博士論文

論文題目　Algebraic structure on the space of intertwining operators
(絡作用素の空間上の代数構造)

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Contents

1 Introduction

The aim of this thesis is to study branching laws of real reductive Lie groups by algebraic methods.

1.1 Irreducible decomposition

The main problem is to analyze the restriction of an irreducible representation of a Lie group (resp. Lie algebra) with respect to a closed subgroup (resp. Lie subalgebra). The problem is called the branching problem. If the given irreducible representation is unitary, the following fact assures us that the restriction has an irreducible decomposition.

Fact 1.1 (Mautner and Teleman)**.** *Let U be a unitary representation of a locally compact group G. Then U has an irreducible decomposition:*

$$U \simeq \int_{\widehat{G}}^{\oplus} m(\pi) V_\pi d\mu(\pi),$$

where $\widehat{G}$ is the unitary dual of G and V_π is a representation space of π.

The measurable function $m : \widehat{G'_{\mathbb{R}}} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is called the multiplicity function. For the case of Lie group representations, R. Goodman showed in the proof of [15, Lemma 3.1] that the direct integral decomposition is compatible with the Lie algebra action.

Fact 1.2 (Goodman). *Let U be a unitary representation of a Lie group G with direct integral decomposition:*

$$U \simeq \int_Z^{\oplus} U_z d\mu(z).$$

Then for any vector $v \in U^\infty$ defined by a section $z \mapsto v(z)$, $v(z)$ belongs to U_z^∞ for μ-almost every z, where U^∞ is the space of smooth vectors with respect to the G-action. Furthermore, for any $X \in \mathcal{U}(\mathfrak{g})$, we have $(Xv)(z) = X(v(z))$ for μ-almost every z.

1.2 Branching problem

Our main concern is the branching problem of unitary representations of real reductive Lie groups. For the branching problem of real reductive Lie groups, we refer to [51, 56]. Let $G_\mathbb{R}$ be a real reductive Lie group and $G'_\mathbb{R}$ a reductive subgroup of $G_\mathbb{R}$. For any irreducible unitary representation V of $G_\mathbb{R}$, the restriction $V|_{G'_\mathbb{R}}$ to $G'_\mathbb{R}$ has an irreducible decomposition by the theorem of Mautner and Teleman:

$$V|_{G'_\mathbb{R}} \simeq \int_{\widehat{G'_\mathbb{R}}}^{\oplus} m(\pi) V_\pi d\mu(\pi). \tag{1.2.1}$$

Since $G'_\mathbb{R}$ is reductive, the irreducible decomposition is unique. The irreducible decomposition is called the branching law of V with respect to $G'_\mathbb{R}$.

The branching problem for compact Lie groups has been studied by many mathematicians, and explicit branching laws have been obtained such as the Clebsh–Gordan formula, the Pieri rule, the branching laws for $(G_\mathbb{R}, G'_\mathbb{R}) = (\mathrm{U}(n), \mathrm{U}(n-1)), (\mathrm{SO}(n), \mathrm{SO}(n-1))$ and $(\mathrm{Sp}(n), \mathrm{Sp}(n-1))$, the Littlewood–Richardson coefficient, Kostant's formula and the Littelmann path model. Conversely, the branching problem for non-compact reductive Lie groups is difficult in general, and branching laws were known at the end of 1980s only for specific unitary representations such as holomorphic discrete series representations [24], [29], [70], [89], the Segal-Shale-Weil representation [25, 26], [35] and K-type formulas.

In the late 1980s, T. Kobayashi discovered discretely decomposable branching laws of Zuckerman derived functor modules $A_\mathfrak{q}(\lambda)$. Let θ be a Cartan involution of $G_\mathbb{R}$ preserving $G'_\mathbb{R}$. Set $K_\mathbb{R} := G_\mathbb{R}^\theta$ and $\mathfrak{g}_\mathbb{R} := \mathrm{Lie}(G_\mathbb{R})$. We denote by K and $\mathfrak{g}$ the complexification of $K_\mathbb{R}$ and $\mathfrak{g}_\mathbb{R}$, respectively. We use a similar notation for $G'_\mathbb{R}$ such as $K'_\mathbb{R}$, K' and $\mathfrak{g}'$. For a representation V of $G_\mathbb{R}$, we write V_K for the subspace of all $K_\mathbb{R}$-finite vectors. In the series of papers [42, 43, 45, 46], Kobayashi initiated and developed the theory of discretely decomposable $(\mathfrak{g}, K)$-modules and gave examples of explicit branching laws for $A_{\mathfrak{q}}(\lambda)$.

Definition 1.3 (T. Kobayashi [45, Definition 1.1])**.** A $(\mathfrak{g}, K)$-module V is said to be *discretely decomposable* if V has a $(\mathfrak{g}, K)$-module filtration $V_0 \subset V_1 \subset \cdots$ such that $\bigcup_i V_i = V$ and each V_i is finite length.

He gave criteria for the discrete decomposability of a restriction of an irreducible $(\mathfrak{g}, K)$-module by the asymptotic K-support [45] and the associated variety [46], and gave necessary and sufficient conditions for the discrete decomposability of a restriction of $A_{\mathfrak{q}}(\lambda)$. An important property is that the discrete decomposability of a $(\mathfrak{g}, K)$-module implies the discrete decomposability of a unitary representation of $G_{\mathbb{R}}$ as follows.

Fact 1.4 (T. Kobayashi [48, Theorem 2.7])**.** *Let V be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that $V_K|_{(\mathfrak{g}', K')}$ is discretely decomposable. Then $V_K|_{(\mathfrak{g}', K')}$ is decomposed into the direct sum of irreducible $(\mathfrak{g}', K')$-modules:*

$$V_K|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\pi \in \widehat{G'_{\mathbb{R}}}} m(\pi)(V_\pi)_{K'},$$

and $V|_{G'_{\mathbb{R}}}$ is decomposed into the direct sum of irreducible unitary representations with the same multiplicity function $m(\pi)$:

$$V|_{G'_{\mathbb{R}}} \simeq \sum_{\pi \in \widehat{G'_{\mathbb{R}}}}^{\oplus} m(\pi) V_\pi.$$

In the framework, discretely decomposable restrictions, explicit branching laws were computed for some unitary representations [8], [16], [42, 43, 45, 46], [58], [74], [78], [83], [95]. The discretely decomposable restrictions of $A_{\mathfrak{q}}(\lambda)$ with respect to symmetric subgroups were classified by T. Kobayashi and Y. Oshima [60], and the branching laws were obtained by Y. Oshima in [82].

1.3 Intertwining operator

The space of all intertwining operators is important to study the branching problem. Let V be an irreducible $(\mathfrak{g}, K)$-module and V' an irreducible $(\mathfrak{g}', K')$-module. Consider the following two vector spaces:

$$\mathrm{Hom}_{\mathfrak{g}', K'}(V, V'),$$
$$\mathrm{Hom}_{\mathfrak{g}', K'}(V', V).$$

An element of $\mathrm{Hom}_{\mathfrak{g}',K'}(V,V')$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ is called an *intertwining operator.*

The two spaces have a natural $\mathcal{U}(\mathfrak{g})^{G'}$-module structure. For the case of compact G', it is well-known that the action on $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ is irreducible. In particular, when $G'_{\mathbb{R}}$ is equal to the maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$, the $\mathcal{U}(\mathfrak{g})^{K}$-module plays an important role in the theory of $(\mathfrak{g},K)$-modules such as Harish-Chandra's subquotient theorem [19]. Remark that for non-compact $G'_{\mathbb{R}}$, the $\mathcal{U}(\mathfrak{g})^{G'}$-modules may be reducible.

The space $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ is deeply related to the discrete decomposability:

Fact 1.5 (T. Kobayashi [46, Lemma 1.5])**.** *Let V be an irreducible $(\mathfrak{g},K)$-module. $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ is non-zero for some irreducible $(\mathfrak{g}',K')$-module if and only if $V|_{(\mathfrak{g}',K')}$ is discretely decomposable.*

In many cases, the restriction of an irreducible $(\mathfrak{g},K)$-module to the sub-pair $(\mathfrak{g}',K')$ is not discretely decomposable, and $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ is zero for any irreducible $(\mathfrak{g}',K')$-module V'. In such cases, any general theories to deal with branching laws are not known. Nevertheless, some geometric and analytic methods work well for some irreducible unitary representations [9], [24], [58], [59], [63], [77], [79], [80], [109].

The space $\mathrm{Hom}_{\mathfrak{g}',K'}(V',V)$ may have many information about branching laws with continuous spectrum. T. Kobayashi proposed the program to construct symmetry breaking operators explicitly. Here a symmetric breaking operators means a continuous $G'_{\mathbb{R}}$-intertwining operator from a continuous irreducible (or finite length) representation of $G_{\mathbb{R}}$ to one of $G'_{\mathbb{R}}$. The explicit construction was obtained for principal series representations and $(G_{\mathbb{R}},G'_{\mathbb{R}}) = (\mathrm{O}(n+1,1),\mathrm{O}(n,1))$ [63], and holomorphic discrete series representations [61, 62]. A relation between symmetry breaking operators and $(\mathfrak{g},K)$-module intertwining operators was discussed in [55], and recent developments and open problems on this topic are in [56].

1.4 Multiplicity-free representation

The concept of a multiplicity-free representation is just as important as that of the discrete decomposability.

Definition 1.6. For a unitary representation V of $G_{\mathbb{R}}$, we denote by $\mathcal{M}_{G_{\mathbb{R}}}(V)$ the essential supremum of the multiplicity function of the irreducible decomposition. V is said to be *multiplicity-free* if $\mathcal{M}_{G_{\mathbb{R}}}(V) = 1$, and to have *uniformly bounded multiplicities* if $\mathcal{M}_{G_{\mathbb{R}}}(V) < \infty$.

We use the same terminology for completely reducible $(\mathfrak{g}, K)$-modules and algebraic representations.

The Fourier transform, the Fourier series expansion and spherical harmonics are classical and important examples of multiplicity-free representations. In the representation theory of Lie groups, many multiplicity-free representations are known such as the Clebsh–Gordan formula, the Pieri rule, the Peter–Weyl theorem, the branching laws for $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{U}(n), \mathrm{U}(n-1))$ and $(\mathrm{SO}(n), \mathrm{SO}(n-1))$, the Cartan–Helgason theorem, the Plancherel formulas for Riemannian symmetric spaces and group manifolds.

A multiplicity-free representation has a 'canonical' irreducible decomposition, and the representation yields some natural transform like the Fourier transform. Therefore, finding a multiplicity-free representation may be related to finding some good analysis and geometry. We refer the reader to [50] for the point of view.

A spherical variety is one of the geometric objects to produce multiplicity-free representations.

Definition 1.7. Let G be a complex reductive algebraic group with Borel subgroup B. A G-variety X is said to be *spherical* if B has an open orbit on X.

By [100, Theorem 2], an affine G-variety X is spherical if and only if the coordinate ring $\mathbb{C}[X]$ of X is a multiplicity-free G-module.

T. Kobayashi introduced the notion of visible action on a complex manifold in [49], and showed the propagation theorem of multiplicity-free property in [44, 54]. Many multiplicity-free representations can be explained in the machinery [50]. An advantage of the notion is that infinite-dimensional unitary representations of any Lie group such as non-reductive Lie groups can be treated in the framework. An example of applications of visible actions is the branching laws of unitary highest weight modules with respect to symmetric subgroups [52].

Let $G_{\mathbb{R}}$ be a connected simple Lie group of Hermitian type with Cartan involution θ and $G'_{\mathbb{R}}$ a symmetric subgroup of $G_{\mathbb{R}}$ preserved by θ. Put $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$ and $K'_{\mathbb{R}} := (G'_{\mathbb{R}})^{\theta}$. Let $\mathfrak{a}'_{\mathbb{R}}$ be a maximal abelian subspace in $\mathfrak{g}^{-\theta} \cap (\mathfrak{g}'_{\mathbb{R}})^{\perp}$. Set $M_{\mathbb{R}} := Z_{K'_{\mathbb{R}}}(\mathfrak{a}'_{\mathbb{R}})$.

Fact 1.8 (T. Kobayashi [50, Theorem 18, Theorem 34], [52, Theorem A])**.** *Let $\mathcal{H}$ be a unitary highest weight module of $G_{\mathbb{R}}$ embedded in $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ for some irreducible unitary representation F of $K_{\mathbb{R}}$. If $F|_{M_{\mathbb{R}}}$ is multiplicity-free, then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free. In particular, if $\mathcal{H}$ is of scalar type, then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free.*

Fact 1.9 (T. Kobayashi [44, Theorem B]). *Retain the notation in the above fact. If $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is a symmetric pair of holomorphic type (i.e. $\mathfrak{g}'$ contains the center of $\mathfrak{k}$), then $\mathcal{H}|_{G'_{\mathbb{R}}}$ has uniformly bounded multiplicities.*

The second fact asserts $\mathcal{M}_{G'_{\mathbb{R}}}(\mathcal{H}) < \infty$. In this case, T Kobayashi stated in [52, Remark 1.5] that using the Howe duality [26], we could relate the multiplicity function and the stable branching coefficients defined by F. Sato [93].

1.5 Stability of multiplicities

If a representation has uniformly bounded multiplicities, the multiplicity function may have a stability property. We state Sato's stability theorem [93] as follows.

Let G be a connected complex semisimple algebraic group and G' a connected reductive subgroup of G. Assume that (G, G') is a spherical pair, that is, there is a Borel subgroup B of G such that BG' is open dense in G. Put

$$L := \{g \in G : gBG' = BG'\} \cap G'.$$

Then L is a reductive subgroup of G' by a Theorem of Brion–Luna–Vust [6]. Note that the set of equivalence classes of irreducible representations of L can be parametrized by a set of characters of $B \cap G' \subset L$. We denote by Λ^+ the set of all dominant integral weights of B. For a weight $\lambda \in \Lambda^+$, we write $F^G(\lambda)$ for the finite-dimensional irreducible representation of G with highest weight λ. Set

$$\Lambda^+(G/G') := \left\{ \lambda \in \Lambda^+ : F^G(\lambda)^{G'} \neq 0 \right\}.$$

Under this setting, F. Sato proved the following theorem.

Fact 1.10 (F. Sato [93, Theorem 3]). *Let F be a finite-dimensional irreducible representation of G'. Then for any $\lambda \in \Lambda^+$, there exists a weight $\nu_0 \in \Lambda^+(G/G')$ such that*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{G'}(F^G(\lambda + \nu_0 + \nu), F)) = \dim_{\mathbb{C}}(\mathrm{Hom}_L(F^L(\lambda|_{B \cap G'}), F))$$

for any $\nu \in \Lambda^+(G/G')$.

The above fact asserts two things: the multiplicity function of $\mathrm{Ind}_{G'}^G(F)$ is invariant by the translation of $\Lambda^+(G/G')$ for enough large parameters; and for such parameters, the multiplicity function can be described by the multiplicity function of the fiber F. The first property is called stability by F. Sato in [93].

For the case of symmetric pairs $(G_{\mathbb{R}}, G'_{\mathbb{R}})$, the stability property appeared in Wallach's book [102, Cor. 8.5.15]. In the case, we can see the phenomena in some literatures [43, Lemma 3.4], [65]. Stable branching coefficients was computed for some concrete compact Lie groups [71], [98].

1.6 Main results

In this section, we state the main theorems in this thesis.

Let $G_{\mathbb{R}}$ be a reductive Lie group with Cartan involution θ and $G'_{\mathbb{R}}$ a reductive subgroup of $G_{\mathbb{R}}$ closed under θ. We put $K_{\mathbb{R}} := G^{\theta}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$ and denote by K and $\mathfrak{g}$ the complexifications of $K_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$, respectively. In a similar way, we define K' and $\mathfrak{g}'$ for $G'_{\mathbb{R}}$.

1.6.1 Direct integral and intertwining operators

For a representation V of $G_{\mathbb{R}}$, we write V_K for the subspace of $K_{\mathbb{R}}$-finite vectors.

Theorem 1.11. *Let V be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that the irreducible decomposition of $V|_{G'_{\mathbb{R}}}$ is as in (1.2.1). Then for almost every $\pi \in \widehat{G'_{\mathbb{R}}}$, there exist a $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $\mathbb{C}^{m(\pi)}$ and a surjective $(\mathfrak{g}', K')$-module and $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:*

$$\phi_\pi : V_K \to (V_\pi)_{K'} \otimes \mathbb{C}^{m(\pi)}$$

such that the vector field $(\pi \mapsto \phi_\pi(v))$ is equal to v in V for any $v \in V_K$.

Remark 1.12. A similar result for the Plancherel formulas on homogeneous spaces is well-known [3].

For the proof of the theorem, we use the reduction theorem by A. E. Nussbaum [76], which is a generalization of von Neumann's reduction theorem for bounded operators to closed operators. Since V_K and $\mathcal{U}(\mathfrak{g})^{G'}$ are at most countable-dimensional, we can define ϕ_π for almost every π. The compatibility with the $\mathfrak{g}'$-action is proved by Fact 1.2. See Theorem 7.11.

Definition 1.13. For a $(\mathfrak{g}, K)$-module V and a $(\mathfrak{g}', K')$-module V', we define

$$H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$$

as the space of all coinvariants of $V \otimes (V')^*_{K'}$. Then $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ has a natural $\mathcal{U}(\mathfrak{g})^{G'}$-module structure.

Remark 1.14. In the context of the Howe duality [26], the space $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ appears as the full theta lift. Hence the $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ is used in the study of the Howe duality [66], [69].

By Theorem 1.11, there is a surjective $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:

$$H_0(\mathfrak{g}', K'; V_K \otimes (V_\pi^*)_{K'}) \to \mathbb{C}^{m(\pi)}$$

for almost every π. This is one of the reasons to study $\mathcal{U}(\mathfrak{g})^{G'}$-modules.
We treat $\mathcal{U}(\mathfrak{g})^{G'}$-modules in Section 7.

1.6.2 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: generalized Verma modules

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ constructed from a semisimple element $H \in \mathfrak{g}'$. Define $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ in a similar way for $\mathfrak{g}'$. We fix a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{l}'$ and extend it to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$.

For a finite-dimensional irreducible $\mathfrak{l}$-module F, we define a generalized Verma module by

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F.$$

The following theorem is needed to consider the branching problem of generalized Verma modules.

Fact 1.15 (T. Kobayashi [53])**.** *Under the setting,* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ *is discretely decomposable and* $\mathfrak{g}'$*-admissible.*

Following Knapp–Vogan's book [40], we recall the notion of the good range. A finite-dimensional irreducible $\mathfrak{l}$-module F is said to be in the *good range* if the infinitesimal character λ of F satisfies

$$\mathrm{Re}(\lambda + \rho(\mathfrak{u}), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Under this setting, the following theorems hold (see Theorem 8.6 and its corollaries).

Theorem 1.16. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module in the good range. Suppose that* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ *is completely reducible and* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$ *is an irreducible direct summand. Then the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module* $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is irreducible.*

Theorem 1.17. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module in the good range. Then the length of the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module* $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is less than or equal to the length of* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}((F')^* \otimes \mathbb{C}_{-2\rho(\mathfrak{u})})$*.*

1.6.3 Outline of the proof of Theorem 1.16

To study the $\mathcal{U}(\mathfrak{g})^{G'}$-modules, we define a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module structure on the space of $\Delta(G')$-finite linear maps as follows (see Section 7).

Definition 1.18. Let V be a $(\mathfrak{g}, K)$-module and V' be a $(\mathfrak{g}', K')$-module. Then $\mathfrak{g}' \oplus \mathfrak{g}$ and $K' \times K$ act on $\mathrm{Hom}_{\mathbb{C}}(V', V)$. $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ is defined as the sum of finite-dimensional $(\Delta(\mathfrak{g}'), \Delta(K'))$-submodules which lift to a representation of $\Delta(G')$. Then $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ becomes a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. We define a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ in the same way.

Remark 1.19. If $G'_{\mathbb{R}}$ is equal to $G_{\mathbb{R}}$, a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module is a Harish-Chandra module of a complex reductive Lie group. In this case, for objects V, V' of the BGG category $\mathcal{O}$, $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G)}$ was studied by many mathematicians because the module is related to primitive ideals of the universal enveloping algebra and principal series representations of complex semisimple Lie groups (e.g. [4], [7], [10, 11], [33]).

An important property of the module is that the $\Delta(G')$-invariant part of the module is equal to the space of all intertwining operators. Hence we can study the $\mathcal{U}(\mathfrak{g})^{G'}$-module through the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. More precisely, the following two propositions hold.

Proposition 1.20. *Retain the settings in the above. Put*

$$\mathcal{I} := \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')} \cap \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})\Delta(\mathfrak{g}').$$

Then there is an algebra isomorphism:

$$\begin{array}{cccc} \alpha: & \mathcal{U}(\mathfrak{g})^{G'} & \simeq & \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')}/\mathcal{I} \\ & \Cup & & \Cup \\ & X & \mapsto & I \otimes X + \mathcal{I}. \end{array}$$

Proposition 1.21. *Let W be a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. Then the length of the $\mathcal{U}(\mathfrak{g})^{G'}$-module on $W^{\Delta(G')}$ is bounded by the length of W. In particular if W is irreducible, then $W^{\Delta(G')}$ is irreducible or zero.*

To prove Theorem 1.17, 1.16 and 1.27, we construct $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules using the Zuckerman derived functor $R^i\Gamma$. Let L' be the analytic subgroup of G' with Lie algebra $\mathfrak{l}'$. For a finite-dimensional irreducible $\mathfrak{l}$-module F with infinitesimal character λ, let $\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}(\lambda)$ be the full subcategory of the relative BGG category $\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}$ whose object V satisfies that $V \otimes F$ can lifts to a representation of L'. We denote by $\mathcal{F}(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$ the category of $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules of finite length. The following theorem is a key result (see Theorem 6.1).

Theorem 1.22. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module with infinitesimal character λ in the good range. Set $S := \dim_{\mathbb{C}}(\mathfrak{u}')$. Then the following functor gives a category embedding:*

$$\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}(\lambda) \ni M \mapsto R^S\Gamma^{\Delta(G')}_{\Delta(L')}(M \otimes \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) \in \mathcal{F}(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G')),$$

that is, the functor is exact and fully faithful, and maps irreducible objects to irreducible objects.

Remark 1.23. If $\mathfrak{g} = \mathfrak{g}'$, the theorem was proved by T. J. Enright [10, Chapter 16] except for the full faithfulness.

Remark 1.24. For a non-symmetric pair $(\mathfrak{g}, \mathfrak{k})$, a $(\mathfrak{g}, \mathfrak{k})$-module with some finiteness conditions is called a generalized Harish-Chandra module by I. Penkov and G. Zuckerman [85, 86, 87].

The proof of the theorem follows the proofs in Knapp–Vogan's book [40], Wallach's book [104] and Penkov–Zuckerman's papers.

Under the settings of Theorem 1.16, using the functor, we can prove the following $\mathcal{U}(\mathfrak{g})^{G'}$-module isomorphism:

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) \simeq R^S\Gamma^{\Delta(G')}_{\Delta(L')}(M \otimes \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))^{\Delta(G')},$$

where M is a unique irreducible quotient of $\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}((F')^* \otimes \mathbb{C}_{-2\rho(\mathfrak{u}')})$. Thus the irreducibility of $\mathrm{Hom}_{\mathfrak{g}',K'}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))$ is reduced to the irreducibility of $R^S\Gamma^{\Delta(G')}_{\Delta(L')}(L \otimes \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))$ by Proposition 1.21.

1.6.4 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: Zuckerman derived functor modules

To apply Theorem 1.16 to Zuckerman derived functor modules, we define a quasi-abelian parabolic subalgebra.

Definition 1.25 (quasi-abelian)**.** $\mathfrak{q}$ is said to be *quasi-abelian* with respect to $\mathfrak{g}'$ if $(\alpha, \beta) \geq 0$ holds for any $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $\beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$.

Remark 1.26. In the case of $G'_{\mathbb{R}} = K_{\mathbb{R}}$, the notion of a quasi-abelian parabolic subalgebra was used by Enright–Parthasarathy–Wallach–Wolf [12] to study Zuckerman derived functor modules.

If $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{g}'$, the completely reducibility always holds as long as F is in the good range. We assume $H \in \mathfrak{k}'$, and set $K_L := Z_K(H)$ and $K'_L := Z_{K'}(H)$. Then the following theorem (Theorem 8.24) holds.

Theorem 1.27. *Let F be a finite-dimensional irreducible $(\mathfrak{l}, K_L)$-module in the good range. Suppose that there exists an ideal $\mathfrak{k}_1$ of $\mathfrak{k}$ such that $H \in \mathfrak{k}_1$ and $\mathfrak{u} \cap \mathfrak{k} \subset \mathfrak{k}_1 \subset \mathfrak{g}'$, and suppose that $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{g}'$. Put $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. Then $R^S\Gamma^K_{K_L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))|_{(\mathfrak{g}',K')}$ is completely reducible and each direct summand is of the form $R^S\Gamma^{K'}_{K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'))$ with F' in the good range. Moreover, $\mathrm{Hom}_{\mathfrak{g}',K'}(R^S\Gamma^{K'}_{K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F')), R^S\Gamma^K_{K_L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)))$ is irreducible as a $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

Remark 1.28. As we mentioned in the introduction, T. Kobayashi gave a necessary and sufficient condition for the discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ (including discrete series representations), and gave some examples of explicit branching laws in [42, 43, 45, 46].

Remark 1.29. One of important cases satisfying the assumptions is the case of discretely decomposable restrictions of discrete series representations with respect to symmetric subgroups. For small discrete series representations and its restrictions to symmetric subgroups, the branching law was computed by Gross–Wallach in [16]. For any discrete series representations and non-symmetric subgroups, Duflo–Vargas gave a formula of the multiplicities like Blattner's formula in [8]. The subgroup K_1 in our setting is the same as in [16] and [8].

Under the assumptions in the theorem, $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)|_{\mathfrak{g}'}$ is completely reducible. Hence the proof is reduced to Theorem 1.16 by the following fact.

Fact 1.30 (Gross–Wallach [16, Lemma 7])**.** *Under the assumptions in Theorem 1.27, $R^S\Gamma^K_{K_L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))$ is isomorphic to $R^S\Gamma^{K_1}_{K_1 \cap L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F))$.*

1.6.5 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: Holomorphic discrete series representations

We study the $\mathcal{U}(\mathfrak{g})^{G'}$-module and $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module arising from the branching law with continuous spectrum.

Following Kanpp's book [38, Chapter VI], we recall holomorphic discrete series representations. Assume that $G_{\mathbb{R}}$ is a connected simple Lie group of Hermitian type. Fix an element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_{\mathbb{R}})$ such that $\mathrm{ad}_{\mathfrak{g}}(H)$ has eigenvalues $-1, 0$ and 1. Then $\mathfrak{g}$ is decomposed into the direct sum of eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(H)$:

$$\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$$

corresponding to eigenvalues $1, 0$ and -1, respectively. We put $\mathfrak{q} := \mathfrak{k} \oplus \mathfrak{p}_+$ and $\overline{\mathfrak{q}} = \mathfrak{k} \oplus \mathfrak{p}_-$. Then $G_{\mathbb{R}}/K_{\mathbb{R}}$ admits a $G_{\mathbb{R}}$-invariant complex structure such that the natural embedding $G_{\mathbb{R}}/K_{\mathbb{R}} \hookrightarrow G/\overline{Q}$ is holomorphic.

Fact 1.31 (Harish-Chandra [20]). *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$. Then $\mathcal{O} \cap L^2(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ is non-zero if and only if F is in the good range with respect to $\mathfrak{q}$, where $\mathcal{O} \cap L^2$ means the space of all holomorphic and L^2 sections. Furthermore, if $\mathcal{O} \cap L^2(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ is non-zero, it is irreducible and unitary as a representation of $G_{\mathbb{R}}$.*

The irreducible unitary representation is called a *holomorphic discrete series representation.*

Assume that $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is a symmetric pair of anti-holomorphic type (i.e. $\mathfrak{g}'_{\mathbb{R}}$ does not contains the center of $\mathfrak{k}_{\mathbb{R}}$) and $G'_{\mathbb{R}}$ satisfies the following condition:

$$\mathrm{Ad}_{\mathfrak{g}}(G'_{\mathbb{R}}) = G' \cap \mathrm{Int}(\mathfrak{g}_{\mathbb{R}}),$$

where G' is the analytic subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}'$.

It is known that the branching law of a holomorphic discrete series representation with respect to $G'_{\mathbb{R}}$ is reduced to the Plancherel formula of the Riemannian symmetric space $G'_{\mathbb{R}}/K'_{\mathbb{R}}$. Hence the irreducible decomposition has a continuous spectrum.

Fact 1.32 (J. Repka [89], R. Howe [24], Ólafsson–Ørsted [77]). *For a holomorphic discrete series representation V of $G_{\mathbb{R}}$ realized in $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, \mathcal{V})$ for a holomorphic $G_{\mathbb{R}}$-equivariant vector bundle $\mathcal{V}$ on $G_{\mathbb{R}}/K_{\mathbb{R}}$, the following isomorphism holds:*

$$V|_{G'_{\mathbb{R}}} \simeq L^2(G'_{\mathbb{R}}/K'_{\mathbb{R}}, \mathcal{V}|_{G'_{\mathbb{R}}/K'_{\mathbb{R}}}).$$

Let $Q'_{\mathbb{R}} = M'_{\mathbb{R}}A'_{\mathbb{R}}N'_{\mathbb{R}}$ be a minimal parabolic subgroup of $G'_{\mathbb{R}}$. Take a Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{m}'$, and put $\mathfrak{h}' := \mathfrak{a}' \oplus \mathfrak{t}'$. Write $I(\delta, \nu)$ for the underlying Harish-Chandra module of the principal series representation induced from $(\delta, V_\delta) \in \widehat{M'_{\mathbb{R}}}$ and $\nu \in (\mathfrak{a}')^*$. We consider 'generic' principal series representations in the following sense (Lemma 9.4).

Lemma 1.33. *Let μ be the infinitesimal character of δ. Assume*

$$\frac{2(-\nu - \mu + \rho(\mathfrak{n}'), \alpha)}{(\alpha, \alpha)} \notin \mathbb{Z} \text{ for any } \alpha \in \Delta(\mathfrak{n}', \mathfrak{h}').$$

Let W be an irreducible subquotient of $I(\delta, \nu)$. Then the following properties hold:

(a) $\mathrm{End}_{\mathbb{C}}(W)_{\Delta(G')}$ *is irreducible as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module;*

(b) *for any finite-dimensional representation F of G', $F \otimes W$ is completely reducible.*

Fix a maximal abelian subspace $\mathfrak{t}_\mathbb{R}$ of $(\mathfrak{g}'_\mathbb{R})^\perp \cap \mathfrak{k}_\mathbb{R}$. Then $\mathfrak{t}_\mathbb{R}$ is a maximal abelian subspace of $(\mathfrak{g}'_\mathbb{R})^\perp$. Choose a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ containing $\Delta(\mathfrak{p}_+, \mathfrak{t})$. Let γ_1 be the highest weight of $\Delta(\mathfrak{g}'_\mathbb{R}, \mathfrak{a}'_\mathbb{R})$ with respect to the parabolic subgroup $Q'_\mathbb{R}$, and β_1 be the highest weight of $\Delta(\mathfrak{p}_+, \mathfrak{t})$. Then the following theorem holds (Theorem 9.13).

Theorem 1.34. *Retain the notation and the assumption in Lemma 1.33. Let $\mathbb{C}_\lambda$ be a one-dimensional representation of $K_\mathbb{R}$ with weight λ. Assume*

$$\pm\frac{(w(\nu - \rho(\mathfrak{n})) + \rho(\mathfrak{n}), \gamma_1)}{(\gamma_1, \gamma_1)} + \frac{(\lambda, \beta_1)}{(\beta_1, \beta_1)} \notin \mathbb{Z}$$

for any $w \in W_{\mathfrak{g}'_\mathbb{R}}$. Then the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ *is irreducible.*

Remark 1.35. In the case of the trivial representation W (the theorem can not apply to this case), $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), \mathbf{1})_{\Delta(G')}$ is isomorphic to a degenerate principal series representation of some real form of G. The irreducibility of degenerate principal series representations can be determined from the data of the K-type decomposition and the $\mathfrak{p}$-action on each K-type. T. Hirai introduced this method to study degenerate principal series representations of Lorentz groups [23]. Many mathematicians computed the composition series of degenerate principal series representations by a similar way such as V. F. Molčanov [73], Klimyk–Gavrilik [37], Johnson–Wallach [32] and Kudla–Rallis [64].

The structure of $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ can be computed by a similar way. Under the assumptions of Theorem 1.34, $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ is completely reducible and multiplicity-free as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module. Hence we can use the irreducible decomposition as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module instead of the K-type decomposition.

It follows from Harish-Chandra's classification of holomorphic discrete series representations (Fact 1.31) that if λ is in the good range, $\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda)$ is isomorphic to the underlying Harish-Chandra module of a holomorphic discrete series representation. We apply the Jantzen–Zuckerman translation functor to $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$, and we obtain the following theorem (Theorem 9.35).

Theorem 1.36. *Let F be an irreducible unitary representation of $K_\mathbb{R}$ in the good range, and let (δ, V_δ) be an irreducible subrepresentation of $F|_{M'_\mathbb{R}}$. Suppose that the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ acts on F by a character λ. Assume that λ, δ and $\nu \in (\mathfrak{a}')^*$ satisfy the conditions of Lemma 1.33 and Theorem 1.34. Let W be an irreducible subquotient of $I(\delta, \nu)$. Then* $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F), W)_{\Delta(G')}$ *is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module, and* $\mathrm{Hom}_{\mathfrak{g}', K'}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F), W)$ *is irreducible as a $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

1.6.6 Stability of multiplicities

We generalize Sato's stability theorem (Fact 1.10) to the case of quasi-affine spherical varieties.

Let X be a quasi-affine spherical variety of a complex connected reductive algebraic group G with Borel subgroup $B = TN$. Then the coordinate ring $\mathbb{C}[X]$ is a multiplicity-free G-module. Fix an open orbit $Bx_0 \subset X$ and put $L := \{g \in G : gx_0 = x_0, gBx_0 \subset Bx_0\}$. Then L is a reductive subgroup of G by the theorem of Brion–Luna–Vust [6, Théorème 3.4]. Consider a finitely generated torsion-free $(\mathbb{C}[X], G)$-module M such as the space of all global sections of a G-equivariant vector bundle on X. Then the following theorem holds (see Theorem 3.18).

Theorem 1.37. *There exists a weight $\lambda_0 \in \Lambda^+(\mathbb{C}[X])$ such that*

$$m^G_M(\lambda + \lambda_0) = m^L_{M/\mathfrak{m}(x_0)M}(\lambda|_{B_{x_0}})$$

for any $\lambda \in \Lambda^+(M)$.

Here $m^G_M(\cdot)$ is the multiplicity function of the G-module M, and $\Lambda^+(M)$ is the set of weights of $B/N \simeq T$ in M^N. We denote by $\mathfrak{m}(x_0)$ the maximal ideal of $\mathbb{C}[X]$ corresponding to x_0.

Remark 1.38. If X is an affine homogeneous variety G/G' of G, the theorem is just Sato's stability theorem (Fact 1.10).

The proof is essentially the same as the proof of Sato's stability theorem [93]. The only difference is that we study some behavior of the evaluation map $M \to M/\mathfrak{m}(x_0)M$ instead of using the reductivity of G'.

As an application of Theorem 1.37, we obtain the following corollary (see Corollary 3.39). Recall the notation in Fact 1.9.

Corollary 1.39. *Let $\mathcal{H}$ be a holomorphic discrete series representation of $G_\mathbb{R}$. Suppose that $G'_\mathbb{R}$ is connected and $(G_\mathbb{R}, G'_\mathbb{R})$ is a symmetric pair of holomorphic type. Then we have*

$$\mathcal{M}_{G'_\mathbb{R}}(\mathcal{H}) = \mathcal{M}_{M_\mathbb{R}}(\mathcal{H}^{\mathfrak{p}_+}_K),$$

where $\mathcal{M}_{G'_\mathbb{R}}(\mathcal{H})$ is the maximal value of the multiplicities. In particular, $\mathcal{H}|_{G'_\mathbb{R}}$ is multiplicity-free if and only if $(\mathcal{H}^{\mathfrak{p}_+}_K)|_{M_\mathbb{R}}$ is multiplicity-free.

Remark 1.40. $\mathcal{M}_{G'_\mathbb{R}}(\mathcal{H}) < \infty$ and the 'if part' of the second assertion were proved by T. Kobayashi (Fact 1.8, 1.9).

We write σ for the involution defining the symmetric pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$. We can reduce the branching law of $\mathcal{H}|_{G'_{\mathbb{R}}}$ to the irreducible decomposition of $S(\mathfrak{p}_-^{-\sigma}) \otimes \mathcal{H}_K^{\mathfrak{p}_+}$ as a K'-module (see [29, Proposition 2.5] and [52, Lemma 8.8]). As in the proof of [52, Theorem 8.3], the irreducible decomposition of $S(\mathfrak{p}_-^{-\sigma}) \otimes \mathcal{H}_K^{\mathfrak{p}_+}$ is considered as the K-type decomposition of some direct sum of holomorphic discrete series representations of the associated symmetric subgroup $G_{\mathbb{R}}^{\theta\sigma}$. Since $S(\mathfrak{p}_-^{-\sigma})$ is multiplicity-free by the Hua–Kostant–Schmid theorem [94], we can apply Theorem 1.37 to $S(\mathfrak{p}_-^{-\sigma}) \otimes \mathcal{H}_K^{\mathfrak{p}_+}$, and this shows the corollary.

1.6.7 Classification of multiplicity-free restrictions of holomorphic discrete series representations

We classify multiplicity-free restrictions of holomorphic discrete series representations with respect to symmetric subgroups.

Let $G_{\mathbb{R}}$ be a connected simple Lie group of Hermitian type with simply-connected complexification G, and σ be an involutive automorphism of $G_{\mathbb{R}}$. Put $G'_{\mathbb{R}} := G_{\mathbb{R}}^{\sigma}$. The following theorem is the classification result (Theorem 10.3).

Theorem 1.41. *Let $\mathcal{H}$ be a holomorphic discrete series representation of $G_{\mathbb{R}}$. Put $F := \mathcal{H}_K^{\mathfrak{p}_+}$. Then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free if and only if F is one-dimensional or the highest weight of $F|_{[\mathfrak{k},\mathfrak{k}]}$ belongs to $\Lambda(\sigma)$ in Table 2 in Section 10.*

Remark 1.42. In the case of scalar type $\mathcal{H}$, the theorem was proved by T. Kobayashi [44, 50] (Fact 1.8). The multiplicity-freeness was shown for $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}}) = (\mathfrak{so}(2, n), \mathfrak{so}(2, n-1))$ by Jakobsen–Vergne [29, Corollary 3.1], and for $(\mathfrak{su}(p, q), \mathfrak{u}(p-1, q))$ by T. Kobayashi [52, Theorem 8.10].

To prove the classification result, the following theorem (Theorem 10.1) is useful. Fix a unitary character $(\zeta, \mathbb{C}_\zeta)$ of $K_{\mathbb{R}}$. For an irreducible unitary representation F of $K_{\mathbb{R}}$ with infinitesimal character λ, we define

$$Z_{hol}(F) := \{z \in \mathbb{Z} : (\lambda + \rho(\mathfrak{p}_+), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\},$$
$$Z_{fin}(F) := \left\{z \in \mathbb{Z} : \frac{2(\lambda + \rho(\mathfrak{p}_+), \alpha)}{(\alpha, \alpha)} \in \{1, 2, \dots\} \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\right\},$$

and let $L(F)$ denote a unique irreducible submodule of $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ (defined in Section 2.2).

Theorem 1.43. *Let F be a unitary irreducible representation of $K_{\mathbb{R}}$. Then the following conditions are equivalent:*

(a) $\mathcal{M}_{G^\sigma_\mathbb{R}}(\overline{\mathrm{pro}^\mathfrak{g}_\mathfrak{q}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for any* $z \in Z_{hol}(F)$*;*

(b) $\mathcal{M}_{G^\sigma_\mathbb{R}}(\overline{\mathrm{pro}^\mathfrak{g}_\mathfrak{q}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for some* $z \in Z_{hol}(F)$*;*

(c) $\mathcal{M}_{G^\sigma}(L(F \otimes \mathbb{C}_{z\zeta})) = 1$ *for any* $z \in Z_{fin}(F)$*;*

(d) $\mathcal{M}_{M_\mathbb{R}}(F) = 1$.

where $M_\mathbb{R}$ *is the subgroup of* $K'_\mathbb{R}$ *defined before Fact 1.8.*

Remark 1.44. For the proof of the theorem, we use the method, analytic continuation of holomorphic discrete series representations [1], [92], [99], [103]. To prove the theorem, we need only that the family of the representations $\mathcal{O}(G_\mathbb{R}/K_\mathbb{R}, G_\mathbb{R} \times_{K_\mathbb{R}} (F \otimes \mathbb{C}_{z\zeta}))_{K_\mathbb{R}}$ depends on z polynomially, that is, any element of $\mathfrak{g}$ acts on the space by a differential operator with polynomial coefficient on z (see Section 4.1).

Remark 1.45. In the branching problem, the method of analytic continuation was used to study symmetry breaking operators [63].

Remark 1.46. The theorem asserts that the sufficient condition for the multiplicity-freeness given by T. Kobayashi (Fact 1.8) is a necessary condition for holomorphic discrete series representations.

By the theorem, the classification of multiplicity-free restrictions of holomorphic discrete series representations is reduced to that of finite-dimensional irreducible representations. In particular, in the case that $G'_\mathbb{R}$ has a one-dimensional center, J. R. Stembridge has classified multiplicity-free restrictions of finite-dimensional irreducible representations with respect to G' in [96]. Thus for such $G'_\mathbb{R}$, the classification is immediately done by Theorem 1.43 and the Stembridge classification.

As a consequence, we obtain the following proposition (Proposition 10.6).

Corollary 1.47. *Let* σ' *be an involutive automorphism of* $G_\mathbb{R}$*. Assume that* G^σ *and* $G^{\sigma'}$ *are conjugate by an inner automorphism of* G*. Then we have* $\Lambda(\sigma) = \Lambda(\sigma')$*.*

Remark 1.48. The theorem asserts that the classification is not depend on a choice of real forms. The similarity of two groups with isomorphic complexifications can be found in many fields in the representation theory and the harmonic analysis. We give several examples:

- the Weyl unitary trick;
- the Flensted-Jensen duality [14];

- non-existence of compact Clifford–Klein forms [57];
- transfer of K-type [16];
- one to one correspondence of infinitesimal characters in the theory of the Howe duality [67], [88].

This thesis is organized as follows. In Section 2, we review some of the standard facts on the representation theory of reductive Lie groups. Section 3 deals with the stability theorem. We give its proof and some examples. In Section 4, we relate the branching laws of infinite-dimensional representations to the branching laws of finite-dimensional representations using the method, analytic continuation. In Section 5, we discuss the Zuckerman derived functor and generalized Verma modules to use Section 6. In Section 6, we construct a category embedding from the BGG category $\mathcal{O}$ to the category of generalized Harish-Chandra modules. Section 7 is devoted to the study of $\mathcal{U}(\mathfrak{g})^{G'}$-modules and $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules. The relation between the direct integral decomposition and $(\mathfrak{g}, K)$-module is in this section. In Section 8, it is shown that $\mathcal{U}(\mathfrak{g})^{G'}$-modules arising from the branching laws of generalized Verma modules and the Zuckerman derived functor modules are irreducible under good conditions. Section 9 contains a discussion of the structure of $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules for holomorphic discrete series representations and principal series representations. In Section 10, we classify multiplicity-free restrictions of holomorphic discrete series representations with respect to symmetric subgroups.

1.7 Notation

In this thesis, any Lie algebra is finite-dimensional. Real Lie groups and their Lie algebras are denoted by Roman alphabets and corresponding German letters with subscript $(\cdot)_{\mathbb{R}}$, respectively. We express complex Lie groups and their Lie algebras by Roman alphabets and corresponding German letters without subscript. For example, the Lie algebras of real Lie groups $G_{\mathbb{R}}, K_{\mathbb{R}}$ and $H_{\mathbb{R}}$ are denoted as $\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}$ with complexification $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$, respectively. We write $\mathcal{U}(\mathfrak{g})$ and $Z(\mathfrak{g})$ for the universal enveloping algebra and its center of a complex Lie algebra $\mathfrak{g}$. For a topological group G, let G_0 denote the identity component of G.

For a compact Lie group, we identify its locally finite representations with rational representations of its complexification. Here we consider the complexification of a compact Lie group as a complex reductive algebraic group. We denote by $F^{\mathfrak{g}}(\lambda)$ (resp. $F^G(\lambda)$) the finite-dimensional representation of a reductive Lie algebra $\mathfrak{g}$ (resp. connected complex reductive algebraic group G) with highest weight λ.

For a vector space V over a field k, we denote by V^* the dual vector space (i.e. $\mathrm{Hom}_k(V,k)$). Let $\mathfrak{a}$ be an abelian subspace of a Lie algebra $\mathfrak{g}$. For an $\mathfrak{a}$-stable subspace $\mathfrak{s}$ of $\mathfrak{g}$ and $\lambda \in \mathfrak{a}^*$, we define

$$\begin{aligned} \mathfrak{s}_\lambda &:= \{X \in \mathfrak{s} : [H,X] = \lambda(H)X \text{ for any } H \in \mathfrak{a}\}, \\ \Delta(\mathfrak{s},\mathfrak{a}) &:= \{\lambda \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{s}_\lambda \neq 0\}. \end{aligned}$$

We denote by $\rho(\mathfrak{s})$ half the sum of elements in $\Delta(\mathfrak{s},\mathfrak{a})$ with multiplicity $\dim(\mathfrak{s}_\lambda)$. Let $\rho_\mathfrak{g}$ denote half the sum of positive roots in $\mathfrak{g}$.

For a Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$, we write $\mathrm{ad}_\mathfrak{g}$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Its group version is written as $\mathrm{Ad}_\mathfrak{g}$. If the Lie algebra is clear from the context, we write ad for $\mathrm{ad}_\mathfrak{g}$. We denote by $\mathfrak{c}(\mathfrak{g})$ the center of a Lie algebra $\mathfrak{g}$.

For a unitary representation V of a real reductive Lie group $G_\mathbb{R}$, $\mathcal{M}_{G_\mathbb{R}}(V)$ stands for the essential supremum of the multiplicity function of V. We use the similar notation for completely reducible representations of a Lie algebra or algebraic group.

For a complex semisimple Lie algebra $\mathfrak{g}$, we denote by $(,)$ the Killing form of $\mathfrak{g}$. For a reductive subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$, whenever we use an invariant bilinear form on $\mathfrak{g}'$ without no mention, we suppose that the bilinear form is the restriction of the Killing form of $\mathfrak{g}$. The dual space $(\mathfrak{g}')^*$ is identified with $\mathfrak{g}'$ by the bilinear form, and admits a $\mathfrak{g}'$-invariant bilinear form induced from the bilinear form on $\mathfrak{g}'$.

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2 Preliminaries

In this section, we summarize without proofs the relevant material on $(\mathfrak{g}, K)$-modules.

2.1 $(\mathfrak{g}, K)$-modules

We define a $(\mathfrak{g}, K)$-module and review its properties. We give a reference [40] for the definition of a pair and $(\mathfrak{g}, K)$-module.

Definition 2.1. Let $\mathfrak{g}$ be a complex Lie algebra and K be a complex reductive algebraic group with Lie algebra $\mathfrak{k}$ which is a subalgebra of $\mathfrak{g}$. Suppose that an algebraic group homomorphism $\phi : K \to \mathrm{Aut}(\mathfrak{g})$ is given. $(\mathfrak{g}, K)$ is said to be a *pair* if the differential of ϕ is equal to the adjoint action $\mathrm{ad}_{\mathfrak{g}}$, and $\phi(k)|_{\mathfrak{k}} = \mathrm{Ad}_{\mathfrak{k}}(k)$ holds for any $k \in K$.

Remark 2.2. If we say that $(\mathfrak{g}, K)$ is a pair, we always assume that the homomorphism ϕ is given implicitly. For simplicity, we write $\mathrm{Ad}_{\mathfrak{g}}$ for the homomorphism ϕ.

A typical example is a pair constructed from a real Reductive Lie group $G_{\mathbb{R}}$. Fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Then $(\mathfrak{g}, K)$ is a pair, where $\mathfrak{g}$ is the complexification of $\mathrm{Lie}(G_{\mathbb{R}})$ and K is the complexification of $K_{\mathbb{R}}$. We define a $(\mathfrak{g}, K)$-module as follows.

Definition 2.3. Let $(\mathfrak{g}, K)$ be a pair and V be a vector space with $\mathfrak{g}$-action and K-action. We will say that V is a $(\mathfrak{g}, K)$-module if

- the K-action on V is algebraic (hence locally finite and completely reducible);
- the action of $\mathfrak{k}$ determined by the differential of the K-action is equal to the restriction of the $\mathfrak{g}$-action;
- $kXv = \mathrm{Ad}_{\mathfrak{g}}(k)(X)kv$ holds for any $k \in K$, $X \in \mathfrak{g}$ and $v \in V$.

We denote by $\mathcal{C}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$-modules. For reductive $\mathfrak{g}$, we write $\mathcal{C}(\mathfrak{g}, K)_\lambda$ for the full subcategory of $\mathcal{C}(\mathfrak{g}, K)$ of which objects have generalized infinitesimal character λ.

We use the similar terminology, $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module for an infinitesimal representation of an infinite covering $G_{\mathbb{R}}$ of a real reductive Lie group. In this case, we replace 'algebraic' by 'locally finite and unitarizable' for $K_{\mathbb{R}}$-action. For an admissible representation V of a real reductive Lie group $G_{\mathbb{R}}$, the space of $K_{\mathbb{R}}$-finite vectors becomes a $(\mathfrak{g}, K)$-module. We denote by V_K the $(\mathfrak{g}, K)$-module.

Let $(\mathfrak{g}, K)$ be a pair, $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$ and K' be a reductive subgroup of K. Suppose $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$. Then $(\mathfrak{g}', K')$ forms a pair. We say that $(\mathfrak{g}', K')$ is a subpair of $(\mathfrak{g}, K)$.

We review the theory of a discretely decomposable $(\mathfrak{g}, K)$-module due to T. Kobayashi [43, 45, 46, 48]. Since we do not use the deep results, we summarize only definitions and fundamental facts.

Definition 2.4 (discretely decomposable $(\mathfrak{g}, K)$-module)**.** Let V be a $(\mathfrak{g}, K)$-module. V is said to be discretely decomposable if there exists a $(\mathfrak{g}, K)$-module filtration $0 = V_0 \subset V_1 \subset \cdots$ such that $\bigcup_i V_i = V$ and each V_i is finite length.

Let $(\mathfrak{g}, K)$ be a pair constructed from a real reductive Lie group $G_{\mathbb{R}}$ and $(\mathfrak{g}', K')$ be a subpair corresponding to a reductive subgroup $G'_{\mathbb{R}}$. The following fact is in [46, Lemma 1.5].

Fact 2.5. *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $V|_{(\mathfrak{g}', K')}$ is discretely decomposable if and only if there exists an irreducible $(\mathfrak{g}', K')$-module V' such that* $\mathrm{Hom}_{\mathfrak{g}', K'}(V', V) \neq 0$.

If V is the underlying Harish-Chandra module of an irreducible unitary representation of $G_{\mathbb{R}}$, the discretely decomposability of $V|_{(\mathfrak{g}', K')}$ is equivalent to the completely reducibility. More precisely, the following facts are known [48, Theorem 2.7]:

Fact 2.6. *Let V be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that $V_K|_{(\mathfrak{g}', K')}$ is discretely decomposable. Then $V_K|_{(\mathfrak{g}', K')}$ is decomposed into the direct sum of irreducible $(\mathfrak{g}', K')$-modules:*

$$V_K|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\pi \in \widehat{G'_{\mathbb{R}}}} m(\pi)(V_\pi)_{K'},$$

and $V|_{G'_{\mathbb{R}}}$ is decomposed into the direct sum of irreducible unitary representations with the same multiplicity function $m(\pi)$ as above:

$$V|_{G'_{\mathbb{R}}} \simeq \sum_{\pi \in \widehat{G'_{\mathbb{R}}}}^{\oplus} m(\pi) V_\pi.$$

2.2 Generalized Verma modules

This section contains a brief summary of generalized Verma modules. The general reference here is the Humphreys book [28]. For the branching problem part, we refer the reader to [53].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix a semisimple element $H \in \mathfrak{g}$ such that $\mathrm{ad}(H)$ has only real eigenvalues. $\mathfrak{l}(H), \mathfrak{u}(H)$ and $\bar{\mathfrak{u}}(H)$ denote the sum of eigenspaces of $\mathrm{ad}(H)$ with zero, positive and negative eigenvalue, respectively. Then $\mathfrak{q}(H) := \mathfrak{l}(H) \oplus \mathfrak{u}(H)$ is a parabolic subalgebra of $\mathfrak{g}$. Similarly, we put $\bar{\mathfrak{q}}(H) := \mathfrak{l}(H) \oplus \bar{\mathfrak{u}}(H)$. If H is clear from the context, we omit '(H)' part. For example, we write $\mathfrak{l}$ for $\mathfrak{l}(H)$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{l}$.

We define generalized Verma modules. Let F be an $\mathfrak{l}$-module. We define

$$\begin{aligned}\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F) &:= \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F,\\ \mathrm{pro}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(F) &:= \mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), F)_{\mathfrak{l}},\end{aligned}$$

where we consider F as a $\mathfrak{q}$- (resp. $\bar{\mathfrak{q}}$-)module letting $\mathfrak{u}$ (resp. $\bar{\mathfrak{u}}$) act on F trivially. If F is a finite-dimensional irreducible $\mathfrak{l}$-module, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is called a *generalized Verma module*. In this case, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is a highest weight module.

