博士論文

論文題目　On codimension two contact embeddings in the standard spheres

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ON CODIMENSION TWO CONTACT EMBEDDINGS IN THE STANDARD SPHERES

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Contents

1. Introduction

A *contact structure* on a $(2n+1)$-dimensional manifold M is a maximally nonintegrable hyperplane field ξ. Locally ξ can be written as $\xi = \ker\alpha$ by a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. If a contact structure ξ is cooriented, we can take α as a global 1-form on M. When M is oriented and $\xi = \ker\alpha$ is cooriented, we say that ξ is *positive* (resp. *negative*) if $\alpha \wedge (d\alpha)^n$ is a positive (resp. negative) volume form on M with respect to the given orientation. An embedding $j\colon M \to N$ between two contact manifolds (M, ξ) and (N, η) is called a *contact embedding* if $Tj(M) \cap \eta|_{j(M)} = dj(\xi)$.

The contact embedding problem of closed contact manifolds of codimension greater than two reduces to homotopy theory by Gromov's h-principle [25] and it is shown that any closed cooriented contact $(2n+1)$-manifold embeds in $(S^{4n+3}, \xi_{\mathrm{std}})$. To construct contact embeddings Ibort, Martínez-Torres and Presas [31] developed the approximately holomorphic methods based on Donaldson's work for symplectic manifolds [9] and Martínez-Torres [35] (and Mori [40] in the case of $n = 1$) also proved that any closed cooriented contact $(2n+1)$-manifold embeds in $(S^{4n+3}, \xi_{\mathrm{std}})$ and the embeddings are compatible with supporting open books.

In this thesis, we consider codimension two contact embeddings of closed contact $(2n+1)$-manifolds in contact $(2n+3)$-manifolds for $n \geq 1$, especially in $(S^{2n+3}, \xi_{\mathrm{std}})$ or in a $(2n+3)$-dimensional Darboux chart. Natural examples of codimension two contact embeddings in the standard spheres are the links of isolated hypersurface singularities [37]. Though there are several other examples, it is not yet well studied whether a given contact $(2n+1)$-manifold can embed in $(S^{2n+3}, \xi_{\mathrm{std}})$.

To construct codimension two contact embeddings in higher dimensions, we focus on *braids* and *braided embeddings* which is a natural generalization in higher dimensions of closed braids in S^3. An m-dimensional manifold M is a *braid* about Y if there is an embedding of $j\colon M \to Y \times D^2$ such that $\mathrm{pr}_1 \circ j$ is a branched covering, where $\mathrm{pr}_1\colon Y \times D^2 \to Y$ is the projection to the first factor. We say that an embedding j of an m-dimensional manifold M in an $(m+2)$-dimensional manifold N is a *braided embedding* if there is an m-dimensional submanifold Y of N with trivial normal bundle and j makes M a braid about Y. A classical 1-dimensional closed braid is naturally a braid about the trivial knot in this sense and thus this definition of braided embeddings seem to be a natural generalization of that of classical closed braids. 2-dimensional braids are well studied. The study on 3-dimensional braids has just begun for example in [3], [4], [5]. Though topological properties of higher dimensional braids are not yet well understood, they are useful to construct contact embeddings.

1.1. **Existence of embeddings of closed contact 3-manifolds in the standard 5-sphere.** Let us consider contact embeddings of closed positive contact 3-manifolds in $(S^5, \xi_{\rm std})$. By Hirsch [29], any closed oriented 3-manifold can embed in S^5 and thus there is no obstruction for smooth embeddings. For codimension two contact embeddings, however, there is one known obstruction other than obstructions for smooth embeddings in a general setting.

Theorem 1.1 (Kasuya [33])**.** *If a closed cooriented contact manifold* (M^{2n+1}, ξ) *embeds in a cooriented contact manifold* (N^{2n+3}, η) *with* $H^2(N^{2n+3}; \mathbf{Z}) = 0$*, then the first Chern class* $c_1(\xi)$ *of* ξ *is trivial.*

Then the following question naturally arises.

Question 1.2. Given a closed oriented 3-manifold M, does a positive contact structure ξ on M embed in $(S^5, \xi_{\rm std})$ if and only if its first Chern class $c_1(\xi)$ is trivial?

In the first part of this thesis we give several partial answers to this question. In the case of S^3, we can embed all the contact structures on S^3 in $(S^5, \xi_{\rm std})$ in nice ways.

Theorem 1.3 (Etnyre and F. [17] Theorem 1.16)**.** *Any positive contact structure on* S^3 *can be embedded in* $(S^5, \xi_{\rm std})$ *so that the embedding is isotopic to the standard one.*

Remark 1.4. By using a T^3 action on S^5, Mori [43] showed that there exists an overtwisted contact structure (and hence infinitely many overtwisted contact structures) on S^3 which embeds in $(S^5, \xi_{\rm std})$ so that the embedding is isotopic to the standard one. But it was not clear whether all the contact structures can embed.

For overtwisted contact structures on any 3-manifold M, we have a partial answer to Question 1.2.

Theorem 1.5 (Etnyre and F. [17] Theorem 1.20)**.** *Let* M *be a closed oriented* 3*-manifold with no* 2*-torsion in its first homology group. Then a positive overtwisted contact structure* ξ *on* M *embeds in* $(S^5, \xi_{\rm std})$ *if and only if* $c_1(\xi) = 0$*.*

For tight contact structures on certain contact 3-manifolds, using the classification of tight contact structures [32] [22] [30], we obtain another partial answer to Question 1.2.

Theorem 1.6 (Etnyre and F. [17] Theorem 1.21)**.** *Let M be one of the following 3-manifolds.*

(1) $S^1 \times S^2$,
(2) T^3, *or*
(3) *a lens space $L(p, q)$ (including S^3) with p odd or with p even and $q = 1$ or $q = p - 1$*

Then a positive contact structure ξ on M embeds in $(S^5, \xi_{\mathrm{std}})$ if and only if its first Chern class $c_1(\xi)$ is trivial.

In this thesis we prove (1) , (2) and (3) for odd p's.

1.2. **The relative Euler numbers of codimension two contact submanifolds and their Seifert hypersurfaces.** One of the main research subjects on transverse knots is the classification problem, more precisely the problem to know whether the isotopy classes of transverse knots in a fixed smooth knot type are determined by their values of the self-linking number. It has been studied, for example, by using braid theory, convex surface theory or Floer type invariants such as knot Heegaard Floer homology and knot contact homology.

In the higher dimensional case, however, the classification problem of codimension two contact embeddings of closed contact manifolds has not been well studied. Some of reasons are that it has been difficult to find contact embeddings and that there have not been systematic ways to calculate invariants of codimension two contact embeddings.

Question 1.7. For positive integers n, are there two contact embeddings of a closed contact manifold (L^{2n+1}, ξ) in $(S^{2n+3}, \xi_{\mathrm{std}})$ which are isotopic as smooth embeddings but not isotopic as contact embeddings?

We partially answer to this question by calculating the *relative Euler number* which is a natural generalization of the self-linking number in higher dimensions. For a null-homologous codimension two positive contact submanifold $(L, \xi|_{TL})$ and its Seifert hypersurface Σ $(L = \partial\Sigma)$ in a positive contact $(2n + 3)$-manifold (M, ξ), we define the *relative Euler number* $e_{\mathrm{rel}}(L, \Sigma)$ of L and Σ by

$$e_{\mathrm{rel}}(L, \Sigma) = -\langle e(\xi, X_\Sigma), [\Sigma, \partial\Sigma] \rangle ,$$

where $e(\xi, X_\Sigma) \in H^{2n+2}(M, L; \mathbf{Z})$ is the relative Euler class relative to an outward vector field X_Σ along L which is tangent to $\xi|_{T\Sigma}$. The self-linking number $sl(L, \Sigma)$ of a transverse knot L and its Seifert surface Σ in a positive contact 3-manifold coincides with $e_{\mathrm{rel}}(L, \Sigma)$; $sl(L, \Sigma) = e_{\mathrm{rel}}(L, \Sigma)$.

The self-linking numbers of transverse knots play important roles in 3-dimensional contact topology. For a null-homologous knot L in a positive tight contact 3-manifold, the Bennequin inequality $sl(L,\Sigma) \leq -\chi(\Sigma)$ holds, where $\chi(\Sigma)$ is the Euler characteristic of Σ [1] [13]. This inequality shows that the set of values of the self-linking numbers of transverse knots reflects tightness of the contact structure. In higher dimensional case, however, Mori showed that we cannot expect the same 'Bennequin inequality'.

Theorem 1.8 (Mori [42])**.** *For any positive integer n, there exists a contact submanifold $L \cong S^1 \times S^{2n}$ and its Seifert hypersurface $\Sigma \cong D^2 \times S^{2n}$ in $(S^{2n+3}, \xi_{\mathrm{std}})$ such that $e_{\mathrm{rel}}(L,\Sigma) > -\chi(\Sigma)$.*

We will give other examples with $e_{\mathrm{rel}}(L,\Sigma) > -\chi(\Sigma)$ in the standard spheres (Theorem 1.10 (1)).

In the second part of this thesis, we calculate the relative Euler numbers of certain codimension two contact submanifolds and their Seifert hypersurfaces obtained by using braided embedding technique [17]. This result generalizes the formula of the self-linking number of the $(k,1)$-cable with $k>0$ of a transverse knot K in a contact 3-sphere in terms of the self-linking number of K.

Theorem 1.9. *Let n be a positive integer. Let M be a $(2n+3)$-dimensional integral homology sphere with a positive contact structure ξ. Let L be a $(2n+1)$-dimensional oriented sphere which is a positive contact submanifold of (M,ξ). Let K be a $(2n-1)$-dimensional oriented sphere which is a positive contact submanifold of $(L, \xi|_{TL})$. For any positive integer k, there exist a contact embedding $j\colon (L_{K,k}, (\xi|_{TL})_{K,k}) \to (M,\xi)$ of the k-fold cyclic covering $(L_{K,k}, (\xi|_{TL})_{K,k})$ of $(L, \xi|_{TL})$ branched along $(K, \xi|_{TK})$ and a Seifert hypersurface $\Sigma_{j(L_{K,k})}$ of $j(L_{K,k})$ such that*

$$e_{\mathrm{rel}}(j(L_{K,k}), \Sigma_{j(L_{K,k})}) = k \cdot e_{\mathrm{rel}}(L) - (k-1) \cdot e_{\mathrm{rel}}(K).$$

Moreover, for any Seifert hypersurfaces Σ_K of K and Σ_L of L, the Seifert hypersurface $\Sigma_{j(L_{K,k})}$ can be taken so that it is diffeomorphic to the k-fold cyclic covering of Σ_L branched along a push-off Σ'_K of Σ_K into Σ_L relative to its boundary corresponding to the $\mathbf{Z}/k\mathbf{Z}$ reduction of the element in $H^1(\Sigma_L \backslash \Sigma'_K; \mathbf{Z})$ dual to the meridian circle of Σ'_K.

We use Theorem 1.9 to give two contact embeddings isotopic but not contact isotopic in Theorem 1.10 (2), which answers to Question 1.7 for any positive even number n.

Recently the study of the flexibility properties of higher dimensional contact manifolds are largely developed by the classification of overtwisted contact structures in higher dimensions [2] and by geometric characterizations of overtwistedness [6]. Combining with these results, Theorem 1.9 gives several informations on the relative Euler numbers of contact submanifolds in the standard sphere which are smooth unknots. Theorem 1.10 (1) gives examples of codimension two contact submanifolds with $e_{\mathrm{rel}}(L,\Sigma) > -\chi(\Sigma)$ as we mentioned.

Theorem 1.10. (1) *Let m be an integer and n be a positive integer. Then there exists a contact embedding of an overtwisted contact structure η_m on S^{2n+1} in $(S^{2n+3},\xi_{\mathrm{std}})$ such that it has the relative Euler number $2m+1$ and it is isotopic to the standard embedding.*

(2) *For any positive even number n, there exists an overtwisted contact structure ξ on S^{2n+1} and infinitely many contact embeddings of ξ in $(S^{2n+3},\xi_{\mathrm{std}})$ with distinct relative Euler numbers which are isotopic to the standard embedding. In particular, for any integer m, there exists a contact embedding of the unique overtwisted contact structure ξ_{ot} on S^5 in (S^7,ξ_{std}) such that it has the relative Euler number $2m+1$ and it is isotopic to the standard embedding.*

1.3. **Plan of the thesis.** In Part 1 consisting of Sections 2–5, we consider the existence of embeddings of closed contact 3-manifolds in (S^5,ξ_{std}). Almost all the contents of Part 1 are in the joint paper with John Etnyre [17]. In Section 2, we review contact embeddings, contact 3-manifolds and branched coverings. In Section 3, we introduce braids and braided embeddings. In Section 4, we consider braids in contact manifolds and give conditions for the braided embeddings to be contact embeddings. In Section 5, we apply braided embeddings to studying contact embeddings of closed contact 3-manifolds in (S^5,ξ_{std}) and prove Theorems 1.3, 1.5 and 1.6.

In Part 2 consisting of Sections 6–8, we consider the relative Euler numbers of codimension two contact submanifolds and its Seifert hypersurfaces. In Section 6, we define the relative Euler number e_{rel} and review overtwisted contact structures in higher dimensions. In Section 7, we prove Theorem 1.9 which gives the relative Euler numbers of contact embeddings of cyclic branched coverings. In Section 8, we apply Theorem 1.9 to proving Theorem 1.10.

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Part 1. Existence of embeddings of closed contact 3-manifolds in the standard 5-sphere

2. Preliminaries to Part 1

For a positive real number c we denote the closed interval $[-c, c]$ by I_c and denote the closed disk with radius r centered at the origin embedded in $\mathbf{R}^2$ or $\mathbf{C}$ by D^2_r.

2.1. **Contact structures and contact embeddings.** A *contact structure* on a $(2n+1)$-dimensional manifold M is a maximally nonintegrable hyperplane field ξ. Locally ξ can be written as $\xi = \ker\alpha$ by a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. If a contact structure ξ is cooriented, we can take α as a global 1-form on M. When M is oriented and $\xi = \ker\alpha$ is cooriented, we say that ξ is *positive* (resp. *negative*) if $\alpha \wedge (d\alpha)^n$ is a positive (resp. negative) volume form on M with respect to the given orientation. An embedding $j\colon M \to N$ between two contact manifolds (M, ξ) and (N, η) is called a *contact embedding* if $Tj(M) \cap \eta|_{j(M)} = dj(\xi)$.

