left(\xi_n^{(n-1)q^{n-1}} Z^{q^{-rn}} \right)^{q^r}$$

$$= (-1)^{\nu-r} \xi_n^{nq^{n-1}M_3(\nu)} z.$$

In particular, (1) follows.

Suppose that we are in the case (2). In particular, we have $\mu = (\nu-1)/2 = r'n + \mu'$ with $r' = r/2 \in \mathbb{Z}$. Similarly to the above computation, since $\delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n) = (-1)^{2r'(n-1)} \delta_{-2r'}(\xi_1, \ldots, Y_i^{q^{-r'n}}, \ldots, Y_j^{q^{-r'n}}, \ldots, \xi_n)$, we have, by Lemma 2.8 (2), $|\delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n) - f_{i,j}| < |\xi_n|^{nq^{n-1}M_3(\nu)}$, where, if $\mu' < j - i < n - \mu'$,

$$f_{i,j} = \left(\xi_n^{(n-2\mu'-2+2\mu'q)q^{n-1}} (Y_i^{q^{-r'n}})^{q^{i-\mu'-1}} (Y_j^{q^{-r'n}})^{q^{j-\mu'-1}} \right)^{q^{2r'}}$$

$$= \left(\xi_n^{(n-2\mu'-2+2\mu'q)q^{n-1}} (\xi_i^{q^{\mu}(q+1)/2}y_i')^{q^{i-\mu-1}} (\xi_j^{q^{\mu}(q+1)/2}y_j')^{q^{j-\mu-1}} \right)^{q^r}$$

$$= \left(\xi_n^{(n-2\mu'-2+2\mu'q)q^{n-1}} \xi_n^{q^{n-1}(q+1)/2}y_i'^{q^{i-\mu-1}} \xi_n^{q^{n-1}(q+1)/2}y_j'^{q^{j-\mu-1}} \right)^{q^r}$$

$$= \xi_n^{(n-\nu'+\nu'q)q^{n-1} \cdot q^r} y_i'^{q^{i-\mu-1+r}} y_j'^{q^{j-\mu-1+r}}$$

$$= \xi_n^{nq^{n-1}M_3(\nu)} y_i y_j$$

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and, otherwise, $f_{i,j} = 0$. The same computation shows that $|\delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n) + \xi_n^{nq^{n-1}M_3(\nu)} y_i y_j| < |\xi_n|^{nq^{n-1}M_3(\nu)}$ if $\mu' < j-1 < n-\mu'$ and $|\delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n)| < |\xi_n|^{nq^{n-1}M_3(\nu)}$ otherwise. Therefore, arguing as in the proof of Proposition 2.9 (1), we have

$$\xi_n^{-nq^{n-1}M_3(\nu)}(\delta(X_1, \ldots, X_n) - t) \equiv \xi_n^{-nq^{n-1}M_3(\nu)}(f + \sum_{\mu' < j-i < n-\mu'} f_{i,j} - \sum_{\mu' < j-1 < n-\mu'} f_{i,j})$$

$$= z^q - z + \sum_{(i,j) \in T(\mu')} y_i y_j$$

modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$. This completes the proof of (2).

Suppose that we are in the case (4). In this case $\mu = \nu/2 = r'n + \mu'$ with $r' = r/2 \in \mathbb{Z}$. Again, by Lemma 2.8 (5), $|\delta(\xi_1, \ldots, Y_i, \ldots, Y_j, \ldots, \xi_n) - g_{i,j}| < |\xi_n|^{nq^{n-1}M_3(\nu)}$, where, if $\mu' < j-i < n-\mu'$,

$$g_{i,j} = \left(\xi_n^{(n-2\mu'-2+(2\mu'-2)q)q^{n-1}} \left(\xi_n^{q^n}(Y_i^{q^{-r'n}})^{q^{i-\mu'-1}} - \xi_n^{q^{n-1}}(Y_i^{q^{-r'n}})^{q^{i-\mu'}} \right) \right.$$

$$\left. \cdot \left(\xi_n^{q^n}(Y_j^{q^{-r'n}})^{q^{j-\mu'-1}} - \xi_n^{q^{n-1}}(Y_j^{q^{-r'n}})^{q^{j-\mu'}} \right) \right)^{q^{2r'}}$$

$$= \left(\xi_n^{(n-2\mu'-2+(2\mu'-2)q)q^{n-1}} \left(\xi_n^{q^n}(\xi_i^{q^\mu} y_i')^{q^{i-\mu-1}} - \xi_n^{q^{n-1}}(\xi_i^{q^\mu} y_i')^{q^{i-\mu}} \right) \right.$$

$$\left. \cdot \left(\xi_n^{q^n}(\xi_j^{q^\mu} y_j')^{q^{j-\mu-1}} - \xi_n^{q^{n-1}}(\xi_j^{q^\mu} y_j')^{q^{j-\mu}} \right) \right)^{q^r}$$

$$= \left(\xi_n^{(n-\nu'+\nu'q)q^{n-1}} \left(y_i'^{q^{i-\mu-1}} - y_i'^{q^{i-\mu}} \right) \left(y_j'^{q^{j-\mu-1}} - y_j'^{q^{j-\mu}} \right) \right)^{q^r}$$

$$= \xi_n^{nq^{n-1}M_3(\nu)} (y_i - y_i^q)(y_j - y_j^q)$$

and

$$g_{i,j} = \begin{cases} -\xi_n^{nq^{n-1}M_3(\nu)} (y_i - y_i^q) y_j^q & \text{if } \mu' = j-i \\ -\xi_n^{nq^{n-1}M_3(\nu)} y_i^q (y_j - y_j^q) & \text{if } j-i = n-\mu' \\ 0 & \text{otherwise.} \end{cases}$$

We also deduce from this the estimate for $\delta(-Y_i^{q^{i-1}}, \xi_2, \ldots, Y_j, \ldots, \xi_n)$ and

thus obtain as before

$$\xi_n^{-nq^{n-1}M_3(\nu)}(\delta(X_1,\ldots,X_n) - t)$$

$$\equiv z - z^q + \sum_{\mu' < j-i < n-\mu'} \left(y_i - y_i^q\right)\left(y_j - y_j^q\right) - \sum_{\mu' < j-1 < n-\mu'} \left(y_i - y_i^q\right)\left(y_j - y_j^q\right)$$

$$- \sum_{\mu' = j-i} \left(y_i - y_i^q\right)y_j^q + \sum_{\mu' = j-1} \left(y_i - y_i^q\right)y_j^q$$

$$- \sum_{j-i=n-\mu'} y_i^q\left(y_j - y_j^q\right) + \sum_{j-1=n-\mu'} y_i^q\left(y_j - y_j^q\right)$$

$$= z - z^q + \sum_{(i,j)\in T(\mu')} \left(y_i - y_i^q\right)\left(y_j - y_j^q\right) - \sum_{\mu' = j-i} \left(y_i - y_i^q\right)y_j^q - \sum_{j-i=n-\mu'} y_i^q\left(y_j - y_j^q\right)$$

$$= z - z^q + \sum_{(i,j)\in T(\mu')} \left(y_i^q - y_i\right)\left(y_j^q - y_j\right) + \sum_{\mu' = j-i} \left(y_i^q - y_i\right)y_j^q + \sum_{j-i=n-\mu'} y_i^q\left(y_j^q - y_j\right).$$

The other cases are treated in the same vein. □

2.5 Stabilizers of the affinoids and their actions on the reductions

To state the main result of this subsection we define several subgroups of $\mathrm{GL}_n(K)$ and $D^\times$.

Let $\mathfrak{I} \subset M_n(K)$ (resp. Let $\mathfrak{P} \subset \mathfrak{I}$) be the inverse image of the set of upper triangular matrices (resp. upper triangular matrices with all the diagonal entries zeros) by the canonical map $M_n(\mathcal{O}) \to M_n(k)$. Note that $\mathfrak{P}^i = \varphi^i\mathfrak{I}$ for all i. We set $U_{\mathfrak{I}} = \mathfrak{I}^\times$ and $U_{\mathfrak{I}}^i = 1 + \mathfrak{P}^i \subset U_{\mathfrak{I}}$ for $i \geq 1$ as usual.

Recall that L is identified with $K(\varphi) \subset M_n(K)$ via i_ξ. We also define C_1 to be the orthogonal complement[7] of $L \subset M_n(K)$ with respect to the non-degenerate symmetric pairing $M_n(K) \times M_n(K) \to K$; $(x, y) \mapsto \mathrm{tr}(xy)$. We set $\mathfrak{P}_{C_1}^i = \mathfrak{P}^i \cap C_1$ and $U_{\mathfrak{I}}^{(i)} = 1 + \mathfrak{P}^i + \mathfrak{P}_{C_1}^{\lfloor(i+1)/2\rfloor}$. Then we have $\mathfrak{P}^i = \mathfrak{p}_L^i \oplus \mathfrak{P}_{C_1}^i$ and therefore $U_{\mathfrak{I}}^{(i)} = 1 + \mathfrak{p}_L^i + \mathfrak{P}_{C_1}^{\lfloor(i+1)/2\rfloor}$. Also, $U_{\mathfrak{I}}^{(i)}$ is a subgroup of $U_{\mathfrak{I}}^{\lfloor(i+1)/2\rfloor}$ containing $U_{\mathfrak{I}}^i$.

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[7]The author learned the importance of C_1 and $U_{\mathfrak{I}}^{(i)}$ in [BW16]. However, these have been studied before elsewhere (see for instance, [BF83, (6.2)]).

Remark 2.11. *As we are working in an equal-characteristic setting, we can describe these groups very explicitly:*

$$U_{\mathfrak{J}}^m = \{1 + (a_{j,i}) \in U_{\mathfrak{J}} \mid a_{j,i}^{(l)} = 0 \text{ if } 0 < i - j + ln < m \,\}$$

$$U_{\mathfrak{J}}^{(m)} = \{1 + (a_{j,i}) \in U_{\mathfrak{J}}^{\lfloor (m+1)/2 \rfloor} \mid \sum_{(i,j,l);i-j+ln=r} a_{j,i}^{(l)} = 0 \text{ for } r < m \,\},$$

where $a_{j,i} = \sum_l a_{j,i}^{(l)} \varpi^l$.

The following can be verified by a simple computation.

Proposition 2.12. *The subgroup $U_{\mathfrak{J}}^{(m+1)} \subset U_{\mathfrak{J}}^{(m)}$ is normal. In the notation of Remark 2.11, the quotient group $S_{1,m} = U_{\mathfrak{J}}^{(m)}/U_{\mathfrak{J}}^{(m+1)}$ is described as follows.*

(1) Suppose that m is odd. Then $S_{1,m}$ is isomorphic to the additive group k:

$$S_{1,m} \to k; \ (1 + a)U_{\mathfrak{J}}^{(m+1)} \mapsto \overline{\mathrm{Tr}\,\varphi^{-m}a} = \sum_{(i,j,l);i-j+ln=m} a_{j,i}^{(l)}$$

(2) Suppose that m is even. Put

$$T_{1,m} = \{(v, (w_i)) \in k \times k^{\mathbb{Z}/n\mathbb{Z}} \mid \sum_i w_i = 0\}.$$

Then the following is a bijection:

$$S_{1,m} \to T_{1,m}; \ (1 + a)U_{\mathfrak{J}}^{(m+1)} \mapsto \left(\overline{\mathrm{Tr}\,\varphi^{-m}a}, (a_i(m/2))_i \right),$$

where, for $1 \leq i \leq n$, we put $a_i(m/2) = a_{j,i}^{(l)}$ with the unique pair (j,l) such that $i - j + ln = m/2$. Under this identification, the induced group operation on $T_{1,m}$ is described as follows:

$$(v, (w_i)) \cdot (v', (w_i')) = (v + v' + \sum_i w_i w_{i+(m/2)}', (w_i + w_i')) \qquad (2.21)$$

For an even m, we identify $S_{1,m}$ with $T_{1,m}$.

We have similar subgroups for $D^\times$. Let $\mathcal{O}_D$ be the maximal order of D and $\mathfrak{P}_D \subset \mathcal{O}_D$ the maximal ideal. Note that $\mathfrak{P}_D^i = \varphi_D^i \mathcal{O}_D$ for all i. We set $U_D = \mathcal{O}_D^\times$ and $U_D^i = 1 + \mathfrak{P}_D^i \subset U_D$ as usual.

Recall that L is identified with $K(\varphi_D) \subset D$ via i_ξ^D. Similarly to the above, we define C_2 to be the orthogonal complement of $L \subset D$ with respect to the non-degenerate symmetric pairing $D \times D \to K$; $(x, y) \mapsto \operatorname{Trd}(xy)$. We set $\mathfrak{P}_{C_2}^i = \mathfrak{P}_D^i \cap C_2$ and $U_D^{(i)} = 1 + \mathfrak{P}_D^i + \mathfrak{P}_{C_2}^{\lfloor (i+1)/2 \rfloor}$. Then we have $\mathfrak{P}_D^i = \mathfrak{p}_L^i \oplus \mathfrak{P}_{C_2}^i$ and $U_D^{(i)} = 1 + \mathfrak{p}_L^i + \mathfrak{P}_{C_2}^{\lfloor (i+1)/2 \rfloor}$. Also, $U_D^{(i)}$ is a subgroup of $U_D^{\lfloor (i+1)/2 \rfloor}$ containing U_D^i.

Remark 2.13. *Again, we can describe these groups very explicitly:*

$$U_D^m = \{d \in U_D \mid d_l = 0 \text{ if } 0 < l < m \}$$
$$U_D^{(m)} = \{d \in U_D^{\lfloor (m+1)/2 \rfloor} \mid \sum_i d_l^{q^{i-1}} = \operatorname{Tr} d_l = 0 \text{ for } l < m \},$$

where $d^{-1} = \sum_l d_l \varphi_D^l$ with $d_l \in k_n$.

As before, we have the following proposition.

Proposition 2.14. *The subgroup $U_D^{(m+1)} \subset U_D^{(m)}$ is normal. In the notation of Remark 2.13, the quotient group $S_{2,m} = U_D^{(m)}/U_D^{(m+1)}$ is described as follows.*

(1) Suppose that m is odd. Then $S_{2,m}$ is isomorphic to the additive group k:

$$S_{2,m} \to k; \ d U_D^{(m+1)} \mapsto \overline{\operatorname{Trd} \varphi_D^{-\nu}(d^{-1} - 1)} = -\overline{\operatorname{Trd} \varphi_D^{-\nu}(d - 1)} = \operatorname{Tr} d_m$$

(2) Suppose that m is even. Put

$$T_{2,m} = \{(v, w) \in k \times k_n \mid \operatorname{Tr} w = 0\}.$$

Then the following is a bijection:

$$S_{2,m} \to T_{2,m}; \ d U_D^{(m+1)} \mapsto (\operatorname{Tr} d_m, d_{m/2})$$

and under this identification, the induced group operation on $T_{2,m}$ is described as follows:

$$(v, w) \cdot (v', w') = (v + v' + \operatorname{Tr}(w^{q^{m/2}} w'), w + w')$$

For even m, we identify $S_{2,m}$ with $T_{2,m}$.