We define the relative BGG category $\mathcal{O}$ denoted by $\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}}$ as follows (see [28, Chapter 9]). $\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}}$ is the full subcategory of $\mathcal{C}(\mathfrak{g})$ whose objects V satisfy:

- V is finitely generated as a $\mathfrak{g}$-module;
- V is a locally finite and completely reducible $\mathfrak{l}$-module;
- the action of $\mathfrak{u}$ on V is locally nilpotent.

Then any generalized Verma module is an object of the category $\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}}$.

In general, a generalized Verma module can be reducible. The following result provide a criterion for the irreducibility of a generalized Verma module. (e.g. [28, Theorem 9.12.])

Fact 2.7. *Let F be an irreducible finite-dimensional $\mathfrak{l}$-module with infinitesimal character λ satisfying*

$$\frac{2(\lambda + \rho(\mathfrak{u}), \alpha)}{(\alpha, \alpha)} \notin \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Then the generalized Verma module $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is irreducible.

If $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is irreducible, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is isomorphic to $\mathrm{pro}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(F)$ because they are irreducible highest weight modules with the same highest weight.

The following definition is a generalization of the standard definition in [28, Section 3.7].

Definition 2.8. Let V be a $\mathfrak{g}$-module. We will say that V has a standard filtration with respect to $\mathfrak{q}$ if there is a filtration $0 = V_0 \subset V_1 \subset \cdots$ such that $\bigcup_i V_i = V$ and each V_{i+1}/V_i is isomorphic to some generalized Verma module $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$.

If the parabolic subalgebra $\mathfrak{q}$ is clear from the context, we simply say that V has a standard filtration. We give several fundamental properties of the standard filtration.

Proposition 2.9. *Let V be a $\mathfrak{g}$-module with standard filtration $V_\cdot$.*

(a) *For any finite-dimensional $\mathfrak{g}$-module F, $V \otimes F$ has a standard filtration.*

(b) *If each V_{i+1}/V_i is irreducible, then V is completely reducible.*

Proof. (a) is clear because it is well-known that if V is a generalized Verma module, $V \otimes F$ has a standard filtration.

To prove (b), we can assume that V is finite length. In fact, if each V_i is completely reducible, V is completely reducible because V can be written as a sum of simple submodules. Since $\mathrm{Ext}^1_{\mathcal{O}^{\mathfrak{g}}_{\mathfrak{q}}}(M, N) = 0$ for any two irreducible generalized Verma modules M and N (see [28, Theorem 3.3 (d)],) the assertion follows. $\square$

We consider the branching law of a generalized Verma module. Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Assume that H is an element of $\mathfrak{g}'$. In [53], $\mathfrak{q}$ is said to be $\mathfrak{g}'$-compatible. Then $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}'$. We write $\mathfrak{u}', \mathfrak{l}'$ and $\bar{\mathfrak{u}}'$ for the intersections of $\mathfrak{u}, \mathfrak{l}$ and $\bar{\mathfrak{u}}$ with $\mathfrak{g}'$, respectively.

An important fact is that any object of $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{q}}$ is discretely decomposable as a $\mathfrak{g}'$-module, and the filtration in the definition of the discrete decomposability can be taken from objects of $\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}$ (see [53, Proposition 3.8]). By the same proof as in [104, Lemma 6.4.4], we obtain

Proposition 2.10. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module. Then $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ has a standard filtration $V_\cdot$ as a $\mathfrak{g}'$-module satisfying*

$$\mathrm{gr}(V_\cdot) \simeq \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes S(\bar{\mathfrak{u}}/\bar{\mathfrak{u}}')),$$

where we consider $\bar{\mathfrak{u}}/\bar{\mathfrak{u}}'$ as a $\mathfrak{q}'$-module by letting $\mathfrak{u}'$ act on $\bar{\mathfrak{u}}/\bar{\mathfrak{u}}'$ trivially.

2.3 Holomorphic discrete series representations

In this section, we review some of the standard facts on a holomorphic discrete series representation. We refer the reader to [38] for the construction, and to [52] for the branching laws.

Let $G_{\mathbb{R}}$ be a connected semisimple Lie group with Cartan involution θ. Assume that each simple factor of $\mathfrak{g}_{\mathbb{R}}$ is of Hermitian type. Then we do not assume that $G_{\mathbb{R}}$ has finite center. Put $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$, and $\mathfrak{p}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}}^{-\theta}$. Fix a *characteristic element* $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_{\mathbb{R}})$ such that eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{p}$ are 1 or -1. Then we have the following $\mathrm{ad}(H)$-eigenspace decomposition:

$$\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$$

with eigenvalue $1, 0$ and -1, respectively. $\mathfrak{q} := \mathfrak{k} \oplus \mathfrak{p}_+$ and $\bar{\mathfrak{q}} := \mathfrak{k} \oplus \mathfrak{p}_-$ are parabolic subalgebras of $\mathfrak{g}$ with abelian nilpotent radical. We fix a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$.

We take a simply-connected connected complex algebraic group G with Lie algebra $\mathfrak{g}$. K, Q and $\overline{Q}$ are the connected subgroups corresponding to $\mathfrak{k}, \mathfrak{q}$ and $\bar{\mathfrak{q}}$. Then there is a natural open embedding:

$$G_{\mathbb{R}}/K_{\mathbb{R}} \hookrightarrow G/\overline{Q}.$$

This embedding induces a $G_{\mathbb{R}}$-invariant complex structure on $G_{\mathbb{R}}/K_{\mathbb{R}}$.

The following fact is due to Harish-Chandra (see e.g. [38, Theorem 6.6])

Fact 2.11. *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$ with infinitesimal character λ satisfying*

$$(\lambda + \rho(\mathfrak{p}_+), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h}).$$

Then $(\mathcal{O} \cap L^2)(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ is non-zero and an irreducible unitary representation of $G_{\mathbb{R}}$, where we denote by $\mathcal{O} \cap L^2$ the space of square-integrable holomorphic sections.

The irreducible unitary representation is called a *holomorphic discrete series representation.*

The underlying Harish-Chandra module of a holomorphic discrete series representation can be written as a generalized Verma module. More generally, the following facts are well-known. For a unitary representation F of $K_{\mathbb{R}}$, we set $M(F) := \mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)_{K_{\mathbb{R}}}$.

Fact 2.12. *Retain the assumptions in Fact 2.11. Then $M(F)$ is the underlying Harish-Chandra module of the holomorphic discrete series representation in Fact 2.11. Hence $M(F)$ is an irreducible $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module.*

Fact 2.13. *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$. Then we have $M(F) \simeq \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F)$. If $M(F)$ is irreducible, then $M(F) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ holds.*

2.4 Zuckerman derived functor

We review the Zuckerman derived functor. We refer the reader to [104, Chapter 6] and [40].

Let $(\mathfrak{g}, K)$ be a pair and M be a reductive subgroup of K. Then $(\mathfrak{g}, M)$ be a subpair of $(\mathfrak{g}, K)$. For simplicity, we assume that M meets every connected component of K, and hence we have $K = MK_0$. We define a functor $\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}$ by

$$\begin{array}{rccc} \Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}: & \mathcal{C}(\mathfrak{g}, M) & \to & \mathcal{C}(\mathfrak{g}, K) \\ & \Uparrow & & \Uparrow \\ & V & \mapsto & V_K, \end{array}$$

where V_K is the sum of finite-dimensional $(\mathfrak{k}, M)$-submodules which lift to a representation of K.

The functor $\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}$ is called the *Zuckerman functor* in [40]. It is obvious that the functor is covariant and left exact. Since $\mathcal{C}(\mathfrak{g}, M)$ has enough injectives, we can define the right derived functors $R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}$. These functors are called the *Zuckerman derived functors* in [40].

Another construction of the Zuckerman derived functors is known as follows (see e.g. [5, I. 8]). Let (τ, V) be a $(\mathfrak{g}, M)$-module. We take a basis $\{X_i\}$ of $\mathfrak{g}$ and the dual basis $\{\lambda_i\}$ of $\mathfrak{g}^*$. For $X \in \mathfrak{g}$, we define

$$\begin{array}{rccc} \mu(X): & V \otimes \mathbb{C}[K] & \to & V \otimes \mathbb{C}[K] \\ & \Uparrow & & \Uparrow \\ & v \otimes f & \mapsto & \sum_i X_i v \otimes \lambda_i(\mathrm{Ad}(k)(X))f. \end{array}$$

We denote by $(L \otimes R, \mathbb{C}[K])$ the regular representation of $K \times K$. Then $\mu(X)$ commutes with $(\tau \otimes L)(k)$ and $(\tau \otimes L)(Y)$ for any $X \in \mathfrak{g}$, $Y \in \mathfrak{k}$ and $k \in M$.

Fact 2.14. *Under the above settings, we have*

$$R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}(V) \simeq H^i(\mathfrak{k}, M; V \otimes \mathbb{C}[K]),$$

where we take the $(\mathfrak{k}, M)$*-cohomology with respect to* $\tau \otimes L$*, and the* $(\mathfrak{g}, K)$*-module structure is induced from* μ *and* $\mathbf{1}_V \otimes R$*.*

By this construction, we can see the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{g}, M) & \xrightarrow{R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}} & \mathcal{C}(\mathfrak{g}, K) \\ {\scriptstyle \mathcal{F}}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathcal{C}(\mathfrak{k}, M) & \xrightarrow[R^i\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}]{} & \mathcal{C}(\mathfrak{k}, K). \end{array}$$

Here $\mathcal{F}$ is the forgetful functor. The following facts are useful to study the $\mathfrak{g}$-action on the Zuckerman derived functor modules (see [5, I. 8. Theorem 8.8] and [104, Lemma 6.3.1]).

Fact 2.15. *Let V be a $(\mathfrak{g}, M)$-module and W be a $(\mathfrak{g}, K)$-module. Then we have*

$$R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}(V \otimes W) \simeq R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}(V) \otimes W.$$

Fact 2.16. *Let V be a $(\mathfrak{g}, M)$-module and S be a $\mathrm{Ad}(K)$-stable subspace of $\mathcal{U}(\mathfrak{g})$. Write*

$$\begin{aligned} &m : S \otimes V \to V, \\ &m' : S \otimes R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}(V) \to R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}(V) \end{aligned}$$

for the multiplication maps. Then we have the following $(\mathfrak{k}, K)$-module commutative diagram:

$$\begin{tikzcd} S \otimes R^i\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}(V) \arrow[r, "m'"] \arrow[d, "\simeq"'] & R^i\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}(V) \\ R^i\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}(S \otimes V) \arrow[ur, "R^i\Gamma(m)"'] & \end{tikzcd}$$

By the above facts, to simplify notation, we write $R^i\Gamma^K_M$ instead of $R^i\Gamma^{\mathfrak{g},K}_{\mathfrak{g},M}$. To prove several vanishing theorems, we need the following lemma.

Lemma 2.17. *Let V be a $(\mathfrak{g}, M)$-module with $(\mathfrak{g}, M)$-module filtration $0 = V_0 \subset V_1 \subset \cdots$ such that $\bigcup_j V_j = V$. If $R^i\Gamma^K_M(V_j) = 0$ for any j, then $R^i\Gamma^K_M(V) = 0$ holds.*

Proof. Consider the standard complex to define $H^i(\mathfrak{k}, M; \cdot)$. Define

$$C^i := \mathrm{Hom}_M(\wedge^i(\mathfrak{k}/\mathfrak{m}), V \otimes \mathbb{C}[K])$$

with $d : C^i \to C^{i+1}$. Take $\omega \in C^i$ such that $d\omega = 0$. Then $\mathrm{Im}(\omega)$ is contained in $V^j \otimes \mathbb{C}[K]$ for some j. Hence by assumption, we can take $\omega' \in C^{i-1}$ such that $d\omega' = \omega$. This shows the assertion. □

2.5 Jantzen–Zuckerman translation functor

In this section, we summarize the facts on the Jantzen–Zuckerman translation functor. For a fuller treatment, we refer the reader to [40, Chapter VII]. We do not use the case of singular infinitesimal characters.

Let $\mathfrak{g}$ be a semisimple Lie algebra with Borel subalgebra $\mathfrak{b}$. Fix a Levi decomposition $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. For $\mathfrak{g}$-module V, we say that V is locally finite $Z(\mathfrak{g})$-module if $Z(\mathfrak{g})v$ is finite-dimensional for any $v \in V$. By Schur's lemma, It is obvious that any discretely decomposable $\mathfrak{g}$-module is locally $Z(\mathfrak{g})$-finite. For a locally $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module and a character χ of $Z(\mathfrak{g})$, we define

$$P^{\mathfrak{g}}_{\chi}(V) := \{v \in V : (z - \chi(z))^n v = 0 \text{ for some } n \text{ depending on } z \in Z(\mathfrak{g})\}.$$

If $\mathfrak{g}$ is clear from the context, we write P_χ for $P^{\mathfrak{g}}_\chi$. Then $P_\chi(V)$ is a $\mathfrak{g}$-submodule, and called the *primary component* corresponding to χ. The following fact is a direct consequence of [40, Proposition 7.20].

Fact 2.18. *Let V be a locally $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module. Then V is the direct sum of its primary components.*

We define the Jantzen–Zuckerman translation functor as follows. Let V be a locally $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module. Then for any finite-dimensional $\mathfrak{g}$-module F, $V \otimes F$ is also locally $Z(\mathfrak{g})$-finite. Take two weights λ, μ of $\mathfrak{t}$. Suppose that μ is algebraically integral. Let F_μ be a finite-dimensional irreducible representation of $\mathfrak{g}$ with extreme weight μ. We write χ_λ for the character of $Z(\mathfrak{g})$ corresponding to λ under the Harish-Chandra isomorphism. We define

$$T^{\lambda}_{\lambda+\mu}(V) := P_{\chi_{\lambda+\mu}}(F_\mu \otimes P_{\chi_\lambda}(V)).$$

The functor $T^{\lambda}_{\lambda+\mu}$ is called the Jantzen–Zuckerman translation functor in [41]. Before we state several properties of the functor, we prepare notation.

Definition 2.19. Let λ be a character of $\mathfrak{t}$. λ is said to be *integrally dominant* with respect to $\mathfrak{b}$ if

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \notin \{-1, -2, \ldots\} \text{ for any } \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}),$$

and is said to be *integrally anti-dominant* with respect to $\mathfrak{b}$ if

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \notin \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}).$$

Here we take the positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ corresponding to $\mathfrak{b}$.

The most important property of the translation functor for us is that the functor gives a category equivalence. We write the same notation $T^{\lambda}_{\lambda+\mu}$ for the restrictions of the translation functor to subcategories of $\mathcal{C}(\mathfrak{g})_\lambda$.

Fact 2.20. *Let λ be a integrally dominant regular weight of $\mathfrak{t}$ and μ be a algebraically integral weight of $\mathfrak{t}$. Suppose that $\lambda + \mu$ is integrally dominant regular.*

- *$T_{\lambda}^{\lambda+\mu}$ sends an irreducible object of $\mathcal{C}(\mathfrak{g})_{\lambda}$ to an irreducible object of $\mathcal{C}(\mathfrak{g})_{\lambda+\mu}$.*
- *$T_{\lambda+\mu}^{\lambda} T_{\lambda}^{\lambda+\mu}(V) \simeq V$ holds for any irreducible object V of $\mathcal{C}(\mathfrak{g})_{\lambda}$.*
- *Let $\mathcal{C}(\mathfrak{g})_{\lambda}^{(1)}$ be the full subcategory of $\mathcal{C}(\mathfrak{g})_{\lambda}$ whose objects have the infinitesimal character λ. Then $T_{\lambda}^{\lambda+\mu}$ gives a category equivalence from $\mathcal{C}(\mathfrak{g})_{\lambda}^{(1)}$ to $\mathcal{C}(\mathfrak{g})_{\lambda+\mu}^{(1)}$.*

We consider the relative BGG category $\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}}$. Recall the notation in Section 2.2. We assume that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{l}$, and do not assume that $\mathfrak{q}$ contains $\mathfrak{b}$. The following fact (see [28, Theorem 7.8]) is used in Section 6.

Fact 2.21. *Let λ be a integrally dominant regular weight of $\mathfrak{t}$ and μ be a algebraically integral weight of $\mathfrak{t}$. Suppose that $\lambda + \mu$ is integrally dominant regular. Then $T_{\lambda}^{\lambda+\mu}$ gives a category equivalence from $(\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}})_{\lambda}$ to $(\mathcal{O}_{\mathfrak{q}}^{\mathfrak{g}})_{\lambda+\mu}$.*

The image of a generalized Verma module by the translation functor is also a generalized Verma module (see [40, Theorem 7.237]).

Fact 2.22. *Let F be a finite-dimensional $\mathfrak{l}$-module with infinitesimal character λ, and μ be a algebraically integral weight of $\mathfrak{t}$. Suppose that $\lambda + \rho(\mathfrak{u})$ and $\lambda + \mu + \rho(\mathfrak{u})$ are integrally dominant and regular with respect to $\mathfrak{b}$. Then we have*

$$T_{\lambda+\rho(\mathfrak{u})}^{\lambda+\mu+\rho(\mathfrak{u})}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(T_{\lambda}^{\lambda+\mu}(F)).$$

At the last, we state the result for the category of $(\mathfrak{g}, K)$-modules. Let $(\mathfrak{g}, K)$ be a pair with semisimple $\mathfrak{g}$ (see Section 2.1). For simplicity, we assume that K is connected. In our usage, this assumption is always true. We denote by $\mathcal{F}(\mathfrak{g}, K)_{\lambda}$ the full subcategory of $\mathcal{C}(\mathfrak{g}, K)$ whose objects are finite length and have the generalized infinitesimal character λ. The following result can be shown by the same proof as in [40, Corollary 7.209]. Remark that we do not assume that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair.

Fact 2.23. *Let λ be a integrally dominant regular weight of $\mathfrak{t}$ and μ be a algebraically integral weight of $\mathfrak{t}$. Suppose that $\lambda+\mu$ is integrally dominant regular, and the finite-dimensional irreducible $\mathfrak{g}$-module F_{μ} with extreme weight μ lifts to a representation of K. Then $T_{\lambda}^{\lambda+\mu}$ gives a category equivalence from $\mathcal{F}(\mathfrak{g}, K)_{\lambda}$ to $\mathcal{F}(\mathfrak{g}, K)_{\lambda+\mu}$.*

Since the Zuckerman derived functor preserves the generalized infinitesimal character and a direct sum decomposition, we obtain the following fact (see the proof of [40, Theorem 7.237]).

Fact 2.24. *Let $(\mathfrak{g}, K)$ be a pair with semisimple $\mathfrak{g}$ and connected K, and M be a reductive subgroup of K. Take a weight λ and algebraically integral weight μ of $\mathfrak{t}$. Suppose that the finite-dimensional irreducible $\mathfrak{g}$-module F_μ with extreme weight μ lifts to a representation of K. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C}(\mathfrak{g}, M)_\lambda \arrow[r, "R^i\Gamma_M^K"] \arrow[d, "T_\lambda^{\lambda+\mu}"'] & \mathcal{C}(\mathfrak{g}, K)_\lambda \arrow[d, "T_\lambda^{\lambda+\mu}"] \\ \mathcal{C}(\mathfrak{g}, M)_{\lambda+\mu} \arrow[r, "R^i\Gamma_M^K"'] & \mathcal{C}(\mathfrak{g}, K)_{\lambda+\mu}. \end{tikzcd}$$

2.6 Polynomial identity

In this section, we discuss the representation theory of an associative algebra with polynomial identity.

We define a invariant of rings called the polynomial identity degree (see [72, Chapter 13]).

Definition 2.25. As a $\mathbb{Z}$-coefficient non-commutative polynomial with n-valuables, we define

$$s_n(X_1, X_2, \ldots, X_n) := \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w) X_{w(1)} X_{w(2)} \cdots X_{w(n)}.$$

Here $\mathfrak{S}_n$ is the symmetric group of degree n and sgn is its signature character. For a ring $\mathcal{R}$, the *polynomial identity degree* of $\mathcal{R}$ is defined by

$$\mathrm{PI.deg}(\mathcal{R}) := \min\{n \in \mathbb{N} : s_{2n} \equiv 0 \text{ on } \mathcal{R}\}.$$

For example, $\mathrm{PI.deg}(\mathcal{R}) = 1$ holds if and only if $\mathcal{R}$ is commutative because $s_2(X, Y) = XY - YX$. The following fact is the key result to control some invariant in the representation theory by the ring invariant.

Fact 2.26 (Amitsur–Levitzki theorem)**.** *Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices. Then we have* $\mathrm{PI.deg}(M_n(\mathbb{C})) = n$.

By the fact, we can see that if a $\mathbb{C}$-algebra $\mathcal{A}$ has the finite polynomial identity degree, then the dimension of any irreducible module of $\mathcal{A}$ can be bounded by the polynomial identity degree. More precisely, we have the following proposition.

Proposition 2.27. *Let $\mathcal{A}$ be an at most countable-dimensional unital associative $\mathbb{C}$-algebra, and $\{(V_\lambda, \pi_\lambda)\}_{\lambda \in \Lambda}$ be a family of irreducible representations of $\mathcal{A}$. Suppose $\bigcap_{\lambda \in \Lambda} \mathrm{Ker}(\pi_\lambda) = 0$. Then* $\mathrm{PI.deg}(\mathcal{A}) = \sup_{\lambda \in \Lambda} \{\dim_{\mathbb{C}}(V_\lambda)\}$ *holds.*

Proof. $\pi_\lambda(\mathcal{A})$ is dense in $\mathrm{End}_{\mathbb{C}}(V_\lambda)$ in the sense of the Jacobson density theorem. Hence $\mathrm{PI.deg}(\pi_\lambda(\mathcal{A})) = \dim_{\mathbb{C}}(V_\lambda)$ holds. By assumption, we have the following injection:

$$\mathcal{A} \hookrightarrow \prod_{\lambda \in \Lambda} \pi_\lambda(\mathcal{A}).$$

Thus we have $\mathrm{PI.deg}(\mathcal{A}) \leq \sup_{\lambda \in \Lambda} \{\dim_{\mathbb{C}}(V_\lambda)\}$. Obviously, $\mathrm{PI.deg}(\mathcal{A}) \geq \mathrm{PI.deg}(\pi_\lambda(\mathcal{A}))$ holds for any $\lambda \in \Lambda$. This shows the converse inequality. □

3 Stability theorem

The aim of this section is to study unitary or algebraic representations with uniformly bounded multiplicities. We treat the irreducible decomposition of torsion-free modules on the coordinate ring of a spherical variety and the branching law of unitary highest weight modules with respect to symmetric subgroups of holomorphic type.

The contents of this section has been published in [36].

3.1 Some algebraic results

In this section, we set up some notation and results about representations of complex algebraic groups.

3.1.1 G-algebra and $(\mathcal{A}, G)$-module

Let G be a connected reductive complex algebraic group. Fix a Borel subgroup B of G. Let $B = TN$ be its Levi decomposition, where T is a maximal torus of G and N is the unipotent radical of B. Let $\Lambda^+ = \Lambda^+_G \subset \mathfrak{t}^*$ be the set of dominant integral weights with respect to B. For each $\lambda \in \Lambda^+$, we denote by $V_\lambda = V_{\lambda,G}$ the irreducible representation of G with highest weight λ.

For an algebraic group H, we say a representation V of H over $\mathbb{C}$ is a *rational representation* if $\mathrm{span}_{\mathbb{C}}\{gv : g \in H\}$ is a finite dimensional algebraic representation of H for any $v \in V$. This implies that any rational representation of G is completely reducible. Given a rational representation V of G, we can decompose V into the direct sum of irreducible representations:

$$V = \bigoplus_{\lambda \in \Lambda^+} m_V^G(\lambda) V_\lambda,$$

where $m_V^G(\lambda)$ is the multiplicity of the irreducible representation V_λ in V. If the group G is obvious, we write $m_V(\lambda) := m_V^G(\lambda)$. We set

$$\Lambda^+(V) := \Lambda_G^+(V) := \{\lambda \in \Lambda^+ : m_V(\lambda) \neq 0\}.$$

We say that a $\mathbb{C}$-algebra $\mathcal{A}$ is a *G-algebra* if $\mathcal{A}$ is a rational representation of G and G acts on $\mathcal{A}$ via $\mathbb{C}$-algebra automorphisms.

Definition 3.1. Let $\mathcal{A}$ be a G-algebra, and M be an $\mathcal{A}$-module and a rational representation of G. Then M is said to be an $(\mathcal{A}, G)$-module if $g(am) = (ga)(gm)$ for any $g \in G$, $a \in \mathcal{A}$ and $m \in M$. Moreover, we will say that an $(\mathcal{A}, G)$-module M is *finitely generated* if M is finitely generated as an $\mathcal{A}$-module.

Let X be a quasi-projective variety over $\mathbb{C}$. We denote by $\mathbb{C}[X]$ the ring of regular functions on X. Suppose that X is a G-variety. The action of G on X induces a rational representation of G on $\mathbb{C}[X]$ as follows:

$$(g \cdot f)(x) = f(g^{-1}x) \text{ for } g \in G, f \in \mathbb{C}[X].$$

We write $\Lambda^+(X) = \Lambda^+(\mathbb{C}[X])$ for short.

3.1.2 Some finiteness results

We prepare some finiteness results for proofs in Section 3.3. Let G be a connected reductive algebraic group over $\mathbb{C}$, and $B = TN$ be a Borel subgroup of G.

Lemma 3.2. *Let $\mathcal{A}$ be a Noetherian G-algebra, and M be a finitely generated $(\mathcal{A}, G)$-module. Then M^G is a finitely generated $\mathcal{A}^G$-module.*

Proof. Observe that $\mathcal{A}M^G \subset M$ is finitely generated as an $\mathcal{A}$-module. In fact, since $\mathcal{A}$ is a Noetherian algebra and M is finitely generated, M is a Noetherian $\mathcal{A}$-module. Thus $\mathcal{A}M^G$ is finitely generated.

Let $\{m_1, m_2, \cdots, m_r\}$ be a generating set of $\mathcal{A}M^G$ as an $\mathcal{A}$-module. We may and do assume $\{m_1, m_2, \ldots, m_r\} \subset M^G$. We show that $\{m_1, m_2, \ldots, m_r\}$ is a generating set of M^G as an $\mathcal{A}^G$-module. Since $\{m_1, m_2, \cdots, m_r\}$ generates $\mathcal{A}M^G$, each $m \in M^G$ can be expressed as $m = f_1m_1 + f_2m_2 + \cdots + f_rm_r$ for some $f_1, f_2, \ldots, f_r \in \mathcal{A}$. Taking the G-invariant part, we have $m = f_1^Gm_1 + f_2^Gm_2 + \cdots + f_r^Gm_r$, where f_i^G is the projection to the G-invariant part of f_i. This shows $\{m_1, m_2, \ldots, m_r\}$ generates M^G as an $\mathcal{A}^G$-module. $\square$

The following result is due to Dž. Hadžiev and F. D. Grosshans [17].

Proposition 3.3. *Let $\mathcal{A}$ be a G-algebra. Then $(\mathcal{A}\otimes\mathbb{C}[G/N])^G$ is isomorphic to $\mathcal{A}^N$ as a $\mathbb{C}$-algebra. Moreover if $\mathcal{A}$ is finitely generated, so is $\mathcal{A}^N$.*

Remark 3.4. For the following lemma, we only define the isomorphism when $\mathcal{A}$ is the regular function ring $\mathbb{C}[X]$ of G-variety X. For $f \in \mathbb{C}[X]^N$, define $\varphi(f)(g,x) := f(g^{-1}x)$. Then φ is an algebra isomorphism between $\mathbb{C}[X]^N$ and $\mathbb{C}[G/N \times X]^G (\simeq (\mathbb{C}[G/N]\otimes\mathbb{C}[X])^G)$.

The following lemma is a key result for the proof of Theorem 3.14. If $\mathcal{A}$ is finitely generated, this result (for arbitrary characteristics) is in [18].

Lemma 3.5. *Let $\mathcal{A}$ be a Noetherian G-algebra, and M be an $(\mathcal{A}, G)$-module. Then M^N is isomorphic to $(M \otimes \mathbb{C}[G/N])^G$ as an $\mathcal{A}^N$-module. Here we consider $(M\otimes\mathbb{C}[G/N])^G$ as an $\mathcal{A}^N$-module via the isomorphism $\mathcal{A}^N \simeq (\mathcal{A}\otimes\mathbb{C}[G/N])^G$ in Proposition 3.3. Moreover, if M is a finitely generated $\mathcal{A}$-module, then M^N is a finitely generated $\mathcal{A}^N$-module.*

Proof. First, observe that the second assertion follows from the first. Indeed, since $\mathbb{C}[G/N]$ is finitely generated, $\mathcal{A}\otimes\mathbb{C}[G/N]$ is a Noetherian algebra by Hilbert's basis theorem. Thus the second assertion follows from the first assertion and Lemma 3.2.

To show the first assertion, it suffices to construct an isomorphism between M^N and $(M \otimes \mathbb{C}[G/N])^G$. Since M is a rational representation of G, $\varphi_m(g) := gm$ is well-defined as an element of $(M \otimes \mathbb{C}[G])^G$ for any $m \in M$. The map $M \ni m \mapsto \varphi_m \in (M \otimes \mathbb{C}[G])^G$ is an isomorphism as an $((\mathcal{A}\otimes\mathbb{C}[G])^G, G)$-module because the map $f \mapsto f(e)$ gives its inverse. Restricting the map to the N-invariant part, we have an $(\mathcal{A}\otimes\mathbb{C}[G/N])^G$-module isomorphism from M^N to $(M \otimes \mathbb{C}[G/N])^G$. This completes the proof. $\square$

3.2 Highest weight modules

In this section, we will review some definitions and facts about unitary highest weight modules of real Lie groups.

Real Lie groups and their Lie algebras are denoted by Roman alphabets and the corresponding German letters, respectively. We express the complexification of a real Lie algebra by writing a subscript $(\cdot)_\mathbb{C}$. For example, the Lie algebras of real Lie groups G, H, and K are denoted as $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{k}$ with complexification $\mathfrak{g}_\mathbb{C}$, $\mathfrak{h}_\mathbb{C}$, and $\mathfrak{k}_\mathbb{C}$, respectively.

For a compact Lie group, we identify its locally finite representations with rational representations of its complexification. Note that the complexification of a compact Lie group is a complex reductive algebraic group. We express the complexification of a compact Lie group by writing a subscript $(\cdot)_{\mathbb{C}}$.

3.2.1 Associated variety and isotropy representation

We define the associated variety and the isotropy representation of a $(\mathfrak{g}, K)$-module.

Let G be a real reductive Lie group, and K be a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ determined by K. Let V be a finitely generated $(\mathfrak{g}, K)$-module. Since V is finitely generated, we can take a K-invariant finite dimensional subspace $W \subset V$ as a generating subspace of V. We put $V_i := \mathcal{U}_i(\mathfrak{g}_{\mathbb{C}})W$ and $V_{-1} := 0$, where $\{\mathcal{U}_i(\mathfrak{g}_{\mathbb{C}})\}$ is the canonical filtration of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. Taking the associated graded module, we have an $(S(\mathfrak{g}_{\mathbb{C}}), K_{\mathbb{C}})$-module

$$\mathrm{gr}(V) := \bigoplus_{i=0}^{\infty} V_i/V_{i-1}.$$

The affine variety determined by $\mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}(V))$ is called the *associated variety of* V, and denoted by $\mathcal{AV}(V) \subset \mathfrak{g}_{\mathbb{C}}^*$. It is well-known that $\mathcal{AV}(V)$ is independent of the choice of W. Since the filtration is $K_{\mathbb{C}}$-stable, $\mathcal{AV}(V)$ is a $K_{\mathbb{C}}$-stable variety contained in $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$. We identify $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ via an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}_{\mathbb{C}}$. By this identification, $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$ corresponds to $\mathfrak{p}_{\mathbb{C}}$, and $\mathcal{AV}(V)$ is a subvariety in the nilpotent cone in $\mathfrak{p}_{\mathbb{C}}$. It is known that the number of $K_{\mathbb{C}}$-orbits in the nilpotent cone in $\mathfrak{p}_{\mathbb{C}}$ is finite. Thus there exists an open $K_{\mathbb{C}}$-orbit in $\mathcal{AV}(V)$.

Following D. Vogan [101], we recall the isotropy representation of V. (See also [108].) To describe the representation effectively, for the rest of this section, we assume that $\mathcal{AV}(V)$ is irreducible. Let I be the defining ideal of $\mathcal{AV}(V)$. By the Hilbert Nullstellensatz, I^n is contained in $\mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}(V))$ for some positive integer n. Since $\mathcal{AV}(V)$ is irreducible, $\mathcal{AV}(V)$ has a unique open dense $K_{\mathbb{C}}$-orbit $\mathcal{O}$. Fix a point $x_0 \in \mathcal{O}$. We denote by $\mathfrak{m}(x_0) \subset \mathbb{C}[\mathfrak{p}_{\mathbb{C}}]$ the maximal ideal corresponding to x_0. We set

$$\mathcal{W} := \bigoplus_{i=0}^{n-1} I^i \mathrm{gr}(V)/\mathfrak{m}(x_0) I^i \mathrm{gr}(V).$$

$\mathcal{W}$ is a finite dimensional rational representation of $(K_{\mathbb{C}})_{x_0}$, where $(K_{\mathbb{C}})_{x_0}$ is the isotropy subgroup of $K_{\mathbb{C}}$ at x_0. The representation $\mathcal{W}$ is called the *isotropy representation* of V. Note that the isotropy representation is dependent on the filtration of V and the point x_0.

3.2.2 Highest weight modules

Next, we review unitary highest weight modules. Throughout the rest of this section, we assume G is a connected non-compact simple Lie group of Hermitian type with finite center. Though the assumption 'finite center' is not essential, we assume this for convenience.

By assumption, $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair (i.e. the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is one-dimensional). We fix a *characteristic element* $Z \in \mathfrak{c}(\mathfrak{k}_{\mathbb{C}})$ (i.e. the eigenvalues of $\mathrm{ad}(Z)$ are $0, \pm 1$), and we write the eigenspace decomposition of $\mathrm{ad}(Z)$ as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-,$$

with the eigenvalues $1, 0, -1$, respectively.

For an irreducible $(\mathfrak{g}, K)$-module V, we will say that V is a *highest weight module* of G if $V^{\mathfrak{p}_+} \neq 0$, where $V^{\mathfrak{p}_+}$ is the space of $\mathfrak{p}_+$-null vectors of V. Moreover, if V is infinitesimally unitary, we will say that V is a *unitary highest weight module.*

Let $\mathcal{H}$ be a highest weight module. Then $\mathcal{H}^{\mathfrak{p}_+}$ is an irreducible representation of K, and $\mathcal{H}^{\mathfrak{p}_+}$ generates $\mathcal{H}$ as a representation of $\mathfrak{g}_{\mathbb{C}}$. We take $\mathcal{H}^{\mathfrak{p}_+}$ as W in Section 3.2.1, and define a filtration of $\mathcal{H}$ by $\mathcal{H}^i := \mathcal{U}_i(\mathfrak{g}_{\mathbb{C}})\mathcal{H}^{\mathfrak{p}_+}$. Since this filtration is stable under $\mathfrak{p}_+$-action, its associated graded module $\mathrm{gr}(\mathcal{H})$ is an $(S(\mathfrak{p}_-), K_{\mathbb{C}})$-module. Thus the associated variety of $\mathcal{H}$ is contained in $\mathfrak{p}_+$.

Since $\mathfrak{p}_-$ is abelian, $\mathcal{U}(\mathfrak{p}_-)$ is isomorphic to $S(\mathfrak{p}_-)$ as an algebra. $\mathcal{H}$ can be considered as a $(S(\mathfrak{p}_-), K_{\mathbb{C}})$-module under this isomorphism. Then the graded module $\mathrm{gr}(\mathcal{H})$ is isomorphic to $\mathcal{H}$ as a $(S(\mathfrak{p}_-), K_{\mathbb{C}})$-module. Hence, we always omit the filtration step for highest weight modules.

About the annihilators of unitary highest weight modules, A. Joseph showed the following result in [34]:

Proposition 3.6. *Let $\mathcal{H}$ be a unitary highest weight module. Then the annihilator $\mathrm{Ann}_{S(\mathfrak{p}_-)}(\mathcal{H})$ is a prime ideal in $S(\mathfrak{p}_-)$, and $\mathrm{Ann}_{S(\mathfrak{p}_-)}(v) = \mathrm{Ann}_{S(\mathfrak{p}_-)}(\mathcal{H})$ for any $v \in \mathcal{H}$.*

By this proposition, the isotropy representation of a unitary highest weight module at $x_0 \in \mathcal{AV}(\mathcal{H})$ is simply written as $\mathcal{W} = \mathcal{H}/\mathfrak{m}(x_0)\mathcal{H}$.

Since $\mathcal{H}^{\mathfrak{p}_+}$ generates $\mathcal{H}$ as a $\mathfrak{g}$-module, we have a canonical surjective homomorphism as a $(\mathfrak{g}, K)$-module:

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+)} \mathcal{H}^{\mathfrak{p}_+} \to \mathcal{H}. \tag{3.6.1}$$

For a finite dimensional representation V of K, we set

$$N^{\mathfrak{g}}(V) := \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+)} V.$$

Definition 3.7. Let $\mathcal{H}$ be a unitary highest weight module. We will say $\mathcal{H}$ is a *holomorphic discrete series representation* if the completion of $\mathcal{H}$ with respect to its Hermitian inner product is a discrete series representation of G.

It is known that if $\mathcal{H}$ is a holomorphic discrete series representation, the homomorphism (3.6.1) is bijective. Therefore, for a holomorphic discrete series representation $\mathcal{H}$, the associated variety $\mathcal{AV}(\mathcal{H})$ is equal to $\mathfrak{p}_+$.

3.2.3 Strongly orthogonal roots

We will describe some structures of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_+$.

We take a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Since $\mathfrak{g}$ is of Hermitian type, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. For each $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, we set

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g}_{\mathbb{C}} : [H, X] = \lambda(H)X \text{ for any } H \in \mathfrak{t}_{\mathbb{C}}\}.$$

For a $\mathfrak{t}_{\mathbb{C}}$-stable subspace $\mathfrak{s} \subset \mathfrak{g}_{\mathbb{C}}$, we define

$$\Delta(\mathfrak{s}, \mathfrak{t}_{\mathbb{C}}) := \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* \backslash \{0\} : \mathfrak{g}_\lambda \cap \mathfrak{s} \neq 0\}.$$

Let $\Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the root system determined by $\mathfrak{t}_{\mathbb{C}}$, and fix a positive system Δ^+ such that $\Delta^+ \supset \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$. Set $\Delta_c^+ := \Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cap \Delta^+$ and $\Delta_n^+ := \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$.

Two roots α, β are said to be *strongly orthogonal* if neither of $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\gamma_1, \gamma_2, \ldots, \gamma_r\} \subset \Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$ as follows:

i) γ_1 is the lowest root in $\Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbb{C}})$,

ii) for $i > 1$, γ_i is the lowest root in the roots that are strongly orthogonal to $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}$.

Fix root vectors $\{X_{\gamma_i}\}_{i=1}^r$ for the roots $\{\gamma_i\}_{i=1}^r$. We set

$$\mathfrak{a} := \bigoplus_{i=1}^r \mathbb{R}(X_{\gamma_i} + \overline{X_{\gamma_i}}),$$
$$\mathfrak{t}_0 := \bigoplus_{i=1}^r \mathbb{C}[X_{\gamma_i}, \overline{X_{\gamma_i}}],$$

where $\overline{\cdot}$ is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. It is known that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Then we have $r = \mathbb{R}\text{-rank}(\mathfrak{g})$.

We state a fact to describe restricted roots of G by the strongly orthogonal roots. For i, j $(1 \le i < j \le r)$, we put

$$\begin{aligned}
C_{ij} &:= \left\{\gamma \in \Delta_c^+ : \gamma|_{\mathfrak{t}_0} = \left(\frac{\gamma_j - \gamma_i}{2}\right)\Big|_{\mathfrak{t}_0}\right\}, \\
C_i &:= \left\{\gamma \in \Delta_c^+ : \gamma|_{\mathfrak{t}_0} = -\left(\frac{\gamma_i}{2}\right)\Big|_{\mathfrak{t}_0}\right\}, \\
C_0 &:= \{\gamma \in \Delta_c^+ : \gamma|_{\mathfrak{t}_0} = 0\}. \\
P_{ij} &:= \left\{\gamma \in \Delta_n^+ : \gamma|_{\mathfrak{t}_0} = \left(\frac{\gamma_j + \gamma_i}{2}\right)\Big|_{\mathfrak{t}_0}\right\}, \\
P_i &:= \left\{\gamma \in \Delta_n^+ : \gamma|_{\mathfrak{t}_0} = \left(\frac{\gamma_i}{2}\right)\Big|_{\mathfrak{t}_0}\right\}, \\
P_0 &:= \{\gamma_1, \gamma_2, \ldots, \gamma_r\}.
\end{aligned}$$

The following fact is due to Moore. (see e.g. [22, Proposition 4.8 in Chapter 5]).

Proposition 3.8. *In the above notation, Δ_c^+ and Δ_n^+ can be decomposed as follows:*

$$\begin{aligned}
\Delta_c^+ &= \left(\bigcup_{1 \le i < j \le r} C_{ij}\right) \cup \left(\bigcup_{1 \le i \le r} C_i\right) \cup C_0, \\
\Delta_n^+ &= \left(\bigcup_{1 \le i < j \le r} P_{ij}\right) \cup \left(\bigcup_{1 \le i \le r} P_i\right) \cup P_0.
\end{aligned}$$

Moreover, the map $\gamma \mapsto \gamma + \gamma_i$ gives bijections from C_{ij} to P_{ij}, from $-C_{ji}$ to P_{ji}, and from C_i to P_i.

Next, we describe $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_+$ by the strongly orthogonal roots. Put $X_i = X_{\gamma_1} + X_{\gamma_2} + \cdots + X_{\gamma_i}$. We set $\mathcal{O}_i := \mathrm{Ad}(K_{\mathbb{C}})X_i$, and $\mathcal{O}_0 = \{0\}$.

Proposition 3.9. *$\mathfrak{p}_+$ is decomposed into $K_{\mathbb{C}}$-orbits as follows:*

$$\mathfrak{p}_+ = \coprod_{i=0}^{n} \mathcal{O}_i.$$

Moreover, for any $1 \le m \le r$, the Zariski closure of $\mathcal{O}_m$ is decomposed into $K_{\mathbb{C}}$-orbits as follows:

$$\overline{\mathcal{O}_m} = \coprod_{i=0}^{m} \mathcal{O}_i.$$

By this proposition, for a highest weight module $\mathcal{H}$, $\mathcal{AV}(\mathcal{H}) = \overline{\mathcal{O}_m}$ for some $m \in \{1, 2, \cdots, r\}$. The irreducible decomposition of $\mathbb{C}[\overline{\mathcal{O}_m}]$ as a representation of $K_{\mathbb{C}}$ is well-known.

Proposition 3.10. *The ring of regular functions on $\overline{\mathcal{O}_m}$ is decomposed as a $K_{\mathbb{C}}$-representation as follows:*

$$\mathbb{C}[\overline{\mathcal{O}_m}] \simeq \bigoplus_{\substack{c_1 \geq c_2 \geq \cdots \geq c_m \geq 0 \\ c_1, c_2, \ldots, c_m \in \mathbb{Z}}} V_{-\sum_{i=1}^m c_i \gamma_i, K_{\mathbb{C}}}.$$

In particular, $\overline{\mathcal{O}_m}$ is a spherical affine $K_{\mathbb{C}}$-variety.

Remark 3.11. The irreducible decomposition of $\mathbb{C}[\mathfrak{p}_+]$ as a representation of $K_{\mathbb{C}}$ is obtained by L. K. Hua [27] (for classical groups), B. Kostant (unpublished) and W. Schmid [94]. The proposition can be obtained by the Hua–Kostant–Schmid theorem and the orbit description.

The decomposition can be considered as the K-type decomposition of irreducible unitary highest weight modules of scalar type. For holomorphic discrete series representations, the Hua–Kostant–Schmid theorem is generalized to the case of restrictions to non-compact symmetric pairs of holomorphic type by T. Kobayashi [44, 52].

3.2.4 Symmetric pairs of holomorphic type

In this section, we review some results about branching laws of unitary highest weight modules. For the following formulation and Table 1, we refer the reader to [47] and [52].

Suppose θ is a Cartan involution of G corresponding to the maximal compact subgroup K, and τ is an involutive automorphism of G commuting with θ. Since $\tau(\mathfrak{k}) = \mathfrak{k}$ and τ is an automorphism, the following two cases are possible:

$$\tau(Z) = Z, \tag{3.11.2}$$

$$\tau(Z) = -Z. \tag{3.11.3}$$

Recall that Z is the characteristic element in $\mathfrak{c}(\mathfrak{k}_{\mathbb{C}})$. We will say $(\mathfrak{g}, \mathfrak{g}^\tau)$ is a *symmetric pair of holomorphic type* if the equation (3.11.2) holds, otherwise we will say $(\mathfrak{g}, \mathfrak{g}^\tau)$ is a *symmetric pair of anti-holomorphic type*.

If $(\mathfrak{g}, \mathfrak{g}^\tau)$ is a symmetric pair of holomorphic type, the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+$ induces a decomposition of $\mathfrak{g}_{\mathbb{C}}^\tau$:

$$\mathfrak{g}_{\mathbb{C}}^\tau = \mathfrak{p}_-^\tau \oplus \mathfrak{k}_{\mathbb{C}}^\tau \oplus \mathfrak{p}_+^\tau,$$

since $Z \in \mathfrak{k}_{\mathbb{C}}^{\tau}$. Suppose $\mathfrak{g}^{\tau} = \bigoplus_{i=1}^{n} \mathfrak{h}_i$ is the direct sum decomposition into simple ideals and abelian ideals. Then $\mathfrak{h}_i$ is contained in $\mathfrak{k}$ if $\mathfrak{h}_i$ is a compact or abelian Lie algebra, and $\mathfrak{h}_i$ is of Hermitian type if $\mathfrak{h}_i$ is a non-compact Lie algebra. Moreover, if $\mathfrak{h}_i$ is of Hermitian type, $\mathfrak{h}_i$ has the decomposition:

$$(\mathfrak{h}_i)_{\mathbb{C}} = (\mathfrak{p}_- \cap (\mathfrak{h}_i)_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{h}_i)_{\mathbb{C}}) \oplus (\mathfrak{p}_+ \cap (\mathfrak{h}_i)_{\mathbb{C}}), \tag{3.11.4}$$

and each summand is nonzero. The following proposition is due to T. Kobayashi [43, 47].

Proposition 3.12. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G, and $(\mathfrak{g}, \mathfrak{g}^{\tau})$ be a symmetric pair of holomorphic type. Then $\mathcal{H}$ is $\mathfrak{g}^{\tau}$-admissible. In particular, $\mathcal{H}$ is completely reducible as a $(\mathfrak{g}^{\tau}, \mathfrak{k}^{\tau})$-module. Moreover, all irreducible components of $\mathcal{H}|_{\mathfrak{g}^{\tau}}$ are holomorphic discrete series representations.*

The following proposition is useful to study the branching law of holomorphic discrete series representations. The proposition is proved by H. P. Jakobsen and M. Vergne [29] (see also [44, 52]).

Proposition 3.13. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G, and $(\mathfrak{g}, \mathfrak{g}^{\tau})$ be a symmetric pair of holomorphic type. Set $H := (G^{\tau})_0$. Then each irreducible component of $\mathcal{H}|_{\mathfrak{g}^{\tau}}$ is a unitary highest weight module of H (more precisely, holomorphic discrete series representation by the above proposition,) and its multiplicity can be described as*

$$m_{\mathcal{H}}^{H}(\lambda) = m_{S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+}}^{H \cap K}(\lambda),$$

where λ is the highest weight of a unitary highest weight module of H.

Table 1: symmetric pairs of holomorphic type

$\mathfrak{g}$	$\mathfrak{g}^{\tau}$
$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(i,j) + \mathfrak{u}(p-i,q-j))$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{su}(n,n)$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{so}^*(2n)$	$\mathfrak{u}(i,n-i)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2i) + \mathfrak{so}^*(2(n-i))$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,n-i) + \mathfrak{so}(i)$
$\mathfrak{so}(2,2n)$	$\mathfrak{u}(1,n)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(i,n-i)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(i,\mathbb{R}) + \mathfrak{sp}(n-i,\mathbb{R})$

$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^*(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(8,2) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(5,1) + \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(4,2) + \mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)} + \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)} + \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(10,2) + \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}^*(12) + \mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6,2)$

3.3 Stability theorem

In this section, we will show a stability theorem for general settings. Throughout this section, G is a connected reductive complex algebraic group, and B is a Borel subgroup of G. Fix a Levi decomposition $B = TN$, where T is a maximal torus and N is the unipotent radical of B.

3.3.1 Stability theorem for general settings

Let X be an irreducible quasi-projective variety over $\mathbb{C}$. We assume the following two conditions:

X is a spherical G-variety (i.e. a Borel subgroup of G has an open dense orbit in X), and (3.13.5)

the quotient field of $\mathbb{C}[X]$ is naturally isomorphic to the rational function field of X. (3.13.6)

The first condition implies that $\mathbb{C}[X]$ is multiplicity-free as a representation of G. Note that the second condition is always true for any irreducible quasi-affine variety X.

Theorem 3.14. *Let M be a finitely generated $(\mathbb{C}[X], G)$-module with no zero divisors:*

$$\bigcup_{m \in M \setminus \{0\}} \mathrm{Ann}_{\mathbb{C}[X]}(m) = 0. \tag{3.14.7}$$

Then there exists a weight $\lambda_0 \in \Lambda^+(X)$ such that

$$m_M(\lambda + \lambda_0) = m_M(\lambda + \mu + \lambda_0)$$

for any $\lambda \in \Lambda^+(M)$ and $\mu \in \Lambda^+(X)$.

This theorem says that the multiplicity function m_M is periodic for sufficiently large parameter λ. This property of the multiplicity function is called stability.

Proof. For the proof of the stability, we will show the uniformly boundedness of $m_M(\lambda)$ for $\lambda \in \Lambda^+$. Since M is a finitely generated $\mathbb{C}[X]$-module, there exists a G-invariant finite dimensional subspace $F \subset M$ that generates M. Then the multiplication map $\mathbb{C}[X] \otimes F \to M(f \otimes m \mapsto fm)$ is a surjective G-intertwining operator. For any $\lambda \in \Lambda^+(M)$, we have

$$\begin{aligned} m_M(\lambda) &\leq m_{\mathbb{C}[X]\otimes F}(\lambda) \\ &= \dim \operatorname{Hom}_G(V_\lambda, \mathbb{C}[X] \otimes F) \\ &= \dim \operatorname{Hom}_G(V_\lambda \otimes F^*, \mathbb{C}[X]). \end{aligned}$$

From [52, Proposition 5.4.1], the number of the irreducible constituents of $V_\lambda \otimes F^*$ is bounded by $\dim(F)$. Therefore, since $\mathbb{C}[X]$ is multiplicity-free, $m_M(\lambda)$ is uniformly bounded by $\dim(F)$. This result is also proved in the proof of Theorem 3.18 (injectivity part).

Next, we will show that $m_M(\cdot)$ is monotone increasing with respect to the translation by $\Lambda^+(X)$. Since $\mathbb{C}[X]$ has no zero divisors in M, the multiplication operator $(m \mapsto fm)$ is injective for any $f \in \mathbb{C}[X]$. In particular, for $\mu \in \Lambda^+(X)$ and $f \in \mathbb{C}[X]^N(\mu)$, f induces an injective linear map

$$f\cdot : M^N(\lambda) \hookrightarrow M^N(\lambda + \mu),$$

where $V(\lambda)$ denotes the weight space of weight λ in a T-representation V. Thus we have

$$m_M(\lambda) \leq m_M(\lambda + \mu) \tag{3.14.8}$$

for any $\mu \in \Lambda^+(X)$.

We show the stability theorem. From Lemma 3.5, M^N is a finitely generated $\mathbb{C}[X]^N$-module. Then we can take a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_r\} \subset \Lambda^+(M)$ such that

$$\Lambda^+(M) = \bigcup_{1 \leq i \leq r} (\Lambda^+(X) + \lambda_i). \tag{3.14.9}$$

By the uniformly boundedness of m_M, for each λ_i, we can find a weight $\lambda_{0,i} \in \Lambda^+(X)$ such that

$$m_M(\lambda_i + \lambda_{0,i}) = \max\{m_M(\lambda_i + \mu) : \mu \in \Lambda^+(X)\}. \tag{3.14.10}$$

We put $\lambda_0 := \lambda_{0,1} + \lambda_{0,2} + \cdots + \lambda_{0,r}$. Observe that λ_0 satisfies the required condition. Take $\lambda \in \Lambda^+(M)$ and $\mu \in \Lambda^+(X)$. By (3.14.9), there exists an $i \in \{1, 2, \ldots, r\}$ such that $\lambda \in \lambda_i + \Lambda^+(X)$. From (3.14.8) and (3.14.10), we have

$$\begin{aligned} m_M(\lambda + \lambda_0) &= \max\{m_M(\lambda_i + \mu) : \mu \in \Lambda^+(X)\} \\ &= m_M(\lambda + \mu + \lambda_0). \end{aligned}$$

This completes the proof. □

3.3.2 Description of multiplicities for large parameters

We describe the multiplicities for sufficiently large parameters by the isotropic representation. Let X be as in the previous section. By the assumption that X is G-spherical (3.13.5), there exists a point $x_0 \in X$ such that Bx_0 is open dense in X.

Put $P = \{g \in G : gBx_0 \subset Bx_0\}$. Then P is a parabolic subgroup of G contains B. The following proposition is due to M. Brion, D. Luna and T. Vust [6].

Proposition 3.15. *In the above settings,*

i) P_{x_0} *is a reductive subgroup of* G*, and*

ii) P_{x_0} *contains the derived group of some Levi subgroup of* P*.*

The following proposition says that B_{x_0} is a 'Borel subgroup' of P_{x_0}.

Proposition 3.16. P_{x_0} *satisfies the following four conditions:*

L-1) $P_{x_0} \subset G_{x_0}$,

L-2) $P_{x_0} \supset B_{x_0}$,

L-3) B_{x_0} *meets every connected component of* P_{x_0},

L-4) the identity component of B_{x_0} *is a Borel subgroup of the identity component of* P_{x_0}.