Example and Notation 2.1. The *standard contact structure* ξ_{std} on $\mathbf{R}^{2n+1}$ is the contact structure given by $\xi_{\mathrm{std}} = \ker(dz - \sum_{i=1}^n y_i dx_i)$. The *standard contact structure* ξ_{std} on the unit sphere S^{2n+1} in $\mathbf{C}^{n+1}$ is a contact structure given by $\xi_{\mathrm{std}} = \ker(-\frac{i}{4}\sum_{j=1}^{n+1}(\bar{z}_j dz_j - z_j d\bar{z}_j)) = \ker(\frac{1}{2}\sum_{j=1}^{n+1} r_j^2 d\theta_j)$, where z_j, $j = 1, 2, \ldots, n+1$, are coordinates on $\mathbf{C}^{n+1}$ and $z_j = r_j e^{i\theta_j}$. We note that $(S^{2n+1}\setminus\{pt\}, \xi_{\mathrm{std}}|_{T(S^{2n+1}\setminus\{pt\})})$ is contactomrophic to $(\mathbf{R}^{2n+1}, \xi_{\mathrm{std}})$. We call $\{z_{n+1} = 0\} \cong S^{2n-1} \subset S^{2n+1}$ the *standard sphere* in S^{2n+1}.

The contact structure on a neighborhood of a contact submanifold is determined by its conformally symplectic normal bundle.

Theorem 2.2 ([21] Theorem 2.5.15)**.** *Let (N_i, η_i), $i = 0, 1$, be contact manifolds with compact contact submanifolds (M_i, ξ_i). Suppose there exists an isomorphism of conformally symplectic normal bundles $\Phi\colon CSN_{N_0}(M_0) \to CSN_{N_1}(M_1)$ that covers a contactomorphism $\phi\colon (M_0, \xi_0) \to (M_1, \xi_1)$. Then ϕ extends to a contactomorphism ψ of suitable neighborhoods $N(M_i)$ of M_i so that $d\psi|_{CSN_{N_0}(M_0)}$ and Φ are homotopic as conformally symplectic bundle isomorphisms.*

Then, for a codimension two contact submanifold with trivial normal bundle, the following holds.

Proposition 2.3. *Let (M, ξ) be a codimension two closed cooriented contact submanifold of (N, η) with trivial normal bundle. Then there* *exists a neighborhood of M which is contactomorphic to $(M \times D^2_\epsilon, \ker(\alpha + r^2 d\theta))$ for some small positive real number ϵ, where α is a contact form for ξ and (r, θ) are polar coordinates on D^2_ϵ.*

Contact embeddings of closed contact manifolds have the isotopy extension property.

Theorem 2.4 ([21] Theorem 2.6.12)**.** *Let $j_t\colon (M, \xi) \to (N, \eta)$, $t \in [0, 1]$, be an isotopy of contact embeddings of a closed contact manifold (M, ξ) in (N, η). Then there is a compactly supported contact isotopy ψ_t of (N, η) such that $\psi_t \circ j_0 = j_t$.*

It is easy to see that the connected sum of contact submanifolds is a contact submanifold of the connected sum of contact manifolds.

Lemma 2.5. *If (M_i, ξ_i) is a contact submanifold of (N_i, η_i) for $i = 1, 2$, then $(M_1 \# M_2, \xi_1 \# \xi_2)$ is a contact submanifold of $(N_1 \# N_2, \eta_1 \# \eta_2)$.*

2.2. **The self-linking numbers of null-homologous transverse knots.** The *self-linking number* of a null-homologous oriented transverse knot L and its Seifert surface Σ_L in a contact 3-manifold is the linking number of L and L' where L' is a push-off of L along a nonvanishing vector field given by a trivialization of ξ over Σ_L. We note that the self-linking number does not depend on the orientation of L.

Proposition 2.6 (Bennequin [1], see also [15] Section 2.6.4)**.** *Let L be a transverse knot which is a closed braid around the z-axis in $(\mathbf{R}^3, \ker \alpha_+)$ or $(\mathbf{R}^3, \ker \alpha_-)$, where $\alpha_+ = dz + \frac{1}{2}(xdy - ydx)$ and $\alpha_- = dz - \frac{1}{2}(xdy - ydx)$. If L is the closure of an element b of the braid group B_k of k strands, then $sl(L) = e(b) - k$ when L is in $(\mathbf{R}^3, \ker \alpha_+)$ and $sl(L) = e(b) + k$ when L is in $(\mathbf{R}^3, \ker \alpha_-)$, where $e(b)$ denotes the sum of the exponents of b with respect to the standard generators.*

The formulas for α_+ and α_- reflect the fact that there is an orientation reversing contactomorphism $(\mathbf{R}^3, \ker \alpha_+) \to (\mathbf{R}^3, \ker \alpha_-)$, $(x, y, z) \mapsto (-x, y, z)$.

2.3. **Homotopy classes of oriented 2-plane fields on oriented 3-manifolds.** For an almost complex 4-manifold (X, J) whose almost complex boundary is (M, ξ), if the first Chern class $c_1(\xi)$ is a torsion class, then we can define an invariant $d_3(\xi) \in \mathbf{Q}$ of ξ by

$$d_3(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)),$$

where $\sigma(X)$ and $\chi(X)$ are the signature of X and the Euler characteristic of X, respectively, see [23, Definition 4.15]. Here, we use the notation $d_3(\xi)$ following [7]. Gompf showed that certain homotopy classes of 2-plane fields on a 3-manifold are classified by the homotopy classes over the 2-skeleton and d_3-invariants in some cases, see [7, Theorem 2.5].

Theorem 2.7 (Gompf [23] Theorem 4.16)**.** *Let ξ_i, $i = 1, 2$, be positive 2-plane fields on a closed oriented connected 3-manifold M such that (M, ξ_i) are the almost complex boundaries of (X_i, J_i), respectively. If ξ_1 and ξ_2 are homotopic on the 2-skeleton and $c_1(\xi_1) = c_1(\xi_2)$ is a torsion class, then ξ_1 and ξ_2 are homotopic on M if and only if $d_3(\xi_1) = d_3(\xi_2)$.*

For S^3, there exists a one-to-one correspondence between the homotopy classes of 2-plane fields on S^3 and $\mathbf{Z} - \frac{1}{2}$ so that $d_3(\xi_{\mathrm{std}}) = -\frac{1}{2}$, see [7].

2.4. **Overtwisted contact structures in dimension three.** Eliashberg [12] defined and classified overtwisted contact structures in dimension three.

Definition 2.8. An embedded disk Δ in a contact 3-manifold (M, ξ) is an *overtwisted disk* if its boundary $\partial\Delta$ is a Legendrian curve whose surface framing coincides with its contact framing and the characteristic foliation Δ_ξ contains a unique singular point in the interior of Δ. A contact 3-manifold is *overtwisted* if it contains an overtwisted disk.

Fix a point p in a 3-manifold M and an embedded disk Δ centered at p in M. Let $\mathfrak{Cont}_{\mathrm{ot}}(M, \Delta)$ be the space of overtwisted positive contact structures on M which have the disk Δ as an overtwisted disk and $\mathfrak{Dist}(M, \Delta)$ be the space of cooriented 2-plane fields on M that are tangent to Δ at p.

Theorem 2.9 (Eliashberg [12])**.** *The inclusion map*

$$j\colon \mathfrak{Cont}_{\mathrm{ot}}(M, \Delta) \to \mathfrak{Dist}(M, \Delta)$$

is a (weak) homotopy equivalence.

For S^3, connected components of $\mathfrak{Dist}(M, \Delta)$ are labelled by d_3-invariants by Theorem 2.7. By Theorem 2.9, the overtwisted contact structure on S^3 with the d_3-invariant $m - \frac{1}{2}$ is well-defined and we denote it by ξ_m.

There is an operation called a *half Lutz twist* along a transverse knot in a contact 3-manifold which makes the given contact structure overtwisted. By a half Lutz twist along a transverse knot in a contact 3-sphere, d_3-invariant changes as follows.

Proposition 2.10 (Ding, Geiges and Stipsicz [8]). *Let ξ be a positive contact structure on S^3. If we obtain a contact structure ξ' from ξ by a half Lutz twist along a transverse knot K, then $d_3(\xi') = d_3(\xi) - sl(K)$.*

2.5. **Branched coverings.** In this paper we use the term *branched covering* in the following sense.

Definition 2.11 (see [21] Definition 7.5.1). Let $\widetilde{L}$ and L be $(n+2)$-dimensional smooth manifolds and K be an n-dimensional smooth submanifold of L. A differentiable map $p\colon \widetilde{L} \to L$ is called a *covering branched along* K or just a *branched covering* if

- $p|_{p^{-1}(L\setminus K)}$ is a covering map of degree m,
- $p^{-1}(x)$ is a set of m' points with $m' < m$ for any $x \in K$,
- there is a neighborhood U_x of $x \in K$ such that for each component U of $p^{-1}(U_x)$ there are a positive integer k and diffeomorphisms $\widetilde{h}\colon D^n \times D^2 \to U$ and $h\colon D^n \times D^2 \to U_x$ such that the diagram
$$\begin{array}{ccc} D^n \times D^2 & \xrightarrow{\widetilde{h}} & U \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p|_U} \\ D^n \times D^2 & \xrightarrow{h} & U_x \end{array}$$
commutes, where $p_k(y,z) = (y, z^k)$ for $(y,z) \in D^n \times D^2 \subset D^n \times \mathbf{C}$.

If $m' = m-1$ and $k = 1$ or 2, then we call $p\colon \widetilde{L} \to L$ a *simple branched covering*. In this thesis, if K is a null-homologous closed oriented submanifold of an oriented manifold L with trivial normal bundle, a *k-fold cyclic branched covering* is a covering determined by the kernel of the composition of maps $\pi_1(L\setminus K) \to H_1(L\setminus K;\mathbf{Z}) \to \mathbf{Z} \to \mathbf{Z}/k\mathbf{Z}$, where the first map is the abelianization and the second map is a homomorphism given by intersection with some chosen compact connect oriented codimension one submanifold bounded by K. If L is a sphere and K is a closed connected oriented submanifold, then a k-fold cyclic branched covering is unique since $H_1(L\setminus K;\mathbf{Z}) \cong \mathbf{Z}$ by the Alexander duality. In this case we denote $\widetilde{L}$ by $L_{K,k}$.

For a covering $p\colon \widetilde{L} \to L$ branched along K, if L is equipped with a contact structure ξ and K is a contact submanifold of (L,ξ), then $\widetilde{L}$ has a natural contact structure induced by p which is unique up to isotopy.

Theorem 2.12 (Geiges [20], Öztürk and Niederkrüger [45]). *Let $p\colon \widetilde{L} \to L$ be a covering branched along a smooth submanifold $K \subset L$. Assume that L has a contact structure $\xi = \ker\alpha$ such that K intersects ξ transversely and $\xi \cap TK$ is a contact structure on K. Then there is a unique* *(up to isotopy) contact structure* ξ_p *on* $\widetilde{L}$. ξ_p *is given by a contact form* β_1 *that can be connected to* $\beta_0 = p^*\alpha$ *by a path* β_t, $t \in [0,1]$, *where* β_t *is a contact form for* $t > 0$ *and* $d\big(\frac{\partial \beta_t}{\partial t}|_{t=0}\big)$ *restricts to a positive area form on each (naturally oriented) fiber of the normal bundle of the singular set in* $\widetilde{L}$.

If L is a sphere and K is a closed connected oriented submanifold and $p\colon \widetilde{L} \to L$ is the k-fold cyclic branched covering as above, we denote the unique induced contact structure up to contactomorphism on $L_{K,k}$ by $\xi_{K,k}$.

For a simple covering of a contact 3-manifold branched along a transverse knot, the induced contact structure is altered when we alter the transverse knot by a (negative) stabilization as in the following.

Proposition 2.13 (see [17]). *Let* $p\colon \widetilde{L} \to L$ *be a simple branched covering of a closed oriented* 3*-manifold with branch locus* $B \subset L$. *Let* ξ *be a contact structure on* L *and* K *be a transverse knot in* (L, ξ) *and* K' *the (negative) stabilization of* K. *Let* ξ_K *and* $\xi_{K'}$ *be contact structures on* $\widetilde{L}$ *induced by a covering* p *branched along* K *and* K', *respectively. Then* $\xi_{K'}$ *is obtained from* ξ_K *by connected sum with the overtwisted contact structure* (S^3, ξ_1), *where* $d_3(\xi_1) = \frac{1}{2}$. *In particular,* $\xi_{K'}$ *is overtwisted, homotopic to* ξ_K *over the* 2*-skeleton, and has* d_3*-invariant (when it is defined)* $d_3(\xi_{K'}) = d_3(\xi_K) + 1$.

3. Topological Braided embeddings

Definition 3.1. Given an n-manifold Y, a *braid about* Y is an embedding of an n-manifold M in $Y \times D^2$

$$e\colon M \to Y \times D^2$$

such that $\pi \circ e\colon M \to Y$ is a branched covering map, where $\pi\colon Y \times D^2 \to Y$ is the projection onto the first factor. Moreover, we say a branched covering $p\colon M \to Y$ can be *braided about* Y if there is a function $f\colon M \to D^2$ such that

$$e\colon M \to Y \times D^2 : x \mapsto (p(x), f(x))$$

is an embedding (and hence represents M as a braid about Y). If Y is embedded in a $(n+2)$-manifold W with trivial normal bundle and M is braided about Y, then clearly M embeds in W and this is called a *braided embedding of* M *in* W (braided about Y).

Theorem 3.2 (Etnyre and F. [17]). *Let* $p\colon M \to Y$ *be a cyclic covering of a closed oriented* n*-manifold* Y *branched along a null-homologous* *submanifold $B \subset Y$ with trivial normal bundle. Then there is a function $f\colon M \to D^2$ such that*

$$e\colon M \to Y \times D^2 : x \mapsto (p'(x), f(x))$$

represents M as a braid about Y, where p' is homotopic to p through cyclic branched coverings.

For 2-fold branched coverings in dimension 2, 3 and 4, this theorem was previously proven in [4].