Theorem 2.15. *Let $\nu > 0$ be an integer. Let $\mathrm{Stab}_\nu \subset G^0$ be the stabilizer of the affinoid $\mathcal{Z}_\nu$.*

(1) *We have $\mathrm{Stab}_\nu = (U_\mathfrak{J}^{(\nu)} \times U_D^{(\nu)} \times \{1\}) \cdot \mathcal{S}$ and the action of Stab_ν on $\mathcal{Z}_\nu$ induces an action on the reduction $\overline{\mathcal{Z}}_\nu$, $U_K^{[\nu/n]}$ and $(Z_\nu)^{\mathrm{perf}}$ in the Cartesian diagram (2.19). The upper-half square of the diagram (2.19) is equivariant for the induced action.*

(2) *In (1), Stab_ν acts on $U_K^{[\nu/n]}$ as the translation by $U_K^{[\nu/n]} \subset K^\times$ via $\mathrm{Stab}_\nu \hookrightarrow G^0 \xrightarrow{N_G} K^\times$.*

(3) *The action of $\varphi_G = (\varphi, \varphi_D, 1) \in \mathcal{S}$ on $(Z_\nu)^{\mathrm{perf}}$ is described as*

$$z \mapsto z, \quad y_1 \mapsto y_n, \quad y_i \mapsto y_{i-1} \quad \text{for } 2 \le i \le n.$$

(4) *The action of $\Delta_\xi(U_L) \subset \mathcal{S}$ on $(Z_\nu)^{\mathrm{perf}}$ is trivial.*

(5) *For $\sigma \in W_L$, set $a_\sigma = \mathrm{Art}_L^{-1}(\sigma) \in L \subset D$, $n_\sigma = v(a_\sigma)$ and $u_\sigma = a_\sigma \varphi_D^{-n_\sigma} \in U_D$. We denote by Frob_q the q-th power geometric Frobenius. Then the action of $(1, a_\sigma^{-1}, \sigma) \in \mathcal{S}$ on $(Z_\nu)^{\mathrm{perf}}$ is described as $\mathrm{Frob}_q^{n_\sigma}$ if ν is even, and as the composite of $\mathrm{Frob}_q^{n_\sigma}$ and the automorphism*

$$z \mapsto z, \quad y_i \mapsto \overline{u}_\sigma^{(q-1)/2} y_i \quad \text{for } 1 \le i \le n$$

if ν is odd.

(6) *The action of $U_\mathfrak{J}^{(\nu)} = \mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)$ on $(Z_\nu)^{\mathrm{perf}}$ factors through $U_\mathfrak{J}^{(\nu)} \to S_{1,\nu}$. If ν is odd, then the induced action of $x \in k = S_{1,\nu}$ is described as*

$$z \mapsto z + x, \quad y_i \mapsto y_i \quad \text{for } 1 \le i \le n.$$

If ν is even, then the induced action of $(v, (w_i)) \in S_{1,\nu}$ is described as

$$z \mapsto z + v + \sum_{i \in \mathbb{Z}/n\mathbb{Z}} w_i y_{i-\mu}, \quad y_i \mapsto y_i + w_i \quad \text{for } i \in \mathbb{Z}/n\mathbb{Z},$$

where we regard $\{y_i\}$ as indexed by $\mathbb{Z}/n\mathbb{Z}$.

(7) *The action of $U_D^{(\nu)} = \mathrm{Stab}_\nu \cap D^\times$ on $(Z_\nu)^{\mathrm{perf}}$ factors through $U_D^{(\nu)} \to S_{2,\nu}$. If ν is odd, then the induced action of $x \in k = S_{2,\nu}$ is described as*

$$z \mapsto z + x, \quad y_i \mapsto y_i \quad \text{for } 1 \le i \le n.$$

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If $\nu = 2\mu$ is even, then the induced action[8] of $(v, w) \in S_{2,\nu}$ is described as

$$z \mapsto z + v + \sum_{i \in \mathbb{Z}/n\mathbb{Z}} w^{q^{r-\nu+i-1}} y_i, \quad y_i \mapsto y_i + w^{q^{r-\mu+i-1}} \quad \text{for } i \in \mathbb{Z}/n\mathbb{Z},$$

where we regard $\{y_i\}$ as indexed by $\mathbb{Z}/n\mathbb{Z}$.

Remark 2.16. *The subgroups and elements appearing in (3) to (7) do not generate the whole group Stab_ν unless $W_L' = W_L$. Later, we shall study the action of Stab_ν on the cohomology in an indirect way (see 4.16).*

The proof[9] of the theorem occupies the rest of this subsection.

We define a rational subset $\mathcal{Z}_0 \subset \mathcal{M}^{\mathrm{ad}}_{\Sigma_0, \infty, \overline{\eta}}$ by $|X_i| = |\xi_i|$ $(1 \leq i \leq n)$. We clearly have $\mathcal{Z}_\nu \subset \mathcal{Z}_0$. Let us begin with the following useful lemma, which follows from the proof of [IT15a, Lemma 3.1].

Lemma 2.17. *Let $(g, d, 1) \in G^0$. Assume that there exists a point $\eta \in \mathcal{Z}_0$ such that $(g, d, 1)^* \eta \in \mathcal{Z}_0$. Then we have $(g, d) \in (\varphi, \varphi_D)^{\mathbb{Z}} \cdot (U_{\mathfrak{J}} \times U_D)$.*

Action of φ_G Let us prove that $\varphi_G \in \mathcal{S}$ stabilizes $\mathcal{Z}_\nu$ and induces the stated action on the reduction. We have

$$\varphi_G^* X_i = \begin{cases} X_n^{q^{n-1}} & \text{if } i = 1 \\ X_{i-1}^{q^{-1}} & \text{otherwise,} \end{cases} \qquad \varphi_G^* \xi_i = \xi_i$$

and hence

$$\varphi_G^* Y_i = \begin{cases} Y_n^{q^{n-1}} & \text{if } i = 1 \\ Y_{i-1}^{q^{-1}} & \text{otherwise,} \end{cases} \qquad \varphi_G^* Z = Z.$$

From this we find that φ_G stabilizes $\mathcal{Z}_\nu$ and acts on the reduction in a manner stated in (3).

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[8]Here, $\nu = rn + s$. Complicated values like $r - \nu + i - 1$ result from our choice of the normalization (2.20).

[9]The argument here is largely inspired by that in [IT15a].

Action of $U_{\mathfrak{J}}^{(\nu)}$ Let us prove that $\mathrm{GL}_n(K) \cap \mathrm{Stab}_\nu = U_{\mathfrak{J}}^{(\nu)}$ and that $U_{\mathfrak{J}}^{(\nu)}$ acts on Z_ν as in the assertion (6). Let $g = (a_{i,j}) \in \mathrm{GL}_n(K)$ stabilize $\check{\mathcal{Z}}_\nu$. Then we see that $g \in U_{\mathfrak{J}}$ by Lemma 2.17. In the notation of (1.2) we have

$$g^*(Y_i) = \sum_{1 \le j \le n} \sum_l a_{j,i}^{(l)} (Y_j + \xi_j)^{q^{ln}} - \xi_i, \tag{2.22}$$

$$g^*(Z) = \sum_{1 \le i \le n} \left(\sum_{1 \le j \le n} \sum_l a_{j,i}^{(l)} (Y_j + \xi_j)^{q^{ln}} - \xi_i \right)^{q^{i-1}}. \tag{2.23}$$

We first consider the condition $g^*\xi \in \mathcal{Z}_\nu$. By (2.22), we must have

$$\left| \sum_{1 \le j \le n} \sum_l a_{j,i}^{(l)} \xi_j^{q^{ln}} - \xi_i \right| \le \begin{cases} |\xi_i|^{q^\mu (q+1)/2} & \text{if } \nu = 2\mu + 1 \text{ is odd} \\ |\xi_i|^{q^\mu} & \text{if } \nu = 2\mu \text{ is even} \end{cases}$$

for all $1 \le i \le n$. Since the valuations of all ξ_i are distinct, we conclude that this is equivalent to

$$a_{i,i}^{(0)} = 1 \text{ for } 1 \le i \le n,$$
$$a_{j,i}^{(l)} = 0 \text{ for } (i,j,l) \text{ such that } 0 < i - j + ln < \lfloor (\nu+1)/2 \rfloor,$$

which in turn is equivalent to $g \in U_{\mathfrak{J}}^{\lfloor (\nu+1)/2 \rfloor}$. Similarly, by (2.23), we must have

$$\sum_{1 \le i \le n} \left(\sum_{1 \le j \le n} \sum_l a_{j,i}^{(l)} \xi_j^{q^{ln}} - \xi_i \right)^{q^{i-1}} \le |\xi_1|^{q^\nu}.$$

Using $\xi_i = \xi_{i+1}^q$, we see that this is equivalent to

$$\sum_{1 \le i \le n} a_{i,i}^{(0)} = n \text{ for } 1 \le i \le n,$$
$$\sum_{(i,j,l);i-j+ln=m} a_{j,i}^{(l)} = 0 \text{ for } 0 < m < \nu.$$

Altogether, the condition $g^*\xi \in \mathcal{Z}_\nu$ amounts to $g \in U_{\mathfrak{J}}^{(\nu)}$. Thus $\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K) \subset U_{\mathfrak{J}}^{(\nu)}$.

Conversely, let $g \in U_{\mathfrak{J}}^{(\nu)}$. Then we have, by (2.22),

$$g^*(Y_i) = Y_i + a_{j,i}^{(l)} \xi_j^{q^{ln}} + \cdots,$$

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where (j,l) is the unique pair such that $i - j + ln = \lfloor (\nu+1)/2 \rfloor$. To be more precise,

$$|g^*(Y_i) - Y_i| < |\xi_i|^{q^\mu (q+1)/2} \quad \text{if } \nu = 2\mu + 1 \text{ is odd}$$

$$\left| g^*(Y_i) - \left(Y_i + a_{j,i}^{(l)} \xi_i^{q^\mu} \right) \right| < |\xi_i|^{q^\mu} \quad \text{if } \nu = 2\mu \text{ is even.}$$

Similarly, we have

$$g^*(Z) = Z + \sum_{(i,j,l); i-j+ln=\nu} a_{j,i}^{(l)} \xi_1^{q^\nu} + \sum_{(i,j,l); i-j+ln=\lfloor (\nu+1)/2 \rfloor} a_{j,i}^{(l)} Y_j^{q^{ln+i-1}} + \cdots$$

by (2.23). Thus

$$\left| g^*(Z) - \left(Z + \sum_{(i,j,l); i-j+ln=\nu} a_{j,i}^{(l)} \xi_1^{q^\nu} \right) \right| < |\xi_1|^{q^\nu} \quad \text{if } \nu \text{ is odd}$$

$$\left| g^*(Z) - \left(Z + \sum_{(i,j,l); i-j+ln=\nu} a_{j,i}^{(l)} \xi_1^{q^\nu} + \sum_{(i,j,l); i-j+ln=\mu} a_{j,i}^{(l)} Y_j^{q^{\mu+j-1}} \right) \right| < |\xi_i|^{q^\nu} \quad \text{if } \nu = 2\mu \text{ is even.}$$

This shows that g stabilizes $\mathcal{Z}_\nu$ and acts on Z_ν in the stated way.

Action of $U_D^{(\nu)}$ The argument is analogous to the above. Let $d \in D^\times$ stabilize $\mathcal{Z}_\nu$ with $d_l \in k_n$. Then we have $d \in U_D$ as $(1,d,1) \in G^0$. In the notation of (1.3) we have

$$d^*(Y_i) = \sum_l d_l (Y_i + \xi_i)^{q^l} - \xi_i, \tag{2.24}$$

$$d^*(Z) = \sum_{1 \le i \le n} \left(\sum_l d_l (Y_i + \xi_i)^{q^l} - \xi_i \right)^{q^{i-1}}. \tag{2.25}$$

By the condition $d^*\xi \in \mathcal{Z}_\nu$, we necessarily have

$$\left| \sum_l d_l \xi_i^{q^l} - \xi_i \right| \le \begin{cases} |\xi_i|^{q^\mu (q+1)/2} & \text{if } \nu = 2\mu + 1 \text{ is odd} \\ |\xi_i|^{q^\mu} & \text{if } \nu = 2\mu \text{ is even} \end{cases}$$

for $1 \le i \le n$ and

$$\left| \sum_{1 \le i \le n} \left(\sum_l d_l \xi_i^{q^l} - \xi_i \right)^{q^{i-1}} \right| \le |\xi_1|^{q^\nu}.$$

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As before, this is seen to be equivalent to $d \in U_D^{(\nu)}$.

Conversely, let $d \in U_D^{(\nu)}$. Then we have, by (2.24),

$$d^*(Y_i) = Y_i + d_{\lfloor (\nu+1)/2 \rfloor} \xi_i^{q^{\lfloor (\nu+1)/2 \rfloor}} + \cdots,$$

which implies

$$\left| d^*(Y_i) - Y_i \right| < |\xi_i|^{q^\mu (q+1)/2} \quad \text{if } \nu = 2\mu + 1 \text{ is odd} \tag{2.26}$$

$$\left| d^*(Y_i) - \left(Y_i + d_\mu \xi_i^{q^\mu} \right) \right| < |\xi_i|^{q^\mu} \quad \text{if } \nu = 2\mu \text{ is even.} \tag{2.27}$$

Likewise, we have

$$d^*(Z) = Z + \sum_{1 \le i \le n} d_\nu^{q^{i-1}} \xi_1^{q^\nu} + \sum_{1 \le i \le n} (d_{\lfloor (\nu+1)/2 \rfloor} Y_i^{q^{\lfloor (\nu+1)/2 \rfloor}})^{q^{i-1}} + \cdots$$

by (2.25). This implies that

$$\left| d^*(Z) - (Z + \sum_{1 \le i \le n} d_\nu^{q^{i-1}} \xi_1^{q^\nu}) \right| < |\xi_1|^{q^\nu} \quad \text{if } \nu \text{ is odd}$$

$$\left| d^*(Z) - \left(Z + \sum_{1 \le i \le n} d_\nu^{q^{i-1}} \xi_1^{q^\nu} + \sum_{1 \le i \le n} (d_\mu Y_i^{q^\mu})^{q^{i-1}} \right) \right| < |\xi_1|^{q^\nu} \quad \text{if } \nu = 2\mu \text{ is even.}$$

From this we conclude that d indeed stabilizes $\mathcal{Z}_\nu$ and acts on Z_ν exactly as in (7).

The inclusion $\mathcal{S} \subset \mathrm{Stab}_\nu$ Let us prove that $\mathcal{S}$ stabilizes $\mathcal{Z}_\nu$ and induces an action on the reduction. We take an element in $\mathcal{S}$ and express it as $(g, d\varphi_D^{-n_\sigma}, \sigma)$, so that $(g, d, 1) \in G^0$. As $(1, \varphi_D^{n_\sigma}, \sigma^{-1})^* \xi \in \mathcal{Z}_0$ and $(g, d, 1)^* \xi = (1, \varphi_D^{n_\sigma}, \sigma^{-1})^* \xi$, we infer that $(g, d) \in (\varphi, \varphi_D)^{\mathbb{Z}} \cdot (U_{\mathfrak{J}} \times U_D)$ by Lemma 2.17. We may also assume that $(g, d) \in U_{\mathfrak{J}} \times U_D$ because we have already proved that $(\varphi, \varphi_D) \in \mathrm{Stab}_\nu$. We have

$$(g, d\varphi_D^{-n_\sigma}, \sigma)^* X_i = (g, d, 1)^* X_i, \quad (g, d\varphi_D^{-n_\sigma}, \sigma)^* \xi_i = \sigma(\xi_i).$$

If we set $f_i(X_1, \ldots, X_n) = (g, d, 1)^* X_i \in \mathcal{O}_{\mathbb{C}_p}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]]$, then $(g, d\varphi_D^{-n_\sigma}, \sigma) \in \mathcal{S}$ amounts to $f_i(\xi_1, \ldots, \xi_n) = \sigma(\xi_i)$. Thus,

$$(g, d\varphi_D^{-n_\sigma}, \sigma)^* Y_i = f_i(Y_1 + \xi_1, \ldots, Y_n + \xi_n) - \sigma(\xi_i)$$
$$= f_i(Y_1, \ldots, Y_n). \tag{2.28}$$

Now since $(g, d) \in U_{\mathfrak{J}} \times U_D$ it is clear that if $|Y_i|$ satisfies the inequality defining $\mathcal{Z}_\nu$, so does $\left| (g, d\varphi_D^{-n_\sigma}, \sigma)^* Y_i \right|$. The arguments for Z proceeds similarly and if follows that $\mathcal{S} \subset \mathcal{Z}_\nu$.