Conversely, if a reductive subgroup L *of* G *satisfies the above four conditions, then we have* $L = P_{x_0}$.

Remark 3.17. If L satisfies the above four conditions, its irreducible representations are parametrized by a subset of characters of B_{x_0}. This is because $V^{N_{x_0}}$ is one-dimensional for any irreducible representation V of L. In fact, since we have the natural injection $B_{x_0}/N_{x_0} \hookrightarrow B/N \simeq T$, we can take a weight vector $v \in V^{N_{x_0}}$ with respect to B_{x_0}/N_{x_0}. Then v generates an irreducible representation V_0 of L_0, where L_0 is the identity component of L. Since B_{x_0} normalizes L_0, V_0 is L-stable. This shows $V_0 = V$. Therefore, $V^{N_{x_0}}$ is one-dimensional.

Proof. For the first assertion, put $L := P_{x_0}$. By definition, L-1) and L-2) are clear. From Proposition 3.15, we can take a Levi subgroup Q of P such that $[Q, Q]$ is contained in L. We have the following commutative diagram.

$$\begin{tikzcd} L/(L \cap B) \arrow[r, hook] & P/B \\ {[Q,Q]/([Q,Q] \cap B)} \arrow[u, hook] \arrow[ur, "\simeq"] & \end{tikzcd}$$

Thus $L/(L \cap B)$ is isomorphic to P/B. Since $P/B \simeq Q/Q \cap B$ is a connected projective variety, $B_{x_0} = L \cap B$ meets every connected component of L, and the identity component of B_{x_0} is a Borel subgroup of the identity component of L. This implies that L satisfies L-3) and L-4).

For the second assertion, suppose L is a reductive subgroup of G satisfying the conditions. From Remark 3.17, we have

$$\begin{aligned} \mathbb{C}[G]^L &= \mathbb{C}[G]^{B_{x_0}} \\ &= \mathbb{C}[G]^{P_{x_0}}. \end{aligned} \tag{3.17.11}$$

For a reductive subgroup H of G, H can be reconstructed from $\mathbb{C}[G]^H$ by the following equation:

$$H = \bigcap_{f \in \mathbb{C}[G]^H} f^{-1}(f(e)).$$

Here e is the identity of G. From this fact and (3.17.11), we have $L = P_{x_0}$. This completes the proof. $\square$

We set $L = P_{x_0}$. We denote by ev_{x_0} the natural quotient map $M \to M_{x_0} (:= M/\mathfrak{m}(x_0)M)$, where $\mathfrak{m}(x_0)$ is the maximal ideal of $\mathbb{C}[X]$ corresponding to x_0. From the inclusion $L \subset G_{x_0}$, ev_{x_0} is an L-intertwining operator from M to M_{x_0}. We describe the stable multiplicities by the representation of L on $M/\mathfrak{m}(x_0)M$.

Theorem 3.18. *Let M be a finitely generated $(\mathbb{C}[X], G)$-module with no zero divisors (see (3.14.7)). We take a weight $\lambda_0 \in \Lambda^+(X)$ as described in Theorem 3.14. Then for any $\lambda \in \Lambda^+(M)$, we have*

$$m_M^G(\lambda + \lambda_0) = m_{M_{x_0}}^L(\lambda|_{B_{x_0}}),$$

where we identify characters of T with characters of B by letting their values be 1 on N.

Remark 3.19. For any $\mu \in \Lambda^+(X)$, $\mu|_{B_{x_0}} = 0$. Thus $\lambda|_{B_{x_0}}$ can be written as $(\lambda + \lambda_0)|_{B_{x_0}}$.

For the proof of Theorem 3.18, we will show two lemmas.

Lemma 3.20. *Under the assumption (3.13.5) and (3.13.6), we have the following equation:*

$$\mathbb{C}[Bx_0] = \mathbb{C}[X]\left[\frac{1}{f_\mu} : \mu \in \Lambda^+(X), f_\mu \in \mathbb{C}[X]^N(\mu)\backslash\{0\}\right].$$

Proof. This lemma is essentially the same as [93, Lemma 2.2]. It is clear that the left hand side contains the right hand side. We will show the converse inclusion.

We take a function $f \in \mathbb{C}[Bx_0]$. Define an ideal by

$$I := \{g \in \mathbb{C}[X] : g \cdot bf \in \mathbb{C}[X] \text{ for any } b \in B\}.$$

Since B acts rationally on $\mathbb{C}[Bx_0]$, $\mathrm{span}_{\mathbb{C}}\{bf : b \in B\}$ is finite dimensional. By the assumption (3.13.6), I is a B-invariant nonzero ideal of $\mathbb{C}[X]$. Since B acts rationally on I, there exists a nonzero B-eigenvector $g \in I$. Therefore, we have $f \in \mathbb{C}[X][1/g]$. This shows the converse inclusion. □

Lemma 3.21. *Let M be a $(\mathbb{C}[X], G)$-module. Suppose $\mathbb{C}[X]$ has no zero divisors in M. Then we have*

$$\bigcap_{y \in Bx_0} (\mathfrak{m}(y)M) = 0.$$

Proof. If M is finitely generated, this lemma is in [108, Corollary 2.1]. Put $N = \bigcap_{y \in Bx_0}(\mathfrak{m}(y)M)$. We assume $N \neq 0$. Since N is a B-invariant subspace, there exists a nonzero B-eigenvector $m \in N$. By definition, m can be written as

$$m = f_1 m_1 + f_2 m_2 + \cdots + f_r m_r \quad (f_i \in \mathfrak{m}(x_0), m_i \in M). \qquad (3.21.12)$$

Let M' be a $(\mathbb{C}[X], G)$-submodule of M generated by $m_1, m_2, \ldots, m_r$. Since M' is a finitely generated $(\mathbb{C}[X], G)$-module with no zero divisors, we have

$$\bigcap_{y \in Bx_0} (\mathfrak{m}(y)M') = 0.$$

From (3.21.12), m is an element of $\mathfrak{m}(x_0)M'$. Since m is a B-eigenvector, we have $m \in \bigcap_{y \in Bx_0}(\mathfrak{m}(y)M')$ and hence $m = 0$. However, this contradicts the assumption that m is nonzero. This completes the proof. $\square$

Proof of Theorem 3.18. Take $\lambda \in \Lambda^+(M)$.

First, we reduce the assertion of the theorem to bijectivity of ev_{x_0} between some B-eigenspace and B_{x_0}-eigenspace. From Remark 3.17, we have

$$m^L_{M_{x_0}}(\lambda|_{B_{x_0}}) = \dim(M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}}))$$

Since ev_{x_0} is a G_{x_0}-intertwining operator, the image of $M^N(\lambda+\lambda_0)$ by ev_{x_0} is contained in $M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$. We denote by the same notation ev_{x_0} the restriction of ev_{x_0} to $M^N(\lambda + \lambda_0)$. Then it suffices to show that ev_{x_0} is a bijection between $M^N(\lambda + \lambda_0)$ and $M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$.

$$\begin{tikzcd}
M \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0} \\
M^N(\lambda+\lambda_0) \arrow[u, hook] \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}}) \arrow[u, hook]
\end{tikzcd}$$

(injectivity). Suppose $m \in M^N(\lambda+\lambda_0)$ and $\mathrm{ev}_{x_0}(m) = 0$. Since m is a B-eigenvector, $m \in \mathfrak{m}(bx_0)M$ for any $b \in B$. Then we have $m \in \bigcap_{y \in Bx_0} \mathfrak{m}(y)M$. Since $\bigcap_{y \in Bx_0} \mathfrak{m}(y)M = 0$ from Lemma 3.21, this implies $m = 0$. This shows ev_{x_0} is injective.

(surjectivity). First, we show the surjectivity for the case that M is a free $\mathbb{C}[X]$-module. Suppose $M \simeq \mathbb{C}[X] \otimes W$ for some finite dimensional rational representation W of G. In this case, ev_{x_0} is actually the evaluation map at x_0. Take $m \in W^{N_{x_0}}(\lambda|_{B_{x_0}})$, and put

$$\varphi(bx_0) = b^{-\lambda-\lambda_0}(bm)$$

for $b \in B$. Then φ is well-defined as an element of $\mathbb{C}[Bx_0] \otimes W$, and φ is a B-eigenvector of weight $\lambda + \lambda_0$. From Lemma 3.20, there exists a B-eigenvector $f_\mu \in \mathbb{C}[X]$ such that $f_\mu \varphi \in \mathbb{C}[X] \otimes W$. Then $f_\mu \varphi$ is in $(\mathbb{C}[X] \otimes W)^N(\lambda+\lambda_0+\mu)$. By Theorem 3.14, the multiplication operator $f_\mu \cdot : (\mathbb{C}[X] \otimes W)^N(\lambda + \lambda_0) \to$

$(\mathbb{C}[X] \otimes W)^N(\lambda + \lambda_0 + \mu)$ is bijective. Thus φ is in $(\mathbb{C}[X] \otimes W)^N(\lambda + \lambda_0)$. Since $\varphi(x_0) = m$, ev_{x_0} is surjective.

Next, we show the surjectivity for general cases. Since M is finitely generated as a $\mathbb{C}[X]$-module, there exists a finite dimensional G-subrepresentation $W \subset M$ such that the $(\mathbb{C}[X], G)$-homomorphism $\times : \mathbb{C}[X] \otimes W \to M$ defined by the multiplication map is surjective. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{C}[X] \otimes W \arrow[r, "\mathrm{ev}_{x_0}"] \arrow[d, "\times"] & W \arrow[d] \\ M \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0}, \end{tikzcd}$$

and all arrows are surjective. Take $\lambda_0' \in \Lambda^+(X)$ described in Theorem 3.14 for $M = \mathbb{C}[X] \otimes W$. By restricting the above diagram to the subspace of B-eigenvectors of weight $\lambda + \lambda_0'$, we have

$$\begin{tikzcd} (\mathbb{C}[X] \otimes W)^N(\lambda + \lambda_0') \arrow[r, "\mathrm{ev}_{x_0}"] \arrow[d] & W^{N_{x_0}}(\lambda|_{B_{x_0}}) \arrow[d] \\ M^N(\lambda + \lambda_0') \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}}). \end{tikzcd}$$

Since G and L are reductive, the vertical arrows are surjective. From the free module case, the above horizontal arrow is surjective. Therefore, $\mathrm{ev}_{x_0} : M^N(\lambda + \lambda_0') \to M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$ is also surjective.

Since $\dim(M^N(\lambda + \lambda_0)) \geq \dim(M^N(\lambda + \lambda_0'))$ by the result of Theorem 3.14, $\mathrm{ev}_{x_0} : M^N(\lambda + \lambda_0) \to M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$ is also surjective. $\square$

Remark 3.22. The injectivity is true in more general settings. For example, suppose X is a projective G-variety that has an open dense Borel orbit Bx_0, and $\pi : \mathcal{V} \to X$ is a G-equivariant algebraic vector bundle over X. Then the global sections $\Gamma(X, \mathcal{V})$ and the evaluation map $\mathrm{ev}_{x_0} : \Gamma(X, \mathcal{V}) \to \pi^{-1}(x_0)$ satisfy the injectivity as in the above proof. This implies that the multiplicity with respect to G can be bounded by the multiplicity with respect to L as in Theorem 3.18. See Section 3.4.2 for example.

We can remove the finiteness of M if we admit that the conclusion becomes weaker.

Corollary 3.23. *Let M be a $(\mathbb{C}[X], G)$-module with no zero divisors. Then we have*

$$\sup_{\mu \in \Lambda^+(X)} \{m_M^G(\lambda + \mu)\} = m_{M_{x_0}}^L(\lambda|_{B_{x_0}}).$$

for any $\lambda \in \Lambda^+(M)$.

Proof. Take a weight $\lambda \in \Lambda^+(M)$. By the same proof as for the injectivity in Theorem 3.18, we have

$$\sup_{\mu \in \Lambda^+(X)} \{m^G_M(\lambda + \mu)\} \leq m^L_{M_{x_0}}(\lambda|_{B_{x_0}}).$$

For any finite dimensional L-subrepresentation $\overline{N} \subset M_{x_0}$, we can take a finitely generated $(\mathbb{C}[X], G)$-submodule N such that $N/(N \cap \mathfrak{m}(x_0)M) \supset \overline{N}$. We may assume $m^G_N(\lambda) \neq 0$. Since the canonical map $N/\mathfrak{m}(x_0)N \to N/(N \cap \mathfrak{m}(x_0)M)$ is surjective, the converse inequality follows from Theorem 3.18. $\square$

For a rational representation V of G, we denote by $\mathcal{M}_G(V)$ the supremum of m^G_V.

Corollary 3.24. *Let M be a $(\mathbb{C}[X], G)$-module with no zero divisors. Then the following equation holds:*

$$\mathcal{M}_G(M) = \mathcal{M}_L(M_{x_0}).$$

In particular, M is multiplicity-free as a representation of G if and only if M_{x_0} is multiplicity-free as a representation of L.

Proof. By Corollary 3.23, $\mathcal{M}_G(M) \leq \mathcal{M}_L(M_{x_0})$ is clear. It suffices to show that any character λ of B_{x_0} satisfying $m^L_{M_{x_0}}(\lambda) \neq 0$ can be extended to a character $\overline{\lambda}$ of B such that $m^G_M(\overline{\lambda}) \neq 0$. As in the proof of the surjectivity in Theorem 3.18, we can assume that M is a free $\mathbb{C}[X]$-module of finite rank, $\mathbb{C}[X] \otimes W$. We take a character λ of B_{x_0} such that $m^L_W(\lambda) \neq 0$, and take $m \in W^{N_{x_0}}(\lambda)$. There exists a character λ' of B such that $\lambda'|_{B_{x_0}} = \lambda$. For $\varphi(bx_0) = b^{-\lambda'}(bm)$, we can find $f_\mu \in \mathbb{C}[X]^N(\mu)$ such that $f_\mu\varphi \in (\mathbb{C}[X] \otimes W)^N(\lambda' + \mu)$. Since $(\lambda' + \mu)|_{B_{x_0}} = \lambda'|_{B_{x_0}} = \lambda$, $\overline{\lambda} := \lambda' + \mu$ satisfies the desired conditions. This completes the proof. $\square$

3.4 Examples of stability theorems

In this section, we will apply the stability theorem to some explicit settings.

3.4.1 Stability theorem for quasi-affine spherical homogeneous spaces

Let G be a connected reductive algebraic group and H be a closed subgroup of G. We assume that (G, H) is a spherical pair (i.e. there exists a Borel subgroup B of G such that BH is open dense in G). The following fact is known as the characterization of quasi-affine homogeneous spaces. (See [97, Theorem 3.12].)

Proposition 3.25. *Let G be a linear algebraic group, and H be a closed subgroup of G. Then the following three conditions are equivalent:*

i) G/H is quasi-affine;

ii) the quotient field of $\mathbb{C}[G/H]$ is equal to the rational function field of G/H;

iii) for any H-representation W, there exists a finite dimensional representation V of G such that W can be embedded in V as a representation of H.

We assume that G/H is a quasi-affine variety. By Proposition 3.25, this assumption is equivalent to the condition (3.13.6). Put $L := \{g \in H : gBH \subset BH\}$.

For a finite dimensional rational representation W of H, we define the *induced representation* of W by

$$\mathrm{Ind}_H^G(W) := (\mathbb{C}[G] \otimes W)^H.$$

$\mathrm{Ind}_H^G(W)$ is a $(\mathbb{C}[G]^H, G)$-module via the left G-action and the multiplication of $\mathbb{C}[G]^H$. Here the H-invariant part is taken via its right action on $\mathbb{C}[G]$.

Applying Theorem 3.14 and Theorem 3.18 to $X = G/H$ and $M = \mathrm{Ind}_H^G(W)$, we have the following theorem. For a connected semisimple subgroup H, this theorem was proved by F. Satō in [93].

Theorem 3.26. *Let W be a finite dimensional rational representation of H. Then there exists a weight $\lambda_0 \in \Lambda^+(G/H)$ such that*

$$m^G_{\mathrm{Ind}_H^G(W)}(\lambda + \lambda_0) = m^L_W(\lambda|_{B\cap H})$$

for any $\lambda \in \Lambda^+(\mathrm{Ind}_H^G(W))$.

Proof. First, we show that $X = G/H$ and $M = \mathrm{Ind}_H^G(W)$ satisfy the conditions of Theorem 3.14 and Theorem 3.18. Recall that (3.13.5), (3.13.6), 'finitely generated' and 'no zero divisors' are the conditions. By definition, the condition (3.13.5) (i.e. X is spherical) is clear. By the assumption that $X = G/H$ is a quasi-affine G-variety and Proposition 3.25, the quotient field of $\mathbb{C}[X]$ coincides with the rational function field of X. This is the condition (3.13.6). It is obvious that $\mathbb{C}[G/H]$ has no zero divisors on $\mathrm{Ind}_H^G(W)$.

Observe that M is finitely generated. From Proposition 3.25, W can be embedded in a finite dimensional representation V of G as a representation of H. The embedding $W \hookrightarrow V$ induces an injection as a $(\mathbb{C}[G/H], G)$-module:

$$\mathrm{Ind}_H^G(W) \hookrightarrow \mathbb{C}[G/H] \otimes V.$$

Since $\mathbb{C}[G/H] \otimes V$ is a Noetherian $\mathbb{C}[G/H]$-module, $\mathrm{Ind}_H^G(W)$ is Noetherian, and hence finitely generated. All conditions are verified.

Next, we show that $M/\mathfrak{m}(eH)M \simeq W$ as a representation of L. We can identify $\mathrm{Ind}_H^G(W)$ with the set of global sections $\Gamma(G/H, G \times_H W)$ of a vector bundle $G \times_H W \to G/H$. Since G/H is a quasi-affine variety, the sheaf constructed from the vector bundle corresponds to the sheaf constructed from $\mathrm{Ind}_H^G(W)$. This shows that the fiber $(\mathrm{Ind}_H^G(W))_{eH}$ is isomorphic to W as a representation of L. This completes the proof. $\square$

3.4.2 Some examples for projective varieties

In this section, we treat flag varieties. Let G be a connected reductive algebraic group, and P be a parabolic subgroup of G. Take a closed connected reductive subgroup H of G such that G/P is a spherical H-variety. Note that if H is a Levi subgroup of G, the classification of such triples (G, H, P) follows from Stembridge's classification [96] of multiplicity-free restrictions of finite-dimensional irreducible representations with respect to H, and if (G, H) is a symmetric pair, such triples (G, H, P) were classified by He–Ochiai–Nishiyama–Oshima [21].

Fix a Borel subgroup B of H. Since G/P is a spherical H-variety, there exists a point $x_0 \in G$ such that Bx_0P is open dense in G. Put $L := \{g \in H_{x_0P} : gBx_0P \subset Bx_0P\}$. The same result as Theorem 3.26 is not true for G/P since G/P is projective (see Example 3.29). However, the theorem can be applied to an 'affine cone' of G/P. Then we have the following theorem.

Theorem 3.27. *Let W be an irreducible representation of P. Then there exists a character λ_0 of P such that*

$$\mathcal{M}_H(\mathrm{Ind}_P^G(W \otimes \mathbb{C}_{\lambda_0+\lambda})) = \mathcal{M}_L(W)$$

for any character λ of P satisfying $\mathrm{Ind}_P^G(\mathbb{C}_\lambda) \neq 0$. Here W is considered as a representation of L via the inclusion ${x_0}^{-1}Lx_0 \subset P$.

Fix a Levi decomposition $P = QN$, where N is the unipotent radical of P. Put $P' := [Q, Q]N$ and $A := Q/[Q, Q]$. By Proposition 3.25, G/P' is a quasi-affine spherical $H \times A$-variety. The action of A on G/P' is given by

$$h \cdot gP' = gh^{-1}P'$$

for $h \in A$ and $g \in G$. Note that $B \times A$ is a Borel subgroup of $H \times A$. For the proof of Theorem 3.27, we show the following lemma.

Lemma 3.28. *Set*

$$L' := \{(g,h) \in H \times A : (g,h)\cdot x_0P' = x_0P', (g,h)\cdot Bx_0P \subset Bx_0P\}.$$

Then there exists a homomorphism $\varphi : L \to A$ such that

$$L' = \{(g,\varphi(g)) \in L \times A : g \in L\}. \tag{3.28.13}$$

Proof. First, we define the homomorphism φ. Take $g \in L$. By definition, we have $gBx_0P \subset Bx_0P$ and $gx_0P = x_0P$. From $gx_0P' \subset x_0P = \sqcup_{l\in A} x_0lP'$, there exists a unique element $\varphi(g) \in A$ such that $gx_0P' = x_0\varphi(g)P'$. It is obvious that φ is a homomorphism from L to A.

Next, we show that φ satisfies the condition. By the definition of L and L', we have $(g,\varphi(g)) \in L'$ for any $g \in L$. For the converse inclusion, we take $(g,h) \in L'$. Since $(g,h) \in L'$, we have $gx_0h^{-1}P' = x_0P'$ and $gBx_0P \subset Bx_0P$. This implies that $g \in L$. Since $x_0P' = gx_0h^{-1}P' = x_0\varphi(g)h^{-1}P'$, we have $\varphi(g) = h$. This completes the proof. □

Proof of Theorem 3.27. We apply Corollary 3.24 to $X = G/P'$ and $M = \mathrm{Ind}_{P'}^G(W)$. Here we replace G in the corollary by $H \times A$, and then L in the corollary is equal to L' in the above lemma.

We will determine the action of L' on $M/\mathfrak{m}(x_0P')M$. $M/\mathfrak{m}(x_0P')M$ is isomorphic to W as a $\mathbb{C}$-vector space. Take $(g,\varphi(g)) \in L'$. For $f \in (\mathbb{C}[G] \otimes W)^{P'}$, we have

$$\begin{aligned}((g,\varphi(g))\cdot f)(x_0) &= \varphi(g)f(g^{-1}x_0\varphi(g)) \\ &= \varphi(g)f(x_0{x_0}^{-1}g^{-1}x_0\varphi(g)) \\ &= \varphi(g)({x_0}^{-1}g^{-1}x_0\varphi(g))^{-1}f(x_0) \\ &= {x_0}^{-1}gx_0f(x_0).\end{aligned}$$

Therefore, the action of L' on $M/\mathfrak{m}(x_0P')M \simeq W$ coincides with the action of L, and we have $\mathcal{M}_{L'}(W) = \mathcal{M}_L(W)$.

From Corollary 3.24, there exists $\lambda' \in \Lambda^+_{H\times A}(\mathrm{Ind}_{P'}^G(W))$ such that

$$m^{H\times A}_{\mathrm{Ind}_{P'}^G(W)}(\lambda') = \mathcal{M}_{L'}(W). \tag{3.28.14}$$

We write $\lambda' = -\lambda_0 + \lambda_1$, where λ_0 is a character of P and λ_1 is a character of B.

We will show that λ_0 satisfies the desired condition. We have the following isomorphisms of representations of H:

$$\begin{aligned}\mathrm{Ind}_{P'}^G(W)(-\lambda_0) &\simeq (\mathrm{Ind}_{P'}^G(W) \otimes \mathbb{C}_{\lambda_0})^A \\ &\simeq ((\mathbb{C}[G] \otimes W)^{P'} \otimes \mathbb{C}_{\lambda_0})^A \\ &\simeq (\mathbb{C}[G] \otimes W \otimes \mathbb{C}_{\lambda_0})^P \\ &\simeq \mathrm{Ind}_P^G(W \otimes \mathbb{C}_{\lambda_0}).\end{aligned}$$

Thus we obtain $\mathcal{M}_H(\mathrm{Ind}_P^G(W \otimes \mathbb{C}_{\lambda_0})) = \mathcal{M}_L(W)$. Again, from the above isomorphisms for $W = \mathbb{C}$, $\mathrm{Ind}_P^G(\mathbb{C}_\lambda) \simeq \mathbb{C}[G/P'](-\lambda)$ is nonzero if and only if there exists a character ν of B such that $-\lambda + \nu \in \Lambda^+_{H\times A}(G/P')$. From this and Theorem 3.18, the proof is completed. □

We give an example where $\mathrm{Ind}_P^G(W)$ is nonzero and we can not take $\lambda_0 = 0$.

Example 3.29. Let $G = \mathrm{GL}(8, \mathbb{C}), H = \mathrm{GL}(4, \mathbb{C}) \times \mathrm{GL}(4, \mathbb{C})$. H is block diagonal in G. Let P be a maximal parabolic subgroup of G containing H and all lower triangular matrices, and B be a Borel subgroup of H containing all upper triangular matrices in H. We take a point

$$x_0 := \begin{pmatrix} I & J \\ 0 & I \end{pmatrix},$$

where J is an anti-diagonal matrix with every anti-diagonal entries 1. Then Bx_0P is open dense in G. In this case, L is of the following form:

$$L = \left\{ (\mathrm{diag}(a_1, a_2, a_3, a_4), \mathrm{diag}(a_4, a_3, a_2, a_1)) \in H : a_1, \ldots, a_4 \in \mathbb{C}^\times \right\},$$

where $\mathrm{diag}(\cdots)$ is a diagonal matrix. Note that L commutes with x_0.

We consider a representation $W = S^2(\bigwedge^2(\mathbb{C}^4))/\bigwedge^4(\mathbb{C}^4)$ of H, where the first factor of $\mathrm{GL}(4, \mathbb{C}) \times \mathrm{GL}(4, \mathbb{C})$ acts on W in standard way and the second factor acts on W trivially. W is an irreducible representation of H with highest weight $(2, 2, 0, \ldots, 0)$ in the standard coordinates. We extend the representation W to P by letting the unipotent radical of P act trivially. Then the induced representation $\mathrm{Ind}_P^G(W)$ is an irreducible representation of G with highest weight $(2, 2, 0, \ldots, 0)$. By the Littlewood–Richardson rule, $\mathrm{Ind}_P^G(W)|_H$ is multiplicity-free, and hence $\mathcal{M}_H(\mathrm{Ind}_P^G(W)) = 1$. However, $W|_L$ is not multiplicity-free. In fact, we can take two weight vectors with weight $(1, 1, 1, 1, 0, 0, 0, 0)$ such as

$$e_1 \wedge e_2 \cdot e_3 \wedge e_4, e_1 \wedge e_3 \cdot e_2 \wedge e_4.$$

Therefore, we have $\mathcal{M}_H(\mathrm{Ind}_P^G(W)) = 1 < 2 = \mathcal{M}_L(W)$.

3.4.3 Stability theorem for highest weight modules

Here we will show a stability theorem for unitary highest weight modules. Let G be a connected simple real Lie group of Hermitian type with finite center. Fix a positive root system Δ^+, strongly orthogonal roots $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ and root vectors $\{X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_r}\}$ as in Section 3.2.3.

Before we state our theorem, we prepare some lemmas relevant to the $K_{\mathbb{C}}$-orbit $\mathcal{O}_i$. Fix $1 \leq m \leq r$. We set $\mathfrak{a}_m := \bigoplus_{i=1}^m \mathbb{R}(X_{\gamma_i} + \overline{X_{\gamma_i}})$, $\mathfrak{t}_m := \bigoplus_{i=1}^m \mathbb{C}[X_{\gamma_i}, \overline{X_{\gamma_i}}]$, $X_m := \sum_{i=1}^m X_{\gamma_i}$, and $L = Z_{K_{\mathbb{C}}}(\mathfrak{a}_m)$. We will show that L satisfies the conditions L-1) $\sim$ L-4) in Proposition 3.16.

Lemma 3.30. *Let B be a Borel subgroup of $K_{\mathbb{C}}$ determined by the positive system Δ_c^+. Then $\mathrm{Ad}(B)X_m$ is open dense in $\overline{\mathcal{O}_m}$.*

Proof. Since $\mathrm{Ad}(K_{\mathbb{C}})X_m$ is open dense in $\overline{\mathcal{O}_m}$, it suffices to show that $[\mathfrak{k}_{\mathbb{C}}, X_m] = [\mathfrak{b}, X_m]$. From Proposition 3.8, we have

$$[\mathfrak{g}_{-\gamma}, X_m] = \begin{cases} 0 & (m < j) \\ \mathfrak{g}_{-\gamma+\gamma_j} & (1 \leq j \leq m) \end{cases} \qquad \text{for any } \gamma \in C_{ij},$$
$$[\mathfrak{g}_{-\gamma}, X_m] = 0 \qquad \text{for any } \gamma \in C_i \cup C_0,$$

for any $i, j (1 \leq i < j \leq r)$. Thus we have

$$\begin{aligned}[\overline{\mathfrak{b}}, X_m] &= [\mathfrak{t}_{\mathbb{C}}, X_m] \oplus \bigoplus_{1 \leq i < j \leq m, \gamma \in C_{ij}} \mathfrak{g}_{-\gamma+\gamma_j} \\ &= [\mathfrak{t}_{\mathbb{C}}, X_m] \oplus \bigoplus_{1 \leq i < j \leq m, \gamma \in P_{ij}} \mathfrak{g}_{\gamma},\end{aligned}$$

where $\overline{\mathfrak{b}}$ is the opposite Borel subalgebra of $\mathfrak{b}$. For any $\gamma \in P_{ij}$, there exists a $\gamma' \in C_{ij}$ such that $\gamma' + \gamma_i = \gamma$. Therefore, we have $[\mathfrak{b}, X_m] \supset [\overline{\mathfrak{b}}, X_m]$. This implies $[\mathfrak{k}_{\mathbb{C}}, X_m] = [\mathfrak{b}, X_m]$. $\square$

Lemma 3.31. *Let B be the same as in the previous lemma. Then the isotropy subgroup B_{X_m} at X_m has the semi-direct product decomposition: $B_{X_m} = (T_{\mathbb{C}})_{X_m} N_{X_m}$, where N is the unipotent radical of B.*

Proof. $B_{X_m} \supset (T_{\mathbb{C}})_{X_m} N_{X_m}$ is obvious.

For the converse inclusion, we take $b \in B_{X_m}$, and write $b = tn$ for $t \in T_{\mathbb{C}}$ and $n \in N$. By Proposition 3.10, B_{X_m} is contained in $\bigcap_{i=1}^m \ker \gamma_i$. Thus we have $t \in \bigcap_{i=1}^m \ker \gamma_i|_{T_{\mathbb{C}}}$. Since X_m is the sum of the eigenvectors of $T_{\mathbb{C}}$ with weight $\gamma_i (1 \leq i \leq m)$, $(T_{\mathbb{C}})_{X_m}$ is equal to $\bigcap_{i=1}^m \ker \gamma_i|_{T_{\mathbb{C}}}$. Therefore, we have $t \in (T_{\mathbb{C}})_{X_m}$ and hence $n \in N_{X_m}$. This shows the converse inclusion. $\square$

Lemma 3.32. *$L (= Z_{K_{\mathbb{C}}}(\mathfrak{a}_m))$ satisfies the conditions L-1) $\sim$ L-4) in Proposition 3.16 for the spherical $K_{\mathbb{C}}$-variety $\mathcal{O}_m$.*

Proof. Recall the conditions:

L-1) $L \subset (K_{\mathbb{C}})_{X_m}$, L-2) $L \supset B_{X_m}$,

L-3) B_{X_m} meets every connected component of L,

L-4) the identity component of B_{X_m} is a Borel subgroup of the identity component of L.

First, we compute the triangular decomposition of $\mathfrak{l}$. For any $g \in L$ and i $(1 \leq i \leq m)$, we have

$$\begin{aligned} X_{\gamma_i} + \overline{X_{\gamma_i}} &= \mathrm{Ad}(g)(X_{\gamma_i} + \overline{X_{\gamma_i}}) \\ &= \mathrm{Ad}(g)(X_{\gamma_i}) + \mathrm{Ad}(g)(\overline{X_{\gamma_i}}). \end{aligned}$$

Since $\mathrm{Ad}(g)(X_{\gamma_i}) \in \mathfrak{p}_+$ and $\mathrm{Ad}(g)(\overline{X_{\gamma_i}}) \in \mathfrak{p}_-$, g stabilizes X_{γ_i} and $\overline{X_{\gamma_i}}$. This implies that

$$\begin{aligned} L &= Z_{K_{\mathbb{C}}}\left(\bigoplus_{i=1}^{m}(\mathfrak{g}_{\gamma_i} \oplus \mathfrak{g}_{-\gamma_i})\right), \qquad (3.32.15) \\ \mathfrak{l} &= Z_{\mathfrak{k}_{\mathbb{C}}}\left(\bigoplus_{i=1}^{m}(\mathfrak{g}_{\gamma_i} \oplus \mathfrak{g}_{-\gamma_i})\right). \end{aligned}$$

From the first equation, the condition L-1) is clear. Since the right hand side of the second equation is stable under the $\mathrm{ad}(\mathfrak{t}_{\mathbb{C}})$-action, so is $\mathfrak{l}$. Thus we have

$$\mathfrak{l} = \overline{(\mathfrak{l} \cap \mathfrak{n})} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathbb{C}}) \oplus (\mathfrak{l} \cap \mathfrak{n}). \qquad (3.32.16)$$

We will show that the Lie algebra $\mathfrak{b}_{X_m}$ of B_{X_m} is a Borel subalgebra of $\mathfrak{l}$. By Proposition 3.8, we have

$$\begin{aligned} &[\mathfrak{g}_\gamma, X_m] = \begin{cases} 0 & (m < i) \\ \mathfrak{g}_{\gamma+\gamma_i} & (1 \leq i \leq m) \end{cases} && \text{for any } \gamma \in C_{ij} \cup C_i, \\ &[\mathfrak{g}_\gamma, X_m] = 0 && \text{for any } \gamma \in C_0, \end{aligned}$$

for any $i, j (1 \leq i < j \leq r)$. This implies that $\mathfrak{b}_{X_m}$ has the following decomposition:

$$\mathfrak{b}_{X_m} = \mathfrak{t}_m^{\perp} \oplus \bigoplus_{\substack{\gamma \in C_{ij} \cup C_i \\ m < i < j \leq r}} \mathfrak{g}_\gamma \oplus \bigoplus_{\gamma \in C_0} \mathfrak{g}_\gamma,$$

where $\mathfrak{t}_m^{\perp}$ is the orthogonal complement of $\mathfrak{t}_m$ in $\mathfrak{t}_{\mathbb{C}}$ with respect to the Killing form. Therefore, we have $\mathfrak{b}_{X_m} = Z_{\mathfrak{b}}(\bigoplus_{i=1}^{m}(\mathfrak{g}_{\gamma_i} \oplus \mathfrak{g}_{-\gamma_i})) = \mathfrak{l} \cap \mathfrak{b}$. From (3.32.16), this shows the condition L-4).

We can show $L = Z_L(\mathfrak{t}_m^\perp)L_0$ by the same proof as [39, Proposition 7.49]. By the equation (3.32.15), $Z_L(\mathfrak{t}_m^\perp)$ is contained in $Z_{K_\mathbb{C}}(\mathfrak{t}_\mathbb{C}) \cap L \subset (T_\mathbb{C})_{X_m}$. This shows that $(T_\mathbb{C})_{X_m}$ meets every connected components of L, and so does B_{X_m}. Thus we have shown the condition L-3).

By the proof in Lemma 3.31, $(T_\mathbb{C})_{X_m}$ is equal to $\bigcap_{i=1}^m \ker \gamma_i|_{T_\mathbb{C}}$. This implies that $(T_\mathbb{C})_{X_m}$ is contained in L. Since $B_{X_m} = (T_\mathbb{C})_{X_m} N_{X_m}$ and N_{X_m} is connected, B_{X_m} is contained in L. This is the condition L-2). All conditions are verified.

□

For a unitary highest weight module $\mathcal{H}$, we consider the $(\mathfrak{g}, K)$-module $\mathcal{H}$ as a $(\mathbb{C}[\mathfrak{p}_+], K_\mathbb{C})$-module, and we set $\mathcal{H}_{X_m} := \mathcal{H}/\mathfrak{m}(X_m)\mathcal{H}$, where $\mathfrak{m}(X_m)$ is the maximal ideal of $\mathbb{C}[\mathfrak{p}_+]$ corresponding to X_m.

Applying Theorem 3.14 and 3.18 to a unitary highest weight module, we have the following theorem.

Theorem 3.33. *Let $\mathcal{H}$ be a unitary highest weight module of G with associated variety $\overline{\mathcal{O}_m}$. Then there exists a weight $\lambda_0 \in \Lambda^+(\overline{\mathcal{O}_m}) = \{-\sum_{i=1}^m c_i\gamma_i : c_1 \geq c_2 \geq \cdots \geq c_m \geq 0, c_i \in \mathbb{Z}\}$ such that*

$$m_\mathcal{H}^{K_\mathbb{C}}(\lambda + \lambda_0) = m_{\mathcal{H}_{X_m}}^L(\lambda|_{T_{X_m}})$$

for any $\lambda \in \Lambda^+(\mathcal{H})$.

Proof. By Proposition 3.6, the annihilator $\mathrm{Ann}_{\mathbb{C}[\mathfrak{p}_+]}(\mathcal{H})$ is equal to the defining ideal of $\overline{\mathcal{O}_m}$. Thus we consider $\mathcal{H}$ as a $(\mathbb{C}[\overline{\mathcal{O}_m}], K_\mathbb{C})$-module.

To apply Theorem 3.18, we will verify the four conditions:

i) $\overline{\mathcal{O}_m}$ is a spherical $K_\mathbb{C}$-variety,

ii) the quotient field of $\mathbb{C}[\overline{\mathcal{O}_m}]$ is equal to the rational function filed of $\overline{\mathcal{O}_m}$,

iii) $\mathcal{H}$ is a finitely generated $\mathbb{C}[\overline{\mathcal{O}_m}]$-module, and

iv) $\mathbb{C}[\overline{\mathcal{O}_m}]$ has no zero divisors in $\mathcal{H}$.

Since $\overline{\mathcal{O}_m}$ is an affine variety, the condition ii) is clear. Since $\mathcal{H}$ is generated by $\mathcal{H}^{\mathfrak{p}^+}$ as a $\mathbb{C}[\overline{\mathcal{O}_m}]$-module, $\mathcal{H}$ is a finitely generated $(\mathbb{C}[\overline{\mathcal{O}_m}], K_\mathbb{C})$-module. This is the condition iii). By Proposition 3.10, $\overline{\mathcal{O}_m}$ is a spherical $K_\mathbb{C}$-variety, and hence the condition i) holds. By Proposition 3.6, we have $\mathrm{Ann}_{\mathbb{C}[\mathfrak{p}_+]}(v) = \mathrm{Ann}_{\mathbb{C}[\mathfrak{p}_+]}(\mathcal{H})$ for any $v \in \mathcal{H}\backslash\{0\}$. This implies the condition iv).

From Proposition 3.10, we obtain the irreducible decomposition of $\mathbb{C}[\overline{\mathcal{O}_m}]$ and $\Lambda^+(\overline{\mathcal{O}_m}) = \{-\sum_{i=1}^m c_i\gamma_i : c_1 \geq c_2 \geq \cdots \geq c_m \geq 0, c_i \in \mathbb{Z}\}$.

We have shown that L satisfies the conditions L-1) $\sim$ L-4) in Lemma 3.32. This completes the proof. □

The following corollary is a direct consequence of Corollary 3.24 and Theorem 3.33.

Corollary 3.34. *Let $\mathcal{H}$ be a unitary highest weight module of G with associated variety $\overline{\mathcal{O}_m}$. Then we have*

$$\mathcal{M}_K(\mathcal{H}) = \mathcal{M}_L(\mathcal{H}_{X_m}).$$

In particular, $\mathcal{H}|_K$ is multiplicity-free if and only if $\mathcal{H}_{X_m}|_L$ is multiplicity-free.

Remark 3.35. In the proof of Theorem 3.33 and Corollary 3.34, we do not use the assumption that $\mathcal{H}$ is irreducible. Thus we can apply the theorem to $\mathcal{H}$ under the assumption that $\mathcal{H}$ is a finite direct sum of unitary highest weight modules with the same associated variety.

Remark 3.36. We review some results about the explicit form of the isotropy representation $\mathcal{H}_{X_m}$.

If $\mathcal{H}$ is a holomorphic discrete series representation, the isotropy representation $\mathcal{H}_{X_m}$ is isomorphic to $\mathcal{H}^{\mathfrak{p}_+}$ as a representation of L. If G is $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{U}(p, q)$ or $\mathrm{SO}^*(2n)$, the isotropy representation $\mathcal{H}_{X_m}$ is computed from the theta correspondence (see [106]). If $\mathcal{H}$ does not appear in the theta correspondence, the explicit form of the isotropy representation was announced by H. Yamashita in [107].

3.4.4 Stability theorem for symmetric pairs of holomorphic type

In this section, we will apply the stability theorem to branching laws of holomorphic discrete series representations with respect to symmetric pairs of holomorphic type.

Let G be a connected simple real Lie group of Hermitian type with finite center, and τ be an involutive automorphism of G commuting with a Cartan involution θ of G. We put $H = (G^\tau)_0$, the identity component of the fixed point group of τ. We assume that $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair of holomorphic type (see Section 3.2.4). Suppose $\mathcal{H}$ is a holomorphic discrete series representation of G.

Before we state a theorem, we set up some notation. We fix a Cartan subalgebra $\mathfrak{t}^\tau$ of $\mathfrak{k}^\tau$, and fix a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}^{\tau\theta}, \mathfrak{t}_\mathbb{C}^\tau)$ such that $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}_\mathbb{C}^\tau) \subset \Delta^+(\mathfrak{g}_\mathbb{C}^{\tau\theta}, \mathfrak{t}_\mathbb{C}^\tau)$. Let $B = TN$ be a Borel subgroup of $(H \cap K)_\mathbb{C}$ corresponding to the positive system $\Delta^+(\mathfrak{g}_\mathbb{C}^{\tau\theta}, \mathfrak{t}_\mathbb{C}^\tau)$.

We will take strongly orthogonal roots $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}_\mathbb{C}^\tau)$ and root vectors $\{X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_r}\}$ as in Section 3.2.3. However, $\mathfrak{g}^{\tau\theta}$ may not be a simple Lie algebra. Suppose $\mathfrak{g}^{\tau\theta} = \bigoplus_{i=1}^n \mathfrak{h}_i$ is the direct sum decomposition into simple ideals and abelian ideals. We set up a lexicographical order on $\Delta(\mathfrak{g}^{\tau\theta}, \mathfrak{t}_{\mathbb{C}}^{\tau})$ such that any element of $\Delta^+(\mathfrak{g}_{\mathbb{C}}^{\tau\theta}, \mathfrak{t}_{\mathbb{C}}^{\tau})$ is greater than zero and $\Delta^+(\mathfrak{h}_i, \mathfrak{t}_{\mathbb{C}}^{\tau}) < \Delta^+(\mathfrak{h}_j, \mathfrak{t}_{\mathbb{C}}^{\tau})$ for any i, j $(i < j)$. Here we write $X < Y$ if $x < y$ for any $x \in X$ and $y \in Y$. Replacing the term 'lowest root' by 'minimum root' in the definition of Section 3.2.3, we take strongly orthogonal roots $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}_{\mathbb{C}}^{\tau})$.

Put $\mathfrak{a} = \bigoplus_{i=1}^{r} \mathbb{R}(X_{\gamma_i} + \overline{X_{\gamma_i}}) \subset \mathfrak{p}^{-\tau}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}^{-\tau}$, and $r = \dim_{\mathbb{R}}(\mathfrak{a}) = \mathbb{R}\text{-rank}(\mathfrak{g}^{\tau\theta})$.

Theorem 3.37. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. We put $L = Z_{H\cap K}(\mathfrak{a})$. Then there exists a weight $\lambda_0 \in \Lambda^+(\mathfrak{p}_+^{-\tau})$ such that*

$$m_{\mathcal{H}}^{H}(\lambda + \lambda_0) = m_{\mathcal{H}^{\mathfrak{p}_+}}^{L}(\lambda|_{Z_T(\mathfrak{a})})$$

for any $\lambda \in \Lambda^+(\mathcal{H})$, where we denote by $m_{\mathcal{H}}^{H}(\lambda)$ the multiplicity of the holomorphic discrete series representation with highest weight λ with respect to $\mathfrak{p}_+^{\tau} \oplus \mathfrak{b}$.

Proof. We will reduce the assertion to Theorem 3.33.

By Proposition 3.13, the decomposition of $\mathcal{H}|_H$ is reduced to the decomposition of $(S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+})|_{H\cap K}$. We have the following isomorphisms as $H \cap K$-representations:

$$\begin{aligned} S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+} &\simeq N^{\mathfrak{g}^{\tau\theta}}(\mathcal{H}^{\mathfrak{p}_+}) \\ &\simeq \mathcal{U}(\mathfrak{g}^{\tau\theta})\mathcal{H}^{\mathfrak{p}_+}(\subset \mathcal{H}). \end{aligned}$$

This implies that $S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+}$ is isomorphic to the finite direct sum of some holomorphic discrete series representations of $(G^{\theta\tau})_0$ as an $H \cap K$-representation. Applying Theorem 3.33 to $N^{\mathfrak{g}^{\tau\theta}}(\mathcal{H}^{\mathfrak{p}_+})$, we obtain the theorem. $\square$

Remark 3.38. T. Kobayashi stated that we could relate the multiplicity function and Satō's stability theorem by the Howe duality in [52, Remark 1.5].

Corollary 3.39. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. We put $L = Z_{H\cap K}(\mathfrak{a})$. Then we have $\mathcal{M}_H(\mathcal{H}) = \mathcal{M}_L(\mathcal{H}^{\mathfrak{p}_+})$. In particular, $\mathcal{H}|_H$ is multiplicity-free if and only if $\mathcal{H}^{\mathfrak{p}_+}|_L$ is multiplicity-free.*

Remark 3.40. The uniformly boundedness of the multiplicity function and the 'if part' of the second assertion was proved by T. Kobayashi in [44] and [50] (see also [52, 54]).

4 Analytic continuation and branching problem

In this section, we relate the branching laws of infinite-dimensional representations to the branching laws of finite-dimensional representations using the analytic continuation. In particular, for holomorphic discrete series representations, we connect two branching laws for two subgroups with isomorphic complexifications.

4.1 General setting

Let $\mathfrak{g}$ be a complex Lie algebra. We consider a family $\{\pi_z, V\}_{z\in\mathbb{C}}$ of representations of $\mathfrak{g}$ with the same representation space V. Such a family can be seen in the representation theory such as generalized Verma modules and principal series representations. We introduce the following definition to deal with the family algebraically.

Definition 4.1. The family $\{(\pi_z, V)\}$ is said to be *dependent on z polynomially* if $\pi_z(X)$ is a polynomial function of z for any $X \in \mathfrak{g}$, namely, there exist $d \in \mathbb{N}$ and $A_0, \ldots, A_d \in \mathrm{End}_{\mathbb{C}}(V)$ such that

$$\pi_z(X) = \sum_{i=0}^{d} z^i A_i.$$

If the family $\{(\pi_z, V)\}$ depends on z polynomially, an algebraic property may be determined by the properties on a Zariski dense subset. To control the family effectively, the following finiteness property is important. Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$.

Definition 4.2. Suppose $\{(\pi_z, V)\}_{z\in\mathbb{C}}$ is a family of $(\mathfrak{g}, \mathfrak{k})$-modules. Then we say that $\{(\pi_z, V)\}_{z\in\mathbb{C}}$ is *admissible* if the following two conditions are satisfied:

- (π_z, V) is locally $\mathfrak{k}$-finite, completely reducible and $\mathfrak{k}$-admissible for any $z \in \mathbb{C}$, that is, $(\pi_z|_{\mathfrak{k}}, V)$ is completely reducible and each isotypic component is finite-dimensional;
- for $z_1, z_2 \in \mathbb{C}$, suppose that the isotypic decompositions with respect to $\pi_{z_1}(\mathfrak{k})$ and $\pi_{z_2}(\mathfrak{k})$ are as follows:

$$V = \bigoplus_{\lambda_1\in\Lambda_1} V_{\lambda_1} = \bigoplus_{\lambda_2\in\Lambda_2} V_{\lambda_2},$$

where Λ's are subsets of $\widehat{\mathfrak{k}}$ such that $V_\lambda \neq 0$ for any $\lambda \in \Lambda_i$. Then there exists a bijection $\sigma : \Lambda_1 \to \Lambda_2$ such that $V_{\lambda_1} = V_{\sigma(\lambda_1)}$ for any $\lambda_1 \in \Lambda_1$.

If $\{(\pi_z, V)\}_{z\in\mathbb{C}}$ is an admissible family, the $\pi_z(\mathfrak{k})$-isotypic decomposition of V does not depend on z. We say that V' is a $\mathfrak{k}$-*isotypic component* of V if V' is a $\pi_z(\mathfrak{k})$-isotypic component of V for some/any z. A subspace V' of V is said to be $\mathfrak{k}$-stable if V' is $\pi_z(\mathfrak{k})$-stable for any z.

Example 4.3. The following three family of $(\mathfrak{g}, \mathfrak{k})$-modules are well-known examples which are admissible and dependent on z polynomially.

- generalized Verma modules;
- the underlying Harish-Chandra module of principal series representations;
- the underlying Harish-Chandra module of holomorphic discrete series representations.

Suppose that $\{(\pi_z, V)\}_{z\in\mathbb{C}}$ is an admissible family of $(\mathfrak{g}, \mathfrak{k})$-modules depending on z polynomially. If conditions $P(z)$ depending on z are satisfied for any z except for finitely many z, we say that $P(z)$ holds for almost all (or almost every) z.

Lemma 4.4. *Retain the above notation. Let V_0 be a $\mathfrak{k}$-stable subspace and V_1 be a $\mathfrak{k}$-isotypic component of V. If $\pi_a(\mathcal{U}(\mathfrak{g}))V_0 \supset V_1$ for some $a \in \mathbb{C}$, then we have $\pi_z(\mathcal{U}(\mathfrak{g}))V_0 \supset V_1$ for almost all $z \in \mathbb{C}$.*

Proof. We take $a \in \mathbb{C}$ such that $\pi_a(\mathcal{U}(\mathfrak{g}))V_0 \supset V_1$. We will show $V_1 \subset \pi_z(\mathcal{U}(\mathfrak{g}))V_1$ for almost all z.

Take a $\pi_a(\mathfrak{k})$-stable complement V_1' of V_1 in V. Since V_1 is a $\mathfrak{k}$-isotypic component, V_1' is $\mathfrak{k}$-stable. By the assumption $\pi_a(\mathcal{U}(\mathfrak{g}))V_0 \supset V_1$, we can take a $\mathfrak{k}$-stable finite-dimensional subspace $U \subset \mathcal{U}(\mathfrak{g})$ such that $\pi_a(U)V_0 \supset V_1$. We have

$$\pi_z(U)V_0 = (\pi_z(U)V_0 \cap V_1) \oplus (\pi_z(U)V_0 \cap V_1') \tag{4.4.1}$$

for any z, because V_1 is a $\mathfrak{k}$-isotypic component and V_1' is its $\mathfrak{k}$-stable complement. Then we define the composition of the multiplication map and the projection from $V = V_1 \oplus V_1'$ to V_1:

$$m_z : U \otimes V_0 \to V \to V_1.$$

By the equation (4.4.1), m_z is surjective if and only if $\pi_z(U)V_0 \supset V_1$ holds.

Since (π_z, V) is dependent on z polynomially, m_z is a $\mathrm{Hom}_{\mathbb{C}}(U \otimes V_0, V_1)$-valued polynomial. Note that U, V_0 and V_1 are finite-dimensional. Therefore, we can take a polynomial $f \in \mathbb{C}[z]$ such that

$$\begin{aligned} f(z) \neq 0 &\iff m_z \text{ has full rank}, \\ &\iff m_z \text{ is surjective.} \end{aligned}$$

Since m_a is surjective, f is a non-zero polynomial. We denote by $N \subset \mathbb{C}$ the zero set of f. Obviously N is a finite set. Then m_z is surjective for any $z \in \mathbb{C} \backslash N$. This implies $\pi_z(U)V_0 \supset V_1$ for any $z \in \mathbb{C} \backslash N$. This shows the lemma. $\square$

Using the above lemma, we prove the following theorem. Roughly speaking, the theorem asserts that the representations (π_z, V) on a Zariski dense subset $\mathcal{S} \subset \mathbb{C}$ have all information about the representation (π_z, V) for any $z \notin \mathcal{S}$.

Theorem 4.5. *Retain the above notation. Take a $\mathfrak{k}$-stable subspace $W \subset V$ and countable subset $\mathcal{S} \subset \mathbb{C}$. Suppose that $\pi_z(\mathcal{U}(\mathfrak{g}))W$ is irreducible as a $\mathfrak{g}$-module for any $z \in \mathcal{S}$. Then the following three conditions are equivalent:*

(a) *for infinitely many $z \in \mathbb{C}$, (π_z, V) is irreducible;*

(b) *for some $z \in \mathbb{C}$, (π_z, V) is irreducible;*

(c) $\bigcup_{z \in \mathcal{S}} \pi_z(\mathcal{U}(\mathfrak{g}))W = V$.

Proof. We set $W_z := \pi_z(\mathcal{U}(\mathfrak{g}))W$.

(a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): We assume that (π_a, V) is irreducible for $a \in \mathbb{C}$. We will show $\bigcup_{z \in \mathcal{S}} W_z = V$.

Take $v \in V$. We can take a subspace $V' \subset V$ such that $v \in V'$ and V' is a finite direct sum of $\mathfrak{k}$-isotypic components of V. Since (π_a, V) is irreducible, we have $V' \subset W_a$. By Lemma 4.4, there exists $z \in \mathcal{S}$ such that $V' \subset W_z$. Therefore, we have $v \in \bigcup_{z \in \mathcal{S}} W_z$. Thus we complete the proof of (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Assume $\bigcup_{z \in \mathcal{S}} W_z = V$. We consider the $\mathfrak{k}$-isotypic decomposition of V:

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

For each $\lambda \in \Lambda$, we will show that there exists a at most countable subset $N_\lambda \subset \mathbb{C}$ satisfying the following two conditions:

(1) $\pi_z(\mathcal{U}(\mathfrak{g}))V_\lambda = V$ for any $z \in \mathbb{C}\backslash N_\lambda$;

(2) V_λ is an irreducible $\pi_z(\mathcal{U}(\mathfrak{k})\mathcal{U}(\mathfrak{g})^{\mathfrak{k}})$-module for any $z \in \mathbb{C}\backslash N_\lambda$.

