博士論文

論文題目
Numerical and mathematical analysis for blow-up phenomena to nonlinear wave equations
(非線形波動方程式の爆発現象に関する数値・数学解析)

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Preface

Blow-up phenomena are one of important problems in the theory of nonlinear partial differential equations (PDEs). Since the behavior of solutions of PDEs near the blow-up time is a meaningful study, the numerical study of them is also crucial from the standpoint of mathematical study. In this paper, we study numerical analysis of blow-up phenomena for nonlinear wave equations focusing on the blow-up time.

In practical applications, it is desirable to use numerical methods which are mathematically guaranteed their validity. This is because it is hard to distinguish the numerical results which exactly simulate blow-up phenomena of PDEs from failure of computations.

Moreover, convergence analysis of numerical method used for the simulations is important for the numerical analysis of blow-up phenomena. In this paper, we consider a splitting method which is a time-discretization numerical method. It is often used for Schrödinger equations.

On the other hand, we analytically show continuous differentiability of the blow-up curve of a wave equation with a nonlinear term involving the derivative of unknown functions by applying the idea of numerical analysis in Chapter 1. We also simulate these results. Moreover, we present numerical results that showed the blow-up curves have singular points.

In Chapter 1, we consider the following wave equation.

$$\begin{cases} u_{tt} - u_{xx} = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{0.1}$$

Here, $S_L = \mathbb{R}/L\mathbb{Z}$ and $p > 1$ is a constant such that the function s^p $(s \geq 0)$ is of class C^4. The solution of (0.1) blows up in finite time if the initial values are large enough. The aims of this Chapter are to construct the numerical method of the blow-up time and to give the error estimates of them. In this paper, we call the approximation of the blow-up time numerical blow-up time. We divide the proof of convergence of the numerical blow-up time into 2 steps.

(Step 1.) Proof of convergence of numerical method for wave equations.
(Step 2.) Proof of convergence of numerical blow-up time.

There are almost no studies on numerical blow-up time for wave equations, while there are lots of such studies for heat equations. In resent years, construction of numerical blow-up time and convergence analysis of it for wave equations were done by Cho [10]. However, the proof of (Step 1.) is still open at present. He proved (Step 2.) holds under the assumption that (Step 1.) holds.

We need to take sufficiently small time increments near the blow-up time in order to compute the blow-up phenomena. That is, we use the variable time increments. There are many results of convergence analysis of numerical methods using variable time increments for heat equations. However, there is no such study for wave equations. The reason is that wave equations have the second derivative by time. Thus, we construct the numerical methods and corresponding numerical blow-up time for (0.1) and prove both (Step 1.) and (Step 2.).

We rewrite (0.1) as the following first order system.

$$\begin{cases} u_t + u_x = \phi, & t > 0,\ x \in S_L, \\ \phi_t - \phi_x = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad \phi(0,x) = u_1(x) + u_0'(x), & x \in S_L. \end{cases} \tag{0.2}$$

We present numerical method using variable time increments for (0.2). We show our numerical methods satisfy (Step 1.) by using the idea of [32]. We also prove our numerical blow-up time satisfies (Step 2.). Moreover, we present numerical results of blow-up time of (0.2).

In Chapter 2, we consider error analysis of semilinear evolution equations. As mentioned above, such study is important from the viewpoint of numerical analysis of blow-up phenomena. Let X be a Hilbert space and let A be an m-dissipative operator in X. For $u_0 \in D(A)$, we consider the following Cauchy problem for semilinear evolution equation:

$$\begin{cases} u_t = Au + F(u), \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{0.3}$$

The splitting method is one of time-discretization methods. Let $S(t)$ be the solution operator of (0.3). The idea behind splitting methods is to approximate the solution $u(t) = S(t)u_0$ of (0.3) by $\Phi_A(t)$ and $\Phi_F(t)$, which are solution operators of $\partial_t v = Av$ and $\partial_t w = F(w)$, respectively. The splitting method is useful when $\Phi_A(t)$ and $\Phi_F(t)$ are easy to compute, while $S(t)u_0$ is difficult to compute. In particular, the approximation $\Psi(t) = \Phi_A(t/2)\Phi_F(t)\Phi_A(t/2)$ is called the Strang-type splitting method. The Strang-type spitting method is numerically known as a second order convergent scheme. In addition, splitting method retains the dissipation or conservation properties of (0.3). Hence their ease of calculation and the dissipation or conservation properties, the splitting method is in common used as a numerical method for solving various differential equations. However, there are many open problems on error analysis of (0.3). In particular, for (0.3), whether the Strang-type splitting method is second order convergent or not was an open question in a rigorous manner.

The splitting method which is split into 2 parts is used on many occasions. On the other hand, sometimes there are cases that we should use the splitting method which is split into 3 parts. Therefore, we demonstrate that the convergence of our Strang-type splitting method which is split into 3 parts is a second order rate.

