HEAVY SUBSETS AND NON-CONTRACTIBLE TRAJECTORIES

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Abstract. Biran, Polterovich and Salamon introduced a relative symplectic capacity which indicates the existence of non-contractible trajectories of certain Hamiltonian isotopies. In the present paper, we give an upper bound of their relative symplectic capacity of certain heavy subsets by using spectral invariants defined in terms of the Hamiltonian Floer theory on contractible trajectories.

1. Introduction

For Hamiltonian isotopies, it is interesting to know whether they have non-contractible (periodic) trajectories. Biran, Polterovich and Salamon introduced a relative symplectic capacity which indicates the existence of non-contractible trajectories of certain Hamiltonian isotopies ([BPS]).

For a compact subset Y of an open symplectic manifold (N,ω) and a free homotopy class $\alpha \in [S^1,N]$, Biran, Polterovich and Salamon [BPS] defined the relative symplectic capacity $C_{BPS}(N,Y;\alpha)$ by

$$C_{BPS}(N,Y;\alpha) = \inf\{K>0; \forall H \in \mathcal{H}_K(N,Y), \mathcal{P}(H;\alpha) \neq \emptyset\},$$

where

$$\mathcal{H}_K(N,Y) = \{H \in C_c^\infty(S^1\times N); \inf_{S^1\times Y} H \geq K\},$$

and $\mathcal{P}(H;\alpha)$ is the set of 1-periodic trajectories of the Hamiltonian isotopy generated by the Hamiltonian function H in the class α.

Biran, Polterovich and Salamon proved the following theorem by showing non-vanishing of the homomorphism from a symplectic homology to a relative symplectic homology.

Theorem 1.1 ([BPS]). *Let N be a connected closed Riemannian manifold and $\alpha \in [S^1,N]$ a non-trivial homotopy class of free loops in N. Assume that N is the n-dimensional torus or has the Riemannian metric whose sectional curvature is negative. Then*

$$C_{BPS}(B^*N,N;\alpha) = l_\alpha,$$

where l_α is the infimum of length of closed geodesics in the class α. Here let (B^*N,ω_N) denote the unit ball subbundle of the cotangent bundle with the standard symplectic form ω_N and let N denote the zero section of B^*N.

After the above work by Biran, Polterovich and Salamon [BPS], Weber [W] proved that Theorem 1.1 holds for any connected closed Riemannian manifold N and Niche [N] gave bounds of Biran-Polterovich-Salamon's capacities for twisted cotangent bundles.

One of the reasons why $C_{BPS}(N, Y)$ is finite in their cases is that the compact subsets Y are non-displaceable in N. Indeed, Biran, Polterovich and Salamon essentially proved the following proposition.

Proposition 1.2 (Proposition 3.3.2 of [BPS]). *Let (N, ω) be a connected open symplectic manifold and Y a compact subset of N. Let α be a non-trivial homotopy class of free loops. Assume that there exists a Hamiltonian function $H \colon S^1 \times N \to \mathbb{R}$ with compact support such that $Y \cap \phi_H^1(Y) = \emptyset$ and $\mathcal{P}(H; \alpha) = \emptyset$. Then $C_{BPS}(N, Y; \alpha) = \infty$. Here $\{\phi_H^t\}$ is the Hamiltonian isotopy generated by H.*

Thus, we would like to know the problem whether Biran-Polterovich-Salamon's capacity is finite or not on non-displaceable subsets in general.

One of the important classes of non-displaceable subsets is the class of heavy subsets. In fact, heavy subsets are known to be non-displaceable, moreover, stably non-displaceable (See Section 2). For example,

$$(\text{Clifford torus of } \mathbb{C}P^n) \times T^n \subset \mathbb{C}P^n \times T^*T^n$$

is a heavy subset and thus non-displaceable.

In the present paper, we give an upper bound of Biran-Polterovich-Salamon's capacity of heavy subsets.

To state our main result, we introduce some notations.

For $R = (R_1, \ldots, R_n) \in (\mathbb{R}_{>0})^n$, let I_R^n be the open subset I_R^n of $\mathbb{R}^n$ defined by

$$I_R^n = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n ; |p_i| < R_i \text{ for } i = 1, \ldots, n\}.$$

We consider the standard symplectic form $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ on $I_R^n \times T^n$ with coordinates $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$, where we regard T^n as $(\mathbb{R}/\mathbb{Z})^n$. We denote the zero-section $\{(p, q) \in I_R^n \times T^n ; p = 0\}$ of $I_R^n \times T^n$ by T^n.

Let (M, ω) be a connected symplectic manifold and X a compact subset of M. For an element $e = (e_1, \ldots, e_n)$ of $\mathbb{Z}^n$ and an element $R = (R_1, \ldots, R_n)$ of $(\mathbb{R}_{>0})^n$, we define the relative symplectic capacity $C(M, X, R; e)$ by

$$C(M, X, R; e) = C_{BPS}(M \times I_R^n \times T^n, X \times T^n; (0_M, e)).$$

Here, we fix the symplectic form $\mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega_0$ on $M \times I_R^n \times T^n$ and we identify the homotopy set $[S^1, I_R^n \times T^n]$ with $\mathbb{Z}^n$ and let 0_M denote the class of constant loops in M.

For a real number λ, a symplectic manifold (M,ω) is said to be *λ-monotone* if $[\omega] = \lambda c_1$ on $\pi_2(M)$ and *monotone* if (M,ω) is λ-monotone for some positive λ. Here c_1 is the first Chern class of TM with respect to an almost complex structure compatible with ω.

Our main theorem is the following one.

Theorem 1.3. *Let (M,ω) be a $2m$-dimensional connected closed λ-monotone symplectic manifold and X a heavy subset of M. Then*

$$C(M,X,R;e) \leq 2\sum_{i=1}^{n} R_i \cdot |e_i| + \max\{0, -\lambda(m+n)\},$$

for any elements $e = (e_1,\dots,e_n)$ and $R = (R_1,\dots,R_n)$ of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively.

We can rewrite Theorem 1.3 in the following form.

Theorem 1.4. *Let X be a heavy subset of a $2m$-dimensional connected closed λ-monotone symplectic manifold (M,ω). Let $e = (e_1,\dots,e_n)$ and $R = (R_1,\dots,R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively. We fix the symplectic form $\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega_0$ on $M \times I_R^n \times T^n$, where $\mathrm{pr}_1 \colon M \times I_R^n \times T^n \to M$ and $\mathrm{pr}_2 \colon M \times I_R^n \times T^n \to I_R^n \times T^n$ are the the projections defined by $\mathrm{pr}_1(x,p,q) = x$ and $\mathrm{pr}_2(x,p,q) = (p,q)$. Let $F \colon S^1 \times M \times I_R^n \times T^n \to \mathbb{R}$ be a Hamiltonian function with compact support such that*

$$F|_{S^1 \times X \times T^n} \geq 2\sum_{i=1}^{n} R_i \cdot |e_i| + \max\{0, -\lambda(m+n)\}.$$

Then the Hamiltonian isotopy $\{\phi_F^t\}_{t\in\mathbb{R}}$ has a 1-periodic trajectory in the free loop homotopy class $(0_M, e) \in [S^1, M \times I_R^n \times T^n]$.

Many of papers which give upper bounds of Biran-Polterovich-Salamon's capacity use the Hamiltonian Floer theory on non-contractible trajectories ([BPS], [W], [N] ,[X]). In the present paper, however, we use (Oh-Schwarz's) spectral invariants ([Sc], [O02] and [O06]) to give an upper bound of Biran-Polterovich-Salamon's capacity. Spectral invariants are defined in terms of the Hamiltonian Floer theory on contractible trajectories, however they behave better when we look at heavy subsets.

For a displaceable compact subset X, we have the following results.

Proposition 1.5. *Let (M,ω) be a connected symplectic manifold and X a displaceable compact subset of M. Let $e = (e_1,\dots,e_n)$ and $R = (R_1,\dots,R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively. Assume that $R_k \cdot |e_k| > E(X)$ for some k, where $E(X)$ denotes the displacement energy of X (see Section 6). Then $C(M,X,R;e) = \infty$.*

Thus we obtain an upper bound of Biran-Polterovich-Salamon's capacity of (Clifford torus of $\mathbb{C}P^n$) $\times T^n$ in $\mathbb{C}P^n \times I_R^n \times T^n$ and a lower bound of the one of (other fiber of the moment map $\mathbb{C}P^n \to \triangle^n$) $\times T^n$ in $\mathbb{C}P^n \times I_R^n \times T^n$.

Example 1.6. Let $(\mathbb{C}P^m, \omega_{FS})$ be the m-dimensional complex projective space with the Fubini-Study form ω_{FS}. Let $\Phi\colon \mathbb{C}P^m \to \mathbb{R}^m$ be the moment map defined by

$$\Phi([z_0 : \dots : z_m]) = (\frac{|z_0|^2}{|z_0|^2 + \cdots + |z_m|^2}, \dots, \frac{|z_m|^2}{|z_0|^2 + \cdots + |z_m|^2}).$$

The Clifford torus $\Phi^{-1}(y_0)$ is a heavy subset of $(\mathbb{C}P^m, \omega_{FS})$ where $y_0 = (\frac{1}{m+1}, \dots, \frac{1}{m+1})$. Since $(\mathbb{C}P^m, \omega_{FS})$ is a monotone symplectic manifold, Theorem 1.3 implies

$$C(\mathbb{C}P^m, \Phi^{-1}(y_0), R; e) \leq 2 \sum_{i=1}^{m} R_i \cdot |e_i|,$$

for any elements $e = (e_1, \dots, e_m)$ and $R = (R_1, \dots, R_m)$ of $\mathbb{Z}^m$ and $(\mathbb{R}_{>0})^m$, respectively.

On the other hand, Lemma 5.1 of [BEP] essentially proved that there exists a positive constant P such that $E(\Phi^{-1}(y)) < P$ for any element $y \neq y_0$ of $\mathbb{R}^m$. Thus for any element y of $\mathbb{R}^m$ with $y \neq y_0$, Proposition 1.5 implies

$$C(\mathbb{C}P^m, \Phi^{-1}(y), R; e) = \infty,$$

for any elements e and R of $\mathbb{Z}^m$ and $(\mathbb{R}_{>0})^m$ such that $R_k \cdot |e_k| > P$ for some k, respectively.