Note that V has at most countable dimension because of $\bigcup_{z\in\mathcal{S}} W_z = V$. Hence Λ is at most countable.

We will prove that for any $\lambda, \mu \in \Lambda$ we can take $a(\lambda,\mu) \in \mathcal{S}$ such that $V_\lambda, V_\mu \subset W_{a(\lambda,\mu)}$. Take $\lambda, \mu \in \Lambda$. By the assumption $\bigcup_{z\in\mathcal{S}} W_z = V$, we have

$$\bigcup_{z\in\mathcal{S}} (W_z \cap (V_\lambda \oplus V_\mu)) = V_\lambda \oplus V_\mu.$$

Since $\mathcal{S}$ is a countable set and $V_\lambda \oplus V_\mu$ is finite-dimensional, there exists $a(\lambda,\mu) \in \mathcal{S}$ satisfying $V_\lambda \oplus V_\mu = W_{a(\lambda,\mu)} \cap (V_\lambda \oplus V_\mu)$. This implies $V_\lambda \oplus V_\mu \subset W_{a(\lambda,\mu)}$.

We will construct a countable subset $N_\lambda^1 \subset \mathbb{C}$ satisfying the condition (1). Take $\mu \in \Lambda$ and $a = a(\lambda,\mu)$. By the assumption of $\mathcal{S}$, W_a is an irreducible $\pi_a(\mathfrak{g})$-module. Therefore, we have $V_\mu \subset \pi_a(\mathcal{U}(\mathfrak{g}))V_\lambda$. By Lemma 4.4, there exists a finite subset $N_{\lambda,\mu} \subset \mathbb{C}$ such that $V_\mu \subset \pi_z(\mathcal{U}(\mathfrak{g}))V_\lambda$ for any $z \in \mathbb{C}\backslash N_{\lambda,\mu}$. We set $N_\lambda^1 := \bigcup_{\mu\in\Lambda} N_{\lambda,\mu}$. Then it is clear that N_λ^1 satisfies the condition (1).

We will construct a countable subset $N_\lambda^2 \subset \mathbb{C}$ satisfying the condition (2). We take $a \in \mathcal{S}$ such that $W_a \supset V_\lambda$. Since W_a is an irreducible $\pi_a(\mathfrak{g})$-module, V_λ is an irreducible $\pi_a(\mathcal{U}(\mathfrak{k})\mathcal{U}(\mathfrak{g})^{\mathfrak{k}})$-module. Since $\dim_{\mathbb{C}}(V_\lambda) < \infty$, we have

$$\pi_a(\mathcal{U}(\mathfrak{k})\mathcal{U}(\mathfrak{g})^{\mathfrak{k}})|_{V_\lambda} = \mathrm{End}_{\mathbb{C}}(V_\lambda)$$

by the Jacobson density theorem. We define a family of algebra homomorphisms:

$$\tau_z := \pi_z(\cdot)|_{V_\lambda} : \mathcal{U}(\mathfrak{k})\mathcal{U}(\mathfrak{g})^{\mathfrak{k}} \to \mathrm{End}_{\mathbb{C}}(V_\lambda).$$

Since τ_z is a polynomial function of z and τ_a is surjective, there exists a finite subset $N_\lambda^2 \subset \mathbb{C}$ such that τ_z is surjective for any $z \in \mathbb{C}\backslash N_\lambda^2$. Then N_λ^2 satisfies the condition (2).

It is obvious that $N_\lambda := N_\lambda^1 \cup N_\lambda^2$ satisfies the two conditions (1) and (2). We set $N := \bigcup_{\lambda\in\Lambda} N_\lambda$. Then N is an at most countable set. We will show that V is an irreducible $\pi_z(\mathfrak{g})$-module for any $z \in \mathbb{C}\backslash N$. Take $z' \in \mathbb{C}\backslash N$ and a non-zero submodule $V' \subset V$. By the irreducibility condition (2), we have $V_\lambda \subset V'$ for some $\lambda \in \Lambda$. By the condition (1), V_λ generates V as a $\pi_{z'}(\mathfrak{g})$-module. Thus we have $V' = V$. This implies that V is irreducible as a $\pi_{z'}(\mathfrak{g})$-module.

We complete the proof of Theorem 4.5. $\square$

4.2 Polynomial identity degree

The polynomial identity degree (see Section 2.6) is defined by polynomials. Hence the polynomial identity degree of $\pi_z(\mathcal{U}(\mathfrak{g}))$ behaves well for the polynomially dependent family of representations. Retain the notation in the previous section.

Lemma 4.6. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{U}(\mathfrak{g})$. Let (π_z, V) be a family of $\mathfrak{g}$-modules depending on z polynomially. Set*

$$n := \sup_{z\in\mathbb{C}} \{\mathrm{PI.deg}(\pi_z(\mathcal{A}))\}.$$

Then for any positive integer $n' < n$, the set $\{z \in \mathbb{C} : \mathrm{PI.deg}(\pi_z(\mathcal{A})) \leq n'\}$ is a finite set. In particular, for any Zariski dense subset $X \subset \mathbb{C}$, we have $n = \sup_{z\in X} \{\mathrm{PI.deg}(\pi_z(\mathcal{A}))\}$.

Proof. Take a positive integer $n' < n$. By the definition of PI.deg (see Definition 2.25), we can take $A_1, A_2, \ldots, A_{2n'} \in \mathcal{A}$ and $a \in \mathbb{C}$ satisfying

$$\pi_a(s_{2n'}(A_1, A_2, \ldots, A_{2n'})) \neq 0.$$

Then $f(z) := \pi_z(s_{2n'}(A_1, A_2, \ldots, A_{2n'}))$ is a non-zero $\mathrm{End}_{\mathbb{C}}(V)$-valued polynomial of z. By the definition of PI.deg, we have

$$\{z \in \mathbb{C} : \mathrm{PI.deg}(\pi_z(\mathcal{A})) \leq n'\} \subset \{z \in \mathbb{C} : f(z) = 0\}.$$

Since $f(z)$ is a polynomial of z, $\{z \in \mathbb{C} : f(z) = 0\}$ is a finite set. This shows the lemma. $\square$

Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$, and (π_z, V) be an admissible family of $(\mathfrak{g}, \mathfrak{k})$-modules depending on z polynomially. Take a $\mathfrak{k}$-stable subspace $W \subset V$ and countable subset $\mathcal{S} \subset \mathbb{C}$. We put

$$W_z := \pi_z(\mathcal{U}(\mathfrak{g}))W.$$

Suppose that W_s is a finite-dimensional irreducible $\mathfrak{g}$-module for any $s \in \mathcal{S}$ and $\bigcup_{s\in\mathcal{S}} W_s = V$. By Theorem 4.5, the second condition is equivalent to the condition that (π_z, V) is irreducible for some $z \in \mathbb{C}$.

For $s \in \mathcal{S}$, since W_s is a finite-dimensional irreducible $\mathfrak{g}$-module, $W_s|_{\mathfrak{g}'}$ is completely reducible. We write $\mathcal{M}_{\mathfrak{g}}(W_s)$ for the maximum value of the multiplicities in $W_s|_{\mathfrak{g}}$.

Theorem 4.7. *Under the above settings, we have*

$$\mathrm{PI.deg}(\pi_z(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})) \leq \sup_{s\in\mathcal{S}} \mathcal{M}_{\mathfrak{g}'}(W_s)$$

for any $z \in \mathbb{C}$. Furthermore, there exists an at most countable subset $N \subset \mathbb{C}$ such that the equality holds for any $z \in \mathbb{C}\backslash N$.

Proof. For $X \subset \mathbb{C}$ and $Y \subset \mathcal{S}$, we define

$$\begin{aligned} PI(X) &:= \sup_{z \in X} \mathrm{PI.deg}(\pi_z(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})), \\ \mathcal{M}(Y) &:= \sup_{s \in Y} \mathcal{M}_{\mathfrak{g}'}(W_s). \end{aligned}$$

Since $\mathcal{S}$ is countable, Lemma 4.6 implies $PI(\mathbb{C}) = PI(\mathcal{S})$. By Lemma 4.6, the two assertions are equivalent to $PI(\mathbb{C}) = \mathcal{M}(\mathcal{S})$. Therefore, it suffices to show $PI(\mathcal{S}) = \mathcal{M}(\mathcal{S})$.

We denote by $(\bar{\pi}_s, W_s)$ the $\mathfrak{g}$-module on W_s. By Proposition 2.27, we have

$$\mathcal{M}(\mathcal{S}) = \sup_{s \in \mathcal{S}} \mathrm{PI.deg}(\bar{\pi}_s(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})).$$

We set $n := \sup_{s \in \mathcal{S}} \mathrm{PI.deg}(\bar{\pi}_s(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'}))$. We will show $n = PI(\mathcal{S})$. It is clear that $n \leq PI(\mathcal{S})$ because the natural homomorphism $\pi_s(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'}) \to \bar{\pi}_s(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})$ is surjective.

For the converse inequality, we take $A_1, A_2, \ldots, A_{2n} \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}'}$. By the definition of PI.deg, we have

$$\bar{\pi}_s(s_{2n}(A_1, A_2, \ldots, A_{2n})) = 0$$

for any $s \in \mathcal{S}$. By Theorem 4.5, for any countable subset $\mathcal{S}' \subset \mathcal{S}$, we have $\bigcup_{s \in \mathcal{S}'} W_s = V$. This implies that $\mathcal{S}_v := \{s \in \mathcal{S} : v \in W_s\}$ is countable for any $v \in V$. Since $f(z) := \pi_z(s_{2n}(A_1, A_2, \ldots, A_{2n}))v$ is a V-valued polynomial of z and $f \equiv 0$ on the countable set $\mathcal{S}_v$, we have $f \equiv 0$ on $\mathbb{C}$. Therefore, we obtain $\pi_z(s_{2n}(A_1, A_2, \ldots, A_{2n})) = 0$ for any $z \in \mathbb{C}$, and hence $PI(\mathcal{S}) \leq n$. We complete the proof. □

Corollary 4.8. *Retain the notation in Theorem* 4.7. *Then the following two conditions are equivalent:*

(a) $\pi_z(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})$ *is commutative for any* $z \in \mathbb{C}$*;*

(b) W_s *is a multiplicity-free* $\mathfrak{g}'$*-module for any* $s \in \mathcal{S}$*.*

Proof. By the definition of PI.deg and $\mathcal{M}_{\mathfrak{g}'}$,

- $\mathrm{PI.deg}(\pi_z(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})) = 1 \iff \pi_z(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}'})$ is commutative, and
- $\mathcal{M}_{\mathfrak{g}'}(W_s) = 1 \iff W_s$ is multiplicity-free as a $\mathfrak{g}'$-module, for $s \in \mathcal{S}$.

Therefore, the assertion is a consequence of Theorem 4.7. □

4.3 Discretely decomposable generalized Verma modules

Let $\mathfrak{g}_{\mathbb{R}}$ be a real simple Lie algebra of Hermitian type with Cartan involution θ. Put $\mathfrak{k}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}}^{\theta}$. We fix a characteristic element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_{\mathbb{R}})$. As in Section 2.3, we construct $\mathfrak{q}, \mathfrak{p}_+, \mathfrak{p}_-$ and so on. Then we have

$$\begin{aligned}\mathfrak{g} &= \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-,\\ \mathfrak{q} &= \mathfrak{p}_+ \oplus \mathfrak{k}.\end{aligned}$$

We fix a unitary character ζ of $\mathfrak{k}_{\mathbb{R}}$ such that $\zeta(H) = 1$.

Let F be a finite-dimensional irreducible unitary representation of $\mathfrak{k}_{\mathbb{R}}$ with infinitesimal character λ. Then we define

$$M(z) := \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F \otimes \mathbb{C}_{z\zeta}).$$

We put

$$\begin{aligned}Z_{hol} &:= \{z \in \mathbb{C} : (\lambda + \rho(\mathfrak{p}_+), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\},\\ Z_{fin} &:= \left\{z \in \mathbb{C} : \frac{2(\lambda + \rho(\mathfrak{p}_+), \alpha)}{(\alpha, \alpha)} \in \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\right\}.\end{aligned}$$

Then if $z \in Z_{hol}$, $M(z)$ is the underlying Harish-Chandra module of a holomorphic discrete series representation, and hence irreducible (see Fact 2.12). $M(z)$ contains a unique irreducible finite-dimensional representation of $\mathfrak{g}$ by the Borel–Weil–Bott theorem. In general, $M(z)$ has a unique irreducible submodule. We denote by $L(z)$ the unique irreducible submodule of $M(z)$.

Next, we consider $M(z)$ as a family of representations depending on z polynomially (see Section 4.1). To do so, we identify the representation spaces $M(z)$ with a space of polynomial functions as follows. We fix an identification of $\mathbb{C}_{z\zeta}$ with $\mathbb{C}$. Then we have the following natural isomorphisms as vector spaces:

$$\begin{aligned}M(z) &= \mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), F \otimes \mathbb{C}_{z\zeta})_{\mathfrak{k}}\\ &\simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{U}(\mathfrak{p}_+), F \otimes \mathbb{C}_{z\zeta})_{\mathfrak{k}}\\ &\simeq \mathbb{C}[\mathfrak{p}_+] \otimes F \otimes \mathbb{C}_{z\zeta}\\ &\simeq \mathbb{C}[\mathfrak{p}_+] \otimes F.\end{aligned}$$

Thus we identify the representation spaces $M(z)$ with $\mathbb{C}[\mathfrak{p}_+] \otimes F$. We denote by π_z the $\mathfrak{g}$-action on $\mathbb{C}[\mathfrak{p}_+] \otimes F$ induced from the action on $M(z)$.

By the above identification, it is easy to see that $(\pi_z, \mathbb{C}[\mathfrak{p}_+] \otimes F)$ is a family of representations depending on z polynomially. More precisely, the

$\mathfrak{p}_+$-action is the derivation on $\mathbb{C}[\mathfrak{p}_+]$, and the $[\mathfrak{k}, \mathfrak{k}]$-action is independent of z. Hence the family $(\pi_z, \mathbb{C}[\mathfrak{p}_+] \otimes F)$ is admissible (Definition 4.2). Using the identification, we consider $L(z)$ as a subspace of $\mathbb{C}[\mathfrak{p}_+] \otimes F$.

Let $\mathfrak{g}'_{\mathbb{R}}$ be a θ-stable reductive subalgebras of $\mathfrak{g}_{\mathbb{R}}$. For a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, we write $\mathfrak{s}'$ for the intersection of $\mathfrak{s}$ with $\mathfrak{g}'$. We assume that $\mathfrak{g}'$ contains H as an element. The branching law of a holomorphic discrete series representation is well-known. The following fact proved by Jakobsen–Vergne [29] is useful to see the branching law (see also [52]).

Fact 4.9. *Suppose $z \in Z_{hol}$. Then $M(z)|_{\mathfrak{g}'}$ is completely reducible and each direct summand is a unitary highest weight module of $\mathfrak{g}'$. Moreover, suppose $\mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F$ has the following irreducible decomposition as a $\pi_z(\mathfrak{k}')$-module:*

$$\mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F \simeq \bigoplus_{(\mu, F_\mu) \in \widehat{\mathfrak{k}'}} m(\mu) F_\mu.$$

Then $M(z)|_{\mathfrak{g}'}$ has the following irreducible decomposition as a $\mathfrak{g}'$-module:

$$M(z)|_{\mathfrak{g}'} \simeq \bigoplus_{(\mu, F_\mu) \in \widehat{\mathfrak{k}'}} m(\mu) \mathrm{pro}^{\mathfrak{g}'}_{\bar{\mathfrak{q}}'}(F_\mu).$$

Remark 4.10. By [47, Corollary 8.7], each irreducible component of $M(z)|_{\mathfrak{g}'}$ is the underlying Harish-Chandra module of a holomorphic discrete series representation.

The correspondence of the two irreducible decompositions in the fact is as follows. Since $M(z)|_{\mathfrak{g}'}$ is the direct sum of irreducible highest weight modules, its irreducible decomposition is completely determined by the $\mathfrak{p}'_+$-invariant part. The $\mathfrak{p}'_+$-action is the derivative on the polynomial $\mathbb{C}[\mathfrak{p}_+] \otimes F$. Thus we have

$$\begin{aligned} M(z)^{\mathfrak{p}'_+} \simeq& (\mathbb{C}[\mathfrak{p}_+] \otimes F)^{\mathfrak{p}'_+} \\ \simeq& \mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F. \end{aligned}$$

Thus the irreducible decomposition of $\mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F$ determines the irreducible decomposition of $M(z)|_{\mathfrak{g}'}$ as in Fact 4.9. By this observation and Theorem 4.5, we have

Theorem 4.11. *Let F' be a direct summand of $\mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F$ as a $\pi_0(\mathfrak{k}')$-module. Fix $z_0 \in Z_{hol}$. Let $L'(z)$ denote the unique irreducible submodule of $\mathrm{pro}^{\mathfrak{g}'}_{\bar{\mathfrak{q}}'}(F' \otimes \mathbb{C}_{z\zeta})$. Then we have*

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'}(M(z_0), \mathrm{pro}^{\mathfrak{g}'}_{\bar{\mathfrak{q}}'}(F' \otimes \mathbb{C}_{z_0\zeta})) \\ =& \max \{\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'}(L(z), L'(z)) : z \in Z_{fin}\}. \end{aligned}$$

Proof. By Theorem 4.5, we have

$$\bigcup_{z \in Z_{fin}} L(z) = \mathbb{C}[\mathfrak{p}_+] \otimes F,$$
$$\bigcup_{z \in Z_{fin}} L(z)^{\mathfrak{p}'_+} = \mathbb{C}[\mathfrak{p}_+/\mathfrak{p}'_+] \otimes F.$$

This implies that

$$\begin{aligned}&\max\left\{\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{k}'}(L(z)^{\mathfrak{p}'_+}, L'(z)^{\mathfrak{p}'_+}) : z \in Z_{fin}\right\}\\ =&\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{k}'}(M(z_0)^{\mathfrak{p}'_+}, L'(z_0)^{\mathfrak{p}'_+}).\end{aligned}$$

Since $L(z)|_{\mathfrak{g}'}$ is completely reducible, its irreducible decomposition is determined by the $\mathfrak{p}'_+$-invariant part. This and Fact 4.9 show the assertion. □

The following corollary is a direct consequence of the theorem. The corollary is useful to classify multiplicity-free restrictions of holomorphic discrete series representations.

Corollary 4.12. *For $z_0 \in Z_{hol}$, we have*

$$\mathcal{M}_{\mathfrak{g}'}(M(z_0)) = \max\left\{\mathcal{M}_{\mathfrak{g}'}(L(z)) : z \in Z_{fin}\right\}.$$

At the last of this section, we compare two branching laws for two subalgebras with isomorphic complexifications. Let $\mathfrak{g}''_{\mathbb{R}}$ be θ-stable reductive subalgebras of $\mathfrak{g}_{\mathbb{R}}$. We assume the following two assumptions:

- $\mathfrak{g}''$ contains H as an element;
- there exists an element c of $\mathrm{Int}(\mathfrak{g})$ such that

$$c(\mathfrak{g}') = \mathfrak{g}''.$$

Any finite-dimensional representation V of $\mathfrak{g}$ can be lifted to a representation of a simply-connected connected complex algebraic group G with Lie algebra $\mathfrak{g}$. By assumption, there is an element $g \in G$ such that $\mathrm{Ad}(g)(\mathfrak{g}') = \mathfrak{g}''$. Hence the branching law of $V|_{\mathfrak{g}'_{\mathbb{R}}}$ is essentially the same as the branching law of $V|_{\mathfrak{g}''_{\mathbb{R}}}$. This shows the following corollary.

Corollary 4.13. *For $z_0 \in Z_{hol}$, we have*

$$\mathcal{M}_{\mathfrak{g}'}(M(z_0)) = \mathcal{M}_{\mathfrak{g}''}(M(z_0)).$$

5 Generalized Verma modules and the Zuckerman derived functor

In this section, we treat modules cohomologically induced from generalized Verma modules. Our main purpose is to show the vanishing theorem (Lemma 5.2) and the existence of a cyclic subspace (Lemma 5.5). The cohomological induction of generalized Harish-Chandra modules is studied by I. Penkov and G. Zuckerman [85, 86, 87]. The proofs in this section are essentially the same as in Knapp–Vogan's book [40, Chapter VIII], Wallach's book [104, Chapter 6] and Penkov–Zuckerman's papers.

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Fix an element $H \in \mathfrak{g}'$ such that H is semisimple in $\mathfrak{g}$ and $\mathrm{ad}(H)$ has real eigenvalues on $\mathfrak{g}$. As in Section 2.2, we define $\mathfrak{u} := \mathfrak{u}(H), \mathfrak{l} := \mathfrak{l}(H)$ and $\bar{\mathfrak{u}} := \bar{\mathfrak{u}}(H)$. We set $\mathfrak{u}' := \mathfrak{u} \cap \mathfrak{g}', \mathfrak{l}' := \mathfrak{l} \cap \mathfrak{g}'$ and $\bar{\mathfrak{u}}' := \bar{\mathfrak{u}} \cap \mathfrak{g}'$. Then $\mathfrak{q} := \mathfrak{u} \oplus \mathfrak{l}$ and $\mathfrak{q}' := \mathfrak{u}' \oplus \mathfrak{l}'$ are parabolic subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. We also define $\bar{\mathfrak{q}} := \bar{\mathfrak{u}} \oplus \mathfrak{l}$ and $\bar{\mathfrak{q}}' := \bar{\mathfrak{u}}' \oplus \mathfrak{l}'$.

We fix a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{l}'$ and a Borel subalgebra $\mathfrak{b}' \subset \mathfrak{q}'$ of $\mathfrak{g}'$ containing $\mathfrak{h}'$. We extend $\mathfrak{h}'$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$. Then we have $H \in \mathfrak{h}', \mathfrak{h}$. Remark that any generalized Verma module $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)$ is $\mathfrak{l}'$-admissible, and hence $\mathfrak{g}'$-admissible (see [53]).

We set $S := \dim_{\mathbb{C}}(\mathfrak{u}')$.

Fix a connected complex reductive algebraic group G' with Lie algebra $\mathfrak{g}'$ and representation on $\mathfrak{g}$ compatible with the adjoint action of $\mathfrak{g}'$. Then $(\mathfrak{g}, G')$ is a pair (see Definition 2.1). Let L' be the centralizer of H in G'. Then L' is a connected reductive algebraic group with Lie algebra $\mathfrak{l}'$.

5.1 Vanishing theorem

We will prove the vanishing theorem of $R^i\Gamma^{G'}_{L'}$. The following fact is in [104, Lemma 6.4.4].

Fact 5.1. *Let M be a $(\mathfrak{g}', L')$-module with standard filtration M. (see Definition 2.8). Then we have $R^d\Gamma^{G'}_{L'}(M) = 0$ for $d < S$.*

The following corollary is a direct consequence of the above fact and Proposition 2.10. The corollary is used to prove the exactness of some functor defined by the Zuckerman derived functor (see Theorem 6.4).

Corollary 5.2. *Let F be an irreducible finite-dimensional $\mathfrak{l}$-module. Suppose that F lifts to a representation of L'. Then we have*

$$R^d\Gamma^{G'}_{L'}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) = 0 \textit{ for } d < S.$$

Moreover, if $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ *is irreducible, then*

$$R^d\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) = 0 \text{ for } d \neq S.$$

Proof. The first assertion is a direct consequence of Proposition 2.10 and Fact 5.1.

Assume $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is irreducible. Then we have $(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)^*)_{\mathfrak{l}'} \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F^*)$. Hence there is a non-degenerate bilinear pairing between $R^d\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ and $R^{2S-d}\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F^*))$. Using the first assertion for $R^{2S-d}\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F^*))$, we have

$$R^{2S-d}\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F^*)) = 0 \text{ for } 2S-d < S.$$

This leads $R^d\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) = 0$ for $d > S$. We have proved the second assertion. □

5.2 Cyclic subspace

In this section, we prove that there is a cyclic subspace in the Zuckerman derived functor module under some dominance condition. The proofs in this section is essentially the same as in [104, Section 6.6]. Retain the notation in the previous section.

Lemma 5.3. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module. Suppose $F|_{\mathfrak{l}'}$ is irreducible and lifts to a representation of L', and $F|_{\mathfrak{l}'}$ has an infinitesimal character λ'. We assume $(\lambda' + \rho(\mathfrak{u}'), \beta) < 0$ for any $\beta \in \Delta(\mathfrak{u}', \mathfrak{h}')$. Then there exists a unique finite-dimensional irreducible G'-subrepresentation W_0 of $R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ with infinitesimal character $\lambda' + \rho(\mathfrak{u}')$.*

Proof. We define a $\mathfrak{g}'$-submodule $W := \mathcal{U}(\mathfrak{g}')(1\otimes F)$ of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$. $R^S\Gamma_{L'}^{G'}(W)$ is a finite-dimensional irreducible representation of G' with infinitesimal character $\lambda' + \rho(\mathfrak{u}')$ because W is isomorphic to $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F)$ and F satisfies the dominance condition.

We will show that W is a unique submodule of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ with generalized infinitesimal character $\lambda'+\rho(\mathfrak{u}')$. If we prove this, we can see that the natural map $R^S\Gamma_{L'}^{G'}(W) \to R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is injective, and we can take the image of the map as W_0 in the assertion.

Set $\bar{\mathfrak{u}}'' := \bar{\mathfrak{u}} \cap (\mathfrak{g}')^{\perp}$ and $\mathfrak{u}'' := \mathfrak{u} \cap (\mathfrak{g}')^{\perp}$. By Proposition 2.10, there exists a $\mathfrak{g}'$-module filtration $M.$ of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ with $\bigcup_i M_i = \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ and

$$\mathrm{gr}(M.) \simeq \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes S(\bar{\mathfrak{u}}'')),$$

where $\mathfrak{u}'$ acts on $\bar{\mathfrak{u}}''$ trivially.

Take a $\mathfrak{g}'$-submodule W' of $\mathrm{gr}(M.)$ with infinitesimal character $\lambda' + \rho(\mathfrak{u}')$. It is enough to show $W' = \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes 1)$ in $\mathrm{gr}(M.)$. Since W' is a submodule of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes S(\bar{\mathfrak{u}}''))$, $\lambda' + \rho(\mathfrak{u}')$ is of the form $\lambda' + \rho(\mathfrak{u}') - Q$ modulo the Weyl group action of $\mathfrak{g}'$. Here Q is a sum of elements of $\Delta(\mathfrak{u}'', \mathfrak{h}')$. Hence we can take an element $s \in W_{\mathfrak{g}'}$ such that

$$\lambda' + \rho(\mathfrak{u}') - Q = s(\lambda' + \rho(\mathfrak{u}')).$$

Since $\lambda' + \rho(\mathfrak{u}')$ is a dominant integral weight with respect to $\Delta^+(\mathfrak{l}', \mathfrak{h}') \cap -\Delta(\mathfrak{u}', \mathfrak{h}')$, we have

$$s(\lambda' + \rho(\mathfrak{u}')) = \lambda' + \rho(\mathfrak{u}') - R + Q',$$

where R is a sum of elements of $\Delta^+(\mathfrak{l}', \mathfrak{h}')$ and Q' is a sum of elements of $\Delta(\mathfrak{u}', \mathfrak{h}')$. By the above two equations, we obtain $Q + Q' - R = 0$. By the definition of H and $\mathfrak{u}$, $R(H) = 0$ and $Q(H), Q'(H) > 0$ holds if Q and Q' are non-zero. Thus we have $Q = Q' = 0$. This implies $W' = \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes 1)$. We have completed the proof. $\square$

The above lemma says that the *bottom-layer* (see [40, Chapter V]) of $R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ does not vanish if F satisfies the dominance condition. We will prove that the subspace W_0 in the lemma generates $R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ as a $\mathfrak{g}$-module under some dominance condition. The following fact is well-known (see e.g. [104, Lemma 6.A.1.3]).

Fact 5.4 (Koszul resolution)**.** *Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$. Let V be a $\mathfrak{g}$-module. Set $n := \dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{g}')$. Then there is an exact sequence of $\mathfrak{g}$-modules:*

$$\begin{aligned} 0 \xrightarrow{\partial_n} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} (\wedge^n(\mathfrak{g}/\mathfrak{g}') \otimes V) &\xrightarrow{\partial_{n-1}} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} (\wedge^{n-1}(\mathfrak{g}/\mathfrak{g}') \otimes V) \\ \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} (\mathfrak{g}/\mathfrak{g}' \otimes V) &\xrightarrow{\partial_0} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} V \xrightarrow{\epsilon} V \to 0. \end{aligned}$$

The last homomorphism ϵ is the multiplication map.

The following lemma is the main result of this section. We use the lemma to prove Theorem 6.1.

Lemma 5.5. *Retain the notation in Lemma* 5.3. *Suppose*

$$(\lambda' + \sum_{\alpha \in E} \alpha + \rho(\mathfrak{u}'), \beta) < 0$$

for any $\beta \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $E \subset \Delta(\mathfrak{q}/\mathfrak{q}', \mathfrak{h}')$. Take W_0 as in Lemma 5.3. *Then W_0 generates $R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ as a $\mathfrak{g}$-module.*

Proof. We apply Fact 5.4 to $\mathfrak{q}$, $\mathfrak{q}'$ and F. Then we have an exact sequence of $\mathfrak{q}$-modules:

$$\begin{aligned}0 \xrightarrow{\partial_n} \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{q}')} (\wedge^n(\mathfrak{q}/\mathfrak{q}') \otimes F) &\xrightarrow{\partial_{n-1}} \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{q}')} (\wedge^{n-1}(\mathfrak{q}/\mathfrak{q}') \otimes F) \\ &\xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{q}')} (\mathfrak{q}/\mathfrak{q}' \otimes F) \xrightarrow{\partial_0} \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{q}')} F \\ &\xrightarrow{\epsilon} F \to 0.\end{aligned}$$

Applying an exact functor $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} (\cdot)$ to the above exact sequence, we obtain an exact sequence of $\mathfrak{g}$-modules:

$$\begin{aligned}0 \xrightarrow{\partial_n} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_n &\xrightarrow{\partial_{n-1}} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_{n-1} \\ \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_1 &\xrightarrow{\partial_0} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_0 \xrightarrow{\epsilon} \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F) \to 0,\end{aligned}$$

where we set $E_i := \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(\wedge^i(\mathfrak{q}/\mathfrak{q}') \otimes F)$. We take $W = \mathcal{U}(\mathfrak{g}')(1 \otimes F) \subset \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ as in the proof of Lemma 5.3. Then we have $\epsilon(1 \otimes E_0) = W$.

We will show that $R^S\Gamma_{L'}^{G'}(\epsilon)$ is surjective. By the dominance condition, Proposition 2.9 and Fact 2.7, E_i is completely reducible. Since $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_i$ is isomorphic to $S((\mathfrak{g}')^\perp) \otimes E_i$ as a $\mathfrak{g}'$-module, Corollary 5.2 implies

$$R^d\Gamma_{L'}^{G'}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_i) \simeq S((\mathfrak{g}')^\perp) \otimes R^d\Gamma_{L'}^{G'}(E_i) = 0$$

for any $d \neq S$. Hence the above exact sequence induces the following exact sequences:

$$\begin{aligned}0 \to R^S\Gamma_{L'}^{G'}(\mathrm{Im}(\partial_0)) \to R^S\Gamma_{L'}^{G'}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} E_0) &\xrightarrow{R^S\Gamma(\epsilon)} R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \\ &\to R^{S+1}\Gamma_{L'}^{G'}(\mathrm{Im}(\partial_0)) \to 0 \\ 0 \to R^d\Gamma_{L'}^{G'}(\mathrm{Im}(\partial_i)) \to R^{d+1}&\Gamma_{L'}^{G'}(\mathrm{Im}(\partial_{i+1})) \to 0 \text{ for } d > S.\end{aligned}$$

The second exact sequence implies $R^d\Gamma_{L'}^{G'}(\mathrm{Im}(\partial_i)) = 0$ for $d > S$. From this and the first exact sequence, $R^S\Gamma_{L'}^{G'}(\epsilon)$ is surjective.

By the isomorphisms $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(g')} E_0 \simeq S((\mathfrak{g}')^\perp) \otimes E_0$ and $R^S\Gamma_{L'}^{G'}(S((\mathfrak{g}')^\perp) \otimes E_0) \simeq S((\mathfrak{g}')^\perp) \otimes R^S\Gamma_{L'}^{G'}(E_0)$, we have the following commutative diagram:

$$\begin{tikzcd} S((\mathfrak{g}')^\perp) \otimes R^S\Gamma_{L'}^{G'}(E_0) \arrow[d, "\simeq"] \arrow[dr, "m"] & \\ R^S\Gamma_{L'}^{G'}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(g')} E_0) \arrow[r, "R^S\Gamma_{L'}^{G'}(\epsilon)"] & R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)), \end{tikzcd}$$

where m is the composition of the multiplication map and the symmetrization mapping (see Fact 2.16). Since $R^S\Gamma_{L'}^{G'}(\epsilon)$ is surjective, we have

$$\mathcal{U}(\mathfrak{g})W_0 = R^S\Gamma_{L'}^{G'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)).$$

This proves the lemma. $\square$

6 Embedding of category

The main concern in this section is to construct a category embedding from the BGG category to the category of generalized Harish-Chandra modules.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be complex reductive Lie algebra. We set $\mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We take a reductive subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that two projections $\mathfrak{g} \to \mathfrak{g}_2$ and $\mathfrak{g} \to \mathfrak{g}_2$ are injective on $\mathfrak{g}'$. In other words, $\mathfrak{g}'$ is a diagonal subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

We fix an element $H \in \mathfrak{g}'$ which is semisimple in $\mathfrak{g}$ with real eigenvalues. We define subalgebras $\mathfrak{q}, \mathfrak{u}, \mathfrak{q}', \mathfrak{u}', \mathfrak{h}, \mathfrak{h}', \ldots$ from H as in Section 5, and we set $\mathfrak{q}_i := \mathfrak{q} \cap \mathfrak{g}_i, \mathfrak{u}_i := \mathfrak{u} \cap \mathfrak{g}_i$ and so on. Then we have a canonical category equivalence between $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{q}}$ and $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1} \otimes \mathcal{O}^{\mathfrak{g}_2}_{\mathfrak{q}_2}$.

Let G' be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}'$ and Lie group action on $\mathfrak{g}$ compatible with the adjoint action of $\mathfrak{g}'$. We write L' for the centralizer of H in G'.

For a finite-dimensional irreducible $\mathfrak{l}_2$-module F with infinitesimal character λ, we define a full subcategory $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$ of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}$ satisfying that M is an object of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$ if and only if $M \otimes F$ lifts to a representation of L'. Then $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$ is closed under taking subquotient modules.

Our main purpose in this section is to show the following theorem. We denote by $\mathcal{L}(M)$ the *lattice of submodules* of M. Set $S := \dim_{\mathbb{C}}(\mathfrak{u}')$.

Theorem 6.1. *Let F be a finite-dimensional irreducible $\mathfrak{l}_2$-module with infinitesimal character λ satisfying*

$$\frac{2(\lambda + \rho(\mathfrak{u}_2), \alpha)}{(\alpha, \alpha)} \notin \{0, 1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2).$$

We denote by T the functor $R^S\Gamma^{G'}_{L'}(\cdot \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F))$ from $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$ to $\mathcal{C}(\mathfrak{g}, G')$. Then the functor T is exact, and induces a lattice isomorphism from $\mathcal{L}(M)$ to $\mathcal{L}(T(M))$ for each object M of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$.

Corollary 6.2. *Under the assumption in Theorem* 6.1*, the functor T is fully faithful, and maps irreducible objects to irreducible objects.*

Proof. Since $\mathcal{L}(M) \simeq \mathcal{L}(T(M))$, the second assertion is clear.

Take two objects M_1, M_2 of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. First, we will prove the faithfulness of T. Take $f \in \mathrm{Hom}_{\mathfrak{g}_1}(M_1, M_2)$ such that $T(f) = 0$. By the exactness of T, we have the following exact sequences:

$$M_1 \xrightarrow{f} \mathrm{Im}(f) \to 0$$
$$T(M_1) \xrightarrow{T(f)} T(\mathrm{Im}(f)) \to 0.$$

Since $T(f) = 0$ and $\mathcal{L}(T(\mathrm{Im}(f))) \simeq \mathcal{L}(\mathrm{Im}(f))$, we obtain $T(\mathrm{Im}(f)) = 0$ and $\mathrm{Im}(f) = 0$. This implies $f = 0$. Therefore, T is faithful.

Next, we will show that T is full. Let p_i be the projection from $M_1 \oplus M_2$ to M_i. Take a homomorphism f from $T(M_1)$ to $T(M_2)$. Consider

$$\mathrm{id}_{T(M_1)} \oplus f : T(M_1) \to T(M_1) \oplus T(M_2).$$

The image of this map is the graph of f. Since $\mathcal{L}(M) \simeq \mathcal{L}(T(M))$, there exists a unique submodule $N \subset M_1 \oplus M_2$ such that $T(N) = \mathrm{Im}(\mathrm{id}_{T(M_1)} \oplus f)$. Then N is a graph. In fact, since T is faithful and $T(p_1|_N)$ is bijective, $p_1|_N$ is also bijective.

We set $f' := p_2 \circ (p_1|_N)^{-1}$. Then we have $T(f') = T(p_2) \circ T((p_1|_N)^{-1}) = T(p_2) \circ (\mathrm{id}_{T(M_1)} \oplus f) = f$. Therefore, T is full. We have proved the corollary. $\square$

6.1 Exactness of T

In this section, we will prove the exactness in Theorem 6.1.

Lemma 6.3. *Let F be an irreducible finite-dimensional $\mathfrak{l}_2$-module with infinitesimal character λ, and V be an object of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. Then we have*

$$R^d\Gamma^{G'}_{L'}(V \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F)) = 0 \textit{ for } d < S.$$

Proof. Fix a non-negative integer $d < S$. By Proposition 2.10, we can take a $\mathfrak{g}'$-module standard filtration $M.$ of $\mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F)$. By Fact 2.17, it suffices to show $R^d\Gamma^{G'}_{L'}(V \otimes M_i) = 0$ for any $i \geq 0$. By induction on i and the exact sequence

$$R^d\Gamma^{G'}_{L'}(V \otimes M_i) \to R^d\Gamma^{G'}_{L'}(V \otimes M_{i+1}) \to R^d\Gamma^{G'}_{L'}(V \otimes (M_{i+1}/M_i)),$$

to prove the assertion, it is enough to show $R^d\Gamma^{G'}_{L'}(V \otimes (M_{i+1}/M_i)) = 0$ for any $i \geq 0$.

Take a non-negative integer i. Since $M.$ is a standard filtration, there is a finite-dimensional irreducible $\mathfrak{l}'$-module F such that $M_{i+1}/M_i \simeq \mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F)$. Then we have $V \otimes (M_{i+1}/M_i) \simeq \mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(V \otimes F)$. Since V is an object of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$, $V \otimes F$ is a completely reducible $\mathfrak{l}'$-module and locally finite $\mathfrak{u}'$-module. Thus $V \otimes (M_{i+1}/M_i)$ has a $\mathfrak{g}'$-module standard filtration. By Lemma 5.1, we have

$$R^d\Gamma^{G'}_{L'}(V \otimes (M_{i+1}/M_i)) = 0.$$

This completes the proof. $\square$

Using the lemma, we prove the exactness in Theorem 6.1.

Theorem 6.4. *Under the assumption of Lemma* 6.3*, if moreover we assume* $\mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$ *is irreducible, then we have*

$$R^d\Gamma_{L'}^{G'}(V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)) = 0 \textit{ for } d \neq S.$$

In particular, $R^S\Gamma_{L'}^{G'}$ *is exact on* $\mathcal{O}_{\mathfrak{q}_1}^{\mathfrak{g}_1}(\lambda) \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$*.*

Proof. We write T^d for the restriction of $R^d\Gamma_{L'}^{G'}$ to $\mathcal{O}_{\mathfrak{q}_1}^{\mathfrak{g}_1}(\lambda) \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$.

By Lemma 6.3, we have $T^d = 0$ for $d < S$. Let us show $T^d = 0$ for $d > S$. By assumption, $\mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$ is irreducible, and hence we have

$$\mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)^*_{\mathfrak{l}_2} \simeq \mathrm{ind}_{\bar{\mathfrak{q}}_2}^{\mathfrak{g}_2}(F^*).$$

Remark that $V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$ is an admissible $\mathfrak{l}'$-module. Therefore, $(V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F))^*_{\mathfrak{l}'}$ is an object of $\mathcal{O}_{\bar{\mathfrak{q}}_1}^{\mathfrak{g}_1} \otimes \mathrm{ind}_{\bar{\mathfrak{q}}_2}^{\mathfrak{g}_2}(F^*)$. Applying Lemma 6.3 to $(V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F))^*_{\mathfrak{l}'}$, we obtain

$$T^{2S-d}((V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F))^*_{\mathfrak{l}'}) = 0 \text{ for } d > S.$$

Since there is a non-degenerate $\mathfrak{g}$-invariant bilinear pairing between $T^d(V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F))$ and $T^{2S-d}((V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F))^*_{\mathfrak{l}'})$, we have

$$T^d(V \otimes \mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)) = 0 \text{ for } d > S.$$

This shows the assertion. □

6.2 Proof of embedding for $\mathfrak{g}_1 \simeq \mathfrak{g}'$

Retain the notation. The exactness of T in Theorem 6.1 has been proved in the previous section. In this section, we will prove the remaining part of the theorem under the assumption $\mathfrak{g}_1 \simeq \mathfrak{g}'$.

The proof is divided into the following four parts:

Step 1 reduction to the case $\mathrm{ind}_{\mathfrak{q}_2}^{\mathfrak{g}_2}(F)$ is of scalar type with enough small infinitesimal character;

Step 2 irreducibility of $T(M)$ for any irreducible object M;

Step 3 $T(M_1) \simeq T(M_2) \Rightarrow M_1 \simeq M_2$ for irreducible objects M_1, M_2;

Step 4 $\mathcal{L}(M) \simeq \mathcal{L}(T(M))$.

For the reduction in Step 1, we need the following lemma. The lemma is the special case of [40, Proposition 4.68].

Lemma 6.5. *The following inequations hold:*

$$(\rho(\mathfrak{u}_2), \alpha) > 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2),$$
$$(\rho(\mathfrak{u}_2)|_{\mathfrak{h}'}, \alpha) > 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}').$$

Proof. We will prove the first inequation. We omit the proof of the second inequation because the proof is the same. Fix $\alpha \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2)$. For $\beta \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2)$ if $(\beta, \alpha) < 0$, $s_\alpha(\beta)$ is in $\Delta(\mathfrak{u}_2, \mathfrak{h}_2)$. Hence we have

$$\rho(\mathfrak{u}_2) = \sum_{\substack{\beta \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2), \\ (\beta, \alpha) < 0}} (\beta + s_\alpha(\beta)) + \sum_{\substack{\beta \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2), \\ (\beta, \alpha) = 0}} \beta + \sum_{\substack{\beta \in \Delta(\mathfrak{u}_2, \mathfrak{h}_2), \\ (\beta, \alpha) > 0, \\ s_\alpha(\beta) \notin \Delta(\mathfrak{u}_2, \mathfrak{h}_2)}} \beta.$$

Note that α appears in the third sum. Since $(\beta + s_\alpha(\beta), \alpha) = 0$, we obtain $(\rho(\mathfrak{u}_2), \alpha) > 0$. $\square$

We will prove the remaining part of the proof of the theorem.

Step 1: reduction to scalar type case

Without loss of generality, we can assume that $\mathfrak{g}_2$ is semisimple and any finite-dimensional $\mathfrak{g}_2$-module can lift to a representation of G'.

Fix a set of positive roots $\Delta^+(\mathfrak{l}_2, \mathfrak{h}_2)$ of $\Delta(\mathfrak{l}_2, \mathfrak{h}_2)$. Then $\Delta^+(\mathfrak{g}_2, \mathfrak{h}_2) := \Delta^+(\mathfrak{l}_2, \mathfrak{h}_2) \cup -\Delta(\mathfrak{u}_2, \mathfrak{h}_2)$ is a set of positive roots of $\Delta(\mathfrak{g}_2, \mathfrak{h}_2)$. Recall that λ is the infinitesimal character of F. We can assume that λ is strictly anti-dominant with respect to $\Delta^+(\mathfrak{l}_2, \mathfrak{h}_2)$. By assumption, $\lambda + \rho(\mathfrak{u}_2)$ is regular and integrally anti-dominant (see Definition 2.19) with respect to $\Delta^+(\mathfrak{g}_2, \mathfrak{h}_2)$.

Set $(\mathfrak{l}_2)_{ss} := [\mathfrak{l}_2, \mathfrak{l}_2]$ and $\mathfrak{h}_{ss} := \mathfrak{h} \cap (\mathfrak{l}_2)_{ss}$. Since $-(\lambda + \rho(\mathfrak{l}_2))|_{\mathfrak{h}_{ss}}$ is a dominant integral weight of $(\mathfrak{l}_2)_{ss}$ and $\mathfrak{l}_2$ is a Levi subalgebra of $\mathfrak{g}_2$, there exists an algebraically integral weight μ of $\mathfrak{g}_2$ such that $\mu|_{\mathfrak{h}_{ss}} = -(\lambda + \rho(\mathfrak{l}_2))|_{\mathfrak{h}_{ss}}$. Hence $T_\lambda^{\lambda+\mu}(F)$ is a one-dimensional $\mathfrak{l}_2$-module. Set $\nu := \lambda + \mu + \rho(\mathfrak{l}_2)$. Then we have $T_\lambda^{\lambda+\mu}(F) \simeq \mathbb{C}_\nu$. Remark that $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) = \mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda + \mu)$ holds because $\mu|_{\mathfrak{h}'}$ is integral with respect to G'. This is because we have assumed that any finite-dimensional $\mathfrak{g}_2$-module lifts to a representation of G'.

By Lemma 6.5, if necessary, replacing μ by $\mu - 2m\rho(\mathfrak{u}_2)$ for enough large integer m, we may assume that $\nu = \lambda + \mu + \rho(\mathfrak{u}_2)$ is regular and integrally anti-dominant with respect to $\Delta^+(\mathfrak{g}_2, \mathfrak{h}_2)$. Using the translation functor $T_{\lambda+\rho(\mathfrak{u}_2)}^{\lambda+\mu+\rho(\mathfrak{u}_2)}$

and Fact 2.22, 2.24, we have

$$T^{\lambda+\mu+\rho(\mathfrak{u}_2)}_{\lambda+\rho(\mathfrak{u}_2)}(\mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F)) \simeq \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu),$$
$$T^{\lambda+\mu+\rho(\mathfrak{u}_2)}_{\lambda+\rho(\mathfrak{u}_2)}(R^S\Gamma^{G'}_{L'}(M \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F))) \simeq R^S\Gamma^{G'}_{L'}(M \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu)),$$
$$T^{\lambda+\rho(\mathfrak{u}_2)}_{\lambda+\mu+\rho(\mathfrak{u}_2)}(R^S\Gamma^{G'}_{L'}(M \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu))) \simeq R^S\Gamma^{G'}_{L'}(M \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F))$$

for any object M of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. Since $T^{\lambda+\rho(\mathfrak{u}_2)}_{\lambda+\mu+\rho(\mathfrak{u}_2)}$ is a left/right adjoint functor of $T^{\lambda+\mu+\rho(\mathfrak{u}_2)}_{\lambda+\rho(\mathfrak{u}_2)}$, the above isomorphisms imply the following category equivalence:

$$T^{\lambda+\mu+\rho(\mathfrak{u}_2)}_{\lambda+\rho(\mathfrak{u}_2)} : R^S\Gamma^{G'}_{L'}(\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F)) \to R^S\Gamma^{G'}_{L'}(\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu)),$$

and the commutative diagram:

$$\begin{tikzcd}
\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) \arrow[r, "T"] \arrow[d, equal] & R^S\Gamma^{G'}_{L'}(\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(F)) \arrow[d, "\simeq"] \\
\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda+\mu) \arrow[r, "T"] & R^S\Gamma^{G'}_{L'}(\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda) \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu)).
\end{tikzcd}$$

Therefore, we may assume that F is a one-dimensional $\mathfrak{l}$-module $\mathbb{C}_\nu$ with

$$(\nu|_{\mathfrak{h}'}, \alpha) \ll 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$$

by Lemma 6.5. In the following proof, replacing ν by enough small one, we can assume that the dominance condition of Lemma 5.5 is always satisfied.

Step 2: irreducibility of $T(M)$

Take an irreducible object M of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. Since M is an irreducible highest weight module, there exist a generalized Verma module $\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_1)$ and a surjective homomorphism $\eta : \mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_1) \to M$.

By the assumption $\mathfrak{g}' \simeq \mathfrak{g}_1$, $(F_1 \otimes \mathbb{C}_\nu)|_{\mathfrak{l}'}$ is irreducible. Since ν is enough small, we can apply Lemma 5.5 to $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F_1 \otimes \mathbb{C}_\nu)$. Then we can take a G'-irreducible subrepresentation $W_0 \subset T(\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_1))$ such that W_0 generates $T(\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_1))$ and $\mathrm{Hom}_{G'}(W_0, T(\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_1))) = 1$. This and the exactness of T imply that $T(\eta)(W_0) \neq 0$ and $T(M)$ is generated by $T(\eta)(W_0)$.

Take a proper submodule X of $T(M)$. Let us show $X = 0$. We consider the dual module $T(M)^*_{G'}$ and a non-degenerate paring between $T(M)$ and $T(M)^*_{G'}$. Since $T(\eta)(W_0)$ generates $T(M)$, X does not contain $T(\eta)(W_0)$, and hence $X^\perp$ contains $T(\eta)(W_0)^*$.

By assumption, M and $\mathrm{ind}^{\mathfrak{g}_2}_{\mathfrak{q}_2}(\mathbb{C}_\nu)$ are irreducible. Hence we have

$$T(M)^*_{G'} \simeq R^S\Gamma^{G'}_{L'}(M^*_{\mathfrak{l}_2} \otimes \mathrm{ind}^{\mathfrak{g}_2}_{\bar{\mathfrak{q}}_2}(\mathbb{C}_{-\nu})).$$

Applying Lemma 5.5 to $T(M)^*_{G'}$, we see that $T(\eta)(W_0)^*$ generates $T(M)^*_{G'}$. This implies $X^\perp = T(M)^*_{G'}$. Therefore, we have $X = 0$. This shows the irreducibility of $T(M)$.

Step 3: $T(M_1) \simeq T(M_2) \Rightarrow M_1 \simeq M_2$ **for irreducible objects** M_1, M_2

Take two irreducible objects M_1 and M_2 of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. We will show that $T(M_1) \simeq T(M_2)$ only if $T_1 \simeq T_2$.

Assume $T(M_1) \simeq T(M_2)$. As in Step 2, we take a generalized Verma module $\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_i)$ such that M_i is a unique irreducible quotient of $\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_i)$. Since ν is sufficiently small, we can apply Lemma 5.3 to $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F_i \otimes \mathbb{C}_\nu)$, and we take W_i as W_0 in the lemma.

Let λ_i be the infinitesimal character of $(F_i \otimes \mathbb{C}_\nu)|_{L'}$. We may assume that λ_i is strictly dominant with respect to $\Delta^+(\mathfrak{l}', \mathfrak{h}')$. By the assumption $T(M_1) \simeq T(M_2)$, we have $\mathrm{Hom}_{G'}(W_1, T(M_2)) = 1$. As in the proof of Lemma 5.3, there exist $s \in W_{\mathfrak{g}'}$ and P_1 a sum of elements of $\Delta(\mathfrak{u}_2, \mathfrak{h}')$ such that

$$\lambda_1 + \rho(\mathfrak{u}') = s(\lambda_2 - P_1 + \rho(\mathfrak{u}')). \tag{6.5.1}$$

Since $\lambda_1 + \rho(\mathfrak{u}')$ is dominant integral weight with respect to $\Delta^+(\mathfrak{l}', \mathfrak{h}') \cup -\Delta(\mathfrak{u}', \mathfrak{h}')$, we have

$$s^{-1}(\lambda_1 + \rho(\mathfrak{u}')) = \lambda_1 + \rho(\mathfrak{u}') - Q_1 \tag{6.5.2}$$

with Q_1 a sum of elements of $\Delta^+(\mathfrak{l}', \mathfrak{h}') \cup -\Delta(\mathfrak{u}', \mathfrak{h}')$. (6.5.1) and (6.5.2) lead to

$$\lambda_1 - \lambda_2 = -P_1 + Q_1.$$

Thus we have $\lambda_1(H) - \lambda_2(H) = -P_1(H) + Q_1(H) \leq 0$. Repeating the above discussion, we obtain $\lambda_2(H) - \lambda_1(H) = -P_2(H) + Q_2(H) \leq 0$. This implies $P_1 = P_2 = 0$ and $\lambda_1 + \rho(\mathfrak{u}') = s(\lambda_2 + \rho(\mathfrak{u}'))$.

Since $\lambda_1 + \rho(\mathfrak{u}')$ and $\lambda_2 + \rho(\mathfrak{u}')$ are strictly dominant with respect to $\Delta^+(\mathfrak{g}', \mathfrak{h}') \cup -\Delta(\mathfrak{u}', \mathfrak{h}')$, we have $s = 0$, and hence $\lambda_1 = \lambda_2$. This implies $F_1 \simeq F_2$. Since M_i is a unique irreducible quotient of $\mathrm{ind}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(F_i)$, we have $M_1 \simeq M_2$.

Step 4: $\mathcal{L}(M) \simeq \mathcal{L}(T(M))$

Take an object M of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}(\lambda)$. For any submodule N of M, we identify $T(N)$ with a submodule of $T(M)$. By the exactness of T, the induced map T :

$\mathcal{L}(M) \to \mathcal{L}(T(M))$ is an injective lattice homomorphism. In fact, for any two submodules $M_1, M_2 \subset M$, we have two exact sequences:

$$0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to M_1 + M_2 \to 0$$
$$0 \to T(M_1 \cap M_2) \to T(M_1) \oplus T(M_2) \to T(M_1 + M_2) \to 0.$$

This implies $T(M_1 \cap M_2) = T(M_1) \cap T(M_2)$ and $T(M_1 + M_2) = T(M_1) + T(M_2)$.