Proof. Let S be a Seifert hypersurface for B, that is, a codimension 1 connected submanifold S of Y such that $B = \partial S$, corresponding to a given cyclic branched covering. We will define in the next paragraph a smooth function $h\colon Y \to \mathbf{C}$ such that 0 is a regular value, $h^{-1}(0) = B$ and for any loop γ in the complement of B its algebraic intersection with S is given by the winding number of $h \circ \gamma$ about $0 \in \mathbf{C}$. Having defined the function h, let

$$X = \{(x, z) \in Y \times \mathbf{C} | z^n = h(x)\}.$$

It is clear that the map $p'\colon X \to Y : (x, z) \to x$ is the n-fold cyclic covering of Y branched along B (indeed it is clearly an n-fold covering map in the complement of the branch locus and unwraps each meridian as desired) that is homotopic to p. Thus X is diffeomorphic to M and the restriction of the projection $Y \times \mathbf{C} \to \mathbf{C}$ to X is the desired function f.

Now we construct h. We use S to provide a framing of the normal bundle of B and we use this framing to identify a tubular neighborhood of B with $N = B \times D^2$, where we are thinking of D^2 as the unit disk in $\mathbf{C}$ and $S \cap N$ agrees with B times the positive real axis. Let $h\colon B \times D^2 \to D^2$ be the projection and extend it to all of Y as follows. Identify a neighborhood of $S \cap (Y - N)$ with $N' = S \times (-\epsilon, \epsilon)$ for some small $\epsilon > 0$ and define h on $S \times (-\epsilon, \epsilon)$ by $h(x, t) = e^{it}$. Notice that we have h defined on $\partial\overline{(Y - N \cup N')}$ so that the image is contained in ∂D^2 minus a neighborhood of 1. That is, the image is contained in an interval in ∂D^2 and hence we can extend h over $Y - (N \cup N')$. We approximate h by a smooth function relative to N and replace h by it. For a C^0-close approximation h, 0 is still a regular value and $B = h^{-1}(0)$. □

4. Contact braided embeddings

In this section we show that braided embeddings about a contact submanifold can be made to be contact embeddings.

Theorem 4.1 (Etnyre and F. [17]). *Let M and Y be closed oriented $(2n+1)$-manifolds and*

$$e\colon M \to Y \times D^2 : x \mapsto (p(x), f(x))$$

a braided embedding of M about Y such that the branched covering $p\colon M \to Y$ whose branch locus $B \subset Y$ is not multiply ramified (that is, at most one component in the pre-image of each component of B is ramified). Then there is an orientation on B with the following property. For a contact structure $\xi = \ker \alpha$ on Y such that B is isotoped to a positive contact submanifold B', we obtain the contact structure ξ' induced on M by the isotoped covering p' branched along the positive contact submanifold B', and e can be isotoped to a contact embedding of (M, ξ') in $(Y \times D^2, \ker(\alpha + r^2\, d\theta))$.

This theorem gives a way to isotope certain embeddings of 3-manifolds in S^5 to be transverse contact embeddings.

Corollary 4.2 (Etnyre and F. [17] Theorem 1.26). *If an embedding of a 3-manifold M in S^5 can be isotoped to be braided about the standardly embedded S^3, then it can be isotoped to be transverse to the standard contact structure ξ_{std} in such a way that the induced plane field on M is a contact structure.*

Proof of Corollary 4.2 [17]. The standard embedding of S^3 in S^5 gives a contact embedding of the standard contact structures. Thus by Proposition 2.3, S^3 has a neighborhood $S^3 \times D^2$ with contact structure given by $\ker(\alpha_{\mathrm{std}} + r^2\, d\theta)$, where α_{std} is a contact form for the standard contact structure on S^3.

To use Theorem 4.1, we show how one can isotope the embedding so that branched covering corresponding to the embedding has a branch locus which is not multiply ramified. Let $\widetilde{B}_1$, $\widetilde{B}_2$ be distinct components of the singular set in M lying above a component B_0 of the branch locus of p. There exist neighborhoods $\widetilde{N}$ of $\widetilde{B}_1$ and N of B_0 such that $\widetilde{N}$ does not contain singular points other than $\widetilde{B}_1$ and $N \cong S^1 \times D^2$ does not contain branch points other than B_0, where B_0 is identified with $S^1 \times \{0\}$. Let ψ_t, $t \in [0,1]$, be an isotopy generated by a vector field with support in N, tangent to the D^2-factors of $N \cong S^1 \times D^2$, and non-zero along B_0. We now define the map $p_t\colon M \to Y$ to be p on $M \setminus \widetilde{N}$ and $\psi_t \circ p$ on $\widetilde{N}$. This is clearly an isotopy of p and and hence induces an isotopy of $e_t = (p_t, f)\colon M \to S^3 \times D^2$ of e through braided embeddings and for $t > 0$ the number of ramified components above B_0 is reduced by one. By repeating this process finitely many times, we can isotope the given e to a braided embedding whose branch locus is not multiply ramified. Since any link in S^3 can be isotoped to be transverse to the standard contact structure on S^3 we can clearly isotope the given embedding to satisfy the hypothesis of Theorem 4.1 and thus the theorem gives the desired isotopy. $\square$

Lemma 4.3 (Etnyre and F. [17])**.** *Let M and Y be closed oriented $(2n+1)$-manifolds and*

$$e\colon M \to Y \times D^2 : x \mapsto (p(x), f(x))$$

a braided embedding of M about Y. Denote the branch locus of p by $B \subset Y$. For a contact structure $\xi = \ker\alpha$ on Y in which B is a positive contact submanifold, let ξ' be the contact structure on M induced by the branched covering p.

Let $\widetilde{B}'$ be the subset of $\widetilde{B} = p^{-1}(B)$ consisting of the singular points. If for any $x \in \widetilde{B}'$ the map $df_x\colon T_xM \to T_{f(x)}D^2$ is orientation preserving when restricted to the fiber $\nu_x(\widetilde{B}')$ of the normal bundle $\nu(\widetilde{B}')$ of $\widetilde{B}'$, where $\nu_x(\widetilde{B}')$ is oriented by the orientations on $\widetilde{B}'$ and M, then for sufficiently small $R > 0$ the embedding

$$e_R\colon M \to Y \times D^2 : x \mapsto (p(x), Rf(x))$$

is a contact embedding of (M, ξ') in $(Y \times D^2, \ker(\alpha + r^2\, d\theta))$.

Proof of Lemma 4.3 [17]. Let $\beta_R = e_R^*(\alpha + r^2\, d\theta) = p^*\alpha + Rf^*(r^2\, d\theta)$. Then

$$\begin{aligned}\beta_R \wedge (d\beta_R)^n = p^*(\alpha \wedge (d\alpha)^n) + R\left(p^*((d\alpha)^n) \wedge f^*(r^2\, d\theta)\right) \\ + 2nR\left(p^*(\alpha \wedge (d\alpha)^{n-1}) \wedge f^*(r\, dr \wedge d\theta)\right).\end{aligned}$$

Away from $\widetilde{B}'$, p is a covering map so the first term is a positive multiple of the volume form. Thus for sufficiently small R, β_R is a contact form on the complement of a neighborhood of $\widetilde{B}'$. On $\widetilde{B}'$ recall that p has rank $2n-1$ and more specifically is a covering map when restricted to $\widetilde{B}'$ and has 0 derivative in the normal directions to $\widetilde{B}'$. Thus the first two terms in the expression for $\beta_R \wedge (d\beta_R)^n$ above are zero. The last term is a positive multiple of the volume form for M. This is clear by the fact that $p^*(\alpha \wedge (d\alpha)^{n-1})$ is a positive volume form on $\widetilde{B}'$ because B is a positive contact submanifold and that by the hypothesis on f in the lemma $f^*(r\, dr \wedge d\theta)$ is a positive area form on the fiber of the normal bundle $\nu(\widetilde{B}')$. Moreover, it is clear from the expression of β_R that β_R gives the contact structure ξ' coming from the covering $p\colon M \to Y$ branched along B. $\square$

By the proof of Theorem 3.2 and Lemma 4.3, we have the following.

Corollary 4.4 (Etnyre and F. [17]). *Let $(Y, \xi = \ker\alpha)$ be a contact $(2n+1)$-manifold and B a codimension two contact submanifold that is null-homologous and has trivial normal bundle. Let (M, ξ') an n-fold cyclic covering branched along B with the contact structure ξ' induced from ξ. Then there is a braided contact embedding of (M, ξ') in $(Y \times D^2, \ker(\alpha + r^2\, d\theta))$.* □

Proof of Theorem 4.1 [17]. Let $e\colon M \to Y \times D^2$ be an embedding in the statement of the theorem. Let $\widetilde{B} = p^{-1}(B)$ and $\widetilde{B}'$ be the subset of $\widetilde{B}$ consisting of the singular points. By hypothesis, p maps bijectively the components of $\widetilde{B}'$ to the components of B.

At a point $x \in \widetilde{B}'$, notice that df_x is an isomorphism from the fiber $\nu_x(\widetilde{B}')$ of the normal bundle of $\widetilde{B}'$ to $T_{f(x)}D^2$. The reason is that the map $de_x\colon T_xM \to T_{e(x)}(Y \times D^2)$ has rank $2n+1$ and $dp_x\colon T_xM \to T_{p(x)}Y$ has only rank $2n-1$. Thus at each point of $\widetilde{B}'$ there is an induced orientation on each fiber $\nu_x(\widetilde{B}')$ and this orients each component of $\widetilde{B}'$, which in turn induce an orientation on B via p.

Now if B can be isotoped to a positive contact submanifold then there is an ambient isotopy $\phi_t\colon Y \to Y$, $t \in [0,1]$, that realizes this isotopy. Thus there is a diffeomorphism of $Y \times D^2$ that takes e to $e'\colon M \to Y \times D^2 : x \to (\phi_1 \circ p(x), f(x))$. Then e' realizes M as braided about Y and the corresponding branch locus is a contact submanifold which is smoothly isotopic to B. The theorem now follows from Lemma 4.3. □

5. Contact embeddings of 3-manifolds in the standard 5-sphere

5.1. **Proof of Theorem 1.3.** In this subsection we prove the following theorem.

Theorem (Theorem 1.3). *Any positive contact structure on S^3 can be embedded in $(S^5, \xi_{\mathrm{std}})$ so that the embedding is isotopic to the standard one.*

To prove this theorem we prepare several lemmas. By [13], it is known that any transverse unknot in the tight S^3 is classified up to isotopy among transverse knots by its self-linking number. By [10] and [16], it is known that any transverse unknot in any overtwisted S^3 is classified up to global contactomorphisms by its self-linking number and it is loose, i.e., its complement is overtwisted. Thus in the following lemma, the transverse unknot U_i ($i \in \mathbf{Z}$) is well-defined.

Lemma 5.1. *Let k be a positive integer and let m and n be integers.*

(1) *The k-fold cyclic covering of $(S^3, \xi_{\mathrm{std}})$ branched along the transverse unknot U_0 with $sl(U_0) = -1$ is $(S^3, \xi_{\mathrm{std}})$. The k-fold cyclic covering of $(S^3, \xi_{\mathrm{std}})$ branched along the transverse unknot U_m with $sl(U_m) = 2m - 1 < -1$ is $(S^3, \xi_{-(k-1)m})$.*
(2) *The k-fold cyclic covering of (S^3, ξ_n) branched along the transverse unknot U_m with $sl(U_m) = 2m - 1$ is $(S^3, \xi_{kn-(k-1)m})$.*

Proof. The first statement of (1) is immediate since U_0 is the binding of the trivial open book which supports $(S^3, \xi_{\mathrm{std}})$. The second statement of (1) is well-known (see [27]). The k-fold cyclic covering of $(S^3, \xi_{\mathrm{std}})$ branched along U_{-1} gives the overtwisted contact structure with $d_3 = -1/2 + k - 1$ and hence it is $\xi_{k-1} = \xi_{-(k-1)\cdot(-1)}$. Since U_{-1} is the negative stabilization of U_0, a negative stabilization of a branch locus contributes to decreasing the self-linking number of the branch locus by 2 and increasing the d_3-invariant of the k-fold cyclic branched covering by $k - 1$. Hence by taking negative stabilizations of U_0 $|m|$ times, we obtain U_m and $\xi_{-(k-1)m}$.

Now we show (2). For any integer n, consider the closure L_n of $\sigma_1^{-n+2} \in B_2$ in $(S^3, \xi_{\mathrm{std}})$, where B_2 is the braid group with two strands generated by σ_1. Since $sl(L_n) = -n + 2 - 2 = -n$, the half Lutz twist along L_n in $(S^3, \xi_{\mathrm{std}})$ gives (S^3, ξ_n) by Proposition 2.10. After operating the half Lutz twist, for any n, the axis of L_n is the transverse unknot U_2 with $sl(U_2) = 3$. The k-fold cyclic covering of (S^3, ξ_n) branched along U_2 is the overtwisted contact structure on S^3 given by the half Lutz twist along the closure L_n^k of $\sigma_1^{(-n+2)k}$. Since $sl(L_n^k) = (-n+2)k - 2 = -\{kn - (k-1)2\}$, this contact structure is $\xi_{kn-(k-1)2}$. We observe that a negative stabilization (resp. destabilization) of a branch locus increases (resp. decreases) the d_3-invariant of the contact structure obtained by the k-fold cyclic branched covering by $k - 1$ and that any transverse unknot in any overtwisted S^3 is loose and hence any cyclic covering of any overtwisted S^3 branched along any transverse unknot is overtwisted. By taking stabilizations or destabilizations $|m|$ times, (2) follows. □

Lemma 5.2 (Etnyre and F. [17]). *There exists a braided embedding*

$$e \colon S^3 \to S^3 \times \mathbf{C} : x \mapsto (p(x), h(x))$$

such that p is the 2-fold cyclic branched covering map with branch locus the unknot and the embedding of $S^3 \to S^5$ coming from e is isotopic to the standard embedding.

Proof of Lemma 5.2 [17]. We think of S^5 as the unit sphere in $\mathbf{C}^3$ and give $\mathbf{C}^3$ coordinates (z_1, z_2, z_3). We then consider the standard embedding of S^3 in S^5 given by $S^3 = \{z_3 = 0\} \cap S^5$ and the unknot in S^3

given by $U = \{z_2 = z_3 = 0\} \cap S^5$. Denote by $U' = \{z_1 = z_2 = 0\}$ the circle in S^5 that is complementary to S^3 (that is one can see S^5 as the join of S^3 and U'). Notice that $C = S^5 - U'$ is diffeomorphic to $S^3 \times \mathbf{C}$ by the diffeomrophism

$$S^3 \times \mathbf{C} \to C : ((z_1, z_2), z_3) \mapsto \left(\frac{z_1}{\sqrt{1 + |z_3|^2}}, \frac{z_2}{\sqrt{1 + |z_3|^2}}, \frac{z_3}{\sqrt{1 + |z_3|^2}} \right)$$

and the map

$$\pi \colon C \to S^3 : (z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{|z_1|^2 + |z_2|^2}}(z_1, z_2)$$

is the projection map to S^3.

