博士論文

論文題目　A moving lemma for algebraic cycles with modulus and contravariance
(モジュラス付き代数的サイクルの移動補題と反変性)

氏名　甲斐　亘

A moving lemma for algebraic cycles with modulus and contravariance

Wataru Kai

Abstract

We prove a moving lemma of algebraic cycles with modulus which implies their contravariance: Bloch-Esnault's additive higher Chow group turns out to be contravariant in smooth affine schemes; Binda-Saito's higher Chow group with modulus proves contravariant in smooth schemes Nisnevich locally. Our moving method is based on parallel translation in the affine space "with modulus" which involves a new integer parameter s.

Contents

1 Introduction

In recent years, the theory of algebraic cycles with modulus has been an attractive subject. It concerns the behavior of algebraic cycles at boundaries; more precisely the intersection property with a chosen effective Cartier divisor (called the *modulus*). The notion of modulus dates back at least to class field theory. Algebro-geometrically this concept probably started in 1952 when Rosenlicht [Ros] introduced the divisor class group relative to a modulus (of a complete nonsingular curve).

The current development has been initiated by Bloch and Esnault [BE] who introduced the additive higher Chow group $\mathrm{TCH}^r(X, n; m)$ (the definitive definition due to Park [Park]) and it has been a fruitful subject over the last decade. It is expected to have a relation to the relative K-groups

$$K_*(X \times \mathbb{A}^1, X \times (m+1)\{0\}),$$

just as Bloch's higher Chow group $\mathrm{CH}^r(X, n)$ is related to the K-group:

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_r \mathrm{CH}^r(X, n)_{\mathbb{Q}}$$

for smooth schemes X over a field.

In the last few years there has been a movement of introducing Chow groups with modulus: Kerz and Saito [KS] studied the Chow group of zero-cycles with modulus for arbitrary algebraic schemes equipped with an effective divisor and showed its remarkable connection to the wildly ramified class field theory of varieties over finite fields. Russell [Rus] defined a slightly different version earlier and studied its relation to his Albanese variety with modulus.

Binda and Saito [BS] then defined the higher Chow group with modulus $\mathrm{CH}^r(X|D, n)$ for an arbitrary pair of an algebraic scheme X and an effective Cartier divisor D on it. It is defined as the homology groups of the cycle complex with modulus $z^r(X|D, \bullet)$. It contains all the groups above (in suitable versions) as particular cases. It is expected as a cycle-theoretic cohomology theory corresponding to the relative K-theory $K_n(X, D)$.

In spite of being a candidate of a nice cohomology theory, it has been unknown if the additive higher Chow group and the higher Chow group with modulus are contravariant for arbitrary morphisms of smooth schemes. In the projective case this has been settled by Krishna and Park [KP, KP2], but in the general case (e.g. affine) the concept "modulus" gets harder to handle.

The aim of this thesis is to provide an affirmative answer to the problem **at least locally** by proving a new moving lemma. The moving lemma assures the contravariance of the additive higher Chow group in smooth **affine** schemes, and that of Nisnevich-localized versions of Binda-Saito's higher Chow group with modulus in pairs (X, D) for which $X \setminus D$ is smooth.

Let us explain our results in slightly more detail:

1.1 Moving lemma

We will often consider pairs (X, D) consisting of an equi-dimensional scheme X over a base field k and an effective Cartier divisor D on it.

For any integer $r \geq 0$, Binda and Saito ([BS, §2.1], recalled in §2) defined a complex of abelian groups $z^r(X|D, \bullet)$ called the codimension r cycle complex of the pair (X, D) as a subcomplex of Bloch's cycle complex $z^r(X, \bullet)$ (cubical version); in particular, elements of $z^r(X|D, n)$ are represented by cycles on $X \times \mathbb{A}^n$ satisfying certain conditions. When $D = \emptyset$ it reduces to Bloch's higher Chow theory.

The complex is contravariant for flat maps. The association

$$(U \xrightarrow{\text{étale}} X) \mapsto z^r(U|D_{|U}, \bullet)$$

defines a complex of étale sheaves on X. We will denote this sheaf simply by $z^r(X|D,\bullet)_{\text{ét}}$, and similarly for weaker topologies (such as Nis., Zar.). We write $\mathrm{CH}^r(X|D,n)_{\mathrm{Nis}}$ for the n-th homology sheaf of $z^r(X|D,\bullet)_{\mathrm{Nis}}$.

Definition 1.1. For a finite collection $\mathcal{W}$ of constructible irreducible subsets of $X \setminus D$, define a subcomplex

$$z^r(X|D,\bullet)_{\mathcal{W}} \subset z^r(X|D,\bullet)$$

as the one which is generated by cycles $V \in z^r(X|D,n)$ such that V (which is by definition a codimension r cycle on $X \times \mathbb{A}^n$) intersects $W \times F$ properly in $X \times \mathbb{A}^n$ for every $W \in \mathcal{W}$ and every face F of $\mathbb{A}^n$. This extends to a subcomplex of étale sheaves

$$z^r(X|D,\bullet)_{\mathcal{W},\text{ét}} \subset z^r(X|D,\bullet)_{\text{ét}}.$$

Theorem 1.2 (Moving Lemma; *see* Theorem 4.11)**.** *Let X be an equi-dimensional k-scheme, D be an effective Cartier divisor on it, and $\mathcal{W}$ be a finite collection of constructible irreducible subsets of $X \setminus D$. Assume $X \setminus D$ is smooth over k. Then the above inclusion is a quasi-isomorphism in the Nisnevich topology:*

$$z^r(X|D,\bullet)_{\mathcal{W},\mathrm{Nis}} \overset{\text{qis}}{\hookrightarrow} z^r(X|D,\bullet)_{\mathrm{Nis}}.$$

Along its proof we establish the following general result:

Theorem 1.3 (Noether's normalization theorem; *see* Theorem 4.6)**.** *Let $X \to B$ be an equidimensional morphism to a regular Noetherian 1-dimensional scheme B of relative dimension d. Then locally in the Nisnevich topology on X and B, there is a finite surjective map*

$$X \to \mathbb{A}^d_B.$$

This explains the need of Nisnevich localization from the technical side.

1.2 Functoriality of motivic cohomology

Binda and Saito defined the motivic complex $\mathbb{Z}(r)_{X|D}$ of a pair (X,D) as

$$\mathbb{Z}(r)_{X|D} := z^r(X|D,\bullet)[2r]$$

where the degree shift is homological. This forms a complex of Nisnevich sheaves $\mathbb{Z}(r)_{X|D,\mathrm{Nis}}$ on $X_{\text{ét}}$. They defined the (Nisnevich) motivic cohomology groups as the hypercohomology groups

$$H^n_{\mathcal{M},\mathrm{Nis}}(X|D,\mathbb{Z}(r)) := \mathbf{H}^n_{\mathrm{Nis}}(X,\mathbb{Z}(r)_{X|D,\mathrm{Nis}}).$$

It is obviously contravariant for flat maps. Our moving lemma, Theorem 1.2, implies its contravariance for **any** map of smooth schemes with effective Cartier divisors:

Theorem 1.4 (Functoriality; *see* Theorem 4.15)**.** *Let $(X, D), (Y, E)$ be pairs of equi-dimensional k-schemes and effective Cartier divisors on them, and assume $Y \setminus E$ is smooth. Let $f\colon X \to Y$ be a map of k-schemes inducing a morphism $D \to E$ of schemes. Then there is a natural map*

$$f^*\colon f^{-1}z^r(Y|E, \bullet)_{\mathrm{Nis}} \to z^r(X|D, \bullet)_{\mathrm{Nis}}$$

in the derived category of complexes of Nisnevich sheaves on X. Consequently there are natural maps of abelian groups

$$H^n_{\mathcal{M},\mathrm{Nis}}(Y|E, \mathbb{Z}(r)) \to H^n_{\mathcal{M},\mathrm{Nis}}(X|D, \mathbb{Z}(r))$$

and Nisnevich sheaves

$$f^{-1}\mathrm{CH}^r(Y|E, n)_{\mathrm{Nis}} \to \mathrm{CH}^r(X|D, n)_{\mathrm{Nis}}.$$

A "projective" variant of Theorem 1.4, without need of Nisnevich localization, has been proved by Krishna and Park [KP2, Th.4.3].

The contravariance in this generality can deduce a natural **product structure**:

$$z^r(X|D, \bullet)_{\mathrm{Nis}} \otimes z^s(X|D', \bullet)_{\mathrm{Nis}} \to z^{r+s}(X|D + D', \bullet)_{\mathrm{Nis}}$$

in the derived category, inducing product structures on Nisnevich motivic cohomology groups and Nisnevich Chow sheaves with modulus.

1.3 Additive higher Chow groups

The higher Chow groups with modulus includes the additive higher Chow groups as special cases. There are some technical simplification in this case and our method yields the following.

Definition 1.5. For schemes X, denote by $Tz^r(X, \bullet; m)$ the complex of abelian groups

$$Tz^r(X, \bullet; m) := z^r(X \times \mathbb{A}^1|(m+1)\{0\},\ \ \bullet -1).$$

Given a finite set $\mathcal{W}$ of irreducible constructible subsets of X, write

$$Tz^r(X, \bullet; m)_{\mathcal{W}} := z^r(X \times \mathbb{A}^1|X \times (m+1)\{0\}, \bullet - 1)_{\mathcal{W}\times\mathbb{A}^1}$$

where $\mathcal{W} \times \mathbb{A}^1 := \{W \times (\mathbb{A}^1 \setminus \{0\}) \mid W \in \mathcal{W}\}$ which is a finite set of irreducible constructible subsets of $X \times (\mathbb{A}^1 \setminus \{0\})$.

Theorem 1.6 (*see* Theorem 3.22)**.** *If X is a smooth affine scheme, then the inclusion of complexes of abelian groups*

$$Tz^r(X, \bullet; m)_{\mathcal{W}} \hookrightarrow Tz^r(X, \bullet; m)$$

is a quasi-isomorphism for any finite set $\mathcal{W}$ of irreducible constructible subsets of X.

Corollary 1.7 (*see* Theorem 3.25)**.** *For any map $f\colon X \to Y$ from an algebraic k-scheme to a smooth* **affine** *k-scheme Y, there is a natural pull-back map*

$$f^*\colon Tz^r(Y,\bullet;m) \to Tz^r(X,\bullet;m)$$

in the derived category of complexes of abelian groups.

Theorem 1.6 for **projective** smooth Y has been proved by Krishna and Park [KP, Th.4.1].

Plan of the paper

In §2 we describe the definition of the cycle complex with modulus $z^r(X|D,\bullet)$ and prove some basic facts used in this paper.

In §3 we treat the case of the additive higher Chow group (Theorem 1.6).

In §4 we treat the case of the higher Chow group with modulus (Theorem 1.2).

Both in §§3 and 4, we follow the traditional strategy used for Bloch's Chow theory ([Lev, Part I, Chap. II, §3.5]), which originates from Chow's proof [Chow]. It consists of the case of the affine space $\mathbb{A}^d$ and the reduction to this case. Our new contribution mainly lies in the proof for the case of the affine space. The treatments are very similar in both sections, but we have opted to write down the details respectively. As a result, each section can be read independently except that we use the facts in §3.2 on linear projection twice.

2 Definitions and basic facts

2.1 Algebraic cycle and pull-back

For an excellent Noetherian equidimensional scheme X (always over a field in this paper), denote by $z^r(X)$ the free abelian group generated by irreducible closed subsets of codimension r in X (also regarded as an integral closed subscheme). Its elements are called **algebraic cycles** on X of codimension r. An algebraic cycle represented by a single irreducible closed subset is called a **prime cycle**. If $V = \sum_i n_i V_i$ is a cycle with non-zero coefficients, its **support** $|V|$ is defined to be the closed subset $|V| := \cup_i V_i$ of X.

Given a flat morphism $f\colon X \to Y$ of excellent Noetherian equidimensional schemes and a prime cycle $V \in z^r(Y)$, we define a cycle $f^*V \in z^r(X)$ by

$$f^*V := \sum_\eta \mathrm{length}_{\mathcal{O}_{X,\eta}}(\mathcal{O}_V \otimes_{\mathcal{O}_Y} \mathcal{O}_X)\overline{\{\eta\}}$$

where η runs through the generic points of irreducible components of $f^{-1}(|V|)$. Since pull-backs of closed subsets by a flat map preserve codimension, every η has codimension r in X. We extend the definition linearly to get

$$f^*\colon z^r(Y) \to z^r(X).$$

This operation is called the **flat pull-back** of algebraic cycles.

Besides flat pull-back of cycles, the following is a useful principle when considering pull-backs of cycles.

Definition 2.1 (Serre's Tor formula)**.** Let $f\colon X \to Y$ be an l.c.i. morphism of equidimensional schemes and $V \in z^r(Y)$ be a codimension r cycle. Suppose $f^{-1}(|V|)$ is a codimension $\geq r$ closed subset of Y (in general it has codimension $\leq r$, if nonempty, by the l.c.i. hypothesis). Then define an element $f^*(V)$ of $z^r(X)$ as follows. First we assume V to be prime (thus V is an integral closed subscheme). Then set:

$$f^*(V) := \sum_{\eta} \left(\sum_i (-1)^i \mathrm{length}_{\mathcal{O}_{X,\eta}} \mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_V, \mathcal{O}_X) \right) \overline{\{\eta\}},$$

where η runs through the generic points of all irreducible components of $f^{-1}(|V|)$. By the condition $\mathrm{codim}_X f^{-1}(|V|) \geq r$, each Tor has finite length, and by the l.c.i. hypothesis it is a finite sum. In the general case, we extend the definition linearly. This operation is functorial whenever the pulled-back cycles involved are defined.

Lemma 2.2 (semi-continuity theorem of Chevalley, [EGA IV$_3$, 13.1.3])**.** *Let $f\colon X \to Y$ be a morphism of finite type of schemes. Then the function $x \mapsto \dim_x(f^{-1}(f(x)))$ is upper-semicontinuous on X.*

Lemma 2.3. *Let $f\colon X \to S$ be a morphism of finite type of Noetherian schemes. Then the function $s \mapsto \dim(f^{-1}(s))$ is constructible on S.*

Proof. This is a consequence of [EGA IV$_3$, (9.5.5)] and can be deduced from the previous lemma as well. □

Variants of the following observation appear in this thesis repeatedly: Let $f\colon X \to Y$ be an l.c.i. morphism of equidimensional schemes. Let $Z^{\geq i}(f) \subset Y$ be the constructible subset consisting of points where the fiber of f has dimension $\geq i$. Then the condition $\mathrm{codim}_X(f^{-1}(|V|)) \geq r$ is equivalent to

$$\dim(V \cap Z^{\geq i}(f)) + i \leq \dim X - r$$

for all $i \geq 0$.

2.2 The cycle complex with modulus

We write $\square^1 = \mathbb{P}^1 \setminus \infty = \mathrm{Spec}\ \mathbb{Z}[z]$ and $\square^n := \mathrm{Spec}\ \mathbb{Z}[z_1, \dots, z_n]$, i.e. the affine space with a coordinate system $\boldsymbol{z} = (z_1, \dots, z_n)$. We will often consider it over a base field k; in that case we mean $\square^n = \mathrm{Spec}\ k[z_1, \dots, z_n]$. We will often consider the compactification $\square^n \subset (\mathbb{P}^1)^n$. Let F_∞ be the Cartier divisor on $(\mathbb{P}^1)^n$ defined by:

$$F_\infty := \sum_{i=1}^n (\mathbb{P}^1)^{i-1} \times \{\overset{i}{\check{\infty}}\} \times (\mathbb{P}^1)^{n-i}.$$

There are distinguished subschemes of $\square^n$, called **faces**. Faces of $\square^n$ are $\{z_i = 0\}$, $\{z_i = 1\}$ $(1 \le i \le n)$ and their finitely many intersections.

Definition 2.4. Let X be an excellent Noetherian equidimensional scheme equipped with an effective Cartier divisor D. Let $\underline{z}^r(X|D,n)$ be the subgroup of $z^r(X \times \square^n)$ consisting of cycles V satisfying the following two conditions:

(1) **(face condition)**: The cycle V meets every face F of $\square^n$ properly, i.e.

$$\mathrm{codim}_{|V|}(|V| \times_{\square^n} F) \ge \mathrm{codim}_{\square^n}(F).$$

(2) **(modulus condition)**: Let $\overline{V}$ be the closure of $|V|$ in $X \times (\mathbb{P}^1)^n$ and $\overline{V}^N$ be its normalization (= the disjoint sum of normalizations of the irreducible components). We have two Cartier divisors on $\overline{V}^N$, the pull-backs of $D \subset X$ and $F_\infty \subset (\mathbb{P}^1)^n$ by the natural projections $\overline{V}^N \to X$, $\overline{V}^N \to (\mathbb{P}^1)^n$. In this notation the condition is: the inequality of Cartier divisors

$$(\text{the pull-back of } D) \le (\text{the pull-back of } F_\infty)$$

holds on $\overline{V}^N$.

If $n = 0$, we read (2) as $|V| \cap D = \emptyset$. Note that the condition (2) always implies $|V| \cap (D \times \square^n) = \emptyset$.

2.2.1

Denote by $\partial_{i,\epsilon} \colon \square^{n-1} \hookrightarrow \square^n$ $(1 \le i \le n,\ \epsilon = 0,1)$ the embedding

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, \overset{i}{\check{\epsilon}}, z_i, \dots, z_{n-1}).$$

By the face condition, we have pull-back maps

$$\partial_{i,\epsilon}^* \colon \underline{z}^r(X|D,n) \to \underline{z}^r(X|D,n-1).$$

Here, the modulus condition is preserved (so that the maps go *into* $\underline{z}^r(X|D,n-1)$) by the following elementary fact.