In Chapter 3, we consider a blow-up curve for the following nonlinear wave equation.

$$\begin{cases} u_{tt} - u_{xx} = F(u), & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{0.4}$$

where $F(u) = |u_t|^p$. Here, $p > 1$ is a constant such that the function s^p $(s \geq 0)$ is of class C^4. It is well known that the solution of (0.4) blows up in finite time if the initial values are large enough. Let R^* and T^* be positive constants. We set $B_{R^*} = \{x \mid |x| < R^*\}$. We consider

$$T(x) = \sup\{t \in (0, T^*) \mid \ |u_t(t,x)| < \infty\} \qquad (x \in B_{R^*}).$$

We call $\Gamma = \{(T(x), x) \mid x \in B_{R^*}\}$ blow-up curve. Below, we will identify Γ with T itself. We have 2 purposes of this Chapter. First, we analytically show that $T \in C^1(B_{R^*})$. Second, we present numerical examples of blow-up curve. We numerically show that the blow-up curve is smooth if the initial values of (0.4) are large and smooth enough. Moreover, we show that the case where the blow-up curve has singular points even the initial values are smooth. In previous study, the cases of $F(u) = |u|^p,\ e^u$ and the following blow-up curve are considered (for example, [6], [7], [18]).

$$\tilde{T}(x) = \sup\{t \in (0, T^*) \mid \ |u(t,x)| < \infty\} \qquad (x \in B_{R^*}).$$

It was shown that $\tilde{T} \in C^1(B_{R^*})$ under suitable initial values. The method introduced by Caffarelli-Friedman [7] are used in the proof of regularity of the blow-up curve. However, we cannot directly apply their method to (0.4) in the case of $F(u) = |u_t|^p$. For these reasons, the mathematical analysis of blow-up curve for the wave equation with a nonlinear term involving the derivative of unknown functions is not well understood.

On the other hand, Ohta-Takamura [30] studied the blow-up curve in the case of $F(u) = (u_t)^2 - (u_x)^2$. The key point of their proof is the transformation $v = e^{-u}$. We see that v satisfies $v_{tt} - v_{xx} = 0$. Thanks to the linearization, we can study the blow-up curve in the case of $F(u) = (u_t)^2 - (u_x)^2$. However, we cannot use this transformation in the case of $F(u) = |u_t|^p$.

Thus, we rewrite (0.4) as the following first order system by using the idea of Chapter 1.

$$\begin{cases} D_-\phi = 2^{-p}|\phi + \psi|^p, & t > 0,\ x \in \mathbb{R}, \\ D_+\psi = 2^{-p}|\phi + \psi|^p, & t > 0,\ x \in \mathbb{R}, \\ \phi(x,0) = f(x), \quad \psi(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $D_-v = v_t - v_x,\ D_+v = v_t + v_x$ and $f = u_1 + \partial_x u_0,\ g = u_1 - \partial_x u_0$. Such rewriting makes it easier to analyze the blow-up curve, not to mention ease of analysis of numerical methods. We also offer an alternative proof of [7] for showing that the blow-up curve of the blow-up limits is an affine function. Our proof is more elementary and easy to read. Moreover, we show some numerical examples of the blow-up curve of (0.4) in the case of $F(u) = |u_t|^p$. From the numerical results, the blow-up curve sometimes has singular points even the initial values are smooth if the initial values are not large. The analytical proof is still open in the case of $F(u) = |u_t|^p$.

In order that we want to readers to avoid to confuse the formulations, we explicitly write the definitions in each chapter. Although multiple same definitions may appear through the thesis, the arguments in each chapter become self contained. This helps readers understand the detailed content of each chapter separately.

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Contents

1 Blow-up of finite-difference solutions to nonlinear wave equations

Finite-difference schemes for computing blow-up solutions of one dimensional nonlinear wave equations are presented. By applying time increments control technique, we can introduce a numerical blow-up time which is an approximation of the exact blow-up time of the nonlinear wave equation. After having verified the convergence of our proposed schemes, we prove that solutions of those finite-difference schemes actually blow up in the corresponding numerical blow-up times. Then, we prove that the numerical blow-up time converges to the exact blow-up time as the discretization parameters tend to zero. Several numerical examples that confirm the validity of our theoretical results are also offered.

1.1 Introduction

The purpose of this paper is to establish numerical methods for computing blow-up solutions of one space dimensional nonlinear wave equations with power nonlinearities. In order to avoid unessential difficulties about boundary conditions, we concentrate our attention to L-periodic functions of x with $L > 0$. That is, setting $S_L = \mathbb{R}/L\mathbb{Z}$, we consider the following initial value problem for the function $u = u(t,x)$ $(t \geq 0,\ x \in S_L)$,

$$\begin{cases} u_{tt} - u_{xx} = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{1.1}$$

Before stating assumptions on nonlinearlity and initial values, we recall a general result for nonlinear wave equations. Set $Q_{T,L} = [0,T] \times S_L$ for $T > 0$.

Proposition 1.1.1. *Let $u_0, u_1 \in C^3(S_L)$ and $f \in C^4(\mathbb{R})$ be given. Then, there exists $T > 0$ and a unique classical solution $u \in C^3(Q_{T,L})$ of*

$$\begin{cases} u_{tt} - u_{xx} = f(u), & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{1.2}$$

Moreover, there exists a positive and continuous function $C_{ml}(\eta)$ of $\eta > 0$ satisfying

$$\left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \leq C_{ml} \left(\|u\|_{L^\infty(Q_{T,L})} \right)$$

for non-negative integers m, l such that $m+l \leq 3$. Furthermore, if $f(s) \geq 0$ for $s \geq 0$ and $u_0(x) \geq 0$, $u_1(x) \geq 0$ for $x \in S_L$, then we have $u(t,x) \geq 0$ for $(t,x) \in Q_{T,L}$.