Consider the complex polynomial $p_{(a,b)}(z_1, z_2, z_3) = az_2 - bz_3^2$, where $(a, b) \in \mathbf{R}^2$. Notice that the zero set transversely intersects S^5 at the sphere $S_{(a,b)}$ for all (a, b) with $a \neq 0$. For $b' \neq 0$, consider the map $p \colon S_{(1,b')} \to S^3$ obtained by restricting π to $S_{(1,b')}$. We claim this is a 2-fold covering map branched along U. To see this, we first note that for each point $(z_1, z_2) \in S^3 - U$, we have $z_2 \neq 0$ so there are precisely 2 square roots $\pm\sqrt{z_2}$ of z_2 and we see that the map $(z_1, z_2) \mapsto \frac{1}{\sqrt{|z_1|^2+|z_2|^2+|z_2/b'|}}(z_1, z_2, \sqrt{z_2/b'})$ is a local section of $p \colon S_{(1,b')} \to S^3$. Thus we see that p is a 2-fold covering map from $S_{(1,b')} - p^{-1}(U)$ to $S^3 - U$. Moreover, for any $(z_1, z_2) \in U$ we see that $z_2 = 0$ so there is a unique square root and the only point in $S_{(1,b')}$ lying above it is $(z_1, 0, 0)$.

Thus we see that $S_{(1,b')}$ is a sphere that is braided about the standardly embedded S^3 in S^5 and realizing a 2-fold cyclic covering branched along the unknot U. Now we see that the spheres $S_{(1,t)}$ for $t \in [0, b']$ provide an isotopy from our braided sphere $S_{(1,b')}$ to the sphere $S_{(1,0)} = \{z_2 = 0\}$ which is the standardly embedded sphere. $\square$

Proof of Theorem 1.3. The standardly embedded (S^3, ξ_{std}) in (S^5, ξ_{std}) has a neighborhood $S^3 \times D^2$ contactomorphic to $(S^3 \times D^2_\epsilon, \ker(\alpha_{\text{std}} + r^2\, d\theta))$ for some small $\epsilon > 0$, where $\xi_{\text{std}} = \ker \alpha_{\text{std}}$. Now consider the transverse unknot U_m with $sl(U_m) = 2m - 1 < -1$ in (S^3, ξ_{std}) and the 2-fold cyclic covering branched along it. By Lemma 5.2, the fact that any oriented knot is isotopic to a positive transverse knot, Theorem 4.1 and Lemma 5.1 (1), there is a contact embedding of $(S^3, \xi_m), m \geq 1$, in (S^5, ξ_{std}) which is isotopic to the standard embedding. By Lemma 5.1 (2), (S^3, ξ_m) for any m is the 2-fold cyclic covering of (S^3, ξ_1) branched along some transverse unknot. Thus arguing similarly we have a contact embedding of (S^3, ξ_m) in $(S^5, \xi_{\mathrm{std}})$ which is isotopic to the standard embedding for any m. $\square$

5.2. **Proof of Theorem 1.5.** In this subsection we prove the following theorem.

Theorem (Theorem 1.5)**.** *Let M be a 3-manifold with no 2-torsion in its first homology group. Then a positive overtwisted contact structure ξ on M embeds in $(S^5, \xi_{\mathrm{std}})$ if and only if $c_1(\xi) = 0$.*

To prove this theorem we use the following fact for topological 3-dimensional braid.

Theorem 5.3 (Hilden, Lozano and Montesinos [28])**.** *Every closed oriented 3-manifold M can be braided about S^3 where the corresponding branched covering is a simple 3-fold branched covering.*

From this we obtain.

Proposition 5.4 (Etnyre and F. [17])**.** *Any closed oriented 3-manifold has some positive contact structure which can embed in $(S^5, \xi_{\mathrm{std}})$.*

Remark 5.5. Given a contact embedding e of a 3-manifold M constructed as a braid about the standardly embedded S^3 as in the proof of Proposition 5.4, by applying the Alexander theorem for transverse links to the branch locus $B \subset S^3$ of the branched covering of the braided embedding, we can isotope B to a positive transverse braid about the transverse unknot. Then we can isotope the given contact embedding among contact embeddings to a contact embedding which is compatible with some supporting open book for the embedded contact 3-manifold and the standard open book which is a supporting open book for $(S^5, \xi_{\mathrm{std}})$.

Proof of Proposition 5.4 [17]. For a closed oriented 3-manifold M, Theorem 5.3 tells us that there is a braided embedding

$$e \colon M \to S^3 \times D^2$$

such that the corresponding branched covering is a simple 3-fold branched covering. Thus e satisfies the hypothesis of Theorem 4.1 and since the branch locus can be isotoped to be a transverse link in $(S^3, \xi_{\mathrm{std}})$ the contact structure ξ' induced on M by this branched covering contact embeds in $(S^3 \times D^2, \ker(\alpha_{\mathrm{std}} + r^2\, d\theta))$, where α_{std} is a contact form for the standard contact structure on S^3.

Since the standard embedding of S^3 in S^5 is also an embedding of the standard contact structures, by Proposition 2.3, S^3 has a neighborhood

$S^3 \times D^2$ in S^5 on which the contact structure is given by $\ker(\alpha_{\mathrm{std}} + r^2\, d\theta)$. Since the contact embedding obtained in Theorem 4.1 can be arranged to be arbitrarily close to $S^3 \times \{(0,0)\}$, we see that M has a contact embedding in $(S^5, \xi_{\mathrm{std}})$ that is arbitrarily close to the embedding of S^3. $\square$

Proof of Theorem 1.5 [17]. By Proposition 5.4 we know that every 3-manifold M has some contact structure ξ that embeds in $(S^5, \xi_{\mathrm{std}})$. By using Lemma 2.5 we know that $\xi \# \xi_n$ embeds for any overtwisted contact structure ξ_n on S^3. Since there is no 2-torsion in the second cohomology of M, all overtwisted contact structures with trivial first Chern class on M is of the form $\xi \# \xi_n$ for some n and thus they all embed. $\square$

5.3. **Proof of Theorem 1.6 (1), (2) and a part of (3).** In some 3-manifolds we can determine the embedability of contact structures by its first Chern classes. In this thesis, we prove the following theorem (1), (2) and (3) in the case of odd p's. The proof of (3) we give here is different from the one in [17] and for the complete proof of (3), see [17].

Theorem (Theorem 1.6)**.** *Let M be one of the following 3-manifolds.*

(1) *$S^1 \times S^2$,*
(2) *T^3, or*
(3) *a lens space $L(p,q)$ (including S^3) with p odd or with p even and $q = 1$ or $q = p - 1$*

Then a positive contact structure ξ on M embeds in $(S^5, \xi_{\mathrm{std}})$ if and only if its first Chern class $c_1(\xi)$ is trivial.

Proof of Theorem 1.6 (1) (2) [17]. There is a unique tight contact structure ξ_{tt} on $S^1 \times S^2$ that is supported by the open book with annulus page and identity monodromy. Thus it is easy to see that it is obtained as the double covering of $(S^3, \xi_{\mathrm{std}})$ branched along the unlink with both components being transverse knots of self-linking number -1. Now Corollary 4.4 allows us to embed ξ_{tt} in $(S^5, \xi_{\mathrm{std}})$. Since there is no 2-torsion in the homology of $S^1 \times S^2$, we see from Theorem 1.5 that all overtwisted contact structures with $c_1 = 0$ also embed.

For T^3 we see that all overtwisted contact structures with $c_1 = 0$ embed in $(S^5, \xi_{\mathrm{std}})$ in a way similar to the previous case. A complete list of tight contact structures on T^3 is given by $\{\xi_n^{T^3}\}_{n \in \mathbf{Z}_{>0}}$, where T^3 is identified with $\mathbf{R}^3/\mathbf{Z}^3$,

$$\xi_n^{T^3} = \ker(\cos 2\pi nz\, dx + \sin 2\pi nz\, dy)$$

and n is a positive integer, see [32]. It is easy to see that $\xi_n^{T^3}$ is an n-fold (ordinary) cyclic covering of $\xi_1^{T^3}$ unwrapped in the direction of the z-coordinate. We notice that if $h\colon T^3 \to S^1$ is the projection onto the z-coordinate thought of as the unit circle in $\mathbf{C}$ then the proof of Theorem 3.2 gives a braided embedding of the n-fold (ordinary) covering of T^3 in $T^3 \times D^2$ and since there is no branch locus to worry about Theorem 4.1 clearly gives a contact embedding of $(T^3, \xi_n^{T^3})$ in $(T^3 \times D^2, \ker(\alpha_1 + r^2\, d\theta))$, where α_1 is the contact form for $\xi_1^{T^3}$. Thus if we can embed $(T^3, \xi_1^{T^3})$ in $(S^5, \xi_{\mathrm{std}})$ then we will have an embedding of all tight contact structures on T^3.

There are many ways to construct Legendrian embeddings of T^2 in $(S^5, \xi_{\mathrm{std}})$. For example, we can use front projections to construct ones, see [11]. A Legendrian T^2 has a neighborhood contactomorphic to a neighborhood of the zero section in $T^*T^2 \times \mathbf{R}$ with the contact structure $\ker(dz - \lambda)$, where λ is the Liouville 1-form on T^*T^2 and z is the coordinate on $\mathbf{R}$. Let S_ϵ be the ϵ-sphere bundle in T^*T^2. It is easy to see that λ restricted to S_ϵ is a contact 1-form defining $\xi_1^{T^3}$ and thus $(T^3, \xi_1^{T^3})$ contact embeds in $(S^5, \xi_{\mathrm{std}})$. $\square$

To prove Theorem 1.6 (3) for odd p's we use invertible Legendrian links. The following definition of *strongly invertible Legendrian links* is the Legendrian version of the strongly invertible links introduced by Montesinos [38].

Definition 5.6. A Legendrian link $L = L_1 \cup L_2 \cup \cdots \cup L_k$ in $(\mathbf{R}^3, \xi_{\mathrm{std}} = \ker(dz - ydx))$ is called *strongly invertible* if L is Legendrian isotopic to a link L' such that each component of L' is invariant under the involution ($\mathbf{Z}/2\mathbf{Z}$-action) generated by $(x, y, z) \mapsto (-x, -y, z)$ and each component has precisely two fixed points under this involution.

We note that this involution extends to an involution of $(S^3, \xi_{\mathrm{std}})$, for example, in the following way. We trivially embed $(\mathbf{R}^3, \xi_{\mathrm{std}})$ in $(\mathbf{R}^2 \times S^1, \ker(dz - ydx))$, where z is the S^1-coordinate. Clearly the involution extends to the involution generated by $(x, y, z) \mapsto (-x, -y, z)$. By the diffeomorphism defined by $(x, y, z) \mapsto (x/2, y, z - xy/2)$ we can identified $(\mathbf{R}^2 \times S^1, \ker(dz - ydx))$ with $(\mathbf{R}^2 \times S^1, \ker(dz + xdy - ydx))$. The latter is completed to $(S^3, \xi_{\mathrm{std}})$ by gluing $S^1 \times D^2$ with a tight contact structure which is invariant in the S^1-direction. Thus the involution extends to $(S^3, \xi_{\mathrm{std}})$.

For $r \in \mathbf{Q} \cup \{\infty\}$, a *contact r-surgery* along a Legendrian knot K is the operation consisting of a Dehn surgery along K whose framing is r with respect to the contact framing of K and an extension by a tight contact structure to the glued-in solid torus. Though it is not well-defined for general $r \in \mathbf{Q}$, when $r = 1/l$, $l \in \mathbf{Z}$, it is well-defined since there is a unique tight contact structure up to isotopy on $S^1 \times D^2$ with a given boundary condition [22] [30]. For more detail about contact surgery, see[7].

Proposition 5.7. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_k$ be a strongly invertible Legendrian link in $(\mathbf{R}^3, \xi_{\text{std}}) \subset (S^3, \xi_{\text{std}})$. We denote the contact 3-manifold obtained by the contact (-1)-surgeries along each L_i's by $(M, \xi(L))$. Then $(M, \xi(L))$ is isotopic to the contact structure obtained by a double covering of (S^3, ξ_{std}) branched along some transverse link.*

Proof. First by isotoping L so that it is invariant under the involution. Moreover, we further isotope L so that near each fixed point L is defined by an arc $\{0\} \times [-\epsilon, \epsilon] \times \{c\} \subset \mathbf{R}^3$ for some small positive real number ϵ and some real number c. For each component L_i there exists a neighborhood $N(L_i)$ such that

(1) $N(L_i)$ is contactomorphic to $(S^1 \times D^2_{\delta_i}, \ker(\cos\theta dx - \sin\theta dy))$ with coordinates $(\theta, x, y) \in S^1 \times D^2$ for some small δ_i and L_i maps to $S^1 \times \{(0,0)\}$,
(2) $N(L_i)$ is invariant under the involution,
(3) the involution on $N(L_i)$ induces the involution generated by $(\theta, x, y) \mapsto (-\theta, x, -y)$ on $S^1 \times D^2_{\delta_i}$.

The condition (3) is satisfied as follows. In the proof of the tubular neighborhood theorem of Legendrian submanifolds (see for example [21] Theorem 2.5.8), first we construct a diffeomorphism induced by the isomorphism of normal bundles of Legendrian submanifolds and then apply the Moser type argument. We first construct a isomorphism from the normal bundle over $[0, \pi] \times \{(0,0)\}$ whose total space is identified with $[0, \pi] \times D^2_{\delta_i}$ to the normal bundle of the 'half' arc in L_i which near each endpoints is identified with a neighborhood of the form $\{(x, y, z); x^2 + z^2 \leq \delta_i', -\epsilon_i' \leq y \leq 0\}$ or $\{(x, y, z); x^2 + z^2 \leq \delta_i', 0 \leq y \leq \epsilon_i'\}$ for some small δ_i' and ϵ_i' so that the induced diffeomorphism is already a strict contactomorphism in these regions. Then we apply the Moser type argument relative to these regions. We can extend this contactomorphism to $[-\pi, 0] \times D^2_{\delta_i}$ by using the involution given by $(\theta, x, y) \mapsto (-\theta, x, -y)$ and then (3) is satisfied.