Lemma 2.5. *Let Y be an integral scheme and D_1, D_2 be two effective Cartier divisors. Let $f \colon Y' \to Y$ be a morphism from an integral scheme, whose image is not contained in $|D_1| \cup |D_2|$; thus Cartier divisors f^*D_1, f^*D_2 on Y' are defined. Suppose an inequality $D_1 \le D_2$ of Cartier divisors holds on Y. Then we have $f^*D_1 \le f^*D_2$ on Y'.*

Proof. Let u_1, u_2 be representatives of D_1, D_2 on some open set of Y respectively. The relation $D_1 \le D_2$ says locally there is a regular function c with

$$u_1 c = u_2.$$

Pulling back this equation by f,

$$f^*u_1 f^*c = f^*u_2.$$

By our hypothesis each term is not a zero-divisor. The equation says $f^*D_1 \leq f^*D_2$ holds on Y'. $\square$

We can organize $\underline{z}^r(X|D,n)$'s to make a complex by the differentials

$$\sum_{i=1}^{n}(-1)^{i-1}(\partial_{i,1}^* - \partial_{i,0}^*)\colon \underline{z}^r(X|D,n) \to \underline{z}^r(X|D,n-1).$$

Let $s_i\colon \square^n \to \square^{n-1}$ $(1 \leq i \leq n)$ be the i-th degenerate map (= collapsing the i-th axis). We define $\underline{z}^r(X|D,n)_{\mathrm{degen}}$ to be the subgroup of $\underline{z}^r(X|D,n)$:

$$\underline{z}^r(X|D,n)_{\mathrm{degen}} := \sum_{1\leq i\leq n} s_i^*\underline{z}^r(X|D,n-1).$$

Then $\underline{z}^r(X|D,\bullet)_{\mathrm{degen}}$ forms a subcomplex of $\underline{z}^r(X|D,\bullet)$. We define

$$z^r(X|D,n) := \underline{z}^r(X|D,n)/\underline{z}^r(X|D,n)_{\mathrm{degen}}.$$

We are principally interested in the quotient complex

$$z^r(X|D,\bullet) = \underline{z}^r(X|D,\bullet)/\underline{z}^r(X|D,\bullet)_{\mathrm{degen}},$$

called the **cycle complex of the pair** (X,D). The homolog groups

$$\mathrm{CH}^r(X|D,n) = H_n(z^r(X|D,\bullet))$$

are called the **higher Chow groups of the pair** (X,D).

If $D = \emptyset$, these definitions reduce to (the cubical version of) Bloch's higher Chow theory.

2.2.2 Additive Chow theory

The case

$$(X,D) = (Y \times \mathbb{A}^1, Y \times (m+1)\{0\})$$

$(m \geq 1)$ had been studied earlier independently and we write

$$Tz^r(Y,n;m) := z^r(Y \times \mathbb{A}^1|Y \times (m+1)\{0\}, n-1),$$

$$\mathrm{TCH}^r(Y,n;m) := \mathrm{CH}^r(Y \times \mathbb{A}^1|Y \times (m+1)\{0\}, n-1),$$

called the additive higher Chow groups of Y with modulus m. Their contravariance in smooth affine Y will be proved in §3.4.

2.2.3 The subcomplex of cycles in good position

Definition 2.6. (1) Let $\mathcal{W}$ be a finite set of irreducible constructible subsets of $X \setminus D$ and $e\colon \mathcal{W} \to \mathbb{N}$ be a map of sets. Define a subgroup

$$\underline{z}^r(X|D,n)_{\mathcal{W},e} \subset \underline{z}^r(X|D,n)$$

to be the subgroup consisting of cycles V satisfying

$$\operatorname{codim}_{W\times F}(|V| \cap (W \times F)) \geq r - e(W)$$

for all $W \in \mathcal{W}$ and faces F of $\square^n$ (we will express this as: $|V|$ and $W \times F$ meet with **excess** $\leq e(W)$). When e is the constant function r, we have

$$\underline{z}^r(X|D,n)_{\mathcal{W},r} = \underline{z}^r(X|D,n).$$

We define

$$\underline{z}^r(X|D,n)_{\mathcal{W}} := \underline{z}^r(X|D,n)_{\mathcal{W},0}.$$

(2) Denote $\underline{z}^r(X|D,n)_{\mathcal{W},e_{\text{degen}}} = \underline{z}^r(X|D,n)_{\mathcal{W},e} \cap \underline{z}^r(X|D,n)_{\text{degen}}$ and define $z^r(X|D,n)_{\mathcal{W},e}$ to be the quotient group:

$$z^r(X|D,n)_{\mathcal{W},e} := \frac{\underline{z}^r(X|D,n)_{\mathcal{W},e}}{\underline{z}^r(X|D,n)_{\mathcal{W},e_{\text{degen}}}}.$$

This is the same as saying $z^r(X|D,n)_{\mathcal{W},e}$ is the image of the natural map

$$\underline{z}^r(X|D,n)_{\mathcal{W},e} \to z^r(X|D,n).$$

The series of groups $\underline{z}^r(X|D,\bullet)_{\mathcal{W},e}$ and $z^r(X|D,\bullet)_{\mathcal{W},e}$ form complexes of abelian groups. We write

$$z^r(X|D,\bullet)_{\mathcal{W}} := z^r(X|D,\bullet)_{\mathcal{W},0}.$$

As a whole, we have the following diagram:

$$\begin{array}{ccccc}
\underline{z}^r(X|D,\bullet)_{\mathcal{W}} := \underline{z}^r(X|D,\bullet)_{\mathcal{W},0} & \subset & \underline{z}^r(X|D,\bullet)_{\mathcal{W},e} & \subset & \underline{z}^r(X|D,\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
z^r(X|D,\bullet)_{\mathcal{W}} := z^r(X|D,\bullet)_{\mathcal{W},0} & \subset & z^r(X|D,\bullet)_{\mathcal{W},e} & \subset & z^r(X|D,\bullet)
\end{array}$$

where the vertical maps are given by “modulo degenerate cycles.”

2.2.4 Cycle complex sheaves

For a scheme X, denote by X_{et} the small étale site over X. Suppose X is excellent Noetherian and equidimensional, and equipped with an effective Cartier divisor D. Then we can consider a presheaf on X_{et}

$$(U \xrightarrow{\text{et}} X) \mapsto \underline{z}^r(U|D_{|U},n)$$

(resp. $z^r(U|D_{|U},n)$) which turns out to be a sheaf. We shall regard it as a Nisnevich sheaf and denote it by $\underline{z}^r(X|D,n)_{\mathrm{Nis}}$ (resp. $z^r(X|D,n)_{\mathrm{Nis}}$).

Given a finite set $\mathcal{W}$ of irreducible constructible subsets of $X \setminus D$ and a function $e\colon \mathcal{W} \to \mathbb{N}$, we consider, for each étale scheme $U \to X$, the set

$$\mathcal{W}_U := \{ \text{ irreducible components of } W \times_X U \mid W \in \mathcal{W}\}$$

of irreducible constructible subsets of U and the function

$$\begin{array}{rcc} e_U\colon \mathcal{W}_U & \to & \mathbb{N} \\ W' & \mapsto & \min_W\{e(W)|W' \text{ is a component of } W \times_X U\}. \end{array}$$

We often omit the subscript $(-)_U$. Then we have presheaves

$$(U \xrightarrow{\text{et}} X) \mapsto \underline{z}^r(U|D_{|U},n)_{\mathcal{W},e}$$

(resp. $z^r(U|D_{|U},n)_{\mathcal{W},e}$) which turns out to form a subsheaf $\underline{z}^r(X|D,n)_{\mathcal{W},e,\mathrm{Nis}}$ of $\underline{z}^r(X|D,n)_{\mathrm{Nis}}$ (resp. $z^r(X|D,n)_{\mathcal{W},e,\mathrm{Nis}}$ of $z^r(X|D,n)_{\mathrm{Nis}}$).

Following the classical theory without modulus, we can define the (Nisnevich) motivic cohomology of the pair (X,D) as

$$H^n_{\mathcal{M},\mathrm{Nis}}(X|D,\mathbb{Z}(r)) := \mathbf{H}^{n-2r}_{\mathrm{Nis}}(X, z^r(X|D,\bullet)_{\mathrm{Nis}}).$$

It will be proved in §4.6 that these motivic cohomology groups are contravariant in pairs (X,D) such that $X \setminus D$ is smooth over the base field.

2.3 Limit and specialization lemmas

Lemma 2.7. *Let X_0 be a Noetherian scheme and D be an effective Cartier divisor on X_0. Let $\{X_i\}_i$ be a filtered system of Noetherian X_0-schemes and assume the transition maps are smooth and affine. Suppose the limit scheme $X := \varprojlim_i X_i$ is Noetherian. Denote the pull-backs of D to X_i or X also by D. Then for each n, the natural map*

$$\varinjlim_i \underline{z}^r(X_i|D,n) \to \underline{z}^r(X|D,n)$$

$$(\text{resp. } \varinjlim_i \underline{z}^r(X_i|D,n)_{\mathcal{W},e} \to \underline{z}^r(X|D,n)_{\mathcal{W},e})$$

is an isomorphism.

Proof. The first statement is a special case of the second. The surjectivity is the nontrivial point. Suppose given a prime cycle $V \in z^r(X|D,n)$. As a cycle it comes from a prime cycle V_i on $X_i \times \square^n$ of codimension r (by the Noetherian hypothesis). Denote by $V_{i'} := V_i \times_{X_i} X_{i'}$ for transition maps $X_{i'} \to X_i$. Let us check V_i will satisfy the face condition, the modulus condition and the intersection condition with $W \in \mathcal{W}$, after replacing i.

First consider the intersection $V_i \cap (X_i \times F)$ where F is a face in $\square^n$. Suppose it contains an irreducible component C of codimension $< r$ in $X_i \times F$. Since

$X \to X_i$ is flat, its inverse image to $X \times F$ would have the same codimension if it were not empty. So the inverse image has to be empty.

We have to show the inverse image of C to some $V_{i'} \times F$ is empty. Let the open set $U_{i'}$ be the image of $X_{i'} \to X_i$. By the condition $C \times_{X_i} X = \emptyset$, we get

$$\bigcap_{i'} (U_{i'} \times F) \cap C = \emptyset.$$

This implies some $U_{i'} \times F$ does not contain the generic point of C. For this i', we have $C \times_{X_i} X_{i'} = \emptyset$.

Next consider the modulus condition. Consider the closed subset $\{D|_{\overline{V_i}^N} > F_\infty|_{\overline{V_i}^N}\}$ on $\overline{V_i}^N$. By the permanence of normality with respect to smooth morphisms, its formation commutes with base changes $X_{i'} \to X_i$. Since V satisfies the modulus condition, it becomes empty in $\overline{V}^N$. Thus similarly to the previous step, V_i will satisfy the modulus condition after pulling back to some $X_{i'}$.

Lastly we consider the condition $V_i \in z^r(X_i|D,n)_{\mathcal{W},e}$. For $W \in \mathcal{W}$, write $W_i = W \times_{X_0} X_i$. Suppose the intersection $V \cap (W_i \times F)$ has an irreducible component C' having codimension $< r - e(W)$ in $W_i \times F$. If $C' \times_{X_i} X$ were nonempty, it should have the same codimension in $W_X \times F$, which contradicts the condition $V \in z^r(X|D,n)_{\mathcal{W},e}$. Therefore $C' \times_{X_i} X$ is empty. If follows $C' \times_{X_i} X_{i'}$ is empty for some i'. Therefore $V_{i'} \in z^r(X_{i'}|D,n)_{\mathcal{W},e}$. ☐

Applying Lemma 2.7 to n and $n-1$ we obtain:

Corollary 2.8. *Under the hypotheses of Lemma* 2.7*, the natural maps*

$$\varinjlim_i \underline{z}^r(X_i|D,n)_{\mathcal{W},e_{\text{degen}}} \to \underline{z}^r(X|D,n)_{\mathcal{W},e_{\text{degen}}}$$

$$\varinjlim_i z^r(X_i|D,n)_{\mathcal{W},e} \to z^r(X|D,n)_{\mathcal{W},e}$$

are isomorphisms.

Lemma 2.9. *Let K/k be a purely transcendental extension of fields. Let $e, e' : \mathcal{W} \to \mathbb{N}$ be two functions satisfying $e \geq e' \geq 0$. Then the natural map*

$$\frac{z^r(X|D,\bullet)_{\mathcal{W},e}}{z^r(X|D,\bullet)_{\mathcal{W},e'}} \to \frac{z^r(X_K|D_K,\bullet)_{\mathcal{W},e}}{z^r(X_K|D_K,\bullet)_{\mathcal{W},e'}}$$

induces injective maps on the homology groups.

Proof. By Lemma 2.7 we may assume K has a finite transcendence degree m over k; it is the function field of the affine space $S = \mathbb{A}^m_k$.

Suppose a cycle $V \in \underline{z}^r(X|D,n)_{\mathcal{W},e}$ represents a homology class in the first complex and maps to the zero class in the second. Then we have:

$$V_K = \mathrm{d}V^2_{(K)} + V^3_{(K)} + Q_{(K)}$$

as cycles on $X_K \times \square^n$, where $V^2_{(K)} \in \underline{z}^r(X_K|D, n{+}1)_{\mathcal{W},e}$, $V^3_{(K)} \in \underline{z}^r(X_K|D, n)_{\mathcal{W},e'}$ and $Q_{(K)}$ is a degenerate cycle in $\underline{z}^r(X_K|D, n)_{\mathcal{W},e}$. By the limit argument (Lemma 2.7 and its corollary), after shrinking S, this formula comes from a formula over S:

$$V_S = \mathrm{d}V^2_{(S)} + V^3_{(S)} + Q_{(S)}.$$

Moreover we may assume every component of these cycles is equidimensional over S by Lemma 2.3.

Now suppose k is an infinite field for a while. Then there is a k-rational point $s \in S$. Pulling back the last formula to s gives a killing relation of V in the homology group of $z^r(X|D, \bullet)_{\mathcal{W},e}/z^r(X|D, \bullet)_{\mathcal{W},e'}$ (pull back to s is possible because cycles are equidimensional over S).

Next, suppose k is a finite field. Pick two prime numbers (say 2 and 3). There is an infinite algebraic extension $k^{(2)}/k$ obtained as the union of finite extensions of 2-power degrees. The class of $V_{k^{(2)}}$ is annihilated by the scaler extension $Kk^{(2)}/k^{(2)}$. Since $k^{(2)}$ is an infinite field, our previous arguments show that the class of $V_{k^{(2)}}$ is already zero. By the limit argument (Lemma 2.7 and its corollary), there is a finite 2-power subextension $k^{(2)}_1$ such that $V_{k^{(2)}_1}$ represents the zero class. Therefore applying finite push-forward by $k^{(2)}_1/k$, we find that $[k^{(2)}_1 : k]V$ represents the zero class, i.e. the class of V is annihilated by a power of 2. Applying the same argument to the prime number 3, we find that the class of V is annihilated by a power of 3. Therefore the class of V must be zero. $\square$

Remark 2.10. Essentially the same proof works for the following more specialized case: Let R be a discrete valuation ring over k with a uniformizer u, and X be an R-scheme of finte type with an effective Cartier divisor D. Let $\mathcal{R}$ be the local ring of the polynomial ring $R[x_1, \ldots, x_m]$ at the height 1 prime ideal (u). Then the map

$$\frac{z^r(X|D, \bullet)_{\mathcal{W},e}}{z^r(X|D, \bullet)_{\mathcal{W},e'}} \to \frac{z^r(X_{\mathcal{R}}|D_{\mathcal{R}}, \bullet)_{\mathcal{W},e}}{z^r(X_{\mathcal{R}}|D_{\mathcal{R}}, \bullet)_{\mathcal{W},e'}}$$

induces injective maps on the homology groups.

3 Theorem for additive higher Chow groups

In this section we prove Theorem 1.6. The basic strategy is as follows: first we prove the statement for the affine space $\mathbb{A}^d$ equipped with a Cartier divisor. We basically use moving by parallel translation on the affine space; however, in order to manage the modulus condition, we have to introduce the moving speed varying depending on the point.

Secondly, we treat the general case. We reduce the problem for a general smooth affine X (equidimensional) to that for the affine space $\mathbb{A}^d$ via finite flat maps $X \to \mathbb{A}^d$, by choosing such maps sufficiently generally.

3.1 The case of affine spaces

Let k be a field. Write $\mathbb{A}^d = \mathrm{Spec}(k[x_1, \dots, x_d])$. Let $u \in k[x_1, \dots, x_d] \setminus \{0\}$ be a nonzero function and $D = (u)$ be the divisor defined by u.

Let $z_1, \dots, z_n$ be the coordinate of $\square^n := \mathrm{Spec}(k[z_1, \dots, z_n])$.

3.1.1 Choosing an integer $s(V)$

Let $V \in \underline{z}^r(\mathbb{A}^d|D, n)$ be a prime cycle. Let $\overline{V}$ be its closure in $\mathbb{A}^d \times (\mathbb{P}^1)^n$ with the reduced scheme structure. We are going to define an integer $s(V) \geq 1$.

Consider a partition $\{1, \dots, d\} = I \sqcup J$. Let U_{IJ} be the open subset of $(\mathbb{P}^1)^n$ which has coordinates $\{z_i\}_{i \in I}$ and $\{1/z_j\}_{j \in J}$. Put $\zeta_j = 1/z_j$. In this region the divisor F_∞ is defined by the function $\zeta_J := \prod_j \zeta_j$.

Choose a finite set of polynomials generating the ideal of the closed subscheme $\overline{V}_{\mathrm{red}} \cap (\mathbb{A}^d \times U_{IJ})$ and write it as

$$\{f_{IJ}^\lambda \in k[x_1, \dots, x_d, z_i \ (i \in I), \zeta_j \ (j \in J)]\}_\lambda.$$

The next lemma is useful to interpret the modulus condition.

Lemma 3.1. *Let A be a commutative ring with 1, $\mathfrak{p}$ be a prime ideal, $\zeta \in A$ and $u \in A \setminus \mathfrak{p}$ be two elements. Then the element ζ/u of $\mathrm{Frac}(A/\mathfrak{p})$ (the residue field of $\mathfrak{p}$) is integral over $A/\mathfrak{p}$ if and only if there is a homogeneous polynomial $E(\alpha, \beta) \in A[\alpha, \beta]$ which is monic in α such that we have*

$$E(\zeta, u) \in \mathfrak{p} \quad \textit{in } A.$$

Proof. If ζ/u is integral over $A/\mathfrak{p}$, there is an equation of the form

$$(\zeta/u)^N + a_1(\zeta/u)^{N-1} + \cdots + a_N = 0 \ (a_i \in A) \quad \text{in } \mathrm{Frac}(A/\mathfrak{p})$$

satisfied by ζ/u. Then we have the equation $\zeta^N + a_1\zeta^{N-1}u + \cdots + a_N u^N = 0$ in $A/\mathfrak{p}$. Therefore we have $\zeta^N + a_1\zeta^{N-1}u + \cdots + a_N u^N \in \mathfrak{p}$ in A. For the converse, read this paragraph backwards. $\square$

Now since V satisfies the modulus condition, we can apply Lemma 3.1 to the elements

$$\zeta_J := \prod_{j \in J} \zeta_j \text{ and } u \in A := k[x_1, \dots, x_d, z_i \ (i \in I), \zeta_j \ (j \in J)]$$

with $\mathfrak{p} := (f_{IJ}^\lambda)_\lambda$. So by Lemma 3.1 we get a homogeneous polynomial

$$E_{IJ}(\alpha, \beta) \in k[x_1, \dots, x_d, z_i \ (i \in I), \zeta_j \ (j \in J)][\alpha, \beta]$$

monic in α satisfying

$$E_{IJ}(\zeta_J, u) \in \sum_\lambda f_{IJ}^\lambda k[x_1, \dots, x_d, z_i \ (i \in I), \zeta_j \ (j \in J)]. \tag{1}$$

By multiplying E_{IJ} by a power of α, we may assume

$$\deg E_{IJ} \geq \deg f_{IJ}^{\lambda}$$

where the first deg is the homogeneous degree and the second deg is the total degree with respect to $x_1, \ldots, x_d$.