This proposition is proved by the standard argument based on the contraction mapping principle (cf. [15, §12.3]) with the aid of the explicit solution formula given as

$$u(t,x) = \frac{1}{2}[u_0(x-t) + u_0(x-t)] \\ + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi + \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(u(s,y))\, dyds.$$

Throughout this paper, we make the following assumptions:

$$f(u) = |u|^p \text{ with } p > 1 \text{ is of class } C^4; \tag{1.3}$$

$$u_0,\ u_1 \in C^3(S_L); \tag{1.4}$$

$$u_0(x) \geq 0, \quad u_1(x) \geq 0, \quad x \in S_L. \tag{1.5}$$

Thanks to Proposition 1.1.1, the problem (1.1) admits a unique non-negative solution $u \in C^3(Q_{T,L})$, which we will call simply a *solution* hereinafter. We note that the condition (1.3) is equivalently written as

$$p = 2 \text{ or } p \text{ is a real number} \geq 4. \tag{1.6}$$

See also Remark 1.2.10.

The supremum of T in Proposition 1.1.1 is called the lifespan of a solution and is denoted by T_∞. If $T_\infty = \infty$, then we say that the solution u of (1.1) exists globally-in-time. On the other hand, if $T_\infty < \infty$, we say that u blows up in finite time and call T_∞ the blow-up time of a solution.

As a readily obtainable consequence of Proposition 1.1.1, we deduce the following proposition.

Proposition 1.1.2. *Let u be the solution of* (1.1). *Then, the following (i) and (ii) are equivalent.*

(i) *u blows up in finite time $T_\infty < \infty$.*

(ii) $\lim_{t \uparrow T_\infty} \|u(t)\|_{L^\infty(S_L)} = \infty$.

Any solution u of (1.1) actually blows up. To verify this fact, the functional

$$K(v) = \frac{1}{L}\int_0^L v(x)\, dx \qquad (v \in C(S_L))$$

plays an important role. Obviously, we have

$$K(v) \leq \|v\|_{L^\infty(S_L)} \quad (0 \leq v \in C(S_L)). \tag{1.7}$$

Proposition 1.1.3. *Assume that*

$$\alpha = K(u_0) \geq 0, \quad \beta = K(u_1) > 0. \tag{1.8}$$

Then, there exists $T_\infty \in (0, \infty)$ *such that the solution* u *of* (1.1) *blows up in finite time* T_∞.

This proposition is not new; however, we briefly review the proof since we will study a discrete analogue of this result in Section 1.4. As a matter of fact, the key point of the proof is that the solution u of (1.1) satisfies, whenever it exists,

$$\frac{d}{dt}K(u(t)) \geq \beta + \int_0^t K(u(s))^p \, ds > 0, \tag{1.9}$$

$$\left[\frac{d}{dt}K(u(t))\right]^2 \geq \frac{2}{p+1}K(u(t))^{p+1} + M_1 \geq 0, \tag{1.10}$$

where $M_1 = \beta^2 - \frac{2}{p+1}\alpha^{p+1}$ and $K(u(t)) = K(u(t, \cdot))$.

These inequalities, together with the following elementary proposition, implies that $K(u(t))$ cannot exist beyond T_K, which is defined below. Thus, $u(t, x)$ blows up in finite time $T_\infty \in (0, T_K]$, which completes the proof of Proposition 1.1.3.

Proposition 1.1.4. *Let a* C^1 *function* $w = w(t)$ *satisfy a differential inequality*

$$\frac{d}{dt}w(t) \geq \sqrt{\frac{2}{p+1}w(t)^{p+1} + M_1} \qquad (t > 0) \tag{1.11}$$

with $w(0) = \alpha \geq 0$. *Then,* $w(t)$ *blows up in finite time* $T_K \in (0, T_1)$, *where*

$$T_1 = \int_\alpha^\infty \left[\beta^2 + \frac{2}{p+1}(s^{p+1} - \alpha^{p+1})\right]^{-\frac{1}{2}} ds < \infty.$$

Inequalities (1.9) and (1.10) are derived in the following manner. First, we derive by using Jensen's inequality

$$\frac{d^2}{dt^2}K(u(t)) \geq K(u(t))^p, \tag{1.12}$$

which gives (1.9). Multiplying the both-sides of (1.12) by $(d/dt)K(u(t))$, we have

$$\frac{d}{dt}K(u(t))\frac{d^2}{dt^2}K(u(t)) \geq \frac{d}{dt}K(u(t))K(u(t))^p.$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d}{dt}K(u(t))\right)^2 - \int_\alpha^{K(u(t))} \xi^p \, d\xi\right] \geq 0.$$

Therefore, we get

$$\left[\frac{d}{dt}K(u(t))\right]^2 \geq \beta^2 + \frac{2}{p+1}\left[K(u(t))^{p+1} - \alpha^{p+1}\right],$$

which implies (1.10).

There are a large number of works devoted to blow-up of positive solutions for nonlinear wave equations. To our best knowledge, the first result was obtained by Kawarada [24]. He studied a nonlinear wave equation

$$u_{tt} - \Delta u = f(u) \quad (x \in \Omega,\ t > 0) \tag{1.13}$$

in a smooth bounded domain Ω in $\mathbb{R}^d$ and proved a positive solution actually blows up in finite time if the initial values are sufficiently large. (He did not consider a positive solution explicitly, but as a readily obtainable corollary of his theorem we could obtain the blow-up of a positive solution.) Those results are referred as "large data blow-up" results. After Kawarada's work, a lost of results have been reported. For example, Glassey's papers [16], [17] are well-known. On the other hand, "small data blow-up" results were presented, for example, F. John ([22]) and T. Kato ([23]). See an excellent survey by S. Alinhac ([2]) for more details on blow-up results for nonlinear hyperbolic equations. In contrast to parabolic equations, it seems that there is a little work devoted to asymptotic profiles and blow-up rates of blow-up solutions for hyperbolic equations. Therefore, numerical methods would be important tools to study blow-up phenomena in hyperbolic equations.