We will prove surjectivity of the lattice homomorphism T. We denote by $\operatorname{Len}(M)$ the length of M. If $\operatorname{Len}(M) = 1$, the surjectivity is obvious from Step 2. Hence we assume $\operatorname{Len}(M) \geq 2$. Note that the exactness and Step 2 imply $\operatorname{Len}(T(M)) = \operatorname{Len}(M) < \infty$.

Assume $T(\mathcal{L}(M)) \subsetneq \mathcal{L}(T(M))$. We consider triples

$$\begin{aligned}\mathcal{T} := \{(M_1, M_2, N) \in \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(T(M)) :\\ T(M_1) \subset N \subset T(M_2), N \notin T(\mathcal{L}(M))\}.\end{aligned}$$

Choose a triple $(M_1, M_2, N) \in \mathcal{T}$ such that $\operatorname{Len}(M_2) - \operatorname{Len}(M_1)$ takes the minimum value throughout $\mathcal{T}$. Replacing M by M_2/M_1 if necessary, we can assume $M_2 = M$ and $M_1 = 0$.

We will show $\operatorname{Len}(M) = 2$. Take a non-zero proper submodule $M' \subset M$. If $T(M') + N \notin T(\mathcal{L}(M))$, then $(M', M, T(M') + N) \in \mathcal{T}$. By the choice of $(0, M, N)$, we have $M' = 0$, and this is contradiction. Thus we have $T(M') + N \in T(\mathcal{L}(M))$ and $(0, T^{-1}(T(M') + N), N) \in \mathcal{T}$. This implies $T(M') + N = T(M)$. Repeating the same discussion for $T(M') \cap N$, we have $T(M') \cap N = 0$. Thus $T(M) = T(M') \oplus N$ holds for any non-zero proper submodule $M' \subset M$. This forces $\operatorname{Len}(M) = 2$ and $T(M)$ is completely reducible.

There are three possibilities for M:

(1) M is a direct sum of two distinct irreducible modules;

(2) M is a direct sum of two isomorphic irreducible modules;

(3) M has a unique non-zero proper submodule.

By Step 2 and 3, if M satisfies (1) or (2), $T(\mathcal{L}(M)) = \mathcal{L}(T(M))$ holds. Hence we can assume that M has a unique submodule M'.

If M' contains a non-zero weight vector with maximal weight of M, replacing M by $M^*_{\mathfrak{l}'}$ and M' by $(M')^\perp$, we can reduce this case to the following case.

If M' does not contain any non-zero weight vector with maximal weight of M, then M is a quotient of a generalized Verma module $\widetilde{M}$. From Lemma

5.5, $T(\widetilde{M})$ has a unique irreducible quotient and hence so does $T(M)$. By the above discussion, $T(M)$ is a direct sum of two irreducible submodules. This is contradiction. Therefore, we have $T(\mathcal{L}(M)) = \mathcal{L}(T(M))$. This completes the proof.

6.3 Extension of embedding

We extend the category embedding proved in the previous section. Keep the notation in the introduction of Section 6. We do not assume $\mathfrak{g}_1 \simeq \mathfrak{g}'$. We denote by $\mathfrak{g}'_1, \mathfrak{q}'_1, \ldots$ the image of $\mathfrak{g}', \mathfrak{q}', \ldots$ by the projection onto the first factor of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. By assumption, $\mathfrak{g}' \simeq \mathfrak{g}'_1$ holds.

Recall that for a $\mathfrak{g}'_1$-module M and an infinitesimal character χ, $P_\chi(M)$ denotes the maximum submodule of M with generalized infinitesimal character χ (see Section 2.5).

By the construction of $\mathfrak{q}_1$ and $\mathfrak{q}'_1$, any module M of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}$ is $\mathfrak{g}'_1$-admissible (see [53]). Hence any object of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}$ has the primary decomposition as a $\mathfrak{g}'_1$-module (Fact 2.18).

Lemma 6.6. *Let M be an object of $\mathcal{O}^{\mathfrak{g}_1}_{\mathfrak{q}_1}$. Then we have a $\mathfrak{g}'_1$-module decomposition:*

$$M = \bigoplus_{\chi} P_\chi(M),$$

where the sum is taken over all infinitesimal characters of $\mathfrak{g}'_1$. Moreover, each $P_\chi(M)$ is finite length and hence an object of $\mathcal{O}^{\mathfrak{g}'_1}_{\mathfrak{q}'_1}$.

Proof. The first assertion is clear from Fact 2.18.

Fix an infinitesimal character χ of $\mathfrak{g}'_1$. Since M is $\mathfrak{g}'_1$-admissible, so is $P_\chi(M)$. Any composition factor of $P_\chi(M)$ is an object of $\mathcal{O}^{\mathfrak{g}'_1}_{\mathfrak{q}'_1}$ because $\mathfrak{u}'_1$-action on $P_\chi(M)$ is locally nilpotent and $\mathfrak{l}'_1$-action is completely reducible. There are only finitely many equivalence classes of irreducible modules of $\mathcal{O}^{\mathfrak{g}'_1}_{\mathfrak{q}'_1}$ with infinitesimal character χ. Therefore, $P_\chi(M)$ is finite length. □

We will prove Theorem 6.1 using this lemma.

proof of Theorem 6.1. We have proved the exactness of T in Theorem 6.4. Then we will show that T induces a lattice isomorphism from $\mathcal{L}(M)$ to $\mathcal{L}(T(M))$. As in the proof in Section 6.2, the induced map $T : \mathcal{L}(M) \to \mathcal{L}(T(M))$ is an injective lattice homomorphism. It remains to prove the surjectivity of T.

Take a module M of $\mathcal{O}_{\mathfrak{q}_1}^{\mathfrak{g}_1}$ and a $\mathfrak{g}$-submodule X of $T(M)$. By Lemma 6.6, we have

$$M = \bigoplus_{\chi} P_{\chi}(M)$$
$$T(M) = \bigoplus_{\chi} T(P_{\chi}(M)).$$

Since the Zuckerman functor preserves generalized infinitesimal characters, $T(P_{\chi}(M)) = P_{\chi}(T(M))$ holds. Here in the right hand side, we consider $T(M)$ as a $\mathfrak{g}_1$-module to apply P_{χ}. Considering X as a $\mathfrak{g}_1'$-module, we obtain

$$X = \bigoplus_{\chi} P_{\chi}(X).$$

Obviously, $P_{\chi}(X)$ is a $\mathfrak{g}_1' \oplus \mathfrak{g}_2$-submodule of $T(P_{\chi}(M))$. Applying Theorem 6.1 for $\mathfrak{g}_1' \oplus \mathfrak{g}_2$ proved in subsection 6.2, we can take a unique $\mathfrak{g}_1'$-submodule N_{χ} of $P_{\chi}(M)$ with $T(N_{\chi}) = P_{\chi}(X)$.

We set $N := \bigoplus_{\chi} N_{\chi} \subset M$. Then we have $T(N) = X$. It is enough to show that N is $\mathfrak{g}_1$-stable. Let $m : \mathfrak{g}_1 \otimes M \to M$ and $m_T : \mathfrak{g}_1 \otimes T(M) \to T(M)$ be the multiplication maps. By Fact 2.16, the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g}_1 \otimes T(M) \arrow[r, "m_T"] \arrow[d, "\simeq"'] & T(M) \\ T(\mathfrak{g}_1 \otimes M) \arrow[ur, "T(m)"'] & \end{tikzcd}$$

We consider $m|_N$ as the composition of the following surjection and inclusion:

$$\begin{tikzcd} \mathfrak{g}_1 \otimes N \arrow[r, two heads] \arrow[rr, bend right, "m|_N"'] & \mathfrak{g}_1 N \arrow[r, hook] & M. \end{tikzcd}$$

Applying T to the above sequence, we have the following commutative diagram:

$$\begin{tikzcd} T(\mathfrak{g}_1 \otimes N) \arrow[r, two heads] \arrow[rr, bend left, "T(m|_N)"] & T(\mathfrak{g}_1 N) \arrow[r, hook] & T(M). \\ \mathfrak{g}_1 \otimes T(N) \arrow[u, "\simeq"] \arrow[urr, "m_T|_{T(N)}"'] & & \end{tikzcd}$$

This implies $T(\mathfrak{g}_1 N) = \mathfrak{g}_1 T(N) \subset X = T(N)$. Therefore, Theorem 6.1 leads $P_{\chi}(\mathfrak{g}_1 N) \subset N_{\chi}$ and hence $\mathfrak{g}_1 N \subset N$. This shows the surjectivity of $T : \mathcal{L}(M) \to \mathcal{L}(T(M))$. We have proved the theorem. $\square$

6.4 Category equivalence in special case

The main purpose in this section is to show that the functor T defined in Theorem 6.1 gives a category equivalence in a special setting.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{b}$. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and H be a closed subgroup of G corresponding to $\mathfrak{h} \subset \mathfrak{g}$. Then $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$ is a pair.

Set $S := \dim_{\mathbb{C}}(\mathfrak{u})$.

Theorem 6.7. *Let λ be a character of $\mathfrak{h}$ with*

$$\frac{2(\lambda + \rho(\mathfrak{u}), \alpha)}{(\alpha, \alpha)} \notin \{0, 1, 2, \dots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Then the following functor gives a category equivalence:

$$\begin{array}{cccc} T: & \mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) & \to & \mathcal{C}(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))_{\lambda+\rho(\mathfrak{u})} \\ & \Cup & & \Cup \\ & M & \mapsto & R^S\Gamma^{\Delta(G)}_{\Delta(H)}(M \otimes \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}}(\mathbb{C}_\lambda)), \end{array}$$

where $\mathcal{C}(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))_{\lambda+\rho(\mathfrak{u})}$ is the full subcategory of $\mathcal{C}(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$ whose object has the infinitesimal character $\lambda + \rho(\mathfrak{u})$ with respect to the action of the second factor of $\mathfrak{g} \oplus \mathfrak{g}$.

Remark 6.8. This theorem was conjectured and partially proved by T. J. Enright in [10].

By Theorem 6.1, the functor T is exact and fully faithful. Hence what we need to show is that T is dense (i.e. for any object N in the codomain, there is an object M in the domain with $T(M) \simeq N$). In fact, a functor F gives a category equivalence if and only if F is fully faithful and dense.

To prove the denseness, we use the following well-known category equivalence (see [4]). For a weight $\lambda' \in \mathfrak{h}^*$, we define the following two covariant functors:

$$\begin{aligned} F(M) &:= \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}}(\mathbb{C}_{\lambda'}), M)_{\Delta(G)} \text{ for } M \in \mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda') \\ G(N) &:= N \otimes_{\mathcal{U}(\mathfrak{g})} \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}}(\mathbb{C}_{\lambda'}) \text{ for } N \in \mathcal{C}(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))_{-\lambda'-\rho(\mathfrak{u})}. \end{aligned}$$

Here to define G, we consider the $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$-module N as a $(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g}))$-bimodule by

$$A \cdot n \cdot B := (A \otimes B)n \text{ for } A, B \in \mathcal{U}(\mathfrak{g}) \text{ and } n \in N,$$

and $(\cdot)_{\Delta(G)}$ means the sum of finite-dimensional $\Delta(\mathfrak{g})$-submodules which can lift to representations of $\Delta(G)$. Then the following fact holds.

Fact 6.9. *Under the above setting, if moreover $\lambda' + \rho(\mathfrak{u})$ is regular integrally dominant (see Definition 2.19,) then F is a category equivalence and G is a quasi-inverse of F.*

Proof of Theorem 6.7. We set $\lambda' = -\lambda - 2\rho(\mathfrak{u})$. Then we have $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda') = \mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$, and λ satisfies the condition in Theorem 6.7 if and only if λ' is regular and integrally dominant. Therefore, by Theorem 6.1 and Fact 6.9, we have an exact fully faithful endofunctor $G \circ T$ on $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$, and $G \circ T$ preserves irreducibility.

We prove that $G \circ T$ is a dense functor. We denote by **Irr** the set of equivalence classes of irreducible modules in $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$. Then $G \circ T$ induces a permutation on **Irr**. Since $G \circ T$ preserves infinitesimal characters, the cardinality of each orbit on **Irr** is bounded by the index of the Weyl group $W_{\mathfrak{g}}$. Thus there exists a positive integer k such that $(G \circ T)^k$ acts on **Irr** trivially. Obviously, $(G \circ T)^k$ is dense if and only if $G \circ T$ is dense.

Set $E := (G \circ T)^k$. We show $E(P) \simeq P$ holds for each projective object P of $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$. Without loss of generality, we may assume that P is indecomposable, and hence a projective cover of an irreducible module L (see [28, Section 3.9]). Then we have a surjective homomorphism $\pi : P \to L$. Remark that L is a unique irreducible quotient of P. Since E is exact, $E(\pi) : E(P) \to E(L)$ is surjective. By the choice of k that E acts on **Irr** trivially, we can identify $E(L)$ with L. Since P is projective, there exists a homomorphism $\tau : P \to E(P)$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\tau"'] \arrow[d] & \\ E(P) \arrow[r, "E(\pi)"'] & E(L) \arrow[r] & 0 \text{ (exact)} \end{tikzcd}$$

Recall that the functor E induces a lattice isomorphism between $\mathcal{L}(P)$ and $\mathcal{L}(E(P))$. Then $E(L)$ is a unique irreducible quotient of $E(P)$, and hence τ is surjective. Since P and $E(P)$ have the same length, we have $P \simeq E(P)$.

Since $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ have enough projectives [28, Section 3.9], any object of $\mathcal{O}^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ can be written as a quotient of a projective object. This implies that E is dense. This completes the proof. □

7 $\mathcal{U}(\mathfrak{g})^{G'}$-module

This section is devoted to the study of $\mathcal{U}(\mathfrak{g})^{G'}$-modules arising from the branching problem. To study $\mathcal{U}(\mathfrak{g})^{G'}$-modules, we define $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-

modules in Section 7.4. The results in this section are used in Section 8 and 9 to prove the irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules.

7.1 General setting

Let $(\mathfrak{g}, K)$ be a pair, and $(\mathfrak{g}', K')$ be a subpair of $(\mathfrak{g}, K)$ (see Definition 2.1). Suppose that $\mathfrak{g}'$ is reductive in $\mathfrak{g}$. Let G' be a subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\mathrm{Ad}_{\mathfrak{g}}(K')$ and the analytic subgroup with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}') \subset \mathfrak{der}(\mathfrak{g})$. Then G' is a complex reductive algebraic group.

For a $(\mathfrak{g}, K)$-module V and a $(\mathfrak{g}', K')$-module V', we define the following three $\mathcal{U}(\mathfrak{g})^{G'}$-modules:

$$\mathrm{Hom}_{\mathfrak{g}',K'}(V', V),\ \mathrm{Hom}_{\mathfrak{g}',K'}(V, V'),\ H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}).$$

Here $H_0(\mathfrak{g}', K'; \cdot)$ means the zeroth relative Lie algebra homology, namely, the space of coinvariants. Since the above modules are the spaces of $(\mathfrak{g}', K')$-(co)invariants on $(\mathfrak{g}, K)$-modules, the $\mathcal{U}(\mathfrak{g})^{G'}$-actions are naturally induced from the $(\mathfrak{g}, K)$-actions. For example, $\mathcal{U}(\mathfrak{g})^{G'}$ acts on $\mathrm{Hom}_{\mathfrak{g}',K'}(V, V')$ via

$$(X \cdot \varphi)(\cdot) = \varphi({}^t X \cdot)$$

for $X \in \mathcal{U}(\mathfrak{g})^{G'}$ and $\varphi \in \mathrm{Hom}_{\mathfrak{g}',K'}(V, V')$. Here ${}^t X$ is the anti-automorphism of $\mathcal{U}(\mathfrak{g})$ with ${}^t X = -X$ for $X \in \mathfrak{g}$.

The three modules are related to each other. To see the relation, we prepare the following well-known lemma (see e.g. [68, Section 2.3.3]).

Lemma 7.1. *Let V be an irreducible $(\mathfrak{g}, K)$-module and W be a vector space. Then for any $(\mathfrak{g}, K)$-submodule U of $V \otimes W$, there exists a unique subspace $W' \subset W$ such that $U = V \otimes W'$.*

Proof. Since V is a completely reducible $(\mathfrak{g}, K)$-module, we can assume that U is irreducible. Take $u \in U$, and write $u = \sum_i v_i \otimes w_i$ such that w's are linearly independent. We can replace W by the linear span of w's. We define p_i by the projection to $V \simeq V \otimes w_i$. By Schur's lemma, $c_{ij} p_i|_U = p_j|_U$ holds for some c_{ij}. Then u can be written as $v_i \otimes \sum_i c_{ij} w_i$. Since U is generated by u, this shows the assertion. □

Proposition 7.2. *Retain the notation in the above. Suppose that V is K-admissible and V' is K'-admissible.*

(a) *The following $\mathcal{U}(\mathfrak{g})^{G'}$-isomorphism holds:*

$$H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})^* \simeq \mathrm{Hom}_{\mathfrak{g}',K'}(V, V').$$

(b) *Set* $U := \bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}',K'}(V,V')} \mathrm{Ker}(\varphi)$. *Then* V/U *is a* $(\mathfrak{g}', K')$*-module and a* $\mathcal{U}(\mathfrak{g})^{G'}$*-module, and there is an injective homomorphism:*

$$V/U \to V' \otimes H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}).$$

Moreover, if V' *is irreducible, the homomorphism is isomorphism.*

(c) *If* V' *is irreducible, then we have a* $\mathcal{U}(\mathfrak{g})^{G'}$*-homomorphism:*

$$\mathrm{Hom}_{\mathfrak{g}',K'}(V, V') \to \mathrm{Hom}_{\mathfrak{g}',K'}(V', V)^*.$$

Moreover, if $V|_{\mathfrak{g}',K'}$ *is completely reducible, the above homomorphism is isomorphism.*

Proof. (a) is clear from

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}), \mathbb{C}) &\simeq \mathrm{Hom}_{\mathfrak{g}',K'}(V \otimes (V')^*_{K'}, \mathbf{1}) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}',K'}(V, V').\end{aligned}$$

The second isomorphism holds because V' is K'-admissible.

To prove (b), we construct the homomorphism. By definition, there is a canonical surjection:

$$\pi : V \otimes (V')^*_{K'} \to H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}).$$

Through the following isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}',K'}(V \otimes (V')^*_{K'}, H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})) \\ \simeq &\mathrm{Hom}_{\mathfrak{g}',K'}(V, \mathrm{Hom}_{\mathbb{C}}((V')^*_{K'}, H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}))_{K'}) \\ \simeq &\mathrm{Hom}_{\mathfrak{g}',K'}(V, V' \otimes H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})),\end{aligned}$$

we define $\pi' \in \mathrm{Hom}_{\mathfrak{g}',K'}(V, V' \otimes H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'}))$ corresponding to π.

We shall show $\mathrm{Ker}(\pi') = U$. For a homomorphism $T \in \mathrm{Hom}_{\mathfrak{g}',K'}(V, V')$, let $\widetilde{T}$ denote the corresponding element of $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})^*$ by the isomorphism in (a). Then it is easy to see $(\mathrm{id}_{V'} \otimes \widetilde{T}) \circ \pi' = T$. This implies

$$\mathrm{Ker}(\pi') = \bigcap_{\widetilde{T}} \mathrm{Ker}((\mathrm{id}_{V'} \otimes \widetilde{T}) \circ \pi') = \bigcap_{T} \mathrm{Ker}(T) = U. \tag{7.2.1}$$

Therefore, we obtain an injection $\pi''$from V/U to $V' \otimes H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$.

Assume that V' is irreducible. Then from Lemma 7.1, there exists a subspace W of $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ satisfying $\mathrm{Im}(\pi'') = V' \otimes W$. Take $\widetilde{T} \in H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})^*$ such that $\widetilde{T}(W) = 0$. Then we have

$$0 = (\mathrm{id}_{V'} \otimes \widetilde{T}) \circ \pi' = T.$$

This implies $\widetilde{T} = 0$, and hence $W = H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$. Therefore, π'' is surjective.

(c) There is a canonical homomorphism:

$$\tau : V' \otimes \mathrm{Hom}_{\mathfrak{g}',K'}(V', V) \to V. \tag{7.2.2}$$

Applying the contravariant functor $\mathrm{Hom}_{\mathfrak{g}',K'}(\cdot, V')$ to this, we have

$$\begin{aligned}\tau' : \mathrm{Hom}_{\mathfrak{g}',K'}(V, V') \to& \mathrm{Hom}_{\mathfrak{g}',K'}(V' \otimes \mathrm{Hom}_{\mathfrak{g}',K'}(V', V), V')\\ \simeq& \mathrm{Hom}_{\mathfrak{g}',K'}(V', V)^*\end{aligned}$$

because the irreducibility of V' implies $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V') \simeq \mathbb{C}$. If $V|_{\mathfrak{g}',K'}$ is completely reducible, (7.2.2) is split and $\mathrm{Hom}_{\mathfrak{g}',K'}(\mathrm{Coker}(\tau), V') = 0$, and hence τ' is an isomorphism.

By construction, it is clear that the homomorphisms constructed in the above are $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphisms. We have proved the proposition. □

Remark 7.3. In the context of the Howe duality [26], $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ appears as the full theta lift.

7.2 Direct integral and $(\mathfrak{g}', K')$-module

In this section, we discuss a relation between the irreducible decomposition of the restriction of an irreducible unitary representation and $\mathrm{Hom}_{\mathfrak{g}',K'}(V, V')$.

Let $G_\mathbb{R}$ be a real reductive Lie group with maximal compact subgroup $K_\mathbb{R}$, and $G'_\mathbb{R}$ be a reductive subgroup of $G_\mathbb{R}$ with maximal compact subgroup $K'_\mathbb{R} = G_\mathbb{R} \cap K_\mathbb{R}$. Construct pairs $(\mathfrak{g}, K)$ and $(\mathfrak{g}', K')$ from $G_\mathbb{R}$ and $G'_\mathbb{R}$. We denote by G' the Zariski closure of $\mathrm{Ad}(G'_\mathbb{R})$ in $\mathrm{Aut}(\mathfrak{g})$.

Let (τ, V) be an irreducible unitary representation of $G_\mathbb{R}$. Then $V|_{G'_\mathbb{R}}$ has a unique irreducible decomposition (see [105, Section 14.9]):

$$V|_{G'_\mathbb{R}} \simeq \int^{\oplus}_{\widehat{G'_\mathbb{R}}} V_\pi \mathbin{\hat{\otimes}} M_\pi d\mu(\pi),$$

where V_π is a representation space of π, M_π is a Hilbert space with trivial $G'_\mathbb{R}$-action and μ is a Borel measure on $\widehat{G'_\mathbb{R}}$.

For μ-almost all $\pi \in \widehat{G'_\mathbb{R}}$, we will construct $\mathcal{U}(\mathfrak{g})^{G'}$-action on a dense subspace M^0_π of M_π, and a $(\mathfrak{g}', K')$-module and a $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:

$$\phi_\pi : V_K \to (V_\pi)_{K'} \otimes M^0_\pi$$

with some compatibility conditions.

To do this, we review some of facts on the reduction theory of closed operators (see [76]). Let $\mathcal{H}$ be a Hilbert space with direct integral decomposition:

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}_z d\nu(z),$$

where Z is a locally compact space, ν a positive Radon measure on Z and $z \to \mathcal{H}_z$ a ν-measurable field of Hilbert spaces. For simplicity, we assume that (Z, ν) is σ-finite. For a operator T in $\mathcal{H}$, we denote by $D(T)$ the domain of T.

Definition 7.4. For a closed operator T in $\mathcal{H}$, we write the projection from $\mathcal{H} \oplus \mathcal{H}$ onto the graph of T as the matrix form:

$$(x, y) \mapsto (P_{11}x + P_{12}y, P_{21}x + P_{22}y),$$

and we call $(P_{ij})_{1 \leq i,j \leq 2}$ the *characteristic matrix* of T.

A field of closed operators $z \to T(z)$ is said to be *measurable* if the field of bounded operators $z \to P_{ij}(z)$ defined by the characteristic matrices of $T(z)$ is measurable.

Fact 7.5. *Let $z \to T(z)$ be a measurable field of closed operators. Let $D(T)$ denote the set of all vectors $v \in \mathcal{H}$ such that $v(z) \in D(T(z))$ for ν-almost all z and the vector field $z \to T(z)v(z)$ is square-integrable. Then the operator T given by $Tv(z) = T(z)v(z)$ for $v \in D(T)$ is closed operator with domain $D(T)$.*

We write $T \sim T(z)$ if a closed operator T is equal to the closed operator defined by a measurable field of closed operators $T(z)$ as in the fact.

Definition 7.6. A closed operator T in $\mathcal{H}$ is decomposable if there exists a measurable field of closed operators $T(z)$ with $T \sim T(z)$.

Fact 7.7. *Let T be a closed decomposable operator in $\mathcal{H}$, and $z \to T(z), S(z)$ be measurable fields of closed operators. Suppose $T \sim T(z)$.*

(1) *If $T \sim S(z)$, then $T(z) = S(z)$ holds for ν-almost every z.*

(2) *$z \to T(z)^*$ is also a measurable field of closed operators, and $T^* \sim T(z)^*$ holds.*

For a operator T and bounded operator B in $\mathcal{H}$, we will say that T commutes with B if $BT \subset TB$. Here $D(BT)$ is equal to $D(T)$, and $D(TB)$ is the set of all vectors $v \in \mathcal{H}$ satisfying $Bv \in D(T)$.

The following proposition is useful to check that given two operators are commutative.

Proposition 7.8. *Let T be a closable operator and B a bounded operator in $\mathcal{H}$.*

(a) *T commutes with B if and only if $BG(T) \subset G(T)$ holds, where $G(T)$ is the graph of T.*

(b) *If $BD(T) \subset D(T)$ and $BT = TB$ holds on $D(T)$, then T commutes with B.*

(c) *If T commutes with B, then so does $\overline{T}$.*

Proof. By definition, (a) and (b) is obvious. If $BG(T) \subset G(T)$, then we have $B\overline{G(T)} \subset \overline{G(T)}$. Hence (a) leads to (c). □

Fact 7.9 (A. E. Nussbaum [76, Corollary 4])**.** *A closed operator T on $\mathcal{H}$ is decomposable if and only if T commutes with every bounded diagonalizable operator.*

We return to the setting of unitary representations. We consider the irreducible decomposition of an irreducible unitary representation (τ, V):

$$V|_{G'_{\mathbb{R}}} \simeq \int_{\widehat{G'_{\mathbb{R}}}}^{\oplus} V_\pi \hat{\otimes} M_\pi d\mu(\pi).$$

In this case, the von Neumann algebra of bounded diagonalizable operators is equal to the center of $\mathrm{End}_{G'_{\mathbb{R}}}(V)$ denoted by $Z_{G'_{\mathbb{R}}}(V)$. Before we state the main theorem in this section, we review the following important fact proved by R. Goodman [15, Lemma 3.1].

Fact 7.10. *Let U be a unitary representation of $G_{\mathbb{R}}$ with direct integral decomposition:*

$$U \simeq \int_Z^{\oplus} U_z d\mu(z).$$

Then for any vector $v \in U^\infty$ defined by a section $z \mapsto v(z)$, $v(z)$ is in U_z^∞ for μ-almost every z, where U^∞ is the space of smooth vectors with respect to the $G_{\mathbb{R}}$-action. Furthermore, for any $X \in \mathcal{U}(\mathfrak{g})$, we have $(Xv)(z) = X(v(z))$ for μ-almost every z.

We write $(\cdot)^*$ for the anti-automorphism of $\mathcal{U}(\mathfrak{g})$ such that $X^* = -\overline{X}$ for any $X \in \mathfrak{g}$, where $\overline{X}$ is the complex conjugate with respect to the real form $\mathfrak{g}_{\mathbb{R}}$. We say that a $\mathcal{U}(\mathfrak{g})^{G'}$-module M is *unitary* if M admits a positive definite invariant Hermitian form with respect to the $(\cdot)^*$-structure. Applying Fact 7.9 and 7.10 to our case, we have

Theorem 7.11. *Let (τ, V) be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that $V|_{G'_{\mathbb{R}}}$ has the following irreducible decomposition:*

$$V|_{G'_{\mathbb{R}}} \simeq \int_{\widehat{G'_{\mathbb{R}}}}^{\oplus} V_\pi \mathbin{\hat{\otimes}} M_\pi d\mu(\pi).$$

Then there exist a μ-null set $\mathcal{N}$ and a family of $(\mathfrak{g}', K')$-homomorphisms

$$\{\phi_\pi : V_K \to (V_\pi)_{K'} \otimes M_\pi\}_{\pi \in \widehat{G'_{\mathbb{R}}} \setminus \mathcal{N}}$$

satisfying the following conditions:

(a) *there exists a dense subspace M_π^0 of M_π such that $\mathrm{Im}(\phi_\pi) = (V_\pi)_{K'} \otimes M_\pi^0$;*

(b) *M_π^0 admits a unique $\mathcal{U}(\mathfrak{g})^{G'}$-module structure such that ϕ_π is a $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism, and the $\mathcal{U}(\mathfrak{g})^{G'}$-module is unitary with respect to $(\cdot)^*$ and the restriction of the inner product of M_π;*

(c) *for any $v \in V_K$, we have $v(\pi) = \phi_\pi(v)$ for μ-almost every π;*

(d) *if $\phi_\pi(v) = 0$ for any $\pi \in \widehat{G'_{\mathbb{R}}} \setminus \mathcal{N}$, then $v = 0$ holds.*

Proof. By the construction of the irreducible decomposition, we can replace the measure space $(\widehat{G'_{\mathbb{R}}}, \mu)$ by a measure space (Z, ν) such that Z is a locally compact space and ν is a positive Radon measure on Z (see [105, Section 14.9]). Then it is enough to show the assertion for the following direct integral decomposition:

$$V|_{G'_{\mathbb{R}}} \simeq \int_Z^{\oplus} V_{\pi(z)} \mathbin{\hat{\otimes}} M_{\pi(z)} d\nu(z),$$

The algebra of bounded diagonalizable operators in the above decomposition is equal to $Z_{G'_{\mathbb{R}}}(V)$.

We consider that the domain of each operator in $\tau(\mathcal{U}(\mathfrak{g}))$ is V^∞. Then any operator in $\tau(\mathcal{U}(\mathfrak{g}))$ is closable.

To apply Fact 7.9, we shall check that the closure of any operator in $\tau(\mathcal{U}(\mathfrak{g}'))$ and $\tau(\mathcal{U}(\mathfrak{g})^{G'})$ commutes with every operator in $Z_{G'_{\mathbb{R}}}(V)$. Take $X \in \tau(\mathcal{U}(\mathfrak{g})^{G'})$. It is obvious that X commutes with any $T \in \tau(G'_{\mathbb{R}})$, and then X commutes with any element of $\mathrm{span}_{\mathbb{C}}\{T \in \tau(G'_{\mathbb{R}})\}$. By Proposition 7.8, $\overline{X}$ commutes with $S \in \mathrm{span}_{\mathbb{C}}\{T \in \tau(G'_{\mathbb{R}})\}$, and we have

$$SG(\overline{X}) \subset G(\overline{X}). \tag{7.11.3}$$

Take $S \in Z_{G'_{\mathbb{R}}}(V) \subset \tau(G'_{\mathbb{R}})''$. Here $\tau(G'_{\mathbb{R}})''$ is the centralizer of $\mathrm{End}_{G'_{\mathbb{R}}}(V)$ in $\mathrm{End}(V)$. By the von Neumann double commutant theorem, $\tau(G'_{\mathbb{R}})''$ is equal to the closure of $\mathrm{span}_{\mathbb{C}}\{T \in \tau(G'_{\mathbb{R}})\}$ in the strong topology. Hence there exists $S_1, S_2, \ldots \in \mathrm{span}_{\mathbb{C}}\{T \in \tau(G'_{\mathbb{R}})\}$ with $\lim_{i\to\infty} S_i = S$ in the strong topology. From (7.11.3), we have

$$SG(\overline{X}) \subset G(\overline{X}),$$

and hence $\overline{X}$ commutes with S. This shows that the closure of any operator in $\tau(\mathcal{U}(\mathfrak{g})^{G'})$ commutes with every operator in $Z_{G'_{\mathbb{R}}}(V)$.

Take $X \in \mathcal{U}(\mathfrak{g}')$ and $T \in \mathrm{End}_{G'_{\mathbb{R}}}(V)$. For $v \in V^\infty$, we have

$$\begin{aligned} T\tau(X)v &= T \lim_{t\to 0} \frac{\tau(\exp(tX))v - v}{t} \\ &= \lim_{t\to 0} \frac{\tau(\exp(tX))Tv - Tv}{t} \\ &= \tau(X)Tv. \end{aligned}$$

From (7.11.3), this implies that $\overline{X}$ commutes with T.

By the above discussion, we can apply Fact 7.9 to any operator in $\tau(\mathcal{U}(\mathfrak{g}'))$ and $\tau(\mathcal{U}(\mathfrak{g})^{G'})$. Fix a basis $\{v_i\}$ of V_K and $\{X_i\}$ of $\mathcal{U}(\mathfrak{g}')\mathcal{U}(\mathfrak{g})^{G'}$, and fix representatives $\{g_k\}$ of connected components of $K'_{\mathbb{R}}$. Suppose that $\{X_i\}$ contains a basis of $\mathcal{U}(\mathfrak{g}')$. For each v_i, we choose a measurable section $z \to v_i(z)$, and for each X_i, take a measurable field $z \to X_i(z)$ of closed operators with $\overline{\tau(X_i)} \sim X_i(z)$. We define a linear map $\phi_z : V_K \to V_{\pi(z)} \hat{\otimes} M_{\pi(z)}$ to be $\phi_z(v_i) = v_i(z)$.

Remark that since V_K and the algebra are at most countable-dimensional, $\{v_i\}$ and $\{X_i\}$ are at most countable sets, and since $K'_{\mathbb{R}}$ is compact, $\{g_k\}$ is a finite set. We put

$$\begin{aligned} \mathcal{N}_0 := &\{z \in Z : \phi_z(\tau(X_i)v_j) = X_i(z)\phi_z(v_j) \text{ for any } i, j\} \\ &\cup \{z \in Z : \phi_z(\tau(g_k)v_j) = \pi(z)(g_k)\phi_z(v_j) \text{ for any } j, k\}. \end{aligned}$$

Fact 7.7 (1) yields that $\mathcal{N}_0$ is a ν-null set. For $z \in Z \backslash \mathcal{N}_0$, we can define a linear map $\tau_z : \mathcal{U}(\mathfrak{g}')\mathcal{U}(\mathfrak{g})^{G'} \to \mathrm{End}_{\mathbb{C}}(\mathrm{Im}(\phi_z))$ to be $\tau_z(X_i) = X_i(z)$. By the definition of $\mathcal{N}_0$, the map τ_z is an algebra homomorphism for $z \in Z \backslash \mathcal{N}_0$.

We will construct $\mathcal{N}$ satisfying the compatibility conditions in the theorem:

(1) $\tau_z(X_i)^* = \tau_z(X_i^*)$ on $\mathrm{Im}(\phi_z)$ for any i;

(2) $\tau_z(X_i) = \pi(z)(X_i)$ on $\mathrm{Im}(\phi_z)$ for any i with $X_i \in \mathcal{U}(\mathfrak{g}')$;

(3) ϕ_z has dense image in $V_{\pi(z)} \hat{\otimes} M_{\pi(z)}$.

For $p \in \{1, 2, 3\}$, we put $\mathcal{N}_p := \{z \in Z : \text{(p) fails for } z\}$. By Fact 7.7 (2), $\mathcal{N}_1$ is a ν-null sets. Fact 7.10 implies $\mathrm{Im}(\phi_z) \subset (V_{\pi(z)} \hat{\otimes} M_{\pi(z)})^\infty$ and $\tau_z(X_i) = \pi(z)(X_i)$ on $(V_{\pi(z)} \hat{\otimes} M_{\pi(z)})^\infty$ for ν-almost every z. Hence $\mathcal{N}_2$ is a ν-null set. Since V_K is a dense subset of V, $\mathcal{N}_3$ is a ν-null set (see [3, Lemma 1.3]).

We put $\mathcal{N} := \bigcup_{0 \leq p \leq 3} \mathcal{N}_p$. Then $\mathcal{N}$ is a ν-null set. Hereafter, we consider $z \in Z \backslash \mathcal{N}$. By the definition of $\mathcal{N}_0$ and $\mathcal{N}_2$, ϕ_z is a $(\mathfrak{g}', K')$-homomorphism. Then we have

$$\mathrm{Im}(\phi_z) \subset (V_{\pi(z)} \hat{\otimes} M_{\pi(z)})_{K'} = (V_{\pi(z)})_{K'} \otimes M_{\pi(z)}.$$

Lemma 7.1 implies that there exists a subspace $M^0_{\pi(z)}$ of $M_{\pi(z)}$ with $\mathrm{Im}(\phi_z) = (V_{\pi(z)})_{K'} \otimes M^0_{\pi(z)}$. Since $\mathrm{Im}(\phi_z)$ is dense in $V_{\pi(z)} \hat{\otimes} M_{\pi(z)}$, $M^0_{\pi(z)}$ is dense in $M_{\pi(z)}$.

Since τ_z is an algebra homomorphism, τ_z defines a $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $(V_{\pi(z)})_{K'} \otimes M^0_{\pi(z)}$. Since the $\mathcal{U}(\mathfrak{g})^{G'}$-action commutes with the $(\mathfrak{g}', K')$-action, $\mathcal{U}(\mathfrak{g})^{G'}$ acts on

$$M^0_{\pi(z)} \simeq \mathrm{Hom}_{\mathfrak{g}', K'}((V_{\pi(z)})_{K'}, (V_{\pi(z)})_{K'} \otimes M^0_{\pi(z)}).$$

The condition (1) implies that this action is unitary.

By the construction of $\mathcal{N}$ and ϕ_z, it is clear that the conditions in the theorem hold. We have proved the theorem. □

By Theorem 7.11, we can relate the branching laws of unitary representations with the branching laws of $(\mathfrak{g}, K)$-modules. The following corollary is an easy consequence of the theorem.

Corollary 7.12. *Retain the setting in Theorem 7.11. Then we have a surjective $\mathcal{U}(\mathfrak{g})^{G'}$-homomorphism:*

$$H_0(\mathfrak{g}', K'; V_K \otimes (V_\pi)^*_{K'}) \to M^0_\pi$$

for μ-almost every $\pi \in \widehat{G'_{\mathbb{R}}}$. In particular, we have

$$\begin{aligned} \mathcal{M}_{G'_{\mathbb{R}}}(V) &\leq \text{ess sup} \left\{ \dim_{\mathbb{C}}(H_0(\mathfrak{g}', K'; V_K \otimes (V_\pi)^*_{K'})) : \pi \in \widehat{G'_{\mathbb{R}}} \right\} \\ &= \text{ess sup} \left\{ \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{g}', K'}(V_K, (V_\pi)_{K'}) : \pi \in \widehat{G'_{\mathbb{R}}} \right\}, \end{aligned}$$

where $\mathcal{M}_{G'_{\mathbb{R}}}(V)$ is the essential supremum of the multiplicity function.

Proof. Since M^0_π is dense in M_π, the second assertion is clear from the first assertion. Theorem 7.11 and Proposition 7.2 (b) show the first assertion. □

Since M^0_π is a unitary module, if M^0_π is finite-dimensional, M^0_π is completely reducible. Using the theory of polynomial identity (Section 2.6), we have

Corollary 7.13. *Retain the setting in Theorem 7.11. We have*

$$\mathrm{PI.deg}(\tau(\mathcal{U}(\mathfrak{g})^{G'})) \leq \mathcal{M}_{G'_{\mathbb{R}}}(V).$$

Proof. If $\mathcal{M}_{G'_{\mathbb{R}}}(V) = \infty$, the assertion is trivial. We assume $\mathcal{M}_{G'_{\mathbb{R}}}(V) < \infty$. Then for almost all π, M^0_π is a completely reducible $\mathcal{U}(\mathfrak{g})^{G'}$-module. Applying Proposition 2.27 to $\{M^0_\pi\}_\pi$ and $\tau(\mathcal{U}(\mathfrak{g})^{G'})$, we obtain the inequation. □

By the above two corollaries, we can estimate $\mathcal{M}_{G_{\mathbb{R}}}(V)$ by algebraic invariants.

7.3 Compact subgroup case

In this section, we review some basic facts about $\mathcal{U}(\mathfrak{g})^{G'}$-modules in the case of compact subgroups G'.

Let $(\mathfrak{g}, K)$ be a pair, and K' be a reductive subgroup of K. The following proposition is well-known, and plays an important role in the theory of $(\mathfrak{g}, K)$-modules (e.g. Harish-Chandra's subquotient theorem [19]).

Proposition 7.14. *Let V be an irreducible $(\mathfrak{g}, K)$-module, and V' be a finite-dimensional irreducible K'-module. Suppose that V is irreducible as a $\mathfrak{g}$-module. Then the $\mathcal{U}(\mathfrak{g})^{K'}$-module $\mathrm{Hom}_{K'}(V', V)$ is irreducible or zero.*

Proof. Assume that $\mathrm{Hom}_{K'}(V', V)$ is non-zero.

Take non-zero elements $T, S \in \mathrm{Hom}_{K'}(V', V)$. Fix a basis $\{v_i\}$ of V'. Since V' is irreducible, $\{T(v_i)\}$ is linearly independent. Hence by the Jacobson density theorem, there exists $X \in \mathcal{U}(\mathfrak{g})$ such that $XT(v_i) = S(v_i)$ for any i, and hence $XT = S$. This implies $\mathcal{U}(\mathfrak{g})T \supset \mathrm{Hom}_{K'}(V', V)$. Since the K'-action on $\mathcal{U}(\mathfrak{g})$ is completely reducible, taking the K'-invariant part, we have

$$\mathcal{U}(\mathfrak{g})^{K'}T = \mathrm{Hom}_{K'}(V', V).$$

This finishes the proof. □

In the proof, it is important that $\mathcal{U}(\mathfrak{g})T = \mathrm{Hom}_{\mathbb{C}}(V', V)$ holds. This observation is useful to generalize Proposition 7.14 to the case of non-compact subgroups G'. We treat the general case in Section 7.4.

As a corollary of Proposition 7.14 and Proposition 2.27, we obtain

Corollary 7.15. *Let (τ, V) be an irreducible $(\mathfrak{g}, K)$-module. Suppose that $V|_{\mathfrak{g}}$ is irreducible. Then we have*

$$\mathrm{PI.deg}(\tau(\mathcal{U}(\mathfrak{g})^{K'})) = \mathcal{M}_{K'}(V).$$

Proof. We denote by $\widehat{K'}$ the equivalence classes of finite-dimensional irreducible representations of K'. Then for any $(\pi, V_\pi) \in \widehat{K'}$, $\mathrm{Hom}_{K'}(V_\pi, V)$ is an irreducible $\mathcal{U}(\mathfrak{g})^{K'}$-module or zero. Applying Proposition 2.27 to $\mathcal{A} = \tau(\mathcal{U}(\mathfrak{g})^{K'})$ and $\{\mathrm{Hom}_{K'}(V_\pi, V)\}_{\pi \in \widehat{K'}}$, we obtain the corollary. $\square$

As variations of the corollary, we state the following two results. These results are our motivation to study $\mathcal{U}(\mathfrak{g})^{G'}$-modules.

Corollary 7.16 (I. Penkov, V. Serganova [84, Theorem 4.3])**.** *Let (τ_1, V_1) and (τ_2, V_2) be irreducible $(\mathfrak{g}, K)$-modules. Suppose that $V_i|_{\mathfrak{g}}$ is irreducible and $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_1) = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_2)$ holds. Then we have*

$$\mathcal{M}_{K'}(V_1) = \mathcal{M}_{K'}(V_2).$$

Proof. Since $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_1) = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_2)$, the algebra $\tau_1(\mathcal{U}(\mathfrak{g})^{K'})$ is isomorphic to $\tau_2(\mathcal{U}(\mathfrak{g})^{K'})$. This implies

$$\mathrm{PI.deg}(\tau_1(\mathcal{U}(\mathfrak{g})^{K'})) = \mathrm{PI.deg}(\tau_2(\mathcal{U}(\mathfrak{g})^{K'})).$$

This and Corollary 7.15 show the assertion. $\square$

Corollary 7.17. *Retain the notation in Corollary 7.15. Let K'' be a reductive subgroup of K. Let G_{ad} denote the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\mathrm{Ad}_{\mathfrak{g}}(K')$ and $\mathrm{Int}(\mathfrak{g})$. Assume that $\mathrm{Ad}_{\mathfrak{g}}(K')$ and $\mathrm{Ad}_{\mathfrak{g}}(K'')$ are conjugate by an inner automorphism of G_{ad}. Then we have*

$$\mathcal{M}_{K'}(V) = \mathcal{M}_{K''}(V).$$

Proof. Since $\mathrm{Ann}_{\mathcal{U}(g)}(V)$ is an $\mathrm{Ad}_{\mathfrak{g}}(K)$-stable and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$-stable subspace of $\mathcal{U}(\mathfrak{g})$, $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ is G_{ad}-stable. By assumption, we can take $g \in G_{\mathrm{ad}}$ such that $g\mathrm{Ad}_{\mathfrak{g}}(K')g^{-1} = \mathrm{Ad}_{\mathfrak{g}}(K'')$. Hence we have

$$\begin{aligned}
\tau(\mathcal{U}(\mathfrak{g})^{K'})) &\simeq (\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))^{\mathrm{Ad}(K')} \\
&\simeq (\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))^{g^{-1}\mathrm{Ad}(K'')g} \\
&\simeq (\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))^{\mathrm{Ad}(K'')} \\
&\simeq \tau(\mathcal{U}(\mathfrak{g})^{K''})).
\end{aligned}$$

Therefore, by Corollary 7.15, we obtain the required equation. $\square$

7.4 $\Delta(G')$-finite linear maps

To generalize the results in the previous section, we define a space of $\Delta(G')$-finite linear maps.

First, we define $\Delta(G')$-finite linear maps. Let $(\mathfrak{g}', K')$ be a pair with reductive $\mathfrak{g}'$, and G' be a complex reductive algebraic group with Lie algebra $\mathfrak{g}'$. Suppose that there is an algebraic group homomorphism $\phi_{K'}$ from K' to G' such that the differential of $\phi_{K'}$ is the inclusion map from $\mathfrak{k}'$ to $\mathfrak{g}'$, and $\phi_{K'}(K')$ and the identity component of G' generate G'.

Let (τ, V) and (τ', V') be $(\mathfrak{g}', K')$-modules. We define

$$(\tau_{\mathrm{ad}}(X)T)(v) = \tau'(X)T(v) - T(\tau(X)v)$$
$$(\tau_{\mathrm{Ad}}(k)T)(v) = \tau'(k)T(\tau(k^{-1})v)$$

for $X \in \mathfrak{g}'$, $k \in K'$, $T \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ and $v \in V$. Then τ_{ad} is a representation of $\mathfrak{g}'$ and τ_{Ad} is a representation of K' as an abstract group. The following equation is the reason why we define τ_{ad} and τ_{Ad} as in the above:

$$\mathrm{Hom}_{\mathbb{C}}(V, V')^{\tau_{\mathrm{ad}}(\mathfrak{g}'), \tau_{\mathrm{Ad}}(K')} = \mathrm{Hom}_{\mathfrak{g}', K'}(V, V').$$

Definition 7.18. $T \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ is said to be $\Delta(G')$-*finite* if there exists an algebraic representation F of G' and $(\mathfrak{g}', K')$-homomorphism with respect to $(\tau_{\mathrm{ad}}, \tau_{\mathrm{Ad}})$:

$$\varphi : F \to \mathrm{Hom}_{\mathbb{C}}(V, V')$$

satisfying $T \in \varphi(F)$. Let $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ denote the space of $\Delta(G')$-finite linear maps.

Then $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ has a G'-module structure via

$$\tau_{\mathrm{Ad}}(g)T = \varphi(g\varphi^{-1}(T)) \text{ for } g \in G' \text{ and } T \in \mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')},$$

where φ is the homomorphism used in Definition 7.18. It is obvious that the definition is well-defined because the G'-action is uniquely determined by the actions of $\tau_{\mathrm{ad}}(\mathfrak{g}')$ and $\tau_{\mathrm{Ad}}(K')$. Hence it is also clear that the G'-action is compatible with the action of $\tau_{\mathrm{ad}}(\mathfrak{g}')$ and $\tau_{\mathrm{Ad}}(K')$.

The following proposition is useful to see the $\Delta(G')$-module structure.

Proposition 7.19. *Retain the notation. Let F be a finite-dimensional representation of G'. Then we have*

$$\mathrm{Hom}_{G'}(F, \mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}) \simeq \mathrm{Hom}_{\mathfrak{g}', K'}(V \otimes F, V').$$

Proof. By the definition of the $\Delta(G')$-module structure, we have

$$\begin{aligned}\mathrm{Hom}_{G'}(F, \mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}) &= \mathrm{Hom}_{\mathfrak{g}',K'}(F, \mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}) \\ &= \mathrm{Hom}_{\mathfrak{g}',K'}(F, \mathrm{Hom}_{\mathbb{C}}(V, V')) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}',K'}(V \otimes F, V').\end{aligned}$$

This shows the assertion. □

Hereafter, we consider $(\mathfrak{g}, K)$-modules. Let $(\mathfrak{g}, K)$ be a pair and $(\mathfrak{g}', K')$ be a subpair. Suppose $\mathfrak{g}'$ is reductive in $\mathfrak{g}$. For simplicity, we assume that $\mathfrak{g}'$ does not intersect with the center of $\mathfrak{g}$. Let (τ, V) be a $(\mathfrak{g}, K)$-module and (τ', V') be a $(\mathfrak{g}', K')$-module.

Let G' denote the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\mathrm{Ad}_{\mathfrak{g}}(K')$ and the analytic subgroup with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}') \subset \mathfrak{Der}(\mathfrak{g})$. Then G' is a complex reductive algebraic group, and satisfies the conditions at the beginning of this subsection.

We consider $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$ as a pair by the diagonal embedding:

$$\Delta : G' \to \mathrm{Aut}(\mathfrak{g}') \times \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g}' \oplus \mathfrak{g}).$$

We define three $\Delta(G')$-modules:

$$\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')} \tag{7.19.4}$$
$$\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')} \tag{7.19.5}$$
$$\Pi^{\mathfrak{g}'\oplus\mathfrak{g},\Delta(G')}_{\mathfrak{g}'\oplus\mathfrak{g},\Delta(K')}((V')^*_{K'} \otimes V), \tag{7.19.6}$$

where $\Pi^{\mathfrak{g}'\oplus\mathfrak{g},\Delta(G')}_{\mathfrak{g}'\oplus\mathfrak{g},\Delta(K')}$ is the Bernstein functor (see [40]). The third module can be written as

$$H_0(\mathfrak{g}', K'; (V')^*_{K'} \otimes V \otimes \mathbb{C}[G']).$$

Then the $\Delta(G')$-invariant subspaces of the modules are equal to

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}',K'}(V, V'),\\ &\mathrm{Hom}_{\mathfrak{g}',K'}(V', V),\\ &H_0(\mathfrak{g}', K'; (V')^*_{K'} \otimes V),\end{aligned}$$

respectively. Note that these spaces are the $\mathcal{U}(\mathfrak{g})^{G'}$-modules defined in Section 7.1.

By definition, the third module has a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module structure. The following proposition ensures that the modules (7.19.4) and (7.19.5) are $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules.

Proposition 7.20. *Retain the above settings.* $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ *is a* $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$*-module by the representation* $(\tau' \otimes \tau^*, \tau_{\mathrm{Ad}})$*, and* $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ *is a* $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$*-module by the representation* $((\tau')^* \otimes \tau, \tau_{\mathrm{Ad}})$*.*

Proof. We shall show that $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ is $\mathfrak{g}' \oplus \mathfrak{g}$-stable. The proof for $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ is the same. Take a G'-stable finite-dimensional subspace $F \subset \mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$. Then there exists a surjective $(\Delta(\mathfrak{g}'), \Delta(K'))$-homomorphism defined by the multiplication:

$$(\mathfrak{g}' \oplus \mathfrak{g}) \otimes F \to (\tau'(\mathfrak{g}') + \tau^*(\mathfrak{g}))F.$$

Since $\mathfrak{g}' \oplus \mathfrak{g}$ is a $\Delta(G')$-module, by definition, $(\tau'(\mathfrak{g}') + \tau^*(\mathfrak{g}))F$ is contained in $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$. This implies that $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ is $\mathfrak{g}' \oplus \mathfrak{g}$-stable.

By definition, the differential representation of τ_{Ad} is equal to the representation τ_{ad} of $\Delta(\mathfrak{g}')$. By the above discussion, the $\mathfrak{g}' \oplus \mathfrak{g}$-action and $\Delta(G')$-action are compatible. □

The study of the $\mathcal{U}(\mathfrak{g})^{G'}$-action on $\mathrm{Hom}_{\mathfrak{g}', K'}(V, V')$ can be reduced to that of the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-action on $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$. To do this, we prove the following two propositions.

Proposition 7.21. *Retain the settings in the above. Put*

$$\mathcal{I} := \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')} \cap \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})\Delta(\mathfrak{g}').$$

Then there is an algebra isomorphism:

$$\begin{array}{cccc} \alpha: & \mathcal{U}(\mathfrak{g})^{G'} & \simeq & \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')}/\mathcal{I} \\ & \Cup & & \Cup \\ & X & \mapsto & I \otimes X + \mathcal{I}. \end{array}$$

Proof. Since $\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})$ is a completely reducible $\Delta(G')$-module, we have

$$\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')}/\mathcal{I} \simeq (\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})/\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})\Delta(\mathfrak{g}'))^{\Delta(G')}.$$

The Poincaré–Birkhoff–Witt theorem leads to

$$(\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})/\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})\Delta(\mathfrak{g}'))^{\Delta(G')} \simeq \mathcal{U}(\mathfrak{g})^{G'}.$$

Therefore, α is an isomorphism. This finishes the proof. □

Proposition 7.22. *Let* W *be a* $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$*-module. Then the length of the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module on* $W^{\Delta(G')}$ *is bounded by the length of* W*. In particular if* W *is irreducible, then* $W^{\Delta(G')}$ *is irreducible or zero.*

Proof. Since the $\Delta(G')$-action on W is completely reducible, the functor which sends W to $W^{\Delta(G')}$ is exact. Hence the first assertion is reduced to the second assertion.