Moreover, we may assume $\deg E_{IJ}$ is the same for all partitions $\{1, \ldots, n\} = I \sqcup J$.

Definition 3.2. For a prime cycle $V \in \underline{z}^r(\mathbb{A}^d|D, n)$, we choose the above data and set

$$s(V) := \deg E_{IJ}.$$

For an arbitrary element $V = \sum_\mu c_\mu V_\mu \in \underline{z}^r(\mathbb{A}^d|D, n)$, we put

$$s(V) := \max_\mu \{s(V_\mu)\}.$$

3.1.2 Construction of homotopy

Choose a vector $\boldsymbol{v} \in \mathbb{A}^d$ and an integer $s \geq 1$. Define a morphism

$$p := p_{\boldsymbol{v},s} \colon \mathbb{A}^d \times \square^n \times \mathbb{A}^1 \to \mathbb{A}^d \times \square^n$$

by

$$(\boldsymbol{x}, \boldsymbol{z}, t) \mapsto (\boldsymbol{x} + tu^s \boldsymbol{v}, \boldsymbol{z}).$$

(It is defined over the residue field of $\boldsymbol{v}$; but we neglect to make the scalar extension explicit in the notation. Or one can make scalar extension to the residue field of $\boldsymbol{v}$ first and assume $\boldsymbol{v}$ is a rational point.) The map p is flat over the open subset $(\mathbb{A}^d \setminus D) \times \square^n$ on the target. So if we have an element $V \in z^r(\mathbb{A}^d|D, n)$, we can define a cycle

$$p^* V \quad \text{on } \mathbb{A}^d \times \square^n \times \mathbb{A}^1 \ (= \mathbb{A}^d \times \square^{n+1})$$

by flat pull-back.

The next is our technical key point.

Proposition 3.3. *Suppose given an element $V \in z^r(\mathbb{A}^d|D, n)$ and define an integer $s(V)$ as in §3.1.1. Consider the cycle $p^*_{\boldsymbol{v},s} V$ on $\mathbb{A}^d \times \square^{n+1}$. If we have $s \geq s(V)$, then $p^*_{\boldsymbol{v},s} V$ satisfies the modulus condition for any choice of $\boldsymbol{v}$.*

Proof. We may assume V is a prime cycle. The $\square^{n+1}$ appearing in the assertion has coordinates $z_1, \ldots, z_n, t$. The compactification $(\mathbb{P}^1)^{n+1}$ of $\square^{n+1}$ is covered by $(\mathbb{P}^1)^n \times \operatorname{Spec}(k[t])$ and

$$\operatorname{Spec}(k[z_i \ (i \in I), \zeta_j \ (j \in J), 1/t])$$

where I, J run through partitions $\{1, \ldots, n\} = I \sqcup J$. We put $\tau = 1/t$. The modulus condition we have to check is

$$D|_{\overline{p^*(V)}^N} \leq F_\infty|_{\overline{p^*(V)}^N}. \tag{2}$$

This can be checked after restricting to the region over each of those open sets.

First consider the open set $(\mathbb{P}^1)^n \times \operatorname{Spec}(k[t])$. The morphism $p_{\boldsymbol{v},s}$ extends to

$$\bar{p} := \bar{p}_{\boldsymbol{v},s} \colon \mathbb{A}^d \times (\mathbb{P}^1)^n \times \operatorname{Spec}(k[t]) \to \mathbb{A}^d \times (\mathbb{P}^1)^n$$

by the same formula, so we have

$$\overline{p^*(V)} \times_{(\mathbb{P}^1)^{n+1}} ((\mathbb{P}^1)^n \times \operatorname{Spec}(k[t])) = \bar{p}^*(\overline{V}).$$

From this we have an induced map

$$\bar{p} \colon \overline{p^*(V)}^N \times_{(\mathbb{P}^1)^{n+1}} ((\mathbb{P}^1)^n \times \operatorname{Spec}(k[t])) \to \overline{V}^N.$$

Thus we can deduce (2) from the corresponding inequality satisfied by V, on this region.

Next consider $U_{IJ} \times \operatorname{Spec}(k[\tau])$. Recall we have put

$$U_{IJ} := \operatorname{Spec}(k[z_i, \zeta_j]_{i\in I, j\in J}).$$

The set $\overline{V} \cap (\mathbb{A}^d \times U_{IJ})$ is defined by equations

$$f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j).$$

So the set $\bar{p}^*(\overline{V}) \cap (\mathbb{A}^d \times U_{IJ} \times \operatorname{Spec}(k[t]))$ is defined by equations

$$f^\lambda_{IJ}(\boldsymbol{x} + tu^s\boldsymbol{v}, z_i, \zeta_j).$$

Therefore the function

$$(1/t^{\deg f^\lambda_{IJ}}) f^\lambda_{IJ}(\boldsymbol{x} + tu^s\boldsymbol{v}, z_i, \zeta_j)$$

vanishes along $\bar{p}^*(\overline{V}) \cap (\mathbb{A}^d \times U_{IJ} \times \operatorname{Spec}(k[t, 1/t]))$. Here $\deg f^\lambda_{IJ}$ is the total degree of f^λ_{IJ} with respect to $\boldsymbol{x}$. Hence the function

$$\varphi^\lambda_{IJ} := \tau^{\deg f^\lambda_{IJ}} f^\lambda_{IJ}(\boldsymbol{x} + (1/\tau)u^s\boldsymbol{v}, z_i, \zeta_j)$$

is regular on $\mathbb{A}^d \times U_{IJ} \times \operatorname{Spec}(k[\tau])$ and vanishes along $\overline{p^*(V)} \cap (\mathbb{A}^d \times U_{IJ} \times \operatorname{Spec}(k[\tau]))$. It has the form

$$\varphi^\lambda_{IJ} = \tau^{\deg f^\lambda_{IJ}} f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j) + u^s g_\lambda$$

for some $g_\lambda \in k[\boldsymbol{x}, z_i, \zeta_j, \tau]_{i\in I, j\in J}$. Suppose the relation (1) in §3.1.1 is explicitly written as (recall $\zeta_J = \prod_{j\in J} \zeta_j$)

$$E_{IJ}(\zeta_J, u) = \sum_\lambda b_\lambda(\boldsymbol{x}, z_i, \zeta_j) f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j) \quad \text{in } k[x_1, \ldots, x_d, z_i, \zeta_j].$$

Combining the last two equations, we get

$$\sum_\lambda \tau^{s(V) - \deg f^\lambda_{IJ}} b_\lambda \varphi^\lambda_{IJ} = \tau^{s(V)} E_{IJ}(\zeta_J, u) + (\sum_\lambda b_\lambda g_\lambda \tau^{s(V) - \deg f^\lambda_{IJ}}) u^s. \qquad (3)$$

By the condition $s(V) \geq \deg f^{\lambda}_{IJ}$, everything in the equation is a polynomial belonging to $k[\boldsymbol{x}, z_i, \zeta_j, \tau]$.

Suppose $E_{IJ}(\alpha, \beta)$ has the form

$$E_{IJ} = \alpha^{s(V)} + c_1 \alpha^{s(V)-1}\beta \cdots + c_{s(V)}\beta^{s(V)}.$$

Then we put

$$\begin{aligned} E'_{IJ} := \alpha^{s(V)} + c_1 \tau \alpha^{s(V)-1}\beta + \cdots + c_{s(V)-1}\tau^{s(V)-1}\alpha\beta^{s(V)-1} \\ + \left(c_{s(V)}\tau^{s(V)} + (\sum_{\lambda} b_\lambda g_\lambda \tau^{s(V)-\deg f^{\lambda}_{IJ}}) u^{s-s(V)} \right) \beta^{s(V)} \end{aligned}$$

which belongs to $k[\boldsymbol{x}, z_i, \zeta_j, \tau][\alpha, \beta]$ by $s \geq s(V)$. Then the equation (3) reads

$$E'_{IJ}(\tau\zeta_J, u) \in \sum_{\lambda} \varphi^{\lambda}_{IJ} k[\boldsymbol{x}, z_i, \zeta_j, \tau].$$

By Lemma 3.1 this shows that the inequality of Cartier divisors

$$D|_{\overline{p^*(V)}^N} \leq F_\infty|_{\overline{p^*(V)}^N}$$

holds on the region over $U_{IJ} \times \mathrm{Spec}(k[\tau])$. □

3.1.3 Proper intersection

We keep the notation. We choose $\boldsymbol{v} := \boldsymbol{v}_{\mathrm{gen}}$ to be the generic point of $\mathbb{A}^d$ in this §3.1.3. Denote the function field of $\mathbb{A}^d$ by k_{gen}.

Lemma 3.4. *Let $\mathcal{W}$ be a finite set of irreducible constructible subsets of $\mathbb{A}^d \setminus D$ and $e \colon \mathcal{W} \to \mathbb{N}$ be a map of sets. Suppose $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{gen}}$. Then for any $s \geq 1$ and for any $V \in z^r(\mathbb{A}^d|D, n)$ we have:*

(1) *The cycle $p^*_{\boldsymbol{v}_{\mathrm{gen}},s}(V)$ on $\mathbb{A}^d_{k_{\mathrm{gen}}} \times \square^{n+1}$ meets all faces properly.*

(2) *The cycle $p^*(V)|_{t=1}$ on $\mathbb{A}^d_{k_{\mathrm{gen}}} \times \square^n$ meets $W_{k_{\mathrm{gen}}} \times F$ properly for every constructible irreducible subset W of $\mathbb{A}^d \setminus D$ (i.e. defined over k) and every face F of $\square^n$.*

(3) *If $V \in z^r(\mathbb{A}^d|D, n)_{\mathcal{W},e}$, the cycle $p^*(V)$ meets $W_{k_{\mathrm{gen}}} \times F$ with excess $\leq e(W)$ for every $W \in \mathcal{W}$ and every face F of $\square^{n+1}$.*

Proof. The assertion (1) is a special case of (3) where $W = \mathbb{A}^d \setminus D$ and $e = 0$. We prove (2) first. We have to prove the intersection $p^*(V)|_{t=1} \cap (W_{k_{\mathrm{gen}}} \times F)$ is proper in $\mathbb{A}^d_{k_{\mathrm{gen}}} \times \square^n$ for any face $F \subset \square^n$.

First suppose the map $u \colon W \to \mathbb{A}^1$ is dominant. Embed $p^*(V)|_{t=1} \cap (W_{k_{\mathrm{gen}}} \times F)$ into $\mathbb{A}^d_{k_{\mathrm{gen}}} \times F \times \mathbb{A}^1$ by the composition

$$p^*(V)|_{t=1} \cap (W \times F) \subset \mathbb{A}^d_{k_{\mathrm{gen}}} \times F \underset{(\mathrm{id}, u^s)}{\hookrightarrow} \mathbb{A}^d_{k_{\mathrm{gen}}} \times F \times \mathbb{A}^1$$

followed by the automorphism

$$\begin{array}{ccc} \mathbb{A}^d_{k_{\rm gen}} \times F \times \mathbb{A}^1 & \to & \mathbb{A}^d_{k_{\rm gen}} \times F \times \mathbb{A}^1 \\ (\boldsymbol{x}, \boldsymbol{z}, \alpha) & \mapsto & (\boldsymbol{x} + \alpha \boldsymbol{v}, \boldsymbol{z}, \alpha). \end{array}$$

Under this embedding, the fiber over a point $\alpha \in \mathbb{A}^1_{k_{\rm gen}}$ looks like:

$$V \cap ((W_\alpha + \alpha \boldsymbol{v}) \times F)) \subset \mathbb{A}^d_{k_{\rm gen}} \times F \ (\subset \mathbb{A}^d_{k_{\rm gen}} \times \square^n).$$

where $W_\alpha = W \times_{\mathbb{A}^1} \alpha$ (we have omitted the base-change notation to the residue field of α in some places). In particular the fiber over $0 \in \mathbb{A}^1$ is empty, since W is given as a subset of $\mathbb{A}^d \setminus D$.

Consider the following subsets of $\mathbb{A}^d \times F \times (\mathbb{A}^1 \setminus \{0\})$:

$$A = W \times F \underset{(\text{incl.}, u^s)}{\hookrightarrow} \mathbb{A}^d \times F \times (\mathbb{A}^1 \setminus \{0\}),$$

$$B = (V \times_{\square^n} F) \times (\mathbb{A}^1 \setminus \{0\}) \ \subset \mathbb{A}^d \times F \times (\mathbb{A}^1 \setminus \{0\}).$$

and the automorphism:

$$\phi \colon \mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\})$$

$$(\boldsymbol{x}, \boldsymbol{z}, \alpha) \mapsto (\boldsymbol{x} + \alpha \boldsymbol{v}, \boldsymbol{z}, \alpha).$$

Then by the observation above, the subset $\phi(A) \cap B$ of $\mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\})$ is exactly the set $p^*(V)|_{t=1} \cap (W_{k_{\rm gen}} \times F)$ embedded into it by the above fashion.

We apply the following lemma to the subsets $\bar{A}, B$ in $\mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\})$, the map

$$\psi \colon \mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^d_{k_{\rm gen}}; \ (\boldsymbol{x}, \boldsymbol{z}, \alpha) \mapsto \alpha \boldsymbol{v}_{\rm gen}$$

and

$$U = \mathbb{A}^d_{k_{\rm gen}} \times F \times (\mathbb{A}^1 \setminus \{0\}).$$

Lemma 3.5 ([Blo, Lem.1.2])**.** *Suppose a connected algebraic k-group G acts on an algebraic k-scheme X. Let A, B be two closed subsets of X and assume the fibers of the map*

$$G \times A \to X; (g, a) \mapsto g \cdot a$$

all have the same dimension and that this map is dominant. Assume given an overfield K of k and a K-morphism $\psi \colon X_K \to G_K$, and there is a nonempty open set $U \subset X$ such that for any point $x \in U_K$ we have

$$\text{tr.deg.}_k(k(\varphi \circ \psi(x), \pi(x))) \geq \dim G,$$

where $\pi \colon X_K \to X$ and $\varphi \colon G_K \to G$. Define

$$\phi \colon X_K \to X_K; x \mapsto \psi(x) \cdot x$$

and assume it is an automorphism. Then the intersection $\phi(A \cap U) \cap B$ is proper in X_K.

(The condition on the map $G \times A \to X$ is satisfied in our case because $u^s \colon W \to \mathbb{A}^1$ is assumed to be dominant.) When we apply Lemma 3.5, note that B is a codimension r subset by the face condition satisfied by V. It yields

$$\dim(p^*(V)|_{t=1} \cap (W \times F)) \quad \leq \dim(W \times F) - r$$

i.e. the intersection is proper.

The case where $u \colon W \to \mathbb{A}^1$ is not dominant is similar and much easier. So we omit the proof for this case.

We prove (3). Let F be a face of $\square^{n+1}$. If F is contained in $\square^n \times \{0\}$, our assertion follows from the assumption $V \in z^r(\mathbb{A}^d|D,n)_{\mathcal{W},e}$ tautologically. The case F is contained in $\square^n \times \{1\}$ was treated in (2). So let us suppose F has the form $F = F' \times \square^1$ where F' is a face of $\square^n$.

We embed $p^*(V) \cap (W \times F)$ into $\mathbb{A}^d_{k_{\mathrm{gen}}} \times F' \times \mathbb{A}^1$ by the inclusion

$$p^*(V) \cap (W \times F) \hookrightarrow \mathbb{A}^d_{k_{\mathrm{gen}}} \times F' \times \square^1$$

followed by the map

$$\mathbb{A}^d_{k_{\mathrm{gen}}} \times F' \times \square^1 \to \mathbb{A}^d_{k_{\mathrm{gen}}} \times F' \times \mathbb{A}^1; (\boldsymbol{x}, \boldsymbol{z}, t) \mapsto (\boldsymbol{x} + tu^s\boldsymbol{v}, \boldsymbol{z}, tu^s).$$

Its fiber over $\alpha \in \mathbb{A}^1_{k_{\mathrm{gen}}}$ looks like

$$V \cap ((W + \alpha\boldsymbol{v}) \times F') \text{ in } \mathbb{A}^d_{k_{\mathrm{gen}}} \times F'.$$

We can apply Lemma 3.5 to the situation

$$A = W \times F' \times \mathbb{A}^1 \subset X = \mathbb{A}^d \times F' \times \mathbb{A}^1,$$

$$B = (V \times_{\square^n} F') \times \mathbb{A}^1 \subset X,$$

$$\psi \colon \mathbb{A}^d_{k_{\mathrm{gen}}} \times F' \times \mathbb{A}^1 \to \mathbb{A}^d_{k_{\mathrm{gen}}};\ (\boldsymbol{x}, \boldsymbol{z}, \alpha) \mapsto \alpha\boldsymbol{v},$$

$$U = X \setminus \{\alpha = 0\}.$$

Then we find that irreducible components of the intersection

$$p^*(V) \cap (W \times F' \times \square^1)$$

which are not contained in $\{t = 0\}$ all have the right dimension. Dimensions of the components contained in $\{t = 0\}$ are bounded by the condition that $V \in z^r(\mathbb{A}^d|D,n)_{\mathcal{W},e}$ because $p^*(V)|_{t=0} = V$. Thus one sees the intersection has excess $\leq e(W)$. □

3.1.4

Here is a consequence of Proposition 3.3 and Lemma 3.4:

Proposition 3.6. *Suppose given $V \in z^r(\mathbb{A}^d|D,n)_{\mathcal{W},e}$. If $\boldsymbol{v} = \boldsymbol{v}_{\text{gen}}$ and $s \geq s(V)$, then we have*

$$p^*_{\boldsymbol{v}_{\text{gen}},s}(V) \in z^r(\mathbb{A}^d_{k_{\text{gen}}}|D,n+1)_{\mathcal{W},e}$$

and

$$p^*_{\boldsymbol{v}_{\text{gen}},s}(V)|_{t=1} \in z^r(\mathbb{A}^d_{k_{\text{gen}}}|D,n)_{\mathcal{W}}.$$

By applying this (partially defined) homotopy operator to arbitrary finitely generated subcomplexes of $z^r(\mathbb{A}^d|D,\bullet)_{\mathcal{W},e}$, we see that the scalar extension map

$$\frac{z^r(\mathbb{A}^d|D,\bullet)_{\mathcal{W},e}}{z^r(\mathbb{A}^d|D,\bullet)_{\mathcal{W}}} \to \frac{z^r(\mathbb{A}^d_{k_{\text{gen}}}|D,\bullet)_{\mathcal{W},e}}{z^r(\mathbb{A}^d_{k_{\text{gen}}}|D,\bullet)_{\mathcal{W}}}$$

induces the zero map on homology groups. On the other hand, since the extension k_{gen}/k is purely transcendental, by a standard specialization argument (Lemma 2.9) the scalar extension map should induce injective maps on homology groups (when the base field is finite we also use a trace argument to reduce the assertion to infinite field case). Therefore the homology groups of the first complex are all zero.