However, the computation of blow-up solutions is a difficult task. We do not state here the detail of those issues; see, for example, [13] and [10]. In order to surmount those obstacles, various techniques for computing blow-up solutions of various nonlinear partial differential equations are developed so far. Among them, variable time-increments Δt_n is of use. The pioneering work is done by Nakagawa [28] in 1976. He considered the explicit Euler/finite difference scheme to a semilinear heat equation $u_t - u_{xx} = u^2$ $(t > 0, 0 < x < 1)$ with $u(t,0) = u(t,1) = 0$. The crucial point of his strategy is that the time increment and the discrete time are given, respectively, as

$$\Delta t_n = \tau \min\left\{1, \frac{1}{\|u_h(t_n)\|_{L^2}}\right\}, \quad t_{n+1} = t_n + \Delta t_n = \sum_{k=0}^{n} \Delta t_k$$

with some $\tau > 0$, where $u_h(t_n)$, h being the size of space grids, denotes the piecewise constant interpolation function of the finite-difference solution at $t = t_n$ and $\|u_h(t_n)\|_{L^2}$ its $L^2(0,1)$ norm. Then, he succeeded in proving that, for a sufficiently large initial value, the finite-difference solution $u_h(t_n)$ actually blows up in finite time

$$T(\tau, h) = \sum_{n=1}^{\infty} \Delta t_n < \infty$$

and

$$\lim_{\tau,h\to 0} T(\tau, h) = T_\infty, \tag{1.14}$$

where τ denotes the size of a time discretization and T_∞ the blow-up time of the equation under consideration. $T(\tau, h)$ is called the *numerical blow-up time*. Later, Nakagawa's result has been extend to several directions; see, for example, Chen [**?**], Abia et al. [1], Nakagawa and Ushijima [29] and Cho et al. [13]. However, those papers are concerned only with parabolic equations. On the other hand, it seems that little is known for hyperbolic equations and C. H. Cho's work ([10]) is the first result on the subject. He studied the initial-boundary value problem for a nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = u^2 \qquad (t > 0,\ x \in (0,1)), \\ u = 0 \quad (t \geq 0,\ x = 0,1), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases}$$

and the explicit Euler/finite-difference scheme

$$\begin{cases} \dfrac{1}{\tau_n}\left(\dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} - \dfrac{u_j^n - u_j^{n-1}}{\Delta t_{n-1}}\right) = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + (u_j^n)^2, \\ u_0^n = u_N^n = 0, \quad u_j^0 = u_0(x_j), \quad u_j^1(x_j) = u_0(x_j) + \Delta t_0 u_1(x_j), \end{cases} \tag{1.15}$$

where the time and space variable are discretized as $t_n = \Delta t_0 + \Delta t_1 + \cdots + \Delta t_{n-1}$, $x_j = j/N$ and $N \in \mathbb{N}$, and u_j^n denotes the approximation of $u(t_n, x_j)$. He proposed the following time-increments control strategy

$$\Delta t_n = \tau \min\left\{1,\ \frac{1}{\|u_h(t_n)\|_{L^2}^{1/2}}\right\}, \qquad \tau_n = \frac{\Delta t_n + \Delta t_{n-1}}{2}. \tag{1.16}$$

Then, he succeeded in proving that (1.14) actually holds true under some assumptions. One of the crucial assumptions in his theorem is convergence of the finite-difference solutions, that is,

$$\lim_{h\to 0} \max_{0\leq t_n \leq T} |u_j^n - u(t_n, x_j)| = 0 \tag{1.17}$$

for any $T \in (0, T_\infty)$. The proof of this convergence result is still open at present. As a matter of fact, we need some a priori estimates or stability in a certain norm in order to prove (1.17). However, as Cho mentioned in [10, page 487], it is quite difficult to prove a stability that remains true even when $\Delta t_n \to 0$.

Recently, K. Matsuya reported some interesting results on global existence and blow-up of solutions of a discrete nonlinear wave equation in [26]. However, it seems that his results are not directly related with approximation of partial differential equations.

This paper is motivated by the paper [10] and devoted to a study of the finite-difference method applied to (1.1). Thus, we propose finite-difference schemes and prove convergence results (cf. Theorems 1.2.4 and 1.2.5) for those schemes even when time-increments approaches to zero. To accomplish this purpose, we rewrite the equation as

$$u_t + u_x = \phi, \quad \phi_t - \phi_x = |u|^p,$$

which is based on the formal factorization $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u = |u|^p$, and then follow the method of convergence analysis proposed by [32] that is originally developed to study time-discretizations for a system of nonlinear Schrödinger equations. Actually, it suffices to prove local stability results in a certain sense (cf. Theorems 1.2.2 and 1.2.3) in order to obtain convergence results. Moreover, we show that discrete analogues of (1.9) and (1.10) holds true, and therefore, we can deduce approximation of blow-up time (1.14) (cf. Theorem 1.2.8).