We assume that W is irreducible. Take a non-zero vector $v \in W^{\Delta(G')}$. By assumption, we have

$$\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})v = W.$$

Taking the $\Delta(G')$-invariant part, we obtain

$$\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')}v = W^{\Delta(G')}.$$

This implies that $W^{\Delta(G')}$ is irreducible as a $\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{G'}$-module. The assertion follows from this and Proposition 7.21. $\square$

Summarizing the above two results, for example, if $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module, then $\mathrm{Hom}_{\mathfrak{g}', K'}(V, V')$ is an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module.

In Section 8 and Section 9, we will estimate the length of $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ for concrete representations.

8 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules: part I

The purpose of this section is to prove that the $\mathcal{U}(\mathfrak{g})^{G'}$-module on the space of $\mathfrak{g}'$-homomorphisms between two generalized Verma modules is irreducible under some assumptions. As an application of this result, we treat the branching laws of discrete decomposable $A_{\mathfrak{q}}(\lambda)$ with quasi-abelian parabolic subalgebra $\mathfrak{q}$, in particular, discrete series representations. The main tool is the results in Section 6.

8.1 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Fix $H \in \mathfrak{g}'$ such that $\mathrm{ad}_{\mathfrak{g}}(H)$ is diagonalizable and has only real eigenvalues on $\mathfrak{g}$. As in Section 5, we construct parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ from H, and fix Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$ and Cartan subalgebras $\mathfrak{h}, \mathfrak{h}'$. Put $S := \dim_{\mathbb{C}}(\mathfrak{u}')$.

Let G be a complex connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. G' (resp. L') is denoted the analytic subgroup of G corresponding to $\mathfrak{g}'$ (resp. $\mathfrak{l}'$).

The following lemma is a key result to relate the module structure of $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ to that of a Zuckerman derived functor module. For an irreducible finite-dimensional $\mathfrak{l}'$-module F, we define

$$d(F) := F^* \otimes \mathbb{C}_{-2\rho(\mathfrak{u}')}.$$

Lemma 8.1. *Let F be an irreducible finite-dimensional $\mathfrak{l}'$-module and L be a quotient module of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F))$. Then $R^S\Gamma_{\Delta(L')}^{\Delta(G')}(L \otimes \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F))$ has a unique $\Delta(G')$-invariant vector up to scalar.*

Proof. There is a unique submodule F' of $F \otimes d(F)$ isomorphic to $\mathbb{C}_{-2\rho(\mathfrak{u}')}$. $\mathcal{U}(\Delta(\mathfrak{g}'))F'$ is an irreducible submodule and we have $R^S\Gamma_{L'}^{G'}(\mathcal{U}(\Delta(\mathfrak{g}'))F') \simeq \mathbf{1}$. We shall show

$$P_{\rho_{\mathfrak{g}'}}^{\Delta(\mathfrak{g}')}(L \otimes \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F)) = \mathcal{U}(\Delta(\mathfrak{g}'))F',$$

where $P_{\rho_{\mathfrak{g}'}}^{\Delta(\mathfrak{g}')}$ is the projection to the maximum submodule with generalized infinitesimal character $\rho_{\mathfrak{g}'}$ (see Fact 2.18). It is obvious that the left hand side contains the right hand side. Hence we may assume $L = \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F))$. Put $M := \mathrm{ind}_{\mathfrak{q}'\oplus\mathfrak{q}'}^{\mathfrak{g}'\oplus\mathfrak{g}'}(d(F) \otimes F)$.

By Proposition 2.10, there exists a $\Delta(\mathfrak{g}')$-module standard filtration $M.$ of M with

$$\mathrm{gr}(M.) \simeq \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(S(\bar{\mathfrak{u}}') \otimes F \otimes d(F))$$

as a $\Delta(\mathfrak{g}')$-module. Thus $\mathrm{gr}(P_{\rho_{\mathfrak{g}'}}^{\Delta(\mathfrak{g}')}(M.))$ can be written as $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(W)$ for some $\mathfrak{l}'$-submodule W of $S(\bar{\mathfrak{u}}') \otimes F \otimes d(F)$.

By the Weyl character formula, the infinitesimal character of an irreducible submodule W' of $W \subset S(\bar{\mathfrak{u}}') \otimes F \otimes d(F)$ is of the form

$$\rho_{\mathfrak{l}'} - 2\rho(\mathfrak{u}') + R - Q,$$

where R is a sum of elements of $\Delta^+(\mathfrak{l}', \mathfrak{h}')$ and Q is a sum of elements of $\Delta(\mathfrak{u}', \mathfrak{h}')$.

By the definition of W,

$$s(\rho_{\mathfrak{l}'} - \rho(\mathfrak{u}')) = \rho_{\mathfrak{l}'} - 2\rho(\mathfrak{u}') + R - Q + \rho(\mathfrak{u}')$$

holds for some $s \in W_{\mathfrak{g}'}$. Since $\rho_{\mathfrak{l}'} - \rho(\mathfrak{u}')$ is dominant integral with respect to $\Delta^+(\mathfrak{l}', \mathfrak{h}') \cup -\Delta(\mathfrak{u}', \mathfrak{h}')$, we have

$$s(\rho_{\mathfrak{l}'} - \rho(\mathfrak{u}')) = \rho_{\mathfrak{l}'} - \rho(\mathfrak{u}') - R' + Q',$$

where R' is a sum of elements of $\Delta^+(\mathfrak{l}', \mathfrak{h}')$ and Q' is a sum of elements of $\Delta(\mathfrak{u}', \mathfrak{h}')$. Thus we obtain $R + R' - Q - Q' = 0$. Since $R + R' - Q - Q'$ is a sum of positive roots of $\mathfrak{g}'$ with respect to $\Delta^+(\mathfrak{l}', \mathfrak{h}') \cup -\Delta(\mathfrak{u}', \mathfrak{h}')$, this implies $R = R' = Q = Q' = 0$. Therefore, we have $W = F'$. This shows the assertion. $\square$

The following corollary is a direct consequence of Lemma 8.1.

Corollary 8.2. *Retain the notation in Lemma 8.1. Let M be a proper $\mathfrak{g}' \oplus \mathfrak{g}'$-submodule of $\mathrm{ind}_{\mathfrak{q}' \oplus \mathfrak{q}'}^{\mathfrak{g}' \oplus \mathfrak{g}'}(F \otimes d(F))$. Then $P_{\rho_{\mathfrak{g}'}}^{\Delta(\mathfrak{g}')}(M) = 0$ holds.*

We give several criteria for the irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules. Let F be a finite-dimensional irreducible $\mathfrak{l}$-module with infinitesimal character λ and F' be a finite-dimensional irreducible $\mathfrak{l}'$-module. Suppose that $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is non-zero. We assume

$$\frac{2(\lambda + \rho(\mathfrak{u}), \alpha)}{(\alpha, \alpha)} \notin \{0, 1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

We write the canonical $\mathfrak{g}'$- and $\mathcal{U}(\mathfrak{g})^{G'}$-homomorphism defined by substitution as:

$$\Phi_{F'}^{F} : \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F') \otimes \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \to \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F).$$

By Lemma 8.1, for any quotient L of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$, we have

$$\begin{aligned} R^S\Gamma_{\Delta(L')}^{\Delta(G')}(L \otimes \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F') \otimes \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)))^{\Delta(G')} \\ \simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)). \end{aligned} \tag{8.2.1}$$

Thus $\Phi_{F'}^{F}$ induces a $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:

$$\widetilde{\Phi_{F'}^{F}} : \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \to R^S\Gamma_{\Delta(L')}^{\Delta(G')}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}. \tag{8.2.2}$$

The submodule lattice of $R^S\Gamma_{\Delta(L')}^{\Delta(G')}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is isomorphic to that of L by Theorem 6.1. Therefore, it is important to know when $\widetilde{\Phi_{F'}^{F}}$ is injective.

Lemma 8.3. *Under the above notation, we have*

$$P_{\rho_{\mathfrak{g}'}}^{\Delta(\mathfrak{g}')}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F')) \otimes \mathrm{Ker}(\Phi_{F'}^{F})) = 0.$$

Proof. Let M be a unique proper maximal submodule of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$. By Corollary 8.2, it is enough to show

$$\mathrm{Ker}(\Phi_{F'}^{F}) \subset M \otimes \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)).$$

Take a non-zero vector $v \in \mathrm{Ker}(\Phi_{F'}^{F})$. We can assume that v is a $\mathfrak{b}'$-eigenvector. We write $v = \sum_{i=0}^{r} v_i \otimes \varphi_i$, where $\{v_i\}_{i=0}^{r} \subset \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$ and $\{\varphi_i\}_{i=0}^{r} \subset \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ such that $\{\varphi_i\}_{i=0}^{r}$ is linearly independent. Then each v_i is a $\mathfrak{b}'$-eigenvector with the same weight μ.

If μ is not the highest weight of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$, v_i is in M, and this shows the assertion. If μ is the highest weight of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$, each v_i is a highest weight vector of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$. Then we may assume $r = 0$. Since $v \in \mathrm{Ker}(\Phi_{F'}^{F})$, we have $\varphi_0(v_0) = 0$. However, this implies $\varphi_0 = 0$ and $v = 0$ because v_0 is a cyclic vector of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$. This contradicts to $v \neq 0$. We have proved the lemma. $\square$

For short, we write $R^S\Gamma_{\Delta(L')}^{\Delta(G')}$ as Γ^S.

Lemma 8.4. *In the above settings, we have $\mathcal{U}(\mathfrak{g})^{G'}$-module isomorphisms:*

$$\begin{aligned}\Gamma^S(L \otimes \mathrm{Dom}(\Phi_{F'}^{F}))^{\Delta(G')} &\simeq \Gamma^S(L \otimes \mathrm{Im}(\Phi_{F'}^{F}))^{\Delta(G')} \\ &\simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)),\end{aligned}$$

where we denote by Dom *the domain of a map.*

Proof. We consider the following $\mathfrak{g}'$- and $\mathcal{U}(\mathfrak{g})^{G'}$-module exact sequence:

$$0 \to L \otimes \mathrm{Ker}(\Phi_{F'}^{F}) \to L \otimes \mathrm{Dom}(\Phi_{F'}^{F}) \to L \otimes \mathrm{Im}(\Phi_{F'}^{F}) \to 0.$$

Applying the Zuckerman derived functor to the exact sequence, we obtain the following exact sequence:

$$\begin{aligned}\Gamma^S(L \otimes \mathrm{Ker}(\Phi_{F'}^{F})) &\to \Gamma^S(L \otimes \mathrm{Dom}(\Phi_{F'}^{F})) \\ &\to \Gamma^S(L \otimes \mathrm{Im}(\Phi_{F'}^{F})) \to \Gamma^{S+1}(L \otimes \mathrm{Ker}(\Phi_{F'}^{F})).\end{aligned}$$

By Lemma 8.3, the $\Delta(G')$-invariant part of $\Gamma^i(L \otimes \mathrm{Ker}(\Phi_{F'}^{F}))$ is trivial for any i. This implies

$$\Gamma^S(L \otimes \mathrm{Dom}(\Phi_{F'}^{F}))^{\Delta(G')} \simeq \Gamma^S(L \otimes \mathrm{Im}(\Phi_{F'}^{F}))^{\Delta(G')}.$$

As in (8.2.1), by Lemma 8.1 we have

$$\Gamma^S(L \otimes \mathrm{Dom}(\Phi_{F'}^{F}))^{\Delta(G')} \simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)).$$

These isomorphisms imply the assertion. $\square$

Lemma 8.5. *Retain the above settings. Suppose* $L \otimes \mathrm{Coker}(\Phi_{F'}^{F})$ *has a standard filtration as a* $\Delta(\mathfrak{g}')$*-module. Then* $\widetilde{\Phi_{F'}^{F}}$ *defined in* (8.2.2) *is injective.*

Proof. We consider the following $\mathfrak{g}'$- and $\mathcal{U}(\mathfrak{g})^{G'}$-module exact sequence:

$$0 \to L \otimes \mathrm{Im}(\Phi_{F'}^{F}) \to L \otimes \mathrm{Codom}(\Phi_{F'}^{F}) \to L \otimes \mathrm{Coker}(\Phi_{F'}^{F}) \to 0.$$

Here we denote by Codom the codomain of a map. The Zuckerman derived functor induces the following exact sequence:

$$\Gamma^{S-1}(L \otimes \mathrm{Coker}(\Phi_{F'}^{F})) \to \Gamma^{S}(L \otimes \mathrm{Im}(\Phi_{F'}^{F})) \to \Gamma^{S}(L \otimes \mathrm{Codom}(\Phi_{F'}^{F})).$$

By the assumption that $L \otimes \mathrm{Coker}(\Phi_{F'}^{F})$ has a standard filtration, Fact 5.1) leads to $\Gamma^{S-1}(L \otimes \mathrm{Coker}(\Phi_{F'}^{F})) = 0$. Hence we have an injection:

$$\Gamma^{S}(L \otimes \mathrm{Im}(\Phi_{F'}^{F})) \hookrightarrow \Gamma^{S}(L \otimes \mathrm{Codom}(\Phi_{F'}^{F})).$$

This and Lemma 8.4 imply that

$$\widetilde{\Phi_{F'}^{F}} : \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \to \Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}$$

is injective. □

Theorem 8.6. *Retain the above settings. Suppose that* L *is a unique irreducible quotient of* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$*, and* $L \otimes \mathrm{Coker}(\Phi_{F'}^{F})$ *has a standard filtration as a* $\Delta(\mathfrak{g}')$*-module. Then* $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is irreducible as a* $\mathcal{U}(\mathfrak{g})^{G'}$*-module.*

Proof. Since $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)(= \mathrm{Codom}(\Phi_{F'}^{F}))$ satisfies the condition of Theorem 6.1, $\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. Therefore, $\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}$ is an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module by Proposition 7.22.

By Lemma 8.5, $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ can be embedded in the irreducible module $\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}$. Hence $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is irreducible as a $\mathcal{U}(\mathfrak{g}')^{G'}$-module. We have proved the theorem. □

The following criterion is useful to prove the irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules.

Corollary 8.7. *Suppose that* L *is a unique irreducible quotient of* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$*, and* $\mathrm{Im}(\Phi_{F'}^{F})$ *is a direct summand of* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$*. (In particular, if* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ *is a completely reducible* $\mathfrak{g}'$*-module, this condition holds.) Then the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module* $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is irreducible.*

Proof. By assumption, we have $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F) \simeq \mathrm{Im}(\Phi_{F'}^{F}) \oplus \mathrm{Coker}(\Phi_{F'}^{F})$. Since $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ has a standard filtration as a $\mathfrak{g}'$-module from Proposition 2.10, $\mathrm{Coker}(\Phi_{F'}^{F})$ has a standard filtration. Thus $L \otimes \mathrm{Coker}(\Phi_{F'}^{F})$ has a standard filtration as a $\Delta(\mathfrak{g}')$-module. This and Theorem 8.6 show the corollary. $\square$

Theorem 8.8. *Under the above settings, the length of the $\mathcal{U}(\mathfrak{g})^{G'}$-module $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is less than or equal to that of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$. In particular, the length is bounded by a constant depending only on $\mathfrak{g}'$.*

Remark 8.9. In general, the dimensions of the spaces of intertwining operators are unbounded. The theorem, however, asserts that the lengths of the $\mathcal{U}(\mathfrak{g})^{G'}$-modules are uniformly bounded by a constant depending only on $\mathfrak{g}'$.

Proof. We set $L = \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$. Then $L \otimes \mathrm{Coker}(\Phi_{F'}^{F})$ has a standard filtration as a $\Delta(\mathfrak{g}')$-module. Hence by Lemma 8.5,

$$\widetilde{\Phi_{F'}^{F}} : \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \to \Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}$$

is injective.

By Proposition 7.22, we have

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')}) \leq \mathrm{Len}_{\mathfrak{g}'\oplus\mathfrak{g},\Delta(G')}(\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))), \quad (8.9.3)$$

where we write Len for the length of a module. Theorem 6.1 implies that the length of $\Gamma^{S}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is equal to that of $L(= \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F')))$. By the injectivity of $\widetilde{\Phi_{F'}^{F}}$ and (8.9.3), we obtain

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))) \leq \mathrm{Len}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))).$$

This is the required inequation. $\square$

The following corollary is clear from the theorem.

Corollary 8.10. *If $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(d(F'))$ is irreducible, $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

8.2 Quasi-abelian parabolic subalgebra

If the parabolic subalgebra $\mathfrak{q}$ satisfies a good condition, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ is completely reducible as a $\mathfrak{g}'$-module. In this section, we review this condition. Retain the notation in the previous section.

We set $\mathfrak{u}'' := \mathfrak{u} \cap (\mathfrak{g}')^{\perp}$ and $\bar{\mathfrak{u}}'' := \bar{\mathfrak{u}} \cap (\mathfrak{g}')^{\perp}$. The following definition is a generalization of the definition in [12].

Definition 8.11 (quasi-abelian). $\mathfrak{q}$ is said to be *quasi-abelian* with respect to $\mathfrak{g}'$ if $(\alpha, \beta) \geq 0$ holds for any $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $\beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$.

If the nilradical of $\mathfrak{q}$ is abelian, $\mathfrak{q}$ is quasi-abelian. More precisely, the following proposition holds.

Proposition 8.12. *If $[\mathfrak{u}', \mathfrak{u}''] = 0$ holds, then $\mathfrak{q}$ is quasi-abelian. In particular, a parabolic subalgebra with abelian nilradical is quasi-abelian.*

Proof. Take $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $\beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$. Assume $(\alpha, \beta) < 0$. Then this implies $\alpha + \beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$ and $[\mathfrak{u}'_\alpha, \mathfrak{u}''_\beta] \neq 0$ because $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}'$ and $(\mathfrak{g}')^\perp$ is a $\mathfrak{g}'$-module. By assumption, $[\mathfrak{u}'_\alpha, \mathfrak{u}''_\beta] \subset [\mathfrak{u}', \mathfrak{u}''] = 0$ holds. This is contradiction. Therefore, $\mathfrak{q}$ is quasi-abelian. □

The following lemma is a generalization of Lemma in [12]. The proof is the same.

Lemma 8.13. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module. Assume that for any irreducible submodule of $F|_{\mathfrak{l}'}$, its infinitesimal character λ' satisfies*

$$\frac{2(\lambda' + \rho(\mathfrak{u}'), \alpha)}{(\alpha, \alpha)} \notin \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}').$$

Suppose that $\mathfrak{q}$ is quasi-abelian. Then $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ is completely reducible.

Proof. By Proposition 2.10, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ has a standard filtration $M.$ with

$$\mathrm{gr}(M.) \simeq \mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F \otimes S(\bar{\mathfrak{u}}'')),$$

where the $\mathfrak{u}'$-action on $\bar{\mathfrak{u}}''$ is trivial.

Take an irreducible $\mathfrak{l}'$-submodule W of $F \otimes S(\bar{\mathfrak{u}}'')$. By the Weyl character formula, the infinitesimal character of W is of the form $\lambda' - R$, where R is a sum of elements of $\Delta(\mathfrak{u}'', \mathfrak{h}')$ and λ' is the infinitesimal character of some irreducible submodule of $F|_{\mathfrak{l}'}$. Since $\mathfrak{q}$ is quasi-abelian, we have

$$\frac{2(R, \alpha)}{(\alpha, \alpha)} \in \{0, 1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}').$$

This and the assumption imply

$$\frac{2(\lambda' - R + \rho(\mathfrak{u}'), \alpha)}{(\alpha, \alpha)} \notin \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}').$$

Thus $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(W)$ is irreducible. From Proposition 2.9, this leads that $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ is completely reducible. We have shown the lemma. □

The following theorem plays an important role in studying the Zuckerman derived functor modules induced from a quasi-abelian parabolic subalgebra.

Theorem 8.14. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module with infinitesimal character λ in the good range, namely,*

$$\mathrm{Re}(\lambda + \rho(\mathfrak{u}), \alpha) < 0 \textit{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Suppose that $\mathfrak{q}$ is quasi-abelian. Then $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ is completely reducible, and each irreducible direct summand is of the form $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$ such that F' is a finite-dimensional irreducible $\mathfrak{l}'$-module in the good range.

To prove the theorem, we prepare several lemmas about the restriction of roots to a subalgebra. For a subset S of a real vector space, $\mathrm{Co}(S)$ denotes the convex hull of S.

Lemma 8.15. *Let $\mathfrak{h}''$ be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{h}$. Consider $(\mathfrak{h}')^*$ as a subspace of $\mathfrak{h}^*$ using the direct sum decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. For $\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}')$, define*

$$\Delta(\alpha) := \{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) : \beta|_{\mathfrak{h}'} = \alpha\}.$$

Then α belongs to $\mathrm{Co}(\Delta(\alpha))$.

Proof. Fix a Cartan involution θ of $\mathfrak{g}$ such that $\mathfrak{g}', \mathfrak{h}$ and $\mathfrak{h}'$ are θ-stable. In fact, since $\mathfrak{g}'$ is reductive in $\mathfrak{g}$, such an involution exists. Then θ is conjugate-linear. For any $\alpha \in \mathfrak{h}^*$, we define $H_\alpha \in \mathfrak{h}$ satisfying

$$\beta(H_\alpha) = (\beta, \alpha) \text{ for any } \beta \in \mathfrak{h}^*.$$

Take $\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}')$ and a non-zero element $X \in \mathfrak{g}'_\alpha$, and write $X = \sum_{\beta \in \Delta(\alpha)} X_\beta$ such that $X_\beta \in \mathfrak{g}_\beta$. Then we have

$$\begin{aligned} H_\alpha &= [X, \theta(X)]/(X, \theta(X)), \\ H_\beta &= [X_\beta, \theta(X_\beta)]/(X_\beta, \theta(X_\beta)) \end{aligned}$$

for any $\beta \in \Delta(\alpha)$ such that $X_\beta \neq 0$. This implies

$$\begin{aligned} H_\alpha &= [X, \theta(X)]/(X, \theta(X)) \\ &= (X, \theta(X))^{-1} \sum_{\beta \in \Delta(\alpha)} [X_\beta, \theta(X_\beta)] \\ &= \sum_{\beta \in \Delta(\alpha)} \frac{(X_\beta, \theta(X_\beta))}{(X, \theta(X))} H_\beta. \end{aligned}$$

By the definition of H_α and H_β, this leads to

$$\alpha = \sum_{\beta\in\Delta(\alpha)} \frac{(X_\beta, \theta(X_\beta))}{(X, \theta(X))}\beta. \tag{8.15.4}$$

Since θ is a Cartan involution, $(X, \theta(X))$ and $(X_\beta, \theta(X_\beta))$ are non-negative. Thus α is a linear combination of elements of $\Delta(\alpha)$ with positive coefficients.

Restricting the equation (8.15.4) to $\mathfrak{h}'$, we obtain

$$\alpha = \sum_{\beta\in\Delta(\alpha)} \frac{(X_\beta, \theta(X_\beta))}{(X, \theta(X))}\alpha.$$

This implies

$$\sum_{\beta\in\Delta(\alpha)} \frac{(X_\beta, \theta(X_\beta))}{(X, \theta(X))} = 1,$$

which is the desired conclusion. $\square$

Lemma 8.16. *Let λ_1 and λ_2 be dominant integral weights of $\mathfrak{l}$. Then we have* $\mathrm{Co}(W_\mathfrak{l}\lambda_1) + \mathrm{Co}(W_\mathfrak{l}\lambda_2) = \mathrm{Co}(W_\mathfrak{l}(\lambda_1 + \lambda_2))$.

Proof. $\mathrm{Co}(W_\mathfrak{l}\lambda_1)+\mathrm{Co}(W_\mathfrak{l}\lambda_2) \supset \mathrm{Co}(W_\mathfrak{l}(\lambda_1+\lambda_2))$ is obvious because $\mathrm{Co}(W_\mathfrak{l}\lambda_1)+\mathrm{Co}(W_\mathfrak{l}\lambda_2) = \mathrm{Co}(W_\mathfrak{l}\lambda_1 + W_\mathfrak{l}\lambda_2)$ holds.

We show the converse inclusion. It is enough to prove $\lambda_1 + s(\lambda_2) \in \mathrm{Co}(W_\mathfrak{l}(\lambda_1 + \lambda_2))$ for any $s \in W_\mathfrak{l}$. If s is the identity, this is trivial. We assume that s is not the identity. By assumption, there is a unique non-zero homomorphism up to scalar multiplication:

$$p : F^\mathfrak{l}(\lambda_1) \otimes F^\mathfrak{l}(\lambda_2) \to F^\mathfrak{l}(\lambda_1 + \lambda_2).$$

We fix a highest weight vector v_i of $F^\mathfrak{l}(\lambda_i)$. Then we have $p(v_1 \otimes v_2) \neq 0$.

Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b} \cap \mathfrak{l}$. There exists $X \in \mathcal{U}(\mathfrak{n})$ such that $Xs(v_2) = v_2$. Since s is not the identity, the constant term of X is 0. Then we have $Xv_1 = 0$ and $X(v_1 \otimes s(v_2)) = v_1 \otimes v_2$. This leads to

$$0 \neq p(v_1 \otimes v_2) = p(X(v_1 \otimes s(v_2))) = Xp(v_1 \otimes s(v_2)).$$

Thus we obtain $p(v_1 \otimes s(v_2)) \neq 0$.

Since any weight of $F^\mathfrak{l}(\lambda_1 + \lambda_2)$ is in $\mathrm{Co}(W_\mathfrak{l}(\lambda_1 + \lambda_2))$, this implies $\lambda_1 + s(\lambda_2) \in \mathrm{Co}(W_\mathfrak{l}(\lambda_1 + \lambda_2))$. This finishes the proof. $\square$

Lemma 8.17. *Under the settings in the above, there exists a character ζ of $\mathfrak{l}'$ such that $\rho(\mathfrak{l}') + \zeta \in \mathrm{Co}(W_\mathfrak{l}\rho(\mathfrak{l}))|_{\mathfrak{h}'}$.*

Proof. We choose a basis of $\mathfrak{h}'$ and extend it to a basis of $\mathfrak{h}$. We consider a lexicographical order on $\mathfrak{h}^*$ induced from the basis and a Borel subalgebra $\mathfrak{b}_{\mathfrak{l}}$ of $\mathfrak{l}$ determined by the order. Then $\mathfrak{b}_{\mathfrak{l}} \cap \mathfrak{l}'$ is a Borel subalgebra of $\mathfrak{l}'$, and $\rho(\mathfrak{l})|_{\mathfrak{h}'}$ is a dominant integral weight of $\mathfrak{l}'$. Since $\mathrm{Co}(W_{\mathfrak{l}}\rho(\mathfrak{l}))|_{\mathfrak{h}'}$ is equal to the convex hull of weights of $F^{\mathfrak{l}}(\rho(\mathfrak{l}))|_{\mathfrak{l}'}$, it suffices to show $\rho(\mathfrak{l}') + \zeta \in \mathrm{Co}(W_{\mathfrak{l}'}(\rho(\mathfrak{l})|_{\mathfrak{h}'}))$ for some character ζ.

To apply Lemma 8.16, we shall prove that $\rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}')$ is dominant integral. Take a simple root $\alpha \in \Delta^+(\mathfrak{g}', \mathfrak{h}')$. By Lemma 8.15, α can be written as

$$\alpha = \sum_{\beta \in \Delta(\alpha)} c_\beta \beta$$

with $\sum c_\beta = 1$ and $c_\beta \geq 0$. By the choice of the Borel subalgebra, $\Delta(\alpha)$ is contained in $\Delta^+(\mathfrak{l}, \mathfrak{h})$. This implies

$$\begin{aligned}
\frac{2(\rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}'), \alpha)}{(\alpha, \alpha)} &= \sum_{\beta \in \Delta(\alpha)} c_\beta \frac{2(\rho(\mathfrak{l}), \beta)}{(\alpha, \alpha)} - 1 \\
&\geq \sum_{\beta \in \Delta(\alpha)} c_\beta \frac{(\beta, \beta)}{(\alpha, \alpha)} - 1 \\
&\geq \sum_{\beta \in \Delta(\alpha)} c_\beta - 1 \\
&= 0.
\end{aligned}$$

The second inequality holds because $\beta|_{\mathfrak{h}'} = \alpha$. Thus we can apply Lemma 8.16 to $\lambda_1 = \rho(\mathfrak{l}')$ and $\lambda_2 = \rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}')$. We obtain

$$\mathrm{Co}(W_{\mathfrak{l}'}\rho(\mathfrak{l}')) + \mathrm{Co}(W_{\mathfrak{l}'}(\rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}'))) = \mathrm{Co}(W_{\mathfrak{l}'}(\rho(\mathfrak{l})|_{\mathfrak{h}'})).$$

Put $\zeta := |W_{\mathfrak{l}'}|^{-1} \sum_{s \in W_{\mathfrak{l}'}} s(\rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}'))$. Then ζ belongs to $\mathrm{Co}(W_{\mathfrak{l}'}(\rho(\mathfrak{l})|_{\mathfrak{h}'} - \rho(\mathfrak{l}')))$, and hence we have $\rho(\mathfrak{l}') + \zeta \in \mathrm{Co}(W_{\mathfrak{l}'}(\rho(\mathfrak{l})|_{\mathfrak{h}'}))$. Since ζ is $W_{\mathfrak{l}'}$-invariant, ζ defines a character of $\mathfrak{l}'$. This gives the lemma. □

Proof of Theorem 8.14. We shall prove that for any irreducible submodule F' of $F|_{\mathfrak{l}'}$, its infinitesimal character λ' satisfies

$$\mathrm{Re}(\lambda' + \rho(\mathfrak{u}'), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}', \mathfrak{h}').$$

By the same proof as Lemma 8.13, if we prove this, the assertion follows.

Any $\mathfrak{h}$-weight of F belongs to $\mathrm{Co}(W_{\mathfrak{l}}(\lambda - \rho(\mathfrak{l})))$. Lemma 8.16 and Lemma 8.17 show

$$\begin{aligned}\lambda' &\in \mathrm{Co}(W_{\mathfrak{l}}(\lambda - \rho(\mathfrak{l})))|_{\mathfrak{h}'} + \rho(\mathfrak{l}') \\ &\subset \mathrm{Co}(W_{\mathfrak{l}}(\lambda - \rho(\mathfrak{l})))|_{\mathfrak{h}'} + \mathrm{Co}(W_{\mathfrak{l}}\rho(\mathfrak{l}))|_{\mathfrak{h}'} + \zeta \\ &= \mathrm{Co}(W_{\mathfrak{l}}(\lambda))|_{\mathfrak{h}'} + \zeta\end{aligned}$$

for some character ζ of $\mathfrak{l}'$. Hence we have

$$\lambda' = \sum_{s \in W_{\mathfrak{l}}} a_s s(\lambda)|_{\mathfrak{h}'} + \zeta$$

with $\sum_{s\in W_{\mathfrak{l}}} a_s = 1$ and $a_s \geq 0$.

We compute $(\lambda' + \rho(\mathfrak{u}'), \alpha)$ for $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$. By Lemma 8.15, α can be written as

$$\alpha = \sum_{\beta \in \Delta(\alpha)} c_\beta \beta$$

with $\sum_{\beta\in\Delta(\alpha)} c_\beta = 1$ and $c_\beta \geq 0$. Since $\alpha(H) > 0$, $\Delta(\alpha)$ is contained in $\Delta(\mathfrak{u}, \mathfrak{h})$. Note that $(\zeta, \alpha) = 0$ holds because ζ is a character of $\mathfrak{l}'$. Then we have

$$\begin{aligned}\mathrm{Re}(\lambda' + \rho(\mathfrak{u}'), \alpha) &= \mathrm{Re}(\sum_{s\in W_{\mathfrak{l}}} a_s s(\lambda)|_{\mathfrak{h}'} + \rho(\mathfrak{u}'), \alpha) \\ &= \sum_{s\in W_{\mathfrak{l}}, \beta\in\Delta(\alpha)} a_s c_\beta \mathrm{Re}(\lambda, s^{-1}(\beta)) + (\rho(\mathfrak{u}'), \alpha) \\ &< \sum_{s\in W_{\mathfrak{l}}, \beta\in\Delta(\alpha)} a_s c_\beta (-\rho(\mathfrak{u}), s^{-1}(\beta)) + (\rho(\mathfrak{u}'), \alpha) \\ &= (-\rho(\mathfrak{u})|_{\mathfrak{h}'} + \rho(\mathfrak{u}'), \alpha) \\ &= -(\rho(\mathfrak{u}''), \alpha) \\ &\leq 0.\end{aligned}$$

The third inequality holds by the assumption of λ. Since $\mathfrak{u}$ is $\mathfrak{l}$-stable, $\rho(\mathfrak{u})$ is $W_{\mathfrak{l}}$-invariant, and hence the firth equality holds. The last inequality is the special case of [40, Proposition 4.68]. We have proved the theorem. □

The following corollary is a direct consequence of Theorem 8.14 and Corollary 8.7.

Corollary 8.18. *Retain the settings in Theorem* 8.14. *Let W be an irreducible direct summand of* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$. *Then* $\mathrm{Hom}_{\mathfrak{g}'}(W, \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is an irreducible* $\mathcal{U}(\mathfrak{g})^{G'}$*-module.*

8.3 Holomorphic discrete series representations

Holomorphic discrete series representations satisfy the conditions of Theorem 8.14. In this section, we summarize the results related to holomorphic discrete series representations.

Let $\mathfrak{g}_{\mathbb{R}}$ be real semisimple Lie algebra with maximal compact subalgebra $\mathfrak{k}_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition with respect to $\mathfrak{k}_{\mathbb{R}}$. We assume that $\mathfrak{g}_{\mathbb{R}}$ is a direct sum of simple Lie algebras of Hermitian type. Fix a characteristic element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_{\mathbb{R}})$ such that $\mathrm{ad}(H)$ has eigenvalues $\{-1, 1\}$ on $\mathfrak{p}$. We construct the subalgebras $\mathfrak{q}, \mathfrak{p}_+, \ldots$ as in Section 2.3. In this case, $\mathfrak{l}$ is equal to $\mathfrak{k}$ and, $\mathfrak{u}$ (resp. $\bar{\mathfrak{u}}$) is equal to $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$).

By construction, the nilpotent radical of $\mathfrak{q}$ is abelian. Then we can apply Theorem 8.14 to $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)$. The following important observation is an easy consequence of Harish-Chandra's classification of holomorphic discrete series representations (see Fact 2.11:) for any irreducible unitary representation F of $\mathfrak{k}_{\mathbb{R}}$, F is in the good range if and only if $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)$ is unitarizable and isomorphic to the underlying Harish-Chandra module of a holomorphic discrete series representations.

Let $\mathfrak{g}'_{\mathbb{R}}$ be a reductive subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with Cartan decomposition $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \cap \mathfrak{g}'_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} \cap \mathfrak{g}'_{\mathbb{R}}$. Suppose $H \in \mathfrak{g}'_{\mathbb{R}}$. We denote by $\mathfrak{q}', \mathfrak{k}', \ldots$ the intersection of $\mathfrak{g}'$ and $\mathfrak{q}, \mathfrak{k}, \ldots$. Let $G_{\mathbb{R}}$ be a simply-connected connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, and $G'_{\mathbb{R}}$ be the analytic subgroup of $G_{\mathbb{R}}$ corresponding to $\mathfrak{g}'_{\mathbb{R}}$.

Proposition 2.27 and Theorem 8.14 show the following theorem.

Theorem 8.19. *Retain the above notation. Let (π, V) be a holomorphic discrete series representation of $G_{\mathbb{R}}$. Then $V|_{G'_{\mathbb{R}}}$ is discretely decomposable, and each irreducible direct summand is a holomorphic discrete series representation of $G'_{\mathbb{R}}$. For an irreducible direct summand W of V, $\mathrm{Hom}_{\mathfrak{g}'}(W_{K'_{\mathbb{R}}}, V_{K_{\mathbb{R}}})$ is an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module. Moreover, we have*

$$\mathrm{PI.deg}(\pi(\mathcal{U}(\mathfrak{g})^{G'})) = \mathcal{M}_{G'_{\mathbb{R}}}(V).$$

Remark 8.20. The first assertion is well-known. In fact, any discrete spectrum of the restriction of a discrete series representation is also a discrete series representation (see [47, Corollary 8.7]). For the discretely decomposable branching laws of holomorphic discrete series representations, we refer the reader to [29], [52] and [53].

In the first assertion, we regard a finite-dimensional unitary representation as a holomorphic discrete series representation.

8.4 Zuckerman derived functor modules

As an application of Theorem 8.14, we consider the branching problem of Zuckerman derived functor modules induced from quasi-abelian parabolic subalgebras.

Let $G_{\mathbb{R}}$ be a connected real semisimple Lie group with finite center and Cartan involution θ. Put $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$. Fix $H \in \sqrt{-1}\mathfrak{k}_{\mathbb{R}}$ and define subalgebras $\mathfrak{q}, \mathfrak{l}, \mathfrak{u}, \ldots$ as in Section 2.2. Then $\mathfrak{q}$ is a θ-stable parabolic subalgebra.

We set $K_L = Z_K(H)$. Then K_L is connected and $(\mathfrak{l}, K_L)$ is a subpair of the pair $(\mathfrak{g}, K)$. For an $(\mathfrak{l}, K_L)$-module V, we define a $(\mathfrak{g}, K)$-module

$$\mathcal{L}_{\mathfrak{q},i}^{\mathfrak{g}}(V) := R^i\Gamma_{K_L}^{K}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)).$$

Remark that in many books and papers (e.g. [40]), this module is defined by the Bernstein functor, and our parametrization is so-called unnormalized version (in [40], $\mathcal{L}$ is written as ${}^{\mathrm{u}}\mathcal{L}$). Put $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. For an $(\mathfrak{l}, K_L)$-module V with infinitesimal character λ, we will say that λ and V are in the good range if λ satisfies

$$\mathrm{Re}(\lambda + \rho(\mathfrak{u}), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Fact 8.21. *Let V be an irreducible $(\mathfrak{l}, K_L)$-module with infinitesimal character λ.*

(a) *$\mathcal{L}_{\mathfrak{q},i}^{\mathfrak{g}}(V)$ has the infinitesimal character $\lambda + \rho(\mathfrak{u})$.*

(b) *If λ is in the good range, $\mathcal{L}_{\mathfrak{q},i}^{\mathfrak{g}}(V)$ is zero for $i \neq S$ and non-zero irreducible for $i = S$.*

Hereafter, we assume that $\mathfrak{k}$ is the direct sum of two non-zero ideals:

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2,$$

H belongs to $\sqrt{-1}(\mathfrak{k}_1)_{\mathbb{R}}$ and $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{k}_1$. This implies that $\mathfrak{k}_L$ contains $\mathfrak{k}_2$. Hence we have $\mathfrak{u} \cap \mathfrak{k} \subset \mathfrak{k}_1$. We denote by K_i the analytic subgroup of K corresponding to $\mathfrak{k}_i$. Remark that the subgroup K_1 is the same as in [16] and [8].

By the following lemma, we can reduce the branching law of $\mathcal{L}_{\mathfrak{q},S}^{\mathfrak{g}}(F)$ to that of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$. The proof is essentially the same as [16, Lemma 7].

Lemma 8.22. *Let V be an irreducible $(\mathfrak{l}, K_L)$-module and K' is a reductive subgroup of K containing K_1. Under the above settings, we have a $(\mathfrak{g}, K')$-module isomorphism:*

$$\mathcal{L}_{\mathfrak{q},i}^{\mathfrak{g}}(V) \simeq R^i\Gamma_{K' \cap K_L}^{K'}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)).$$

Proof. Put $K'_L := K' \cap K_L$. By Fact 2.14, we have

$$\mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},i}(V) \simeq H^i(\mathfrak{k}, K_L; \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K]).$$

We consider the following complex with cohomology $H^i(\mathfrak{k}, K_L; \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K])$ (see [5]:)

$$C^i(\mathfrak{k}, K_L; \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K]) = \mathrm{Hom}_{K_L}(\wedge^i(\mathfrak{k}/\mathfrak{k}_L), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K]).$$

By the assumption $K' \supset K_1$ and $K_L \supset K_2$, the right hand side is isomorphic to

$$\begin{aligned}
&\mathrm{Hom}_{K_1}(\wedge^i(\mathfrak{k}'/\mathfrak{k}'_L), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K_2\backslash K])\\
\simeq &\mathrm{Hom}_{K_1}(\wedge^i(\mathfrak{k}'/\mathfrak{k}'_L), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[(K' \cap K_2)\backslash K'])\\
\simeq &\mathrm{Hom}_{K' \cap K'_L}(\wedge^i(\mathfrak{k}'/\mathfrak{k}'_L), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K']).
\end{aligned}$$

Taking its cohomology, we obtain

$$\begin{aligned}
H^i(\mathfrak{k}, K_L; \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K]) &\simeq H^i(\mathfrak{k}', K'_L; \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V) \otimes \mathbb{C}[K'])\\
&\simeq R^i\Gamma^{K'}_{K'_L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(V)).
\end{aligned}$$

This gives the lemma. □

Proposition 8.23. *Let $\mathfrak{g}'$ be a θ-stable reductive subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}' \supset \mathfrak{k}_1$. Then $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{g}'$.*

Proof. Note that $\mathfrak{g}'$ contains H as an element because $H \in \mathfrak{k}_1$. Fix a fundamental Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}'$ which is contained in $\mathfrak{g}' \cap \mathfrak{l}$ and contains H. Define $\mathfrak{u}' := \mathfrak{u} \cap \mathfrak{g}'$ and $\mathfrak{u}'' := \mathfrak{u} \cap (\mathfrak{g}')^{\perp}$. Then by assumption, we have $\mathfrak{u}'' \subset \mathfrak{k}^{\perp}$.

Assume that $\mathfrak{q}$ is not quasi-abelian with respect to $\mathfrak{g}'$. Then there exist $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $\beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$ such that $(\alpha, \beta) < 0$. Since $(\mathfrak{g}')^{\perp}$ is a $\mathfrak{g}'$-module and $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}'$, this implies $[\mathfrak{u}'_\alpha, \mathfrak{u}''_\beta] \neq 0$. There is two possibilities:

(a) $\mathfrak{u}'_\alpha \subset \mathfrak{k}^{\perp}$;

(b) $(\mathfrak{u}'_\alpha \oplus \mathfrak{u}'_{\theta(\alpha)}) \cap \mathfrak{k} \neq 0$.

Assume (a). Using $[\mathfrak{u}', \mathfrak{u}''] \subset \mathfrak{u}''$, $\mathfrak{u}'' \subset \mathfrak{k}^{\perp}$ and $[\mathfrak{k}^{\perp}, \mathfrak{k}^{\perp}] \subset \mathfrak{k}$, we have

$$[\mathfrak{u}'_\alpha, \mathfrak{u}''_\beta] \subset \mathfrak{u}'' \cap \mathfrak{k} \subset \mathfrak{k}^{\perp} \cap \mathfrak{k} = 0.$$

This contradicts $[\mathfrak{u}'_\alpha, \mathfrak{u}''_\beta] \neq 0$.

Assume (b). We put $\mathfrak{h}'_{\mathfrak{k}} := \mathfrak{h}' \cap \mathfrak{k}$. Remark that $\mathfrak{h}'_{\mathfrak{k}}$ contains a Cartan subalgebra of $\mathfrak{k}_1$. Since $\mathfrak{u}''_\beta \subset \mathfrak{k}^\perp$, β is θ-invariant. Thus we have

$$(\alpha|_{\mathfrak{h}'_{\mathfrak{k}}}, \beta|_{\mathfrak{h}'_{\mathfrak{k}}}) = (\alpha, \beta) < 0. \tag{8.23.5}$$

By the assumption (b), $\alpha|_{\mathfrak{h}'_{\mathfrak{k}}}$ belongs to $\Delta(\mathfrak{u}\cap\mathfrak{k}, \mathfrak{h}'_{\mathfrak{k}})$. Therefore, the inequation (8.23.5) contradicts the assumption that $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{k}_1$. This finishes the proof. $\square$

By Proposition 8.23, we can apply Theorem 8.14 to any reductive subalgebra containing $\mathfrak{k}_1$. Let $G'_{\mathbb{R}}$ be a connected reductive subgroup of $G_{\mathbb{R}}$ closed under θ. Assume $\mathfrak{g}' \supset \mathfrak{k}_1$. Put $K'_{\mathbb{R}} := (G'_{\mathbb{R}})^\theta$ and define subalgebras $\mathfrak{q}', \mathfrak{l}', \mathfrak{u}', \ldots$ as in Section 2.2. We set $K'_L := K' \cap K_L$. Under these settings, $S(= \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})) = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k}')$ holds.

Theorem 8.24. *Let F be a finite-dimensional irreducible $(\mathfrak{l}, K_L)$-module with infinitesimal character λ. Suppose that λ is in the good range. Then $\mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F)|_{(\mathfrak{g}',K')}$ is decomposed into a direct sum of irreducible modules of the form $\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F')$ with finite-dimensional irreducible $(\mathfrak{l}', K'_L)$-module F' in the good range. Moreover, $R^S\Gamma^{K'}_{K'_L}$ induces the following $\mathcal{U}(\mathfrak{g})^{G'}$-isomorphism:*

$$\mathrm{Hom}_{\mathfrak{g}',K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) \simeq \mathrm{Hom}_{\mathfrak{g}',K'}(\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F'), \mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F)),$$

and $\mathrm{Hom}_{\mathfrak{g}',K'}(\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F'), \mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F))$ *is zero or irreducible as a $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

Proof. Lemma 8.22 gives a $(\mathfrak{g}, K')$-module isomorphism:

$$\mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F) \simeq R^S\Gamma^{K'}_{K'_L}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)).$$

By Theorem 8.14, $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)|_{\mathfrak{g}',K'}$ is completely reducible, and any irreducible $(\mathfrak{g}', K')$-submodule of $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)$ is of the form $\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F')$ with F' in the good range. Then we have

$$\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)|_{(\mathfrak{g}',K'_L),\mathcal{U}(\mathfrak{g})^{G'}} \simeq \bigoplus_{F'} \mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F') \otimes \mathrm{Hom}_{\mathfrak{g}',K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)),$$

and hence

$$\mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F)|_{(\mathfrak{g}',K'),\mathcal{U}(\mathfrak{g})^{G'}} \simeq \bigoplus_{F'} \mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F') \otimes \mathrm{Hom}_{\mathfrak{g}',K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)).$$

The sum is taken over all finite-dimensional irreducible $(\mathfrak{l}', K'_L)$-modules in the good range.

By Fact 8.21, each $\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F')$ is non-zero irreducible, and for $F'' \not\simeq F'$ in the good range, $\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F'')$ is not isomorphic to $\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F')$ because their infinitesimal characters are different. This implies

$$\mathrm{Hom}_{\mathfrak{g}',K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) \simeq \mathrm{Hom}_{\mathfrak{g}',K'}(\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{q}',S}(F'), \mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F)).$$

This isomorphism is induced by the functor $R^S\Gamma^{K'}_{K'_L}$. The irreducibility of the $\mathcal{U}(\mathfrak{g})^{G'}$-module follows from Corollary 8.18. We have proved the theorem. □

8.5 Discrete series representations

One of important examples of Theorem 8.24 is discrete series representations.

Let $G_\mathbb{R}$ be a connected real simple Lie group with finite center and Cartan involution θ, and $G'_\mathbb{R}$ be a θ-stable connected non-compact reductive subgroup of $G_\mathbb{R}$. We put $K_\mathbb{R} := G^\theta_\mathbb{R}$ and $K'_\mathbb{R} := (G'_\mathbb{R})^\theta$. We assume that $(G_\mathbb{R}, G'_\mathbb{R})$ is a symmetric pair and $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{k})$.

Fix a Cartan subgroup $H_\mathbb{R}$ of $K_\mathbb{R}$ satisfying that $H_\mathbb{R} \cap K'_\mathbb{R}$ is a Cartan subgroup of $K'_\mathbb{R}$. Then $H_\mathbb{R}$ is a Cartan subgroup of $G_\mathbb{R}$. Take a θ-stable Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let $\mathfrak{n}$ denote the nilpotent radical of $\mathfrak{b}$. We set $S' := \dim_\mathbb{C}(\mathfrak{n})$.

Under these settings, for a unitary character $\mathbb{C}_\lambda$ of $H_\mathbb{R}$ in the good range with respect to $\mathfrak{b}$, $\mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda)$ is a underlying Harish-Candra module of a discrete series representation of $G_\mathbb{R}$. By the classification ([60]) of discretely decomposable $A_\mathfrak{q}(\lambda)$, we can see the following fact (see also [8] and [82, Section 9]).

Fact 8.25. *Let $\mathbb{C}_\lambda$ be a unitary character of $H_\mathbb{R}$ in the good range with respect to $\mathfrak{b}$. Suppose that $\mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda)|_{(\mathfrak{g}',K')}$ is discretely decomposable. Then there exists an element $H \in \mathfrak{g}' \cap \sqrt{-1}\mathfrak{h}_\mathbb{R}$ such that the corresponding θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ satisfies the following conditions:*

(a) $\mathfrak{q} \supset \mathfrak{b}$;

(b) $\mathfrak{l} \subset \mathfrak{k}$;

(c) $\mathfrak{q}$ *is quasi-abelian with respect to* $\mathfrak{k}$;

(d) *there exists an ideal $\mathfrak{k}_1$ of $\mathfrak{k}$ such that $\mathfrak{u} \cap \mathfrak{k} \subset \mathfrak{k}_1, H \in \mathfrak{k}_1$ and $\mathfrak{k}_1 \subset \mathfrak{g}'$;*

(e) *there exists an irreducible unitary representation F of $\mathfrak{l}_\mathbb{R}$ in the good range such that*

$$\mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda) \simeq \mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(F),$$

where $S = \dim_\mathbb{C}(\mathfrak{u})$.

Remark 8.26. If the first four conditions hold, the last condition is automatically satisfied for $F = \mathcal{L}^{\mathfrak{l}}_{\mathfrak{b}\cap\mathfrak{l},S'-S}(\mathbb{C}_\lambda)$. In fact, by induction in stages (see [40, Corollary 11.86]) and the vanishing theorem, we have

$$\mathcal{L}^{\mathfrak{g}}_{\mathfrak{q},S}(\mathcal{L}^{\mathfrak{l}}_{\mathfrak{b}\cap\mathfrak{l},S'-S}(\mathbb{C}_\lambda)) \simeq \mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S}(\mathbb{C}_\lambda).$$

For convenience, we construct the parabolic subalgebra $\mathfrak{q}$ in the above fact. Define $H \in \sqrt{-1}\mathfrak{h}_{\mathbb{R}}$ such that

$$\alpha(H) = \begin{cases} 0 & \text{if } \alpha \text{ is a compact simple root,} \\ 1 & \text{if } \alpha \text{ is a non-compact simple root.} \end{cases}$$

We construct subalgebras $\mathfrak{q}, \mathfrak{u}, \mathfrak{l}$ as in Section 2.2. Then it is easy to see $\mathfrak{q} \supset \mathfrak{b}, \mathfrak{l} \subset \mathfrak{k}$ and $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{n} \cap \mathfrak{p}$.

We set $\mathfrak{u}_n := \mathfrak{u} \cap \mathfrak{p}$ and $\mathfrak{u}_c := \mathfrak{k} \cap \mathfrak{u}$. By construction, the set of simple roots of $[\mathfrak{l}, \mathfrak{l}]$ is equal to the set of compact simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. This implies

$$[\mathfrak{u}_n, \mathfrak{u}_n] = \mathfrak{u}_c.$$

Let $\mathfrak{k}_1$ be the ideal of $\mathfrak{k}$ generated by $\mathfrak{u}_c$ and H. The theorem by M. Duflo and J. A. Vargas in [8] asserts that $\mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda)|_{(\mathfrak{g}',K')}$ is discretely decomposable if and only if $\mathfrak{g}'$ contains $\mathfrak{k}_1$. Hence by assumption, $\mathfrak{g}'$ contains $\mathfrak{k}_1$. This is the condition (d).

We shall show that $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{k}$. By Proposition 8.12, it is enough to show $[\mathfrak{u}_c, \mathfrak{u}_n] = 0$. We write σ for the involution defining the symmetric pair $(\mathfrak{g}, \mathfrak{g}')$. Then we have

$$[\mathfrak{u}_n^\sigma, \mathfrak{u}_n^{-\sigma}] \subset \mathfrak{u}_c^{-\sigma} \subset \mathfrak{k}_1^{-\sigma} = 0. \tag{8.26.6}$$

Let $\mathfrak{i}'$ denote the subalgebra of $\mathfrak{g}'$ generated by $\mathfrak{p}^\sigma$. Then $\mathfrak{i}'$ is an ideal of $\mathfrak{g}'$, and $\mathfrak{g}'$ is the direct sum of $\mathfrak{i}'$ and an ideal $\mathfrak{j}'$ of $\mathfrak{k}'$. We need the following lemma to prove $[\mathfrak{u}_n^\sigma, \mathfrak{u}_n^\sigma] = \mathfrak{u}_c$.

Lemma 8.27. $[\mathfrak{u}_n^\sigma, \mathfrak{k}^{-\sigma}] = \mathfrak{u}_n^{-\sigma}$.

Proof. $[\mathfrak{u}_n^\sigma, \mathfrak{k}^{-\sigma}] \subset \mathfrak{u}_n^{-\sigma}$ is obvious because $\mathfrak{k}^{-\sigma} \subset \mathfrak{l}$. For the converse inclusion, we assume $[\mathfrak{u}_n^\sigma, \mathfrak{k}^{-\sigma}] \neq \mathfrak{u}_n^{-\sigma}$. Then there exists a non-zero element $Z \in \bar{\mathfrak{u}}_n^{-\sigma}$ such that $(Z, [\mathfrak{u}_n^\sigma, \mathfrak{k}^{-\sigma}]) = 0$.