The present paper is organized as follows. We review the definitions in symplectic geometry in Section 2 and spectral invariants in Section 3 which are needed to prove Theorem 1.4 in Section 4. We discuss in Section 5 the existence of periodic trajectories of period not more than 1. In Section 6 and 7, we look at the capacity of displaceable subsets and prove Proposition 1.5. In Section 8, we discuss generalizations of our main Example 1.6. In Sections 9 and 10, we discuss exact values of the capacities, where we show that our capacity of a certain non-displaceable but not stably non-displaceable subsets is infinite.

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2. Preliminaries

In this section, we review several definitions in symplectic geometry in order to fix the terminology.

Let (M, ω) be a symplectic manifold. For a Hamiltonian function $H\colon M \to \mathbb{R}$ with compact support, we define the *Hamiltonian vector field* X_H associated with H by

$$\omega(X_H, V) = -dH(V) \text{ for any } V \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M.

Let S^1 denote $\mathbb{R}/\mathbb{Z}$. For a (time-dependent) Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ with compact support and for $t \in S^1$, we define $H_t\colon M \to \mathbb{R}$ by $H_t(x) = H(t, x)$. We denote the Hamiltonian vector field associated with H_t by X^t_H and denote by $\{\phi^t_H\}_{t\in\mathbb{R}}$ the isotopy generated by X^t_H such that $\phi^0_H = \mathrm{id}$. ϕ^1_H is called *the Hamiltonian diffeomorphism generated by the Hamiltonian function* H and denoted by ϕ_H. For a symplectic manifold (M, ω), we denote by $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω). For $x \in M$, we denote by $\gamma^x_H\colon [0, 1] \to M$ the path defined by $\gamma^x_H(t) = \phi^t_H(x)$.

A subset X of M is *displaceable* if $\bar{X} \cap \phi^1_H(X) = \emptyset$ for some Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$, where $\bar{X}$ is the topological closure of X. X is *non-displaceable* otherwise. A subset X of a symplectic manifold M is *stably displaceable* if $X \times T^1$ is displaceable in $M \times T^*T^1$. X is *stably non-displaceable* otherwise. If X is stably non-displaceable, then X is non-displaceable.

We denote the free loop space $C^\infty(S^1, M)$ of M by $\mathcal{L}M$. For $z \in \mathcal{L}M$, we denote its free homotopy class by $[z] \in [S^1, M]$. Let $\mathrm{ev}\colon \mathcal{L}M \to M$ be the evaluation map defined by $\mathrm{ev}(z) = z(0)$. For a given class $\alpha \in [S^1, M]$, we define the subset $\mathcal{L}_\alpha M$ of $\mathcal{L}M$ by $\mathcal{L}_\alpha M = \{z \in \mathcal{L}M; [z] = \alpha\}$. For a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$, we define the set of 1-periodic trajectories of $\{\phi^t_H\}_{t\in\mathbb{R}}$ in the class α by

$$\mathcal{P}(H; \alpha) = \{z \in \mathcal{L}_\alpha M; \dot{z}(t) = X^t_H(z(t))\}.$$

We define the covering space $\tilde{\mathcal{L}}_{0_M}(M)$ of $\mathcal{L}_{0_M}(M)$ by

$$\tilde{\mathcal{L}}_{0_M}(M) = \{u \in C^\infty(D^2, M); u|_{\partial D^2} \in \mathcal{L}_{0_M}(M)\}/\sim .$$

Here $u \sim u'$ if $u|_{\partial D^2} = u'|_{\partial D^2}$, $\omega(\bar{u}\sharp u') = 0$ and $c_1(\bar{u}\sharp u') = 0$, where $\sharp$ denotes the map from the sphere obtained from u with the reversed orientation and u' by gluing along their common boundary. We also define the covering space $\tilde{\mathcal{P}}(H)$ of $\mathcal{P}(H; 0_M)$ by

$$\tilde{\mathcal{P}}(H) = \{[z, u] \in \mathcal{P}(H; 0_M) \times C^\infty(D^2, M); u|_{\partial D^2} = z\}/\sim .$$

Here $[z,u] \sim [z',u']$ if $z = z'$, $\omega(\bar{u}\sharp u') = 0$ and $c_1(\bar{u}\sharp u') = 0$.

3. Spectral invariants and heavy subsets

In this section, we review spectral invariants which we use in the proof of our results.

3.1. **Spectral invariants.** For a $2m$-dimensional closed connected symplectic manifold (M,ω), we define

$$\Gamma = \frac{\pi_2(M)}{\mathrm{Ker}(c_1) \cap \mathrm{Ker}([\omega])}.$$

The Novikov ring Λ of the closed symplectic manifold (M,ω) is defined as follows:

$$\Lambda = \left\{ \sum_{A \in \Gamma} a_A A; a_A \in \mathbb{Z}_2, \#\{A; a_A \neq 0, \int_A \omega < R\} < \infty \text{ for any real number } R \right\}.$$

The quantum homology $QH_*(M,\omega)$ is a Λ-module isomorphic to $H_*(M;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda$ and $QH_*(M,\omega)$ has a ring structure with the multiplication called the *quantum product* ([O06]).

For a Hamiltonian function $H \colon S^1 \times M \to \mathbb{R}$, the action functional $\mathcal{A}_H \colon \tilde{\mathcal{L}}_{0_M} M \to \mathbb{R}$ is given by

$$\mathcal{A}_H([z,u]) = \int_0^1 H(t,z(t))dt - \int_{D^2} u^*\omega.$$

Then we regard $\tilde{\mathcal{P}}(H)$ as the set of critical points of $\mathcal{A}_H$.

We define the *non-degeneracy* of Hamiltonian functions as follows:

Definition 3.1. A Hamiltonian function $H \colon S^1 \times M \to \mathbb{R}$ is said to be *non-degenerate* if for any element z of $\mathcal{P}(H; 0_M)$, 1 is not an eigenvalue of the differential $(d\phi_H^1)_{z(0)}$.

When H is non-degenerate, the Floer chain complex $CF_*(H)$ is generated by $\tilde{\mathcal{P}}(H)$ as a module over $\mathbb{Z}_2$. Since there exists a natural action of Λ on $CF_*(H)$, we regard $CF_*(H)$ as a module over Λ. The chain complex $CF_*(H)$ is graded by the Conley-Zehnder index $\mathrm{ind}_{\mathrm{CZ}}$([SZ]). Note that $\mathrm{ind}_{\mathrm{CZ}}([z, u\sharp A]) = \mathrm{ind}_{\mathrm{CZ}}([z,u]) + 2c_1(A)$ for any map $A \in \pi_2(M)$. We obtain the boundary homomorphism of this chain complex by counting isolated negative gradient flow lines of $\mathcal{A}_H$ formally. Let $F \colon M \to \mathbb{R}$ be a Morse function on M and x a critical point of F. Assume that dF is C^1-small near x. Then $\mathrm{ind}_{\mathrm{Morse}}(x) = \mathrm{ind}_{\mathrm{CZ}}([x, c_x])$, where c_x is a trivial capping disk and $\mathrm{ind}_{\mathrm{Morse}}$ is the Morse index. There exists a natural isomorphism $\Phi \colon QH_*(M,\omega) \to HF_*(M,\omega)$. We call this isomorphism the PSS isomorphism ([PSS]).

Given an element $A = \sum_i a_i[z_i, u_i]$ of $CF_*(H)$, we define the action level $l_H(A)$ of A by

$$l_H(A) = \max\{\mathcal{A}_H([z_i, u_i]); a_i \neq 0\}.$$

For a non-zero element a of $QH_*(M, \omega)$, we define the spectral invariant associated to a non-degenerate Hamiltonian function H and a by

$$c(a, H) = \inf\{l_H(A); [A] = \Phi(a)\}.$$

The following proposition summarizes the properties of spectral invariants which we need to show our result.

Proposition 3.2 ([O06]). *The spectral invariant has the following properties.*

- **(1)*Lipschitz property*:** *The map $H \mapsto c(a, H)$ is Lipschitz on $C^\infty(S^1 \times M)$ with respect to the C^0-norm,*
- **(2)*Homotopy invariance*:** *Assume that Hamiltonian functions $F, G\colon S^1 \times M \to \mathbb{R}$ are normalized i.e. $\int_M F_t(x)\omega^m = 0, \int_M G_t(x)\omega^m = 0$ for any $t \in S^1$ and satisfy $\phi_F^1 = \phi_G^1$ and that their Hamiltonian isotopies $\{\phi_F^t\}$ and $\{\phi_G^t\}$ are homotopic relative to endpoints. Then $c(a, F) = c(a, G)$,*
- **(3)*Triangle inequality*:** *$c(a*b, F\sharp G) \leq c(a, F) + c(b, G)$ for any Hamiltonian functions $F, G\colon S^1 \times M \to \mathbb{R}$, where $*$ denotes the quantum product. Here the Hamiltonian function $F\sharp G\colon S^1 \times M \to \mathbb{R}$ is defined by*

$$(F\sharp G)(t, x) = F(t, x) + G(t, (\phi_F^t)^{-1}(x)),$$

whose Hamiltonian isotopy is $\{\phi_F^t \phi_G^t\}$.

For a general Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ (which might be degenerate) , we define the spectral invariant $c(a, H)$ by the Lipschitz property for spectral invariants (Proposition 3.2 (1)). Then the spectral invariant defined for general Hamiltonian functions also satisfy the properties in Proposition 3.2.

3.2. **Heaviness.** Entov and Polterovich ([EP]) defined the notion of *heaviness* of compact subsets in closed symplectic manifolds.

For an idempotent a of the quantum homology $QH_*(M, \omega)$, we define the functional $\zeta_a\colon C^\infty(M) \to \mathbb{R}$ to be the stabilization of $c(a, \cdot)$;

$$\zeta_a(H) = \lim_{l \to \infty} \frac{c(a, lH)}{l}.$$

Definition 3.3 ([EP]). Let (M, ω) be a $2m$-dimensional closed symplectic manifold and a an idempotent of the quantum homology $QH_*(M, \omega)$. A compact subset X of M is said to be *a-heavy* if

$$\zeta_a(H) \geq \inf_X H,$$

for any (time-independent) Hamiltonian function $H\colon M \to \mathbb{R}$. A compact subset X of M is said to be *heavy* if X is a-heavy for some idempotent a of $QH_*(M, \omega)$.

Entov and Polterovich [EP] proved that every heavy subset is non-displaceable ([EP] Theorem 1.4).

Example 3.4. On the torus $T^n_R \times T^n = \mathbb{R}/2R_1\mathbb{Z} \times \cdots \times \mathbb{R}/2R_n\mathbb{Z} \times (\mathbb{R}/\mathbb{Z})^n$ with coordinates $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$, we fix the standard symplectic form $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$. Entov and Polterovich [EP] proved that for any element $R = (R_1, \ldots, R_n)$ of $(\mathbb{R}_{>0})^n$, $\{0\} \times T^n$ is a heavy subset of $T^n_R \times T^n$.