We have shown:

Theorem 3.7. *Let $D \subset \mathbb{A}^d$ be an effective Cartier divisor in an affine space over a field k. For any $\mathcal{W}$ and $e\colon \mathcal{W} \to \mathbb{N}$, the inclusion of the complexes*

$$z^r(\mathbb{A}^d|D,\bullet)_{\mathcal{W}} \subset z^r(\mathbb{A}^d|D,\bullet)_{\mathcal{W},e}$$

is a quasi-isomorphism.

3.2 Generalities on linear projection

In this §3.2 we review techniques involving linear projection, which is used to prove Theorem 1.6 out of Theorem 3.7. Everything is known and has been used to prove the corresponding result for Bloch's higher Chow theory in [Lev] (explicitly and implicitly).

We work over a base field k.

3.2.1 Terminologies

The projective terminology. Let $L \subset \mathbb{P}^N$ be a linear subscheme of codimension $d+1$. The Grassmannian variety parametrizing linear subschemes L' of codimension d in $\mathbb{P}^N$ containing L is a projective space of dimension d. Let us denote it by P_d.

Let $\mathcal{Q} \subset \mathbb{P}^N \times P_d$ be the incidence correspondence

$$\mathcal{Q} = \{(x,L') \mid x \in L'\} \subset \mathbb{P}^N \times P_d.$$

The first projection $\mathrm{pr}_1 \colon \mathcal{Q} \to \mathbb{P}^N$ is an isomorphism over the open set $\mathbb{P}^N \setminus L$. We define the **linear projection from** L as the composite

$$\pi_L \colon \mathbb{P}^N \setminus L \xleftarrow[\cong]{\mathrm{pr}_1} \mathcal{Q}_{|\mathbb{P}^N \setminus L} \xrightarrow{\mathrm{pr}_2} P_d.$$

Given a rational point $x \in \mathbb{P}^N \setminus L$, the fiber of π_L containing x is the linear subspace spanned by L and x.

Choosing a system of equations $F_i \in k[X_0, \ldots, X_N]$ $(0 \leq i \leq d)$ for L gives a trivialization $P_d \cong \mathbb{P}^d$, and π_L is written as:

$$\pi_L = (F_0 : \cdots : F_d) \colon \mathbb{P}^N \setminus L \to \mathbb{P}^d.$$

The affine terminology. Let $M \subset \mathbb{A}^N$ be a linear subspace of codimension d, corresponding to a linear subspace of the vector space k^N having codimension d. Let A_d be the affine space corresponding to the d-dimensional vector space k^N/M. Then we have the following map, called the **linear projection**

$$\pi_M \colon \mathbb{A}^N \to A_d.$$

The fiber of it containing a rational point $x \in \mathbb{A}^N$ is the affine linear subspace $x + M$.

If we choose a system of equations $f_1, \ldots, f_d \in k[x_1, \ldots, x_N]$ defining M, it determines a trivialization $A_d \cong \mathbb{A}^d$ and the linear projection is written as:

$$(f_1, \ldots, f_d) \colon \mathbb{A}^N \to \mathbb{A}^d.$$

The relation between the projective and the affine terminologies. Embed $\mathbb{A}^N$ into $\mathbb{P}^N$ naturally. Their coordinates are related by $x_i = X_i/X_0$. Write $\mathbb{P}^{N-1}_\infty = \mathbb{P}^N \setminus \mathbb{A}^N = \{X_0 = 0\}$, the hyperplane at the infinity. Choosing a linear subspace $M \subset \mathbb{A}^N$ of codimension d is equivalent to choosing a linear subspace $L \subset \mathbb{P}^N$ of codimension $d+1$ contained in $\mathbb{P}^{N-1}_\infty$ (the codimension in $\mathbb{P}^{N-1}_\infty$ is d).

By assigning to an affine linear subspace of the form $x + M \subset \mathbb{A}^N$ a linear subspace spanned by x and L in $\mathbb{P}^N$, we get a map $A_d \to P_d$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^N \setminus L & \xrightarrow{\pi_L} & P_d \\ \uparrow & & \uparrow \\ \mathbb{A}^N & \xrightarrow{\pi_M} & A_d. \end{array}$$

Moreover this diagram is Cartesian as one can easily see from the description by coordinates below.

Choosing a system f_i $(1 \leq i \leq d)$ of equations for M is equivalent to choosing a system of equations for L in the form

$$(X_0, F_1, \ldots, F_d)$$

where f_i and F_i are related by $f_i = F_i/X_0$. Then the above diagram takes the form:

$$\begin{array}{ccc} \mathbb{P}^N \setminus L & \xrightarrow{\pi_L=(X_0:F_1:\cdots:F_d)} & \mathbb{P}^d \\ \uparrow & & \uparrow \\ \mathbb{A}^N & \xrightarrow{\pi_M=(f_1,\ldots,f_d)} & \mathbb{A}^d. \end{array}$$

The following will be useful when we reduce the problems to a larger field.

Lemma 3.8. *Let X be a scheme of finite type over a field k. Let K be an overfield of k and suppose given a dense open subset $U_{(K)}$ of X_K. Then there is a dense open subset U of X such that for any algebraic closure $\bar{k}$ of k and any common overfield Ω of $\bar{k}$ and K,*

$$\begin{array}{ccc} \bar{k} & \subset & \Omega \\ \cup & & \cup \\ k & \subset & K \end{array}$$

we have the inclusion of subsets in $X(\Omega)$:

$$U(\bar{k}) \subset U_{(K)}(\Omega).$$

Proof. First we may assume K/k is a finitely generated field extension. Indeed, the open set $U_{(K)}$ is defined over some subfield K' of K finitely generated over k. Let us denote the dense open set by $U_{(K')} \subset X_{K'}$. Then if an open set $U \subset X$ solves the problem for $U_{(K')}$, it solves the problem for $U_{(K)}$.

So let us assume K is finitely generated over k throughout the proof. First suppose K is algebraic over k; it is a finite extension of k. The projection $p_{K/k} \colon X_K \to X$ is finite. Therefore the set $Z := p_{K/k}(X_K \setminus U_{(K)}) \subset X$ is a closed subset of X containing no generic point. Thus the set $U := X \setminus Z$ is a dense open subset of X. Since we have $p_{K/k}^{-1}(U) \subset U_{(K)}$, it clearly solves our problem.

Next we will observe that if our assertion is true for algebraically closed base fields, then our assertion is true in general. So let us assume our assertion is true for $\bar{k}$ and verify the assertion for k. The algebra $(\bar{k} \otimes_k K)_{\mathrm{red}}$ is a finite product of fields, say

$$(\bar{k} \otimes_k K)_{\mathrm{red}} \cong \prod_i K_i.$$

Put $U_{(K_i)} = U_{(K)} \times_K K_i \subset X_{K_i}$. By the assumption we can take a dense open subset $U_{(\bar{k})} \subset X_{\bar{k}}$ which solves the problem for all $U_{(K_i)}$; for any common overfield Ω of $\bar{k}$ and K_i (for an i), we have

$$U_{(\bar{k})}(\bar{k}) \subset U_{(K_i)}(\Omega) \tag{4}$$

in $X(\Omega)$.

Now $U_{(\bar{k})}$ is defined over some finite subextention k' of $\bar{k}/k$, say $U_{(k')} \subset X_{k'}$. Set $U := X \setminus \big(p_{k'/k}(X_{k'} \setminus U_{(k')})\big)$, a dense open subset of X as we have seen above. Then we have $p^{-1}_{\bar{k}/k}(U) \subset U_{(\bar{k})}$, *a fortiori* we have

$$U(\bar{k}) \subset U_{(\bar{k})}(\bar{k}). \tag{5}$$

We can check this U solves the problem: suppose given a common overfield Ω of $\bar{k}$ and K. There is an index i such that the inclusion maps $\bar{k} \to \Omega$ and $K \to \Omega$ factor through an inclusion $K_i \to \Omega$. Then we have a relation (4). Combined with (5), we get

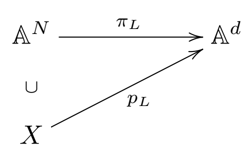
$$U(\bar{k}) \subset U_{(K)}(\Omega).$$

We deal with the case $k = \bar{k}$. In this case every closed point of X is a k-rational point. Write K as the function field of an integral scheme V over k with the generic point η. $U_{(K)}$ is the restriction of an open subset $U_{(V)} \subset X \times_k V$. Let $\mathrm{pr}_1 \colon X \times_k V \to X$ be the first projection. It is a flat surjective map, so $U := \mathrm{pr}_1(U_{(V)}) \subset X$ is a dense open subset. We will check this U solves our problem. Indeed, suppose given an overfield Ω of K. We have to show $U(\bar{k}) \subset U_{(K)}(\Omega)$ in $X(\Omega)$. It suffices to show $U(\bar{k}) \subset U_{(K)}(K)$. Let $x \in X(\bar{k})$ be a $\bar{k}$-rational point. Its image in $X(K)$ is represented by $(x, \eta) \in X \times_{\bar{k}} \eta$. Suppose it does not belong to $U_{(K)}$. Then the closure of the point $(x, \eta)^- = x \times_{\bar{k}} V$ in $X \times_{\bar{k}} V$ is contained in the closed subset $(X \times_{\bar{k}} V) \setminus U_{(V)}$. Therefore x does not lie in the image of $U_{(V)}$ by the projection $X \times V \to X$. Thus we have shown $U(\bar{k})$ maps into $U_{(K)}(K)$. $\square$

3.2.2 Notation

We will keep the following notation throughout the rest of §3.2: Let X be an affine equidimensional scheme embedded in an affine space $\mathbb{A}^N$. Let $\mathbb{A}^N \subset \mathbb{P}^N$ be the natural open embedding. Write $\mathbb{P}^{N-1}_\infty = \mathbb{P}^N \setminus \mathbb{A}^N$. Let us allow ourselves mixed usage of projective and affine terminologies on linear projection.

Let $d \leq N$ be a positive integer. For $L \in \mathrm{Gr}(N-d-1, \mathbb{P}^{N-1}_\infty)$ (which we may think defined by d linear functions on $\mathbb{A}^N$), a surjective linear morphism $\pi_L \colon \mathbb{A}^N \to \mathbb{A}^d$ is defined (by the d functions) well-defined up to linear automorphism on the target. Denote by p_L its restriction to X. The fiber of π_L passing a rational point $x \in \mathbb{A}^N$ is equal to $x + \ker \pi_L$.



When we say some property holds **for a general** $L \in \mathrm{Gr}(N-d-1, \mathbb{P}^{N-1}_\infty)$, let us mean the property holds on the set of closed points of a dense open subset of $\mathrm{Gr}(N-d-1, \mathbb{P}^{N-1}_\infty)$. Our assertions made below does not depend on the ambiguity of linear automorphism on the target (or you may use the affine space A_d as the target to avoid any ambiguity).

3.2.3 Avoidance

Lemma 3.9. *Suppose given a point $x \in \mathbb{A}^N$ and a closed subset $Z \subset \mathbb{A}^N$ of dimension $< d$ not containing x. Then for a general L, we have $\pi_L(x) \notin \pi_L(Z)$. (Consequently, finitely many distinct points with closure dimension $< d$ have distinct images for a general L.)*

Proof. By Lemma 3.8, after scalar extension, we may assume x is a k-rational point and k is algebraically closed. A general codimension d linear subspace passing through x does not meet Z by the assumption on dimension. □

Definition 3.10. Let $\overline{X}$ be the closure of X in $\mathbb{P}^N$. Denote by $\mathcal{U}_X$ the open subset of $\mathrm{Gr}(N-d-1, \mathbb{P}^{N-1}_\infty)$ consisting of linear subspaces L of $\mathbb{P}^{N-1}_\infty$ which does not meet $\overline{X}$. (If $\dim \leq d$, it is a dense open subset. Otherwise it is empty.)

If $L \in \mathcal{U}_X$, then the map $p_L\colon X \to \mathbb{A}^d$ is finite and surjective. It is flat on the Cohen-Macaulay locus of X [EGA IV_2, (6.1.5)].

3.2.4 Smoothness

Let $m \geq 1$ be an integer. Denote by V the vector space of polynomials in $(x_1, \ldots, x_d)$ of degree $\leq m$. It defines an affine space over k, also denoted by V. The **Veronese embedding of degree** m is the closed embedding

$$\mathbb{A}^N \hookrightarrow V$$

corresponding to the inclusion (which is a k-linear map from a vector space to a k-algebra)

$$V \hookrightarrow k[x_1, \ldots, x_d].$$

The **Veronese reembedding** of degree m of an affine embedding refers to the composite of a given embedding $X \subset \mathbb{A}^N$ followed by the Veronese embedding $\mathbb{A}^N \subset V$ of degree m.

Lemma 3.11. *Suppose* $\dim X \geq d$ *and given* $x \in X$*. After any Veronese reimbedding of degree ≥ 2 of the original affine embedding, a general p_L is smooth on the subset $p_L^{-1} p_L(x) \cap X_{\mathrm{sm}}$.*

Proof. By the flat descent of smoothness and Lemma 3.8, we may assume k is algebraically closed and x is a closed point. A general L satisfies the condition that $x + \ker \pi_L$ meets X properly. In this case the map p_L is equidimensional around $p_L^{-1} p_L(x)$. Therefore it is flat on X_{sm} by [EGA IV_2, (6.1.5)]. By Bertini's smoothness theorem [SGA4, XI, 2.1], the fiber $p_L^{-1} p_L(x) \cap X_{\mathrm{sm}}$ is smooth. This completes the proof. □

3.2.5 Birationality

Lemma 3.12. *Suppose* $\dim X \leq d$ *and let* $x \in X_{\mathrm{sm}}$ *be a point whose closure has dimension $< d$, and set $y := p_L(x)$. Then the induced map $k(y) \to k(x)$ is an isomorphism for a general L.*

Proof. By Lemma 3.11, a general p_L is unramified at x. Therefore the extension $k(x)/k(y)$ is finite and separable. We have to prove it is also purely inseparable for a general L.

Let $\overline{k(x)}$ be an algebraic closure of $k(x)$. The points of $x \times_k \overline{k(x)} \subset X \times_k \overline{k(x)}$ are generic points of several irreducible subsets of dimension $< d$. By Lemma 3.9, for a general $L \in \mathcal{U}_{X_{\overline{k(x)}}}$, these points have distinct images by p_L. Therefore by Lemma 3.8, a general $L \in \mathcal{U}_X$, the map p_L induces an injection on the set of $\overline{k(x)}$-valued points over $x \to y$. This implies that the extension $k(x)/k(y)$ is purely inseparable. □

3.2.6 Chow's moving lemma

Definition 3.13. Let $L \in \mathcal{U}_X$. For a constructible subset Z of X, define a new constructible subset

$$L^+ Z := \left((p_L^{-1} p_L(Z)) \setminus Z\right)^-$$

where we take the closure $(-)^-$ inside $p_L^{-1} p_L(Z)$.

Definition 3.14. For irreducible constructible subsets A, B of X, set

$$i(A,B) = \max\{-1, \dim A + \dim B - \dim X\}$$

(the "ideal dimension" for the intersection $A \cap B$) and

$$e(A,B) = \max\{0, \dim(A \cap B) - i(A,B)\}$$

(the **excess dimension** of the intersection $A \cap B$) where we set $\dim(\emptyset) = -1$ by convention. When A and B are constructible subsets not necessarily irreducible, we define $e(A,B)$ as the maximum of componentwise $e(-,-)$'s.

Lemma 3.15 (Chow's moving lemma)**.** *Keep the notation in* §3.2.2. *Let* $d = \dim X$. *Let* Z, W *two irreducible constructible subsets of* X. *Assume* X *is smooth at each generic point of* $Z \cap W$. *Then for a general* L *we have:*

$$e(L^+ Z, W) \leq \max\{0, e(Z,W) - 1\}.$$

Proof. This is essentially known, cf. [Lev, Part I, Chap. II, 3.5.4]. We may restrict ourselves to those L's such that $p_L : X \to \mathbb{A}^d$ is finite. Let $\mathrm{ram}(p_L)$ be the closed subset of X where p_L is not étale.

As we have tautologically

$$L^+(Z) \cap W \ \subset\ \left[L^+(Z) \cap W \setminus (\mathrm{ram}(p_L) \cap Z \cap W)\right] \ \cup\ \left[\mathrm{ram}(p_L) \cap Z \cap W\right],$$

it suffices to control the dimensions of $L^+(Z) \cap W \setminus (\mathrm{ram}(p_L) \cap Z \cap W)$ and $(\mathrm{ram}(p_L) \cap Z \cap W)$.

For the second one, if $\mathrm{ram}(p_L)$ does not contain any generic point of $Z \cap W$, it has dimension $\leq \dim(Z \cap W) - 1$ or is the empty set. Such L's form a dense open subset of the Grassmannian because of our smoothness assumption (Lemma 3.11).