This paper is organized as follows. In Section 1.2, after having stated our finite-difference schemes, we mention stability and convergence results for our schemes (Theorems 1.2.2, 1.2.3, 1.2.4 and 1.2.5). Therein, approximation of blow-up time is also mentioned (Theorem 1.2.8). Section 1.3 is devoted to the proofs of Theorems 1.2.2, 1.2.3, 1.2.4 and 1.2.5. The proof of Theorem 1.2.8 is given in Section 1.4. We conclude this paper by examining several numerical examples in Section 1.5.

Notation

For $\boldsymbol{v} = (v_1, \dots, v_J)^{\mathrm{T}} \in \mathbb{R}^J$, we set $\|\boldsymbol{v}\| = \max_{1\le j\le J} |v_j|$, where $\cdot^{\mathrm{T}}$ indicates the transpose of a matrix. We write $\boldsymbol{v} \geq \boldsymbol{0}$ if and only if $v_i \geq 0$ $(1 \leq i \leq J)$. We use the matrix ∞ norm

$$\|E\| = \max_{\boldsymbol{v}\in\mathbb{R}^J} \frac{\|E\boldsymbol{v}\|}{\|\boldsymbol{v}\|} = \max_{1\le i\le J} \sum_{j=1}^{J} |E_{ij}|$$

for a matrix $E = (E_{ij}) \in \mathbb{R}^{J\times J}$. Moreover, we write $E \geq O$ if and only if $E_{i,j} \geq 0$ $(1 \leq i, j \leq J)$. The set of all positive integers is denoted by $\mathbb{N}$.

1.2 Schemes and main results

Introducing a new variable $\phi = u_t + u_x$, we first convert (1.1) into the first order system as follows:

$$\begin{cases} u_t + u_x = \phi & (t,x) \in Q_{T,L}, \\ \phi_t - \phi_x = |u|^p & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad \phi(0,x) = u_1(x) + u_0'(x), & x \in S_L. \end{cases} \tag{1.18}$$

Take a positive integer J and set $x_j = jh$ with $h = L/J$. As a discretization of the time variable, we take positive constants $\Delta t_0, \Delta t_1, \dots$ and set

$$t_0 = 0, \quad t_n = \sum_{k=0}^{n-1} \Delta t_k = t_{n-1} + \Delta t_{n-1} \quad (n \geq 1).$$

Then, our explicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^n \approx \phi(t_n, x_j) \qquad (1 \leq j \leq J,\ t \geq 0)$$

reads as

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{u_j^n - u_{j-1}^n}{h} = \phi_j^n \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t_n} - \dfrac{\phi_{j+1}^n - \phi_j^n}{h} = |u_j^{n+1}|^p \end{cases} \qquad (1 \leq j \leq J,\ n \geq 0) \tag{1.19}$$

where u_0^n and ϕ_{J+1}^n are set as $u_0^n = u_J^n$ and $\phi_{J+1}^n = \phi_1^n$.

We also consider an implicit scheme for the purpose of comparison. However, we do not prefer fully implicit schemes since we need iterative computations for solving resulting nonlinear system. Instead, we consider a linearly-implicit scheme by introducing dual time grids

$$t_{n+\frac{1}{2}} = \frac{\Delta t_0}{2} + t_n \quad (n \geq 0). \tag{1.20}$$

Then, our implicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^{n+\frac{1}{2}} \approx \phi(t_{n+\frac{1}{2}}, x_j) \qquad (1 \leq j \leq J,\ n \geq 0)$$

reads as

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{1}{2}\left(\dfrac{u_j^{n+1} - u_{j-1}^{n+1}}{h} + \dfrac{u_j^n - u_{j-1}^n}{h}\right) = \phi_j^{n+\frac{1}{2}}, \\ \dfrac{\phi_j^{n+\frac{3}{2}} - \phi_j^{n+\frac{1}{2}}}{\Delta t_n} - \dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}} - \phi_j^{n+\frac{3}{2}}}{h} + \dfrac{\phi_{j+1}^{n+\frac{1}{2}} - \phi_j^{n+\frac{1}{2}}}{h}\right) = |u_j^{n+1}|^p, \end{cases}$$
$$(1 \leq j \leq J,\ n \geq 0), \tag{1.21}$$

where u_0^n and $\phi_{J+1}^{n+\frac{1}{2}}$ are set as $u_0^n = u_J^n$ and $\phi_{J+1}^{n+\frac{1}{2}} = \phi_1^{n+\frac{1}{2}}$.

Remark 1.2.1. It is possible to take

$$t_{\frac{1}{2}} = \frac{\Delta t_0}{2}, \quad t_{n+\frac{1}{2}} = \frac{\Delta t_0}{2} + \sum_{k=1}^{n} \tau_k \quad (n \geq 1)$$

as dual time grids instead of (1.20), where $\tau_k = (\Delta t_{k-1} + \Delta t_k)/2$. With this choice, the implicit scheme is modified as

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{1}{2}\left(\dfrac{u_j^{n+1} - u_{j-1}^{n+1}}{h} + \dfrac{u_j^n - u_{j-1}^n}{h}\right) = \phi_j^{n+\frac{1}{2}}, \\ \dfrac{\phi_j^{n+\frac{3}{2}} - \phi_j^{n+\frac{1}{2}}}{\tau_n} - \dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}} - \phi_j^{n+\frac{3}{2}}}{h} + \dfrac{\phi_{j+1}^{n+\frac{1}{2}} - \phi_j^{n+\frac{1}{2}}}{h}\right) = |u_j^{n+1}|^p, \end{cases}$$
$$(1 \leq j \leq J,\ n \geq 0). \tag{1.22}$$

Then, we can deduce all the results presented below with obvious modifications.