4. Proof of Theorem 1.4

To prove Theorem 1.4, we give an upper bound of the spectral invariant associated to a Hamiltonian function $F\colon S^1 \times M \times I^n_{R(2\epsilon)} \times T^n \to \mathbb{R}$ such that $\mathcal{P}(F; (0_M, e)) = \emptyset$. Here, for $R = (R_1, \ldots, R_n) \in (\mathbb{R}_{>0})^n$ and a positive real number ϵ with $\epsilon < \min\{R_1, \ldots, R_n\}$, let $R(\epsilon)$ denote $(R_1 - \epsilon, \ldots, R_n - \epsilon) \in (\mathbb{R}_{>0})^n$.

Proposition 4.1. *Let (M, ω) be a $2m$-dimensional connected closed λ-monotone symplectic manifold. Let $e = (e_1, \ldots, e_n)$ and $R = (R_1, \ldots, R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively. For a positive real number ϵ with $2\epsilon < \min\{R_1, \ldots, R_n\}$, let U_ϵ be the open subset of $T^n_R \times T^n$ defined by*

$$U_\epsilon = \{(p, q) \in T^n_R \times T^n; p \in I_{R(2\epsilon)}\}.$$

We fix the symplectic form $\mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega_0$ on $M \times T^n_R \times T^n$, where $\mathrm{pr}_1\colon M \times T^n_R \times T^n \to M$ and $\mathrm{pr}_2\colon M \times T^n_R \times T^n \to T^n_R \times T^n$ are the projections defined by $\mathrm{pr}_1(x, p, q) = x$ and $\mathrm{pr}_2(x, p, q) = (p, q)$. Then for any Hamiltonian function $F\colon S^1 \times M \times U_\epsilon \to \mathbb{R}$ with compact support such that $\mathcal{P}(F; (0_M, e)) = \emptyset$,

$$c([M \times T^n_R \times T^n], F) < 2 \sum_{i=1}^n R_i \cdot |e_i| + \max\{0, -\lambda(m + n)\}.$$

To prove Proposition 4.1, we use the following proposition. Let 0_T denote the free homotopy class of constant loops in $T^n_R \times T^n$.

Proposition 4.2. *Let W be an open subset of a $2w$-dimensional connected closed λ-monotone symplectic manifold $(\hat{W}, \omega)$ and $\alpha \in [S^1, \hat{W}]$ a non-trivial homotopy class of free loops on $\hat{W}$. Assume that a Hamiltonian function $H\colon W \to \mathbb{R}$ satisfies the following conditions.*

- *for any point x in W, $\phi^1_H(x) = x$ and $[\gamma^x_H] = -\alpha$,*

- *H is a Morse function and* $\mathrm{ev}(\mathcal{P}(H; 0_{\hat{W}})) = \mathrm{Crit}(H)$*,*
- $\mathrm{ind}_{\mathrm{Morse}}(x) = \mathrm{ind}_{\mathrm{CZ}}([x, c_x])$ *for any point x in* $\mathrm{Crit}(H)$*,*

where $0_{\hat{W}}$ denotes the class of constant loops in $\hat{W}$.

Then for any Hamiltonian function $F\colon S^1 \times W \to \mathbb{R}$ with compact support such that $\mathcal{P}(F; \alpha) = \emptyset$,

$$c([\hat{W}], F) \leq 2\|H\|_{C^0} + \max\{0, -\lambda w\},$$

Proof. To give an upper bound of the spectral invariant associated to F, we consider the concatenation of ϕ_F^1 and a Hamiltonian diffeomorphism ϕ_H^1 with trajectories in $-\alpha$. We can choose a smooth function $\chi\colon [0, \frac{1}{2}] \to [0, 1]$ satisfying the following conditions.

- $\frac{\partial \chi}{\partial t}(t) \geq 0$ for any $t \in [0, \frac{1}{2}]$, and
- $\chi(t) = 0$ for any $t \in [0, \frac{1}{5}]$ and $\chi(t) = 1$ for any $t \in [\frac{2}{5}, \frac{1}{2}]$.

Let $L\colon S^1 \times \hat{W} \to \mathbb{R}$ be a Hamiltonian function defined by

$$L(t, x) = \begin{cases} \frac{\partial \chi}{\partial t}(t) H(\chi(t), x) & \text{when } t \in [0, \frac{1}{2}], \\ \frac{\partial \chi}{\partial t}(t - \frac{1}{2}) F(\chi(t - \frac{1}{2}), x) & \text{when } t \in [\frac{1}{2}, 1]. \end{cases}$$

We claim

$$c([\hat{W}], L) \leq \|H\|_{C^0} + \max\{0, -\lambda w\}.$$

Let $[z, u]$ be an element of $\tilde{\mathcal{P}}(H)$ and define x by $x = \mathrm{ev}(z)$. If $x \in W$, by the assumption of H, $[\gamma_H^x] = \mathcal{L}_{-\alpha}(W)$. Since the path γ_L^x is the concatenation of the paths γ_H^x and $\gamma_F^{\phi_H(x)}$ up to parameter change, $\mathcal{P}(F; \alpha) = \emptyset$ implies $\gamma_L^x \notin \mathcal{L}_{0_{\hat{W}}}(\hat{W})$ for any $x \in W$. If $x \notin W$, then $\phi_H(x) \notin W$. Thus γ_L^x is equal to γ_H^x up to parameter change and $\int_0^1 H(t, \gamma_H^x(t)) dt = \int_0^1 L(t, \gamma_L^x(t)) dt$. Therefore we see that there exists a natural inclusion map $\iota\colon \tilde{\mathcal{P}}(L) \to \tilde{\mathcal{P}}(H)$ which preserves values of the action functionals and the Conley-Zehnder indices.

We give an estimate of the critical value of the action functional $\mathcal{A}_L$ which attains the fundamental class. Since every element of $\mathcal{P}(H; 0_{\hat{W}})$ is a constant loop, every element of $\mathcal{P}(L; 0_{\hat{W}})$ is also a constant loop. Since $\mathcal{P}(L; 0_{\hat{W}})$ is a finite set and $(\hat{W}, \omega)$ is monotone, $\mathcal{A}_L(\tilde{\mathcal{P}}(H))$ is a discrete subset of $\mathbb{R}$. Thus $c([\hat{W}], L)$ is attained by a 1-periodic trajectory of the Conley-Zehnder index $2w$ that is the dimension of the fundamental class. Since every element of $\mathcal{P}(L; 0_{\hat{W}})$ is a constant loop, there exists a point x in $\hat{W}$ and $A \in \Gamma$ such that $\mathrm{ind}_{\mathrm{CZ}}([x, c_x \sharp A]) = 2w$ and $c([\hat{W}], L) = \mathcal{A}_L([x, c_x \sharp A])$.

Then, by the assumption,

$$\begin{aligned}&\mathrm{ind}_{\mathrm{Morse}}(x)+2c_1(A)\\&=\mathrm{ind}_{\mathrm{CZ}}([x,c_x])+2c_1(A)\\&=\mathrm{ind}_{\mathrm{CZ}}([x,c_x\sharp A])\\&=2w.\end{aligned}$$

Since $0\leq \mathrm{ind}_{\mathrm{Morse}}(x)\leq 2w$,

$$0\leq c_1(A)\leq w.$$

Thus

$$\begin{aligned}\mathcal{A}_L([x,c_x\sharp A])&=\mathcal{A}_H([x,c_x\sharp A])\\&=H(x)-\omega(A)\\&=H(x)-\lambda c_1(A),\end{aligned}$$

and therefore $c([\hat{W}],L)\leq \|H\|_{C^0}+\max\{0,-\lambda w\}$. By $\|\bar{H}\|_{C^0}=\|H\|_{C^0}$, the Lipschitz property and the homotopy invariance for spectral invariants (Proposition 3.2 (1) and (2)) imply

$$\begin{aligned}c([\hat{W}],F)&\leq c([\hat{W}],L)+\|\bar{H}\|_{C^0}\\&\leq(\|H\|_{C^0}+\max\{0,-\lambda(m+n)\})+\|H\|_{C^0}\\&=2\|H\|_{C^0}+\max\{0,-\lambda(m+n)\},\end{aligned}$$

□

The idea of using a Hamiltonian function H satisfying the above conditions comes from Irie's paper [I]. Seyfaddini's techniques of using the monotonicity assumption [Se] is useful in our proof.

To prove Proposition 4.1, we construct the Hamiltonian function H in Proposition 4.2 by using $H^{R,\epsilon,e}$ given by the following lemma.

Lemma 4.3. *Let R, ϵ be positive real numbers such that $2\epsilon<R$. Let w_1, w_2, w_3 and w_4 denote points $(R-\epsilon,0)$, $(R-\epsilon,\frac{1}{2})$, $(R+\epsilon,0)$ and $(R+\epsilon,\frac{1}{2})$ in $T^1_R\times T^1$, respectively. For an integer e, there exists a Hamiltonian function $H^{R,\epsilon,e}\colon T^1_R\times T^1\to\mathbb{R}$ which satisfies the following conditions.*

- $H^{R,\epsilon,e}(p,q)=-ep$ *on* $U_\epsilon=(-R+2\epsilon,R-2\epsilon)\times T^1$,
- $\mathrm{Crit}(H^{R,\epsilon,e})=\{w_1,w_2,w_3,w_4\}$,
- $H^{R,\epsilon,e}$ *is a Morse function,*
- $\|H^{R,\epsilon,e}\|_{L^\infty}<(R-\epsilon)\cdot|e|$,
- $\mathrm{ev}(\mathcal{P}(H^{R,\epsilon,e};0_T))=\mathrm{Crit}(H^{R,\epsilon,e})$,
- $\mathrm{ind}_{\mathrm{Morse}}(w_i)=\mathrm{ind}_{\mathrm{CZ}}([w_i,c_{w_i}])$ *for any* $i\in\{1,2,3,4\}$.

Here $\mathrm{Crit}(H^{R,\epsilon,e})$ *is the set of critical points of* $H^{R,\epsilon,e}$.