To handle the first one, we introduce some notation. Let $l(Z,W) \subset \mathrm{Gr}(1,\mathbb{P}^N)$ be the constructible subset consisting of lines l such that there are points $z \in Z$ and $w \in W$, $z \neq w$, on l. Clearly $\dim l(Z,W) \leq \dim Z + \dim W$. For each integer $j \geq 0$, let S_j be the set of points $x \in \mathbb{P}^{N-1}_\infty$ whose fiber of the following map ("direction")

$$\begin{array}{cccc} \delta\colon & l(Z,W) & \to & \mathbb{P}^{N-1}_\infty \\ & l & \mapsto & l \cap \mathbb{P}^{N-1}_\infty \end{array}$$

has dimension j. Let $S_j = \bigcup_\lambda S_j^{(\lambda)}$ be its irreducible decomposition. Note that one has for any $j \geq 0$ and λ:

$$j + \dim S_j^{(\lambda)} \leq \dim l(Z,W) \leq \dim Z + \dim W. \tag{6}$$

Let us recall:

Lemma 3.16 ([Rob, Lem.6]). *If a geometric point $x \in X(\overline{k})$ lands on $L^+(Z) \setminus (\mathrm{ram}(p_L) \cap Z)$, there is a $y \in Z(\overline{k})$, different from x, such that $p_L(x) = p_L(y)$.*

Therefore if a geometric point $x \in X(\overline{k})$ lands on $(L^+(Z) \setminus (\mathrm{ram}(p_L) \cap Z)) \cap W$, there is an $l \in l(Z,W)$ with $x \in l$ and $l \cap L \neq \emptyset$. So we can consider the diagram of constructible subsets of schemes

$$\begin{array}{ccc} X \supset \text{ (image)} & \overset{\text{Lem.3.16}}{\supset} & (L^+(Z) \setminus \mathrm{ram}(p_L) \cap Z) \cap W \\ \mathrm{pr}_1 \nwarrow & & \\ & \left\{ (x,l) \;\middle|\; \begin{array}{c} x \in X,\ x \in l \in l(Z,W) \\ \text{and } \delta\colon l(Z,W) \to \mathbb{P}^{N-1}_\infty \text{ maps} \\ l \text{ into } L \subset \mathbb{P}^{N-1}_\infty \end{array} \right\} & \\ & \downarrow \mathrm{pr}_2 & \\ & l(Z,W) \supset \quad \delta^{-1}(L) \xrightarrow{\delta} L & \end{array}$$

where the map pr_2 is quasi-finite because p_L is finite on X. So we have

$$\dim \big((L^+(Z) \setminus \mathrm{ram}(p_L) \cap Z) \cap W \big) \leq \dim \delta^{-1}(L).$$

At this point we consider the condition:

- L should meet every $S_j^{(\lambda)}$ properly in $\mathbb{P}^{N-1}_\infty$.

This condition is true on a dense open subset of $\mathrm{Gr}(N-1-d, \mathbb{P}^{N-1}_\infty)$. For such an L we have

$$\begin{aligned} \dim(\delta^{-1}(L)) &= \sup_{j,\lambda} \left\{ \dim(\delta^{-1}(L \cap S_j^{(\lambda)})) \right\}, \\ \dim(\delta^{-1}(L \cap S_j^{(\lambda)})) &= \dim(L \cap S_j^{(\lambda)}) + j \\ &\leq \dim(S_j^{(\lambda)}) - d + j \quad \text{(because } L \text{ meets } S_j^{(\lambda)} \text{ properly)} \\ &\leq \dim Z + \dim W - d \quad \text{(by (6))} \end{aligned}$$

Therefore for such an L we have

$$\dim\left[(L^+(Z)\setminus(\mathrm{ram}(p_L)\cap Z)\cap W)\right]\le \dim Z+\dim W-d,$$

the right hand side being the *right* dimension for the intersection of Z and W. This handles the first one and completes the proof. $\square$

3.2.7 Higher Chow-like situations

Keep the notation in §3.2.2 and let $d=\dim X$.

For $L\in\mathcal{U}_X$ we will denote $p_L\times \mathrm{id}_{\square^n}: X\times\square^n\to\mathbb{A}^d\times\square^n$ also by p_L when no confusion can arise. For a constructible subset V of $X\times\square^n$, we will denote by L^+V the set

$$(p_L^{-1}p_LV\setminus V)^-$$

where the closure is taken in $p_L^{-1}p_LV$. It is a constructible subset of $X\times\square^n$.

Lemma 3.17. *Let $V'\subset X\times\square^n$ be an irreducible constructible set with generic point η. Let $Z\subset X\times\square^n$ be a constructible set containing no generic point of $X\times_k\mathrm{pr}_2(\eta)$. Assume $X\times_k\mathrm{pr}_2(\eta)$ is smooth over $\mathrm{pr}_2(\eta)$ at each generic point of $(V'\times_{\square^n}\mathrm{pr}_2(\eta))\cap(Z\times_{\square^n}\mathrm{pr}_2(\eta))$ (for example X is smooth).*

Then for a general $L\in\mathcal{U}_X$, no irreducible component of L^+V' is contained in Z.

Proof. We apply Chow's moving lemma 3.15 to constructible subsets

$$V'\times_{\square^n}\mathrm{pr}_2(\eta)\text{ and }Z\times_{\square^n}\mathrm{pr}_2(\eta)\quad\text{ of }X\times_k\mathrm{pr}_2(\eta).$$

Taking into account the fact that Z contains no generic point of $X_{\mathrm{pr}_2(\eta)}$, we find that a general $L\in\mathcal{U}_X\times_k\mathrm{pr}_2(\eta)$ satisfies the condition that no component of $L^+(V'\times_{\square^n}\mathrm{pr}_2(\eta))$ is contained in $Z\times_{\square^n}\mathrm{pr}_2(\eta)$. Using Lemma 3.17, this holds for a general $L\in\mathcal{U}_X$. Since the generic points of L^+V' are all on $L^+(V'\times_{\square^n}\mathrm{pr}_2(\eta))$, we get our assertion. $\square$

Lemma 3.18. *Let $V\subset X\times\square^n$ be a prime cycle with generic point η and assume η is not a generic point of $X\times_k\mathrm{pr}_2(\eta)$. Assume moreover that X is smooth at $\mathrm{pr}_1(\eta)\in X$. Then for a general L, the support of $p_L^{-1}p_{L*}V-V$ is $L^+|V|$. (I.e. the component V appears in $p_L^{-1}p_{L*}V$ with multiplicity 1.)*

Proof. The assertion will follow from properties that the map $X\times\square^n\to\mathbb{A}^d\times\square^n$ is étale at η, and that the composite

$$V\subset X\times\square^n\to\mathbb{A}^d\times\square^n$$

is a birational morphism to the image.

The first follows from Lemma 3.11 applied to $\mathrm{pr}_1(\eta)\in X$.

We consider the second property. Consider $\eta\in X\times_k\mathrm{pr}_2(\eta)$. Then by Lemma 3.12, the property holds for a general $L\in\mathcal{U}_{X_{\mathrm{pr}_2(\eta)}}$. By Lemma 3.8 the property also holds for a general $L\in\mathcal{U}_X$. $\square$

Definition 3.19. Given a closed subset $V \subset X \times \square^n$, an integer $i \geq 0$ and a face of $F \subset \square^n$, let $V_F^{\geq i} \subset V$ be the set of $y \in V \times_{\square^n} F$ such that the fiber of $\mathrm{pr}_1 \colon V \times_{\square^n} F \to X$ has dimension $\geq i$ around y. By the semi-continuity theorem of Chevalley on fiber dimensions 2.2, $V_F^{\geq i}$ is a closed subset. Put $Z_F^{\geq i}(V) := \mathrm{pr}_1(V_F^{\geq i})$, a constructible subset of X. It is the set of $x \in X$ such that

$$\dim (V \times_{X \times \square^n} (x \times F)) \geq i.$$

Lemma 3.20. *Assume X is smooth (but see Remark 3.21 below). Let $V \subset X \times \square^n$ be an irreducible closed subset with generic point η. Assume η is not a generic point of $X \times_k \mathrm{pr}_2(\eta)$. Let $i \geq 0$ be an integer and assume $Z_F^{\geq i}(V)$ is not dense in X. Then for a general $L \in \mathcal{U}_X$, we have the next inclusion of subsets of X:*

$$Z_F^{\geq i}(L^+V) \subset L^+(Z_F^{\geq i}(V)).$$

Moreover, the first is dense in the second.

Proof. First we find $p_L^{-1}p_L(V_F^{\geq i}) = (p_L^{-1}p_LV)_F^{\geq i}$ because p_L is finite. By Lemma 3.17 we may assume no irreducible component of $L^+(V_F^{\geq i})$ is contained in V by taking L sufficiently generally. Then $p_L^{-1}p_L(V_F^{\geq i}) \setminus V$ and $(p_L^{-1}p_LV)_F^{\geq i} \setminus V = (p_L^{-1}p_LV \setminus V)_F^{\geq i}$ has the same set of generic points. So we have an equality of their closures: $(L^+V)_F^{\geq i} = L^+(V_F^{\geq i})$. Thus the problem is whether or not we have

$$\mathrm{pr}_1(L^+(V_F^{\geq i})) \overset{?}{\subset} L^+(Z_F^{\geq i}(V)),$$

i.e. whether pr_1 maps $L^+(V_F^{\geq i})$ into $L^+(Z_F^{\geq i}(V))$ (and the density of the image).

Let us denote $Z = Z_F^{\geq i}(V)$. Using the trivial fact $X \times \square^n = X \times_{\mathbb{A}^d} (\mathbb{A}^d \times \square^n)$ we have

$$\mathrm{pr}_1 p_L^{-1}p_L(V_F^{\geq i}) = p_L^{-1}p_L(Z).$$

Therefore we have

$$\begin{array}{ccc} p_L^{-1}p_L(V_F^{\geq i}) \setminus \mathrm{pr}_1^{-1}(Z) & \subset & p_L^{-1}p_L(V_F^{\geq i}) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ p_L^{-1}p_L(Z) \setminus (Z) & \subset & p_L^{-1}p_L(Z) \end{array}$$

By Lemma 3.17, we may assue no irreducible component of $L^+(V_F^{\geq i})$ is contained in $\mathrm{pr}_1^{-1}(Z)$. Hence $L^+(V_F^{\geq i})$ equals to the closure of $p_L^{-1}p_L(V_F^{\geq i}) \setminus \mathrm{pr}_1^{-1}(Z)$ in $p_L^{-1}p_L(V_F^{\geq i})$. Therefore taking into account that pr_1 is a continuous map we find that pr_1 maps $L^+(V_F^{\geq i})$ into L^+Z, and the image is dense. $\square$

Remark 3.21. From the proof, it is clear that we only needed the smoothness of X at some finitely many points.

3.3 The general case

Theorem 3.22. *Let X be a smooth affine scheme. Suppose given a finite set $\mathcal{W}$ of irreducible constructible subsets of X and a function $e\colon \mathcal{W} \to \mathbb{N}$. Let $D \subset \mathbb{A}^m$ be any effective Cartier divisor. Denote by $\mathcal{W}\times\mathbb{A}^m$ the set of irreducible constructible subsets $W \times \mathbb{A}^m$ ($W \in \mathcal{W}$) of $X \times \mathbb{A}^m$. It is in bijection with $\mathcal{W}$. Denote the function*

$$\mathcal{W} \times \mathbb{A}^m \overset{\cong}{\leftarrow} \mathcal{W} \xrightarrow{e} \mathbb{N}$$

also by the letter e. Then the inclusion

$$z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e} \subset z^r(X \times \mathbb{A}^m | X \times D, \bullet)$$

is a quasi-isomorphism.

Proof. Embed X into any affine space $\mathbb{A}^N$ as a closed subscheme. Put $d = \dim X$. By a trace argument we may assume the base field k is infinite. We will apply the techniques recalled in §3.2 (regarding $\mathbb{A}^m \times \square^n$ as the affine space $\square^{m+n}$).

So, for a face $F \subset \square^n$, an integer $i \geq 0$ and a cycle V on $X \times \mathbb{A}^m \times \square^n$, put

$$V_F^{\geq i} \subset |V|$$

to be the closed set of points $y \in |V| \cap (X \times \mathbb{A}^m \times F)$ around which the fiber of the projection

$$|V| \cap (X \times \mathbb{A}^m \times F) \to X$$

has dimension $\geq i$. Let $Z_F^{\geq i}(V) \subset X$ be its image.

Lemma 3.23. *Let $V \in \underline{z}^r(X \times \mathbb{A}^m | X \times D, n)$. It belongs to $\underline{z}^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W}\times\mathbb{A}^m, e}$ if and only if we have*

$$\dim(Z_F^{\geq i}(V) \cap W) + i \leq \dim(W \times \mathbb{A}^m \times F) - r + e(W) \tag{7}$$

for all $i \geq 0$, faces $F \subset \square^n$ and $W \in \mathcal{W}$.

[Let $Z_F^{\geq i}(V) = \cup_\mu Z_F^{\geq i}(V)^\mu$ be the irreducible decomposition. We will use (7) in the equivalent form

$$e(Z_F^{\geq i}(V)^\mu, W) \leq \dim(X \times \mathbb{A}^m) + \dim F + e(W) - (\dim Z_F^{\geq i}(V)^\mu + r + i)$$

for all $i \geq 0$, faces F, $W \in \mathcal{W}$ and components μ.]

Proof of Lemma. Suppose $V \in \underline{z}^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W}\times\mathbb{A}^m, e}$; it says

$$\dim(|V| \cap (W \times \mathbb{A}^m \times F)) \leq \dim(W \times \mathbb{A}^m \times F) - r + e(W).$$

Then because we have inclusion $V_F^{\geq i} \subset |V| \cap (X \times \mathbb{A}^m \times F)$, we have

$$\dim(V_F^{\geq i} \cap (W \times \mathbb{A}^m \times F)) \leq \dim(W \times \mathbb{A}^m \times F) - r + e(W).$$

Since the surjection $V_F^{\geq i} \cap (W \times \mathbb{A}^m \times F) \twoheadrightarrow Z_F^{\geq i}(V) \cap W$ has fibers of dimension $\geq i$, we have

$$\dim(Z_F^{\geq i}(V) \cap W) + i \leq \dim(V_F^{\geq i} \cap (W \times \mathbb{A}^m \times F)).$$

The last two inequalities imply (7).

Next, suppose we have inequalities (7). Let η be any generic point of $|V| \cap (W \times \mathbb{A}^m \times F)$. Let $i \geq 0$ be the dimension of the fiber of the projection

$$|V| \cap (W \times \mathbb{A}^m \times F) \to X$$

around η. Then η lands on $Z_F^{\geq i}(V) \cap W$ by the projection to X. Hence the irreducible component of $|V| \cap (W \times \mathbb{A}^m \times F)$ represented by η has dimension

$$\leq \dim(Z_F^{\geq i}(V) \cap W) + i.$$

By (7) this is

$$\leq \dim(W \times \mathbb{A}^m \times F) - r + e(W).$$

Therefore we have $V \in \underline{z}^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W} \times \mathbb{A}^m, e}$. $\square$

We are going to prove Theorem 3.22 by showing the complex

$$\frac{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W} \times \mathbb{A}^m, e}}{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W} \times \mathbb{A}^m, e-1}}$$

is acyclic for any $\mathcal{W}$ and e. Here the function $e - 1 \colon \mathcal{W} \to \mathbb{N}$ is defined by $(e - 1)(W) = \max\{0, e(W) - 1\}$. Take any finitely generated subcomplex $z^r(\bullet)'_{e/e-1}$ of it and fix a finite generating set; we may assume the generating set consists of prime cycles by enlarging $z^r(\bullet)'_{e/e-1}$ a little. We can make general position arguments appearing below work for all of these generators simultaneously.

Suppose given a prime cycle $V \in z^r(X \times \mathbb{A}^m | X \times D, n)$ on $X \times \mathbb{A}^m \times \square^n$ and suppose it is an irreducible component of a closed subset of type

$$X \times T$$

where T is an irreducible closed subset of $\mathbb{A}^m \times \square^n$. (In this case T necessarily meets faces of $\square^n$ properly.) Then the cycle V belongs to $z^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W}}$, because for any $W \in \mathcal{W}$ and $F \subset \square^n$ we have

$$\begin{aligned} \dim(V \cap (W \times \mathbb{A}^m \times F)) &\leq \dim(W \times ((\mathbb{A}^m \times F) \cap T)) \\ &\leq \dim(W \times \mathbb{A}^m \times F) - r \end{aligned}$$

Therefore we may assume our generating set doesn't include this kind of cycles.

Now take a cycle V from our generating set.

Claim 3.24. For a general $L \in \mathcal{U}_X$, we have

$$p_L^* p_{L*} V - V \in z^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W} \times \mathbb{A}^m, e-1}.$$

Proof of Claim. Note that by Lemma 3.18 and the assumption we've just made, the support of $p_L^* p_{L*} V - V$ is $L^+|V|$ for a general L. So for proving the assertion, according to Lemma 3.23, we have to show

$$e(Z_F^{\geq i}(L^+|V|)^\nu, W) \leq \dim(X \times \mathbb{A}^m) + \dim F + (e-1)(W) - (\dim Z_F^{\geq i}(L^+|V|)^\nu + r + i) \quad (8)$$

for all $i \geq 0$, faces F, $W \in \mathcal{W}$ and components ν, where we have taken irreducible decomposition

$$Z_F^{\geq i}(L^+|V|) = \bigcup_\nu Z_F^{\geq i}(L^+|V|)^\nu.$$

Take any i, F, W, ν. First suppose $Z_F^{\geq i}(L^+|V|)^\nu$ is dense in X; then the left hand side is 0 and there is nothing to prove.

So let us assume $Z_F^{\geq i}(L^+|V|)^\nu$ is not dense in X. Since p_L is finite, $Z_F^{\geq i}(V)$ is not dense either. By $V \in z^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W} \times \mathbb{A}^m, e}$ and Lemma 3.23 we have

$$e(Z_F^{\geq i}(V)^\mu, W) \leq \dim(X \times \mathbb{A}^m) + \dim F + e(W) - (\dim Z_F^{\geq i}(V)^\mu + r + i)$$

for any irreducible component μ of $Z_F^{\geq i}(V)$. By Chow's moving lemma 3.15 we have

$$e(L^+(Z_F^{\geq i}(V)^\mu), W) \leq \dim(X \times \mathbb{A}^m) + \dim F + (e-1)(W) - (\dim Z_F^{\geq i}(V)^\mu + r + i).$$

By Lemma 3.20 and Remark 3.21 we know $Z_F^{\geq i}(L^+|V|) \subset L^+(Z_F^{\geq i}(V))$ and it is a dense inclusion. Therefore for any component ν of $Z_F^{\geq i}(L^+|V|)$, there is a component μ of $L^+(Z_F^{\geq i}(V))$ containing it and sharing the generic point. Thus the previous inequality implies:

$$e((Z_F^{\geq i}(L^+|V|)^\nu), W) \leq \dim(X \times \mathbb{A}^m) + \dim F + (e-1)(W) - (\dim Z_F^{\geq i}(L^+|V|)^\nu + r + i).$$

By Lemma 3.23, it says $p_L^{-1} p_L V - V$ belongs to the smaller subgroup:

$$p_L^{-1} p_L V - V \in z^r(X \times \mathbb{A}^m | X \times D, n)_{\mathcal{W} \times \mathbb{A}^m, e-1}.$$

This proves Claim 3.24. □

Using Lemma 3.9, choosing L generally, we may assume $p_L(W)$ are different subsets of $\mathbb{A}^d$ for different $W \in \mathcal{W}$. Then the definitions

$$p_* \mathcal{W} = \{p(W) \subset \mathbb{A}^d | W \in \mathcal{W}\}$$

(a set of irreducible constructible subsets of $\mathbb{A}^d$) and

$$p_*e\colon p_*\mathcal{W} \to \mathbb{N};\ p(W) \mapsto e(W)$$

make sense.