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Action of $\Delta_\xi(U_L)$ Let $u \in U_L$. To show that the induced action of u on Z_ν is trivial, we may assume that u lies in U_L^1 since $\Delta_\xi(K^\times)$ acts trivially on $\mathcal{M}_{\Sigma_0,\infty,\bar\eta}^{\mathrm{ad}}$. Since $\Delta_\xi(u) \in \mathcal{S}$, we have, in the notation of (1.2), (1.3)

$$\Delta_\xi(u)^*(Y_i) = \sum_{1 \le j \le n} a_{j,i}^{(m)} \sum_m \sum_l d_l Y_j^{q^{l+mn}},$$

$$\Delta_\xi(u)^*(Z) = \sum_{1 \le i \le n} \sum_{1 \le j \le n} a_{j,i}^{(m)} \sum_m \sum_l d_l Y_j^{q^{l+mn+i-1}}.$$

Note that $d_l \in k$ for all l by assumption. Now, since $a_{i,i}^{(0)} = 1$ for all $1 \le i \le n$ and $d_0 = 1$, we see that

$$|\Delta_\xi(u)^*(Y_i) - Y_i| < |Y_i|.$$

For Z, we argue as follows:

$$\Delta_\xi(u)^* Z = \sum_{1 \le j \le n} \sum_{c>0} \sum_{(i,l,m);l+mn+i-1=c} a_{j,i}^{(m)} d_l Y_j^{q^c}$$
$$= Z,$$

where, in the last equality, we used the relation

$$\sum_{(i,l,m);l+mn+i-1=c} a_{j,i}^{(m)} d_l = \delta_{c,0},$$

which follows from the assumption. Hence, we conclude that $\Delta_\xi(u)$ acts trivially on Z_ν.

Action of $(1, a_\sigma^{-1}, \sigma)$ Let $\sigma \in W_K$ and set $a_\sigma, n_\sigma, u_\sigma$ as usual. We also set $u_\sigma = \sum_{j \ge 0} d_j \varphi_D^j \in U_D$ with $d_j \in k$. Since $(1, a_\sigma^{-1}, \sigma) \in \mathcal{S}$, we have, in the notation of (1.2), (1.3)

$$(1, a_\sigma^{-1}, \sigma)^* Y_i = \sum_j d_j Y_i^{q^j},$$

$$(1, a_\sigma^{-1}, \sigma)^* Z = \sum_j d_j Z^{q^j},$$

$$(1, a_\sigma^{-1}, \sigma)^* \xi_n = \sum_j d_j \xi_i^{q^j}.$$

Suppose that $\nu = 2\mu + 1$ is odd. Then it follows from the above and (2.1) that

$$(1, a_\sigma^{-1}, \sigma)^* y_i' = \left(\sum_j d_j \xi_i^{q^j}\right)^{-q^\mu(q+1)/2} \cdot \left(\sum_j d_j Y_i^{q^j}\right)$$

$$= \left(\sum_j d_j \xi_i^{q^j}\right)^{-q^\mu(q+1)/2} \cdot \left(\sum_j d_j (\xi_i^{q^\mu(q+1)/2} y_i')^{q^j}\right).$$

Thus we see that

$$(1, a_\sigma^{-1}, \sigma)^* y_i' \equiv d_0^{-(q-1)/2} y_i'$$

modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$. By (2.20) we infer that the induced action on y_i is as stated in (5) in this case. The computations for the actions on z and in the even ν case are similar and easier.

The inclusion $\mathrm{Stab}_\nu \subset (U_{\mathfrak{J}}^{(\nu)} \times U_D^{(\nu)} \times \{1\}) \cdot \mathcal{S}$ To prove the inclusion $\mathrm{Stab}_\nu \subset (U_{\mathfrak{J}}^{(\nu)} \times U_D^{(\nu)} \times \{1\}) \cdot \mathcal{S}$, we take an element in Stab_ν and write it as $(g, d\varphi_D^{-n_\sigma}, \sigma)$ with $(g, d, 1) \in G^0$ and $\sigma \in W_K$. Since $(1, \varphi_D^{n_\sigma}, \sigma^{-1})^* \xi \in \mathcal{Z}_0$, we have $(g, d) \in U_{\mathfrak{J}} \times U_D$ by Lemma 2.17.

Let us first show $\sigma \in W_{L'}$. There exists an element $\zeta_n \in \mu_n(\overline{K})$ such that $\sigma(\varphi_L) = \zeta_n \varphi_L$. We are to prove that $\zeta_n \in k$. The fact that $(g, d\varphi_D^{-n_\sigma}, \sigma)$ stabilizes ξ implies

$$\left| \sum_{j,m} a_{j,i}^{(m)} \sum_l d_l \xi_j^{q^{l+mn}} - \sigma(\xi_i) \right| \leq \begin{cases} |\xi_i|^{q^\mu(q+1)/2} & \text{if } \nu = 2\mu + 1 \text{ is odd} \\ |\xi_i|^{q^\mu} & \text{if } \nu = 2\mu \text{ is even} \end{cases}$$

for all $1 \leq i \leq n$. In particular,

$$\left| a_{i,i}^0 d_0 \xi_i - \sigma(\xi_i) \right| < |\xi_i|$$

for $1 \leq i \leq n$. As $\xi_i = \xi_{i+1}^q$ for $1 \leq i \leq n-1$, we deduce that $a_{i,i}^0 d_0 = a_{i+1,i+1}^0 d_0^q$, and hence $d_0^{q-1} \in k^\times$. On the other hand, by Proposition 2.4 (4), we have $|\sigma(\xi_1) - \zeta \xi_1| < |\xi_1|$ for some element $\zeta \in \overline{K}$ such that $\zeta^{q-1} = \zeta_n$. Therefore,

$$\zeta_n = \zeta^{q-1} = (a_{i,i}^0 d_0)^{q-1} = d_0^{q-1} \in k,$$

as desired.

Now it remains to prove the claim under the assumption $\sigma = 1$. We shall prove that, for $(g, d, 1) \in G^0$,

$$(g, d)^* \xi \in \mathcal{Z}_\nu \text{ implies } (g, d)^* \xi \in (U_{\mathfrak{J}}^{(\nu)} \times U_D^{(\nu)})^* \xi,$$

where we drop the factor of Weil group to ease the notation. Clearly, this is exactly what we need to prove.

The Y_i-coordinates and the Z-coordinate of $(g, d)^* \xi$ are

$$\sum_r \left(\sum_{(j,m,l);\ l+mn+i-j=r} a_{j,i}^{(m)} d_l \right) \xi_i^{q^r} - \xi_i, \quad \sum_r \left(\sum_{(i,j,m,l);\ l+mn+i-j=r} a_{j,i}^{(m)} d_l^{q^{i-1}} \right) \xi_1^{q^r} - n\xi_1,$$

respectively. Accordingly, we put

$$\alpha_{i,r}(g, d) = \sum_{(j,m,l);\ l+mn+i-j=r} a_{j,i}^{(m)} d_l, \quad \beta_r(g, d) = \sum_r \left(\sum_{(i,j,m,l);\ l+mn+i-j=r} a_{j,i}^{(m)} d_l^{q^{i-1}} \right) \xi_1^{q^r}.$$

As $\beta_r(g, d) = \sum_i (\alpha_{i,r}(g, d))^{q^{i-1}}$ by definition, we need to prove the inclusion

$$\left\{ (\alpha_{i,r}(g, d))_{i,r} \,\middle|\, (g, d, 1) \in \text{Stab}_\nu \right\} \subset \left\{ (\alpha_{i,r}(g, d))_{i,r} \,\middle|\, (g, d) \in U_{\mathfrak{J}}^{(\nu)} \times U_D^{(\nu)} \right\}.$$

Let $(g, d, 1) \in \text{Stab}_\nu$. To construct $(h, e) \in U_{\mathfrak{J}}^{(\nu)} \times U_D^{(\nu)}$ such that $(\alpha_{i,r}(g, d)) = (\alpha_{i,r}(h, e))$, we set

$$h = (b_{j,i}), \quad b_{j,i} = \sum_m b_{j,i}^{(m)} \varpi^m, \quad e^{-1} = \sum_l e_l \varphi_D^l$$

as usual. For $1 \le i \le n$ and $r \in \mathbb{Z}$ we put $a_i(r) = a_{j,i}^{(m)}$ for the unique pair (j, m) such that $i - j + mn = r$ and similarly define $b_i(r)$. Note that we must have $b_{i,i}^{(0)} = e_0 = 1$ and $b_i(r) = e_l = 0$ for $0 < r < \lfloor (\nu + 1)/2 \rfloor$.

From the condition on $\alpha_{i,r}(g, d)$, we infer by induction that

$$d_r \in k \text{ and } a_i(r) = a_{i'}(r) \text{ for } 0 \le r < \lfloor (\nu + 1)/2 \rfloor \text{ and } 1 \le i, i' \le n. \quad (2.29)$$

For $\lfloor (\nu + 1)/2 \rfloor \le r < \nu$, we divide sums;

$$\alpha_{i,r}(g, d) = \sum_{0 \le m < \lfloor (\nu+1)/2 \rfloor} a_i(r - m) d_m + \sum_{\lfloor (\nu+1)/2 \rfloor \le m \le r} a_i(r - m) d_m.$$

Here, we have $d_m \in k$ in the first summation and $a_i(r - m)$ is independent of i in the second summation by (2.29) and $r - \lfloor(\nu+1)/2\rfloor < \nu - \lfloor(\nu+1)/2\rfloor \leq \lfloor(\nu+1)/2\rfloor$. From the last inequality we have, for (h, e),

$$\alpha_{i,r}(h, e) = b_i(r) + d_r.$$

By $\sum_i(\alpha_{i,r}(g, d))^{q^{i-1}} = \beta_r(g, d) = 0$, it is now clear that, for instance,

$$b_i(r) = \sum_{0 \leq m < \lfloor(\nu+1)/2\rfloor} a_i(r - m)d_m - n^{-1}\,\mathrm{Tr}\left(\sum_{\lfloor(\nu+1)/2\rfloor \leq m \leq r} a_i(r - m)d_m\right)$$

$$d_r = \sum_{\lfloor(\nu+1)/2\rfloor \leq m < \nu} a_i(r - m)d_m + n^{-1}\,\mathrm{Tr}\left(\sum_{\lfloor(\nu+1)/2\rfloor \leq m \leq r} a_i(r - m)d_m\right),$$

satisfy the required conditions. As there are no conditions to consider for $r \geq \nu$, this completes the proof of the claim and also of the theorem.

2.6 Alternative description of the reductions in terms of algebraic groups and quadratic forms

In [BW16], algebraic varieties obtained by the reduction of affinoids are described in terms of the Lang torsors of certain algebraic groups. Motivated by their observation, we give here an alternative description of Z_ν using algebraic groups $\mathcal{G}_\nu$ and quadratic forms Q_ν for $\nu > 0$ not divisible by n. It suffices to treat the cases where $0 < \nu < 2n$.

Suppose first that $1 \leq \nu < 2n$ is odd. We put $\mathcal{G}_\nu = \mathbb{G}_a$, considered over k. We define a quadratic form $Q_\nu(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ by

$$Q_\nu(y_1, \ldots, y_n) = \begin{cases} -\sum_{\mu < j-i < n-\mu} y_i y_j & \text{if } 1 \leq \nu < n \\ \sum_{n-\mu \leq j-i \leq \mu} y_i y_j & \text{if } n+1 \leq \nu < 2n. \end{cases}$$

If we denote by F_q the q-th power Frobenius endomorphism, then the Lang torsor $L_\mathcal{G}$ of an algebraic group $\mathcal{G}$ over k is defined by

$$L_\mathcal{G} \colon \mathcal{G} \to \mathcal{G}; \ x \mapsto F_q(x) \cdot x^{-1}.$$

In this case the Lang torsor $L_{\mathcal{G}_\nu}$ of $\mathcal{G}_\nu$ is nothing but the Artin-Schreier map:

$$L_{\mathcal{G}_\nu} \colon \mathcal{G}_\nu \to \mathcal{G}_\nu; \ x \mapsto x^q - x.$$

Then it is clear from Theorem 2.10 (2), (3) that Z_ν is isomorphic to the base change to $\bar{k}$ of $Z_{\nu,0}$ defined by the following Cartesian diagram:

$$
\begin{array}{ccc}
Z_{\nu,0} & \longrightarrow & \mathcal{G}_\nu \\
\downarrow & & \downarrow{\scriptstyle L_{\mathcal{G}_\nu}} \\
V & \xrightarrow[Q_\nu|_V]{} & \mathcal{G}_\nu,
\end{array}
$$

where $V = V(y_1 + \cdots + y_n) \subset \mathbb{A}_k^n = \operatorname{Spec} k[y_1, \ldots, y_n]$ is a closed subscheme defined by $y_1 + \cdots + y_n = 0$ and Q_ν is considered as a morphism $\mathbb{A}_k^n \to \mathcal{G}$. Note that the action of $S_{1,\nu}$ (and also $S_{2,\nu}$) agrees with the action of $\mathcal{G}_\nu(k)$ induced by the Lang torsor.

Suppose that ν is even. In this case we first define an auxiliary algebraic group $\widetilde{\mathcal{G}}_\nu$. We set $\widetilde{\mathcal{G}}_\nu = \mathbb{A}_k^{n+1}$ as a scheme and define a structure of an algebraic group by the same formula as (2.21):

$$
(v, (w_i)) \cdot (v', (w_i')) = \left(v + v' + \sum_i w_i w_{i+(\nu/2)}', (w_i + w_i')\right)
$$

for any k-algebra R and for any $(v, (w_i)) \in \widetilde{\mathcal{G}}_\nu(R)$. We define

$$
\mathcal{G}_\nu = \operatorname{Ker}\left(\widetilde{\mathcal{G}}_\nu \to \mathbb{G}_a; \ (v, (w_i)) \mapsto \sum_i w_i\right).
$$

By definition, we have $\mathcal{G}_\nu(k) = S_{1,\nu}$. We put

$$
Q_\nu(y_1, \ldots, y_n) = \begin{cases} -\sum_{\mu < j-i < n-\mu} y_i y_j & \text{if } 1 \le \nu < n \\ -\sum_{n-\mu < j-i < \mu} y_i y_j & \text{if } n+1 \le \nu < 2n \end{cases}
$$

and put $f = (Q_\nu, \operatorname{id}) \colon \mathbb{A}_k^n \to \widetilde{\mathcal{G}}_\nu$.

If $1 \le \nu < n$, then we define $Z_{\nu,0}$ by the following Cartesian diagrams:

$$
\begin{array}{ccc}
Z_{\nu,0} & \longrightarrow & \mathcal{G}_\nu \\
\downarrow & & \downarrow{\scriptstyle (\cdot)^{-1} \circ L_{\mathcal{G}_\nu}} \\
V & \longrightarrow & \mathcal{G}_\nu \\
\downarrow & & \downarrow \\
\mathbb{A}_k^n & \xrightarrow[f]{} & \widetilde{\mathcal{G}}_\nu.
\end{array}
$$

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If $n+1 \le \nu < 2n$, then we define $Z_{\nu,0}$ by the following Cartesian diagrams:

$$
\begin{array}{ccc}
Z_{\nu,0} & \longrightarrow & \mathcal{G}_\nu \\
\downarrow & & \downarrow {\scriptstyle L_{\mathcal{G}_\nu}} \\
V & \longrightarrow & \mathcal{G}_\nu \\
\downarrow & & \downarrow \\
\mathbb{A}^n_k & \xrightarrow{\ f\ } & \widetilde{\mathcal{G}}_\nu,
\end{array}
$$

Note that the Lang torsor (or its composite with $(\cdot)^{-1}$) induces an action of $S_{1,\nu} = \mathcal{G}_\nu(k)$ on $Z_{\nu,0}$ in each case.