Proof. We realize a 2-torus T^2 in $\mathbb{R}^3$ as

$$T^2 = \{(x, y, z) \in \mathbb{R}^3; (\sqrt{x^2 + z^2} - 3)^2 + y^2 = 1\}.$$

Define the (time-independent) Hamiltonian function $H\colon T^2 \to \mathbb{R}$ by $H(x, y, z) = z$. Note that the set of critical points of H is

$$\{(0, 0, 2), (0, 0, 4), (0, 0, -2), (0, 0, -4)\}.$$

We can take a diffeomorphism $f\colon T^1_R \times T^1 \to T^2$ which maps w_1, w_2, w_3 and w_4 to $(0, 0, 2), (0, 0, 4), (0, 0, -2)$ and $(0, 0, -4)$, respectively and satisfies $H(f(p, q)) = \frac{p}{R}$ for any $p \in I_{R(2\epsilon)}$. Let $u^{R,\epsilon,e}\colon \mathbb{R} \to \mathbb{R}$ be a function such that

- $du^{R,\epsilon,e}(x) < 0$ for any real number x,
- $u^{R,\epsilon,e}(x) = -eRx$ if $|x| \leq 1 - \frac{2\epsilon}{R}$,
- $|u^{R,\epsilon,e}(x)| < (R - \epsilon) \cdot |e|$ if $|x| < 4$,

Define the Hamiltonian function $H^{R,\epsilon,e}\colon T^1_R \times T^1 \to \mathbb{R}$ by $H^{R,\epsilon,e} = u^{R,\epsilon,e} \circ H \circ f$. Assume that $(du^{R,\epsilon,e})_x$ is sufficiently C^1-small for any x with $2 \leq |x| \leq 4$. Then the Yorke estimate ([Y]) implies $\mathrm{ev}(\mathcal{P}(H^{R,\epsilon,e}; 0_T)) = \mathrm{Crit}(H^{R,\epsilon,e})$. Since $2 \leq |H(f(w_i))| \leq 4$ for any i, $dH^{R,\epsilon,e}$ is sufficiently C^1-small near $\mathrm{Crit}(H^{R,\epsilon,e})$ and hence $\mathrm{ind}_{\mathrm{Morse}}(w_i) = \mathrm{ind}_{\mathrm{CZ}}([w_i, c_{w_i}])$ for any i. □

Proof of Proposition 4.1. To use Proposition 4.2, we construct the Hamiltonian function H. Define the Hamiltonian function $H'\colon T^n_R \times T^n \to \mathbb{R}$ by

$$H'(p, q) = \sum_{i=1}^{n} H^{R_i,\epsilon_i,e_i}(p_i, q_i).$$

Then $\gamma^x_{H'} \in \mathcal{L}_{-e}(T^n_R \times T^n)$ for any $x \in U_\epsilon$. Thus we can take a neighborhood W of U_ϵ such that

$$\mathrm{ev}(\mathcal{P}(H'; (0_M, 0_T))) \cap \bar{W} = \emptyset.$$

In order to compute the spectral invariant, we take a generic perturbation of H'. Let $\rho\colon T^n_R \times T^n \to [0, 1]$ be a function such that

$$\rho(p, q) = \begin{cases} 1 & \text{for any } (p, q) \in T^n_R \times T^n \setminus W, \\ 0 & \text{for any } (p, q) \in U_\epsilon. \end{cases}$$

Let $G\colon M \to \mathbb{R}$ be a Morse function and define the Hamiltonian function $H\colon M \times T^n_R \times T^n \to \mathbb{R}$ by

$$H(y, p, q) = H'(p, q) + \rho(p, q) \cdot G(y).$$

If the Morse function G is sufficiently C^2-small, then

- $\mathrm{ev}(\mathcal{P}(H; (0_M, 0_T))) \cap (M \times W) = \emptyset$, and
- there exist only finitely many points $y_1, \ldots, y_k$ in M such that $\mathrm{Crit}(G) = \mathrm{ev}(\mathcal{P}(tG; 0_M)) = \{y_1, \ldots, y_k\}$ for any $t \in (0, 1]$.

Thus

$$\mathrm{ev}(\mathcal{P}(H;(0_M,0_T))) = \{(y_i,(w_{j_1},\dots,w_{j_n}))\}_{i\in\{1,\dots,k\},j_1,\dots,j_n\in\{1,2,3,4\}} = \mathrm{Crit}(H).$$

By Proposition 4.3,

$$\mathrm{ind}_{\mathrm{Morse}}(x) = \mathrm{ind}_{\mathrm{CZ}}([x,c_x]),$$

for any point x in $\mathrm{Crit}(H)$.

Hence H satisfies the conditions of Proposition 4.2 and thus we apply Proposition 4.2.

By Proposition 4.2 and $\|\bar{H}\|_{C^0} = \|H\|_{C^0}$, the Lipschitz property and the homotopy invariance for spectral invariants (Proposition 3.2 (1) and (2)) imply

$$\begin{aligned} c([M\times T^n_R\times T^n],F) &\leq< 2\|H\|_{C^0} + \max\{0,-\lambda(m+n)\} \\ &< 2(\sum_{i=1}^n (R_i-\epsilon)\cdot|e_i| + \|G\|_{C^0}) + \max\{0,-\lambda(m+n)\}, \end{aligned}$$

If the Morse function G is sufficiently C^2-small,

$$c([M\times T^n_R\times T^n],F) < 2\sum_{i=1}^n R_i\cdot|e_i| + \max\{0,-\lambda(m+n)\}.$$

□

To prove Theorem 1.4, we use the following theorem and proposition by Entov and Polterovich ([EP]).

Theorem 4.4 ([EP] Theorem 1.7). *Let (N_1,ω_1) and (N_2,ω_2) be closed symplectic manifolds. Assume that for $i=1,2$, Y_i is a heavy subset of (N_i,ω_i). Then the product $Y_1\times Y_2$ is a heavy subset of $N_1\times N_2$.*

Proposition 4.5 ([EP] Theorem 1.4). *Let (N,ω) be a closed symplectic manifold. Assume that Y is a heavy subset of (N,ω). Then Y is $[N]$-heavy.*

Proof of Theorem 1.4. Fix a positive real number ϵ with $\epsilon < \min\{R_1,\dots,R_n\}$ and take a Hamiltonian function $F\colon S^1\times M\times I^n_{R(\epsilon)}\times T^n\to\mathbb{R}$ with compact support such that $F|_{S^1\times X\times T^n} \geq 2\sum_{i=1}^n R_i\cdot|e_i| + \max\{0,-\lambda(m+n)\}$. Assume $\mathcal{P}(F;(0_M,e)) = \emptyset$. By $\mathcal{P}(F;(0_M,e)) = \emptyset$, Proposition 4.1 and the triangle inequality imply

$$\zeta_{[M\times T^n_{R(\epsilon)}\times T^n]}(F) < 2\sum_{i=1}^n R_i\cdot|e_i| + \max\{0,-\lambda(m+n)\}.$$

Note that Example 3.4 and Theorem 4.4 imply that $X\times T^n$ is a heavy subset. Since Proposition 4.5 implies that $X\times T^n$ is $[M\times T^n_{R(\epsilon)}\times T^n]$-heavy, by Definition 3.3,

$$\zeta_{[M\times T^n_{R(\epsilon)}\times T^n]}(F) \geq 2\sum_{i=1}^{n} R_i\cdot|e_i| + \max\{0, -\lambda(m+n)\}.$$

These two inequalities contradicts. Since any Hamiltonian function $F\colon S^1\times M\times I^n_R\times T^n\to\mathbb{R}$ with compact support has support in $S^1\times M\times I^n_{R(\epsilon)}\times T^n$ for some ϵ, we complete the proof of Theorem 1.4. □

As we mentioned in Introduction, Theorem 1.4 gives the inequality

$$C(\mathbb{C}P^m, \Phi^{-1}(y_0), R; e) \leq 2\sum_{i=1}^{m} R_i\cdot|e_i|.$$

We have another example.

Example 4.6. Since $\pi_2(T^n_R\times T^n) = 0$, by applying Theorem 1.3 to Example 3.4, we attain the inequality $C(T^n_R\times T^n, T^n, R; e)\leq 2\sum_{i=1}^{m} R_i\cdot|e_i|$ for any elements $e = (e_1,\ldots,e_m)$ and $R = (R_1,\ldots,R_m)$ of $\mathbb{Z}^m$ and $(\mathbb{R}_{>0})^m$, respectively.

5. Non-contractible trajectories on non-monotone symplectic manifolds

When we replace the existence problem of 1-periodic trajectories by the existence problem of periodic orbits whose period is not more than 1, we have the following result which does not need the assumption of monotonicity.

Theorem 5.1. *Let X be a heavy subset of a connected closed symplectic manifold (M,ω). Let $e = (e_1,\ldots,e_n)$ and $R = (R_1,\ldots,R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively. For any (time-independent) Hamiltonian function $F\colon M\times I^n_R\times T^n\to\mathbb{R}$ with compact support such that $F|_{X\times T^n}\geq 2\sum|e_i|R_i$, the Hamiltonian flow $\{\phi^t_F\}_{t\in\mathbb{R}}$ has a periodic orbits $(1,e)$ whose period is not more than 1 in the free loop homotopy class $(0_M, e)$.*

To prove Theorem 5.1, we give an upper bound of the spectral invariant for a Hamiltonian function $F\colon S^1\times M\times U_\epsilon\to\mathbb{R}$ such that its Hamiltonian isotopy $\{\phi^t_F\}$ has no trajectories in the free loop homotopy class $(0_M, e)$ whose period is not more than 1. For $R = (R_1,\ldots,R_n)\in(\mathbb{R}_{>0})^n$ and a positive real number ϵ with $\epsilon<\min\{R_1,\ldots,R_n\}$, let $R(\epsilon)$ denote $(R_1-\epsilon,\ldots,R_n-\epsilon)\in(\mathbb{R}_{>0})^n$, as before.

Proposition 5.2. *Let (M,ω) be a connected closed symplectic manifold. Let $e = (e_1,\ldots,e_n)$ and $R = (R_1,\ldots,R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively. For a positive real number ϵ with $2\epsilon<\min\{R_1,\ldots,R_n\}$, we define the open subset U_ϵ of $T^n_R\times T^n$ as in Proposition 4.1. Then for any Hamiltonian function $F\colon S^1\times M\times U_\epsilon\to\mathbb{R}$ with compact support such that* *its Hamiltonian isotopy* $\{\phi_F^t\}$ *has no trajectories in the free loop homotopy class* $(0_M, e)$ *whose period is not more than 1,*

$$c([M \times T_R^n \times T^n], F) < 2\sum_{i=1}^{n} R_i \cdot |e_i|.$$

To prove Proposition 5.2, we use the following proposition which is a slight modification of an argument in [I].