To perform the contact (-1)-surgery, first we remove this $N(L_i)$, reglue $S^1 \times D^2$ and then extend the induced contact structure near the boundary to the whole glued-in $S^1 \times D^2$ by a tight contact structure. We first give a diffeomorphism f from a product neighborhood of $\partial(S^1 \times D^2)$ to a product neighborhood of $\partial N(L_i)$ (which is identified with

$\partial(S^1 \times D^2_{\delta_i}))$ given by $(\Theta, R, \Phi) \mapsto (-\Phi + \Theta, R, 2\Phi - \Theta) = (\theta, r, \phi)$ where (r, ϕ) are polar coordinates on $D^2_{\delta_i}$. Since $\partial(S^1 \times D^2_{\delta_i})$ is convex with respect to the contact vector field $r\partial_r$ and has two dividing curves with slope -1, f makes $\partial(S^1 \times D^2)$ convex with respect to $R\partial_R$ and to have two dividing curves with slope ∞. Thus the extension to the glued-in $S^1 \times D^2$ by a tight contact structure is unique up to isotopy.

The problem is whether we can take this extension so that it is invariant under the involution generated by $(\Theta, R, \Phi) \mapsto (-\Theta, R, -\Phi)$. Now for a positive real number c, we take a smooth function $f_c(R)$ on $\mathbf{R}_{\geq 0}$ which has the support on $(c, 5c)$, is equal to 1 on $[2c, 4c]$, is nondecreasing on $[c, 2c]$ and is nonincreasing on $[4c, 5c]$. We define a family of 1-forms $\{\beta_{c,t}\}_{t\in[0,1]}$ on $S^1 \times \mathbf{R}^2$ by

$$\begin{aligned}\beta_{c,t} =& d(R\cos\Phi) + R\sin\Phi d\Theta + (1 - tf_c(R))\cos\Phi dR - R\sin\Phi d\Phi \\ =& 2d(R\cos\Phi) + R\sin\Phi d\Theta - tf_c(R)\cos\Phi dR.\end{aligned}$$

Then

$$d\beta_{c,t} = \sin\Phi dR \wedge d\Theta + R\cos\Phi d\Phi \wedge d\Theta - tf_c(R)\sin\Phi dR \wedge d\Phi,$$

$$\beta_{c,t} \wedge d\beta_{c,t} = (2 - tf_c(R))RdR \wedge d\Phi \wedge d\Theta.$$

Thus $\{\beta_{c,t}\}_{t\in[0,1]}$ is a family of contact forms on $S^1 \times \mathbf{R}^2$ which are fixed on $\{0 \leq R \leq c, R \geq 5c\}$. Moreover, $\ker\beta_{c,0}$ is a tight contact structure, and hence $\ker\beta_{c,t}$ is tight for any $t \in [0, 1]$. The above glued-in $S^1 \times D^2$ has a contact structure near its boundary given by

$$\begin{aligned}&\cos(-\Phi + \Theta)d\{R\cos(2\Phi - \Theta)\} - \sin(-\Phi + \Theta)d\{R\sin(2\Phi - \Theta)\} \\ &= d(R\cos\Phi) + R\sin\Phi d\Theta - R\sin\Phi d\Phi.\end{aligned}$$

Thus by taking $\beta_{\delta_i/3,1}$, we can extend the given contact structure near the boundary to the whole glued-in $S^1 \times D^2$ by a tight contact structure which is invariant under the given involution.

Since the double covering of $(S^3, \xi_{\mathrm{std}})$ branched along the standard unknot is $(S^3, \xi_{\mathrm{std}})$ and the above involution induces this branched covering, $(S^3 \backslash (\cup_{i=1}^k N(L_i)), \xi_{\mathrm{std}})$ is obtained from the double covering of $(S^3, \xi_{\mathrm{std}})$ branched along the standard unknot by removing the preimage of the union of k tight 3-balls. On the other hand, by taking smaller δ_i's if necessary, $(S^1 \times D^2_{2\delta_i}, \ker(\beta_{\delta_i/3,1}))$ can be identified with a neighborhood of the standard Legendrian unknot (which is in the strongly invertible position) in $(S^3, \xi_{\mathrm{std}})$ equivariantly. Thus the glued-in $S^1 \times D^2$ is obtained from the double covering of $(S^3, \xi_{\mathrm{std}})$ branched along the standard unknot as the preimage of a tight 3-ball. By these observations $(M, \xi(L))$ can be seen as a double branched covering of a contact manifold obtained from $(S^3, \xi_{\mathrm{std}})$ by removing k tight 3-balls and re-gluing k tight 3-balls. Thus it is the double covering of $(S^3, \xi_{\rm std})$ branched along a transverse link. $\square$

For integers p, q with $p > q > 0$, we have the unique continued fraction expansion of $-p/q$

$$-\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 \cdots - \cfrac{1}{r_k}}}$$

with $r_i \leq -2$ for $i = 0, 1, \ldots, k$. We denote this continued fraction by $[r_0, r_1, \ldots, r_k]$.

Theorem 5.8 (Giroux [22], Honda [30])**.** *Let $L(p, q)$ be a lens space and assume that $-p/q$ has the continued fraction representation $[r_0, r_1, \ldots, r_k]$. Then there exist $|(r_0+1)(r_1+1)\cdots(r_k+1)|$ tight contact structures up to isotopy all of which are distinguished by their homotopy classes as 2-plane fields and are Stein fillable. Moreover, their homotopy classes are distinguished by their first Chern classes when p is odd.*

We can draw a surgery diagram for any of the above tight contact structures by using a chain $L = L_1 \cup L_2 \cup \cdots \cup L_k$ of Legendrian unknots L_i's, $i = 1, 2, \ldots, k$, and the surgery coefficient is given by $r_i = \mathrm{tb}(L_i) - 1$ for each component L_i, where $\mathrm{tb}(K)$ is the Thurston-Bennequin number of a Legendrian knot K. Thus it is a contact (-1)-surgery along L, see [30, Figure 16]. The first Chern class of the contact structure is calculated from this surgery diagram in terms of rotation numbers; $c_1(\xi) = \sum_{i=0}^{k} \mathrm{rot}(L_i)\mu_i$, where $\mu_i \in H^2(L(p, q); \mathbf{Z})$ is a cohomology class which is Poincaré dual to the oriented meridian circle of each component L_i of L, see [23, Proposition 2.3].

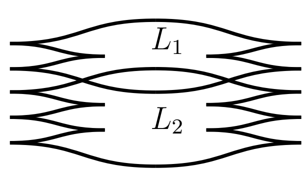

FIGURE 1. A Legendrian surgery diagram of a Stein fillable (hence tight) contact structure. It is a two component Legendrian link $L = L_1 \cup L_2$ with $\mathrm{tb}(L_1) = -3$, $\mathrm{rot}(L_1) = 0$, $\mathrm{tb}(L_2) = -5$ and $\mathrm{rot}(L_2) = 0$. Hence, $-\frac{p}{q} = (-3-1) - \frac{1}{-5-1} = -\frac{23}{6}$ and it represents a tight contact structure on $L(23, 6)$.

Proof of Theorem 1.6 (3) for odd p's. First we note that the homotopy class of a plane field is determined by its first Chern class and d_3-invariant since $H^2(L(p,q);\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$ has no 2-torsion. Since by Theorem 1.5 we know that any overtwisted contact structure on $L(p,q)$ with trivial first Chern class embeds in $(S^5, \xi_{\mathrm{std}})$, it suffices to show that we can embed any tight contact structure with trivial first Chern class. We note that there may exist no such tight contact structures. If there exists a tight contact structure with trivial first Chern class, then all the coefficients r_i's are even. Indeed, if $\xi = \ker\alpha$ is a positive tight contact structure, then $\xi' = \ker(-\alpha)$ is also a positive tight contact structure with $c_1(\xi) = -c_1(\xi')$. Hence, by Theorem 5.8, if there exists a tight contact structure with trivial first Chern class on $L(p,q)$ for odd p, then the number of tight contact structures must be odd and thus all the r_i's must be even. In this case we can draw a surgery digram of the tight contact structure with trivial first Chern class given by a strongly invertible Legendrian link along each component of which we do the contact (-1)-surgery, see for example Figure 1. Since by Corollary 4.4 any contact structure which is obtained as a double branched covering of $(S^3, \xi_{\mathrm{std}})$ embeds in $(S^5, \xi_{\mathrm{std}})$, combining with Proposition 5.7, the statement follows. $\square$

Part 2. The relative Euler numbers of codimension two contact submanifolds and their Seifert hypersurfaces

6. Preliminaries to Part 2

6.1. **The relative Euler number.** In this subsection we define the relative Euler number, see also [42, Section 1].

Definition 6.1. Let n be a nonnegative integer. Let $(M, \xi = \ker \alpha)$ be a $(2n+3)$-dimensional positive contact manifold and $(L, \xi|_{TL})$ be a $(2n+1)$-dimensional orientable closed contact submanifold of (M, ξ) which is null-homologous in M. We take a codimension one compact connected smooth oriented submanifold Σ bounded by L whose orientation induces the orientation of $L = \partial\Sigma$ given by $\xi|_{TL}$. We refer Σ as *a Seifert hypersurface* of L. We take an outward vector field X_Σ along $\partial\Sigma = L$ which is tangent to $\xi|_{T\Sigma}$ and then we get the relative Euler class $e(\xi, X_\Sigma) \in H^{2n+2}(M, L; \mathbf{Z})$ of ξ relative to the section X_Σ as the obstruction to extend X_Σ to a nonvanishing section to ξ over $(2n+2)$-dimensional skeleton of a cell decomposition of M relative to L. We define the *relative Euler number* $e_{\mathrm{rel}}(L, \Sigma)$ of L and Σ by $e_{\mathrm{rel}}(L, \Sigma) = -\langle e(\xi, X_\Sigma), [\Sigma, \partial\Sigma]\rangle$.

When the relative Euler number is independent of the choice of Seifert hypersurfaces, we denote it by $e_{\mathrm{rel}}(L)$. For example, when M is a $(2n+3)$-dimensional integral homology sphere and L is an oriented S^{2n+1}, the relative Euler number is independent of the choice of Seifert hypersurfaces since the Euler class $e(\xi) \in H^{2n+2}(M, \mathbf{Z}) \cong 0$ is trivial and the vector field X_Σ is unique up to homotopy among normal vector fields to L which are tangent to ξ. We note that X_Σ can be seen as a section of the conformally symplectic normal bundle of L which is trivial and whose total space is identified with a tubular neighborhood of L. Then the homotopical uniqueness of X_Σ follows from $H^1(L; \mathbf{Z}) \cong 0$ when $\dim L \geq 3$ and from the fact that the *Seifert framing* is determined uniquely up to homotopy when $\dim L = 1$.

Let Ω be a positive volume form on Σ. We define the characteristic foliation Σ_ξ on Σ by the flow line of the vector field Y defined by $\iota_Y \Omega = (\alpha \wedge (d\alpha)^n)|_{T\Sigma}$ (see [21] Definition 2.5.18 and Lemma 2.5.20). The line field $Y|_L$ with a suitable orientation can be used as the vector field X_Σ in the above definition. When singularities of the characteristic foliation Σ_ξ on Σ are isolated, the relative Euler number can be calculated as follows.

$$e_{\mathrm{rel}}(L, \Sigma) = -\sum_{q \in \mathrm{Sing}_+(\Sigma_\xi)} \mathrm{Ind}(q) + \sum_{q \in \mathrm{Sing}_-(\Sigma_\xi)} \mathrm{Ind}(q),$$

where $\mathrm{Sing}_{\pm}(\Sigma_\xi)$ denote the set of positive and negative singularities of Σ_ξ, respectively, and $\mathrm{Ind}(q)$ denotes the index of the isolated singularity q.

In the case where $\dim M = 3$ and $\dim L = 1$, the relative Euler number of a null-homologous positive transverse knot L and its Seifert surface Σ_L is equal to the self-linking number determined by L and Σ_L, that is, $e_{\mathrm{rel}}(L, \Sigma_L) = sl(L, \Sigma_L)$.

6.2. **Almost contact structures.** An *almost contact structure* on a $(2n+1)$-dimensional orientable manifold M is the reduction of the structure group $GL(2n+1, \mathbf{R})$ of the tangent bundle TM to $1 \times U(n)$. A cooriented contact structure on M induces a compatible almost contact structure which is unique up to homotopy. The existence of an almost contact structure on M is equivalent to the existence of a section of the $SO(2n+1)/U(n)$-bundle over M which is associated with TM. Two almost contact structures are homotopic if and only if the two associated sections are homotopic.

We set $F_{2n} = SO(2n)/U(n)$ and $F_{2n+1} = SO(2n+1)/U(n)$ and we note that the homogeneous space F_{2n+1} is diffeomorphic to F_{2n+2} [24, Corollary 3.1.3]. $\pi_{2n+1}(F_{2n+1})$ was calculated by Harris and Massey.

Proposition 6.2 (Harris [26], Massey [36] Lemma 1)**.**

$$\pi_{2n+1}(SO(2n+1)/U(n)) = \begin{cases} \mathbf{Z}/n!\mathbf{Z} & n \equiv 0 \pmod 4, \\ \mathbf{Z} & n \equiv 1 \pmod 4, \\ \mathbf{Z}/\frac{n!}{2}\mathbf{Z} & n \equiv 2 \pmod 4, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & n \equiv 3 \pmod 4. \end{cases}$$

6.3. **Overtwisted contact structures in higher dimensions.** Recently Borman, Eliashberg and Murphy [2] defined and classified overtwisted contact structures in higher dimensions and Casals, Murphy and Presas [6] gave several characterizations of overtwisted contact structures. An overtwisted contact structure is defined in [2] as a contact structure which contains a piecewise smooth codimension one disk with a certain germ of contact structure, which is not easy to describe here. We give one of characterizations in [6] as the definition of an overtwisted contact structure.