Proposition 2.18. *There exists a natural isomorphism between Z_ν and the base change to $\overline{k}$ of $Z_{\nu,0}$ which respects the actions of $S_{1,\nu}$.*

Proof. This can be verified by a computation. Note that if we set $\wp(x) = x^q - x$ and $\nu = 2\mu$, then we have

$$
L_{\mathcal{G}_\nu}(v, (w_i)) = \left(\wp(v) - \sum_i \wp(w_{i-\mu}) w_i, \, \wp(w_1), \dots, \wp(w_n) \right)
$$

$$
L_{\mathcal{G}_\nu}(v, (w_i))^{-1} = \left(-\wp(v) + \sum_i \wp(w_{i-\mu}) w_i^q, \, -\wp(w_1), \dots, -\wp(w_n) \right)
$$

on valued points. $\qquad\square$

Remark 2.19. *It is interesting that the complicated defining equation of Z_ν simplifies with the introduction of $\mathcal{G}_\nu$ when ν is even. On the other hand, we do not use this description in the computation of the cohomology.*

3 Cohomology of the reductions

3.1 Quadratic forms and ℓ-adic cohomology

Let V be a k-vector space of dimension m and $Q = Q(x_1, \dots, x_m)$ a quadratic form on V.

Suppose that $p \ne 2$. We define the associated symmetric bilinear form $b_Q \colon V \times V \to k$ by

$$
b_Q(v_1, v_2) = 2^{-1}(Q(v_1 + v_2) - Q(v_1) - Q(v_2)).
$$

Then Q is non-degenerate if and only if b_Q is non-degenerate. In this case, we put $\det Q = \det b_Q \pmod{k^{\times 2}}$. In general, we have an orthogonal decomposition $(V, Q) = (V_{\mathrm{nd}}, Q_{\mathrm{nd}}) \oplus (V_0, Q_0)$, where Q_{nd} is non-degenerate and Q_0 is zero. We simply write $Q = Q_{\mathrm{nd}} \oplus Q_0$. Let ψ be a non-trivial character of k. To treat quadratic exponential sums associated to quadratic forms by reducing to the standard one, we put

$$\mathfrak{g}(\psi) = \sum_{x \in k} \psi(x^2) = \sum_{x \in k} \left(\frac{x}{q}\right) \psi(x).$$

Suppose that $p = 2$. We define the associated alternating bilinear form $a_Q \colon V \times V \to k$ by

$$a_Q(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2).$$

For $a, b \in k$ we define a quadratic form $Q_{a,b}$ on k^2 by $Q_{a,b}(x, y) = ax^2 + xy + by^2$. For $c \in k$ we define a quadratic form Q_c on k by $Q_c(z) = cz^2$. It is well-known that there exists an orthogonal decomposition

$$(V, Q) \simeq \bigoplus_{1 \leq i \leq r'} (k^2, Q_{a_i, b_i}) \oplus (k, Q_c)^{\oplus \varepsilon} \oplus (k, Q_0)^{\oplus s} \tag{3.1}$$

where $r' \geq 0$, $\varepsilon \in \{0, 1\}$, $s \geq 0$ and $a_i, b_i \in k, c \in k^\times$. This is called a quasi-diagonalization of Q. Here r', ε, s do not depend on the choice of decomposition and we call $r = 2r'$ the rank of Q. We denote by $(V_{\mathrm{nd}}, Q_{\mathrm{nd}})$ (resp. $(V_{\mathrm{ql}}, Q_{\mathrm{ql}})$) the quadratic space corresponding to a non-degenerate subspace $\bigoplus_{1 \leq i \leq r'} (k^2, Q_{a_i, b_i})$ (resp. a "quasi-linear" subspace $(k, Q_c)^{\oplus \varepsilon} \oplus (k, Q_0)^{\oplus s}$) and simply write $Q = Q_{\mathrm{nd}} \oplus Q_{\mathrm{ql}}$. We put $\mathrm{Arf}(Q_{\mathrm{nd}}) = \sum_{1 \leq i \leq r'} a_i b_i \pmod{\wp(k)}$, where $\wp(k) = \{x^2 + x \mid x \in k\}$. This is also an invariant of Q, called the Arf invariant.

The following can be found, for instance, in [Lic97, Definition 10.2]

Proposition 3.1. *Let (V, Q) be as above. Suppose that Q is non-degenerate (so that $m = 2r'$ is even) and $k = \mathbb{F}_2$. Then the number of elements of the fiber $Q^{-1}(1)$ is either $2^{m-1} - 2^{r'-1}$ or $2^{m-1} + 2^{r'-1}$. We have $\mathrm{Arf}(Q) = 0$ in the first case, and $\mathrm{Arf}(Q) = 1$ in the second case.*

We record an elementary computation of the cohomology of certain varieties associated to quadratic forms.

Proposition 3.2. *Let Q and k be as above. Let X_0 be the algebraic variety defined by the Cartesian diagram;*

$$
\begin{array}{ccc}
X_0 & \longrightarrow & \mathbb{A}^1_k \\
\downarrow & & \downarrow {\scriptstyle \wp_k} \\
\mathbb{A}^m_k & \xrightarrow{\ Q\ } & \mathbb{A}^1_k,
\end{array}
$$

where $\wp_k$ is the Artin-Schreier map $\wp_k(x) = x^q - x$ and Q is the morphism induced by the quadratic form. Denote by X the base change of X_0 to the algebraic closure $\overline{k}$. Take a prime number $\ell \neq p$ and put

$$
H^i_c = H^i_c(X, \overline{\mathbb{Q}}_\ell),
$$

which carries the actions of the additive group k and $\Omega = \mathrm{Gal}(\overline{k}/k)$. Then the following assertions hold.

(1) *Suppose that $p \neq 2$ and $r > 0$. Let r be the rank of Q and express Q as an orthogonal sum $Q = Q_{\mathrm{nd}} \oplus Q_0$ as before. We denote the quadratic residue symbol of k by $\left(\frac{\cdot}{q}\right)$. Then we have*

$$
H^i_c \simeq \begin{cases}
\bigoplus_{\psi \in k^\vee \setminus \{1\}} V_\psi & \text{if } i = 2m - r \\
\overline{\mathbb{Q}}_\ell(m) & \text{if } i = 2m \\
0 & \text{otherwise,}
\end{cases}
$$

where V_ψ is a one-dimensional vector space on which k acts via ψ and the q-th power geometric Frobenius Frob_q acts as multiplication by the scalar

$$
(-1)^{2m-r} \left(\frac{\det Q_{\mathrm{nd}}}{q}\right) \mathfrak{g}(\psi)^r q^{m-r},
$$

and $\overline{\mathbb{Q}}_\ell(m)$ is a one-dimensional vector space on which k acts trivially and Frob_q acts as multiplication by the scalar q^m.

(2) *Suppose that $p = 2$ and $r > 0$. Let r be the rank of Q and express Q as an orthogonal sum $Q = Q_{\mathrm{nd}} \oplus Q_{\mathrm{ql}}$ as before. Let $\varepsilon \in \{0,1\}$ be as in (3.1). We denote by ψ_0 the unique non-trivial character of $\mathbb{F}_2$. Then we have*

$$
H^i_c \simeq \begin{cases}
\bigoplus_{\psi \in k^\vee \setminus \{1\}} V_\psi & \text{if } i = 2m - 2\varepsilon - r \\
\overline{\mathbb{Q}}_\ell(m) & \text{if } i = 2m \\
0 & \text{otherwise,}
\end{cases}
$$

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where V_ψ is a one-dimensional vector space on which k acts via ψ and Frob_q *acts as multiplication by the scalar*

$$(-1)^{2m-2\varepsilon-r}\psi_0\left(\mathrm{Tr}_{k/\mathbb{F}_2}\mathrm{Arf}\left(Q_{\mathrm{nd}}\right)\right)q^{m-\varepsilon-r/2}$$

(which is independent of ψ), and $\overline{\mathbb{Q}}_\ell(m)$ is a one-dimensional vector space on which k acts trivially and Frob_q *acts as multiplication by the scalar* q^m.

Proof. For any additive character ψ of k, let $\mathscr{L}_\psi$ denote the Artin-Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_k$ associated to ψ, which is equal to $\mathfrak{F}(\psi)$ in the notation of [Del77, Sommes trig. 1.8 (i)]. No matter whether $p=2$ or not, we have an isomorphism

$$H^i_c \simeq \bigoplus_{\psi \in k^\vee} H^i_c(\mathbb{A}^m_k, Q^*\mathscr{L}_\psi)$$

as representations of $k \times \Omega$.

Suppose that $p \neq 2$. The assertion in this case is well-known. Diagonalizing Q and applying the Künneth formula, we are reduced to computing $H^i_c(\mathbb{A}^m_k, Q^*\mathscr{L}_\psi)$ for $m=1$ and $Q(x) = ax^2, (a \in k)$. As the pull-back of $\mathscr{L}_\psi$ by the zero map is the constant sheaf, giving rise to $\overline{\mathbb{Q}}_\ell(1)[2]$, it suffices to show the proposition in $r=m=1$ case, which is done by the Grothendieck-Ogg-Shafarevich formula and the Grothendieck-Lefschetz trace formula.

Suppose now that $p=2$. Again, by quasi-diagonalizing Q, we are reduced to computing $H^i_c(\mathbb{A}^m_k, Q^*\mathscr{L}_\psi)$ for either $m=1$ and $Q(x) = ax^2, (a \in k)$, or $m=2$ and $Q(x,y) = ax^2 + xy + by^2, (a, b \in k)$. As the computation is easier if ψ is trivial, we assume that ψ is non-trivial. In the first case, if $a=0$, then the cohomology is $\overline{\mathbb{Q}}_\ell(1)[2]$ as above and if $a \neq 0$, then Q is a morphism of additive group schemes $\mathbb{G}_a \simeq \mathbb{A}^1_k$, which implies that the cohomology vanishes in all degrees by [Del77, Sommes trig. Théorème 2.7*]. In the remaining case we may assume that ψ is non-trivial. Recall the isomorphism $k \xrightarrow{\sim} k^\vee;\ x \mapsto \psi_{0,x}$, where $\psi_{0,x}(y) = \psi_0\left(\mathrm{Tr}_{k/\mathbb{F}_2}(xy)\right)$, and take $c \in k^\times$ such that $\psi = \psi_{0,c}$. Then we see that $\psi_{0,c}(ax^2) = \psi_{0,d}(x)$ for all $x \in k$ with

$d = (ca)^{1/2} \in k$. Now we can turn an elementary manipulation

$$
\begin{aligned}
\sum_{x,y \in k} \psi_{0,c}(Q(x,y)) &= \sum_{x,y} \psi_{0,c}(ax^2 + xy + by^2) \\
&= \sum_{y} \psi_{0,c}(by^2) \sum_{x} \psi_{0,c}(ax^2 + xy) \\
&= \sum_{y} \psi_{0,c}(by^2) \sum_{x} \psi_{0,d+cy}(x) \\
&= \psi_{0,c}\left(b(-d/c)^2\right) q \\
&= \psi_0\left(\mathrm{Tr}_{k/\mathbb{F}_2} \mathrm{Arf}(Q)\right) q
\end{aligned}
$$

into the desired cohomological statement as in the proof of [Boya13, Prop. 2.10]. □

3.2 Representations of a cyclic group in finite classical groups

In [BF83] and [BH05b], one is naturally led to consider orthogonal and symplectic representations of a cyclic group over a finite field in order to compute subtle invariants of certain representations. We use the theory in our analysis of the cohomology of Z_ν. Thus we summarize parts of [BH05b, Section 4] in this subsection.

We put $\Omega = \mathrm{Gal}(\overline{k}/k)$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$, where n is assumed to be coprime to p as always.

Let Ω act on $\widehat{\Gamma} = \mathrm{Hom}(\Gamma, \overline{k}^\times)$ via its natural action on the target. For $\chi \in \widehat{\Gamma}$, we define a $k[\Gamma]$-module V_χ in the following way: the underlying vector space is the field $k[\chi] \subset \overline{k}$ generated by the values of χ and Γ acts via the character $\chi \colon \Gamma \to k[\chi]^\times$.

Proposition 3.3. *The $k[\Gamma]$-module V_χ is simple and its isomorphism class depends only on the Ω-orbit of χ. Moreover, we have a bijection between the set of Ω-orbits of $\widehat{\Gamma}$ and the set of isomorphism classes of simple $k[\Gamma]$-modules induced by $\chi \mapsto V_\chi$. In particular, the following decomposition holds;*

$$
k[\Gamma] = \bigoplus_{\chi \in \Omega \backslash \widehat{\Gamma}} V_\chi.
$$

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As we need to treat $p = 2$ case, we define *an orthogonal representation* (V, Q) of Γ over k to be a $k[\Gamma]$-module V endowed with a non-degenerate Γ-invariant quadratic form Q. If $p \neq 2$, non-degenerate quadratic forms Q correspond to non-degenerate symmetric bilinear forms b_Q and thus this notion coincides with the usual one.

Proposition 3.4. *Assume that $p \neq 2$. Let (V, Q) be an orthogonal representation of Γ over k.*

(1) Suppose that (V, Q) is indecomposable. Then exactly one of the following holds.

   (i) The underlying $k[\Gamma]$-module V is simple and isomorphic to V_χ with $\chi^2 = 1$.

   (ii) V is isomorphic to $U \oplus U^\vee$, where U is a simple $k[\Gamma]$-module which is not isomorphic to its contragredient $U^\vee$.

   (iii) $V = V_\chi$ is simple, isomorphic to its contragredient, but $\chi^2 \neq 1$.

   Moreover, the isometry class of (V, Q) as an orthogonal representation is determined by the isomorphism class of V in the last two cases.

(2) There exists a decomposition of (V, Q) into an orthogonal sum of indecomposable orthogonal representations. In particular, $\det Q$ is determined by the restriction of Q to the subspace of V fixed by $\{\gamma^2 \mid \gamma \in \Gamma\}$.

Example 3.5. *Let a be a positive divisor of n and let $\Gamma^a \subset \Gamma$ denote the unique subgroup of order n/a. The regular $k[\Gamma]$-module $k[\Gamma]$ has canonical Γ-submodules*

$$k[\Gamma/\Gamma^a] = k[\Gamma]^{\Gamma^a} = \bigoplus_{\chi \in \Omega \backslash \widehat{\Gamma}, \chi^a = 1} V_\chi, \quad I_k(\Gamma; a, n) = \bigoplus_{\chi \in \Omega \backslash \widehat{\Gamma}, \chi^a \neq 1} V_\chi.$$

Then $I_k(\Gamma; a, n)$ is the unique complement of $k[\Gamma/\Gamma^a]$ in $k[\Gamma]$ as a Γ-submodule. Similarly, if $a \mid b \mid n$, we define $I_k(\Gamma; a, b)$ by

$$I_k(\Gamma; a, n) = I_k(\Gamma; b, n) \oplus I_k(\Gamma; a, b).$$

Suppose now that $p \neq 2$. Let $\varepsilon \colon k[\Gamma] \to k$ be the k-linear map sending $1 \in \Gamma$ to $1 \in k$ and $1 \neq \gamma \in \Gamma$ to $0 \in k$, and let $x \mapsto \overline{x}$ be the standard k-linear involution on $k[\Gamma]$ such that $\overline{\gamma} = \gamma^{-1}$ for $\gamma \in \Gamma$. Then $Q_\Gamma(x) = \varepsilon(x\overline{x})$,

$x \in k[\Gamma]$ *defines a* Γ*-invariant non-degenerate quadratic form on* $k[\Gamma]$ *such that* $\det Q_\Gamma = 1$. *We see that*

$$Q_\Gamma|_{k[\Gamma/\Gamma^a]} = a^{-1}n Q_{\Gamma/\Gamma^a}$$

and hence

$$\det(Q_\Gamma|_{k[\Gamma/\Gamma^a]}) = (a^{-1}n)^a, \quad \det(Q_\Gamma|_{I_k(\Gamma;a,n)}) = (a^{-1}n)^a \pmod{k^{\times 2}}.$$

As usual, by *a symplectic representation* of Γ over k we mean a pair (V,b) consisting of a $k[\Gamma]$-module V and a non-degenerate Γ-invariant alternating form b.