Proposition 5.3. *Let* W *be an open subset of a* $2w$*-dimensional connected closed symplectic manifold* $(\hat{W}, \omega)$ *and* $\alpha \in [S^1, \hat{W}]$ *a non-trivial homotopy class of free loops on* $\hat{W}$*. Assume that a Hamiltonian function* $H\colon W \to \mathbb{R}$ *satisfies the following conditions.*

- *for any point* x *in* W*,* $\phi_H^1(x) = x$ *and* $[\gamma_H^x] = -\alpha$*,*
- H *is non-degenerate,*

Let $0_{\hat{W}}$ *denote the class of constant loops in* $\hat{W}$*.*

Let $F\colon S^1 \times W \to \mathbb{R}$ *be a Hamiltonian function with compact support such that its Hamiltonian isotopy* $\{\phi_F^t\}$ *has no trajectories in the free loop homotopy class* $(0_{\hat{W}}, e)$ *whose period is not more than 1. Then*

$$c([\hat{W}], F) \leq 2\|H\|_{C^0},$$

For a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ with compact support, let $\mathrm{Spec}(H)$ denote the set of critical values of the action functional $\mathcal{A}_H$ *i.e.* $\mathcal{A}_H(\tilde{\mathcal{P}}(H))$. To prove Proposition 5.3, we use the following theorem.

Theorem 5.4 ([U], non-degenerate spectrality)**.** *Let* (M, ω) *be a closed symplectic manifold and* a *be an element of* $QH_*(M, \omega)$*. Then for any non-degenerate Hamiltonian function* $F\colon S^1 \times M \to \mathbb{R}$ *with compact support,* $c(a, F) \in \mathrm{Spec}(F)$*.*

Proof of Proposition 5.3. We give an upper bound of the spectral invariant associated to F by using the concatenation with ϕ_H^t.

For a real number s with $s \in [0, 1]$, we define the new Hamiltonian function $L^s\colon S^1 \times \hat{W} \to \mathbb{R}$ as follows:

$$L^s(t, x) = \begin{cases} \frac{\partial\chi}{\partial t}(t) H(\chi(t), x) & \text{when } t \in [0, \frac{1}{2}], \\ s\frac{\partial\chi}{\partial t}(t - \frac{1}{2}) F(s\chi(t - \frac{1}{2}), x) & \text{when } t \in [\frac{1}{2}, 1], \end{cases}$$

where $\chi\colon [0, \frac{1}{2}] \to [0, 1]$ is the function defined in the proof of Proposition 4.1. Since $\frac{\partial\chi}{\partial t} = 0$ on neighborhoods of $t = 0$ and $t = \frac{1}{2}$, L^s is a smooth Hamiltonian function.

We claim $\mathrm{Spec}(L^s) \subset \mathrm{Spec}(H)$ for a real number s with $s \in [0, 1]$. Let $F^s\colon S^1 \times \hat{W} \to \mathbb{R}$ denote the Hamiltonian function defined by $F^s(t, x) = s\frac{\partial\chi}{\partial t}(\frac{t}{2}) F(s\chi(\frac{t}{2}), x)$. Let $[z, u]$ be an element of $\tilde{\mathcal{P}}(H)$ and define x by $x = \mathrm{ev}(z)$. If $x \in W$, by the definition of H, $\gamma_H^x \in \mathcal{L}_{0_{\hat{W}}}(W)$. Since the path

$\gamma^x_{L^s}$ is the concatenation of the paths γ^x_H and $\gamma^{\phi_H(x)}_{F^s}$ up to parameter change and $\{\phi^t_F\}$ has no trajectories in the free loop homotopy class $0_{\hat{W}}$ whose period is not more than 1, $\gamma^x_{L^s} \notin \mathcal{L}_{0_{\hat{W}}}(\hat{W})$ for any $x \in W$. If $x \notin W$, then $\phi_H(x) \notin W$. Thus $\gamma^x_{L^s}$ is equal to γ^x_H up to parameter change and $\int_0^1 H(t,\gamma^x_H(t))dt = \int_0^1 L(t,\gamma^x_{L^s}(t))dt$. Therefore we see that there exists a natural inclusion map $\iota\colon \tilde{\mathcal{P}}(L^s) \to \tilde{\mathcal{P}}(H)$ which preserves values of the action functionals, and hence $\mathrm{Spec}(L^s) \subset \mathrm{Spec}(H)$. Since H is a non-degenerate Hamiltonian function, L^s is also non-degenerate, and hence Theorem 5.4 implies $c([\hat{W}], L^s) \in \mathrm{Spec}(H)$.

By the Lipschitz property for spectral invariants (Proposition 3.2 (1)), $c([\hat{W}], L^s)$ depends continuously on s. Since $\mathrm{Spec}(H)$ is a measure-zero set (Lemma 2.2 of [O02]), $c([\hat{W}], L^s)$ is a constant function of s. The homotopy invariance for spectral invariants (Proposition 3.2 (2)) implies

$$c([\hat{W}], L^0) = c([\hat{W}], H)$$

hence for any $s \in [0,1]$,

$$c([\hat{W}], L^s) = c([\hat{W}], H).$$

Then $c([\hat{W}], F)$ is estimated as follows.

$$\begin{aligned} c([\hat{W}], F) &\leq c([\hat{W}], L^1) + \|\bar{H}\|_{C^0} \\ &= c([\hat{W}], H) + \|H\|_{C^0} \\ &< 2(\sum_{i=1}^{n}(R_i - \epsilon)\cdot|e_i| + \|G\|_{C^0}). \end{aligned}$$

□

Proof of Proposition 5.2. Let G be a Morse function on M and $H\colon \hat{W} \to \mathbb{R}$ the Hamiltonian function defined in the proof of Proposition 4.1.

As we explained in the proof of Proposition 4.1, if the Morse function G is sufficiently C^2-small, then

$$\mathrm{ev}(\mathcal{P}(H;(0_M,0_T))) = \{(y_i,(w_{j_1},\dots,w_{j_n}))\}_{i\in\{1,\dots,k\},j_1,\dots,j_n\in\{1,2,3,4\}} = \mathrm{Crit}(H).$$

In particular, H is a non-degenerate Hamiltonian function. Since

$$\|\bar{H}\|_{C^0} = \|H\|_{C^0} \leq \sum_{i=1}^{n}(R_i - \epsilon)\cdot|e_i| + \|G\|_{C^0},$$

Proposition 5.3 implies

$$c([M\times T^n_R\times T^n], F) < 2(\sum_{i=1}^{n}(R_i - \epsilon)\cdot|e_i| + \|G\|_{C^0}).$$

If the Morse function G is sufficiently C^2-small,

$$c([M \times T^n_R \times T^n], F) < 2\sum_{i=1}^{n} R_i \cdot |e_i|.$$

□

The idea of using the Hamiltonian function H comes from [I].

Proof of Theorem 5.1. Fix a positive real number ϵ with $\epsilon < \min\{R_1, \dots, R_n\}$ and take a Hamiltonian function $F\colon S^1 \times M \times I^n_{R(\epsilon)} \times T^n \to \mathbb{R}$ with compact support such that $F|_{S^1 \times X \times T^n} \geq 2\sum_{i=1}^n R_i \cdot |e_i|$. Assume that $\{\phi^t_F\}$ has no trajectories in the free loop homotopy class $(0_M, e)$ whose period is not more than 1. Since $\{\phi^t_F\}$ has no trajectories in the free loop homotopy class $(0_M, e)$ whose period is not more than 1, Proposition 5.2 and the triangle inequality for spectral invariants (Proposition 3.2 (3)) imply $\zeta_{[M \times T^n_{R(\epsilon)} \times T^n]}(F) < 2\sum_{i=1}^n R_i \cdot |e_i|$.

By applying Theorem 4.4 to Example 3.4, we see that $X \times T^n$ is a heavy subset. Then Proposition 4.5 implies that $X \times T^n$ is $[M \times T^n_{R(\epsilon)} \times T^n]$-heavy, and hence $\zeta_{[M \times T^n_{R(\epsilon)} \times T^n]}(F) \geq \inf_{X \times T^n} F \geq 2\sum_{i=1}^n R_i \cdot |e_i|$ by Definition 3.3.

These two inequalities contradict and we proved that $\{\phi^t_F\}$ has a trajectory in the free loop homotopy class $(0_M, e)$ whose period is not more than 1. Since any Hamiltonian function $F\colon S^1 \times M \times I^n_R \times T^n \to \mathbb{R}$ with compact support has support in $S^1 \times M \times I^n_{R(\epsilon)} \times T^n$ for some ϵ, we complete the proof of Theorem 5.1. □

6. Displaceable subsets and non-contractible trajectories

For a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ with compact support on a symplectic manifold M, we define *the Hofer length* $\|H\|$ of H by

$$\|H\| = \int_0^1 \|H_t\|_{C^0} dt.$$

For a subset X of M, we define *the displacement energy* of X by

$$E(X) = \inf\{\|H\|; H \in C^\infty_c(S^1 \times M), \bar{X} \cap \phi^1_H(X) = \emptyset\},$$

where $\bar{X}$ is the topological closure of X. If X is non-displaceable, we define $E(X) = \infty$.

Proof of Proposition 1.5. To use Proposition 1.2, we construct a Hamiltonian function $\hat{H}\colon S^1 \times M \times I^n_R \times T^n \to \mathbb{R}$ such that $(X \times T^n) \cap \phi^1_{\hat{H}}(X \times T^n) = \emptyset$ and $\mathcal{P}(\hat{H}; (0_M, e)) = \emptyset$. Fix a sufficiently small positive real number ϵ. We take a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ with compact support such that $\|H\| < E(X) + \epsilon$ and $X \cap \phi^1_H(X) = \emptyset$. Since $|e_k| \cdot R_k > E(X)$ and ϵ is sufficiently small, we can take a function $\rho_k \in C^\infty_c(-R_k, R_k)$ such that

- $\rho_k = 1$ in a neighborhood of $\{0\}$,
- $|\dot{\rho}_k(x)| < |e_k| \cdot (E(X) + \epsilon)^{-1}$ for any $x \in (-R_k, R_k)$.