By Claim 3.24 we have a diagram

$$z^r(\bullet)'_{e/e-1} \xrightarrow{p_{L*}} \frac{z^r(\mathbb{A}^d \times \mathbb{A}^m | \mathbb{A}^d \times D)_{p_*\mathcal{W}\times\mathbb{A}^m, p_*e}}{z^r(\mathbb{A}^d \times \mathbb{A}^m | \mathbb{A}^d \times D)_{p_*\mathcal{W}\times\mathbb{A}^m, p_*e-1}} \xrightarrow{p_L^*} \frac{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e}}{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e-1}}.$$

Since the middle term is acyclic by Theorem 3.7, the composite $p_L^* p_{L*}$ induces the zero map on homology. Also by Claim 3.24 the cycle $p_L^* p_{L*} V - V$ is zero as an element of the last term.

Therefore the map

$$\text{incl.} = p_L^* p_{L*} - [p_L^* p_{L*} - \text{incl.}]\colon z^r(\bullet)'_{e/e-1} \to \frac{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e}}{z^r(X \times \mathbb{A}^m | X \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e-1}} \quad (9)$$

induces the zero map on homology. Since this holds for any finitely generated subcomplex $z^r(\bullet)'_{e/e-1}$ of the right hand side, the right hand side is acyclic. This completes the proof of Theorem 3.22. □

3.4 Functoriality

Let $D \subset \mathbb{A}^m_k$ be an effective Cartier divisor. Let $f\colon X \to Y$ be a map from an equidimensional algebraic scheme over k to a smooth affine equidimensional k-scheme. Let $Z^{\geq i}(f) \subset Y$ be the constructible subset consisting of points where the fiber of f has dimension $\geq i$.

Suppose given $V \in \underline{z}^r(Y \times \mathbb{A}^m | Y \times D, n)$ and let us consider if we can define a cycle $f^*V \in \underline{z}^r(X \times \mathbb{A}^m | X \times D, n)$. (By abuse of notation we wrote f for $f \times \mathrm{id}_{\mathbb{A}^m \times \square^n}\colon X \times \mathbb{A}^m \times \square^n \to Y \times \mathbb{A}^m \times \square^n$.)

First we'd like to have $\mathrm{codim}_{X\times\mathbb{A}^m\times\square^n}(f^{-1}(|V|)) \geq r$ to have a well-defined cycle f^*V on $X \times \mathbb{A}^m \times \square^n$. Furthermore the cycle f^*V has to satisfy the face condition. The modulus condition will be automatically true.

The first and the face conditions are summarized as follows: for any face F of $\square^n$, we have

$$\mathrm{codim}_{X\times\mathbb{A}^m\times F}(f^{-1}(|V| \times_{\square^n} F)) \geq r.$$

This is equivalent to:

$$\dim(|V| \times_{Y\times\square^n} \left(Z^{\geq i}(f) \times F\right)) + i \leq \dim(X \times \mathbb{A}^m \times F) - r$$

for all $i \geq 0$ and F. If $Z^{\geq i}(f) = \cup_\nu Z^{\geq i}(f)^\nu$ is the irreducible decomposition, it can be written, using excess of intersections of subsets:

$$e(|V|, Z^{\geq i}(f) \times \mathbb{A}^m \times F) \leq \dim X - \dim(Z^{\geq i}(f)^\nu) - i.$$

So, if we set $\mathcal{W} := \{Z^{\geq i}(f)^\nu\}_{i,\nu}$ and define a function e by:

$$e\colon \mathcal{W} \to \mathbb{N};\ Z^{\geq i}(f)^\nu \mapsto \dim X - \dim(Z^{\geq i}(f)^\nu) - i,$$

the condition is equivalent to

$$V \in \underline{z}^r(Y \times \mathbb{A}^m | Y \times D, n)_{\mathcal{W}\times\mathbb{A}^m, e}.$$

Therefore the pull-back operation f^* is well-defined on this complex. By Theorem 3.22, our subcomplex

$$z^r(Y \times \mathbb{A}^m | Y \times D, \bullet)_{\mathcal{W}\times\mathbb{A}^m, e} \subset z^r(Y \times \mathbb{A}^m | Y \times D, \bullet)$$

is quasi-isomorphic to the whole.

Thus we have proven:

Theorem 3.25. *Let $D \subset \mathbb{A}^m_k$ be an effective Cartier divisor and let $f\colon X \to Y$ be a map from an equidimensional algebraic scheme over k to a smooth affine equidimensional k-scheme.*

Then there is a pull-back map in the derived category of complexes of abelian groups:

$$f^*\colon z^r(Y \times \mathbb{A}^m | Y \times D, \bullet) \to z^r(X \times \mathbb{A}^m | X \times D, \bullet).$$

4 Theorem for higher Chow groups with modulus

In this section we prove Theorem 1.2. The basic strategy is the same as in §3. In this case the technique of linear projection to the affine space causes a new trouble; for an affine equidimensional scheme X equipped with an effective Cartier divisor D, there might be no finite surjective map $X \to \mathbb{A}^d$ such that the divisor D is the pull-back of some divisor on $\mathbb{A}^d$. Fortunately, it turns out that such maps can be canstructed after Ninevich localization (§4.2). That is why Theorem 1.2 involves Nisnevich topology.

4.1 The case of affine spaces over a discrete valuation ring

Let R be a discrete valuation ring over k and u be a uniformizer. Denote by K the fraction field of R. Denote by $\kappa = R/(u)$ the residue field of R.

Theorem 4.1. *Let $D = \mathbb{A}^d_\kappa = (u)$ be the divisor on $\mathbb{A}^d_R = \operatorname{Spec}(R[x_1, \dots, x_d])$ defined by u. Let $\mathcal{W}$ be a finite set of irreducible constructible subsets of $\mathbb{A}^d_K = \mathbb{A}^d_R \setminus D$ and $e\colon \mathcal{W} \to \mathbb{N}$ be a map of sets. Then the inclusion of complexes*

$$z^r(\mathbb{A}^d_R | D, \bullet)_{\mathcal{W},e} \subset z^r(\mathbb{A}^d_R | D, \bullet)$$

is a quasi-isomorphism.

Section 4.1 is devoted to the proof of Theorem 4.1, though it is similar to that of Theorem 3.7.

4.1.1

Let $V \in \underline{z}^r(\mathbb{A}^d_R|D,n)$ be a prime cycle. Let $\overline{V}$ be its closure in $\mathbb{A}^d_R \times (\mathbb{P}^1)^n$. We are defining an integer $s(V) \geq 1$.

For a partition $\{1,\ldots,n\} = I \sqcup J$, we had defined open sets $U_{IJ} = \mathrm{Spec}(k[z_i,\zeta_j]_{i\in I, j\in J})$ of $(\mathbb{P}^1)^n$. Choose a finite set of generators of the ideal of $\overline{V}_{\mathrm{red}} \cap \mathbb{A}^d_R \times U_{IJ}$:

$$\left\{ f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j) \in R[\boldsymbol{x}, z_i, \zeta_j] \right\}_\lambda$$

Since V satisfies the modulus condition, we can apply Lemma 3.1 to get a homogeneous polynomial (in α, β)

$$E_{IJ}(\alpha,\beta) \in R[\boldsymbol{x}, z_i, \zeta_j][\alpha,\beta]$$

monic in α, satisfying

$$E_{IJ}(\zeta_J, u) \in \sum_\lambda f^\lambda_{IJ} R[\boldsymbol{x}, z_i, \zeta_j]. \tag{10}$$

By multiplying E_{IJ} by a power of α, we may assume

$$\deg E_{IJ} \geq \deg f^\lambda_{IJ}$$

where the first deg is the homogeneous degree of E_{IJ} and the second is the total degree of f^λ_{IJ} with respect to $\boldsymbol{x}$. Furthermore, we may assume $\deg E_{IJ}$ are the same for all partition I, J. Under these choices we set $s(V) := \deg E_{IJ}$.

For an element $V \in z^r(\mathbb{A}^d_R|D,n)$, choose a representative $\sum_\mu c_\mu V_\mu$ and put $s(V) := \max_\mu\{s(V_\mu)\}$.

Remark 4.2. Our definition is almost the same as §3.1.1, but be aware that u is a scalar now, not a polynomial.

4.1.2 Construction of homotopy

Let R'/R be a faithfully flat extension of discrete valuation rings and a vector $\boldsymbol{v} \in \mathbb{A}^d_R(R')$. Let $s \geq 1$ be an integer.

Define a morphism

$$p = p_{\boldsymbol{v},s} \colon \mathbb{A}^d_{R'} \times \square^n \times \mathbb{A}^1 \to \mathbb{A}^d_R \times \square^n$$

by

$$(\boldsymbol{x}, \boldsymbol{z}, t) \mapsto (\boldsymbol{x} + tu^s\boldsymbol{v}, \boldsymbol{z}).$$

Given an element $V \in z^r(\mathbb{A}^d_R|D,n)$, we can define a cycle $p^*(V)$ on $\mathbb{A}^d_{R'} \times \square^n \times \mathbb{A}^1$ $(= \mathbb{A}^d_{R'} \times \square^{n+1})$.

Proposition 4.3. *Suppose given a $V \in z^r(\mathbb{A}^d_R|D,n)$ and define $s(V) \geq 1$ by the procedure in §4.1.1. Consider the cycle $p^*_{\boldsymbol{v},s}(V)$ on $\mathbb{A}^d_{R'} \times \square^{n+1}$. If we have $s \geq s(V)$, then $p^*_{\boldsymbol{v},s}(V)$ satisfies the modulus condition for any $\boldsymbol{v}$.*

Proof. The modulus condition can be checked after restricting $\overline{p^*(V)} \subset \mathbb{A}^d_{R'} \times (\mathbb{P}^1)^{n+1}$ to the regions over the open subsets

$$(\mathbb{P}^1)^n \times \mathbb{A}^1 \text{ and } U_{IJ} \times \operatorname{Spec}(R[\tau]) \text{ of } (\mathbb{P}^1)^{n+1}.$$

On $(\mathbb{P}^1)^n \times \mathbb{A}^1$ it is very easy. Let us consider $U_{IJ} \times \operatorname{Spec}(R'[\tau])$. In this region the ideal of $\overline{p^*(V)}$ contains functions

$$\varphi^\lambda_{IJ} = \tau^{\deg f^\lambda_{IJ}}(\boldsymbol{x} + (1/\tau)u^s\boldsymbol{v}, z_i, \zeta_j) \in R'[\boldsymbol{x}, z_i, \zeta_j, \tau].$$

It has the form

$$\varphi^\lambda_{IJ} = \tau^{\deg f^\lambda_{IJ}} f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j) + u^s g$$

for some $g \in R'[\boldsymbol{x}, z_i, \zeta_j, \tau]$. Write the relation (10) explicitly as

$$E_{IJ}(\zeta_J, u) = \sum_\lambda b_\lambda(\boldsymbol{x}, z_i, \zeta_j) f^\lambda_{IJ}(\boldsymbol{x}, z_i, \zeta_j).$$

From these two equations we get

$$\sum_\lambda \tau^{s(V) - \deg f^\lambda_{IJ}} b_\lambda \varphi^\lambda_{IJ} = \tau^{s(V)} E_{IJ}(\zeta_J, u) + u^s g_2(\boldsymbol{x}, z_i, \zeta_j, \tau) \tag{11}$$

for some polynomial g_2.

Suppose $E_{IJ}(\alpha, \beta)$ has the form

$$E_{IJ} = \alpha^{s(V)} + c_1\alpha^{s(V)-1}\beta \cdots + c_{s(V)}\beta^{s(V)}.$$

Then we put

$$\begin{aligned} E'_{IJ} := \alpha^{s(V)} + \tau c_1 \alpha^{s(V)-1}\beta + \cdots + c_{s(V)-1}\tau^{s(V)-1}\alpha\beta^{s(V)-1} \\ + (c_{s(V)}\tau^{s(V)} + u^{s-s(V)}g_2)\beta^{s(V)} \end{aligned}$$

which belongs to $R'[\boldsymbol{x}, z_i, \zeta_j, \tau][\alpha, \beta]$ by $s \geq s(V)$. Then the equation (11) reads

$$E'_{IJ}(\tau\zeta_J, u) \in \sum_\lambda \varphi^\lambda_{IJ} R'[\boldsymbol{x}, z_i, \zeta_j, \tau].$$

By Lemma 3.1 this shows the inequality of Cartier divisors

$$D|_{\overline{p^*(V)}^N} \leq F_\infty|_{\overline{p^*(V)}^N}$$

holds on the region over $U_{IJ} \times \operatorname{Spec}(R'[\tau])$. Thus Proposition 4.3 has been proved. □

4.1.3 Proper intersection

Here in §4.1.3 we specify our choice of $\boldsymbol{v}$. Let R_{gen} be the local ring of $\mathbb{A}^d_R$ at the generic point of $\mathbb{A}^d_\kappa$. Its fraction field is $K_{\mathrm{gen}} := K(x_1, \ldots, x_d)$ and its residue field is $\kappa(x_1, \ldots, x_d)$. Let $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{gen}} \in \mathbb{A}^d(R_{\mathrm{gen}})$ be the vector corresponding to the inclusion $\mathrm{Spec}(R_{\mathrm{gen}}) \hookrightarrow \mathbb{A}^d_R$.

Lemma 4.4. *Let $\mathcal{W}$ be a finite set of irreducible constructible subsets of $\mathbb{A}^d_R \setminus D$ and $e\colon \mathcal{W} \to \mathbb{N}$ be a map of sets. Suppose $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{gen}}$. Then for any $s \geq 1$ and for any $V \in z^r(\mathbb{A}^d_R|D, n)$ we have:*

(1) *The cycle $p^*_{\boldsymbol{v}_{\mathrm{gen}},s}(V)$ on $\mathbb{A}^d_{R_{\mathrm{gen}}} \times \square^{n+1}$ meets every face of $\square^{n+1}$ properly.*

(2) *The cycle $p^*(V)|_{t=1}$ on $\mathbb{A}^d_{R_{\mathrm{gen}}} \times \square^n$ meets $W_{K_{\mathrm{gen}}} \times F$ properly for every irreducible constructible set W of $\mathbb{A}^d_K$ (i.e. defined over K) and face F of $\square^n$.*

(3) *If $V \in z^r(\mathbb{A}^d_R|D, n)_{\mathcal{W},e}$, the cycle $p^*(V)$ meets $W_{K_{\mathrm{gen}}} \times F$ with excess $\leq e(W)$ for every $W \in \mathcal{W}$ and every face F of $\square^{n+1}$.*

Proof. The assertion (1) is a special case of (3). We will prove (2) first.

The cycle $p^*(V)|_{t=1} \cap (W_{K_{\mathrm{gen}}} \times F)$ equals to

$$(V - u^s\boldsymbol{v}) \cap (W_{K_{\mathrm{gen}}} \times F).$$

After translated by the automorphism $+u^s\boldsymbol{v}$ of $\mathbb{A}^d_{K_{\mathrm{gen}}} \times F$, it looks like

$$V_{K_{\mathrm{gen}}} \cap ((W + u^s\boldsymbol{v}) \times F) \subset \mathbb{A}^d_{K_{\mathrm{gen}}} \times F.$$

We can apply the following lemma to:

$$A = W \times F \subset X = \mathbb{A}^d_K \times F,$$

$$B = V \times_{\square^n} F \subset X.$$

Lemma 4.5 ([Blo, Lem.1.1])**.** *Let X be a scheme of finite type over a field k and G a connected algebraic k-group acting on X. Let $A, B \subset X$ be two closed subsets, and assume the fibers of the map*

$$G \times A \to X;\ (g, a) \mapsto g \cdot a$$

all have the same dimension, and that this map is dominant. Then there exists an open set $\emptyset \neq U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.

This completes the proof of (2).

We prove (3). Let F be a face of $\square^{n+1}$. If F is contained in $\square^n \times \{0, 1\}$, then the assertion follows respectively from the assumption $V \in z^r(\mathbb{A}^d_R|D, n)_{\mathcal{W},e}$ and from (2). So assume F has the form $F = F' \times \square^1$. Embed $p^*(V) \cap (W_{K_{\mathrm{gen}}} \times F)$ into $\mathbb{A}^d_{K_{\mathrm{gen}}} \times F' \times \mathbb{A}^1$ by the inclusion

$$p^*(V) \cap (W_{K_{\mathrm{gen}}} \times F) \hookrightarrow \mathbb{A}^d_{K_{\mathrm{gen}}} \times F' \times \square^1$$

followed by the isomorphism

$$\mathbb{A}^d_{K_{\mathrm{gen}}} \times F' \times \square^1 \cong \mathbb{A}^d_{K_{\mathrm{gen}}} \times F' \times \mathbb{A}^1$$

$$(\boldsymbol{x}, \boldsymbol{z}, t) \mapsto (\boldsymbol{x} + tu^s \boldsymbol{v}, \boldsymbol{z}, tu^s).$$

Its fiber over $\alpha \in \mathbb{A}^1_{K_{\mathrm{gen}}}$ looks like

$$V \cap ((W + \alpha \boldsymbol{v}) \times F') \quad \text{in } X \times F'$$

(of course, everything base-changed to the residue field of α). We apply Lemma 3.5 to our situation:

$$A := W \times F' \times \mathbb{A}^1 \overset{\text{incl.}}{\hookrightarrow} X := \mathbb{A}^d_K \times F' \times \mathbb{A}^1$$

embedded by the above map,

$$B := (V \times_{\square^n} F') \times \mathbb{A}^1 \subset X,$$

$$\psi \colon \mathbb{A}^d_{K_{\mathrm{gen}}} \times F' \times \mathbb{A}^1 \to \mathbb{A}^d_{K_{\mathrm{gen}}};\ (\boldsymbol{x}, \boldsymbol{z}, \alpha) \mapsto \alpha \boldsymbol{v},$$

$$U = \{\alpha \neq 0\} \subset X.$$

then we find that the intersection $p^*_{\boldsymbol{v}_{\mathrm{gen}},e}(V) \cap (W_{K_{\mathrm{gen}}} \times F)$ in $\square^{n+1}$ is proper away from the closed subset $\{t = 0\}$. The dimensions of components contained in this closed subset is bounded by the fact that we originally started with $V \in z^r(\mathbb{A}^d_R|D, n)_{\mathcal{W},e}$. This completes the proof. $\square$

4.1.4

By Proposition 4.3 and Lemma 4.4, the canonical map

$$\frac{z^r(\mathbb{A}^d_R|D, \bullet)_{\mathcal{W},e}}{z^r(\mathbb{A}^d_R|D, \bullet)_{\mathcal{W}}} \to \frac{z^r(\mathbb{A}^d_{R_{\mathrm{gen}}}|D_{R_{\mathrm{gen}}}, \bullet)_{\mathcal{W},e}}{z^r(\mathbb{A}^d_{R_{\mathrm{gen}}}|D_{R_{\mathrm{gen}}}, \bullet)_{\mathcal{W}}}$$

induces the zero map on homology. But by a standard specialization argument (Remark 2.10) it should induce an injective map on homology. Therefore we conclude the first complex is acyclic. Thus we have shown Theorem 4.1.