Proposition 3.6. [10] *Let* (V,b) *be a symplectic representation of* Γ *over* k.

(1) Suppose that (V,b) *is indecomposable. Then exactly one of the following holds.*

    (i) The underlying $k[\Gamma]$*-module* V *is isomorphic to* $U \oplus U^\vee$, *where* U *is either isomorphic to* V_χ *for some* χ *with* $\chi^2 = 1$, *or is a simple* $k[\Gamma]$*-module which is not isomorphic to its contragredient.*

    (ii) $V = V_\chi$ *is simple, isomorphic to its contragredient, but* $\chi^2 \neq 1$.

    Moreover, the isometry class of (V,b) *as a symplectic representation is determined by the isomorphism class of* V *in all the three cases.*

(2) There exists a decomposition of (V,b) *into an orthogonal sum of indecomposable symplectic representations.*

Remark 3.7. *Suppose that* $p = 2$. *Let* (V,Q) *be an orthogonal representation of* Γ *over* k. *Considering the orthogonal decomposition of the alternating bilinear form* a_Q *associated to* Q, *we may separately study the isotypic components of* V *appearing in Proposition 3.6 to compute the Arf invariant of* Q.

---

[10]In [BF83][BH05b], this proposition is stated under the assumption that $k = \mathbb{F}_p$. However, it plays a role only in the discussion of the computation of trace invariants and this proposition remains true without the assumption.

3.3 Certain Heisenberg groups and their representations

Let H be a finite group and $Z \subset H$ its center. We say that a finite group H is a Heisenberg group if it is not abelian and $Q = H/Z$ is abelian. If H is a Heisenberg group, the map $H \times H \to Z$; $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ induces an alternating bilinear map $[\cdot] \colon Q \times Q \to Z$.

The following is well-known; see for instance [Bum98, Exercises 4.1.4-4.1.7].

Proposition 3.8. *Let H, Z, Q be as above. Let $\psi \colon Z \to \overline{\mathbb{Q}}_\ell^\times$ be a character of Z. Assume that $\psi \circ [\cdot] \colon Q \times Q \to \overline{\mathbb{Q}}_\ell^\times$ is non-degenerate. Then there exists an irreducible representation ρ_ψ of H, unique up to isomorphism, whose central character is ψ. Moreover, $\dim \rho_\psi = \sqrt{c}$, where c is the order of H.*

The following proposition can be verified easily.

Proposition 3.9. *Let $1 < \nu = 2\mu < 2n$ be an even integer. Let $S_{1,\nu}$ (resp. $S_{2,\nu}$) be the group defined in Proposition 2.12 (resp. in Proposition 2.14). Assume that n and ν are coprime.*

(1) The group $S_{1,\nu}$ is a Heisenberg group. In the notation of Proposition 2.12 (2), the center $Z(S_{1,\nu})$ is

$$Z(S_{1,\nu}) = k = \{(v, (w_i)) \in k \times k^{\mathbb{Z}/n\mathbb{Z}} \mid w_i = 0 \quad \text{for all } i\} \subset S_{1,\nu}.$$

Moreover, a character ψ of k induces a non-degenerate alternating form $\psi \circ [\cdot] \colon S_{1,\nu}/Z(S_{1,\nu}) \times S_{1,\nu}/Z(S_{1,\nu}) \to \overline{\mathbb{Q}}_\ell^\times$ if and only if ψ is non-trivial.

(2) The group $S_{2,\nu}$ is a Heisenberg group. In the notation of Proposition 2.14 (2), the center $Z(S_{2,\nu})$ is

$$Z(S_{2,\nu}) = k = \{(v, w) \in k \times k_n \mid w = 0\} \subset S_{2,\nu}.$$

Moreover, a character ψ of k induces a non-degenerate alternating form $\psi \circ [\cdot] \colon S_{2,\nu}/Z(S_{2,\nu}) \times S_{2,\nu}/Z(S_{2,\nu}) \to \overline{\mathbb{Q}}_\ell^\times$ if and only if ψ is non-trivial.

If n and ν are coprime, then, by Propositions 3.8 and 3.9, there exists a unique irreducible representation $\rho_{1,\psi}$ (resp. $\rho_{2,\nu}$) of $S_{1,\nu}$ (resp. of $S_{2,\nu}$) with the central character ψ, for any non-trivial character ψ of k.

3.4 Cohomology of the reductions

Proposition 3.10. [11] *Let $1 \leq \nu < 2n$ be an integer. Suppose that ν is odd and write $\nu = 2\mu + 1$. Put $d = \gcd(n, \nu)$. Let Z_ν be the algebraic variety defined in Theorem 2.10 and Q_ν the quadratic form defined in 2.6. Put*

$$H_c^i = H_c^i(Z_\nu, \overline{\mathbb{Q}}_\ell).$$

If $p = 2$, we denote by $\left(\frac{\cdot}{m}\right)$ the Jacobi symbol for any positive odd integer m.

(1) We have $H_c^i = 0$ unless $i = 2(n-1), n + d - 2$.

(2) If $d \neq n$, then

$$H_c^i \simeq \begin{cases} \bigoplus_{\psi \in k^\vee \backslash \{1\}} W_\psi & \text{if } i = n + d - 2 \\ \overline{\mathbb{Q}}_\ell(n-1) & \text{if } i = 2(n-1), \end{cases}$$

where W_ψ is a one-dimensional vector space on which k acts via ψ and Frob_q acts as multiplication by the scalar

$$\begin{cases} \left(\frac{q}{n/d}\right) q^{\frac{n+d-2}{2}} & \text{if } n \text{ is odd and } p = 2 \\ \left(\frac{n/d}{q}\right) \mathfrak{g}(\psi)^{n-d} q^{d-1} & \text{if } n \text{ is odd and } p \neq 2 \\ -\left(\frac{-1}{q}\right)^\mu \left(\frac{2}{q}\right) \left(\frac{n/d}{q}\right) \mathfrak{g}(\psi)^{n-d} q^{d-1} & \text{if } n \text{ is even (and hence } p \neq 2), \end{cases}$$

and k acts trivially on $\overline{\mathbb{Q}}_\ell(n-1)$.

(3) If $d = n$, then

$$H_c^{2(n-1)} \simeq \overline{\mathbb{Q}}_\ell[k] \boxtimes \overline{\mathbb{Q}}_\ell(n-1)$$

as a representation of $k \times \Omega$.

(4) Consider the following natural action of the standard generators of $\Gamma = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ on Z_ν;

$$(z, y_1, y_2, \ldots, y_n) \mapsto (z, y_n, y_1, \ldots, y_{n-1}),$$
$$(z, y_1, \ldots, y_n) \mapsto (z, -y_1, \ldots, -y_n)$$

---

[11]Although we compute the cohomology of Z_ν for any ν (not divisible by $2n$) in this proposition, only the cases where n and ν are coprime are relevant in our main theorem; we find the result interesting nonetheless.

respectively (cf. 2.5). Let ψ be a character of k. Then both actions induce, on the ψ-isotypic component of $\bigoplus_i H_c^i$, multiplication by the scalar

$$\begin{cases} (-1)^{n-1} & \text{if } \psi \text{ is non-trivial} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We first treat (1) to (3). In view of Proposition 3.2, we need to compute various invariants of the quadratic form Q_ν defining Z_ν. For this, we follow the approach of [BH05b, 8.3] and exploit the Γ-invariance of Q_ν. Denoting the standard generator of Γ by γ, we regard $Q_\nu = Q_\nu(y_1, \dots, y_n)$ as a quadratic form on $k[\Gamma] = \left\{ \sum_{1 \le i \le n} y_i \gamma^i \mid y_i \in k \right\}$.

First we claim, with no assumption on the parity of p, that $I_k(\Gamma; 1, n) \subset k[\Gamma]$ admits an orthogonal decomposition

$$I_k(\Gamma; 1, n) = I_k(\Gamma; d, n) \oplus I_k(\Gamma; 1, d) \tag{3.2}$$

in the notation of Example 3.5, where (the restriction of) Q_ν is non-degenerate (or zero) on $I_k(\Gamma; d, n)$ (resp. on $I_k(\Gamma; 1, d)$). Suppose that $1 \le \nu < n$. Then, for $x \in k[\Gamma]$,

$$Q_\nu(x) = \begin{cases} -\varepsilon \left(\sum_{\mu+1 \le i \le (n-1)/2} x \overline{(\gamma^{-i} x)} \right) & \text{if } n \text{ is odd} \\ -\varepsilon \left(\sum_{\mu+1 \le i \le n/2-1} x \overline{(\gamma^{-i} x)} + 2^{-1} x \overline{(\gamma^{-n/2} x)} \right) & \text{if } n \text{ is even (and hence } p \ne 2\text{).} \end{cases}$$

By Proposition 3.4 and Remark 3.7 we may separately consider each isotypic component underlying some indecomposable orthogonal (or symplectic, if $p = 2$) representation of Γ. Also, to prove non-degeneracy or triviality of Q_ν, we may assume that all characters of Γ take values in $k^\times$.

For a character χ of Γ, let

$$e_\chi = n^{-1} \sum_{1 \le i \le n} \chi(\gamma^i) \gamma^{-i}$$

be the corresponding idempotent, so that $V_\chi = e_\chi k[\Gamma]$. First let χ be a character such that $\chi^2 \ne 1$ and consider $Q_\nu|_{V_\chi \oplus V_{\chi^{-1}}}$. Put $\alpha = \chi(\gamma)$. If n is odd, then we have, for $y, z \in k$,

$$Q_\nu(ye_\chi + ze_{\chi^{-1}}) = -\varepsilon \left(\sum_{\mu+1 \le i \le (n-1)/2} (ye_\chi + ze_{\chi^{-1}})(\alpha^{-i} ye_{\chi^{-1}} + \alpha^i ze_\chi) \right)$$

$$= -n^{-1} yz \left(\sum (\alpha^i + \alpha^{-i}) \right) = -n^{-1} yz \left(\alpha^{n-\mu} - \alpha^{\mu+1} \right) / (\alpha - 1).$$

Similarly, if n is even, we see that

$$Q_\nu(ye_\chi + ze_{\chi^{-1}}) = -n^{-1}yz \left(\sum_{\mu+1 \le i \le n/2-1} (\alpha^i + \alpha^{-i}) + \alpha^{n/2} \right)$$
$$= -n^{-1}yz \left(\alpha^{n-\mu} - \alpha^{\mu+1} \right) / (\alpha - 1).$$

Thus, $Q_\nu|_{V_\chi \oplus V_{\chi^{-1}}}$ is trivial if $\chi^d = \mathrm{id}$ and is non-degenerate otherwise.

Next let χ be of order two (so that n is even) and consider the restriction of Q_ν on $V_\chi \subset I_k(\Gamma; d, n)$. Then

$$Q_\nu(e_\chi) = -n^{-1} \left(\sum_{\mu+1 \le i \le n/2-1} (-1)^i + 2^{-1}(-1)^{n/2} \right) = (2n)^{-1}(-1)^\mu \quad (3.3)$$

and Q_ν is non-degenerate on V_χ. Hence we have proved the claim (3.2) if $1 \le \nu < n$. The case where $n+1 \le \nu < 2n$ can be reduced to the above case by noting that $Q_\nu = -Q_{2n-\nu}$. In particular, we find that

$$Q_\nu(e_\chi) = (2n)^{-1}(-1)^\mu \quad (3.4)$$

for the character χ of order two, if n is even.

Now suppose that $p \ne 2$. Then we need to show that

$$\det Q_\nu|_{I_k(\Gamma;d,n)} = \begin{cases} n/d & \text{if } n \text{ is odd} \\ (-1)^\mu 2n/d & \text{if } n \text{ is even} \end{cases} \quad (\bmod \ k^{\times 2}).$$

We compare Q_ν with the standard quadratic form Q_Γ on $k[\Gamma]$. If n is even and χ is the character of order two, then the determinant of $Q_\Gamma|_{V_\chi}$ is $1/n$. By Proposition 3.4, (3.3), (3.4), we infer that the determinants of $Q_\nu|_{I_k(\Gamma;d,n)}$ and $Q_\Gamma|_{I_k(\Gamma;d,n)}$ differ by a factor of $(-1)^\mu 2$ if n is even and coincide if n is odd. Now the determinant of $Q_\Gamma|_{I_k(\Gamma;d,n)}$ is indeed n/d, as is seen from Example 3.5.

Suppose that $p = 2$ (and hence n is odd). Then we are to prove that

$$\psi_0 \left(\mathrm{Tr}_{k/\mathbb{F}_2} \, \mathrm{Arf} \left(Q_\nu|_{I_k(\Gamma;d,n)} \right) \right) = \left(\frac{q}{n/d} \right).$$

However, as Q_ν is clearly defined over $\mathbb{F}_2$ and $I_k(\Gamma; d, n) = I_{\mathbb{F}_2}(\Gamma; d, n) \otimes_{\mathbb{F}_2} k$, the additivity of the Arf invariant and the multiplicativity of the Jacobi

symbol reduce the computation to $k = \mathbb{F}_2$ case. In fact, in the course of computation below we shall see that if a $k[\Gamma]$-module V with no non-zero fixed vectors admits an invariant non-degenerate quadratic form, then the Arf invariant is uniquely determined by V.

Let $m > 1$ be a divisor of n and set

$$V_m = \bigoplus V_\chi,$$

where the sum is taken over all the orbits $\Omega\chi \in \Omega\backslash\Gamma^\vee$ of order m. Take a prime divisor l_m of m. Suppose that a non-degenerate quadratic form Q on V_m is invariant under the action of Γ. Now every Γ-orbit of V_m except for $\{0\}$ is of length m. Thus if $\mathrm{Arf}(Q) = 0$ (or $= 1$), then $2^{\varphi(m)/2-1} + 2^{\varphi(m)-1} \equiv 1 \mod m$ (resp. $\equiv 0 \mod m$) by Proposition 3.1. Since the two congruences in the latter condition never occur together and $\mathrm{Arf}(Q) \in \{0,1\}$, the two conditions are in fact equivalent. The congruences are further equivalent to $2^{\varphi(m)/2} \equiv 1 \mod m$ (resp. $\equiv -1 \mod m$). We find that these in turn are equivalent to the same congruence $\mod l_m$, again by observing that $2^{\varphi(m)/2}$ can only be congruent to 1 or $-1 \mod m$. Now an elementary calculation shows that $2^{\varphi(m)/2} \equiv 2^{l_m-1} \mod l_m$ if m is a prime power and $2^{\varphi(m)/2} \equiv 1 \mod l_m$ otherwise. Therefore,

$$\psi_0\left(\mathrm{Arf}(Q)\right) = \begin{cases} \left(\frac{2}{l_m}\right) & \text{if } m \text{ is a prime power} \\ 1 & \text{otherwise,} \end{cases}$$

from which we conclude

$$\psi_0\left(\mathrm{Arf}\left(Q_\nu|_{I_k(\Gamma;d,n)}\right)\right) = \left(\frac{2}{n/d}\right)$$

as desired.