For $i \neq k$, we take a function $\rho_i \in C_c^\infty(-R_i, R_i)$ such that $\rho_i = 1$ in a neighborhood of $\{0\}$. we define the Hamiltonian function $\hat{H}\colon S^1 \times M \times I_R^n \times T^n \to \mathbb{R}$ by

$$\hat{H}(t, x, p, q) = \prod_i \rho_i(p_i) \cdot H(t, x).$$

Then

$$(X_{\hat{H}}^t)_{(x,p,q)} = (\prod_i \rho_i(p_i) \cdot (X_H^t)_x, 0, \dots, 0, \dot{\rho}_1(p_1) \cdot H(t, x), \dots, \dot{\rho}_n(p_n) \cdot H(t, x)).$$

Since $\rho_i = 1$ in a neighborhood of $\{0\}$, $(X \times T^n) \cap \phi_{\hat{H}}^1(X \times T^n) = \emptyset$. Since $|\dot{\rho}_k| < |e_k| \cdot (E(X) + \epsilon)^{-1}$ and $\int_0^1 ||H_t||_{C^0} dt = ||H|| < E(X) + \epsilon$, $\int_0^1 |\dot{\rho}_k(p_k)| \cdot |H(t, x)| dt$ is smaller than $|e_k|$ and hence $\mathcal{P}(\hat{H}; (0_M, e)) = \emptyset$. Thus Proposition 1.2 implies

$$C(M, X, R; e) = C_{BPS}(M \times I_R^n \times T^n, X \times T^n; (0_M, e)) = \infty.$$

□

7. Non-Lagrangian submanifolds and non-contractible trajectories

For a compact non-Lagrangian submanifold X, we have the following result.

Proposition 7.1. *Let (M, ω) be a $2m$-dimensional connected symplectic manifold and X an m-dimensional compact non-Lagrangian submanifold of M. Let $e = (e_1, \dots, e_n)$ and $R = (R_1, \dots, R_n)$ be elements of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$ with $e \neq 0$, respectively. Then $C(M, X, R; e) = \infty$.*

To prove Proposition 7.1, we use the following theorem which follows from an argument similar to that in the proof of Theorem C of [BPS].

Theorem 7.2. *Let (N, ω) be a $2m$-dimensional connected symplectic manifold, Y an m-dimensional compact non-Lagrangian submanifold of N and α a non-trivial homotopy class of free loops in N. Assume that the normal fibre bundle ν of Y in N has a non-vanishing section. Then $C_{BPS}(N, Y; \alpha) = \infty$.*

To prove Theorem 7.2, we use the following theorem.

Proposition 7.3 ([P95], [LS]). *Let (N, ω) be a $2m$-dimensional connected symplectic manifold and Y an m-dimensional compact non-Lagrangian submanifold of N. Assume that the normal fibre bundle ν of Y in N has a non-vanishing section. Then there exists a Hamiltonian function $H\colon N \to \mathbb{R}$ with compact support such that $(X_H)_y \notin T_y Y$ for any point y in Y.*

Proof of Theorem 7.2. To use Proposition 1.2, we construct a Hamiltonian function $\hat{H}\colon S^1 \times M \times I_R^n \times T^n \to \mathbb{R}$ such that $(X \times T^n) \cap \phi_{\hat{H}}^1(X \times T^n) = \emptyset$ and $\mathcal{P}(\hat{H}; (0_M, e)) = \emptyset$.

By the assumption, Proposition 7.3 implies that we can take a Hamiltonian function $H\colon N \to \mathbb{R}$ with compact support such that $(X_H)_y \notin T_yY$ for any point y in Y. Then there exists a positive real number ϵ such that $Y \cap \phi_{tH}^1(Y) = \emptyset$ and $\mathcal{P}(tH; \alpha) = \emptyset$ for any real number t with $0 < t < \epsilon$ and any integer e with $e \neq 0$.

Thus Proposition 1.2 implies

$$C(M, X, R; e) = C_{BPS}(M \times I_R^n \times T^n, X \times T^n; (0_M, e)) = \infty,$$

for any integer e with $e \neq 0$ □

Proof of Proposition 7.1. Since X is an m-dimensional non-Lagrangian submanifold and the normal fibre bundle ν of $X \times T^n$ in $M \times I_R^n \times T^n$ has a non-vanishing section, Theorem 7.2 implies

$$C(M, X, R; e) = C_{BPS}(M \times I_R^n \times T^n, X \times T^n; (0_M, e)) = \infty,$$

for any e with $e \neq 0$ □

8. Compressible Hamiltonian torus action and non-contractible trajectories

We have a family of examples similar to Example 1.6. Let (M, ω) be a closed symplectic manifold. We consider the case when a moment map $\Phi = (F^1, \dots, F^k)\colon M \to \mathbb{R}^k$ induces a Hamiltonian torus action *i.e.* $\phi_{F^i}^1 = \mathrm{id}$ for $i = 1, \dots, k$ and $\{F^i, F^j\} = 0$ for $i \neq j$. Then there exists a natural inclusion map $\iota\colon T^k \to \mathrm{Ham}(M, \omega)$. A Hamiltonian action induced by Φ is *compressible* if the image of the map $\iota_*\colon \pi_1(T^k) \to \pi_1(\mathrm{Ham}(M, \omega))$ is finite, where ι_* is induced by ι.

Entov and Polterovich proved the following theorem.

Theorem 8.1 ([EP])**.** *Let (M, ω) be a $2m$-dimensional connected closed symplectic manifold and $\Phi = (F^1, \dots, F^k)\colon M \to \mathbb{R}^k$ a moment map which induces a compressible Hamiltonian torus action. Assume that F^i is normalized as a Hamiltonian function for any i. Then*

(1): $\Phi^{-1}(0)$ *is heavy, thus stably non-displaceable,*

(2): $\Phi^{-1}(y)$ *is stably displaceable for any point y in $\Phi(M)$ with $y \neq 0$.*

We have the corresponding result on the existence problem of non-contractible trajectories.

Theorem 8.2. *Let (M, ω) be a connected closed λ-monotone symplectic manifold and $\Phi = (F^1, \dots, F^k)\colon M \to \mathbb{R}^k$ be a moment map which induces a compressible Hamiltonian torus action. Assume that F^i is normalized as* *a Hamiltonian function for any i. Then there exists a positive real number E such that*

- **(1):** $C(M, \Phi^{-1}(0), R; e) \leq 2\sum_{i=1}^{n} R_i \cdot |e_i| + \max\{0, -\lambda(m+n)\}$ *for any elements* $e = (e_1, \ldots, e_n)$ *and* $R = (R_1, \ldots, R_n)$ *of* $\mathbb{Z}^n$ *and* $(\mathbb{R}_{>0})^n$*, respectively,*
- **(2):** $C(M, \Phi^{-1}(y), R; e) = \infty$ *for any point* y *in* $\Phi(M)$ *with* $y \neq 0$ *and for any elements* $e = (e_1, \ldots, e_n)$ *and* $R = (R_1, \ldots, R_n)$ *of* $\mathbb{Z}^n$ *and* $(\mathbb{R}_{>0})^n$ *with* $R_l > E$ *for some* l *and* $e \neq 0$*, respectively.*

(1) of Theorem 8.2 follows immediately from Theorem 1.3 and (1) of Theorem 8.1.

To prove (2) of Theorem 8.2, we use the following theorem which is a slight modification of Theorem 2.1 of [EP]. Note that we can identify T^*T^1 with $\mathbb{R} \times T^1$ with the coordinate (p, q).

Proposition 8.3. *Let* X *be a compact subset of a closed symplectic manifold* M*. Assume that there exists a normalized Hamiltonian function* $F\colon S^1 \times M \to \mathbb{R}$ *generating a loop* $\{\phi_F^t\}_{t\in[0,1]}$ *in* $\mathrm{Ham}(M, \omega)$ *which is homotopic to the trivial isotopy relative to endpoints and* $F(t, x) \neq 0$ *for any* t *and any point* x *with* $x \notin X$*. Then there exists a Hamiltonian function* $H\colon S^1 \times M \times T^*T^1 \to \mathbb{R}$ *with compact support such that* $(X \times T^1) \cap \phi_H^1(X \times T^1) = \emptyset$ *and* $|\frac{\partial H}{\partial p}(t, x, p, q)| < 1$ *for any point* (t, x, p, q) *in* $S^1 \times M \times T^*T^1$*.*

Proof. Let $\{f_t^s\}_{s,t\in[0,1]}$ be a homotopy of loop $\{\phi_F^s\}_{s\in[0,1]}$ to the constant loop *i.e.* $f_0^s = \mathrm{id}$ and $f_1^s = \phi_F^s$. Let $F^t\colon S^1 \times M \to \mathbb{R}$ denote the normalized Hamiltonian function generating the isotopy $\{f_t^s\}_{s\in[0,1]}$. Consider the family of diffeomorphisms Ψ_t of $M \times T^*T^1$ defined by

$$\Psi_t(x, p, q) = (f_t^q x, p - F^t(q, f_t^q x), q).$$

By Theorem 6.1.B of [P01], Ψ_t is a Hamiltonian isotopy. Let $\hat{H}\colon S^1 \times M \times T^*T^1 \to \mathbb{R}$ be a Hamiltonian function generating Ψ_t. Note that $\hat{H}$ does not depend on the coordinate p since $\mathrm{pr}_3(\frac{d\Psi_t}{ds}) = 0$, where $\mathrm{pr}_3\colon M \times T^*T^1 \to T^1$ is the projection defined by $\mathrm{pr}_3(x, p, q) = q$.

We can take a function $\rho\colon \mathbb{R} \to \mathbb{R}$ with compact support such that

- $\rho = 1$ in $\bigcup_t(\mathrm{pr}_2(\Psi_t(X)))$, where $\mathrm{pr}_2\colon M \times T^*T^1 \to \mathbb{R}$ is the projection defined by $\mathrm{pr}_2(x, p, q) = p$,
- $|\dot{\rho}(x)| < \inf_t \|\hat{H}_t\|_{C^0}^{-1}$ for any $x \in \mathbb{R}$.

Let $H\colon S^1 \times M \times T^*T^1 \to \mathbb{R}$ a Hamiltonian function defined by

$$H(t, x, p, q) = \rho(p) \cdot \hat{H}(t, x, p, q).$$

Since $\hat{H}$ does not depend on the coordinate p,

$$\mathrm{pr}_{3*}((X_H^t)_{(x,p,q)}) = \dot{\rho}(p) \cdot \hat{H}_t(x, p, q).$$

Since $|\dot{\rho}(x)| < \inf_t \|\hat{H}_t\|^{-1}$, $|\frac{\partial H}{\partial p}(t,x,p,q)| < 1$. Since $\rho = 1$ in $\bigcup_t(\mathrm{pr}_2(\Psi_t(X)))$, $(X \times T^1) \cap \phi_H^1(X \times T^1) = \emptyset$.

□

The construction of Ψ_t appeared in [P01] and [EP].