4.2 Noether's normalization theorem over a Dedekind base

Noether's normalization theorem asserts that a d-dimensional integral affine scheme of finite type over a field k admits a finite map to the d-dimensional affine space $\mathbb{A}^d_k$. This theorem is often convenient to reduce a problem on an affine scheme to the case of affine spaces. In this section we show that this statement holds over the spectrum of a Dedekind domain, locally in the Nisnevich topology. Namely, we prove the following variant of [Lev2, 10.2.2] customized for our use. The proof is similar to that in *loc. cit.*

Theorem 4.6. *Let B be the spectrum of a Dedekind ring and $\varphi\colon X \to B$ be a B-scheme of finite type and equi-dimensional with d-dimensional fibers. Let $x_0 \in X$ and $b_0 = \varphi(x_0) \in B$. Suppose the residue field at b_0 is infinite. Then there are affine Nisnevich neighborhoods $(Y, y_0) \to (X, x_0)$ and $(B', b'_0) \to (B, b_0)$ and a commutative diagram:*

$$\begin{array}{ccc} (Y, y_0) & \longrightarrow & (X, x_0) \\ \downarrow & & \downarrow \varphi \\ (B', b'_0) & \longrightarrow & (B, b_0) \end{array}$$

such that the following holds: there is a closed embedding $Y \hookrightarrow \mathbb{A}^n_{B'}$ such that if we denote by $\overline{Y}$ the closure of Y in $\mathbb{P}^n_{B'}$, then Y is fiberwise dense in $\overline{Y}$ over B', i.e. for any $b' \in B'$ the open subset $Y_{b'} \subset \overline{Y}_{b'}$ is dense.

A Nisnevich neighborhood of a point $s \in S$ on a scheme refers to an étale S-scheme equipped with a point having the same residue field as that of s.

Corollary 4.7. *Keep the notation from Theorem* 4.6*. Then after a further (Zariski) localization of B', there is a finite surjective B'-morphism $Y \to \mathbb{A}^d_{B'}$.*

Proof. By the conclusion of Theorem 4.6, the closed subset $\overline{Y} \cap \mathbb{P}^{n-1}_{\infty B'}$ of $\mathbb{P}^{n-1}_{\infty B'}$ is equidimensional over B' with $(d-1)$-dimensional fibers. Therefore after localizing B', there is a linear subspace $L_{B'}$ of $\mathbb{P}^{n-1}_{\infty B'}$ (relatively over B') having codimension d which misses $\overline{Y} \cap \mathbb{P}^{n-1}_{\infty B'}$. Then the linear projection from $L_{B'}$ restricts to a finite surjective map $Y \to \mathbb{A}^d_{B'}$. $\square$

Proof of Theorem 4.6. We may assume X is embedded into a projective space $\mathbb{P}^N_B$ as a locally closed subscheme and take the closure $\overline{X} \subset \mathbb{P}^N_B$. Note since B is the spectrum of a Dedekind ring $\overline{X}$ is still equi-dimensional with d-dimensional fibers. Take a codimension d linear subspace $L_B \subset \mathbb{P}^N_B$ relative to B, meeting $\overline{X}$ fiberwise properly (i.e. the intersection $L_B \cap \overline{X}$ is finite over B), and missing x_0. For this we may have to shrink B. By "shrink" we will always mean to take an open neighborhood of a marked point. Shrinking X if necessary, we may assume L_B does not meet X. Let $\tilde{\mathbb{P}}^N_B$ be the blow-up of $\mathbb{P}^N_B$ by L_B. Denote by $\tilde{X}$ the strict transform of $\overline{X}$. Linear projection from L_B gives the diagram:

$$\begin{array}{ccc} X & \searrow & \\ \cap & & \\ \overline{X} & \longleftarrow & \tilde{X} \\ \cap & & \cap \\ \mathbb{P}^N_B & \xleftarrow{\text{blow-up}} & \tilde{\mathbb{P}}^N_B \\ & & \downarrow p \\ & & T := \mathbb{P}^{d-1}_B \end{array} \tag{12}$$

Write $T = \mathbb{P}_B^{d-1}$ and set $t_0 = p(x_0) \in T$. The map $p_{\tilde{X}}\colon \tilde{X} \to T$ is equidimensional with 1-dimensional fibers.

Choose any projective embedding $\tilde{X} \hookrightarrow \mathbb{P}_T^{N'}$ and take a hypersurface H_{t_0} of $\mathbb{P}_{t_0}^{N'}$ such that:

(i) [In case x_0 is a closed point in $p^{-1}(p(x_0))$] H_{t_0} passes x_0;

(ii) Write $\tilde{X}_{t_0} = \tilde{X} \times_T t_0$. Then $\tilde{X}_{t_0}$ and H_{t_0} meet properly in $\mathbb{P}_{t_0}^{N'}$.

(iii) Denote by $(X_{t_0})^-$ the closure of $X_{t_0} = X \times_{\mathbb{P}_B^{d-1}} t_0$ in $\tilde{X}_{t_0}$. Then we have

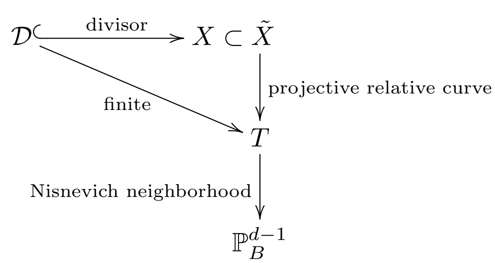
$$H_{t_0} \cap (X_{t_0})^- \subset X_{t_0}.$$

[Indeed, a general hypersurface of some high degree satisfying (i) satisfies (ii)(iii). We used the assumption that the residue field is infinite.]

Shrinking T if necessary, there exists a hypersurface H_T of $\mathbb{P}_T^{N'}$ relative to T which specializes to H_{t_0} at t_0. We replace $\tilde{X}$ with the restriction $\tilde{X} \times_{\mathbb{P}_B^{d-1}} T$. Condition (ii) holds with t_0 replaced by $t \in T$ if we shrink T further, since $H_T \cap \tilde{X}$ is proper over T.

Set $D = H_T \cap \tilde{X}$. It is an effective divisor on $\tilde{X}$ and the morphism $D \to T$ is finite, by (ii).

We let T vary among affine Nisnevich neighborhoods of $(\mathbb{P}_B^{d-1}, t_0)$ towards the henselian local scheme and replace $\tilde{X}$ by its base change; then $D \subset \tilde{X}$ becomes a direct sum of (eventually) local components finite over T. Let $\mathcal{D}$ be the sum of those components meeting X_{t_0}. Shrinking T if necessary, we may assume $\mathcal{D}$ is contained in X by the properness.



Lemma 4.8. *For any sufficiently large $m > 0$, the map*

$$\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D})) \to \Gamma(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\mathcal{D})) \cong \Gamma(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$$

is surjective.

Proof. Let $s \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\mathcal{D}))$ be the canonical section (i.e. the one corresponding to the inclusion $\mathcal{O}_{\tilde{X}} \subset \mathcal{O}_{\tilde{X}}(\mathcal{D})$). We have exact sequences

$$0 \to \mathcal{O}_{\tilde{X}}((m-1)\mathcal{D}) \xrightarrow{\times s} \mathcal{O}_{\tilde{X}}(m\mathcal{D}) \to \mathcal{O}_{\tilde{X}}(m\mathcal{D}) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\mathcal{D}} \to 0$$

which yield the long exact sequences

$$\begin{aligned} & H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D})) \to H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\mathcal{D})) \to \\ \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}((m-1)\mathcal{D})) \to \quad & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D})) \to H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\mathcal{D})) = 0. \end{aligned}$$

The last vanishing is because $\mathcal{D}$ is affine. Write $T = \operatorname{Spec}(A)$. Since $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D}))$ are Noetherian A-modules, the series of surjections

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}((m-1)\mathcal{D})) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D})) \to \cdots$$

eventually stabilizes. Then $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\mathcal{D})) \to H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\mathcal{D}))$ becomes surjective. This proves Lemma 4.8. $\square$

Thus we take an m sufficiently large so that $\mathcal{O}_{\tilde{X}}(m\mathcal{D})$ is generated by the canonical section s_0 and another section s_1 (which maps to an invertible element of $\Gamma(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\mathcal{D})) \cong \Gamma(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$). The pair (s_0, s_1) defines a morphism

$$f \colon \tilde{X} \to \mathbb{P}^1_T.$$

As $\mathcal{D}$ is very ample on $(X_{t_0})^-$ by condition (iii) on H_{t_0}, f is quasi-finite on X_{t_0}; replacing X by an open subset containing X_{t_0}, we may assume f is quasi-finite on X because the quasi-finite locus of a morphism is open [EGA IV$_3$, 13.1.4].

Shrinking T further, we may continue assuming $\mathcal{D} \subset X$, equivalently $(\tilde{X} \setminus X) \cap \mathcal{D} = \emptyset$. Thus $W := f(\tilde{X} \setminus X) \subset \mathbb{P}^1_T$ is contained in the open subset $\mathbb{P}^1_T \setminus \{s_0 = 0\} \cong \mathbb{A}^1_T$. Moreover W is proper over T. Therefore W is finite over T.

The morphism

$$f|_{\mathbb{P}^1_T \setminus W} \colon \tilde{X} \setminus f^{-1}(W) \to \mathbb{P}^1_T \setminus W$$

is finite because it is proper and f is quasi-finite on X (here, needless to say, $\tilde{X} \setminus f^{-1}(W) \subset X$). We write $X' := X \setminus f^{-1}(W)$. Note that $x_0 \in X'$ by condition (i) on H_{t_0}.

Now, by induction on the relative dimension d of X over B, we may assume T has a projective compactification $\overline{T}$ such that $T \subset \overline{T}$ is fiberwise dense over B after possibly Nisnevich-localizing B and T.

Since $f \colon \tilde{X} \to \mathbb{P}^1_T$ is projective it factors as the composite $\tilde{X} \hookrightarrow \mathbb{P}^M_T \times_T \mathbb{P}^1_T \to \mathbb{P}^1_T$ of a closed immersion and the projection.

$$\begin{array}{ccccc} X' & \overset{\circ}{\hookrightarrow} & \tilde{X} & \cdots\cdots & \tilde{X}' \\ \cap & & \cap & & \vdots \\ \mathbb{P}^M_T \times_T (\mathbb{P}^1_T \setminus W) & \overset{\circ}{\hookrightarrow} & \mathbb{P}^M_T \times_T \mathbb{P}^1_T & \overset{\circ}{\hookrightarrow} & \mathbb{P}^M_{\overline{T}} \times_{\overline{T}} \mathbb{P}^1_{\overline{T}} \\ & & \downarrow & & \downarrow \\ & & T & \overset{\circ}{\hookrightarrow} & \overline{T} \\ & & & & \downarrow \\ & & & & B \end{array}$$

Let $\tilde{X}'$ be the closure of $\tilde{X}$ in $\mathbb{P}^M_{\overline{T}} \times_{\overline{T}} \mathbb{P}^1_{\overline{T}}$. Take the Stein factorization [EGA III_1, §4.3] of the proper morphism $f' \colon \tilde{X}' \to \mathbb{P}^1_{\overline{T}}$ to get

$$\tilde{X}' \to \overline{Y} \xrightarrow[\text{finite}]{} \mathbb{P}^1_{\overline{T}}.$$

Since f' is already finite over the open subset $\mathbb{P}^1_T \setminus W$ of $\mathbb{P}^1_{\overline{T}}$ we have a canonical isomorphism $\overline{Y} \times_{\mathbb{P}^1_{\overline{T}}} \mathbb{P}^1_T \setminus W \cong X'$.

$$\begin{array}{ccccc} \tilde{X}' & \longrightarrow & \overline{Y} & \xrightarrow{\text{finite}} & \mathbb{P}^1_{\overline{T}} \\ & \nwarrow & \cup & \square & \cup \\ & & X' & \xrightarrow[f]{\text{finite}} & \mathbb{P}^1_T \setminus W \end{array}$$

Now we check that the open immersion $X' \subset \overline{Y}$ is fiberwise dense over B. Take any point $b \in B$ and any irreducible component P of $\overline{Y}_b$. We have to show $P \cap X'_b$ is nonempty.

First consider the fiber above a generic point ξ of B. Then $X'_\xi \subset \overline{Y}_\xi$ is dense because $X' \subset \overline{Y}$ is dense by construction. Moreover since X'_ξ is purely d-dimensional it follows by the semi-continuity theorem of Chevalley 2.2 that any irreducible component of the fibers of $\overline{Y} \to B$ has dimension $\geq d$.

Back to the general case, since the composite

$$P \hookrightarrow \overline{Y}_b \to \mathbb{P}^1_{\overline{T}_b}$$

is finite and the last scheme has pure dimension d by the induction hypothesis, P must have dimension exactly d, and it dominates an irreducible component of $\mathbb{P}^1_{\overline{T}_b}$. Note that $\mathbb{P}^1_{T_b} \setminus W_b$ is a dense open subset of $\mathbb{P}^1_{\overline{T}_b}$ because $T_b \subset \overline{T}_b$ is a dense open subset by induction, and W is finite over T. Therefore it follows that $P \times_{\mathbb{P}^1_{\overline{T}_b}} \mathbb{P}^1_{T_b} \setminus W_b = P \cap X'_b$ is nonempty. Therefore $X' \subset \overline{Y}$ is fiberwise dense.

Choose any closed embedding $\overline{Y} \hookrightarrow \mathbb{P}^n_B$. Then since $X' \subset \overline{Y}$ is fiberwise dense, there exists a hypersurface H of $\mathbb{P}^n_B$ relative to B which contains the closed set $\overline{Y} \setminus X'$, misses $x_0 \in X'$, and meets $\overline{Y}$ fiberwise properly, at least after shrinking B. By composing the projective embedding with the m-fold Veronese embedding where m is the degree of H relative to B, we may think H is a hyperplane. By linear automorphism of $\mathbb{P}^n_B$ we can take $H = \mathbb{P}^{n-1}_{\infty,B}$, the hyperplane at the infinity. Then the neighborhood $Y = X' \setminus \mathbb{P}^{n-1}_{\infty,B} \subset \mathbb{A}^n_B$ of x_0 makes our assertion true. This completes the proof. $\square$

4.3 Choosing a good defining equation

Proposition 4.9. *Let X be a finite-type scheme over an infinite field k, $x \in X$ be a point, and U be an open subset of X which is smooth over k. Then there is an open neighborhood V of x in X and a morphism $v \colon V \to \mathbb{G}_m$ which is smooth on $V \cap U$.*

Proof. We may assume x is a closed point. If $x \in U$, the statement is plainly true. Let us suppose to the contrary. We may assume X has a locally closed embedding $X \hookrightarrow \mathbb{P}^N$.

Fix a degree n and consider the scheme $P = \mathbb{P}(\Gamma(\mathbb{P}^N, \mathcal{O}(n)))$ parametrizing the hypersurfaces in $\mathbb{P}^N$ of degree n. There is a linear closed subset Q of P consisting of hypersurfaces containing x. By [AK, Th.1], if n is large enough, there is a dense open subset P^o of P meeting Q, consisting of hypersurfaces transversal to U.

Take a rational curve l inside P^o which meets Q but not contained in Q. It defines a pencil — an open set $X^o \subset X$ and a morphism $v\colon X^o \to l$. Its axis does not contain the point x. Since l is contained in P^o the morphism $v\colon X^o \to l$ restricted to $U \cap X^o$ is smooth because it has smooth fibers and is flat (being a map to a Dedekind scheme).

Our assertion now follows after setting a coordinate on l and shrinking X^o if necessary so that v becomes a map into $\mathbb{G}_m$. □

4.3.1

Assume k be an infinite field. Suppose given a k-scheme X of finite type and an effective principal Cartier divisor D on it; it is defined by a function u on X. Assume $X \setminus D$ is smooth over k. Let $x \in D$ be a point and $\{x'_i\}_i \subset X \setminus D$ be finitely many points generalizing x.

Proposition 4.10. *Keep the notation and assumptions in* 4.3.1. *Then we can replace X with an open neighborhood of x and replace the defining equation u of D so that the morphism $u\colon X \to \mathbb{A}^1$ is smooth at each x'_i.*

Proof. By Proposition 4.9 we may assume there is an invertible function $v\colon X \to \mathbb{G}_m$ which is a smooth morphism on $X \setminus D$. Given a scalar $\alpha \in k$, we may assume $v + \alpha$ is also an invertible function (unless $\alpha = v(x)$) by shrinking X. Consider the function $(v+\alpha)u\colon X \to \mathbb{A}^1$. Let us see its differential at x'_i:

$$d[(v+\alpha)u](x'_i) = dv(x'_i) \cdot u(x'_i) + (v(x'_i) + \alpha)du(x'_i).$$

The first term has a nonzero value in the cotangent space $T^*_{x'_i}X$ because of the smoothness of v on $X \setminus D$. Hence the whole formula has a nonzero value in $T^*_{x'_i}X$ (for each i simultaneously) for all but finitely many $\alpha \in k$. Choosing α away from such a finite subset, we take $(v+\alpha)u$ as the new defining equation of D. This proves our assertion. □

4.4 Statement of the general case

We are proving the following:

Theorem 4.11. *Let (X, D) be a pair of an equi-dimensional scheme over a base field k and an effective Cartier divisor on it, $\mathcal{W}$ be a finite collection of constructible irreducible subsets of $X \setminus D$ and $e\colon \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a map of sets.*

Assume $X \setminus D$ is smooth over k. Then for any integer $r \geq 0$ the inclusion of complexes of Nisnevich sheaves

$$z^r(X|D,\bullet)_{\mathcal{W},e,\mathrm{Nis}} \hookrightarrow z^r(X|D,\bullet)_{\mathrm{Nis}}$$

is a quasi-isomorphism.