Since $[Z, \mathfrak{u}_n^\sigma] \subset \mathfrak{k}^{-\sigma}$, this implies $[Z, \mathfrak{u}_n^\sigma] = 0$. This and $[Z, \bar{\mathfrak{u}}_n^\sigma] = 0$ (8.26.6) show $Z \in N_{\mathfrak{g}}(\mathfrak{i}')$. Hence we have $N_{\mathfrak{g}}(\mathfrak{i}') \supset \mathfrak{i}' \oplus \mathfrak{j}' \oplus \mathbb{C}Z \supsetneq \mathfrak{g}'$. Since a symmetric subalgebra of a simple Lie algebra is a maximal reductive subalgebra, we obtain $N_{\mathfrak{g}}(\mathfrak{i}') = \mathfrak{g}$. However, $N_{\mathfrak{g}}(\mathfrak{i}')$ has the nontrivial ideal $\mathfrak{i}'$. This contradicts that $\mathfrak{g}$ is simple. We have proved the lemma. $\square$

Replacing σ by $\theta\sigma$ in the lemma, we obtain $[\mathfrak{u}_n^{-\sigma}, \mathfrak{k}^{-\sigma}] = \mathfrak{u}_n^{\sigma}$. By this lemma, we have

$$\begin{aligned}[\mathfrak{u}_n^{-\sigma}, \mathfrak{u}_n^{-\sigma}] &= [[\mathfrak{u}_n^{\sigma}, \mathfrak{k}^{-\sigma}], \mathfrak{u}_n^{-\sigma}] \\ &= [\mathfrak{u}_n^{\sigma}, [\mathfrak{k}^{-\sigma}, \mathfrak{u}_n^{-\sigma}]] \\ &= [\mathfrak{u}_n^{\sigma}, \mathfrak{u}_n^{\sigma}].\end{aligned}$$

The second equality follows from (8.26.6). This shows $[\mathfrak{u}_n^{\sigma}, \mathfrak{u}_n^{\sigma}] = [\mathfrak{u}_n^{-\sigma}, \mathfrak{u}_n^{-\sigma}] = \mathfrak{u}_c$. Since $[\mathfrak{u}_n^{\sigma}, \mathfrak{u}_n^{-\sigma}] = 0$, this implies $[\mathfrak{u}_n, \mathfrak{u}_c] = 0$. We have constructed the parabolic subalgebra $\mathfrak{q}$ in Fact 8.25.

From the construction, we can see that $\mathfrak{u}$ is 2-step nilpotent and $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{k}$. A discrete series representation induced from such a parabolic subalgebra is said to be small (see [16]). By Fact 8.25, we can apply Theorem 8.24 to discretely decomposable discrete series representations. We set $\mathfrak{b}' := \mathfrak{b} \cap \mathfrak{g}'$, $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$ and so on. Put $S'' := \dim_{\mathbb{C}}(\mathfrak{n}')$.

Corollary 8.28. *Retain the notation in Fact* 8.25. $\overline{\mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda)}|_{G'_{\mathbb{R}}}$ *is decomposed into a direct sum of discrete series representations of* $G'_{\mathbb{R}}$*, and any irreducible submodule is isomorphic to* $\overline{\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{b}',S''}(\mathbb{C}_{\lambda'})}$ *for some unitary character* λ' *of* $\mathfrak{h}'_{\mathbb{R}}$ *in the good range. Moreover, we have*

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{b}',S''}(\mathbb{C}_{\lambda'}), \mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda)) \simeq \mathrm{Hom}_{\mathfrak{g}',K'_L}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{q}'}(F'), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)),$$

where F and F' are irreducible modules corresponding to $\mathbb{C}_\lambda$ and $\mathbb{C}_{\lambda'}$ as in Remark 8.26*. In particular,* $\mathrm{Hom}_{\mathfrak{g}',K'}(\mathcal{L}^{\mathfrak{g}'}_{\mathfrak{b}',S''}(\mathbb{C}_{\lambda'}), \mathcal{L}^{\mathfrak{g}}_{\mathfrak{b},S'}(\mathbb{C}_\lambda))$ *is an irreducible* $\mathcal{U}(\mathfrak{g})^{G'}$*-module.*

Remark 8.29. The first study of discretely decomposable Zuckerman derived functor modules is due to T. Kobayashi. He gave a necessary and sufficient condition for the discretely decomposability of Zuckerman derived functor modules and gave examples of explicit branching laws in [42, 43, 45, 46]. The branching laws of discretely decomposable Zuckerman derived functor modules including discrete series representations have been studied by several mathematicians. In the case of algebraic $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$, B. Gross and N. Wallach computed the branching laws of discrete series representations in [16] using the technique so-called K-type transfer. For non-symmetric subgroups, M. Duflo and J. A. Vargas announced a multiplicity formula using partition function like Blattner's formula in [8]. Y. Oshima obtained the branching laws for any discretely decomposable restrictions of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ with respect to symmetric subgroups using $\mathcal{D}$-modules [82].

9 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-modules: part II

The purpose of this section is to study the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module on the space of $\Delta(G')$-finite linear maps from holomorphic discrete series representations to principal series representations of a symmetric subgroup of anti-holomorphic type. In the main theorem in this section, we prove the irreducibility of the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module with generic parameter.

9.1 Setting

Let $\mathfrak{g}_\mathbb{R}$ be a real semisimple Lie algebra with Cartan involution θ and $G_\mathbb{R}$ be a simply-connected connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$. Take an involution σ commuting with θ. Let G' denote the identity component of $\mathrm{Aut}(\mathfrak{g}_\mathbb{R})^\sigma$. We put $K_\mathbb{R} := G_\mathbb{R}^\theta, \mathfrak{p} := \mathfrak{g}^{-\theta}, G'_\mathbb{R} := \mathrm{Ad}^{-1}(G' \cap \mathrm{Int}(\mathfrak{g}_\mathbb{R})) \subset G_\mathbb{R}$ and $K'_\mathbb{R} := G'_\mathbb{R} \cap K_\mathbb{R}$. Remark that $K_\mathbb{R}$ may be non-compact. Nevertheless, $G_\mathbb{R}$ and $G'_\mathbb{R}$ have the 'Cartan decomposition':

$$G_\mathbb{R} = K_\mathbb{R} \times \exp(\mathfrak{p}_\mathbb{R}),$$
$$G'_\mathbb{R} = K'_\mathbb{R} \times \exp(\mathfrak{p}^\sigma_\mathbb{R}).$$

We assume that $(\mathfrak{g}_\mathbb{R}, \mathfrak{g}'_\mathbb{R})$ is one of the following cases:

- $\mathfrak{g}_\mathbb{R}$ is a simple Lie algebra of Hermitian type and $(\mathfrak{g}_\mathbb{R}, \mathfrak{g}'_\mathbb{R})$ is of anti-holomorphic type;
- $(\mathfrak{g}_\mathbb{R}, \mathfrak{g}'_\mathbb{R}) = (\mathfrak{g}''_\mathbb{R} \oplus \mathfrak{g}''_\mathbb{R}, \Delta(\mathfrak{g}''_\mathbb{R}))$, where $\mathfrak{g}''_\mathbb{R}$ is a simple Lie algebra of Hermitian type;

By assumption, $G_\mathbb{R}$ is of Hermitian type. We fix a characteristic element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_\mathbb{R})$ such that $\mathrm{ad}(H)$ has eigenvalues $\{1, -1\}$ on $\mathfrak{p}$. We choose H satisfying $H \in \mathfrak{g}^{-\sigma}$. This holds automatically if $\mathfrak{g}_\mathbb{R}$ is simple because $(\mathfrak{g}_\mathbb{R}, \mathfrak{g}'_\mathbb{R})$ is of anti-holomorphic type. We have the $\mathrm{ad}(H)$-eigenspace decomposition:

$$\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$$

with eigenvalue $1, 0, -1$, respectively. Then $\mathfrak{q} := \mathfrak{p}_+ \oplus \mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{g}$.

Our purpose is to study the branching law of $\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F)|_{(\mathfrak{g}', K')}$. By the assumption $H \notin \mathfrak{g}'$, $\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F)|_{(\mathfrak{g}', K')}$ is not discretely decomposable (see [46, Theorem 5.3]). However, the $(\mathfrak{g}', K')$-module structure is not complicated. In fact, $\sigma(H) = -H$ implies $\sigma(\mathfrak{p}_+) = \mathfrak{p}_-$ and this leads to

$$\mathfrak{p}^\sigma + \mathfrak{q} = \mathfrak{g}.$$

Hence by the Poincaré–Birkhoff–Witt theorem, we have

$$\begin{aligned}\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{(\mathfrak{g}',K')} &= (\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F)|_{(\mathfrak{g}',K')} \\ &\simeq \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{k})} F.\end{aligned}$$

From this, the following lemma is clear.

Lemma 9.1. *Let F be a finite-dimensional representation of $\mathfrak{k}$ and V' be a $(\mathfrak{g}', K')$-module. Then we have a natural isomorphism*

$$\operatorname{Hom}_{\mathfrak{g}',K'}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F), V') \simeq \operatorname{Hom}_{K'}(F, V')$$

as a vector space.

We will study the $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $\operatorname{Hom}_{\mathfrak{g}',K'}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F), V')$ and the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module structure on $\operatorname{Hom}_{\mathbb{C}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F), V')_{\Delta(G')}$. These modules was defined in Section 7.

The discussion in the after section is true if we divide $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ by a subgroup of the center of $G_{\mathbb{R}}$ and replace $G'_{\mathbb{R}}$ by its open subgroup as long as $\operatorname{Ad}_{\mathfrak{g}}(G'_{\mathbb{R}}) = G' \cap \operatorname{Int}(\mathfrak{g}_{\mathbb{R}})$ holds. $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{Sp}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{R})_0)$, for example, does not satisfy this condition. To check the condition, the following proposition is useful.

Proposition 9.2. *Suppose that there exists a connected complex algebraic group G such that the Lie algebra of G is $\mathfrak{g}$ and G contains $G_{\mathbb{R}}$. Let $\widetilde{G'}$ be the analytic subgroup of G with Lie algebra $\mathfrak{g}'$. If $G'_{\mathbb{R}} = \widetilde{G'} \cap G_{\mathbb{R}}$, then $\operatorname{Ad}_{\mathfrak{g}}(G'_{\mathbb{R}}) = G' \cap \operatorname{Int}(\mathfrak{g}_{\mathbb{R}})$ holds.*

Proof. Put $Z := \operatorname{Ker}(\operatorname{Ad}_{\mathfrak{g}}) \subset G$. Since G' is connected, we have $\operatorname{Ad}_{\mathfrak{g}}(\widetilde{G'}) = G'$. By definition, $\operatorname{Ad}_{\mathfrak{g}}(G_{\mathbb{R}}) = \operatorname{Int}(\mathfrak{g}_{\mathbb{R}})$ is obvious.

Since $G_{\mathbb{R}}$ is of Hermitian type, $G_{\mathbb{R}}$ has a compact Cartan subgroup $T_{\mathbb{R}}$. The center Z of G is contained in the complexification T of $T_{\mathbb{R}}$. Since Z is a finite group, Z is contained in $T_{\mathbb{R}}$ and $G_{\mathbb{R}}$. Thus we have

$$\begin{aligned}G' \cap \operatorname{Int}(\mathfrak{g}_{\mathbb{R}}) &= \operatorname{Ad}_{\mathfrak{g}}(\widetilde{G'}) \cap \operatorname{Ad}_{\mathfrak{g}}(G_{\mathbb{R}}) \\ &= \operatorname{Ad}_{\mathfrak{g}}(\widetilde{G'}Z \cap G_{\mathbb{R}}Z) \\ &= \operatorname{Ad}_{\mathfrak{g}}(\widetilde{G'}Z \cap G_{\mathbb{R}}) \\ &= \operatorname{Ad}_{\mathfrak{g}}((\widetilde{G'} \cap G_{\mathbb{R}})Z) \\ &= \operatorname{Ad}_{\mathfrak{g}}(G'_{\mathbb{R}}).\end{aligned}$$

This shows the proposition. ∎

In particular, if $G_{\mathbb{R}}$ is a real form of a simply-connected connected complex algebraic group G, $G'_{\mathbb{R}} = G^{\sigma}_{\mathbb{R}}$ satisfies the condition $\mathrm{Ad}(G'_{\mathbb{R}}) = G' \cap \mathrm{Int}(\mathfrak{g}_{\mathbb{R}})$.

We return to the case of simply-connected $G_{\mathbb{R}}$. We discuss the connected components of $M'_{\mathbb{R}}$. Let $\mathfrak{a}'_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{p}'_{\mathbb{R}}$. We put $M'_{\mathbb{R}} := Z_{K'_{\mathbb{R}}}(\mathfrak{a}'_{\mathbb{R}})$. Under our assumptions, $M'_{\mathbb{R}}$ is not complicated.

Lemma 9.3. *Let $Z_{M'_{\mathbb{R}}}$ be the center of $M'_{\mathbb{R}}$. Then we have $M'_{\mathbb{R}} = Z_{M'_{\mathbb{R}}} \cdot (M'_{\mathbb{R}})_0$.*

Proof. Put $Z := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{g}})$. Then Z is a subgroup of the center of $G_{\mathbb{R}}$. First, we show that $\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}}) = Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})} \cdot \mathrm{Ad}_{\mathfrak{g}}((M'_{\mathbb{R}})_0)$ implies the assertion.

Assume $\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}}) = Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})} \cdot \mathrm{Ad}_{\mathfrak{g}}((M'_{\mathbb{R}})_0)$. Since $M'_{\mathbb{R}} \supset Z$, this implies

$$\begin{aligned} M'_{\mathbb{R}} &= \mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})}) \cdot (M'_{\mathbb{R}})_0 Z \\ &= \mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})}) \cdot (M'_{\mathbb{R}})_0. \end{aligned}$$

Since $Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})}$ is contained in a maximal torus $T_{\mathbb{R}}$ of $\mathrm{Ad}_{\mathfrak{g}}(K_{\mathbb{R}})$, $\mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})})$ is contained in $\mathrm{Ad}_{\mathfrak{g}}^{-1}(T_{\mathbb{R}})$. $\mathrm{Ad}_{\mathfrak{g}}^{-1}(T_{\mathbb{R}})$ is abelian because $K_{\mathbb{R}} = Z_{K_{\mathbb{R}}} \times [K_{\mathbb{R}}, K_{\mathbb{R}}]$. Hence $\mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})})$ is abelian. Since the adjoint action of $\mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})})$ on $(M'_{\mathbb{R}})_0$ is trivial, $\mathrm{Ad}_{\mathfrak{g}}^{-1}(Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})})$ is contained in $Z_{M'_{\mathbb{R}}}$. This shows the assertion.

Thus it is enough to show $\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}}) = Z_{\mathrm{Ad}_{\mathfrak{g}}(M'_{\mathbb{R}})} \cdot \mathrm{Ad}_{\mathfrak{g}}((M'_{\mathbb{R}})_0)$. Hence we can assume that $G'_{\mathbb{R}} \subset \mathrm{Int}(\mathfrak{g})$. By assumption, $G'_{\mathbb{R}}$ is contained in the connected complex reductive algebraic group G'. Put $L' := Z_{G'}(\mathfrak{a}')$ and $M' := Z_{K'}(\mathfrak{a}')$. Then L' is a Levi subgroup of some parabolic subgroup of G'. Hence L' is connected and so is $[L', L']$. Since $\mathfrak{a}'_{\mathbb{R}}$ is maximal abelian in $\mathfrak{p}'_{\mathbb{R}}$, $[L', L']$ is contained in K' and hence contained in M'. This implies that there exists a subgroup Z' of Z_L such that $M' = Z' \cdot [L', L']$. Thus we have $M' = Z_{M'} \cdot M'_0$. Taking real valued points, we obtain the desired equation. □

9.2 Principal series representations

We consider principal series representations of $G'_{\mathbb{R}}$. Now $G'_{\mathbb{R}}$ may have countably many connected components. Since Lemma 9.3 holds, however, this is not a serious problem. In this section, we treat only $G'_{\mathbb{R}}$. Then we remove primes of $G'_{\mathbb{R}}$, $K'_{\mathbb{R}}$ and so on such as $G_{\mathbb{R}}, K_{\mathbb{R}}$.

Let $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{p}_{\mathbb{R}}$. Fix a set of positive roots $\Delta^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}})$ of the restricted root system. We write $\mathfrak{n}_{\mathbb{R}}$ for the sum of root spaces of positive roots. We put $M_{\mathbb{R}} := Z_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}), A_{\mathbb{R}} := \exp(\mathfrak{a}_{\mathbb{R}})$ and $N_{\mathbb{R}} := \exp(\mathfrak{n}_{\mathbb{R}})$. Then $Q_{\mathbb{R}} := M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ is a minimal parabolic subgroup of $G_{\mathbb{R}}$. We denote by $W_{\mathfrak{g}_{\mathbb{R}}}$ the little Weyl group for $\mathfrak{g}_{\mathbb{R}}$.

We fix a maximal torus $T_{\mathbb{R}}$ of $M_{\mathbb{R}}$. Then $H_{\mathbb{R}} := T_{\mathbb{R}} A_{\mathbb{R}}$ is a Cartan subgroup of $G_{\mathbb{R}}$. By Lemma 9.3, $H_{\mathbb{R}}$ is abelian.

For an irreducible unitary representation (δ, V_δ) of $M_{\mathbb{R}}$ and a (non-unitary) character $(e^\nu, \mathbb{C}_\nu)$ of $\mathfrak{a}_{\mathbb{R}}$, we define

$$I_{Q_{\mathbb{R}}}^{G_{\mathbb{R}}}(\delta, \nu) := C^\infty(G_{\mathbb{R}}/Q_{\mathbb{R}}, G_{\mathbb{R}} \times_{Q_{\mathbb{R}}} (V_\delta \otimes \mathbb{C}_\nu))_{K_{\mathbb{R}}},$$

letting $N_{\mathbb{R}}$ act on $V_\delta \otimes \mathbb{C}_\nu$ trivially. We write $I(\delta, \nu)$ for short if the groups are clear from the context.

To study an algebraic structure of $I(\delta, \nu)$, we give an algebraic realization as follows (see [40, Proposition 11.57]). Taking derivatives at e, we have

$$\begin{array}{cccc} r: & I(\delta,\nu) & \rightarrow & \mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(\mathcal{U}(\mathfrak{g}), V_\delta \otimes \mathbb{C}_\nu) \\ & \Psi & & \Psi \\ & f & \mapsto & (X \mapsto (Xf)(e)), \end{array}$$

where the $\mathcal{U}(\mathfrak{q})$-action on $\mathcal{U}(\mathfrak{g})$ is the natural right action. We denote by $\mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(\mathcal{U}(\mathfrak{g}), V_\delta \otimes \mathbb{C}_\nu)_{K_{\mathbb{R}}}$ the sum of irreducible $(\mathfrak{k}_{\mathbb{R}}, M_{\mathbb{R}})$-submodules which lift to a unitary representation of $K_{\mathbb{R}}$. Then r is a $(\mathfrak{g}, K)$-module isomorphism from $I(\delta, \nu)$ to $\mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(\mathcal{U}(\mathfrak{g}), V_\delta \otimes \mathbb{C}_\nu)_{K_{\mathbb{R}}}$.

$\mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(\mathcal{U}(\mathfrak{g}), V_\delta \otimes \mathbb{C}_\nu)_{K_{\mathbb{R}}}$ can be written as:

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})^*_{K_{\mathbb{R}}}.$$

Using this expression, we show the following lemma.

Lemma 9.4. *Let λ be the infinitesimal character of $\delta|_{(M_{\mathbb{R}})_0}$. Suppose*

$$\frac{2(-\nu - \lambda + \rho(\mathfrak{n}), \alpha)}{(\alpha, \alpha)} \notin \mathbb{Z} \text{ for any } \alpha \in \Delta(\mathfrak{n}, \mathfrak{h}).$$

(a) *For any finite-dimensional representation F of G, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^*$ is completely reducible as a $\mathfrak{g}$-module and*

(b) *if $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^*$ has the irreducible decomposition $\bigoplus_i \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta_i}^* \otimes \mathbb{C}_{-\nu_i})$ as a $(\mathfrak{g}, M_{\mathbb{R}})$-module, then $I(\delta, \nu) \otimes F \simeq \bigoplus_i I(\delta_i, \nu_i)$ holds.*

(c) *Let W be a submodule of $I(\delta, \nu)$ with $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}}}(W, I(\delta, \nu)) = 1$. Then $\mathrm{End}_{\mathbb{C}}(W)_{\Delta(G)}$ is irreducible as a $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$-module.*

Remark 9.5. Lemma 9.3 implies that the restriction of any finite-dimensional irreducible $M_{\mathbb{R}}$-representation to $(M_{\mathbb{R}})_0$ is irreducible. Hence the irreducibility of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})$ as a $(\mathfrak{g}, M_{\mathbb{R}})$-module is equivalent to the irreducibility as a $\mathfrak{g}$-module.

Proof. By the Mackey isomorphism (see [40, Theorem 2.103]), we have

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^* \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu} \otimes F^*).$$

Hence $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^*$ has a standard filtration $V_\cdot$. V_i/V_{i-1} is of the form $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F')$, where F' is an irreducible $(\mathfrak{m} \oplus \mathfrak{a}, M_\mathbb{R})$-submodule of $V_\delta^* \otimes \mathbb{C}_{-\nu} \otimes F^*$. Fact 2.7 and the assumption imply that V_i/V_{i-1} irreducible. Thus by Proposition 2.9, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^*$ is completely reducible. This shows (a).

(b) is proved by (a) and the following isomorphism:

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})^*_{K_\mathbb{R}} \otimes F \simeq (\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F^*)^*_{K_\mathbb{R}}.$$

To prove (c), we prepare a few lemmas. □

In the following lemmas, we retain the notation and the assumptions in Lemma 9.4.

Lemma 9.6. *There exists an injective map:*

$$\varphi : \mathrm{End}_\mathbb{C}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)} \to \mathrm{End}_\mathbb{C}(I(\delta, \nu))_{\Delta(G)},$$

and φ *is* $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$*-equivariant in the following sense:*

$$\varphi(X \otimes Y \cdot) = Y \otimes X \varphi(\cdot).$$

Proof. Since $I(\delta, \nu) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})^*_{K_\mathbb{R}}$, we have

$$\begin{aligned} \mathrm{End}_\mathbb{C}(I(\delta,\nu))_{\Delta(G)} &= \mathrm{Hom}_\mathbb{C}(I(\delta,\nu), I(\delta,\nu))_{\Delta(G)} \\ &\simeq \mathrm{Hom}_\mathbb{C}(I(\delta,\nu) \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}), \mathbf{1})_{\Delta(G)} \\ &\simeq \mathrm{Hom}_\mathbb{C}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}), I(\delta,\nu)^*_{\mathfrak{q}, M_\mathbb{R}})_{\Delta(G)}. \end{aligned}$$

For the second and third isomorphism, we used the fact that a $\Delta(G)$-finite linear map preserves locally-finiteness (see Proposition 7.19). Since $I(\delta, \nu) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})^*_{K_\mathbb{R}}$ again, there is a natural homomorphism:

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \to I(\delta,\nu)^*_{\mathfrak{q}, M_\mathbb{R}}.$$

By assumption, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})$ is irreducible, and hence the above homomorphism is injective. Thus we have an injective homomorphism:

$$\varphi : \mathrm{End}_\mathbb{C}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)} \to \mathrm{End}_\mathbb{C}(I(\delta, \nu))_{\Delta(G)}.$$

This is the desired homomorphism. □

Lemma 9.7. $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}$ *is an irreducible* $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$*-module.*

Proof. Fix a set of positive roots $\Delta^+(\mathfrak{m}, \mathfrak{t})$ such that $-\lambda$ is strictly dominant. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ corresponding to $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Then $\mu := -\lambda - \rho(\mathfrak{m}) - \nu$ satisfies

$$\frac{2(\mu + \rho_{\mathfrak{g}}, \alpha)}{(\alpha, \alpha)} \notin \{0, -1, -2, \ldots\} \text{ for any } \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}).$$

From Fact 6.9, the functor $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\mu), \cdot)_{\Delta(G)}$ is an exact functor and preserves irreducibility. Hence $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\mu), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}$ is irreducible. Since $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})$ has the same highest weight as $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\mu)$, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})$ is a unique irreducible quotient of $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\mu)$. Thus there is an injection:

$$\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\mu), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}.$$

This implies that $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}$ is irreducible. □

The following lemma is an analogue of [40, Proposition 7.199].

Lemma 9.8. *Let F be a finite-dimensional representation of G and χ be a infinitesimal character of $\mathfrak{g}$. Suppose*

$$P_\chi(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}) \otimes F) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta'}^*, \mathbb{C}_{-\nu'}) \otimes \mathbb{C}^r$$

for some $\delta' \in \widehat{M_{\mathbb{R}}}, \nu' \in \mathfrak{a}^*$ and $r \in \mathbb{Z}$. Then we have an algebra isomorphism:

$$\begin{aligned} &P_\chi \circ (\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(F)) \circ P_\chi \\ \simeq &\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta'}^*, \mathbb{C}_{-\nu'}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r). \end{aligned}$$

Proof. It is easy to see

$$\begin{aligned} &P_\chi \circ (\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(F)) \circ P_\chi \\ \simeq &P_\chi \circ \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}) \otimes F)_{\Delta(G)} \circ P_\chi \\ \simeq &\mathrm{End}_{\mathbb{C}}(P_\chi(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}) \otimes F))_{\Delta(G)} \\ \simeq &\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta'}^*, \mathbb{C}_{-\nu'}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r). \end{aligned}$$

□

Lemma 9.9. *Retain the settings in Lemma 9.8. Let W be an irreducible subquotient of $I(\delta, \nu)$. Then $W \otimes F$ is completely reducible and*

$$P_\chi(W \otimes F) \simeq W' \otimes \mathbb{C}^r$$

for some irreducible $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module W'.

Proof. The completely reducibility follows from the second assertion and Lemma 9.4 (a).

Since $W \otimes F$ is an irreducible $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^*, \mathbb{C}_{-\nu}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(F)$-module, $P_\chi(W \otimes F)$ is an irreducible $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta'}^*, \mathbb{C}_{-\nu'}))_{\Delta(G)} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r)$-module by Lemma 9.8.

Since $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^r)$ is a finite-dimensional simple algebra, $P_\chi(W \otimes F)$ can be written as $W' \otimes \mathbb{C}^r$ for some $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module W'. By Lemma 9.7, $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta'}^*, \mathbb{C}_{-\nu'}))_{\Delta(G)}$ is surjective. Thus W' is irreducible as a $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module. $\square$

By the lemma, we can see that W' is dependent only on χ and, in particular, independent of F. Hence W' can be written as an image of the Jantzen–Zuckerman translation functor of W.

Lemma 9.10. *Let V_1, V_2 be finite-dimensional irreducible representations of $M_{\mathbb{R}} A_{\mathbb{R}}$ and F be a finite-dimensional representation of G. Suppose*

$$\mathrm{Hom}_{M_{\mathbb{R}} A_{\mathbb{R}}}(V_1, V_2 \otimes F) \neq 0.$$

Then we have $\mathrm{Hom}_{(M_{\mathbb{R}})_0 A_{\mathbb{R}}}(V_1, V_2 \otimes F) = \mathrm{Hom}_{M_{\mathbb{R}} A_{\mathbb{R}}}(V_1, V_2 \otimes F)$.

Proof. Recall the homomorphism $\mathrm{Ad} : G_{\mathbb{R}} \to G$. $\mathrm{Ad}(M_{\mathbb{R}} A_{\mathbb{R}})$ is contained in a Levi subgroup L with Lie algebra $\mathfrak{m} \oplus \mathfrak{a}$. Since G is connected, L is connected. Note that we have a canonical isomorphism $\mathrm{Hom}_{M_{\mathbb{R}} A_{\mathbb{R}}}(V_1, V_2 \otimes F) \simeq \mathrm{Hom}_{M_{\mathbb{R}} A_{\mathbb{R}}}(V_1 \otimes V_2^*, F)$. If we show that $V_1 \otimes V_2^*$ has an L-module structure compatible with the diagonal $K_{\mathbb{R}} A_{\mathbb{R}}$-action, the assertion follows because

$$\begin{aligned}
\mathrm{Hom}_{(M_{\mathbb{R}})_0 A_{\mathbb{R}}}(V_1 \otimes V_2^*, F) &= \mathrm{Hom}_{\mathfrak{m} \oplus \mathfrak{a}}(V_1 \otimes V_2^*, F) \\
&= \mathrm{Hom}_L(V_1 \otimes V_2^*, F) \\
&\subset \mathrm{Hom}_{M_{\mathbb{R}} A_{\mathbb{R}}}(V_1 \otimes V_2^*, F) \\
&\subset \mathrm{Hom}_{(M_{\mathbb{R}})_0 A_{\mathbb{R}}}(V_1 \otimes V_2^*, F).
\end{aligned}$$

By assumption, $V_1 \otimes V_2^*$ has a non-zero $M_{\mathbb{R}} A_{\mathbb{R}}$-submodule W with L-module structure. Since V_1 and V_2^* are irreducible as $(\mathfrak{m} \oplus \mathfrak{a})$-modules, we have $V_1 \otimes V_2^* = \mathcal{U}(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{a}) W$. This implies that the diagonal $M_{\mathbb{R}} A_{\mathbb{R}}$-action on $V_1 \otimes V_2^*$ lifts to an L-action. $\square$

Proof of (c) in Lemma 9.4. We have constructed an injective homomorphism:

$$\varphi' : \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)} \to \mathrm{End}_{\mathbb{C}}(I(\delta, \nu))_{\Delta(G)}$$

in Lemma 9.6. Then we have a homomorphism:

$$\begin{array}{cccc} \varphi: & \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)} & \to & \mathrm{End}_{\mathbb{C}}(W)_{\Delta(G)} \\ & \cup & & \cup \\ & f & \to & \varphi(f)|_W \end{array}$$

This is well-defined because $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}$ is generated by id and $\varphi(\mathrm{id}) = \mathrm{id}$ holds by construction.

By Lemma 9.7, $\mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}$ is irreducible. Hence it suffices to show that φ is surjective.

We compare multiplicities of each $\Delta(G)$-type. Take a finite-dimensional irreducible $\Delta(G)$-module F. By Lemma 9.4 (a), the irreducible decomposition of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F$ can be written as:

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}) \otimes F \simeq \bigoplus_i \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta_i}^* \otimes \mathbb{C}_{-\nu_i}).$$

By Lemma 9.4 (b), we have

$$I(\delta, \nu) \otimes F^* \simeq \bigoplus_i I(\delta_i, \nu_i).$$

Then Proposition 7.19 gives

$$\begin{aligned} \mathrm{Hom}_{\Delta(G)}(F, \mathrm{End}_{\mathbb{C}}(W)_{\Delta(G)}) &\simeq \mathrm{Hom}_{\mathfrak{g}, K_{\mathbb{R}}}(W, W \otimes F^*) \\ &\subset \mathrm{Hom}_{\mathfrak{g}, K_{\mathbb{R}}}(W, I(\delta, \nu) \otimes F^*) \\ &\simeq \bigoplus_i \mathrm{Hom}_{\mathfrak{g}, K_{\mathbb{R}}}(W, I(\delta_i, \nu_i)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\mathrm{Hom}_{\Delta(G)}(F, \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}) \\ \simeq &\bigoplus_i \mathrm{Hom}_{\mathfrak{g}, K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_{\delta_i}^* \otimes \mathbb{C}_{-\nu_i}), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu})). \end{aligned}$$

By the proof of Lemma 9.4 (a), Lemma 9.10 and the assumption of ν, $I(\delta_i, \nu_i)$ has the same infinitesimal character as $I(\delta, \nu)$ if and only if $(\delta_i, \nu_i) \simeq (\delta, \nu)$. Thus we obtain

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta(G)}(F, \mathrm{End}_{\mathbb{C}}(W)_{\Delta(G)}) \\ \leq &\# \{i : (\delta_i, \nu_i) \simeq (\delta, \nu)\} \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}}}(W, I(\delta, \nu)) \\ = &\# \{i : (\delta_i, \nu_i) \simeq (\delta, \nu)\} \\ = &\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta(G)}(F, \mathrm{End}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V_\delta^* \otimes \mathbb{C}_{-\nu}))_{\Delta(G)}). \end{aligned} \tag{9.10.1}$$

We used the assumption $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(W, I(\delta,\nu)) = 1$ for the third line. Since φ is injective, (9.10.1) implies that φ is surjective. This finishes the proof. $\square$

A typical example of W in Lemma 9.4 (c) is a submodule of a spherical principal series representation generated by a non-zero $K_{\mathbb{R}}$-invariant vector. We state a lemma about a relation between principal series representations and irreducible $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-modules with one-dimensional $K_{\mathbb{R}}$-type. The following lemma is a direct consequence of the proof of Harish-Chandra's subquotient theorem (see [104, Section 3.5]).

Lemma 9.11. *Let δ be the restriction of a unitary character $(\gamma, \mathbb{C}_\gamma)$ of $K_{\mathbb{R}}$ to $M_{\mathbb{R}}$, and ν be a character of $\mathfrak{a}_{\mathbb{R}}$. Let V be a unique subquotient of $I(\delta,\nu)$ with $K_{\mathbb{R}}$-type γ. Then V is a subquotient of $I(\delta, w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}))$ for any $w \in W_{\mathfrak{g}_{\mathbb{R}}}$.*

Proof. By the Harish-Chandra isomorphism, there is an exact sequence (see [104, Theorem 3.6.6]):

$$0 \to \mathcal{U}(\mathfrak{g})^{K_{\mathbb{R}}} \cap \sum_{X\in\mathfrak{k}} \mathcal{U}(\mathfrak{g})(X-\gamma(X)) \to \mathcal{U}(\mathfrak{g})^{K_{\mathbb{R}}} \to \mathcal{U}(\mathfrak{a})^{W_{\mathfrak{g}_{\mathbb{R}}}} \to 0.$$

For two irreducible $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-module V_1, V_2, if $\mathrm{Hom}_{K_{\mathbb{R}}}(\mathbb{C}_\gamma, V_1)$ is isomorphic to $\mathrm{Hom}_{K_{\mathbb{R}}}(\mathbb{C}_\gamma, V_2)$ as a $\mathcal{U}(\mathfrak{g})^{K_{\mathbb{R}}}$-module, we have $V_1 \simeq V_2$ (see [104, Proposition 3.5.4]). By assumption and the Harish-Chandra isomorphism, we obtain the $\mathcal{U}(\mathfrak{g})^{K_{\mathbb{R}}}$-module isomorphism:

$$\mathrm{Hom}_{K_{\mathbb{R}}}(\mathbb{C}_\gamma, I(\delta,\nu)) \simeq \mathrm{Hom}_{K_{\mathbb{R}}}(\mathbb{C}_\gamma, I(\delta, w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}))).$$

Thus V is a subquotient of $I(\delta, w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}))$. $\square$

Remark 9.12. δ in the lemma is $W_{\mathfrak{g}_{\mathbb{R}}}$-invariant. In fact, $\delta(kmk^{-1}) = \gamma(k)\delta(m)\gamma(k)^{-1} = \delta(m)$ holds for any $m \in M_{\mathbb{R}}$ and $k \in N_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$.

9.3 $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module structure

We return to the branching problem of the pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ (see Section 9.1).

We take a minimal parabolic subgroup $Q'_{\mathbb{R}}$ of $G'_{\mathbb{R}}$ as in the previous section. Let $(e^\lambda, \mathbb{C}_\lambda)$ be a unitary character of $K_{\mathbb{R}}$. We denote by γ (resp. δ) the restriction of e^λ to $K'_{\mathbb{R}}$ (resp. $M'_{\mathbb{R}}$). For a character $\nu \in \mathfrak{a}^*$, we write $W(\nu)$ for a submodule of $I^{G'_{\mathbb{R}}}_{Q'_{\mathbb{R}}}(\delta,\nu)$ generated by the $K'_{\mathbb{R}}$-type γ, and write

$\overline{W}(\nu)$ for a unique irreducible quotient of $W(\nu)$. Hereafter, we consider the problem to determine the structure of

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')}$$

for $\nu \in \mathfrak{a}^*$ satisfying the condition of Lemma 9.4.

We will prove the following theorem.

Theorem 9.13. *Suppose that δ and ν satisfy the condition of Lemma 9.4 and ν is generic in some sense (see Theorem 9.35). Then the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')}$ is irreducible.*

Remark 9.14. The proof is divided into two parts: computation of the irreducible decomposition of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')}|_{(\mathfrak{g}'\oplus\mathfrak{g}', \Delta(G'))}$; and computation of the $(\mathfrak{g}')^{\perp}$-action on each irreducible component. This is an analogue of the method introduced by T. Hirai to study degenerate principal series representations of Lorentz groups [23]. A similar method is used in [73], [37], [32], [64], etc.

To prove this theorem, we prepare several lemmas. Henceforth, we assume that δ and ν satisfy the condition of Lemma 9.4. For $\mu \in (\mathfrak{a}')^*$, we denote by χ_μ the infinitesimal character of $I(\delta, \mu)$. We define

$$\Lambda := \{\mu \in (\mathfrak{a}')^* : \mathbb{C}_\mu \text{ lifts to a character of } L'/M'\}.$$

In other words, Λ is the set of characters of L' trivial on $M'_{\mathbb{R}}$.

Lemma 9.15. *We have*

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), I(\delta, \nu)) \\ =&\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W(\nu)) \\ \xrightarrow{\simeq}&\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu)).\end{aligned}$$

Moreover, the spaces are one-dimensional.

Proof. The assertion is clear from Lemma 9.1 because

$$\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W) \simeq \mathrm{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\lambda, W)$$

for any $(\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}})$-module W. □

Lemma 9.16. *Let μ be an element of Λ and W a submodule of $I(\delta, \nu)$. Suppose that $\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu), W)_{\Delta(G')}$ is non-zero. Then we have*

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu), I(\delta, \nu))_{\Delta(G')} \\ \simeq&\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu), W)_{\Delta(G')}.\end{aligned}$$

Moreover, the $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-modules are irreducible.

Proof. Since there is a natural injection

$$\operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), W)_{\Delta(G')} \hookrightarrow \operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), I(\delta,\nu))_{\Delta(G')},$$

it is enough to show that $\operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), I(\delta,\nu))_{\Delta(G')}$ is irreducible.

By Lemma 9.4 (b) and the Jantzen–Zuckerman translation functor, this is equivalent to the irreducibility of $\operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), I(\delta,\nu+\mu))_{\Delta(G')}$. By the same proof as that of Lemma 9.4 (c), $\operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), I(\delta,\nu+\mu))_{\Delta(G')}$ is irreducible. This finishes the proof. □

Lemma 9.17. *Let W be a submodule of $I(\delta,\nu)$. The following $(\mathfrak{g}'\oplus\mathfrak{g}', \Delta(G'))$-homomorphism defined by the composition of maps:*

$$\begin{aligned}\varphi : \bigoplus_{\mu\in\Lambda} \operatorname{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W(\nu+\mu)) \otimes \operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), W)_{\Delta(G')} \\ \to \operatorname{Hom}_{\mathbb{C}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W)_{\Delta(G')}.\end{aligned}$$

gives an isomorphism.

Proof. By Lemma 9.15, $\operatorname{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W(\nu+\mu))$ is one-dimansional. Hence the injectivity of φ is clear because the infinitesimal characters of $\operatorname{Hom}_{\mathbb{C}}(W(\nu+\mu), W)_{\Delta(G')}$ are mutually different and any non-zero element of $\operatorname{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W(\nu+\mu))$ is surjective.

To prove the surjectivity of φ, we compare the multiplicities of each $\Delta(G')$-type. We denote by $\chi_{\nu+\mu}$ the infinitesimal character of $I(\delta,\nu+\mu)$. Take an irreducible $\Delta(G')$-module F. Then we have

$$\begin{aligned}&\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta(G')}(F, \operatorname{Dom}(\varphi)) \\ =&\sum_{\mu\in\Lambda} \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(W(\nu+\mu), W\otimes F^*) \\ =&\sum_{\mu\in\Lambda} \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(W(\nu+\mu), P_{\chi_{\nu+\mu}}(W\otimes F^*)) \\ =&\sum_{\mu\in\Lambda} \dim_{\mathbb{C}} \operatorname{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, P_{\chi_{\nu+\mu}}(W\otimes F^*)) \\ =&\dim_{\mathbb{C}} \operatorname{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, W\otimes F^*) \\ =&\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W\otimes F^*) \\ =&\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta(G')}(F, \operatorname{Codom}(\varphi)).\end{aligned}$$

The third equality follows from $W\otimes F^* \subset \bigoplus_i I(\delta_i,\nu_i)$ (see Lemma 9.4 (b)). This shows the surjectivity of φ. □

Since $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda)|_{(\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}})} \simeq \mathcal{U}(\mathfrak{g}')\otimes_{\mathcal{U}(\mathfrak{k}')}\mathbb{C}_\lambda$, $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\cdot)_{\Delta(G')}$ is an exact functor. The following result is a direct consequence of the above lemma.

Lemma 9.18. *Let $\overline{W}$ be a subquotient of $I(\delta,\nu)$ The following $(\mathfrak{g}'\oplus\mathfrak{g}',\Delta(G'))$-homomorphism defined by the composition of maps:*

$$\overline{\varphi}:\bigoplus_{\mu\in\Lambda}\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),W(\nu+\mu))\otimes\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}$$
$$\to\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W})_{\Delta(G')}.$$

gives an isomorphism.

Proof. The proof of the injectivity is the same as Lemma 9.17.

Let p be a quotient map $W\to\overline{W}$ for some submodule W of $I(\delta,\nu)$. Then we have $\overline{\varphi}(f\otimes g)=p\circ(\varphi(f\otimes g))$. Since $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\cdot)_{\Delta(G')}$ is an exact functor,

$$p\circ:\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),W)_{\Delta(G')}\to\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W})_{\Delta(G')}$$

is surjective. Thus $\overline{\varphi}$ is surjective. □

Corollary 9.19. *Let W be a submodule of $I(\delta,\nu)$ and $\overline{W}$ be a quotient of W. The natural homomorphism*

$$\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),W)_{\Delta(G')}\to\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}$$

is surjective. Moreover, if $\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}$ is non-zero, the homomorphism is an isomorphism and the modules are irreducible.

Proof. The surjectivity is clear from Lemma 9.18. The second assertion follows from Lemma 9.16. □

By the above discussion, we can consider the $(\mathfrak{g}'\oplus\mathfrak{g}',\Delta(G'))$-isomorphism:

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W})_{\Delta(G')}\\ \simeq&\bigoplus_{\mu\in\Lambda}\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),W(\nu+\mu))\otimes\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}\end{aligned}$$

as the 'K-type decomposition'. For any $\mu\in\Lambda$, Lemma 9.15 leads to $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),W(\nu+\mu))=1$. Hence we identify

$$\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}},K_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),W(\nu+\mu))\otimes\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}$$

with $\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}$. Then we have

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W})_{\Delta(G')}=\bigoplus_{\mu\in\Lambda}\overline{\varphi}(\mathrm{Hom}_{\mathbb{C}}(W(\nu+\mu),\overline{W})_{\Delta(G')}).$$

We give another 'K-type decomposition.'

Definition 9.20. Take $\mu \in \Lambda$. Then there is a finite-dimensional representation F of G' such that $P_{\chi_{\nu+\mu}}(\overline{W}(\nu) \otimes F) \neq 0$. By Lemma 9.9, $P_{\chi_{\nu+\mu}}(\overline{W}(\nu) \otimes F)$ is a direct sum of some copies of an irreducible module. We write $\overline{W}_\nu(\nu+\mu)$ for the irreducible module.

By Lemma 9.9, $\overline{W}_\nu(\nu+\mu)$ does not depend on the choice of F. Remark that $\overline{W}_\nu(\nu+\mu)$ is an irreducible subquotient of $I(\delta, \nu+\mu)$. Hence if $\overline{W}_\nu(\nu+\mu)$ has the $K'_{\mathbb{R}}$-type γ, $\overline{W}_\nu(\nu+\mu)$ is isomorphic to $\overline{W}(\nu+\mu)$. By definition, $\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}(\nu))_{\Delta(G')}$ is non-zero. More precisely, the following lemma holds.

Lemma 9.21. $\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}(\nu))_{\Delta(G')}$ *is non-zero irreducible.*

Proof. Using the Jantzen–Zuckerman translation functor, we can assume $\mu = 0$. Since $\mathrm{Hom}_{\mathbb{C}}(\overline{W}(\nu), \overline{W}(\nu))_{\Delta(G')}$ is embedded in $\mathrm{Hom}_{\mathbb{C}}(W(\nu), \overline{W}(\nu))_{\Delta(G')}$, Corollary 9.19 gives the assertion. □

Lemma 9.22. *The following* $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$*-homomorphism defined by the composition of maps:*

$$\varphi' : \bigoplus_{\mu \in \Lambda} \mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}_\nu(\nu+\mu)) \otimes \mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}(\nu))_{\Delta(G')}$$
$$\to \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')}$$

is an isomorphism.

Proof. As in the proof of Lemma 9.17, the injectivity follows. We compare the two $\Delta(G')$-type decompositions. Take a finite-dimensional irreducible representation F of $\Delta(G')$. Then we have

$$\begin{aligned}
&\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta(G')}(F, \mathrm{Dom}(\varphi')) \\
=&\sum_{\mu \in \Lambda} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}_\nu(\nu+\mu)) \\
&\cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\overline{W}_\nu(\nu+\mu), \overline{W}(\nu) \otimes F^*) \\
=&\dim_{\mathbb{C}} \mathrm{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, \overline{W}(\nu) \otimes F^*) \\
=&\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta(G')}(F, \mathrm{Codom}(\varphi')).
\end{aligned}$$

This shows the lemma. □

9.4 $\mathfrak{g}^{-\sigma}$-action

In the previous section, we have obtained the 'K-type decomposition' of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')}$. In this section, we study the $\mathfrak{g}^{-\sigma}$-action on each 'K-type' $\varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}(\nu))_{\Delta(G')})$. Recall that $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair determined by σ. Then $\mathfrak{g}^{-\sigma}$ is equal to $\mathfrak{g}'^{\perp}$.

We define

$$\begin{aligned} \Lambda_\nu &:= \left\{\mu \in \Lambda : \mathrm{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, \overline{W}_\nu(\nu+\mu)) \neq 0\right\} \\ &= \left\{\mu \in \Lambda : \overline{W}_\nu(\nu+\mu) \simeq \overline{W}(\nu+\mu)\right\}. \end{aligned}$$

For each $\mu \in \Lambda$, we fix a non-zero element

$$\Phi_{\nu+\mu} \in \mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}', K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu+\mu)),$$

and identify $\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu+\mu))$ with $\mathbb{C}$. Then the isomorphism in Lemma 9.22 can be rewritten as

$$\varphi' : \bigoplus_{\mu \in \Lambda_\nu} \mathrm{Hom}_{\mathbb{C}}(\overline{W}(\nu+\mu), \overline{W}(\nu))_{\Delta(G')} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}(\nu))_{\Delta(G')},$$

and

$$\varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}(\nu+\mu), \overline{W}(\nu))_{\Delta(G')}) = \mathrm{Hom}_{\mathbb{C}}(\overline{W}(\nu+\mu), \overline{W}(\nu))_{\Delta(G')} \circ \Phi_{\nu+\mu},$$

where $\circ$ means the composition.

Lemma 9.23. *Let $\mu, \mu' \in \Lambda$. Then we have*

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}_\nu(\nu))_{\Delta(G')} \circ \mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu+\mu'), \overline{W}_\nu(\nu+\mu))_{\Delta(G')} \\ =&\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu+\mu'), \overline{W}_\nu(\nu))_{\Delta(G')}. \end{aligned}$$

Proof. By the definition of $\overline{W}_\nu(\cdot)$, there exists a finite-dimensional $\Delta(G')$-module F such that $\overline{W}_\nu(\nu+\mu)$ is a direct summand of $\overline{W}_\nu(\nu) \otimes F$. This implies that there exists an injective map in $\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu), \overline{W}_\nu(\nu))_{\Delta(G')}$. Hence the left hand side of the desired equation is non-zero.

By Lemma 9.21, $\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\nu+\mu+\mu'), \overline{W}_\nu(\nu))_{\Delta(G')}$ is irreducible. This shows the assertion. □

Remark that for $\mu \in \Lambda_\nu$, $\Lambda_{\nu+\mu}+\mu = \Lambda_\nu$ holds. Then putting $\Lambda'_\nu := \Lambda_\nu + \nu$, we have $\Lambda'_\mu = \Lambda'_{\mu'}$ for any $\mu, \mu' \in \Lambda'_\nu$, and

$$\overline{W}_{\nu'}(\mu) \simeq \overline{W}_{\nu''}(\mu)$$

for any $\nu', \nu'', \mu \in \Lambda'_\nu$.

Lemma 9.24. *Let $\mu \in \Lambda'_\nu$. Suppose*

$$\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g}')\mathfrak{g}^{-\sigma}\Phi_\mu = \bigoplus_{\alpha \in D(\mu)} \varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\mu(\mu+\alpha), \overline{W}_\mu(\mu))_{\Delta(G')})$$

in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}_\mu(\mu))_{\Delta(G')}$ *for some subset* $D(\mu) \subset \Lambda_\mu$. *Then we have*

$$\begin{aligned} &\mathfrak{g}^{-\sigma}\varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu), \overline{W}_\nu(\nu))_{\Delta(G')}) \\ &= \bigoplus_{\alpha \in D(\mu)} \varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu+\alpha), \overline{W}_\nu(\nu))_{\Delta(G')}). \end{aligned}$$

Proof. φ' is the restriction of the following homomorphism defined by the composition of maps:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}_\nu(\mu))_{\Delta(G')} \otimes \mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu), \overline{W}_\nu(\nu))_{\Delta(G')} \\ \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \overline{W}_\nu(\nu))_{\Delta(G')}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\mathfrak{g}^{-\sigma}\varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu), \overline{W}_\nu(\nu))_{\Delta(G')}) \\ =&\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu), \overline{W}_\nu(\nu))_{\Delta(G')} \circ (\mathfrak{g}^{-\sigma}\Phi_\mu) \\ =&\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu), \overline{W}_\nu(\nu))_{\Delta(G')} \circ \bigoplus_{\alpha \in D(\mu)} \mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu+\alpha), \overline{W}_\nu(\mu))_{\Delta(G')} \circ \Phi_{\mu+\alpha} \\ =&\bigoplus_{\alpha \in D(\mu)} \mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu+\alpha), \overline{W}_\nu(\nu))_{\Delta(G')} \circ \Phi_{\mu+\alpha} \\ =&\bigoplus_{\alpha \in D(\mu)} \varphi'(\mathrm{Hom}_{\mathbb{C}}(\overline{W}_\nu(\mu+\alpha), \overline{W}_\nu(\nu))_{\Delta(G')}). \end{aligned}$$

We used Lemma 9.23 to show the third equality. We have proved the lemma. □

By the lemma, to see the $\mathfrak{g}^{-\sigma}$-action, it is enough to study the decomposition of $\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g}')\mathfrak{g}^{-\sigma}\Phi_\mu$. Since the decomposition depends only on μ, we can assume $\mu = \nu$. The decomposition of $\mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g}')\mathfrak{g}^{-\sigma}\Phi_\nu$ in the lemma always exists.

By definition, $\alpha \in D(\nu)$ holds only if $\overline{W}_\nu(\nu+\alpha)$ is a direct summand of $\overline{W}_\nu(\nu) \otimes \mathfrak{g}^{-\sigma}$. We denote by D_0 the set of weights of $\mathfrak{a}'$ in $(\mathfrak{g}^{-\sigma})^{M'_{\mathbb{R}}}$. Then we have $D(\nu) \subset D_0 \subset \Lambda$ by the proof of Lemma 9.4 (a) and (b). We will determine the condition in which $D(\nu)$ is equal to $D_0 \cap \Lambda_\nu$.

Lemma 9.25. *Let $\{X_i\}$ be a basis of $\mathfrak{g}^{-\sigma}$ and $\{Y_i\}$ be the dual basis of $\{X_i\}$ with respect to the Killing form. Put $\Phi'_\nu := \sum_i Y_i \otimes X_i \Phi_\nu$. For $X \in \mathfrak{g}^{-\sigma}$, define $\iota_X : \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}^{-\sigma} \otimes \overline{W}(\nu), \overline{W}(\nu))$ by*

$$\iota_X(a \otimes w) := (X, a)w \text{ for } a \in \mathfrak{g}^{-\sigma} \text{ and } w \in \overline{W}(\nu).$$

Then we have

$$\begin{aligned} &\Phi'_\nu \in \mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), \mathfrak{g}^{-\sigma} \otimes \overline{W}(\nu)), \\ &\iota_X \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}^{-\sigma} \otimes \overline{W}(\nu), \overline{W}(\nu))_{\Delta(G')}, \\ &\iota_X \circ \Phi'_\nu = X\Phi_\nu. \end{aligned}$$

The proof of the lemma is straightforward. From the above lemma, it is enough to determine the image of Φ'_ν. Since $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \simeq \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{k}')} \mathbb{C}_\delta$, Φ'_ν is reproduced by the value at $1 \in \mathbb{C}_\lambda$. Then we will compute $\Phi'_\nu(1)$ using a good basis of $\mathfrak{g}^{-\sigma}$.

We fix $\Psi_\nu \in \mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda), W(\nu))$, define Ψ'_ν by the same way as Φ'_ν and define

$$D'(\nu) := \left\{\alpha \in \Lambda : \mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}, K'_{\mathbb{R}}}(\mathcal{U}(\mathfrak{g}')\Psi'_\nu(1), W(\nu + \alpha)) \neq 0\right\}.$$

Since the quotient map induces the following surjection:

$$\mathrm{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, \mathcal{U}(\mathfrak{g}')\Psi'_\nu(1)) \to \mathrm{Hom}_{K'_{\mathbb{R}}}(\mathbb{C}_\gamma, \mathcal{U}(\mathfrak{g}')\Phi'_\nu(1)),$$

we have $D'(\nu) \cap \Lambda_\nu = D(\nu)$.