Proof of (2) of Theorem 8.2. Let e be an element of $(\mathbb{Z}_{>0})^n$ with $e_k \neq 0$. To use Proposition 1.2, we construct a Hamiltonian function $\hat{H}\colon S^1 \times M \times I_R^n \times T^n \to \mathbb{R}$ such that

$$(\Phi^{-1}(y) \times T^n) \cap \phi_{\hat{H}}^1(\Phi^{-1}(y) \times T^n) = \emptyset,$$

and $\mathcal{P}(\hat{H}; (0_M, e)) = \emptyset$.

First, we prepare some Hamiltonian functions $H^l\colon S^1 \times M \times T^*T^1 \to \mathbb{R}$ ($l = 1, \dots, k$). Since the action induced by Φ is compressible, for any l there exists a sufficient large positive integer N_l such that the Hamiltonian function $N_l F^l$ generates a contractible Hamiltonian circle action on M.

Since $N_l F^l$ generates a contractible Hamiltonian circle action on M for any l, Proposition 8.3 implies that there exist Hamiltonian functions $H^l\colon S^1 \times M \times T^*T^1 \to \mathbb{R}$ ($l = 1, \dots, L$) with compact support such that

$$(\Phi^{-1}(y) \times T^1) \cap \phi_{H^l}^1(\Phi^{-1}(y) \times T^1) = \emptyset$$

for any y with $y_l \neq 0$ and $|\frac{\partial H^l}{\partial p}(x,p,q)| < 1$ for any point (x,p,q) in $M \times T^*T^1$.

Define the projection $\mathrm{pr}_2\colon M \times T^*T^1 \to \mathbb{R}$ by $\mathrm{pr}_2(x,p,q) = p$ and put $E = \max_l \sup\{|r|; r \in \mathrm{pr}_2(\cup_{t\in[0,1]} \mathrm{Supp}(H_t^l))\}$.

Fix a point $y = (y_1, \dots, y_k)$ of $\Phi(M)$ with $y \neq 0$. There exists some l such that $y_l \neq 0$. Let $R = (R_1, \dots, R_n)$ be an element of $(\mathbb{R}_{>0})^n$ with $R_k > E$. For $i \neq k$, we take a function $\rho_i\colon (-R_i, R_i) \to [0,1]$ with compact support such that $\rho_i = 1$ in a neighborhood of $\{0\}$. Let $\hat{H}^l\colon S^1 \times M \times I_R^n \times T^n \to \mathbb{R}$ be a Hamiltonian function defined by

$$\hat{H}^l(t,x,p,q) = \prod_{i\neq k} \rho_i(p_i) \cdot H^l(t,x,p_k,q_k).$$

Since $R_k > E$, $\hat{H}^l$ has compact support in $S^1 \times M \times I_R^n \times T^n$. Then

$$\mathrm{pr}_*((X_{\hat{H}^l}^t)_{(x,p,q)}) = \prod_{i\neq k} \rho_i(p_i) \cdot \frac{\partial H^l}{\partial p}(t,x,p_k,q_k),$$

where $\mathrm{pr}\colon M \times I_R^n \times T^n \to T^1$ is the projection defined by $\mathrm{pr}(x,p,q) = q_k$

Since

$$(\Phi^{-1}(y) \times T^1) \cap \phi_{H^l}^1(\Phi^{-1}(y) \times T^1) = \emptyset,$$

and $\rho_i = 1$ in a neighborhood of $\{0\}$,

$$(\Phi^{-1}(y) \times T^n) \cap \phi_{\hat{H}^l}^1(\Phi^{-1}(y) \times T^n) = \emptyset.$$

Since $|\frac{\partial H^l}{\partial p}(t,x,p,q)| < 1$ for any point (t,x,p,q) in $S^1 \times M \times T^*T^1$ and the image of ρ_i is in $[0,1]$, $\mathcal{P}(\hat{H}^l;(0_M,e)) = \emptyset$.

Thus Proposition 1.2 implies

$$C(M,X,R;e) = C_{BPS}(M \times I^n_R \times T^n, X \times T^n; (0_M,e)) = \infty,$$

for any e with $e \neq 0$. □

9. Stably non-displaceable subsets and non-contractible trajectories

For a positive integer n, a subset X of a symplectic manifold M is *n-stably displaceable* if $X \times T^n$ is displaceable in $M \times T^*T^n$. X is *n-stably non-displaceable* otherwise. If X is n-stably non-displaceable, then X is stably non-displaceable.

By our Theorem 1.4, we have estimates on $C(M,X,R;e)$ for heavy subsets. However, we expect better estimates. We would like to pose the following problems.

Problem 9.1. Let X be an n-stably non-displaceable compact subset of a closed symplectic manifold (M,ω). Show that the inequality

$$C(M,X,R;e) \leq \sum_{i=1}^{n} R_i \cdot |e_i|$$

holds for any elements $e = (e_1,\ldots,e_n)$ and $R = (R_1,\ldots,R_n)$ of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively.

Problem 9.2. Let N be a connected closed Riemannian manifold and $\alpha \in [S^1,N]$ a non-trivial free homotopy class. Let X be a stably non-displaceable compact subset of a closed symplectic manifold (M,ω). Show that the inequality

$$C_{BPS}(M \times B^*N, X \times N; \alpha) = l_\alpha,$$

holds, where l_α is the infimum of geodesic length in the class α.

Weber [W] gave the positive answer to Problem 9.2 when M is one point set $\{*\}$ and $M = X$.

Since heavy subsets are stably non-displaceable, the positive answer to Problem 9.1 is a generalization of Theorem 1.3. In this subsection, we give several supporting observations related to Problem 9.1.

The argument in [BPS] shows the following estimate from below.

Proposition 9.3. *Let X be any subset of a closed symplectic manifold (M,ω). Then*

$$C_{BPS}(M \times B^*N, X \times N; \alpha) \geq l_\alpha.$$

In Section 10, we introduce a relative symplectic capacity C^P which is defined in terms of invariant measures of (time-independent) Hamiltonian flow and satisfies $C^P(M,X,R;e) \leq C(M,X,R;e)$.

Theorem 9.4. *Let (M,ω) be a closed symplectic manifold and X an n-stably non-displaceable compact subset of M. Then*

$$C^P(M,X,R;e) \leq \sum_{i=1}^{n} R_i \cdot |e_i|,$$

for any elements $e=(e_1,\dots,e_n)$ and $R=(R_1,\dots,R_n)$ of $\mathbb{Z}^n$ and $(\mathbb{R}_{>0})^n$, respectively.

We prove Theorem 9.4 in Section 10.

We cannot replace the assumption that X is n-stably non-displaceable in Problem 9.1 by that X is non-displaceable. We have the following example.

Proposition 9.5. *Let S^2 be a 2-sphere $\{(x,y,z)\in\mathbb{R}^3; x^2+y^2+z^2=1\}$ with the standard area (symplectic) form. For a positive real number ϵ, we define a subset C_ϵ of S^2 by $C_\epsilon=\{(x,y,z)\in S^2; z=\pm\epsilon\}$. Then C_ϵ is stably displaceable for any positive real number ϵ and there exists a positive real number E such that*

$$C(S^2,C_\epsilon,R;e)=\infty,$$

for any positive real number ϵ and any elements R and e of $(\mathbb{R}_{>0})^n$ and $\mathbb{Z}^n$ with $R_k>E$ and $e_k\neq 0$ for some k, respectively.

Remark 9.6. Let A_ϵ and B_ϵ be the subsets of S^2 defined by $A_\epsilon=\{(x,y,z)\in S^2; |z|\leq\epsilon\}$ and $B_\epsilon=\{(x,y,z)\in S^2; z>\epsilon\}$, respectively. If $\epsilon<\frac{1}{3}$, then $\mathrm{Area}(A_\epsilon)<\mathrm{Area}(B_\epsilon)$. Since any Hamiltonian diffeomorphism is area-preserving, C_ϵ is non-displaceable.

Professor Kaoru Ono told the author that C_ϵ for $\epsilon<\frac{1}{3}$ is an example due to Professor Polterovich of a non-displaceable subset which is stably non-displaceable.

Proof of Proposition 9.5. Let e be an element of $(\mathbb{Z}_{>0})^n$ with $e_k\neq 0$. To use Proposition 1.2, we construct a Hamiltonian function $\hat{H}\colon S^1\times S^2\times I_R^n\times T^n\to\mathbb{R}$ such that $(C_\epsilon\times T^n)\cap\phi_{\hat{H}}^1(C_\epsilon\times T^n)=\emptyset$ and $\mathcal{P}(\hat{H};(0_M,e))=\emptyset$.

Let $F\colon S^2\to\mathbb{R}$ be a Hamiltonian function defined by $F(x,y,z)=4\pi z$. The isotopy $\{\phi_F^t\}_{t\in[0,1]}$ is homotopic to the trivial isotopy relative to end-points.

Thus Proposition 8.3 implies that there exists a Hamiltonian function $H\colon S^1\times S^2\times T^*T^1\to\mathbb{R}$ with compact support such that $(C_\epsilon\times T^1)\cap\phi_H^1(C_\epsilon\times T^1)=\emptyset$ and $|\frac{\partial H}{\partial p}(x,y,z,p,q)|<1$ for any point (x,y,z,p,q) in $S^2\times T^*T^1$.

Define the projection $\mathrm{pr}_4\colon S^2\times T^*T^1\to\mathbb{R}$ by $\mathrm{pr}_4(x,y,z,p,q)=p$ and put $E=\sup\{|r|; r\in\mathrm{pr}_4(\cup_{t\in[0,1]}\mathrm{Supp}(H_t))\}$. Let $R=(R_1,\dots,R_n)$ be an element of $(\mathbb{R}_{>0})^n$ with $R_k > E$. For $i \neq k$, we take a function $\rho_i\colon (-R_i, R_i) \to [0,1]$ with compact support such that $\rho_i = 1$ in a neighborhood of $\{0\}$. Let $\hat{H}\colon S^1 \times S^2 \times I^n_R \times T^n \to \mathbb{R}$ be a Hamiltonian function defined by

$$\hat{H}(t,x,y,z,p,q) = \prod_{i\neq k} \rho_i(p_i) \cdot H(t,x,y,z,p_k,q_k).$$

Since $R_k > E$, $\hat{H}$ has compact support in $S^1 \times S^2 \times I^n_R \times T^n$. Then

$$\mathrm{pr}_*((X^t_{\hat{H}})_{(x,y,z,p,q)}) = \prod_{i\neq k} \rho_i(p_i) \cdot \frac{\partial H}{\partial p}(t,x,y,z,p_k,q_k),$$

where $\mathrm{pr}\colon S^2 \times I^n_R \times T^n \to T^1$ is the projection defined by $\mathrm{pr}(x,y,z,p,q) = q_k$

Since $\rho_i = 1$ in a neighborhood of $\{0\}$, $(X \times T^n) \cap \phi^1_{\hat{H}}(X \times T^n) = \emptyset$. Since $|\frac{\partial H}{\partial p}(t,x,y,z,p,q)| < 1$ for any point (t,x,y,z,p,q) in $S^1 \times S^2 \times T^*T^1$ and the image of ρ_i is in $[0,1]$, $\mathcal{P}(\hat{H};(0_M,e)) = \emptyset$.