For $n \geq 0$, we set

$$\boldsymbol{u}^n = (u_1^n, \ldots, u_J^n)^{\mathrm{T}} \in \mathbb{R}^J,$$
$$\boldsymbol{\phi}^n = (\phi_1^n, \ldots, \phi_J^n)^{\mathrm{T}} \in \mathbb{R}^J, \qquad \boldsymbol{\phi}^{n+\frac{1}{2}} = (\phi_1^{n+\frac{1}{2}}, \ldots, \phi_J^{n+\frac{1}{2}})^{\mathrm{T}} \in \mathbb{R}^J.$$

Theorem 1.2.2 (Local stability of the explicit scheme)**.** *Let $\tau = \gamma h$ with some $\gamma \in (0,1)$ and assume that $\Delta t_n \leq \tau$ for $n \geq 0$. Let $\boldsymbol{a} \geq \boldsymbol{0}, \boldsymbol{b} \geq \boldsymbol{0} \in \mathbb{R}^J$. Then,* *the solution* $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ *of the explicit scheme* (1.19) *with* $\boldsymbol{u}^0 = \boldsymbol{a}$ *and* $\boldsymbol{\phi}^0 = \boldsymbol{b}$ *satisfies* $\boldsymbol{u}^n \geq \mathbf{0}$ *and* $\boldsymbol{\phi}^n \geq \mathbf{0}$ *for* $n \geq 1$*. Furthermore, for any* $N \in \mathbb{N}$*, there exists a constants* $h_{R,N} > 0$ *depending only on* N *and* $R = \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ *such that, if* $h \in (0, h_{R,N}]$*, we have*

$$\sup_{1\leq n\leq N} (\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^n\|) \leq 2R. \tag{1.23}$$

Theorem 1.2.3 (Well-posedness and local stability of the implicit scheme)**.** *Let* $\tau = 2\gamma h$ *with some* $\gamma \in (0,1)$ *and assume that* $\Delta t_n \leq \tau$ *for* $n \geq 0$*. Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^J$*. Then, the implicit scheme* (1.21) *admits a unique solution* $(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})$ *for any* $n \geq 1$*, where* $\boldsymbol{u}^0 = \boldsymbol{a}$ *and* $\boldsymbol{\phi}^{\frac{1}{2}} = \boldsymbol{b}$*. Moreover, if* $\boldsymbol{a} \geq \mathbf{0}$ *and* $\boldsymbol{b} \geq \mathbf{0}$*, then we have* $\boldsymbol{u}^n \geq \mathbf{0}$ *and* $\boldsymbol{\phi}^{n+\frac{1}{2}} \geq \mathbf{0}$ *for* $n \geq 1$*. Furthermore, for any* $N \in \mathbb{N}$*, there exists a constants* $h_{R,N} > 0$ *depending only on* N *and* $R = \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ *such that, if* $h \in (0, h_{R,N}]$*, we have*

$$\sup_{1\leq n\leq N} \left(\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\|\right) \leq 2R. \tag{1.24}$$

In order to state convergence results, we introduce $\boldsymbol{e}^n = (e_j^n)$, $\boldsymbol{\varepsilon}^n = (\varepsilon_j^n)$ and $\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = (\varepsilon_j^{n+\frac{1}{2}})$ which are given as

$$e_j^n = u(t_n, x_j) - u_j^n, \quad \varepsilon_j^n = \phi(t_n, x_j) - \phi_j^n, \quad \varepsilon_j^{n+\frac{1}{2}} = \phi(t_{n+\frac{1}{2}}, x_j) - \phi_j^{n+\frac{1}{2}}.$$

Recall that T_∞ denotes the blow-up time of the solution $u(t,x)$ of (1.1).

Theorem 1.2.4 (Convergence of the explicit scheme)**.** *Let* $\tau = \gamma h$ *with some* $\gamma \in (0,1)$ *and assume that* $\Delta t_n \leq \tau$ *for* $n \geq 0$*. Suppose that* $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ *is the solution of the explicit scheme* (1.19) *for* $n \geq 1$*, where* $(\boldsymbol{u}^0, \boldsymbol{\phi}^0)$ *is defined as*

$$u_j^0 = u_0(x_j), \quad \phi_j^0 = u_1(x_j) + u_0'(x_j) \quad (1 \leq j \leq J). \tag{1.25}$$

Let $T \in (0, T_\infty)$ *be arbitrarily. Then, there exists positive constants* h_0 *and* M_0 *which depend only on*

$$p, \quad T, \quad \gamma, \quad M = \max_{0\leq m+l\leq 3} \left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \tag{1.26}$$

such that we have

$$\max_{0\leq t_n\leq T} (\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^n\|) \leq M_0(\tau + h)$$

for any $h \in (0, h_0]$*.*

Theorem 1.2.5 (Convergence of the implicit scheme)**.** *Let* $\tau = 2\gamma h$ *with some* $\gamma \in (0,1)$ *and assume that* $\Delta t_n \leq \tau$ *for* $n \geq 0$*. Suppose that* $(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})$ *is the solution of the implicit scheme* (1.21) *for* $n \geq 1$*, where* $(\boldsymbol{u}^0, \boldsymbol{\phi}^{\frac{1}{2}})$ *is defined as*

$$u_j^0 = u_0(x_j), \quad \phi_j^{\frac{1}{2}} = u_1(x_j) + u_0'(x_j) \quad (1 \leq j \leq J). \tag{1.27}$$

Let $T \in (0, T_\infty)$ be arbitrarily. Then, there exists positive constants h_0 and M_0, which depend only on (1.26)*, such that we have*

$$\max_{0 \le t_{n+1} \le T} \left(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \right) \le M_0(\tau + h) \tag{1.28}$$

for any $h \in (0, h_0]$.

