Discrete integrable equations associated with cluster algebra and its extension (summary)

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In this article, we deal with cluster algebras, which were introduced by Fomin and Zelevinsky [1,2]. A cluster algebra is a commutative ring described by cluster variables and coefficients. A generating set of the cluster algebra is defined by mutation, which is a transformation of a seed consisting of a set of cluster variables, coefficients, and a quiver. Cluster variables and coefficients obtained from a mutation of an initial seed satisfy some difference equations. It is known that cluster variables can satisfy the discrete KdV equation [3] and the Hirota-Miwa equation [4], when the initial seed includes suitable quivers with infinite vertices [5]. These quivers have the property that an infinite number of mutations gives a permutation of its vertices. This property is called `mutation-period' and a quiver with this property is called a mutation-periodic quiver. Several results concerning mutation-periodic quivers have been reported in [6]. All the mutation-periodic quivers in which a permutation of its vertices is achieved by a single mutation have already obtained. The quiver which gives the discrete KdV equation is obtained from a transformation of the quiver of the Hirota-Miwa equation. This transformation corresponds to a reduction from the Hirota-Miwa equation to the discrete KdV equation. In this paper, we construct the cluster algebras whose variables and coefficients satisfy the discrete mKdV equation, the discrete Toda equation [7], and some q-discrete Painlevé equations [8]. We introduce the quiver which generalizes the one that corresponds to the discrete KdV equation and the Hirota-Miwa equation. Quivers of q-Painlevé I,II equations and their higher order analogues have been obtained in [5,9]. We shall introduce the quivers for the q-Painlevé III,VI equations, which are mutation-periodic and are obtained from transformations of quivers for the discrete KdV equation and the discrete mKdV equation. We deal with the Laurent phenomenon algebras introduced by Lam and Pylyavskyy [10]. The Laurent phenomenon algebra is a generalization of the cluster algebra with Laurent phenomenon of cluster variables. A Laurent phenomenon algebra is a commutative ring described by cluster variables. A generating set of the Laurent phenomenon algebra is defined by mutation, which is a transformation of a seed consisting of a set of cluster variables and exchange polynomials. The Laurent phenomenon is the property that all cluster variables obtained by mutations of an initial seed are Laurent polynomials in cluster variables of the initial seed. Difference equations obtained by a cluster algebra are restricted to the form:

$$x_k' x_k = \prod_{i\in I} x_i^{a_i} + \prod_{j\in J} x_j^{b_j}.$$

However, the discrete BKP equation

$$x_{n+1}^{m+1,l+1} x_n^{m,l} = x_n^{m+1,l+1} x_{n+1}^{m,l} + x_{n+1}^{m+1,l} x_n^{m,l+1} x_{n+1}^{m,l+1} x_n^{m+1,l}$$

does not have the form given by cluster algebra. Cluster variables of a Laurent phenomenon algebra obtained from a mutation of an initial seed can satisfy the difference equations of more general form as

$$x_k' x_k = P \in \mathbf{Z}[x_i | i \in I].$$

It is known that cluster variables of a Laurent phenomenon algebra can satisfy the Somos-6 [11] and some related difference equations, when the initial seed includes suitable exchange polynomials [10,12]. These seeds have the property called 'mutation-period'. This property is an analogue of the mutation-periodic quiver of cluster algebras. Several results concerning period-1 seeds have been reported in [12].

In this paper, we construct the Laurent phenomenon algebras whose cluster variables satisfy the discrete BKP equation [4], the discrete Sawada-Kotera equation [13], the Somos-7, and several other difference equations. The seed which gives the discrete BKP equation has infinite rank. This seed has the property of a generalization of the mutation-period. The seeds which give the Somos-6, Somos-7, and related 2-dimensional difference equations are obtained from reductions of the seed that gives the discrete BKP equation. This reduction corresponds to a reduction from the discrete BKP equation to these difference equations.

References

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