Definition 6.3 (Casals, Murphy and Presas [6] Theorem 1.1.)**.** Let (Y, ξ) be a contact manifold of dimension $2n-1 > 3$ and α_{ot} be a contact form for an overtwisted contact structure on $\mathbf{R}^3$. A contact manifold (Y, ξ) is overtwisted if and only if there is a contact embedding of $(\mathbf{R}^3 \times \mathbf{C}^{n-2}, \ker(\alpha_{\mathrm{ot}} + \frac{1}{2}\sum_{i=1}^{n-2} r_i^2 d\theta_i))$ in (Y, ξ), where (r_i, θ_i)'s, $i = 1, 2, \ldots n-2$, are polar coordinates.

Given a contact manifold (M, ξ_0) and a closed subset A in M, let $\mathfrak{Cont}_{\mathrm{ot}}(M; A, \xi_0)$ be the space of contact structures on M that are overtwisted on $M \backslash A$ and coincide with ξ_0 on an open neighborhood $\mathcal{O}pA$ of A and let $\mathfrak{cont}_{\mathrm{ot}}(M; A, \xi_0)$ be the space of almost contact structures on M that coincide with ξ_0 on $\mathcal{O}pA$.

Theorem 6.4 (Borman, Eliashberg and Murphy [2])**.** *The inclusion map*

$$j \colon \mathfrak{Cont}_{\mathrm{ot}}(M; A, \xi_0) \to \mathfrak{cont}_{\mathrm{ot}}(M; A, \xi_0)$$

induces an isomorphism

$$j_* \colon \pi_0(\mathfrak{Cont}_{\mathrm{ot}}(M; A, \xi_0)) \to \pi_0(\mathfrak{cont}_{\mathrm{ot}}(M; A, \xi_0)).$$

When $M = S^{2n+1}$ and $A = \emptyset$, since homotopy classes $\pi_0(\mathfrak{cont}_{\mathrm{ot}}(S^{2n+1}))$ of almost contact structures on S^{2n+1} are classified by the homotopy group $\pi_{2n+1}(F_{2n+1})$, this theorem implies that isotopy classes of overtwisted contact structures on S^{2n+1} are also classified by this group, see Proposition 6.2.

7. Proof of Theorem 1.9

In this section we construct a braided embedding and its smooth Seifert hypersurface. Then we prove Theorem 1.9 by looking at singularities of the characteristic foliation. Though the construction of a braided embedding was appeared in [17], a smooth Seifert hypersurface was not constructed there. The construction of a smooth Seifert hypersurface is based on the idea in [44]. Instead of taking a cyclic branched covering of the whole manifold as in [44], we take a branched covering of a tubular neighborhood of a submanifold and embed it again to the same tubular neighborhood. Then we can calculate indices of singularities of the characteristic foliation.

Let M be a $(2n+3)$-dimensional integral homology sphere and L be a $(2n+1)$-dimensional oriented sphere in M. Let K be a $(2n-1)$-dimensional oriented sphere in L. We take a small tubular neighborhood $N(L)$ of L of the form $L \times I^2_{\epsilon_L}$ with $\epsilon_L < 1$. Take a Seifert hypersurface Σ_K of K and a small product neighborhood $N(\Sigma_K)$ of Σ_K of the form $\Sigma_K \times I_{\epsilon_K}$ with $\epsilon_K < 1$. We take a small tubular neighborhood $N(K)$ of the form $K \times I^2_{\epsilon_K}$ so that $(K \times I^2_{\epsilon_K}) \cap \Sigma_K$ is given by $\{(x, x_K, y_K) \in K \times I^2_{\epsilon_K} | x_K \leq 0, y_K = 0\}$ and product structures are compatible.

7.1. **Construction of a copy of a contact submanifold and its Seifert hypersurface.**
In this subsection we construct a submanifold

L_s in $N(L)$ which is isotopic to L and intersects L transversely along K and also take the Seifert hypersurface of L_s in $N(L)$.

We take a positive real number ϵ with $5\epsilon < \epsilon_K$ and consider a smooth function $f = (f_{x_L}, f_{y_L})\colon L \to \mathbf{R}^2$, where $\mathbf{R}^2$ has coordinates (x_L, y_L), so that

- $f(L) \subset I_4^2$,
- For $(x, x_K, y_K) \in K \times I_{3\epsilon}^2$, $f(x, x_K, y_K) = (x_K/\epsilon, y_K/\epsilon)$,
- For $(x', y_K) \in \Sigma_K \times I_{2\epsilon} \backslash K \times I_{3\epsilon}^2$, $f(x', y_K) = (g(x')/\epsilon, y_K/\epsilon)$, where g is a nonincreasing smooth function on $\mathbf{R}_{\leq -3\epsilon}$ with $g(r) = r$ near $r = -3\epsilon$ and $g(r) = -7\epsilon/2$ for $r \leq -4\epsilon$,
- For $x'' \in L\backslash(K \times I_{3\epsilon}^2 \cup \Sigma_K \times I_{2\epsilon})$, $f(x'') \cap f(K \times I_\epsilon^2 \cup \Sigma_K \times I_\epsilon) = \emptyset$.

Take a number s such that $0 < 5s < \epsilon_L$. Then we set $L_s \subset L \times I_{\epsilon_L}^2$ by

$$L_s = \{(x, x_L, y_L) \in L \times I_{5s}^2 | x_L = s f_{x_L}(x), y_L = s f_{y_L}(x)\}$$

and $S_{L_s} \subset L \times I_{\epsilon_L}^2$ by

$$\begin{aligned} S_{L_s} =& \{(x, x_L, y_L) \in L \times I_{5s}^2 | x_L \geq s f_{x_L}(x), y_L = s f_{y_L}(x)\} \\ =& \{(x, z_L) \in L \times I_{5s}^2 | \arg(z_L - s f(x)) = 0\}, \end{aligned}$$

where $z_L = x_L + i y_L \in \mathbf{C}$ and we regard f as a $\mathbf{C}$-valued function. L_s and S_{L_s} have the following properties.

(1) L_s is isotopic to $L \times \{(0,0)\}$ and S_{L_s} is diffeomorphic to $L \times I$.
(2) S_{L_s} is transverse to $\{x\} \times \mathbf{R}^2$ for any $x \in L$.
(3) $S_{L_s} \cap (L \times \{(0,0)\}) = \Sigma_K \times \{(0,0)\}$ because $x \in L$ satisfies $f_{x_L}(x) \leq 0$ and $f_{y_L}(x) = 0$ if and only if $x \in \Sigma_K$.
(4) $L_s \cap (L \times I_s^2) = \{(x, x_K, y_K, x_L, y_L) \in K \times I_\epsilon^2 \times I_s^2 | x_K = \epsilon x_L/s, y_K = \epsilon y_L/s\}$.
(5) $S_{L_s} \cap (L \times I_s^2) = \{(x, x_K, y_K, x_L, y_L) \in K \times I_\epsilon^2 \times I_s^2 | x_K \leq \epsilon x_L/s, y_K = \epsilon y_L/s\} \cup \{(x', y_K, x_L, y_L) \in (\Sigma_K \times I_\epsilon \backslash K \times I_\epsilon^2) \times I_s^2 | y_K = \epsilon y_L/s\}$.

We orient S_{L_s} such that its orientation induces the given orientation of L. We set $\partial^e S_{L_s} = -(\partial S_{L_s} \backslash L_s)$, where the orientation of $\partial^e S_{L_s}$ is opposite to that induced as the boundary of S_{L_s}. The following lemma follows from the properties (1) and (2) of L_s and S_{L_s}.

Lemma 7.1. *Let $\xi = \ker \alpha_L$ be a positive contact structure on L. For sufficiently small s, L_s and $\partial^e S_{L_s}$ are positive contact submanifolds of $(L \times \mathbf{R}^2, \ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L)))$ which are isotopic to $L \times \{(0,0)\}$ as contact submanifolds.*

7.2. **Construction of a braided embedding and its Seifert hypersurface.** Let k be a positive integer. Let $\pi_k\colon L \times \mathbf{R}^2 \to L \times \mathbf{R}^2$ be a smooth map given by

$$\pi_k(x, x'_L, y'_L) = (x, \mathrm{Re}((x'_L + iy'_L)^k), \mathrm{Im}((x'_L + iy'_L)^k)),$$

where $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ denote the real part and the imaginary part of a complex number w, respectively. For a real number c with $0 < c < 1$, let $m_c\colon L \times \mathbf{R}^2 \to L \times \mathbf{R}^2$ be a diffeomorphism given by

$$m_c(x, x'_L, y'_L) = (x, c \cdot x'_L, c \cdot y'_L).$$

In this subsection we take t so that $t(\sqrt{2}\epsilon_L)^{1/k} < \epsilon_L$ and we consider $m_t(\pi_k^{-1}(S_{L_s})) \subset L \times I^2_{\epsilon_L}$.

By the properties (4) and (5) of L_s and S_{L_s}, the following holds.

Lemma 7.2. (1) *$m_t(\pi_k^{-1}(S_{L_s}))$ is a smooth submanifold of $L \times I^2_{\epsilon_L}$ which is diffeomorphic to the k-fold cyclic covering of S_{L_s} branched along $\Sigma_K \times \{(0,0)\}$ and has $k+1$ boundary components.*

(2) *$m_t(\pi_k^{-1}(L_s))$ is diffeomorphic to $L_{K,k}$ and $\mathrm{pr}_1|_{m_t(\pi_k^{-1}(L_s))}$ is a k-fold cyclic covering branched along K, where $\mathrm{pr}_1\colon L \times I^2_{\epsilon_L} \to L$ is the projection to the first factor.*

(3) *$m_t(\pi_k^{-1}(\partial^e S_{L_s}))$ is diffeomorphic to a disjoint union of k-copies of L.*

Moreover, we have the following by Lemma 4.3.

Lemma 7.3. *Let $\xi = \ker \alpha_L$ be a positive contact structure on L and K be a positive contact submanifold of (L, ξ). We take sufficiently small s so that Lemma 7.1 holds. Then for sufficiently small t the followings hold.*

(1) *$m_t(\pi_k^{-1}(L_s))$ is a positive contact submanifold of $(L \times I^2_{\epsilon_L}, \ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L)))$ which is contactomorphic to $(L_{K,k}, (\xi|_{TL})_{K,k})$.*

(2) *$m_t(\pi_k^{-1}(\partial^e S_{L_s}))$ is a positive contact submanifold of $(L \times I^2_{\epsilon_L}, \ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L)))$ each component of which is isotopic to $L \times \{(0,0)\}$ through contact submanifolds.*

To construct a smooth Seifert hypersurface of $m_t(\pi_k^{-1}(L_s))$ we glue a codimension one oriented smooth submanifold in $M \setminus (m_t(\pi_k^{-1}(L \times I^2_{5\epsilon})))$ which has no closed components and whose boundary is $m_t(\pi_k^{-1}(\partial^e S_{L_s}))$. Moreover, if we choose a Seifert hypersurface Σ_L of L then we can take the above glued submanifold so that it is diffeomorphic to the disjoint union of k copies of Σ_L. In this case the resulting Seifert hypersurface is diffeomorphic to the k-fold cyclic covering of Σ_L branched along a push-off Σ'_K of Σ_K into Σ_L relative to its boundary corresponding to the $\mathbf{Z}/k\mathbf{Z}$ reduction of the element in $H^1(\Sigma_L \setminus \Sigma'_K; \mathbf{Z})$ dual to the meridian circle of Σ'_K.

The following two observations are immediate but essential for our construction.

Lemma 7.4. *For $p \in S_{L_s} \subset L \times I^2_{\epsilon_L}$, then $|\mathrm{pr}_2 \circ m_t(\pi_k^{-1}(p))| = t|\mathrm{pr}_2(p)|^{1/k}$, where $\mathrm{pr}_2 \colon L \times I^2_{\epsilon_L} \to I^2_{\epsilon_L}$ is the projection to the second factor and $|\cdot|$ denotes the standard norm on $I^2_{\epsilon_L}$.*

Lemma 7.5. *We assume that $k \geq 2$ and $0 < t < 1$. Then $tr^{1/k} < r$ for $0 < t^{k/(k-1)} < r \leq 1$ and $tr^{1/k} > r$ for $0 < r < t^{k/(k-1)} < t$.*

By these two lemmas we have the following.

Lemma 7.6. *If we assume that $k \geq 2$ and $t < s < s^{(k-1)/k} \leq 1$, then, for any $p \in S_{L_s} \subset L \times I^2_{\epsilon_L}$ with $|\mathrm{pr}_2(p)| \geq s$, $|\mathrm{pr}_2 \circ m_t(\pi_k^{-1}(p))| < |\mathrm{pr}_2(p)|$.*

By combining Lemma 7.6 with the property (2) of S_{L_s}, we have the following.

Lemma 7.7. *Let $\xi = \ker \alpha_L$ be a positive contact structure on L. For sufficiently small t, $m_t(\pi_k^{-1}(S_{L_s}))$ is transverse to $\ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L))$ on $|(x_L, y_L)| \geq ts^{1/k}$.*

By the property (5) of S_{L_s},

$$S_{L_s} \cap (L \times D^2_s) = (S_{L_s} \cap (K \times I^2_\epsilon \times D^2_s)) \cup (S_{L_s} \cap ((\Sigma_K \times I_\epsilon \backslash K \times I^2_\epsilon) \times D^2_s))$$

and we denote this by $A \cup B$: $A = (S_{L_s} \cap (K \times I^2_\epsilon \times D^2_s))$ and $B = (S_{L_s} \cap ((\Sigma_K \times I_\epsilon \backslash K \times I^2_\epsilon) \times D^2_s))$. By $m_t \circ \pi_k^{-1}$, $A \cup B$ is sent into $L \times D^2_{ts^{1/k}}$. We have the following.