Remark 1.2.6. If taking constant time-increments $\Delta t_n = \tau$ and suitable initial value $\boldsymbol{\phi}^{\frac{1}{2}}$, we can prove

$$\max_{0 \le t_{n+1} \le T} \left(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \right) \le M_0(\tau^2 + h)$$

instead of (1.28).

By using the solutions of the explicit scheme (1.19) and the implicit scheme (1.21), we can calculate the blow-up time T_∞ of the solution of (1.1). To this purpose, we fix

$$1 \le q < \infty, \qquad 0 < \gamma < 1 \tag{1.29}$$

and choose the time increments $\Delta t_0, \Delta t_1, \ldots$ as

$$\Delta t_n = \tau \cdot \min\left\{1,\ \frac{1}{\|\boldsymbol{u}^n\|^q}\right\} \qquad (n \ge 0), \tag{1.30}$$

where τ is taken as

$$\tau = \begin{cases} \gamma h & \text{for the explicit scheme (1.19)} \\ 2\gamma h & \text{for the implicit scheme (1.21).} \end{cases} \tag{1.31}$$

Definition 1. Let $\boldsymbol{u}^n$ be the solution of the explicit scheme (1.19) or the implicit scheme (1.21) with the time increment control (1.30) and (1.31). Then, we set

$$T(h) = \sum_{n=0}^{\infty} \Delta t_n.$$

If $T(h) < \infty$, we say that $\boldsymbol{u}^n$ blows up in finite time $T(h)$.

Remark 1.2.7. The blow-up of $\boldsymbol{u}^n$ implies that $\lim_{t_n \to T(h)} \|\boldsymbol{u}^n\| = \lim_{n \to \infty} \|\boldsymbol{u}^n\| = \infty$.

We are now in a position to state numerical blow-up results.

Theorem 1.2.8 (Approximation of the blow-up time)**.** *Let $\boldsymbol{u}^n$ be the solution of the explicit scheme* (1.19) *or the implicit scheme* (1.21) *with the time increment control* (1.30) *and* (1.31)*, where the initial value is defined as* (1.25) *or* (1.27)*, respectively. In addition to the basic assumptions* (1.4) *and* (1.5) *on initial values, assume that $u_1(x)$ is so large that*

$$u_1(x) + u_0'(x) \ge 0, \not\equiv 0 \quad (x \in S_L). \tag{1.32}$$

Then, we have the following:

(i) $\boldsymbol{u}^n \geq 0$ *and* $\boldsymbol{\phi}^n \geq \mathbf{0}$ *(or* $\boldsymbol{\phi}^{n+\frac{1}{2}} \geq \mathbf{0}$*) for all* $n \geq 0$*.*

(ii) *If* (1.8) *holds true,* $\boldsymbol{u}^n$ *blows up in finite time* $T(h)$ *and*

$$T_\infty \leq \liminf_{h\to 0} T(h). \tag{1.33}$$

(iii) *In addition to* (1.8)*, we assume that*

$$\lim_{t\to T_\infty} K(u(t)) = \infty, \tag{1.34}$$

then we have

$$T_\infty = \lim_{h\to 0} T(h). \tag{1.35}$$

Remark 1.2.9. The assumption (1.34) is somewhat restrictive. Essentially the same assumption is considered in [10]. However, we are unable to remove it at present. To find the sufficient condition for (1.34) to hold is an interesting open question.

Remark 1.2.10. All results presented above remain valid for $f(u) = u|u|^2$, since it is a C^4 function on $\mathbb{R}$.

1.3 Proofs of Theorems 1.2.2, 1.2.3, 1.2.4 and 1.2.5

We rewrite the explicit scheme (1.19) and the implicit scheme (1.21), respectively, as

$$\begin{cases} \boldsymbol{u}^{n+1} = M_n \boldsymbol{u}^n + \Delta t_n \boldsymbol{\phi}^n \\ \boldsymbol{\phi}^{n+1} = N_n \boldsymbol{\phi}^n + \Delta t_n \boldsymbol{f}(\boldsymbol{u}^{n+1}) \end{cases} \quad (n \geq 0), \tag{1.36}$$

and

$$\begin{cases} A_n \boldsymbol{u}^{n+1} = B_n \boldsymbol{u}^n + \Delta t_n \boldsymbol{\phi}^{n+\frac{1}{2}} \\ C_n \boldsymbol{\phi}^{n+\frac{3}{2}} = D_n \boldsymbol{\phi}^{n+\frac{1}{2}} + \Delta t_n \boldsymbol{f}(\boldsymbol{u}^{n+1}) \end{cases} \quad (n \geq 0), \tag{1.37}$$

where

$$M_n = P(-\gamma_n), \quad N_n = P(-\gamma_n)^{\mathrm{T}},$$
$$A_n = P(\delta_n), \quad B_n = P(-\delta_n), \quad C_n = P(\delta_n)^{\mathrm{T}}, \quad D_n = P(-\delta_n)^{\mathrm{T}},$$
$$\gamma_n = \frac{\Delta t_n}{h}, \quad \delta_n = \frac{\Delta t_n}{2h},$$
$$P(\mu) = \begin{pmatrix} 1+\mu & 0 & \cdots & -\mu \\ -\mu & 1+\mu & 0 & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & -\mu & 1+\mu \end{pmatrix},$$
$$\boldsymbol{f}(\boldsymbol{v}) = (|v_1|^p, \ldots, |v_J|^p)^{\mathrm{T}} \quad \text{for} \quad \boldsymbol{v} = (v_1, \ldots, v_J)^{\mathrm{T}}.$$