Below, pull-backs of the divisor D to other schemes will be denoted by the same letter. By a trace argument we may assume k is an infinite field.

4.4.1

We are going to prove that for any $\mathcal{W}, e$ the quotient complex

$$\frac{z^r(X|D,\bullet)_{\mathcal{W},e}}{z^r(X|D,\bullet)_{\mathcal{W},e-1}}$$

is acyclic locally in the Nisnevich topology, i.e. for any point $x \in X$, the map

$$\frac{z^r(X|D,\bullet)_{\mathcal{W},e}}{z^r(X|D,\bullet)_{\mathcal{W},e-1}} \to \varinjlim_{X' \to X} \frac{z^r(X'|D,\bullet)_{\mathcal{W},e}}{z^r(X'|D,\bullet)_{\mathcal{W},e-1}}$$

is zero on homology, where $X' \to X$ runs through Nisnevich neighborhoods of x.

Take any finitely generated subcomplex $\underline{z}^r(X|D,\bullet)'_{\mathcal{W},e}$ of $\underline{z}^r(X|D,\bullet)_{\mathcal{W},e}$. Let $\{V_\lambda\}_\lambda$ be the finite set of prime cycles that appear as components of elements of $\underline{z}^r_{\mathcal{W},e}(X|D,\bullet)'$. Let $z^r_{\mathcal{W},e}(X|D,\bullet)'/\sim$ be the quotient such that an injection

$$z^r(X|D,\bullet)'_{\mathcal{W},e}/\sim \;\hookrightarrow \frac{z^r(X|D,\bullet)_{\mathcal{W},e}}{z^r(X|D,\bullet)_{\mathcal{W},e-1}}$$

is induced. It suffices to find a Nisnevich neighborhood X' of x such that the induced map

$$z^r(X|D,\bullet)'_{\mathcal{W},e}/\sim \;\to \frac{z^r(X'|D,\bullet)_{\mathcal{W},e}}{z^r(X'|D,\bullet)_{\mathcal{W},e-1}}$$

is zero on homology.

By the limit argument (Corollary 2.8), we are allowed to take X' to be a limit of Nisnevich neighborhoods.

Definition 4.12. For a closed subset V of $X \times \square^n$, an integer $i \geq 0$ and a face $F \subset \square^n$, denote by $V_F^{\geq i}$ the closed subset of V consisting of points y such that, denoting $x := \mathrm{pr}_1(y) \in X$, one has

$$\dim_y\left(\{x\} \times_X \times V \times_{\square^n} F\right) \geq i.$$

By the semi-continuity theorem on fiber dimensions (Lemma 2.2=[EGA IV$_3$, (13.1.3)]), it is a closed subset of $V \times_{\square^n} F$.

Put $Z_F^{\geq i}(V) := \mathrm{pr}_1(V_F^{\geq i}) \subset X$. It equals the set of points $x \in X$ where

$$\dim\left(\{x\} \times_X \times V \times_{\square^n} F\right) \geq i.$$

It is a constructible subset of X.

Let $Z_{\underline{F}}^{\geq i}(V) = \cup_\mu Z_{\underline{F}}^{\geq i}(V)^\mu$ be the irreducible decomposition. The next has been essentially proved in Lemma 3.23.

Lemma 4.13. *Suppose given $V \in \underline{z}^r(X|D,n)$. It belongs to $\underline{z}^r(X|D,n)_{\mathcal{W},e}$ if and only if*

$$e(Z_{\underline{F}}^{\geq i}(V)^\mu, W) \leq \dim X + \dim F + e(W) - (\dim Z_{\underline{F}}^{\geq i}(V)^\mu + r + i)$$

for all components μ, $i \geq 0$, $W \in \mathcal{W}$ and faces $F \subset \square^n$.

4.4.2

Since our assertion is local, we may assume the divisor D is defined by a function u on X. By Proposition 4.10, we may assume $u\colon X \to \mathbb{A}^1$ is smooth at finitely many chosen points on $X \setminus D$; we choose those points so that the general position arguments (using the results from §3.2) appearing below are all valid.

Let B^h be the henselization of $\mathbb{A}^1$ at 0. Let us denote the base change of $u\colon X \to \mathbb{A}^1$ with $B^h \to \mathbb{A}^1$ by the same letter $u\colon X' \to B^h$.

By Theorem 4.6, after Nisnevich localizing X' we can choose a closed embedding

$$X' \hookrightarrow \mathbb{A}^N_{B^h}$$

such that if we denote by $X'^c \subset \mathbb{P}^N_{B^h}$ its closure and $\mathbb{P}^{N-1}_{\infty B^h} = \mathbb{P}^N_{B^h} \setminus \mathbb{A}^N_{B^h}$, the set $X'^c \cap \mathbb{P}^{N-1}_{\infty B^h}$ has pure relative dimension $(\dim X - 2)$ over B^h. (Without knowing Theorme 4.6, the special fiber of $X'^c \cap \mathbb{P}^{N-1}_{\infty B^h}$ might have dimension $\geq \dim X - 1$.) This will be used to ensure the set $\mathcal{U}_{X'}$ which we now introduce has a nonempty special fiber.

Let $\mathrm{Gr}(N - \dim X, \mathbb{P}^{N-1}_{\infty B^h})$ be the Grassmannian variety parametrizing linear subspaces of $\mathbb{P}^{N-1}_{\infty B^h}$ of relative dimension $(N - \dim X)$ over B^h (= codimension $(\dim X - 1)$). A morphism $L\colon B' \to \mathrm{Gr}(N - \dim X, \mathbb{P}^{N-1}_{\infty B^h})$ from a scheme B' determines a linear map $\mathbb{A}^N_{B'} \to \mathbb{A}^{\dim X - 1}_{B'}$ up to linear automorphism on the target.

Let $\mathcal{U}_{X'}$ be the open subset of $\mathrm{Gr}(N - \dim X, \mathbb{P}^{N-1}_{\infty B^h})$ consisting of L's which miss $X'^c \cap \mathbb{P}^{N-1}_{\infty B^h}$. By our choice of the affine embedding $X' \subset \mathbb{A}^N_{B^h}$ it has a nonempty special fiber. A section $L\colon B^h \to \mathcal{U}_{X'}$ defines a linear map which restricts to a finite surjective map $p_L\colon X' \to \mathbb{A}^{\dim X - 1}_{B^h}$. It is also flat on the Cohen-Macaulay locus of X' [EGA IV$_2$, (6.1.5)].

By the flatness (on $X' \setminus D$) and the finiteness of p_L and the fact that it is a B^h-morphism we have push-forward and pull-back maps of cycles with modulus (where π denotes a uniformizer of B^h)

$$p_{L*}\colon z^r(X'|D, \bullet) \rightleftarrows z^r(\mathbb{A}^{\dim X - 1}_{B^h}|(\pi), \bullet)\colon p_L^*.$$

4.5 The proof of the general case

Keep the notation in §4.4. Denote by K^h the function field of B^h. We will apply Chow's moving lemma §3.2.6 for the scheme $X'_{K^h} = X' \setminus D$ over K^h of dimension $d = \dim X - 1$. We will choose a section $L\colon B^h \to \mathcal{U}_{X'} \subset \mathrm{Gr}(N - \dim X, \mathbb{P}^{N-1}_{\infty B^h})$ sufficiently generally so that in particular we will be able to use the facts recalled in §3.2. When we say some property holds for a sufficiently general L, we will mean there is a dense open subset of $\mathcal{U}_{X'} \times_{B^h} K^h$ such that if $L \times_{B^h} K^h$ ($\in \mathcal{U}_{X'}(K^h)$) belongs to it, the property holds. (We shall always deal with properties depending only on $L \times_{B^h} K^h$.)

For a function $e\colon \mathcal{W} \to \mathbb{N}$, we define a new function $e-1$ by $(e-1)(W) = \max\{0, e(W) - 1\}$.

Claim 4.14. In this setting, if L is sufficiently general, $p_L^* p_{L*} V - V$ belongs to $z^r_{\mathcal{W},e-1}(X'|D,n)$ for all $V \in \{V_\lambda\}_\lambda$.

Proof. Let $V \in \{V_\lambda\}_\lambda$. The case where the generic point of V is a generic point of the projection $\mathrm{pr}_2\colon X \times \square^n \to \square^n$ is exceptionally easy and we omit it.Otherwise, for a sufficiently general L the support of $p_L^* p_{L*} V - V$ is $L^+ V$ by Lemma 3.18. By Lemma 3.23 we have to show the inequality

$$e(Z_F^{\geq i}(V)^\mu, W) \leq \dim X + \dim F + e(W) - (\dim Z_F^{\geq i}(V)^\mu + r + i)$$

for all μ, F, i, W.

Suppose $Z_F^{\geq i}(L^+V)$ is dense in X. Then the inequality automatically holds using the fact that $p_L^* p_{L*} V$ meets faces properly. Otherwise, choosing L sufficiently general, we deduce for each i a dense inclusion

$$Z(L^+V)_F^{\geq i} \subset L^+\left(Z(V)_F^{\geq i}\right). \tag{13}$$

Let $Z(L^+V)_F^{\geq i} = \cup_\nu Z(L^+V)_F^{\geq i,\nu}$ be the irreducible decomposition.

Now since $V \in \underline{z}^r(X|D,n)_{\mathcal{W},e}$ we know for any face F of $\square^n$ and i and μ:

$$e(Z_F^{\geq i,\mu}, W) \leq \dim X + \dim F + e(W) - (\dim Z_F^{\geq i,\mu} + r + i)$$

by Lemma 3.23. By Chow's Moving Lemma 3.15, we have

$$e(L^+(Z_F^{\geq i,\mu}), W) \leq \dim X + \dim F + (e-1)(W) - (\dim L^+(Z_F^{\geq i,\mu}) + r + i)$$

By the knowledge of the inclusion (13), something similar holds for $Z(L^+V)_F^{\geq i}$, i.e.

$$e(Z(L^+V)_F^{\geq i,\nu}, W) \leq \dim X + \dim F + (e-1)(W) - (\dim Z(L^+V)_F^{\geq i,\nu} + r + i)$$

for any ν. Therefore it follows that $p_L^* p_{L*} V - V$ belongs to $\underline{z}^r(X'|D,n)_{\mathcal{W},e-1}$. $\square$

By Claim 4.14, it also follows that $p_L^* p_{L*} V \in z^r(X'|D,n)_{\mathcal{W},e}$. If V happens to be in the smaller subgroup $z^r(X|D,n)_{\mathcal{W},e-1}$, we have more strongly $p_L^* p_{L*} V \in z^r(X'|D,n)_{\mathcal{W},e-1}$.

We may assume our L moreover satisfies:

- No two subsets $p_L(W) \subset \mathbb{A}^{\dim X-1}_{K^h}$ ($W \in \mathcal{W}$) are the same; in particular the map
$$\begin{array}{rcl} p\mathcal{W} := \{p_L(W)\}_{W\in\mathcal{W}} & \to & \mathbb{Z}_{\geq 0} \\ p_L(W) & \mapsto & e(W) \end{array}$$
is well-defined, which we denote by pe.

Then it follows that $p_{L*}V \in z^r(\mathbb{A}^{\dim X-1}_{B^h}|(\pi), n)_{p\mathcal{W},pe}$ by the projection formula (of subsets):
$$|p_{L*}V| \cap p_L(W) = p_L(|p_L^* p_{L*}V| \cap W).$$
If V happens to $\in z^r(X|D,n)_{\mathcal{W},e-1}$, then $p_{L*}V \in z^r(\mathbb{A}^{\dim X-1}_{B^h}|(\pi), n)_{p\mathcal{W},pe-1}$.

Therefore we have maps of complexes:

$$z^r(X|D,\bullet)'_{\mathcal{W},e}/\sim \quad \xrightarrow{p_{L*}} \frac{z^r(\mathbb{A}^{\dim X-1}_{B^h}|(\pi),\bullet)_{p\mathcal{W},pe}}{z^r(\mathbb{A}^{\dim X-1}_{B^h}|(\pi),\bullet)_{p\mathcal{W},pe-1}} \xrightarrow{p_L^*} \frac{z^r(X'|D,\bullet)_{\mathcal{W},e}}{z^r(X'|D,\bullet)_{\mathcal{W},e-1}}.$$

Consider the equality of operations (where the map can. is the canonical map)

$$\text{can.} = p_L^* p_{L*} - [p_L^* p_{L*} - \text{can.}] \colon z^r_{\mathcal{W},e}(X|D,\bullet)'/\sim \quad \to \frac{z^r(X'|D,\bullet)_{\mathcal{W},e}}{z^r(X'|D,\bullet)_{\mathcal{W},e-1}}.$$

The first term $p_L^* p_{L*}$ is zero on homology because it factors through an acyclic complex (Theorem 4.1). The second term $[p_L^* p_{L*} - \text{can.}]$ is zero by Claim 4.14. Therefore the operator can. is also zero on homology. This completes the proof (recall 4.4.1).

4.6 Functoriality

Suppose given pairs $(X,D),(Y,E)$ of equidimensional algebraic k-schemes and effective Cartier divisors, and a k-morphism $f\colon X \to Y$ inducing a morphism $D \to E$. Let $V \in \underline{z}^r(Y|E,n)$ and consider if a pulled-back cycle f^*V in $\underline{z}^r(X|D,n)$ is defined via the construction in Definition 2.1.

First, the closed subset $(f\times \mathrm{id}_{\square^n})^{-1}(|V|)$ of $X\times\square^n$ have to have codimension $\geq r$.

Set $Z^{\geq i}(f) \subset Y \setminus E$ to be the constructible subset consisting of points where the fibers of f have dimension $\geq i$, and let $Z^{\geq i}(f) = \cup_\mu Z^{\geq i}(f)^\mu$ be the irreducible decomposition. Then the previous condition is equivalent to

$$\dim(|V| \cap (Z^{\geq i}(F) \times \square^n)) + i \leq \dim(X \times \square^n) - r$$

for all $i \geq 0$.

Furthermore, the pulled-back cycle $(f \times \mathrm{id}_{\square^n})^*V$ on $X \times \square^n$ has to meet faces of $\square^n$ properly. This is equivalent to

$$\dim(|V| \cap (Z^{\geq i}(F) \times F)) + i \leq \dim(X \times F) - r \tag{14}$$

for all faces F of $\square^n$. (The cycle $(f \times \mathrm{id})^*V$ satisfies the modulus condition automatically from the fact that f restricts to a morphism $D \to E$.)

The condition (14) can be stated equivalently as follows:

$$\mathrm{codim}_{Z^{\geq i}(f)^{\mu}\times F}(|V|\cap(Z^{\geq i}(f)^{\mu}\times F))\geq r-(\dim X-dimZ^{\geq i}(f)^{\mu}-i).$$

Therefore if we put

$$\mathcal{W}=\{Z^{\geq i}(f)^{\mu}\}_{i,\mu},$$

a finite set of irreducible constructible subsets of Y and set a function on $\mathcal{W}$ as

$$e(Z^{\geq i}(f)^{\mu})=\dim X-\dim Z^{\geq i}(f)^{\mu}-i\ (\geq 0),$$

then our condition is equivalent to

$$V\in \underline{z}^r(Y|E,n)_{\mathcal{W},e}.$$

So we have a diagram of complexes

$$z^r(Y|E,\bullet)\supset z^r(Y|E,\bullet)_{\mathcal{W},e}\xrightarrow{f^*} z^r(X|D,\bullet)$$

which extends to a diagram of sheaves

$$z^r(Y|E,\bullet)_{\mathrm{Nis}}\supset z^r(Y|E,\bullet)_{\mathcal{W},e,\mathrm{Nis}}\xrightarrow{f^*} f_*z^r(X|D,\bullet)_{\mathrm{Nis}}$$

on Y (equivalently, a diagram

$$f^{-1}z^r(Y|E,\bullet)_{\mathrm{Nis}}\supset f^{-1}z^r(Y|E,\bullet)_{\mathcal{W},e,\mathrm{Nis}}\xrightarrow{f^*} z^r(X|D,\bullet)_{\mathrm{Nis}}$$

on X). By Theorem 4.11 the left two complexes are quasi-isomorphic in the Nisnevich topology. Thus we have shown:

Theorem 4.15. *Suppose given pairs $(X,D),(Y,E)$ of equidimensional algebraic k-schemes and effective Cartier divisors, and a k-morphism $f\colon X\to Y$ inducing a morphism $D\to E$.*

Then there is a natural pull-back map

$$f^{-1}z^r(Y|E,\bullet)_{\mathrm{Nis}}\xrightarrow{f^*} z^r(X|D,\bullet)_{\mathrm{Nis}}$$

in the derived category of complexes of Nisnevich sheaves on X.

Therefore we have the contravariance of the motivic cohomology groups $\mathbf{H}^n(X,z^r(X|D,\bullet)_{\mathrm{Nis}})$.

Using this we can deduce a product structure on them:

Lemma 4.16. *For any pairs $(X,D),(Y,E)$ of equidimensional k-schemes and effective Cartier divisors, there are obvious external product maps*

$$\boxtimes\colon \underline{z}^r(X|D,m)\times\underline{z}^s(Y|E,n)\to\underline{z}^{r+s}(X\times Y|(D\times Y)+(X\times E),m+n)$$

$$(V,W)\mapsto\ \text{(the cycle associated with) } V\times W.$$

Proof. It follows directly from the definitions. □

An appropriate signed sum of the above maps gives a map of complexes

$$\boxtimes\colon z^r(X|D,\bullet)\otimes z^s(Y|E,\bullet)\to z^{r+s}(X\times Y|(D\times Y)+(X\times E),\bullet).$$

Corollary 4.17. *Let X be an equidimensional k-scheme and let D, D' be two effective Cartier divisors on X such that $X\setminus(|D|\cap|D'|)$ is smooth. Then there is a natural intersection product map*

$$z^r(X|D,\bullet)_{\mathrm{Nis}}\otimes z^s(X|D',\bullet)_{\mathrm{Nis}}\to z^{r+s}(X|D+D',\bullet)_{\mathrm{Nis}}$$

in the derived category of complexes of Ninevich sheaves on X.

Proof. We use the previous lemma and the pull-back by the diagonal $X\hookrightarrow X\times X$ which exists Nisnevich locally by Theorem 4.15. □

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