Lemma 7.8. *Let $\xi = \ker \alpha_L$ be a positive contact structure on L and K be a positive contact submanifold of (L, ξ). Assume that ξ is given by $\ker(\alpha_K + \frac{1}{2}(x_K dy_K - y_K dx_K))$ on $K \times I^2_{\epsilon_K}$ for some contact form α_K for $\xi|_{TK}$. Then for sufficiently small t, the followings hold.*

(1) *$m_t(\pi_k^{-1}(A))$ is transverse to $\ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L))$.*

(2) *$m_t(\pi_k^{-1}(B))$ is given by*

$$\{(x', y_K, x_L, y_L) \in (\Sigma_K \times I_\epsilon \backslash K \times I^2_\epsilon) \times D^2_{ts^{1/k}})| \\ y_K = \epsilon/s \cdot \mathrm{Im}(t^{-k}(x_L + iy_L)^k)\}.$$

Proof. (1) A direct calculation shows that

$$\begin{aligned} &m_t(\pi_k^{-1}(A)) \\ =&\{(x, x_K, y_K, x_L, y_L) \in K \times I^2_\epsilon \times D^2_{ts^{1/k}}| \\ &\mathrm{Re}((t^{-1}(x_L + iy_L))^k) \geq s x_K/\epsilon, \mathrm{Im}((t^{-1}(x_L + iy_L))^k) = s y_K/\epsilon\}. \end{aligned}$$

Since $K \times \{pt\} \subset K \times I^2_\epsilon \subset K \times I^2_{\epsilon_K}$ is a contact submanifold of (L, ξ), the statement follows.

(2) It follows from the direct calculation as above. $\square$

7.3. **The characteristic foliation on a Seifert hypersurface of a braided embedding.** Now we calculate the indices of isolated singularities of the characteristic foliation on the surface in $\mathbf{R}^3$ given by a graph of a function on $\mathbf{R}^2$. We use this calculation to prove Theorem 1.9.

Proposition 7.9. *Let k be a positive integer. We consider the positive contact form $\alpha_+ = dy_K + \frac{1}{2}(x_L dy_L - y_L dx_L)$ and the negative contact form $\alpha_- = -dy_K + \frac{1}{2}(x_L dy_L - y_L dx_L)$ on $\mathbf{R}^3$ in the coordinates (y_K, x_L, y_L). For $C > 1$ and $0 < t < 1$ consider an oriented surface Σ given by the graph $(g(x_L, y_L), x_L, y_L)$ of the function $g(x_L, y_L) = C \cdot \mathrm{Im}(t^{-k}(x_L + iy_L)^k)$ on $\mathbf{R}^2$ whose orientation is compatible, through the projection $\mathbf{R}^3 \to \mathbf{R}^2$ to the last two coordinates, with the standard orientation of $\mathbf{R}^2$ given by the ordered basis $(\partial_{x_L}, \partial_{y_L})$. Then the followings hold.*

1. *When $k = 2$, $\ker \alpha_+$ (resp. $\ker \alpha_-$) defines the characteristic foliation on Σ with one positive (resp. negative) hyperbolic singularity at $(0, 0, 0)$.*
2. *When $k \neq 2$, $\ker \alpha_+$ defines the characteristic foliation on Σ with one positive elliptic singularity at $(0, 0, 0)$ and k positive hyperbolic singularities at*
$$\Sigma \cap \{(r_L, \theta_L) = ((t^k/(2kC))^{1/(k-2)}, (2j + 1)\pi/k) | j = 0, 1, \ldots, k - 1\}.$$
3. *When $k \neq 2$, $\ker \alpha_-$ defines the characteristic foliation on Σ with one negative elliptic singularity at $(0, 0, 0)$ and k negative hyperbolic singularities at*
$$\Sigma \cap \{(r_L, \theta_L) = ((t^k/(2kC))^{1/(k-2)}, 2j\pi/k) | j = 0, 1, \ldots, k - 1\}.$$

Proof. Consider the polar coordinates (r_L, θ_L) corresponding to the product coordinates (x_L, y_L). Since
$$\pm g^*(\alpha_\pm) = (Ct^{-k} k r_L^{k-1} \sin k\theta_L) r_L dr_L + (Ct^{-k} k r_L^{k-1} \cos k\theta_L \pm \frac{1}{2} r_L) r_L d\theta_L,$$
the vector fields $X_\pm$ given by $\iota_{X_\pm}(r_L dr_L \wedge d\theta_L) = \pm g^* \alpha_\pm$ which integrate to the characteristic foliations can be written as
$$X_\pm = (Ct^{-k} k r_L^{k-1} \cos k\theta_L \pm \frac{1}{2} r_L) \partial_{r_L} - (Ct^{-k} k r_L^{k-2} \sin k\theta_L) \partial_{\theta_L}.$$
Thus the statement follows. $\square$

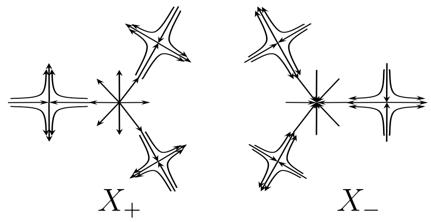

FIGURE 2. Schematic pictures for the characteristic foliations given by X_+ and X_- for $k = 3$.

Proof of Theorem 1.9. In the case of $k = 1$ the statement is clear and hence we assume that $k \geq 2$.

We take a neighborhood of $(L, \xi|_{TL})$ in (M, ξ) of the form $(L \times I^2_{\epsilon_L}, \ker(\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L)))$, where α_L is a contact form for $\xi|_{TL}$ and $\epsilon_L < 1$. We take a Seifert hypersurface Σ_K and a product neighborhood $\Sigma_K \times I_{\epsilon_K}$ of Σ_K with $\epsilon_K < 1$. We take a neighborhood of K in L of the form $K \times I^2_{\epsilon_K}$ such that $(K \times I^2_{\epsilon_K}) \cap \Sigma_K = \{x_K \leq 0, y_K = 0\}$. We may assume that the neighborhood of $(K, \xi|_{TK})$ is of the form $(K \times I^2_{\epsilon_K}, \ker(\alpha_K + \frac{1}{2}(x_K dy_K - y_K dx_K)))$. We may also assume that the characteristic foliation $(\Sigma_K)_\xi$ on Σ_K has isolated singularities and that for each small neighborhood $N(q)$ of the singularity q of $(\Sigma_K)_\xi$, $\ker(\alpha_L|_{T(N(q) \times I_{\epsilon_K})})$ is invariant in the I_{ϵ_K}-direction since $N(q) \times I_{\epsilon_K}$ is a neighborhood of a transverse curve. By reparametrizing y_K near each q and taking smaller ϵ_K if necessary, we may assume that $\alpha_L|_{\{q\} \times I_{\epsilon_K}} = \pm dy_K$. Moreover, by taking smaller ϵ_K if necessary, we may assume that outside of this neighborhood $\xi|_{TL}$ is transverse to $\Sigma_K \times \{y_K\}$ for any $y_K \in I_{\epsilon_K}$.

We perform the above construction in subsections 7.1 and 7.2 by taking ϵ, s and t as follows: First we take small ϵ with $5\epsilon < \epsilon_K$. Second, we take sufficiently small s so that $s < \epsilon$ and Lemmas 7.1 holds. Third, we take sufficiently small t so that Lemma 7.3, 7.7 and 7.8 hold. Moreover, when $k \geq 3$, we take t so that $(t^k/(2k\epsilon/s))^{1/(k-2)} < ts^{1/k}$ holds. Then $m_t(\pi_k^{-1}(L_s))$ is a contact submanifold which is isotopic to $(L_{K,k}, (\xi|_{TL})_{K,k})$ by Lemma 7.3 (1) and we define j so that its image is $m_t(\pi_k^{-1}(L_s))$. By Lemma 7.3 (2) and the statement after it, we can take a Seifert hypersurface which is diffeomorphic to a manifold obtained by attaching k-copies of Σ_L along the boundary of $\Sigma = m_t(\pi_k^{-1}(S_{L_s}))$. By Lemma 7.2 (1) it is diffeomorphic to the branched covering of Σ_L branched along a push-off of Σ_K into Σ_L. We denote this Seifert hypersurface $\Sigma_{j(L_{K,k})}$. By perturbing $\Sigma_{j(L_{K,k})}$ relative to Σ, we may assume that the characteristic foliation $(\Sigma_{j(L_{K,k})})_\xi$ of $\Sigma_{j(L_{K,k})}$ has isolated singularities and the characteristic foliation is given by the integral curves of a vector field X which directs outward along $j(L_{K,k}) = \partial\Sigma_{j(L_{K,k})}$.

Since the contact structure on $L \times I^2_{\epsilon_L}$ is given by $\alpha_L + \frac{1}{2}(x_L dy_L - y_L dx_L)$, if the contact plane at $(x, x_L, y_L) \in L \times I^2_{\epsilon_L}$ is tangent to $T_{(x,x_L,y_L)}\Sigma$, then $\ker \alpha_L$ has to be tangent to $T_{(x,x_L,y_L)}\Sigma \cap T_{(x,x_L,y_L)}(L \times \{(x_L, y_L)\})$. Thus by Lemmas 7.7 and 7.8, the singularities of Σ_ξ are on

$$\begin{aligned}&\{(x', y_K, x_L, y_L) \in (\Sigma_K \times I_\epsilon \backslash K \times I^2_\epsilon) \times D^2_{ts^{1/k}})| \\ &\quad x' \in \mathrm{Sing}((\Sigma_K)_\xi), y_K = \epsilon/s \cdot \mathrm{Im}(t^{-k}(x_L + iy_L)^k)\}.\end{aligned}$$

By using the calculation of Proposition 7.9 for $C = \epsilon/s$, we can calculate the relative Euler number of $j(L_{K,k})$ and $\Sigma_{j(L_{K,k})}$ as follows.

$$\begin{aligned}
&e_{\mathrm{rel}}(j(L_{K,k}), \Sigma_{j(L_{K,k})}) \\
=&-\left\langle e(\xi, X|_{\partial\Sigma_{j(L_{K,k})}}), [\Sigma_{j(L_{K,k})}, \partial\Sigma_{j(L_{K,k})}]\right\rangle \\
=&k \cdot e_{\mathrm{rel}}(L) - \{\sum_{q\in \mathrm{Sing}_+(\Sigma_\xi)} \mathrm{Ind}(q) - \sum_{q\in \mathrm{Sing}_-(\Sigma_\xi)} \mathrm{Ind}(q)\} \\
=&k \cdot e_{\mathrm{rel}}(L) \\
&- \{(1-k) \sum_{q\in \mathrm{Sing}_+((\Sigma_K)_\xi)} \mathrm{Ind}(q) - (1-k) \sum_{q\in \mathrm{Sing}_-((\Sigma_K)_\xi)} \mathrm{Ind}(q)\} \\
=&k \cdot e_{\mathrm{rel}}(L) - (k-1) \cdot e_{\mathrm{rel}}(K).
\end{aligned}$$

Since any Seifert hypersurface Σ_K of K can be isotoped relative to K so that Σ_K satisfies the above assumption, by construction of $\Sigma_{j(L_{K,k})}$ the latter statement follows. □

8. The relative Euler numbers of smooth unknots in the standard spheres

Let L be a codimension two positive contact submanifold of (M, ξ) and Σ be a smooth Seifert hypersurface of L. Recall that when the singularities of the characteristic foliation Σ_ξ on Σ are isolated, the relative Euler number and the Euler characteristic $\chi(\Sigma)$ of Σ can be calculated as follows.

$$e_{\mathrm{rel}}(L, \Sigma) = -\sum_{q\in \mathrm{Sing}_+(\Sigma_\xi)} \mathrm{Ind}(q) + \sum_{q\in \mathrm{Sing}_-(\Sigma_\xi)} \mathrm{Ind}(q),$$

$$\chi(\Sigma) = \sum_{q\in \mathrm{Sing}_+(\Sigma_\xi)} \mathrm{Ind}(q) + \sum_{q\in \mathrm{Sing}_-(\Sigma_\xi)} \mathrm{Ind}(q).$$

Thus the sum of the relative Euler number $e_{\mathrm{rel}}(L, \Sigma)$ of a contact submanifold L and its Seifert hypersurface Σ and the Euler characteristic $\chi(\Sigma)$ of the same Seifert hypersurface Σ is an even number. In particular, if the smooth knot type of a contact submanifold is the unknot in a sphere, then the relative Euler number becomes an odd number which is independent of the choice of Seifert hypersurfaces.

8.1. **Contact 3-spheres in the standard S^5.** By the obstruction theory, we can observe the following.

Theorem 8.1 ([17]). *Let $j_i\colon (S^3, \xi) \to (S^5, \xi_{\mathrm{ot}})$, $i = 1, 2$, be two contact embeddings of a contact structure ξ on S^3 in the overtwisted contact structure ξ_{ot} on S^5 such that the contact structure on the complements of their images are overtwisted. If j_1 is isotopic to j_2 as smooth embeddings, then there is a contactomorphism $\phi\colon (S^5, \xi_{\mathrm{ot}}) \to (S^5, \xi_{\mathrm{ot}})$ such that $j_2 = \phi \circ j_1$.*

We have the following proposition by this theorem.

Proposition 8.2. *Let η be a positive contact structure on S^5. Let $j_i\colon (S^3, \xi) \to (S^5, \eta)$, $i = 1, 2$, be two contact embeddings such that they are isotopic as smooth embeddings. Then $e_{\mathrm{rel}}(j_1(S^3)) = e_{\mathrm{rel}}(j_2(S^3))$.*

Proof. Since j_1 and j_2 are codimension two embeddings of a closed manifold, the trace of an isotopy between j_1 and j_2 has measure zero in S^5 and thus it avoids a neighborhood of some point $p \in S^5$. By taking the connected sum at p with the overtwisted S^5, the resulting embeddings j_i''s in the overtwisted S^5 are isotopic as smooth embeddings. By Theorem 8.1, there exists a self-contactomorphism ϕ of the overtwisted S^5 so that $j_2' = \phi \circ j_1'$. Thus $e_{\mathrm{rel}}(j_1'(S^3)) = e_{\mathrm{rel}}(j_2'(S^3))$ holds. We can calculate $e_{\mathrm{rel}}(j_i'(S^3))$'s by using Seifert hypersurfaces contained in $S^5 \backslash \{p\}$ before taking the connected sum and thus $e_{\mathrm{rel}}(j_1(S^3)) = e_{\mathrm{rel}}(j_1'(S^3)) = e_{\mathrm{rel}}(j_2'(S^3)) = e_{\mathrm{rel}}(j_2(S^3))$. $\square$

Now we describe the relative Euler numbers of contact embeddings of contact 3-spheres in $(S^5, \xi_{\mathrm{std}})$ which are isotopic to the standard embedding in terms of their d_3-invariants.

Theorem 8.3. *Any contact structure on S^3 has a contact embedding in $(S^5, \xi_{\mathrm{std}})$ so that it is isotopic to the standard embedding. Moreover, for any contact embedding $j\colon (S^3, \xi) \to (S^5, \xi_{\mathrm{std}})$ which is isotopic to the standard embedding, $e_{\mathrm{rel}}(j(S^3)) = 2d_3(\xi)$ holds.*

Remark 8.4. We note that the first statement has already been shown in [17].