Finally, let us prove (4). By [DL76, Theorem 3.2][12], we have

$$\sum_i (-1)^i \operatorname{tr}\left(\gamma x \mid H_c^i\right) = \sum_i (-1)^i \operatorname{tr}\left(x \mid H_c^i(Z_\nu^\gamma, \overline{\mathbb{Q}}_\ell)\right),$$

where $x \in k$ and Z_ν^γ denotes the fixed point variety with respect to the action of the generator $\gamma \in \Gamma$. Since Z_ν^γ is clearly a discrete set of points indexed

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[12]The author learned the idea of applying the Deligne-Lusztig fixed point formula in [BW13, 4.4].

by k, the right-hand side equals the trace of the regular representation of k. Now we find the required action of γ by applying the idempotents of the group ring corresponding to each character of k. The action of $\mathbb{Z}/2\mathbb{Z}$ is treated in exactly the same way because the fixed point variety remains the same (unless $p = 2$, in which case the statement is trivial). $\qquad\square$

Proposition 3.11. *Let $1 \leq \nu < 2n$ be an integer. Suppose that ν is even and write $\nu = 2\mu$. Assume that n and ν are coprime. Let Z_ν be the algebraic variety defined in Theorem 2.10. Put*

$$H_c^i = H_c^i(Z_\nu, \overline{\mathbb{Q}}_\ell),$$

which carries the actions of $S_{1,\nu} \times S_{2,\nu}$ and $\Omega = \mathrm{Gal}(\overline{k}/k)$ in the notation of 2.5.

(1) We have $H_c^i = 0$ unless $i = 2(n-1), n-1$.

(2) We have

$$H_c^{n-1} \simeq \bigoplus_{\psi \in k^\vee \setminus \{1\}} \rho_{1,\psi} \boxtimes \rho_{2,\psi}$$

as a representation of $S_{1,\nu} \times S_{2,\nu}$, where $\rho_{1,\psi}$ $(\rho_{2,\psi})$ is the unique irreducible representation of $S_{1,\nu}$ (resp. of $S_{2,\nu}$) with the central character ψ (cf, Proposition 3.9), and

$$\mathrm{tr}\left(\mathrm{Frob}_q \mid \rho_{1,\psi} \boxtimes \rho_{2,\psi}\right) = q^{n-1}.$$

(3) Consider the following natural action of the standard generator $\gamma \in \Gamma = \mathbb{Z}/n\mathbb{Z}$ on Z_ν;

$$(z, y_1, y_2, \ldots, y_n) \mapsto (z, y_n, y_1, \ldots, y_{n-1}).$$

Let ψ be a character of k and $H_{c,\psi}^\bullet$ the ψ-isotypic component of $\bigoplus_i H_c^i$. Then we have

$$\mathrm{tr}\left(\gamma^j \mid H_{c,\psi}^\bullet\right) = 1$$

for any j coprime to n.

Proof. Let us prove the assertions (1), (2). As the case where $n < \nu < 2n$ is settled in exactly the same way, we only treat the case $\nu < n$. We denote by $\widetilde{P}_\nu(y_1, \ldots, y_n)$ the polynomial appearing in the right-hand side of the second

equation in 2.10 (4). We put $P_\nu(y_2, \ldots, y_n) = \widetilde{P}_\nu(-(y_2 + \cdots + y_n), y_2, \ldots, y_n)$.
Then for any i we have the following decomposition:

$$H_c^i \simeq \bigoplus_{\psi \in k^\vee} H_c^i(\mathbb{A}_{\overline{k}}^{n-1}, P_\nu^* \mathscr{L}_\psi)$$

as a representation of $S_{1,\nu} \times S_{2,\nu} \times \Omega$. It suffices to prove

$$\dim H_c^i(\mathbb{A}_{\overline{k}}^{n-1}, P_\nu^* \mathscr{L}_\psi) = \begin{cases} q^{n-1} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

for any non-trivial $\psi \in k^\vee$. Indeed, $\psi = 1$ non-trivially contributes to the
above decomposition only if $i = 2(n-1)$, and we have $\dim \rho_{1,\psi} = \dim \rho_{2,\psi} = q^{(n-1)/2}$ if ψ is non-trivial by Proposition 3.8. Also, as $\psi(x^q - x) = 1$ for
any $x \in k$ and any $\psi \in k^\vee$, the statement for the Frobenius trace immedi-
ately follows from the Grothendieck-Lefschetz trace formula and the above
vanishing.

Our basic strategy is to apply [Del80, (3.7.2.3)][13]:

Let $P \in k[T_1, \ldots, T_m]$ be a polynomial of degree d. Suppose that
d is coprime to p and that the homogeneous part $P^{(d)}$ of degree
d of P defines a smooth hypersurface in $\mathbb{P}_k^{m-1}$. Then

$$\dim H_c^i(\mathbb{A}_{\overline{k}}^m, P^* \mathscr{L}_\psi) = \begin{cases} (d-1)^m & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

Although the polynomial P_ν is of degree $2q$, we may replace each mono-
mial of the form $y_i^q y_j^q$ with $y_i y_j$, because $f^* \mathscr{L}_\psi$ is a constant sheaf if $f = g^q - g$
for some polynomial g. We denote by $P_\nu' \in k[y_2, \ldots, y_n]$ the polynomial
obtained by applying the above procedures to all monomials of the form
$y_i^q y_j^q$. We similarly denote by $\widetilde{P}_\nu' \in k[y_1, \ldots, y_n]$ the polynomial obtained
from $\widetilde{P}_\nu$ in the same way. Then we have $\deg P_\nu' = \deg \widetilde{P}_\nu' = q + 1$ and
$P_\nu'(y_2, \ldots, y_n) = \widetilde{P}_\nu'(-(y_2 + \cdots + y_n), y_2, \ldots, y_n)$. Thus it suffices to show
that $P_\nu'^{(q+1)}$ defines a smooth hypersurface in $\mathbb{P}_k^{n-2}$. As this hypersurface is

---

[13]In fact, it also asserts that the cohomology in degree m is pure of weight m. However,
we only need the dimension assertions in what follows.

isomorphic to the projective variety V defined by $\widetilde{P}'^{(q+1)}_\nu$ and $y_1 + \cdots + y_n$ in $\mathbb{P}^{n-1}_{\overline{k}}$, we are reduced to proving that the Jacobian

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{\partial}{\partial y_1} \widetilde{P}'^{(q+1)}_\nu & \frac{\partial}{\partial y_2} \widetilde{P}'^{(q+1)}_\nu & \cdots & \frac{\partial}{\partial y_n} \widetilde{P}'^{(q+1)}_\nu
\end{pmatrix}
$$

is of rank two at every $\overline{k}$-valued point $[Y_1 : \cdots : Y_n]$ of V. Regarding $\{y_i\}$ as indexed by $\mathbb{Z}/n\mathbb{Z}$ we easily verify that

$$
\widetilde{P}'^{(q+1)}_\nu(y_1, \ldots, y_n) = - \sum_{1 \le i \le n} y_i \sum_{\mu \le d < n-\mu} y^q_{i+d}
$$

and hence

$$
\frac{\partial}{\partial y_i} \widetilde{P}'^{(q+1)}_\nu(y_1, \ldots, y_n) = - \sum_{\mu \le d < n-\mu} y^q_{i+d}.
$$

The rank of the Jacobian is not maximal if and only if these partial derivatives are all equal, that is, $Y^q_i = Y^q_j$ for all $1 \le i, j \le n$, by the assumption that n and $\nu = 2\mu$ are coprime. Together with $Y_1 + \cdots + Y_n = 0$, this implies $Y_1 = \cdots = Y_n = 0$, as required.

Now the assertion (3) follows from [DL76, Theorem 3.2] exactly as in Proposition 3.10 (4). $\qquad\square$

Given the preceding propositions, the cohomology of the reduction $\overline{\mathscr{Z}}_\nu$ is computed by exploiting the periodicity of Z_ν with respect to ν and the following proposition.

Proposition 3.12. *Let $\nu > 0$ be an integer, not divisible by n. Then we have the following isomorphism*

$$
H^i_c(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi \in \left(U^{\lceil \nu/n \rceil}_K\right)^\vee} H^i_c(Z_\nu, \overline{\mathbb{Q}}_\ell) \otimes (\chi \circ N_G).
$$

of representations of Stab_ν *for any i.*

Proof. The proposition follows from Theorem 2.15 (1) in the same way as [BW16, Corollary 3.6.2]. $\qquad\square$

4 Realization of correspondences

4.1 Special cases of essentially tame local Langlands and Jacquet-Langlands correspondences

Let us allow n to be divisible by p *only in this subsection.*

Definition 4.1. ([BH05a, 1, 2]) Let ρ be an n-dimensional irreducible smooth representation of W_K. Let $t(\rho)$ be the number of unramified characters χ of $K^\times$ such that $\chi \otimes \rho \simeq \rho$. Then $t(\rho)$ divides n and ρ is said to be *essentially tame* if p does not divide $n/t(\rho)$.

We denote by $\mathcal{G}_n^{et}(K)$ the set of isomorphism classes of n-dimensional (irreducible) essentially tame representations of W_K.

Similarly, let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(K)$ or an irreducible smooth representation of $D^\times$. We say that π is *essentially tame* if p does not divide $n/t(\pi)$, where $t(\pi)$ is the number of unramified characters χ of $K^\times$ such that $\chi\pi \simeq \pi$.

Definition 4.2. ([BH05a, 3. Definition]) *An admissible pair* (of degree n) is a pair $(F/K, \xi)$ in which F/K is a tamely ramified extension of degree n and ξ is a character of $F^\times$ such that

(1) if ξ factors through the norm map $N_{F/E} \colon F^\times \to E^\times$ for a subextension $K \subset E \subset F$, then $F = E$, and

(2) if $\xi|_{U_F^1}$ factors through $N_{F/E} \colon F^\times \to E^\times$ for a subextension $K \subset E \subset F$, then F/E is unramified.

Two admissible pairs $(F_1/K, \xi_1)$, $(F_2/K, \xi_2)$ are said to be K-isomorphic if there exists a K-isomorphism $i : F_1 \xrightarrow{\sim} F_2$ such that $\xi_1 = \xi_2 \circ i$.

We denote by $P_n(K)$ the set of K-isomorphism classes of admissible pairs of degree n.

The proof of the following is found in [BH05a, A.3 Theorem].

Proposition 4.3. *The following map is a bijection:*

$$P_n(K) \to \mathcal{G}_n^{\mathrm{et}}(K); \ (F/K, \xi) \mapsto \mathrm{Ind}_{F/K}\, \xi = \mathrm{Ind}_{W_F}^{W_K}\, \xi.$$

In [BH05a], [BH05b], [BH10], a canonical bijection $(F/K, \xi) \mapsto \pi_\xi$ between $P_n(K)$ and the set of isomorphism classes of essentially tame representations of $\mathrm{GL}_n(K)$ is constructed, and the existence and an explicit description of tamely ramified characters ${}_K\mu_\xi$ are established such that $\mathrm{Ind}_{F/K} \xi \mapsto \pi_{K\mu_\xi\xi}$ is the local Langlands correspondence. Likewise, a special case of the main result of [BH11] yields a canonical bijection $(F/K, \xi) \mapsto \pi_\xi^D$ between $P_n(K)$ and the set of isomorphism classes of essentially tame representations of $D^\times$, and the description of tamely ramified characters ${}_D\iota_\xi$ such that $\pi_{D\iota_\xi\xi} \mapsto \pi_\xi^D$ is the local Jacquet-Langlands correspondence.

In what follows, we review the construction of π_ξ and π_ξ^D and the description of ${}_K\mu_\xi$ and ${}_D\iota_\xi$ for certain admissible pairs $(F/K, \xi)$ that are relevant to our results.

Minimal pairs

Definition 4.4. Let $i \geq 0$ be an integer. An admissible pair $(F/K, \xi)$ is said to be *minimal with the jump at i* if $\xi|_{U_F^{i+1}}$ factors through the norm map $N_{F/K}$ and $\xi|_{U_F^i}$ does not factor through $N_{F/E}$ for any subextension $K \subset E \subsetneq F$.

We say that an admissible pair is *minimal*[14] if it is minimal with the jump at i for some $i \geq 0$.

Remark 4.5. *(1) If n is a prime, then any admissible pairs are minimal.*

(2) If $(F/K, \xi)$ is a minimal pair with the jump at i, then there exists a decomposition

$$\xi|_{U_F^i} = \left(\varphi \otimes (\chi \circ N_{F/K})\right)|_{U_F^i}, \tag{4.1}$$

where φ is a character of $F^\times$ trivial on U_F^{i+1} and χ is a character of $K^\times$.

(3) If $(F/K, \xi)$ is a minimal pair with the jump at i, then the ramification index of F/K is coprime to i.

(4) In this paper we are interested in minimal pairs $(F/K, \xi)$ with the jump at ν in which F/K is totally ramified (hence, $p \nmid n$ and ν is coprime to n). We will see that the cohomology of each reduction $\overline{\mathscr{Z}}_\nu$, with ν coprime to n, realizes the local Langlands and Jacquet-Langlands correspondences for representations parametrized by such minimal pairs (for a specific F).

---

[14]Note that some authors further impose the triviality of $\xi|_{U_F^{i+1}}$ in the definition of minimality. This definition is taken from [BH05a, 2.2] (except that i is assumed to be positive there). They also discuss jumps of possibly not minimal pairs.

(5) *Many of the preceding results treat the representations parametrized by minimal pairs;*

- *The cohomology of each reduction in [BW16] deals with minimal pairs $(F/K, \xi)$ with the jump at some $i \geq 1$ in which F/K is unramified.*

- *That of [IT15a] deals with minimal pairs $(F/K, \xi)$ with the jump at 1 in which F/K is totally ramified. These representations are (character twists of) the simple epipelagic representations if p does not divide n. Imai and Tsushima announced that they constructed corresponding affinoids also in the cases where p divides n. In these cases simple epipelagic representations are not essentially tame.*

- *In [Wei14], it is assumed that $n = 2$ and $p \neq 2$, in which case all representations involved are parametrized by minimal pairs.*

Construction of π_ξ and π_ξ^D in special cases In the rest of this subsection we assume that $(F/K, \xi)$ is a minimal pair with the jump at i and F/K is totally ramified.

Fix a character ψ of K [15] which is trivial on $\mathfrak{p}$, but not on $\mathcal{O}_K$. For $\alpha \in F$, define a function ψ_α^F by $\psi_\alpha^F(u) = \psi(\mathrm{Tr}_{F/K}\, \alpha(u-1))$, $(u \in F)$. Then by the tame ramification assumption we have

$$\mathfrak{p}_F^{-s}/\mathfrak{p}_F^{-r} \xrightarrow{\sim} (U_F^{r+1}/U_F^{s+1})^\vee; \ \alpha + \mathfrak{p}_F^{-r} \mapsto \psi_\alpha^F \tag{4.2}$$

for any integers r, s such that $0 \leq r < s \leq 2r + 1$. Similarly, for $\beta \in M_n(K)$ and $\gamma \in D$, set

$$\psi_\beta(g) = \psi(\mathrm{tr}\, \beta(g-1)), \ (g \in M_n(K)) \text{ and } \psi_\gamma^D(d) = \psi(\mathrm{Trd}\, \gamma(d-1)), \ (d \in D).$$

Let us construct π_ξ and π_ξ^D.

First suppose that ξ is trivial on U_F^{i+1}. Then there exists an $\alpha \in F$ with $v_F(\alpha) = -i$ such that $\xi|_{U_F^{\lfloor i/2 \rfloor +1}} = \psi_\alpha^F$. Take a K-embedding $F \to M_n(K)$ (resp. $F \to D$) and regard F as a K-subalgebra $F \subset M_n(K)$ (resp. $F \subset D$). Let $\mathfrak{I} = \mathfrak{I}_\xi \subset M_n(K)$ be the unique hereditary $\mathcal{O}_K$-order normalized by $F^\times$. (Later we will always arrange it to be the standard Iwahori order; see Remark 4.8). Denote by $\mathfrak{P}_\mathfrak{I} = \mathrm{rad}\, \mathfrak{I}$ the Jacobson radical of $\mathfrak{I}$ and set

---

[15]Thus, we change notation here; in Section 3 ψ generally denote a character of k.