Thus Proposition 1.2 implies

$$C(M,X,R;e) = C_{BPS}(M \times I^n_R \times T^n, X \times T^n; (0_M,e)) = \infty$$

for any e with $e \neq 0$. □

10. Polterovich's invariant measure and Proof of Theorem 9.4

First, we review several definitions in order to fix the terminology.

Definition 10.1. Let N be a manifold and X a smooth vector field on N generating a flow ϕ^t. For an invariant Borel measure μ of ϕ^t with compact support, its *rotation vector* $\rho(\mu, X)$ is an element of 1-dimensional homology $H_1(N;\mathbb{R})$ defined by

$$\langle \mathbf{l}^*, \rho(\mu, X)\rangle = \int_N \lambda(X)\mu,$$

for any cohomology class $\mathbf{l}^*$ of $H^1(N;\mathbb{R})$, where λ is a closed 1-form representing $\mathbf{l}^*$.

We can easily verify that $\int_N \lambda(X)\mu$ does not depend on the choice of λ.

We define relative symplectic capacities C^P_{BPS} and C^P. For a manifold N and the vector field X on N generating a flow ϕ^t, let $\mathfrak{M}(N,X)$ denote the set of invariant Borel measures of ϕ^t with compact support.

Definition 10.2. Let Y be a compact subset of an open symplectic manifold (N,ω) and $\alpha \in [S^1, N]$ a free homotopy class in N. For a cohomology class $\mathbf{l}^* \in H^1(N;\mathbb{R})$, we define the relative symplectic capacity $C^P_{BPS}(N,Y;\mathbf{l}^*,\alpha)$

by

$$\begin{aligned}&C^P_{BPS}(N, Y; \mathbf{l}^*, \alpha)\\&= \inf\{K > 0; \forall H \in C^\infty(N) \text{ such that } H|_Y \geq K,\\&\exists \mu \in \mathfrak{M}(N, X_H) \text{ such that } |\langle \mathbf{l}^*, \rho(\mu, X_H)\rangle| \geq \mathbf{l}^*(\alpha)\}.\end{aligned}$$

We define the relative symplectic capacity $C^P_{BPS}(N, Y; \alpha)$ by

$$C^P_{BPS}(N, Y; \alpha) = \sup_{\mathbf{l}^* \in H^1(N;\mathbb{R})} C^P_{BPS}(N, Y; \mathbf{l}^*, \alpha).$$

Let X be a compact subset of a closed symplectic manifold (M, ω). For an element $e = (e_1, \ldots, e_n)$ of $\mathbb{Z}^n$ and an element $R = (R_1, \ldots, R_n)$ of $(\mathbb{R}_{>0})^n$, we define the relative symplectic capacity $C^P(M, X, R; e)$ by

$$C^P(M, X, R; e) = C^P_{BPS}(M \times I^n_R \times T^n, X \times T^n; (0_M, e)).$$

Note that for any positive real number s, $C^P_{BPS}(N, Y; s\mathbf{l}^*, \alpha) = C^P_{BPS}(N, Y; \mathbf{l}^*, \alpha)$. Since every 1-periodic orbit representing a non-trivial homology class $\mathbf{a}$ determines an invariant measure with the rotation vector $\mathbf{a}$, we see that $C^P_{BPS}(N, Y, \alpha) \leq C_{BPS}(N, Y; \alpha)$ and $C^P(M, X, R; e) \leq C(M, X, R; e)$.

A diffeomorphism ψ of M is said to be *a symplectomorphism* if $\psi^*\omega = \omega$ and an isotopy $\{\psi^t\}_{t\in[0,1]}$ of M is said to be *a symplectic isotopy* if $\psi^0 = \mathrm{id}$ and $(\psi^t)^*\omega = \omega$ for any t. Let $\mathrm{Symp}(M, \omega)$ denote the group of symplectomorphisms of (M, ω) with compact support. Let $\widetilde{\mathrm{Symp}}_0(M, \omega)$ denote the universal covering of the identity component of $\mathrm{Symp}(M, \omega)$ and its element is the homotopy class of a symplectic isotopy $\{\psi^t\}_{t\in[0,1]}$ of M relative to the end points $\psi^0 = \mathrm{id}$ and $\psi^1 = \psi$.

Definition 10.3. The *flux homomorphism* $\mathrm{Flux}\colon \widetilde{\mathrm{Symp}}_0(M, \omega) \to H^1(M; \mathbb{R})$ is defined by

$$\mathrm{Flux}([\{\psi^t\}_{t\in[0,1]}]) = \int_0^1 \iota_{X^t}\omega dt,$$

for any element $[\{\psi^t\}_{t\in[0,1]}]$ of $\widetilde{\mathrm{Symp}}_0(M, \omega)$, where X^t is the (time-dependent) vector field which generates $\{\psi^t\}_t$.

The flux homomorphism is known to be surjective.

To prove Theorem 9.4, we explain Polterovich's result in [P14].

Theorem 10.4 ([P14])**.** *Let Y_1 and Y_2 be non-displaceable compact subsets of a closed symplectic manifold (N, ω). Assume that $Y_1 \cap Y_2 = \emptyset$ and there exists a symplectic isotopy $\{\psi^t\}_{t\in[0,1]}$ such that $\psi^1(Y_1) = Y_2$. Put $\mathbf{l}^* = \mathrm{Flux}(\{\psi^t\})$. Then for any positive real number p and any Hamiltonian* *function* $F\colon N \to \mathbb{R}$ *such that* $F|_{Y_1} \leq 0$ *and* $F|_{Y_2} \geq p$, $\{\phi_F^t\}$ *possesses an invariant measure* μ *such that* $\mathrm{Supp}(\mu) \subset \mathrm{Supp}\, F$ *and*

$$|\langle \mathbf{l}^*, \rho(\mu, X_F)\rangle| \geq p.$$

Let $\mathrm{pr}_1\colon M \times I_R^n \times T^n \to M$ denote the projections defined by $\mathrm{pr}_1(y, p, q) = y$. Define the subset S_R of $\mathbb{R}^3$ by $S_R = \partial \bar{I}_R^n$, more precisely,

$$\begin{aligned} S_R &= (\{-R_1, R_1\} \times [-R_2, R_2] \times \cdots \times [-R_n, R_n]) \\ &\quad \cup ([-R_1, R_1] \times \{-R_2, R_2\} \times \cdots \times [-R_n, R_n]) \\ &\quad \cup \cdots \cup ([-R_1, R_1] \times [-R_2, R_2] \times \cdots \times \{-R_n, R_n\}). \end{aligned}$$

Proof of Theorem 9.4. Fix a cohomology class

$$\mathbf{l}^* = \mathrm{pr}_1^* \mathbf{b}^* + a_1[dq_1] + \cdots + a_n[dq_n] \neq 0 \in H^1(M \times I_R^n \times T^n; \mathbb{R}),$$

where $a_1, \dots, a_n$ are real numbers and $\mathbf{b}^* \in H^1(M; \mathbb{R})$ is a cohomology class of M. To use Theorem 10.4, we prepare the symplectic isotopy $\{\psi^t\}_{t\in[0,1]}$. Since $\mathbf{l}^* \neq 0$, there exists a unique positive real number K such that $(Ka_1, \dots, Ka_n) \in S_R$. Then we regard $I_R^n \times T^n$ as a subset of $T_{2R}^n \times T^n$. Fix a point x_0 in M. Since the flux homomorphism is surjective, there exists a symplectic isotopy $\{\psi_0^t\}_{t\in[0,1]}$ of M such that $\mathrm{Flux}(\{\psi_0^t\}_{t\in[0,1]}) = K\mathbf{b}^*$. Let $\{\psi^t\}$ be the symplectic isotopy of $M \times T_{2R}^n \times T^n$ defined by

$$\psi^t(x, p_1, \dots, p_n, q_1, \dots, q_n) = (\psi_0^t(x), p_1 + Ka_1 t, \dots, p_n + Ka_n t, q_1, \dots, q_n).$$

Then

$$\mathrm{Flux}(\{\psi^t\}_{t\in[0,1]}) = Ka_1[dq_1] + \cdots + Ka_n[dq_n] + K\,\mathrm{pr}_1^* \mathbf{b}^* = K\mathbf{l}^*.$$

Assume that a Hamiltonian function $H\colon M \times I_R^n \times T^n \to \mathbb{R}$ satisfies $H|_{X\times T^n} \geq \sum_{i=1}^n R_i \cdot |e_i|$. We regard F as a Hamiltonian function on $M \times T_{2R}^n \times T^n$. Since $\psi^1(X \times \{-R\} \times T^n) = X \times \{0\} \times T^n$ and $F|_{X\times\{(-Ka_1,\dots,-Ka_n)\}\times T^n} = 0$, Theorem 10.4 implies that there exists an invariant measure μ on $M \times I_R^n \times T^n$ such that

$$|\langle R_1[dq_1] + \cdots + R_n[dq_n], \rho(\mu, X_F)\rangle| \geq \sum_{i=1}^n R_i \cdot |e_i|.$$

Since $(Ka_1, \dots, Ka_n) \in S_R$, $K\mathbf{l}^*(e) \leq \sum_{i=1}^n R_i \cdot |e_i|$. Thus, for any cohomology class $\mathbf{l}^*$ with $\mathbf{l}^* \neq 0$,

$$\begin{aligned} &C_{BPS}^P(M \times I_R^n \times T^n, X \times T^n; \mathbf{l}^*, (0_M, e)) \\ &= C_{BPS}^P(M \times I_R^n \times T^n, X \times T^n; K\mathbf{l}^*, (0_M, e)) \leq \sum_{i=1}^n R_i \cdot |e_i|. \end{aligned}$$

Since $C^P_{BPS}(M \times I^n_R \times T^n, X \times T^n; 0, (0_M, e)) = 0$, $C^P(M, X, R; e) \leq \sum_{i=1}^n R_i \cdot |e_i|$. □

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