Proof of Theorem 8.3. We note that there is a unique tight contact structure on S^3 up to isotopy and there is a unique overtwisted contact structure on S^3 with a given d_3-invariant up to isotopy. Thus by Theorem 8.1 and Gray's stability theorem, it is enough to show that for any contact structure ξ on S^3 there is a contact embedding of (S^3, ξ) in $(S^5, \xi_{\mathrm{std}})$ which is isotopic to the standard embedding and whose relative Euler number is given by $2d_3(\xi)$. For ξ_{std}, since the standard sphere bounds a 4-ball whose characteristic foliation has one positive elliptic singularity, the statement follows. For any overtwisted ξ_m with $m \geq 1$, Theorem 1.9 and Lemma 5.1 (1) imply that, by considering a braided embedding about $(S^3, \xi_{\mathrm{std}})$ defined by the double covering branched along the unknot U_{-m} with $sl(U_{-m}) = -2m - 1$, there is a contact embedding of (S^3, ξ_m) whose relative Euler number is equal to $-2 - (-2m - 1) = 2m - 1 = 2d_3(\xi_m)$. Moreover, this embedding can be taken so that it is isotopic to the standard embedding. Next we take an embedding of (S^3, ξ_1) obtained, for example, in the previous step, and then for any $m' \geq 2$ we consider a braided embedding about this contact submanifold given by the double covering branched along the transverse unknot $U_{m'}$ with $sl(U_{m'}) = 2m' - 1$. Then by Theorem 1.9 and Lemma 5.1 (2), for any $2 - m' \leq 0$ we obtain a contact embedding of $(S^3, \xi_{2-m'})$ whose relative Euler number is equal to $2 - (2m' - 1) = 3 - 2m' = 2d_3(\xi_{2-m'})$ and which is isotopic to the standard embedding. $\square$

8.2. **Connected sum of codimension two contact spheres.** Given two unlinked $(2n+1)$-dimensional positive contact spheres in a $(2n+3)$-dimensional positive contact sphere, by embedding a part of $(2n+2)$-dimensional symplectic 1-handle we can get a contact sphere which is smoothly the connected sum of given two spheres, which is contactomorphic to the connected sum of given two contact spheres and whose relative Euler number is calculated in the natural way. It can be considered as a higher dimensional generalization of the connected sum of two transverse knots by using a positive band, see [18] for transverse knots case where the connected sum operation is defined by using the front projection.

Proposition 8.5. *Let S_1 and S_2 be two unlinked $(2n+1)$-dimensional positive contact spheres in a positive contact sphere (S^{2n+3}, ξ). Then there exists a contact submanifold S_3 in (S^{2n+3}, ξ) such that S_3 is the connected sum of S_1 and S_2 as an oriented submanifold, $(S_3, \xi|_{TS_3})$ is contactomorphic to the connected sum of $(S_1, \xi|_{TS_1})$ and $(S_2, \xi|_{TS_2})$, and $e_{\mathrm{rel}}(S_3) = e_{\mathrm{rel}}(S_1) + e_{\mathrm{rel}}(S_2) + 1$.*

Remark 8.6. Since in $(S^3, \xi_{\mathrm{std}})$ there exists a transverse unknot whose self-linking number is any negative odd number, we can take a transverse knot S_3 which is a connected sum of S_1 and S_2 as a smooth knot and has $sl(S_3) = sl(S_1) + sl(S_2) - (2k-1)$ for any nonnegative integer k.

A local model of the connected sum is described in the following way. We consider $(\mathbf{R}^{2n+3}, \ker\alpha)$ with $\alpha = dz - y_1 dx_1 + \frac{1}{2}\sum_{i=2}^{n+1} r_i^2 d\theta_i$, where $(x_1, y_1, z, r_2, \theta_2, \ldots, r_{n+1}, \theta_{n+1})$ are coordinates on $\mathbf{R}^{2n+3}$ and (r_i, θ_i)'s are polar coordinates. Consider an embedding $j_{l,\epsilon}\colon I_l \times I_\epsilon^{2n+1} \to \mathbf{R}^{2n+3}$ for positive real numbers l and ϵ given by

$$\begin{aligned} &j_{l,\epsilon}(s, t, r_2, \theta_2, \ldots, r_{n+1}, \theta_{n+1}) \\ &= \left(s, t\cos\left(\frac{\pi}{2l}s\right), -t\sin\left(\frac{\pi}{2l}s\right), r_2, \theta_2, \ldots, r_{n+1}, \theta_{n+1}\right). \end{aligned}$$

Then

$$d(j^*\alpha) = \cos(\pi s/2l) ds \wedge dt + \sum_{i=2}^{n+1} r_i dr_i \wedge d\theta_i,$$

$$(d(j^*\alpha))^{n+1} = (n+1)!\cos(\pi s/2l) ds \wedge dt \wedge \prod_{i=2}^{n+1} r_i dr_i \wedge d\theta_i.$$

If we set $\Omega = (d(j^*\alpha))^{n+1}$, then it is a volume form for $I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\}$ and hence $d(j^*\alpha)$ is a symplectic form for $I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\}$. We orient $I_l \times I_\epsilon^{2n+1}$ by Ω. Now consider the characteristic foliation on $I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\}$ as the integral curve of the vector field Y defined by $\iota_Y \Omega = j^*(\alpha \wedge (d\alpha)^n)$;

$$Y = -\frac{1}{n+1}\tan\left(\frac{\pi}{2l}s\right)\partial_s + \frac{1}{n+1}\left(\frac{\pi}{2l} + 1\right) t\partial_t + \frac{1}{2(n+1)}\sum_{i=2}^{n+1} r_i \partial_{r_i}.$$

By definition, Y is a Liouville vector field on $(I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\}, \Omega)$ and Y has a unique singular point of index -1 at the origin. Now consider contact submanifolds $j_{l,\epsilon}(\{\pm l\} \times I_\epsilon^{2n+1})$. Let $A_\pm$ be $\{\pm l\} \times I_\epsilon^{2n+1}$, respectively. Consider two submanifolds $A_{f_\pm}$ of $I_l \times I_\epsilon^{2n+1}$ of the form $\{(f_\pm(R), t, r_2, \theta_2, \ldots, r_{n+1}, \theta_{n+1})\}$, where $R = \sqrt{t^2 + \sum_{i=2}^{n+1} r_i^2}$, for some smooth functions $f_\pm$ such that

- $f_+(R)$ is a nondecreasing function and $f_-(R)$ is a nonincreasing function,
- $f_+(R) > 0$ and $f_-(R) < 0$,
- near $\partial I_\epsilon^{2n+1}$, $f_+(R) = l$ and $f_-(R) = -l$,
- on $\{R/\epsilon \leq 1/4\}$, $f_\pm/l = \pm\sqrt{(R/\epsilon)^2 + 1/16}$, respectively,
- $f_\pm(I_\epsilon^{2n+1}) \cap \mathrm{int}(I_l \times I_\epsilon^{2n+1})$ are transverse to Y.

Then $j_{l,\epsilon}(A_{f_+})$ (resp. $j_{l,\epsilon}(A_{f_-})$) is contact isotopic to $j_{l,\epsilon}(A_+)$ (resp. to $j_{l,\epsilon}(A_-)$) relative to its boundary. We consider a submanifold B of $I_l \times I_\epsilon^{2n+1}$ such that

- B is obtained by connecting $(A_{f_+} \cup A_{f_-}) \backslash (I_{\sqrt{2}l/4} \times I_{\epsilon/4})$ smoothly by a tube $I \times S^{2n}$ in $I_{\sqrt{2}l/4} \times I_{\epsilon/4}$,
- each slice $B \cap \{s = c\}$, $c \in I_{\sqrt{2}l/4}$, is a sphere centered at the origin on $\{s = c\}$,
- $B \cap \mathrm{int}(I_l \times I_\epsilon^{2n+1})$ is transverse to Y.

Since $(I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\}, \Omega)$ can be considered as a subset containing the origin of $\mathbf{R}^{2n+2}$ with the standard symplectic structure, $j_{l,\epsilon}(B)$ is a positive contact submanifold obtained by taking the connected sum of $j_{l,\epsilon}(A_+)$ and $j_{l,\epsilon}(A_-)$ relative to their boundaries.

Proof of Proposition 8.5. Given S_1 and S_2, take a disjoint balls B_1 and B_2 so that $S_1 \subset B_1$ and $S_2 \subset B_2$. Moreover, we take Seifert hypersurfaces Σ_{S_1} of S_1 in B_1 and Σ_{S_2} of S_2 in B_2. We take an isotropic arc γ connecting S_1 and S_2 such that $\gamma \cap (\Sigma_{S_1} \cup \Sigma_{S_2}) = \partial\gamma$. Indeed we can take such an isotropic arc by Gromov's h-principle for isotropic submanifolds. By the neighborhood theorem of codimension two contact submanifolds and isotropic submanifolds, there is a neighborhood $N(\gamma)$ of γ which is orientation preserving contactomorphic to $I_{l+\epsilon'} \times I_\epsilon^{2n+2} \subset (\mathbf{R}^{2n+3}, \ker\alpha)$ for some positive real numbers l, ϵ and ϵ' such that γ is sent to $I_l \times \{(0, 0, \ldots, 0)\}$, $S_1 \cap N(\gamma)$ (resp. $S_2 \cap N(\gamma)$) is sent to $j_{l,\epsilon}(A_-)$ (resp. $j_{l,\epsilon}(A_+)$) and $\Sigma_{S_1} \cap N(\gamma)$ (resp. $\Sigma_{S_2} \cap N(\gamma)$) is sent to $[-l - \epsilon', -l] \times \{0\} \times I_\epsilon^{2n+1}$ (resp. $[l, l + \epsilon'] \times \{0\} \times I_\epsilon^{2n+1}$). If we do a surgery as above, we obtain a new contact submanifold $S_3 = (S_1 \backslash j_{l,\epsilon}(A_-)) \cup (S_2 \backslash j_{l,\epsilon}(A_+)) \cup j_{l,\epsilon}(B)$ of (S^{2n+3}, ξ) which is contactomorphic to the connected sum of $(S_1, \xi|_{TS_1})$ and $(S_2, \xi|_{TS_2})$ and is the connected sum of S_1 and S_2 as an oriented submanifold of S^{2n+3}. We can take a Seifert hypersurface of S_3 which is diffeomorphic to a boundary connected sum of Σ_{S_1} and Σ_{S_2}. The statement for the relative Euler number follows from the fact that $\ker\alpha$ positively tangent to $j_{l,\epsilon}(I_l \times I_\epsilon^{2n+1} \backslash \{s = \pm l\})$ at the origin and Y has a unique isolated singularity of index -1 at the origin. □

8.3. **Contact S^{2n+1}'s in the standard S^{2n+3}.** The relative Euler number of a contact embedding of a contact 3-sphere in $(S^5, \xi_{\mathrm{std}})$ is determined by the isotopy class of a smooth embedding and the contact structure on the domain, however, in some other dimensions, for example, the case of the self-linking number of a transverse knot in a contact 3-manifold, it is not determined by these data.

In the light of the classification of overtwisted contact structures given by Theorem 6.4 and Lemma 6.2, we know that there exist only finitely many overtwisted contact structures on S^{2n+1} up to isotopy when n is an even number.

Proposition 8.7. *For any integer n with $n \geq 1$, there exists an overtwisted S^{2n+1} in S^{2n+3} which is ambient isotopic to the standard sphere.*

This is a corollary of the following theorem in [6].

Theorem 8.8 (Casals, Murphy and Presas [6] Theorem 4.1)**.** *Let (M^{2n+3}, ξ) be a contact manifold with $n \geq 1$ and $(L^{2n+1}, \xi|_{TL})$ a codimension two overtwisted contact submanifold. A k-fold contact cyclic covering of (M, ξ) branched along $(L, \xi|_{TL})$ is overtwisted for k large enough.*

Proof of Proposition 8.7. By Theorem 1.9, for any positive integer k, the k-fold contact cyclic branched covering of a contact $(S^{2n+1}, \ker\alpha)$ along a contact S^{2n-1} which is ambient isotopic the standard S^{2n-1} can be embedded in $(S^{2n+1} \times D^2_\epsilon, \ker(\alpha + \frac{1}{2}r^2 d\theta))$ for any sufficiently small ϵ. Thus if there exists an overtwisted S^{2n-1} in a contact S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$ which is ambient isotopic to the standard S^{2n-1}, then there exists an overtwisted S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$ by Theorem 8.8. Moreover, we can construct an overtwisted S^{2n+1} which is ambient isotopic to the standard S^{2n+1}. Since by Theorem 8.3 any overtwisted S^3 can be embedded in (S^5, ξ_{std}) so that it is ambient isotopic to the standard S^3, by the induction on dimensions, the statement follows. $\square$

Proof of Theorem 1.10. Let (M, ξ) be $(S^{2n+3}, \xi_{\text{std}})$ and L be the standard S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$. If there is a contact S^{2n-1} denoted by K in L which is ambient isotopic to the standard S^{2n-1} among smooth submanifolds and whose relative Euler number is equal to $-2m-3$, then by applying Theorem 1.9 with $k = 2$ to the above M, L and K, there exists a contact S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$ which is ambient isotopic to the standard S^{2n+1} and whose relative Euler number is equal to $-2-(-2m-3) = 2m+1$. Theorem 8.3, by induction on dimensions, implies that for any positive integer n and any integer m, there exists a contact S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$ which is ambient isotopic to the standard S^{2n+1} and whose relative Euler number is equals to $2m+1$. Since Proposition 8.7 implies that there exists at least one overtwisted S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$, we take one of these overtwisted S^{2n+1}. By Proposition 8.5 and taking the connected sum with this overtwisted S^{2n+1}, we can see that, for any positive integer n and any integer m, there exists an overtwisted S^{2n+1} in $(S^{2n+3}, \xi_{\text{std}})$ which is ambient isotopic to the standard S^{2n+1} and whose relative Euler number is equals to $2m+1$. Thus by Gray's stability theorem (1) follows.

Since for any positive even number n, there exist finitely many overtwisted contact structures on S^{2n+1} up to isotopy and in particular there exists a unique overtwisted contact structure on S^5 up to isotopy, by Gray's stability theorem (2) follows. □

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