9.5 Root decomposition

To see the structure of $D'(\nu), D_0$, we review the root decomposition of $\mathfrak{g}$ (see [13, Part II]). The root decomposition comes from the structure theory of Jordan algebras and Jordan triple systems. Jordan algebras and Jordan triple systems are used in the study of degenerate principal series representations [30, 31], [75], [81], [90, 91]. Following them, we prepare notations.

We define $\overline{\sigma}(X) := \sigma(\overline{X})$ for $X \in \mathfrak{g}$. Fix a maximal abelian subspace $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{k}_{\mathbb{R}}^{-\sigma}$. Since the characteristic element H belongs to $\sqrt{-1}\mathfrak{k}_{\mathbb{R}}^{-\sigma}$, we have $H \in \mathfrak{t}$. This implies that $\mathfrak{t}_{\mathbb{R}}$ is a maximal abelian subspace of $\mathfrak{g}_{\mathbb{R}}^{-\sigma}$. Then for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, $\mathfrak{g}_\alpha$ is $\overline{\sigma}$-stable. We fix a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ containing $\Delta(\mathfrak{p}_+, \mathfrak{t})$.

For two roots α, β, we will say that α and β are *strongly orthogonal* if $\alpha \pm \beta$ are not roots. We take a maximal set of strongly orthogonal roots $\{\beta_1, \beta_2, \ldots, \beta_r\} \subset \Delta(\mathfrak{p}_+, \mathfrak{t})$ as follows:

(i) β_1 is the highest root in $\Delta(\mathfrak{p}_+, \mathfrak{t})$;

(ii) for each $i > 0$, β_i is the highest root in the roots that are strongly orthogonal to $\beta_1, \beta_2, \dots \beta_{i-1}$.

Then β's have the same length. For each $1 \leq i \leq r$, we choose an element $X_i \in (\mathfrak{p}_+)^{\overline{\sigma}}_{\beta_i}$ such that $2(X_i, \sigma(X_i)) = (\beta_i, \beta_i)$. Put $Y_i := \sigma(X_i), H_i := [X_i, Y_i]$. Then $\{X_i, Y_i, H_i\}$ is a $\mathfrak{sl}_2$-triple.

Define

$$\begin{aligned} H'_i &:= X_i + Y_i \in \mathfrak{p}'_{\mathbb{R}}, \\ X'_i &:= \frac{1}{2}(H_i + Y_i - X_i) \in \sqrt{-1}\mathfrak{g}_{\mathbb{R}}^{-\sigma}, \\ Y'_i &:= \frac{1}{2}(H_i - Y_i + X_i) \in \sqrt{-1}\mathfrak{g}_{\mathbb{R}}^{-\sigma}, \\ \mathfrak{a}'_{\mathbb{R}} &:= \bigoplus_{1 \leq i \leq r} \mathbb{R}(X_i + Y_i). \end{aligned}$$

Then $\mathfrak{a}'_{\mathbb{R}}$ is a maximal abelian subspace of $\mathfrak{p}'_{\mathbb{R}}$, and $\{X'_i, Y'_i, H'_i\}$ forms a $\mathfrak{sl}_2$-triple. We define $\gamma_i \in (\mathfrak{a}')^*$ by $\gamma_i(H'_j) = 2\delta_{ij}$. Then X'_i is in $(\mathfrak{g}^{-\sigma})_{\gamma_i}$ and Y'_i is in $(\mathfrak{g}^{-\sigma})_{-\gamma_i}$. We replace $\mathfrak{a}'_{\mathbb{R}}$ in the previous section by this $\mathfrak{a}'_{\mathbb{R}}$, and define a lexicographical order in $(\mathfrak{a}'_{\mathbb{R}})^*$ by the following ordered basis

$$\gamma_1 > \gamma_2 > \cdots > \gamma_r.$$

We take $N'_{\mathbb{R}}$ determined by this ordering.

Lemma 9.26. *X''s and Y''s are $M'_{\mathbb{R}}$-invariant.*

Proof. By definition, we have

$$\begin{aligned} [H, H'_i] &= [H, X_i + Y_i] \\ &= X_i - Y_i \\ &= -X'_i + Y'_i. \end{aligned}$$

Since $[H, H'_i]$ is $M'_{\mathbb{R}}$-invariant, so is $-X'_i + Y'_i$. X'_i and Y'_i have mutually different weights of $\mathfrak{a}'$. Thus X''s and Y''s are $M'_{\mathbb{R}}$-invariant. □

The root system $\Delta(\mathfrak{g}, \mathfrak{a}')$ is of type C_r or BC_r. More precisely, the following fact is known (see [13, Proposition II.2.1]).

Fact 9.27. *$\Delta(\mathfrak{g}, \mathfrak{a}')$ is of the form:*

$$\left\{ \frac{\pm\gamma_i \pm \gamma_j}{2} : 1 \leq i, j \leq r \right\} \setminus \{0\} \ \ \textit{or}$$
$$\left\{ \frac{\pm\gamma_i \pm \gamma_j}{2}, \pm\frac{\gamma_i}{2} : 1 \leq i, j \leq r \right\} \setminus \{0\}.$$

By the classification [13, Part II, Table 5], we can see that $\Delta(\mathfrak{g}',\mathfrak{a}')$ is of type B_r or BC_r if $\Delta(\mathfrak{g},\mathfrak{a}')$ is of type BC_r, and $\Delta(\mathfrak{g}',\mathfrak{a}')$ is of type A_{r-1}, C_r or D_r if $\Delta(\mathfrak{g},\mathfrak{a}')$ is of type C_r. Therefore, if $\Delta(\mathfrak{g}',\mathfrak{a}')$ is of type A_{r-1}, the $W_{\mathfrak{g}'_{\mathbb{R}}}$-orbits on $\{\pm\gamma_i : 1 \leq i \leq r\}$ are $\{\gamma_i : 1 \leq i \leq r\}$ and $\{-\gamma_i : 1 \leq i \leq r\}$, otherwise $\{\pm\gamma_i : 1 \leq i \leq r\}$ contains one $W_{\mathfrak{g}'_{\mathbb{R}}}$-orbit.

In our setting, $\mathfrak{g}'_{\mathbb{R}}$ is either semisimple or reductive with one-dimensional non-compact center. In the first case, $\mathfrak{g}^{-\sigma}$ is irreducible, and in the second case, $\mathfrak{g}^{-\sigma}$ is a direct sum of non-isomorphic two irreducible submodules as a $\mathfrak{g}'$-module. Summarizing these facts, we have

Lemma 9.28. *The following conditions are equivalent:*

(a) $\Delta(\mathfrak{g}',\mathfrak{a}')$ *is of type* A_{r-1}*;*

(b) $\mathfrak{g}^{-\sigma}$ *is a direct sum of non-isomorphic two irreducible submodules.*

Proof. If $\Delta(\mathfrak{g}',\mathfrak{a}')$ is of type A_{r-1}, $(\mathfrak{g}^{-\sigma})_{\gamma_1}$ and $(\mathfrak{g}^{-\sigma})_{-\gamma_r}$ are contained in $(\mathfrak{g}^{-\sigma})^{M'_{\mathbb{R}}N'_{\mathbb{R}}}$. Hence (b) follows.

Assume (b). Then $\mathfrak{g}'_{\mathbb{R}}$ has a one-dimensional non-compact center. Hence $\{\pm\gamma_i : 1 \leq i \leq r\}$ is divided into at least two $W_{\mathfrak{g}'_{\mathbb{R}}}$-orbits. This implies (a). $\square$

Lemma 9.29. *For any* i*,* $(\mathfrak{g}^{-\sigma})_{\pm\gamma_i}$ *are one-dimensional.*

Proof. Assume $\Delta(\mathfrak{g}',\mathfrak{a}')$ is not of type A_{r-1}. Then $\mathfrak{g}^{-\sigma}$ is an irreducible $\mathfrak{g}'$-module. By the explicit root decomposition, $(\mathfrak{g}^{-\sigma})_{\gamma_1}$ is contained in $(\mathfrak{g}^{-\sigma})^{M'_{\mathbb{R}}N'_{\mathbb{R}}}$. Since $\mathfrak{g}^{-\sigma}$ is an irreducible $\mathfrak{g}'$-module, $(\mathfrak{g}^{-\sigma})_{\gamma_1}$ is one-dimensional. Since the $W_{\mathfrak{g}'_{\mathbb{R}}}$-action on $\{\pm\gamma_i\}$ is transitive, this implies that $(\mathfrak{g}^{-\sigma})_{\pm\gamma_i}$ is one-dimensional for any i.

For type A_{r-1}, we can apply the same discussion to each irreducible summand of $\mathfrak{g}^{-\sigma}$. $\square$

Lemma 9.30. *We have*

$$D_0 \backslash \{0\} = \{\pm\gamma_i : 1 \leq i \leq r\},$$
$$\Lambda = \bigoplus_{1 \leq i \leq r} \mathbb{Z}\gamma_i,$$

where D_0 *is the set of weights in* $(\mathfrak{g}^{-\sigma})^{M'_{\mathbb{R}}}$ *and* Λ *is the set of characters of* L' *trivial on* $\mathrm{Ad}(M'_{\mathbb{R}})$ *(defined in the previous section). In particular, as an additive group,* Λ *is generated by* D_0*.*

Proof. We prove that for each $\alpha \in \Delta(\mathfrak{g}^{-\sigma},\mathfrak{a}')\backslash\{\pm\gamma_i : 1 \leq i \leq r\}$, $(\mathfrak{g}^{-\sigma})^{M'_{\mathbb{R}}}_{\alpha}$ is zero.

For each i, we define $g_i := \exp(\pi\sqrt{-1}\mathrm{ad}(H'_i))$. Let $\phi_i : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Int}(\mathfrak{g})$ be the homomorphism determined by the $\mathfrak{sl}_2$-triple $\{H'_i, X'_i, Y'_i\}$. Then we have $\phi_i(-I) = g_i$. This implies that g_i is an element of $G' \cap \mathrm{Int}(\mathfrak{g}_\mathbb{R}) = \mathrm{Ad}(G'_\mathbb{R})$, and hence $\mathrm{Ad}(M'_\mathbb{R})$. Take $\alpha \in \Delta(\mathfrak{g}^{-\sigma}, \mathfrak{a}') \backslash \{\pm\gamma_i\}_{i=0}^r$ and $X \in (\mathfrak{g}^{-\sigma})_\alpha$. By Fact 9.27, α is equal to $(\pm\gamma_i \pm \gamma_j)/2$ or $(\pm\gamma_i)/2$ for some $i \neq j$. Hence we have

$$g_i^\alpha = \exp(\pi\sqrt{-1}\alpha(H'_i)) = -1.$$

This shows $(\mathfrak{g}^{-\sigma})_\alpha^{M'_\mathbb{R}} = 0$.

It is obvious that Λ contains $\bigoplus_{1\leq i\leq r} \mathbb{Z}\gamma_i$ because D_0 is contained in Λ. The above discussion shows the converse inclusion. $\square$

9.6 Computation of $D(\nu)$ and $D'(\nu)$

Using the results in the previous section, we will compute $D'(\nu)$ and prove the main theorem 9.13. To do so, the following lemma is useful. Recall

$$D'(\nu) \backslash \{0\} \subset D_0 \backslash \{0\} = \{\pm\gamma_i : 1 \leq i \leq r\}.$$

Lemma 9.31. *$-\gamma_1 \in D'(\nu)$ if and only if*

$$(\Psi'_\nu(1)(e), X'_1) \neq 0.$$

If $\Delta(\mathfrak{g}', \mathfrak{a}')$ is of type A_{r-1}, the same statement holds for γ_r.

Proof. Recall

$$D'(\nu) = \left\{\alpha \in \Lambda : \mathrm{Hom}_{\mathfrak{g}_\mathbb{R}, K'_\mathbb{R}}(\mathcal{U}(\mathfrak{g}')\Psi'_\nu, W(\nu+\alpha))\right\}.$$

We consider the projection $p : \mathfrak{g}^{-\sigma} \otimes I(\delta,\nu) \to I(\delta, \nu - \gamma_1)$ and the following isomorphism:

$$\iota : \mathfrak{g}^{-\sigma} \otimes I(\delta,\nu) \simeq C^\infty(G'_\mathbb{R}/Q'_\mathbb{R}, G'_\mathbb{R} \times_{Q'_\mathbb{R}} ((\mathbb{C}_\gamma \otimes \mathbb{C}_\nu \otimes \mathbf{1}_{N'_\mathbb{R}}) \otimes \mathfrak{g}^{-\sigma}))_{K'_\mathbb{R}}.$$

Then under this isomorphism, the projection p is induced from the quotient map

$$p' : \mathfrak{g}^{-\sigma} \to \mathbf{1}_{M'_\mathbb{R}} \otimes \mathbb{C}_{-\gamma_1} \otimes \mathbf{1}_{N'_\mathbb{R}}.$$

Then we have

$$p(X \otimes f)(e) = p'(\iota(X \otimes f)(e)).$$

for any $f \in I(\delta,\nu)$ and $X \in \mathfrak{g}^{-\sigma}$. Since $\iota(X \otimes f)(g) = \mathrm{Ad}(g^{-1})(X) \otimes f(g)$ for $g \in G'_{\mathbb{R}}$, we obtain

$$p(X \otimes f)(e) = p'(\iota(X \otimes f)(e)) = p'(X \otimes f(e)) = (X, X'_1)Y'_1 \otimes f(e).$$

Since Ψ'_ν is a relative $K'_{\mathbb{R}}$-invariant in $(\mathfrak{g}^{-\sigma} \otimes W(\nu))$, $p(\Psi'_\nu)$ is non-zero if and only if $p(\Psi'_\nu)(e)$ is non-zero. This implies that $-\gamma_1 \in D'(\nu)$ if and only if

$$(\Psi'_\nu(1)(e), X'_1) \neq 0.$$

The proof for type A_{r-1} is the same. □

By the above lemma, we can see

Lemma 9.32. *$-\gamma_1 \in D'(\nu)$ if and only if*

$$\frac{(\nu,\gamma_1)}{(\gamma_1,\gamma_1)} + \frac{(\lambda,\beta_1)}{(\beta_1,\beta_1)} \neq 0.$$

Suppose $\Delta(\mathfrak{g}',\mathfrak{a}')$ is of type A_{r-1}. Then $\gamma_r \in D'(\nu)$ if and only if

$$\frac{(\nu,\gamma_r)}{(\gamma_r,\gamma_r)} - \frac{(\lambda,\beta_r)}{(\beta_r,\beta_r)} \neq 0.$$

Remark 9.33. The value of $(\lambda,\beta_i)/(\beta_i,\beta_i)$ is independent of i. This is because the Weyl group acts on $\{\beta_i : 1 \leq i \leq r\}$ transitively and λ is a character of $\mathfrak{k}$.

Proof. To compute Ψ'_ν, we take a basis of $\mathfrak{g}^{-\sigma}$ such that each vector is weight vector of $\mathfrak{a}'$. Then we have

$$(\Psi'_\nu(1)(e), X'_1) = -\Psi_\nu(X'_1 \cdot 1)(e).$$

Since $\mathfrak{p}^\sigma \oplus \mathfrak{p}_+ = \mathfrak{p}^{-\sigma} \oplus \mathfrak{p}_+$, we have

$$\begin{aligned} X'_1 &= \frac{1}{2}(X'_1 + \theta(X'_1)) + \frac{1}{2}(X'_1 - \theta(X'_1)) \\ &= \frac{1}{2}(X'_1 + Y'_1) + \frac{1}{2}(X'_1 - Y'_1) \\ &\sim \frac{1}{2}(X'_1 + Y'_1) - [Z, \frac{1}{2}(X'_1 - Y'_1)] \pmod{\mathfrak{p}_+} \\ &= \frac{1}{2}(X'_1 + Y'_1) + \frac{1}{2}H'_1. \end{aligned}$$

Hence we obtain

$$\begin{aligned}\Psi_\nu(X_1'\cdot 1)(e) &= \frac{1}{2}H_1'\Psi_\nu(1)(e) + \Psi_\nu\left(\frac{1}{2}(X_1'+Y_1')\cdot 1\right)(e)\\ &= \frac{\nu(H_1')}{2}\Psi_\nu(1)(e) + \frac{\lambda(H_1)}{2}\Psi_\nu(1)(e)\\ &= \frac{(\nu,\gamma_1)}{(\gamma_1,\gamma_1)}\Psi_\nu(1)(e) + \frac{(\lambda,\beta_1)}{(\beta_1,\beta_1)}\Psi_\nu(1)(e)\end{aligned}$$

This shows the assertion. □

Recall $D(\nu) = D'(\nu)\cap\Lambda_\nu$. To know the structure of $D(\nu)$, let us study Λ_ν. We define

$$\Lambda^- := \left\{\sum_{1\le i\le r} m_i\gamma_i \in \Lambda : m_1 \le m_2 \le \cdots \le m_r \le 0\right\}.$$

By Fact 9.27, any element of Λ^- is the lowest restricted weight of some irreducible G'-module.

Lemma 9.34. *There exists an element $w \in W_{\mathfrak{g}_\mathbb{R}}$ such that $\overline{W}(\nu)$ is an irreducible submodule of $I(\delta, w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}))$. Moreover, for such w, $\Lambda_\nu + w^{-1}(\Lambda^-)$ is contained in Λ_ν.*

Proof. By the Casselman subrepresentation theorem, $\overline{W}(\nu)$ can be embedded in $I(\delta',\nu')$ for some δ',ν'. Since $\overline{W}(\nu)$ has the one-dimensional $K'_\mathbb{R}$-type γ, δ' is isomorphic to δ. As in the proof of Lemma 9.11, comparing two $\mathcal{U}(\mathfrak{g}')^{K'_\mathbb{R}}$-actions, we can take an element $w \in W_{\mathfrak{g}_\mathbb{R}}$ such that $\nu' = w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n})$. This proves the first assertion.

Since $\overline{W}(\nu)$ is isomorphic to $\overline{W}(w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}))$, $\Lambda_{w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n})} = w(\Lambda_\nu)$ holds. Hence we can assume $w = e$. Then $\overline{W}_\nu(\nu+\mu)$ is an irreducible submodule of $I(\delta,\nu+\mu)$ for any $\mu\in\Lambda$. Take $\alpha\in\Lambda^-$. Then $I(\mathbf{1},\alpha)$ contains a unique spherical finite-dimensional representation F. We take non-zero $\phi \in F^{K'_\mathbb{R}}$.

Take $\mu\in\Lambda_\nu$. Then $\overline{W}_\nu(\nu+\mu) = W(\nu+\mu)$ holds. Since any function of $I(\delta,\nu+\mu)$ is real analytic, $\phi\cdot W(\nu+\mu)$ is a non-zero subspace of $I(\delta,\nu+\mu+\alpha)$ and has the $K'_\mathbb{R}$-type γ. Hence $F\otimes W(\nu+\mu)$ contains $W(\nu+\mu+\alpha)$. Thus we have

$$\overline{W}_\nu(\nu+\mu+\alpha) = W(\nu+\mu+\alpha) = \overline{W}(\nu+\mu+\alpha).$$

This shows $\alpha+\mu\in\Lambda_\nu$. □

Theorem 9.35. *Suppose that δ and ν satisfy the condition of Lemma 9.4, and suppose*

$$\pm\frac{(w(\nu-\rho(\mathfrak{n}))+\rho(\mathfrak{n}),\gamma_1)}{(\gamma_1,\gamma_1)}+\frac{(\lambda,\beta_1)}{(\beta_1,\beta_1)}\notin\mathbb{Z}$$

for any $w\in W_{\mathfrak{g}'_{\mathbb{R}}}$. Then $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W}(\nu))_{\Delta(G')}$ is irreducible as a $(\mathfrak{g}'\oplus\mathfrak{g},\Delta(G'))$-module.

Proof. For simplicity, define $w\circ\mu:=w(\mu-\rho(\mathfrak{n}))+\rho(\mathfrak{n})$. We assume that $\Delta(\mathfrak{g}',\mathfrak{a}')$ is not of type A_{r-1}. For the case of type A_{r-1}, the proof is the same.

First, we show $D(\mu)\setminus\{0\}=(D_0\setminus\{0\})\cap\Lambda_\mu$ for any $\mu\in\Lambda'_\nu$. Take $\mu\in\Lambda'_\nu$. By assumption, we can apply Lemma 9.32 to $w\circ\mu$ for any $w\in W_{\mathfrak{g}'_{\mathbb{R}}}$. Then we have $D'(w\circ\mu)\ni-\gamma_1$. From the following relation:

$$w(D(\mu))=D(w\circ\mu)=D'(w\circ\mu)\cap\Lambda_{w\circ\mu},$$

$-\gamma_1\in\Lambda_{w\circ\mu}$ if and only if $-\gamma_1\in w(D(\mu))$. This implies that $-w^{-1}(\gamma_1)\in D(\mu)$ if and only if $-w^{-1}(\gamma_1)\in\Lambda_\mu$. Thus since $W_{\mathfrak{g}'_{\mathbb{R}}}$ acts on $\{\pm\gamma_i:1\leq i\leq r\}$ transitively, we obtain

$$D(\mu)\setminus\{0\}=(D_0\setminus\{0\})\cap\Lambda_\mu. \tag{9.35.2}$$

For $\mu,\mu'\in\Lambda'_\nu$, μ is said to be adjacent to μ' if $\mu'-\mu\in D(\mu)$ holds. By (9.35.2), we can see that μ is adjacent to μ' if and only if μ' is adjacent to μ. Then there is a bijection between the set of submodules of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W}(\nu))_{\Delta(G')}$ and the set of connected components of Λ'_ν with respect to this adjacent relation. Hence Λ'_ν is connected if and only if $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda),\overline{W}(\nu))_{\Delta(G')}$ is irreducible.

Take $\mu,\mu'\in\Lambda'_\nu$. We show that μ and μ' are in the same connected component. By Lemma 9.34, there exists an element $w\in W_{\mathfrak{g}'_{\mathbb{R}}}$ such that $\mu+w(\Lambda^-)$ and $\mu'+w(\Lambda^-)$ are contained in Λ'_ν. By the definition of Λ^-, $\mu+w(\Lambda^-)$ is contained in one connected component. We can assume $w=e$.

We prove that $(\mu+\Lambda^-)\cap(\mu'+\Lambda^-)$ is non-empty. Replacing μ by $\mu-\mu'$, we can assume that $\mu'=0$ and μ belongs to Λ. Write

$$\mu=\sum_{i=0}^{r}c_i\gamma_i$$

for some $c_i\in\mathbb{Z}$. For convenience, we set $c_{r+1}=0$. We define

$$a_i:=\sum_{j=1}^{i}\max(c_j-c_{j+1},0),$$

$$\alpha:=\sum_{i=1}^{r}a_i\gamma_i.$$

Then it is clear that $\alpha \in \Lambda^-$ and $\mu + \alpha \in \Lambda^-$. This implies $\mu + \alpha \in (\mu + \Lambda^-) \cap \Lambda^-$. We have proved the theorem. $\square$

Using the Jantzen–Zuckerman translation functor, we obtain the following corollary.

Corollary 9.36. *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$ in the good range with respect to $\mathfrak{q}$, and let (δ, V_δ) be an irreducible subrepresentation of $F|_{M'_{\mathbb{R}}}$. Suppose that $\mathfrak{c}(\mathfrak{k})$ acts on F by a character λ. Assume that λ, δ and $\nu \in (\mathfrak{a}')^*$ satisfy the condition of Lemma 9.4 and Theorem 9.35. Let W be an irreducible subquotient of $I(\delta, \nu)$. Then $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F), W)_{\Delta(G')}$ is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module.*

Proof. Let $\widetilde{G}$ be a simple-connected connected algebraic group with Lie algebra $\mathfrak{g}$ and $\widetilde{G_{\mathbb{R}}}$ the real form of G corresponding to $\mathfrak{g}_{\mathbb{R}}$. Then there are surjective homomorphism:

$$G_{\mathbb{R}} \to \widetilde{G_{\mathbb{R}}}.$$

The image of a subgroup of $G_{\mathbb{R}}$ under the above homomorphism is denoted by the same Roman alphabet with tilde such as $\widetilde{G'_{\mathbb{R}}}$.

We denote by χ_1 (resp. χ'_1) the infinitesimal character of $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ (resp. $I(\delta, \nu)$). As in the proof of Theorem 6.1, there exists a unitary character $\mathbb{C}_\alpha$ of $K_{\mathbb{R}}$ in the good range such that $\mathbb{C}_{-\alpha} \otimes F$ reduces to a representation of $\widetilde{K_{\mathbb{R}}}$. Hence we can take a finite-dimensional irreducible $\widetilde{G}$-module V and a infinitesimal character χ_2 such that

$$T_{\chi_2}^{\chi_1}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\alpha)) = P_{\chi_1}(V \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\alpha)) \simeq \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F).$$

Since $V_\delta \otimes \mathbb{C}_{-\alpha}$ reduces to a representation of $\widetilde{M'}$, there exist a finite-dimensional irreducible $\widetilde{G'}$-module V', an infinitesimal character χ'_2 and $\nu' \in (\mathfrak{a}')^*$ such that

$$T_{\chi'_2}^{\chi'_1}(I(\alpha, \nu')) = P_{\chi'_1}(V' \otimes I(\alpha, \nu')) \simeq I(\delta, \nu).$$

In this setting, $\mathrm{Hom}_{\mathbb{C}}(V, V')$ is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(\widetilde{G'}))$-module, and has a non-zero $M'_{\mathbb{R}}$-invariant vector. Hence $\mathrm{Hom}_{\mathbb{C}}(V, V')$ is a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. Thus $\mathrm{Hom}_{\mathbb{C}}(V, V')$ defines the translation functor $T_{(\chi'_2, \chi_2)}^{(\chi'_1, \chi_1)}$ in the category $\mathcal{C}(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$. This implies that $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F), W)_{\Delta(G')}$ is irreducible if and only if $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\alpha), T_{\chi'_1}^{\chi'_2}(W))_{\Delta(G')}$ is irreducible.

Considering the action of g_i defined in Lemma 9.30 on $M'_{\mathbb{R}}$-invariant vectors, we obtain

$$\pm\frac{(w(\nu'-\nu),\gamma_1)}{(\gamma_1,\gamma_1)}+\frac{(\alpha-\lambda,\beta_1)}{(\beta_1,\beta_1)}\in\mathbb{Z}$$

for any $w \in W_{\mathfrak{g}'_{\mathbb{R}}}$. Therefore, Theorem 9.35 shows the assertion. □

By the above corollary, we obtain the following results.

Corollary 9.37. *Retain the setting in Corollary 9.36. Then the $\mathcal{U}(\mathfrak{g})^{G'}$-module $\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F), W)$ is irreducible.*

Corollary 9.38. *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$ in the good range with respect to $\mathfrak{q}$. Let π denote the algebra homomorphism of $\mathcal{U}(\mathfrak{g})$ defining $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)$. Then we have*

$$\mathrm{PI.deg}(\pi(\mathcal{U}(\mathfrak{g})^{G'})) = \mathcal{M}_{G'_{\mathbb{R}}}(\overline{\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)}) = \mathcal{M}_{M'_{\mathbb{R}}}(F),$$

where $\overline{\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)}$ is the Hilbert completion with respect to an invariant inner product.

Proof. The first equation is proved by Corollary 9.37, 7.12 and 7.13.

For an irreducible unitary representation δ of $M'_{\mathbb{R}}$, Proposition 9.1 leads to

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F), I(\delta,\nu)) &\simeq \mathrm{Hom}_{K'_{\mathbb{R}}}(F, I(\delta,\nu))\\ &\simeq \mathrm{Hom}_{M'_{\mathbb{R}}}(F,\delta).\end{aligned}$$

Since $I(\delta,\nu)$ is irreducible for generic ν by a theorem of Bruhat (see [38, Theorem 7.2]), this and Corollary 7.12 imply

$$\mathcal{M}_{G'_{\mathbb{R}}}(\overline{\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)}) \leq \mathcal{M}_{M'_{\mathbb{R}}}(F).$$

Since $\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{R}},K'_{\mathbb{R}}}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F), I(\delta,\nu))$ is an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module for generic ν, Proposition 2.27 shows

$$\mathrm{PI.deg}(\pi(\mathcal{U}(\mathfrak{g})^{G'})) \geq \mathcal{M}_{M'_{\mathbb{R}}}(F).$$

Therefore, the second equation holds. □

10 Application: classification of multiplicity-free holomorphic discrete series representations

In this section, we classify multiplicity-free restrictions of holomorphic discrete series representations with respect to symmetric subgroups.

10.1 Setting

Let $G_\mathbb{R}$ is a connected real simple Lie group of Hermitian type with Cartan involution θ. Assume that $G_\mathbb{R}$ is a subgroup of simply-connected connected complex simple Lie group G with Lie algebra $\mathfrak{g}$. Put $K_\mathbb{R} := G_\mathbb{R}^\theta$. Take an involutive automorphism σ of G commuting with θ.

As in Section 2.3, fix a characteristic element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k})$, and construct subalgebras $\mathfrak{q}, \bar{\mathfrak{q}}, \mathfrak{p}_+, \mathfrak{p}_-$. We fix a unitary character ζ of $K_\mathbb{R}$ such that $2(\zeta,\alpha)/(\alpha,\alpha) = 1$ for a unique non-compact simple root $\alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})$. For an irreducible unitary representation F of $K_\mathbb{R}$ with infinitesimal character λ, we define

$$Z_{hol}(F) := \{z \in \mathbb{Z} : (\lambda + \rho(\mathfrak{p}_+), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\},$$
$$Z_{fin}(F) := \left\{ z \in \mathbb{Z} : \frac{2(\lambda + \rho(\mathfrak{p}_+), \alpha)}{(\alpha,\alpha)} \in \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h}) \right\},$$

and let $L(F)$ denote a unique irreducible submodule of $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F)$.

Summarizing the results in the previous sections, we obtain

Theorem 10.1. *Let $M_\mathbb{R}$ denote the centralizer in $K_\mathbb{R}^\sigma$ of a maximal abelian subspace $\mathfrak{a}'_\mathbb{R} \subset \mathfrak{p}_\mathbb{R}^{-\sigma}$. For an irreducible unitary representation F of $K_\mathbb{R}$, the following conditions are equivalent:*

(a) $\mathcal{M}_{G_\mathbb{R}^\sigma}(\overline{\mathrm{pro}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for any* $z \in Z_{hol}(F)$*;*

(b) $\mathcal{M}_{G_\mathbb{R}^\sigma}(\overline{\mathrm{pro}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for some* $z \in Z_{hol}(F)$*;*

(c) $\mathcal{M}_{G^\sigma}(L(F \otimes \mathbb{C}_{z\zeta})) = 1$ *for any* $z \in Z_{fin}(F)$*;*

(d) $\mathcal{M}_{M_\mathbb{R}}(F) = 1$.

Proof. Note that $G_\mathbb{R}^\sigma$ is connected in the above settings if $(\mathfrak{g}_\mathbb{R}, {\mathfrak{g}_\mathbb{R}}^\sigma)$ is of holomorphic type. If $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of holomorphic type, the equivalence of (a), (b) and (d) has been proved in Corollary 3.39, and the equivalence of (a) and (c) has been proved in Theorem 4.12.

Assume that $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is of anti-holomorphic type. Note that $M_\mathbb{R}$ is equal to the centralizer in $K_\mathbb{R}^\sigma$ of some maximal abelian subspace of $\mathfrak{p}_\mathbb{R}^\sigma$. In fact, $[H, \mathfrak{a}'_\mathbb{R}]$ is a maximal abelian subspace of $\mathfrak{p}_\mathbb{R}^\sigma$, and $Z_{K_\mathbb{R}^\sigma}([H, \mathfrak{a}'_\mathbb{R}]) = Z_{K_\mathbb{R}^\sigma}(\mathfrak{a}'_\mathbb{R})$.

Corollary 9.38 shows the equivalence of (a), (b), (d). We denote by π_z the representation map of $\mathrm{pro}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes \mathbb{C}_{z\zeta})$. We use Theorem 4.7. The condition (c) is equivalent to the condition that $\pi_z(\mathcal{U}(\mathfrak{g})^{G^\sigma})$ is commutative for any z. The condition (a) is also equivalent to the same condition. Thus (a) and (c) are equivalent. $\square$

Remark 10.2. The theorem holds for the universal covering of $G_\mathbb{R}$.

10.2 Classification

Using Theorem 10.1, we classify multiplicity-free restrictions of holomorphic discrete series representations with respect to $G_{\mathbb{R}}^{\sigma}$.

Theorem 10.3. *Let $\mathcal{H}$ be a holomorphic discrete series representation of $G_{\mathbb{R}}$. Put $F := \mathcal{H}_K^{\mathfrak{p}_+}$. Then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free if and only if F is one-dimensional or the highest weight of $F|_{[\mathfrak{k},\mathfrak{k}]}$ belongs to $\Lambda(\sigma)$ in Table 2.*

Remark 10.4. The multiplicity-freeness for special cases are known:

- $\dim_{\mathbb{C}}(F) = 1$ (by T. Kobayashi [44, 50] (Fact 1.8));
- $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}}) = (\mathfrak{so}(2,n), \mathfrak{so}(2,n-1))$ (by Jakobsen–Vergne [29, Corollary 3.1]);
- $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}}) = (\mathfrak{su}(p,q), \mathfrak{u}(p-1,q))$ (by T. Kobayashi [52, Theorem 8.10]).

Remark 10.5. The classification of irreducible symmetric pairs was obtained by M. Berger [2]. We refer the reader for the table to [47, Table I, II].

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}_{\mathbb{R}}^{\sigma}$	$\pm$	st.	$\Lambda(\sigma)$
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p_1,q_1)+\mathfrak{su}(p_2,q_2)+\mathfrak{t}$	h	○	$(p_1+q_1=1, p+q-1)$ any
				$(p_1+q_1=2, p+q-2)$ $m\omega_i$
				ω_i, $m\omega_i (i=1, p\pm1, p+q-1)$
	$\mathfrak{so}(p,q)$	a		ω_i
	$\mathfrak{sp}(p/2,q/2)$	a		$m\omega_i (i=1, p\pm1, p+q-1)$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$	h		ω_i
	$\mathfrak{sp}(n,\mathbb{R})$	h		$(n=2)$ any
				$(n\le 4)$ $m\omega_i$
				$m\omega_i (i=1, n\pm1, 2n-1)$
	$\mathfrak{sl}(n,\mathbb{C})+\mathbb{R}$	a	○	$(n=2)$ $m\omega_i$
				ω_i, $m\omega_i (i=1, n\pm1, 2n-1)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2p)+\mathfrak{so}^*(2n-2p)$	h	○	$(\min(p,n-p)=1)$ $m\omega_i$
				ω_2, ω_n
	$\mathfrak{u}(p,n-p)$	h	○	$m\omega_i (i=2,n)$
				$(n=4, p\text{:odd})$ $m\omega_i (i=2,3,4)$
	$\mathfrak{so}(n,\mathbb{C})$	a		ω_2, ω_n
	$\mathfrak{su}^*(n)+\mathbb{R}$	a	○	$m\omega_i (i=2,n)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,p)+\mathfrak{so}(n-p)$	h		$(p=n-1)$ any
				$(n\text{:odd})$ ω_1
			○	$(n\text{:even } p=0, n-2)$ $m\omega_1, m\omega_2$
				$(n\text{:even})$ ω_1, ω_2

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}_{\mathbb{R}}^{\sigma}$	$\pm$	st.	$\Lambda(\sigma)$
	$\mathfrak{so}(1,p)+\mathfrak{so}(1,n-p)$	a		the same as above
	$\mathfrak{u}(1,n/2)$	h	○	$m\omega_i (i=1,2,n/2)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(p,\mathbb{R})+\mathfrak{sp}(n-p,\mathbb{R})$	h		$(\min(p,n-p)=1)\ m\omega_i$
				ω_2, ω_n
	$\mathfrak{u}(p,n-p)$	h	○	ω_i
	$\mathfrak{sp}(n/2,\mathbb{C})$	a		ω_2, ω_n
	$\mathfrak{sl}(n,\mathbb{R})+\mathbb{R}$	a	○	ω_i
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)+\mathfrak{t}$	h	○	$m\omega_6$
	$\mathfrak{so}^*(10)+\mathfrak{t}$	h	○	
	$\mathfrak{so}(2,8)+\mathfrak{t}$	h	○	
	$\mathfrak{su}(5,1)+\mathfrak{sl}(2,\mathbb{R})$	h		none
	$\mathfrak{su}(4,2)+\mathfrak{su}(2)$	h		none
	$\mathfrak{f}_{4(-20)}$	a		$m\omega_2$, $m\omega_3$
	$\mathfrak{sp}(2,2)$	a		ω_6
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)}+\mathfrak{t}$	h	○	none
	$\mathfrak{e}_{6(-14)}+\mathfrak{t}$	h	○	none
	$\mathfrak{so}(10,2)+\mathfrak{sl}(2,\mathbb{R})$	h		none
	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	h		none
	$\mathfrak{su}(6,2)$	h		none
	$\mathfrak{e}_{6(-26)}+\mathbb{R}$	a	○	none
	$\mathfrak{su}^*(8)$	a		none

Table 2: the classification of multiplicity-free restrictions
of holomorphic discrete series representations

The symbol $\mathfrak{t}$ (resp. $\mathbb{R}$) means that the Lie algebra has the one-dimensional compact (resp. non-compact) center. The circle of the column with title 'st' means that its classification can be reduced to the Stembridge classification, that is, $G'_{\mathbb{R}}$ has a one-dimensional center. The column with title '$\pm$' means that if the value is 'h', the symmetric pair is of holomorphic type, and if the value is 'a', the symmetric pair is of anti-holomorphic type. ω's are fundamental weights corresponding to simple roots given later. $m\omega_i$ means that $m\omega_i$ is in $\Lambda(\sigma)$ for any m.

In [96], Stembridge classified multiplicity-free restrictions of irreducible finite-dimensional representations with respect to Levi subgroups. If $G_{\mathbb{R}}^{\sigma}$ has a one-dimensional center, G^{σ} is a Levi subgroup of G. Therefore, by Theorem 10.1, the desired classification for the case is immediately obtained from Stembridge's classification.

The following two propositions are useful to prove the theorem.

Proposition 10.6. *Let σ' be an involutive automorphism of $G_{\mathbb{R}}$. Assume that G^σ and $G^{\sigma'}$ are conjugate in G. Then $\Lambda(\sigma) = \Lambda(\sigma')$ holds.*

Proof. The assertion is a direct consequence of Theorem 10.1. $\square$

The proposition asserts that the classification is independent of a choice of real forms.

Proposition 10.7. *Let $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F)$ be the underlying Harish-Candra module of a holomorphic discrete series representation of $G_{\mathbb{R}}$. Suppose that $(G_{\mathbb{R}}, G_{\mathbb{R}}^\sigma)$ is of holomorphic type (see Section 3.2.4). Then $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{(\mathfrak{g}^\sigma, K^\sigma)}$ is multiplicity-free if and only if $\mathrm{pro}_{\mathfrak{q}^{\theta\sigma}}^{\mathfrak{g}^{\theta\sigma}}(F)|_{K^{\theta\sigma}}$ is multiplicity-free.*

Proof. Since $\mathfrak{p}_+/\mathfrak{p}_+^\sigma \simeq \mathfrak{p}_+^{\theta\sigma}$, Fact 4.9 shows the proposition. $\square$

Remark 10.8. The reduction to the K-type formula of the associated symmetric subalgebra $\mathfrak{g}_{\mathbb{R}}^{\theta\sigma}$ is used in the proof of the Hua–Kostant–Schmid–Kobayashi theorem [44, Theorem C] (see also [52, Theorem 8.3 and Lemma 8.8]).

By the proposition, we can reduce the classification for symmetric pairs of holomorphic type to the K-type case of $\mathfrak{g}^{\theta\sigma}$. As mentioned above, the classification of K-type multiplicity-free holomorphic discrete series representations is obtained from Stembridge's classification.

Lemma 10.9. *Let Λ^+ be the set of dominant integral weights of $\mathfrak{k}_{ss} := [\mathfrak{k}, \mathfrak{k}]$. If $\lambda \notin \Lambda(\sigma)$, $(\lambda + \Lambda^+) \cap \Lambda(\sigma) = \emptyset$ holds.*

Proof. Take $\mu \in \Lambda^+$. We can take $z, z' \in \mathbb{C}$ such that $\lambda + z\zeta$ and $\mu + z'\zeta$ are dominant integral weights of G. For a dominant integral weight ν of G, we denote by $F^G(\nu)$ the finite-dimensional irreducible representation of G.

By Theorem 10.1 and the assumption $\lambda \notin \Lambda(\sigma)$, replacing z by larger one, we can assume that $F^G(\lambda + z\zeta)|_{G^\sigma}$ is not multiplicity-free. Fix a Borel subgroup $B = TN$ of G^σ (since G is simply-connected and connected, G^σ is connected), where N is the unipotent radical of B.

We take a non-zero T-weight vector $v \in F^G(\mu+z'\zeta)^N$. Then by the Borel–Weil construction, the multiplication of two sections induces a T-module injection $F^G(\lambda + z\zeta)^N \otimes \mathbb{C}v \to F^G(\lambda + \mu + (z + z')\zeta)^N$. Since $F^G(\lambda + z\zeta)$ is not multiplicity-free, this implies that $F^G(\lambda+\mu+(z+z')\zeta)$ is not multiplicity-free. Therefore, we have $\lambda + \mu \notin \Lambda(\sigma)$. $\square$

Remark 10.10. A similar result for a tensor product of two finite-dimensional irreducible representations is used in Stembridge's classification [96, Corollary 2.10].

For fundamental weights, we can easily check the condition $\omega_i \notin \Lambda(\sigma)$ because for classical cases, the corresponding finite-dimensional irreducible representation $F^G(\omega_i)$ can be realized in the exterior product of a natural representation.

Lemma 10.11. *Assume* $\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$*. Then we have* $\Lambda(\sigma) \subset \Lambda(\theta)$*.*

Proof. By assumption, $\mathfrak{a}'_{\mathbb{R}}$ is a maximal abelian subspace of $\mathfrak{p}_{\mathbb{R}}$, and $Z_{K_{\mathbb{R}}^{\sigma}}(\mathfrak{a}'_{\mathbb{R}}) \subset Z_{K_{\mathbb{R}}}(\mathfrak{a}'_{\mathbb{R}})$ holds. Therefore, Theorem 10.1 shows the assertion. □

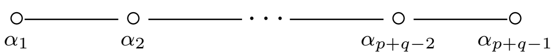
Since the set $\Lambda(\theta)$ can be computed by Stembridge's classification, we can narrow candidates of $\Lambda(\sigma)$.

10.3 Proof of Theorem 10.3

Let Λ^+ be the set of all dominant integral weights of $[\mathfrak{k}, \mathfrak{k}]$. We identify the restriction of a fundamental weight ω_i with ω_i. For $\lambda \in \Lambda^+$, we denote by $M_{\mathfrak{g}}(\lambda)$ the underlying Harish-Chandra module of one of holomorphic discrete series representations with highest weight $\lambda + z\zeta$. We write $F(\lambda)$ for the $\mathfrak{p}_+$-invariant part of $M_{\mathfrak{g}}(\lambda)$.

10.3.1 $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$

The Dynkin diagram of $\mathfrak{g}$ is as follows:



, and α_p is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}(p, q)$ Assume $p \leq q$. $M_{\mathbb{R}}$ is isomorphic to $\mathrm{O}(1) \times \mathrm{O}(1) \times \cdots \times \mathrm{O}(1) \times \mathrm{SO}(q-p)$, the direct product of $\mathrm{SO}(q-p)$ and p copies of $\mathrm{O}(1)$. By a straightforward computation for $\bigwedge^i(\mathbb{C}^p)$ and $\bigwedge^i(\mathbb{C}^q)$, we have $\Lambda(\sigma) \supset \{\omega_i : 1 \leq i < p+q, i \neq p\}$. By the explicit branching laws for $(\mathrm{U}(n), \mathrm{U}(n-1))$ and $(\mathrm{U}(n), \mathrm{O}(n))$, $F^K(\omega_i + \omega_j)|_{M_{\mathbb{R}}}$ is not multiplicity-free for any i, j. Thus we obtain

$$\Lambda(\sigma) = \{\omega_i : 1 \leq i < p+q, i \neq p\}.$$

$\mathfrak{g}^\sigma_\mathbb{R} = \mathfrak{sp}(p/2, q/2)$ $M_\mathbb{R}$ is isomorphic to $\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\cdots\times\mathrm{Sp}(1)\times\mathrm{Sp}(q-p)$, the direct product of $\mathrm{Sp}(q-p)$ and p copies of $\mathrm{Sp}(1)$. By a direct computation for $\bigwedge^i(\mathbb{C}^p)|_{M_\mathbb{R}}$ and $\bigwedge^i(\mathbb{C}^q)|_{M_\mathbb{R}}$, we have

$$\omega_i \in \Lambda(\sigma) \iff i = 1, p \pm 1, p+q-1.$$

It is easy to see that $\omega_i + \omega_j \notin \Lambda(\sigma)$ if $i, j \subset \{1, p\pm 1, p+q-1\}$ and $i \neq j$ by an explicit computation. This implies

$$\Lambda(\sigma) \subset \{m\omega_i : i = 1, p\pm 1, p+q-1\}.$$

Since $m\omega_i$ for $i = 1, p\pm 1, p+q-1$ is isomorphic to a symmetric product of a natural representation, it is easy to see that their restriction to $M_\mathbb{R}$ are multiplicity-free. Then we obtain

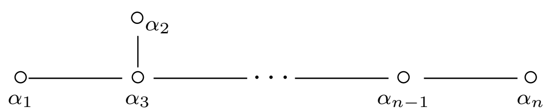
$$\Lambda(\sigma) = \{m\omega_i : i = 1, p\pm 1, p+q-1\}.$$

$\mathfrak{g}^\sigma_\mathbb{R} = \mathfrak{so}^*(2n), p = q = n$ We divide the highest weight λ into two parts λ_1, λ_2 corresponding to $\mathfrak{k}_{ss} = \mathfrak{su}(p) \oplus \mathfrak{su}(q)$. By Proposition 10.7, $M_\mathfrak{g}(\lambda_1 + \lambda_2)|_{\mathfrak{g}^\sigma}$ is multiplicity-free if and only if $M_{\mathfrak{g}^{\sigma\theta}}(\lambda_1) \otimes M_{\mathfrak{g}^{\sigma\theta}}(\lambda_2)$ is multiplicity-free. By Stembridge's classification, we have $\lambda_1 = \omega_i, \lambda_2 = 0$ or $\lambda_1 = 0, \lambda_2 = \omega_i$ for some i.

$\mathfrak{g}^\sigma_\mathbb{R} = \mathfrak{sp}(n, \mathbb{R}), p = q = n$ In this case, we can do the classification by the same way as the case $\mathfrak{g}^\sigma_\mathbb{R} = \mathfrak{so}^*(2n)$.

10.3.2 $\mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(2n)$

The Dynkin diagram of $\mathfrak{g}$ is as follows:



, and α_1 is a unique non-compact simple root.

$\mathfrak{g}^\sigma_\mathbb{R} = \mathfrak{so}^*(2p) + \mathfrak{so}^*(2n-2p)$

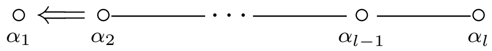
min$(p, n-p) \neq 1$ **case.** Using Proposition 10.7, we reduce the classification to the case of $(\mathfrak{g}_\mathbb{R}, \mathfrak{k}_\mathbb{R}) = (\mathfrak{su}(p,q), \mathfrak{su}(p) + \mathfrak{su}(q) + \mathfrak{t})$.

min$(p, n-p) = 1$ **case.** In this case, $\mathfrak{so}^*(2p) \simeq \mathfrak{u}(1)$ or $\mathfrak{so}^*(2n-2p) \simeq \mathfrak{u}(1)$. Thus we have $\Lambda(H) = \{m\omega_i : i \in \{2, 3, \ldots, n\}\}$ by Stembridge's classification.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}(n, \mathbb{C})$ Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. In this case, $M_{\mathbb{R}}$ is a maximal torus of $\mathrm{SO}(n)$. Hence it is easy to see that $\Lambda(\sigma) = \{\omega_2, \omega_n\}$.

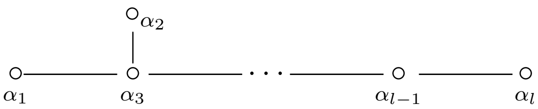
10.3.3 $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2, n)$

If n is odd, the Dynkin diagram of $\mathfrak{g}$ is as follows:



, where $l = (n+1)/2$. α_l is a unique non-compact simple root.

If n is even, the Dynkin diagram of $\mathfrak{g}$ is as follows:



, where $l = n/2 + 1$. α_l is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}(2, p) + \mathfrak{so}(n-p)$

$p = 0$ **or** $p = n-2$ **case** In this case, the classification is reduced to Stembridge's classification.

$p = n-1$ **case** Using a well-known fact that the restriction of any irreducible representation of $\mathrm{SO}(n+2, \mathbb{C})$ with respect to $\mathrm{SO}(n+1, \mathbb{C})$ is multiplicity-free. Therefore, we have $\Lambda(\sigma) = \Lambda^+$.

$p \neq 0, n-2, n-1$ **case** Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. Hence we have

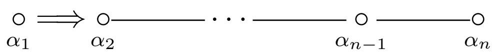
$$\Lambda(\sigma) \subset \begin{cases} \{\omega_1\} & \text{if } n \text{ is odd} \\ \{m\omega_1, m\omega_2\} & \text{if } n \text{ is even} \end{cases}$$

By straightforward computation, the assertion follows.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}(1, p) + \mathfrak{so}(1, n-p)$ The classification is the same as above by Proposition 10.6.

10.3.4 $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$

The Dynkin diagram of $\mathfrak{g}$ is as follows:



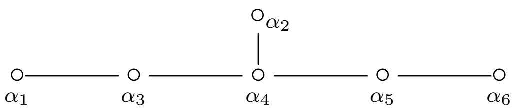
, and α_1 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{sp}(p, \mathbb{R}) + \mathfrak{sp}(n-p, \mathbb{R})$ By Proposition 10.7, the classification is the same as $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}^{\sigma}) = (\mathfrak{so}^*(2n), \mathfrak{so}^*(2p) + \mathfrak{so}^*(2q))$ case.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{sp}(n/2, \mathbb{C})$ The classification is the same as above by Proposition 10.6.

10.3.5 $\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{6(-14)}$

The Dynkin diagram of $\mathfrak{g}$ is as follows:



, and α_1 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{su}(5, 1) + \mathfrak{sl}(2, \mathbb{R})$ Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. Thus we have $\Lambda(\sigma) \subset \mathbb{Z}_{\geq 0}\omega_6$. Note $\mathfrak{g}_{\mathbb{R}}^{\theta\sigma} \simeq \mathfrak{so}^*(10) + \mathfrak{t}$.

Consider $\lambda = \omega_6$. We compute the branching law using Proposition 10.7. We can see that $\mathrm{pro}_{\bar{\mathfrak{q}}^{\theta\sigma}}^{\mathfrak{g}^{\theta\sigma}}(F^K(\omega_6))|_{K^\sigma}$ is isomorphic to

$$\bigoplus_{a \geq b \geq 0} F^{K^\sigma}((a, a, b, b, 0)) \otimes (F^{K^\sigma}((1, 0, 0, 0, 0)) \oplus F^{K^\sigma}((0, 0, 0, 0, -1)))$$

and this is not multiplicity-free. Therefore, we have $\Lambda(\sigma) = \{0\}$.

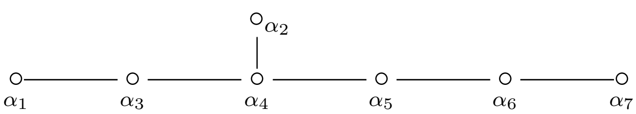
$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{su}(4, 2) + \mathfrak{su}(2)$ The classification is the same as above by Proposition 10.6.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{f}_{4(-20)}$ In this case, $M_{\mathbb{R}}$ is isomorphic to $\mathrm{Spin}(7)$. Computing explicit branching laws for $(\mathrm{Spin}(10), \mathrm{Spin}(7))$, we have the classification.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{sp}(2, 2)$ Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. Hence $\Lambda(\sigma) \subset \{m\omega_6\}$. $M_{\mathbb{R}}$ is isomorphic to $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. It is easy to see that $F(\omega_6)|_{M_{\mathbb{R}}} \simeq \mathbb{C}^{10}|_{M_{\mathbb{R}}}$ is not multiplicity-free.

10.3.6 $\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{7(-25)}$

The Dynkin diagram of $\mathfrak{g}$ is as follows:



, and α_7 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}(10,2) + \mathfrak{sl}(2,\mathbb{R})$ By Proposition 10.7, we can reduce the classification to the K-type decomposition for $\mathfrak{g} = \mathfrak{e}_6$. $\mathfrak{g}_{\mathbb{R}}^{\sigma\theta} \cap \mathfrak{k}_{ss} = \mathfrak{so}(10) + \mathfrak{t}$ is a Levi subgroup of $\mathfrak{k}_{ss}$. Consider $(\mathfrak{k}_{ss})_{\mathbb{C}} = \mathfrak{e}_6 = \mathfrak{p}'_+ \oplus \mathfrak{k}^{\sigma} \oplus \mathfrak{p}'_-$ and the $\mathfrak{p}'_{\pm}$-action on $F(\lambda)$. Then there exists a submodule F' of $F(\lambda)$ such that $\mathrm{pro}_{\mathfrak{q}^{\sigma\theta}}^{\mathfrak{g}^{\sigma\theta}}(F')|_{\mathfrak{k}^{\sigma}}$ is not multiplicity-free.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{so}^*(12) + \mathfrak{su}(2)$ The classification is the same as above by Proposition 10.6.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{su}(6,2)$ Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. Thus we have $\Lambda(\sigma) = \{0\}$.

$\mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{su}^*(8)$ Since $\mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}^{\sigma\theta})$, $\Lambda(\sigma) \subset \Lambda(\theta)$ holds by Lemma 10.11. Thus we have $\Lambda(\sigma) = \{0\}$.

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