博士論文（要約）

論文題目　Affinoids in the Lubin-Tate perfectoid space and special cases of the local Langlands correspondence in positive characteristic

（Lubin-Tate パーフェクトイド空間のアフィノイドと正標数の局所 Langlands 対応の特別な場合について）

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Affinoids in the Lubin-Tate perfectoid space and special cases of the local Langlands correspondence in positive characteristic

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Abstract

Following Weinstein, Boyarchenko-Weinstein and Imai-Tsushima, we construct a family of affinoids in the Lubin-Tate perfectoid space and their formal models such that the cohomology group of the reduction of each formal model realizes the local Langlands correspondence and local Jacquet-Langlands correspondence for certain representations. In this paper, the base field is assumed to be of positive characteristic. In the terminology of the essentially tame local Langlands correspondence, the representations treated here are characterized as being parametrized by minimal admissible pairs involving totally ramified extensions of the base field.

1 Background

Let K be a non-archimedean local field with residue field k of characteristic $p > 0$. Let W_K be the Weil group of K and D the central division algebra over K of invariant $1/n$. Denote by $\mathcal{O}_K \subset K$ the valuation ring and by $\mathfrak{p} \subset \mathcal{O}_K$ the maximal ideal. We fix an algebraic closure $\overline{k}$ of k. Let $n \geq 1$ be an integer. Then the Lubin-Tate spaces are defined to be deformation spaces of a one-dimensional formal $\mathcal{O}_K$-module over $\overline{k}$ of height n with level structures. The Lubin-Tate spaces naturally form a projective system, called the Lubin-Tate tower, and the non-abelian Lubin-Tate theory asserts that the cohomology group of the Lubin-Tate tower, which admits a natural action of a large subgroup of $\mathrm{GL}_n(K) \times D^\times \times W_K$, realizes the local Langlands correspondence for $\mathrm{GL}_n(K)$ and the local Jacquet-Langlands correspondence simultaneously.

However, as the proofs of this fact ([Boye99], [HT01]) make heavy use of the theory of automorphic representations and the global geometry, the geometry of the Lubin-Tate spaces and its relation to representations are not yet fully understood.

Among the studies on the geometry of the Lubin-Tate spaces is a work [Yos10] of Yoshida. There he constructed a semistable model of the Lubin-Tate space of level $\mathfrak{p}$ and proved that an affine open subscheme of the reduction is isomorphic to a Deligne-Lusztig variety for $\mathrm{GL}_n(k)$. The appearance of the Deligne-Lusztig variety reflects the fact that some irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ can be constructed from irreducible cuspidal representations of $\mathrm{GL}_n(k)$. We note that this open subscheme can also be obtained as the reduction of an affinoid subspace in the Lubin-Tate space by considering its tube.

More recently, Weinstein showed in [Wei14] that a certain limit space of the Lubin-Tate tower makes sense as a perfectoid space. While it is no longer an ordinary finite-type analytic space, the Lubin-Tate perfectoid space has a simpler geometry; with coordinates not available on the individual Lubin-Tate spaces, the defining equation is simpler and the group actions can be made more explicit. Taking advantage of these properties, Weinstein[Wei14], Boyarchenko-Weinstein[BW16] and Imai-Tsushima[IT15a] constructed families of affinoid subspaces in the Lubin-Tate perfectoid space and their formal models such that the cohomology group of the reduction of each formal model realizes the local Langlands and Jacquet-Langlands correspondences for some representations. The aim of this paper is to establish the existence of such a family of affinoids related to certain other representations, under a simplifying assumption that K is of characteristic $p > 0$.

2 Main result

Let $\ell \neq p$ be a prime number. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Set $G = \mathrm{GL}_n(K) \times D^\times \times W_K$. Here is our main theorem:

Theorem. *Suppose that K is of equal-characteristic and that p does not divide n. Let $\nu > 0$ be an integer which is coprime to n. Let L/K be a totally ramified extension of degree n. Then there exist an affinoid $\mathcal{Z}_\nu$ in the Lubin-Tate perfectoid space and a formal model $\mathscr{Z}_\nu$ of $\mathcal{Z}_\nu$ such that the following hold.*

(1) The stabilizer Stab_ν *of* $\mathcal{Z}_\nu$ *naturally acts on* $\mathscr{X}_\nu$ *and hence on the reduction* $\overline{\mathscr{X}}_\nu$.

(2) For an irreducible smooth representation π *of* $\mathrm{GL}_n(K)$*, we have*

$$\mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\text{c-Ind}^G_{\mathrm{Stab}_\nu} H_c^{n-1}\left(\overline{\mathscr{X}}_\nu, \overline{\mathbb{Q}}_\ell\right)((1-n)/2), \pi\right) \neq 0$$

if and only if the image τ *of* π *under the local Langlands correspondence is a character twist of an* n*-dimensional irreducible smooth representation of the form* $\mathrm{Ind}_{L/K}\,\xi$ *for a character* ξ *of* $L^\times$ *which is non-trivial on* U_L^ν*, but trivial on* $U_L^{\nu+1}$*. Moreover, if the above space is non-zero, it is isomorphic to* $\rho \boxtimes \tau$ *as a representation of* $D^\times \times W_K$*, where* ρ *is the image of* π *under the local Jacquet-Langlands correspondence.*

Here, ξ is identified with a character of the Weil group W_L of L via the Artin reciprocity map and $\mathrm{Ind}_{L/K}$ denotes the smooth induction from W_L to W_K.