$U_{\mathfrak{J}} = \mathfrak{J}^\times$, $U_{\mathfrak{J}}^i = 1 + \mathfrak{P}_{\mathfrak{J}}^i$ for $i \geq 1$ as usual. Define a character θ_ξ (resp. θ_ξ^D) of $H_\xi^1 = U_F^1 U_{\mathfrak{J}}^{\lfloor i/2 \rfloor + 1}$ (resp. of $H_\xi^{1,D} = U_F^1 U_D^{\lfloor i/2 \rfloor + 1}$) by

$$\theta_\xi|_{U_F^1} = \xi|_{U_F^1}, \quad \theta_\xi|_{U_{\mathfrak{J}}^{\lfloor i/2 \rfloor + 1}} = \psi_\alpha|_{U_{\mathfrak{J}}^{\lfloor i/2 \rfloor + 1}},$$
$$\theta_\xi^D|_{U_F^1} = \xi|_{U_F^1}, \quad \theta_\xi^D|_{U_D^{\lfloor i/2 \rfloor + 1}} = \psi_\alpha^D|_{U_D^{\lfloor i/2 \rfloor + 1}}.$$

Set $J_\xi^1 = U_F^1 U_{\mathfrak{J}}^{\lfloor (i+1)/2 \rfloor}$, $J_\xi^{1,D} = U_F^1 U_D^{\lfloor (i+1)/2 \rfloor}$, $J_\xi = F^\times J_\xi^1$ and $J_\xi^D = F^\times J_\xi^{1,D}$. To construct an irreducible smooth representation Λ_ξ (resp. Λ_ξ^D) of J_ξ (resp. of J_ξ^D), we use the following lemma.

Lemma 4.6. *Let $\theta = \theta_\xi$ or $\theta = \theta_\xi^D$. Accordingly, set $H^1 = H_\xi^1$ (resp. $H_\xi^{1,D}$), $J^1 = J_\xi^1$ (resp. $J_\xi^{1,D}$) and $J = J_\xi$ (resp. J_ξ^D).*

(1) *The conjugation by $F^\times$ stabilizes θ. Thus, the cyclic group $\Gamma = F^\times/K^\times U_F^1$ acts on the finite p-group $\mathcal{Q} = J^1/\mathrm{Ker}\,\theta$. The center $\mathcal{Z}$ of $\mathcal{Q}$ is the cyclic group $\mathcal{Z} = H^1/\mathrm{Ker}\,\theta$, which is also the Γ-fixed part $\mathcal{Z} = \mathcal{Q}^\Gamma$.*

(2) *There exists a unique irreducible smooth representation η of $\mathcal{Q}$ whose central character is θ.*

(3) *There exists a unique irreducible smooth representation $\tilde\eta$ of $\Gamma \ltimes \mathcal{Q}$ such that $\tilde\eta|_{\mathcal{Q}} \simeq \eta$ and $\det \tilde\eta|_\Gamma = 1$.*

(4) *There exists a constant $\epsilon \in \{\pm 1\}$ such that $\mathrm{tr}\,\tilde\eta(\gamma u) = \epsilon \xi(u)$ for any generator $\gamma \in \Gamma$ and $u \in U_F^1$.*

(5) *There exists a unique irreducible smooth representation Λ of J such that $\Lambda|_{J^1} \simeq \eta$ and*

$$\mathrm{tr}\,\Lambda(h) = \epsilon \xi(h) \tag{4.3}$$

for any $h \in F^\times$ whose image in Γ is a generator.

Proof. This follows from [BH05a, (4.1.4) and Lemma 4.1] and [BH11, 5.2 Lemma 1], where the construction of Λ using $\tilde\eta$ is given.

Note that the statements are trivial if i is odd, in which case $H^1 = J^1$. In fact, then Λ is one-dimensional, $\Lambda|_{F^\times} = \xi$ and $\epsilon = 1$. Note also that the existence of η (if i is even) is a consequence of Proposition 3.8 and that $\Lambda|_{U_F}$ is a sum of ξ by (4.3). $\qquad\square$

According to whether $\theta = \theta_\xi$ or $\theta = \theta_\xi^D$, we denote the sign ϵ appearing in the proposition by ϵ_ξ (resp. ϵ_ξ^D) and similarly denote the representation Λ by Λ_ξ (resp. Λ_ξ^D). We set

$$\pi_\xi = \text{c-Ind}_{J_\xi}^{\text{GL}_n(K)} \Lambda_\xi, \quad \pi_\xi^D = \text{Ind}_{J_\xi^D}^{D^\times} \Lambda_\xi^D.$$

Finally, if ξ is not trivial on U_F^{i+1}, we take a decomposition of ξ as in (4.1) and put

$$\pi_\xi = \chi\pi_\varphi, \quad \pi_\xi^D = \chi\pi_\varphi^D.$$

The representation π_ξ (resp. π_ξ^D) is irreducible and supercuspidal (resp. irreducible). The isomorphism classes of π_ξ and π_ξ^D only depend on the K-isomorphism class of $(F/K, \xi)$.

We need an explicit description of the signs ϵ_ξ and ϵ_ξ^D.

Proposition 4.7. *Suppose that i is even. In the situation of Lemma 4.6 the sign ϵ equals the Jacobi symbol*

$$\epsilon = \left(\frac{q}{n}\right).$$

Proof. By [BF83, (8.6.1)], the sign ϵ is determined by the symplectic representation $(\mathcal{Q}/\mathcal{Z}, h_\theta)$ of Γ induced by $h_\theta \colon (x, y) \mapsto \theta(xyx^{-1}y^{-1})$ and hence by the $k[\Gamma]$-module $\mathcal{Q}/\mathcal{Z}$ (cf. Proposition 3.6). Also it is multiplicative with respect to orthogonal sums of symplectic representations. We have $\mathcal{Q}/\mathcal{Z} \simeq I_k(\Gamma; 1, n) \simeq I_{\mathbb{F}_p}(\Gamma; 1, n) \otimes_{\mathbb{F}_p} k$, no matter whether $\theta = \theta_\xi$ or $\theta = \theta_\xi^D$. Therefore, the assertion is reduced to $k = \mathbb{F}_p$ case, which is treated in [BF83, (9.3.5)] $\qquad\qquad\square$

Remark 4.8. *Let us temporarily return to the situation of Theorem 2.15. There $L = K(\varphi_L)$ is a totally tamely ramified extension of K of degree n and it is considered as a K-subalgebra of $M_n(K)$ (resp. D) via a fixed embedding $\varphi_L \mapsto \varphi$ (resp. $\varphi_L \mapsto \varphi_D$) arising from the fixed CM point. It is easily seen that $L^\times$ does normalize the standard Iwahori order, which is denoted there again by $\mathfrak{I} \subset M_n(K)$. We apply the preceding constructions with respect to this field, these embeddings and the order. Note also the equalities $L^\times U_{\mathfrak{I}}^{(\nu)} = L^\times U_{\mathfrak{I}}^{\lfloor(\nu+1)/2\rfloor}$ and $L^\times \cap U_{\mathfrak{I}}^{(\nu)} = U_L^\nu$, and the analogous equalities for $U_D^{(\nu)}$.*

Description of $_K\mu_\xi$ **and** $_D\iota_\xi$ **in special cases** Let us first define some invariants attached to F/K, ψ and ξ. We set

$$R_{F/K} = \mathrm{Ind}_{F/K}\, 1_K, \quad \delta_{F/K} = \det R_{F/K},$$

where 1_K denotes the trivial representation of W_K. We define the Langlands constant $\lambda_{F/K}(\psi)$ by

$$\lambda_{F/K}(\psi) = \frac{\varepsilon(R_{F/K}, 1/2, \psi)}{\varepsilon(1_F, 1/2, \psi \circ \mathrm{Tr}_{F/K})},$$

where the denominator and the numerator denote the Langlands-Deligne local constants (see [BH06, Section 30] for these two constants).

Take $\alpha = \alpha(\xi) \in F$ as in the construction of π_ξ, so that $v_F(\alpha) = -i$ and $\varphi|_{U_F^{\lfloor i/2 \rfloor + 1}} = \psi_\alpha^F$ where φ is as in the decomposition (4.1). Note that $\alpha(\xi)U_F^1$ only depends on ξ by (4.2). For any uniformizer $\varpi_F \in F$, we define $\zeta(\varpi_F, \xi) \in F$ as the unique root of unity satisfying

$$\zeta(\varpi_F, \xi) \equiv \varpi_F^i \alpha(\xi) \pmod{U_F^1}.$$

In the case at hand, [BH05b, Theorem 2.1] reads as follows.

Theorem 4.9. *Let $(F/K, \xi)$ be an admissible pair as above, i.e. it is minimal with the jump at i and F/K is totally ramified. Then the image of $\mathrm{Ind}_{F/K}\, \xi$ under the local Langlands correspondence is $\pi_{_K\mu_\xi \xi}$, where $_K\mu_\xi$ is a character of $F^\times$ defined below.*

(1) If n is odd, then $_K\mu_\xi$ is unramified and, for any uniformizer $\varpi_F \in F$,

$$_K\mu_\xi(\varpi_F) = \lambda_{F/K}(\psi).$$

(2) If n is even, then $_K\mu_\xi$ is determined by the following conditions

$$_K\mu_\xi|_{U_F^1} = 1, \quad _K\mu_\xi|_{K^\times} = \delta_{F/K},$$

$$_K\mu_\xi(\varpi_F) = \left(\frac{\zeta(\varpi_F, \xi)}{q}\right)\left(\frac{-1}{q}\right)^{(i-1)/2} \lambda_{F/K}(\psi)$$

for any uniformizer $\varpi_F \in F$.

Remark 4.10. *In both cases, $_K\mu_\xi$ does not depend on the choice of ψ (see [BH05b, Remark 2.1.3]).*

Similarly, we need the following special case of [BH11, Theorem 5.3].

Theorem 4.11. *Let $(F/K, \xi)$ be as above. Then the image of π_ξ^D under the local Jacquet-Langlands correspondence is $\pi_{D \iota_\xi \xi}$, where $\iota = {}_D\iota_\xi$ is the unramified character of $F^\times$ sending uniformizers to $(-1)^{n-1}$.*

We record here some of the explicit values used later.

Proposition 4.12. *Let the notation be as above.*

(1) Suppose that n is odd. Then $\lambda_{F/K}(\psi) = \left(\frac{q}{n}\right)$.

(2) Suppose that n is even. Then $\delta_{F/K}(u) = \left(\frac{\overline{u}}{q}\right)$ for $u \in U_K$.

Proof. The assertion (1) (resp. (2)) is part of [BH05b, Lemma 1.5(2)] (resp. part of [IT15a, Lemma 5.3]). □

4.2 Realization of correspondences

[16] Let ψ be the additive character of K fixed in the previous subsection and denote by $\overline{\psi}$ the non-trivial additive character of k obtained as the reduction of $\psi|_{\mathcal{O}_K}$. We also denote by $\overline{\psi}_\zeta$ the character $\overline{\psi}_\zeta \colon k \to \overline{\mathbb{Q}}_\ell^\times$; $x \mapsto \overline{\psi}(\overline{\zeta}x)$ for any $\zeta \in \mu_{q-1}(K)$.

Let $\nu > 0$ be an integer and assume that it is coprime to n. We return to our analysis of the cohomology in Section 3. Put $H_\nu = H_c^{n-1}(Z_\nu, \overline{\mathbb{Q}}_\ell)((1-n)/2)$ and $\Pi_\nu = H_c^{n-1}(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell)((1-n)/2)$. We denote by $H_{\nu,\zeta}$ the $\overline{\psi}_\zeta$-isotypic component of H_ν and set

$$\Pi_{\nu,\zeta} = \bigoplus_{\chi \in \left(U_K^{[\nu/n]}\right)^\vee} H_{\nu,\zeta} \otimes (\chi \circ N_G).$$

Then we have $H_\nu = \bigoplus_{\zeta \in \mu_{q-1}(K)} H_{\nu,\zeta}$ and $\Pi_\nu = \bigoplus_{\zeta \in \mu_{q-1}(K)} \Pi_{\nu,\zeta}$ by Theorem 2.10, Proposition 3.10 (2), Proposition 3.11 (2) and Proposition 3.12.

Lemma 4.13. *Let π be an irreducible smooth representation of $GL_n(K)$. Set $G_1 = GL_n(K)$ and $G_2 = D^\times \times W_K$. Denote by $\overline{\mathrm{Stab}}_\nu \subset G_2$ the image of Stab_ν under the projection $G \to G_2$. Then we have a canonical isomorphism*

$$\mathrm{Hom}_{G_1}\left(\text{c-Ind}_{\mathrm{Stab}_\nu}^G \Pi_\nu, \pi\right) \simeq \mathrm{Ind}_{\overline{\mathrm{Stab}}_\nu}^{G_2} \mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\Pi_\nu, \pi)$$

[16] The reasoning in this subsection is motivated by that in [IT15a].

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of representations of G_2, where the action of $(d, \sigma) \in \overline{\mathrm{Stab}}_\nu$ on $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\Pi_\nu, \pi)$ is given by the composition of the action of $(g, d, \sigma)^{-1} \in \mathrm{Stab}_\nu$ on the source and that of $g \in G_1$ on the target for some lift $(g, d, \sigma) \in \mathrm{Stab}_\nu$ of $(d, \sigma) \in \overline{\mathrm{Stab}}_\nu$.

Proof. This is straightforward; one only needs to check the action of G_2 on the right-hand side of the following isomorphism

$$\mathrm{Hom}_{G_1}\left(\text{c-Ind}_{\mathrm{Stab}_\nu}^{G}\Pi_\nu, \pi\right) \simeq \bigoplus_{\overline{\mathrm{Stab}}_\nu \backslash G_2} \mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\Pi_\nu, \pi)$$

induced by the Mackey decomposition

$$(\text{c-Ind}_{\mathrm{Stab}_\nu}^{G}\Pi_\nu)|_{G_1} \simeq \bigoplus_{\mathrm{Stab}_\nu\, gG_1 \in \mathrm{Stab}_\nu \backslash G / G_1} \text{c-Ind}_{\mathrm{Stab}_\nu^g \cap G_1}^{G_1}\Pi_\nu^g$$

$$\simeq \bigoplus_{\overline{\mathrm{Stab}}_\nu \backslash G_2} \text{c-Ind}_{\mathrm{Stab}_\nu \cap G_1}^{G_1}\Pi_\nu$$

and the Frobenius reciprocity. $\square$

Proposition 4.14. *Let (π, V) be an irreducible smooth representation of $GL_n(K)$. Let $\zeta \in \mu_{q-1}(K)$ be a $(q-1)$-st root of unity. We have*

$$\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu, \zeta}, \pi) \neq 0$$

if and only if π is an essentially tame (supercuspidal) representation parametrized by a minimal admissible pair $(L/K, \xi)$ such that $\xi|_{U_L^\nu} = \psi^L_{\zeta \varphi_L^{-\nu}}$. Moreover, if this space is non-zero, then we have

$$\dim \mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu, \zeta}, \pi) = \begin{cases} 1 & \text{if } \nu \text{ is odd} \\ q^{(n-1)/2} & \text{if } \nu \text{ is even.} \end{cases}$$

Proof. Define irreducible representations ρ_1 and ρ_2 of $S_{1,\nu}$ and $S_{2,\nu}$ by expressing $H_{\nu, \zeta} \simeq \rho_1 \boxtimes \rho_2$ as a representation of $S_{1,\nu} \times S_{2,\nu}$, so that

$$\dim \rho_1 = \dim \rho_2 = \begin{cases} 1 & \text{if } \nu \text{ is odd} \\ q^{(n-1)/2} & \text{if } \nu \text{ is even.} \end{cases}$$

Then we need to determine the condition for ρ_1 to occur in π and prove that the multiplicity is (at most) one.