Let us compare Theorem with the preceding results. The affinoid $\mathcal{Z}_1$ and the formal model $\mathscr{X}_1$ in Theorem are essentially identical to those constructed in [IT15a]. Also, in [Wei14], the affinoids and the formal models in Theorem are constructed when $n = 2$ and $p \neq 2$, along with those related to the unramified case in a suitable sense. Thus, Theorem generalizes [IT15a][1] and partially [Wei14], in the equal-characteristic setting. In the terminology of Definition 4.4, which is essentially taken from [BH05b], the above condition for π to occur in the compact induction is equivalent to being parametrized by a minimal admissible pair $(L/K, \eta)$ with the jump at ν. Let F/K be an unramified extension of degree n. The affinoids and the formal models constructed in [BW16] are related, in the same way as in Theorem, to irreducible supercuspidal representations π parametrized by minimal admissible pairs $(F/K, \eta)$ with the jump at some ν. The author learned from Imai and Tsushima that they had previously constructed what should be $\mathcal{Z}_2$ and $\mathscr{X}_2$ in our notation, computed the reduction and verified the non-triviality of the cohomology group in the degree $n-1$. Although this unannounced result preceded ours, our result was obtained independently. On a related note, in a recent article [IT15b], the corresponding affinoids in the Lubin-Tate space of level $\mathfrak{p}^2$ are studied when K is of equal-characteristic, $p \geq 5$ and $n = 3$.

[1] However, Imai and Tsushima announced that they also obtained a corresponding result for n divisible by p.

We note some properties of the affinoids $\mathcal{Z}_\nu$ and the reductions $\overline{\mathscr{X}}_\nu$. While only those with ν coprime to n are relevant to Theorem, the affinoids $\mathcal{Z}_\nu$ and the formal models $\mathscr{X}_\nu$ are constructed for any $\nu > 0$ in a certain uniform way. The reductions $\overline{\mathscr{X}}_\nu$ are related to the perfections of algebraic varieties Z_ν, which turn out to be periodic in ν with period $2n$. They are quite different, according to whether ν is odd or even. If ν is odd, Z_ν is the variety obtained by pulling back the Artin-Schreier covering $\mathbb{A}^1_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$ (that is, the base change of the Lang torsor of the additive group scheme $\mathbb{G}_a = \mathbb{A}^1_k$) by a morphism $\mathbb{A}^{n-1}_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$ corresponding to a quadratic form depending on ν. If ν is even, the defining equation of Z_ν is more involved. However, it can be described in terms of the Lang torsor of a more general algebraic group $\mathcal{G}_\nu$ over k and a morphism related to a quadratic form. Here we imitated a similar description found in [BW16], but the analogy is not so straightforward; the relevant algebraic groups are not the same and no quadratic forms occur there.

3 Organization of the paper

In Section 1, we review some basic facts on the Lubin-Tate perfectoid space and a formal model, following [Wei14], [BW16] and [IT15a]. In particular, a power series δ is defined which essentially serves as the defining equation of the formal model of the Lubin-Tate perfectoid space. Also the actions of the relevant groups on the formal model are described explicitly.

In Section 2, a family of affinoids and formal models is constructed, and the reductions are studied along with the induced actions of the stabilizers. Building on the notion of CM points and related facts found in [BW16], [IT15a], which are recalled in Subsection 2.1, we construct affinoids $\mathcal{Z}_\nu$ in Subsection 2.2. The construction of formal models $\mathscr{X}_\nu$ and the computation of the reductions $\overline{\mathscr{X}}_\nu$ given in Subsection 2.4 are based on the behavior of the power series δ under a certain change of coordinates, which is the subject of Subsection 2.3. While motivated by that in [Wei14], our argument is more intricate. Thus we give a rather detailed account. In Subsection 2.5 we compute the stabilizers Stab_ν of the affinoids $\mathcal{Z}_\nu$ and the induced actions on the reductions $\overline{\mathscr{X}}_\nu$. The algebraic groups $\mathcal{G}_\nu$ appearing in the alternative description of Z_ν given in Subsection 2.6 are modeled on the actions of $\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)$.

In Section 3 we compute the cohomology of $\overline{\mathscr{X}}_\nu$ together with the relevant group actions. This is reduced to the corresponding computation for Z_ν and is treated separately according to whether ν is odd or even. If ν is odd, we compute the cohomology groups for any ν. In particular, it turns out that the cohomology group in the degree $n-1$ is non-trivial if and only if ν is coprime to n. If ν is even, our understanding is not as complete and we restrict to the cases where ν is coprime to n; this suffices for the proof of the main theorem. Subsections 3.1, 3.2 (resp. Subsection 3.3) contain key ingredients for the computations for the odd (resp. even) cases.

In Section 4 we prove the main theorem described above. To this end, we apply the theory developed in [BH05a], [BH05b], [BH10], [BH11], which explicitly describes the local Langlands and Jacquet-Langlands correspondences for essentially tame representations. Subsection 4.1 provides a review of this theory in the special cases that we need. In Subsection 4.2 we combine the theory of Bushnell-Henniart and the results obtained in the previous sections to finally deduce the main theorem.

4 Remarks

(1) Although the equal-characteristic assumption is in force throughout the paper, it plays only a minor role in Sections 3 and 4. It seems reasonable to expect very similar varieties to appear as the reductions of suitable affinoids also in the mixed-characteristic setting. On the other hand, our computation of the reductions and the stabilizers in Section 2 heavily relies on this assumption, especially on the particularly simple expression of δ and the group actions. We hope to consider the problem of extending our results to the mixed-characteristic setting in a future work.

(2) We do not discuss the relation between the cohomology of each reduction $\overline{\mathscr{X}}_\nu$ and that of the Lubin-Tate tower. Thus we cannot directly deduce any consequences about the non-abelian Lubin-Tate theory from our result. However, we believe that the result is interesting nonetheless and that the varieties and the representations occurring in this paper, or some variants of them, should appear in the corresponding study on the geometry of the Lubin-Tate spaces.

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