Suppose ρ_1 occurs in π. Note first that $L^\times$ normalizes C_1 (appearing in the definition of $U_{\mathfrak{J}}^{(\nu)}$) and the central character of ρ_1, which in turn implies that it normalizes $(U_{\mathfrak{J}}^{(\nu)}, \rho_1)$. Thus $L^\times U_{\mathfrak{J}}^{(\nu)} = L^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}$ acts on the ρ_1-isotypic part V^{ρ_1} of π. Take an irreducible subrepresentation $\Lambda \subset V^{\rho_1}$ The restriction $\Lambda|_{U_{\mathfrak{J}}^\nu}$ is clearly a sum of characters $\psi_{\zeta\varphi_L^{-\nu}}$. Now the discussions in [Car84, 5.6, 5.7], which treat more general cases, yield in this case the classification of irreducible representations Ξ of $K(\zeta\varphi_L^{-\nu})^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor} = L^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}$ containing $\psi_{\zeta\varphi_L^{-\nu}}$ when restricted to $U_{\mathfrak{J}}^\nu$. In particular,

- $\dim \Xi = \dim \rho_1$.

- $\Xi|_{K \times U_L^1}$ is a sum of characters.

- An irreducible representation of $K^\times U_L^1 U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}$ containing $\psi_{\zeta\varphi_L^{-\nu}}|_{U_{\mathfrak{J}}^\nu}$ admits exactly n extensions to $L^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}$.

It can be readily verified that $\Lambda \simeq \Lambda_\xi$ for some ξ as in the statement. Hence we obtain a homomorphism $\pi_\xi = \text{c-Ind}\,\Lambda_\xi \to \pi$, which is an isomorphism by the irreducibility of π. The converse being easy, we deduce the desired condition for the occurrence of ρ.

Since $\dim \Lambda_\xi = \dim \rho$ as above, the claim about the multiplicity is reduced to certain multiplicity one statement in the theory of types. We can argue as follows. Take once again a subrepresentation Λ' in V^{ρ_1}. By the above argument we have $\Lambda' \simeq \Lambda_{\xi'}$ for some ξ' and $\pi_\xi \simeq \pi_{\xi'}$. Then $\xi^\sigma = \xi'$ for some $\sigma \in \text{Aut}(L/K)$ by the injectivity of the parametrization, which implies $\zeta\varphi_L^{-\nu} = \zeta(\varphi_L^{-\nu})^\sigma$ and hence $\sigma = \text{id}$. This shows that V^{ρ_1} is Λ_ξ-isotypic. Therefore,

$$\dim \text{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\rho_1, \pi) = \dim \text{Hom}_{L^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}}(\Lambda_\xi, \pi) = \dim \text{End}_{\text{GL}_n(K)}(\pi) = 1$$

as desired (see also [BH06, 15.7 Proposition (3)]). □

Proposition 4.15. *Let π be as above. Suppose that $\text{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_\nu, \pi) \neq 0$, so that $\pi \simeq \pi_\xi$ for some minimal admissible pair $(L/K, \xi)$ by Proposition 4.14. Then this space contains*
$$\Lambda_{\iota\xi}^D \boxtimes {}_K\mu_\xi^{-1}\xi$$
as a representation of $L^\times U_D^{\lfloor(\nu+1)/2\rfloor} \times W_L \subset \overline{\text{Stab}_\nu}$.

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Proof. By Proposition 4.14 we have $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\pi)\neq 0$ for some $\zeta\in$ $\mu_{q-1}(K)$ such that $\xi|_{U_L^\nu}=\psi^L_{\zeta\varphi_L^{-\nu}}$. We claim that this subspace $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\pi)\subset$ $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_\nu,\pi)$ is isomorphic to the representation $\Lambda^D_{\iota\xi}\boxtimes{}_K\mu_\xi^{-1}\xi$ appearing in the assertion.

First it is indeed stable under the action of $L^\times U_D^{\lfloor(\nu+1)/2\rfloor}\times W_L\subset\overline{\mathrm{Stab}_\nu}$ because the action on the source of $U_D^{(\nu)}\times W_L$ clearly commutes with that of $U_{\mathfrak{J}}^{(\nu)}$ and $L^\times\subset\mathrm{GL}_n(K)$ normalizes $U_{\mathfrak{J}}^{(\nu)}$.

By the proof of Proposition 4.14, $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\pi_\xi)=\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\Lambda_\xi)$ is isomorphic to $\rho_2^\vee$ (inflated via $U_D^{(\nu)}\to S_{2,\nu}$) as a representation of $U_D^{(\nu)}$. Therefore we may set $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\pi)=\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(H_{\nu,\zeta},\Lambda_\xi)=\Lambda'\boxtimes\xi'$ with some irreducible smooth representation Λ' of $L^\times U_D^{\lfloor(\nu+1)/2\rfloor}$ whose restriction to $U_D^{(\nu)}$ is isomorphic to $\rho_2^\vee$ and some smooth character ξ' of W_L.

In the first part of the argument to follow, we do not divide cases; we make no assumption on the parity of ν. Let us first show $\Lambda'\simeq\Lambda^D_{\iota\xi}$. The action of $x\in L^\times\subset L^\times U_D^{\lfloor(\nu+1)/2\rfloor}$ is given by the composition of the action of $(x,x,1)^{-1}\in\mathrm{Stab}_\nu$ on the source and that of $x\in L^\times U_{\mathfrak{J}}^{\lfloor(\nu+1)/2\rfloor}$ on the target. As $(x,x,1)\in\mathrm{Stab}_\nu$ $(x\in U_L)$ acts trivially by Theorem 2.15 (4) and $\Lambda_\xi|_{U_L}$ is a sum of ξ by Lemma 4.6, we find that U_L acts via the character ξ on Λ', as desired. Thus, to conclude $\Lambda'\simeq\Lambda^D_{\iota\xi}$, we need to show

$$\mathrm{tr}\,\Lambda'(\varphi_D^j)=\epsilon^D_{\iota\xi}(\iota\xi)(\varphi_L^j)\qquad(4.4)$$

for any j coprime to n by (4.3). Put $Q=U_{\mathfrak{J}}^{(\nu)}/U_{\mathfrak{J}}^{(\nu+1)}$. By a standard argument we find that

$$\mathrm{tr}\,\Lambda'(\varphi_D^j)=|Q|^{-1}\sum_{x\in Q}\mathrm{tr}\left(\left(x\varphi^j,\varphi_D^j,1\right)^{-1}\Big|H_{\nu,\zeta}\right)\mathrm{tr}\,\Lambda_\xi(x\varphi^j).$$

To further compute, we quote a result from the representation theory of finite groups[17]. Regard $\overline{\psi}_\zeta$ as a character of the center Z of Q and set $Q'=Q/\mathrm{Ker}\,\overline{\psi}_\zeta$, $Z'=Z/\mathrm{Ker}\,\overline{\psi}_\zeta$. Then $H_{\nu,\zeta}$ (resp. Λ_ξ) is inflated from a representation $H'_{\nu,\zeta}$ (resp. Λ'_ξ) of Q'. By [BH99, Lemma A1.3] and a counting argument we infer that for every $x\in Q'$ there exist unique elements

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[17]The quoted result is trivial if ν is odd and hence $Z=Q$.

$y \in Q'/Z'$ and $z \in Z'$ such that $x\varphi^j = yz\varphi^j y^{-1}$. Hence

$$\operatorname{tr} \Lambda'(\varphi_D^j) = |Q|^{-1} |\operatorname{Ker} \overline{\psi}_\zeta| \sum_{(y,z) \in Q'/Z' \times Z'} \operatorname{tr} \left(\left(yz\varphi^j y^{-1}, \varphi_D^j, 1 \right)^{-1} \Big| H'_{\nu,\zeta} \right) \operatorname{tr} \Lambda'_\xi (yz\varphi^j y^{-1})$$

$$= |Q|^{-1} |\operatorname{Ker} \overline{\psi}_\zeta| |Q'/Z'||Z'| \operatorname{tr} \left(\left(\varphi^j, \varphi_D^j, 1 \right)^{-1} \Big| H_{\nu,\zeta} \right) \operatorname{tr} \Lambda_\xi (\varphi^j)$$

$$= (-1)^{n-1} \epsilon_\xi \xi(\varphi_L^j),$$

where we use Theorem 2.15 (3), Proposition 3.10 (4), Proposition 3.11 (3) and (4.3) in the last equality. Now the equality (4.4) follows since $\epsilon_{\iota\xi}^D = \epsilon_\xi$ for any ν by Proposition 4.7.

Let us prove $\xi' = {}_K \mu_\xi^{-1} \xi$ by checking

$$\operatorname{tr} \left((a_\sigma^{-1}, \sigma) \Big| \operatorname{Hom}_{U_{\mathfrak{J}}^{(\nu)}} (H_{\nu,\zeta}, \Lambda_\xi) \right) = \operatorname{tr} \left(\Lambda_{\iota\xi}^D \boxtimes {}_K \mu_\xi^{-1} \xi \right) (a_\sigma^{-1}, \sigma) \qquad (4.5)$$

for any $\sigma \in W_L$ with $n_\sigma = v(a_\sigma) = -1$.

Now we proceed by cases. First suppose that ν is even. Noting the twist and the multiplicity of Λ_ξ in $H_{\nu,\zeta}$, we see

$$\operatorname{tr} \left((a_\sigma^{-1}, \sigma) \Big| \operatorname{Hom}_{U_{\mathfrak{J}}^{(\nu)}} (H_{\nu,\zeta}, \Lambda_\xi) \right) = 1$$

by Theorem 2.15 (5) and Proposition 3.11 (2). By Theorems 4.9, 4.11 and (4.3), we have

$$\operatorname{tr} \left(\Lambda_{\iota\xi}^D \boxtimes {}_K \mu_\xi^{-1} \xi \right) (a_\sigma^{-1}, \sigma) = (-1)^{n-1} \epsilon_\xi \lambda_{L/K}(\psi).$$

As n is odd, the equality (4.5) follows from Proposition 4.7 and Proposition 4.12 (1).

Suppose next that ν is odd. Put $u_\sigma = a_\sigma \varphi \in U_L$ and $\mathfrak{m}(\overline{\psi}_\zeta) = q^{-1/2} \mathfrak{g}(\overline{\psi}_\zeta) = \left(\frac{\zeta}{q} \right) \mathfrak{m}(\overline{\psi})$. We have

$$\operatorname{tr} \left((a_\sigma^{-1}, \sigma) \Big| \operatorname{Hom}_{U_{\mathfrak{J}}^{(\nu)}} (H_{\nu,\zeta}, \Lambda_\xi) \right)$$

$$= \begin{cases} \left(\frac{q}{n} \right) & \text{if } p = 2 \text{ and } n \text{ is odd} \\ \left(\frac{n}{q} \right) \mathfrak{m}(\overline{\psi}_\zeta)^{n-1} & \text{if } p \neq 2 \text{ and } n \text{ is odd} \\ -\left(\frac{\overline{u}_\sigma}{q} \right)^{n-1} \left(\frac{-1}{q} \right)^{(\nu-1)/2} \left(\frac{2}{q} \right) \left(\frac{n}{q} \right) \mathfrak{m}(\overline{\psi}_\zeta)^{n-1} & \text{if } p \neq 2 \text{ and } n \text{ is even,} \end{cases}$$

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by Theorem 2.15 (5) and Proposition 3.10 (2), (4). On the other hand, we have

$$\mathrm{tr}\left(\Lambda_{\iota\xi}^{D} \boxtimes {}_{K}\mu_{\xi}^{-1}\xi\right)(a_{\sigma}^{-1},\sigma) = (-1)^{n-1}{}_{K}\mu_{\xi}^{-1}(a_{\sigma})$$

$$= \begin{cases} \lambda_{L/K}(\psi) & \text{if } n \text{ is odd} \\ -\delta_{L/K}(u_{\sigma}^{-1})\left(\frac{\varsigma}{q}\right)\left(\frac{-1}{q}\right)^{(\nu-1)/2}\lambda_{L/K}(\psi) & \text{if } n \text{ is even,} \end{cases}.$$

by Theorems 4.9, 4.11 and Lemma 4.6. Now the equality (4.5) follows from Proposition 4.12 (1) if $p = 2$, and from Proposition 4.12 (2) and the equalities

$$\lambda_{L/K}(\psi) = \begin{cases} \left(\frac{n}{q}\right)\mathfrak{m}(\overline{\psi})^{n-1} & \text{if } n \text{ is odd} \\ \left(\frac{2}{q}\right)\left(\frac{n}{q}\right)\mathfrak{m}(\overline{\psi})^{n-1} & \text{if } n \text{ is even,} \end{cases}$$

which appear in [IT15a, (5.22)], if $p \neq 2$. □

Let $\mathrm{LJ}(\pi)$ (resp. $\mathrm{LL}(\pi)$) be the image of π under the local Jacquet-Langlands correspondence (resp. the local Langlands correspondence).

Theorem 4.16. *Let π be as above. We have*

$$\mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\text{c-Ind}_{\mathrm{Stab}_\nu}^{G}\Pi_\nu, \pi\right) \neq 0$$

if and only if π is parametrized by a minimal admissible pair $(L/K, \xi)$ with the jump at ν. Moreover, if non-zero, this space is isomorphic to $\mathrm{LJ}(\pi) \boxtimes \mathrm{LL}(\pi)$ as a representation of $D^\times \times W_K$.

Proof. The first assertion follows immediately from Lemma 4.13, Proposition 3.12, Proposition 4.14 and the Frobenius reciprocity.

To prove the second assertion let $\pi \simeq \pi_\xi$ occur in c-$\mathrm{Ind}_{\mathrm{Stab}_\nu}^{G}\Pi_\nu$. By Theorems 4.9, 4.11 it suffices to show that the following morphism, induced by the Frobenius reciprocity, is an isomorphism;

$$\mathrm{Ind}_{L^\times U_D^{\lfloor(\nu+1)/2\rfloor} \times W_L}^{\overline{\mathrm{Stab}}_\nu}\left(\Lambda_{\iota\xi}^{D} \boxtimes {}_{K}\mu_{\xi}^{-1}\xi\right) \rightarrow \mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\Pi_\nu, \pi).$$

Since the source is irreducible, we only need to show the equality of the dimensions. We have

$$\dim \mathrm{Ind}_{L^\times U_D^{\lfloor(\nu+1)/2\rfloor} \times W_L}^{\overline{\mathrm{Stab}}_\nu}\left(\Lambda_{\iota\xi}^{D} \boxtimes {}_{K}\mu_{\xi}^{-1}\xi\right) = \dim \Lambda_{\iota\xi}^{D} \cdot [\overline{\mathrm{Stab}}_\nu : L^\times U_D^{\lfloor(\nu+1)/2\rfloor} \times W_L]$$

$$= \dim \Lambda_{\iota\xi}^{D} \cdot [W_{L'} : W_L].$$

By Proposition 4.14 we are reduced to showing that $[W_{L'} : W_L]$ equals the number of $\zeta \in \mu_{q-1}(K)$ such that $\mathrm{Hom}_{U_{\mathfrak{J}}^{(\nu)}}(\Pi_{\nu,\zeta}, \pi) \neq 0$. This can be done readily by the injectivity of the paramatrization of essentially tame representations. $\qquad\square$

Now recalling that a totally ramified extension L/K is arbitrarily given after the Lemma 2.3, we obtain the main theorem described in Introduction.

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