Lemma 1.3.1. (i) *$P(\mu)$ is non-singular, $P(\mu)^{-1} \geq O$ and $\|P(\mu)^{-1}\| \leq 1$ if $\mu > 0$.*

(ii) $P(-\mu) \geq O$ *and* $\|P(-\mu)\| = 1$ *if* $0 < \mu \leq 1$.

Proof. (i) Let $\mu > 0$. The matrix $P(\mu)$ is expressed as $P(\mu) = (1+\mu)(I-G)$, where

$$G = \frac{\mu}{1+\mu}\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Since $\|G\| = \mu(1+\mu)^{-1} < 1$, the matrix $I-G$ is non-singular, $(I-G)^{-1} = \sum_{l=0}^{\infty} G^l \geq O$ and $\|(I-G)^{-1}\| \leq 1/(1-\|G\|) = 1+\mu$. Hence, $P(\mu)$ is also non-singular, $P(\mu)^{-1} = (1+\mu)^{-1}\sum_{l=0}^{\infty} G^l \geq O$ and $\|P(\mu)^{-1}\| \leq (1+\mu)^{-1}\|(I-G)^{-1}\| = 1$.
(ii) Let $0 < \mu \leq 1$. Then, $P(-\mu) \geq O$ is obvious. We further have

$$\|P(-\mu)\| = \max_{1\leq i\leq J}\sum_{j=1}^{J}|p_{ij}| = (1-\mu)+\mu = 1,$$

where $P(\mu) = (p_{ij})$, which completes the proof. $\square$

Now, we can state the following proofs.

Proofs of Theorems 1.2.2 and 1.2.3. According to Lemma 1.3.1, we have $M_n, N_n, B_n, D_n \geq O$ and $\|M_n\| = \|N_n\| = \|B_n\| = \|D_n\| = 1$. Moreover, A_n, C_n are non-singular, $A_n^{-1}, C_n^{-1} \geq O$ and $\|A_n^{-1}\|, \|C_n^{-1}\| \leq 1$. Therefore, the unique existence and non-negativity of solutions of (1.19) and (1.21) are direct consequences of the expressions (1.36) and (1.37), respectively.

Below we are going to show local stability results (1.23) and (1.24). We only state the proof of (1.24); that of (1.23) could be done in the same way. Recall that we are assuming that $\Delta t_j \leq \tau$ for all j and $\tau = 2\gamma h$ with some $\gamma \in (0,1)$. Choose $N \in \mathbb{N}$ arbitrarily and fix it.

Now we can prove (1.24) by induction on n. First, note that $\|\boldsymbol{u}^0\| + \|\boldsymbol{\phi}^{\frac{1}{2}}\| = \|\boldsymbol{a}\| + \|\boldsymbol{b}\| = R$. Assume that

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 2R \tag{1.38}$$

for $0 \leq n \leq N-1$. Since $\boldsymbol{u}^{n+1}$ and $\boldsymbol{\phi}^{n+\frac{3}{2}}$ are given as

$$\boldsymbol{u}^{n+1} = H_n \cdots H_0 \boldsymbol{a} + \sum_{j=0}^{n} \Delta t_{n-j} H_n \cdots H_{n-j+1} A_{n-j}^{-1} \boldsymbol{\phi}^{n-j+\frac{1}{2}},$$

$$\boldsymbol{\phi}^{n+\frac{3}{2}} = L_n \cdots L_0 \boldsymbol{b} + \sum_{j=0}^{n} \Delta t_{n-j} L_n \cdots L_{n-j+1} C_{n-j}^{-1} \boldsymbol{f}(\boldsymbol{u}^{n-j+1})$$

with $H_n = A_n^{-1}B_n$ and $L_n = C_n^{-1}D_n$, we have

$$\|\boldsymbol{u}^{n+1}\| \leq \|\boldsymbol{a}\| + \tau \sum_{j=0}^{n} \|\boldsymbol{\phi}^{n-j+\frac{1}{2}}\| \leq \|\boldsymbol{a}\| + N\tau(2R),$$
$$\|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq \|\boldsymbol{b}\| + \tau \sum_{j=0}^{n} \|\boldsymbol{u}^{n-j+1}\|^p \leq \|\boldsymbol{b}\| + N\tau(2R)^p$$

for $0 \leq n \leq N-1$. Hence,

$$\|\boldsymbol{u}^{n+1}\| + \|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq R + N\tau[2R + (2R)^p] \tag{1.39}$$

for $0 \leq n \leq N-1$.

At this stage, we define $\tau_{R,N}$ and $h_{R,N}$ as

$$\tau_{R,N} = \frac{R}{N[2R + (2R)^p]}, \qquad h_{R,N} = \frac{\tau_R}{2\gamma}$$

and suppose $h \in (0, h_{R,N}]$.

Then, by (1.39), we get

$$\|\boldsymbol{u}^{n+1}\| + \|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq 2R.$$

This completes the proof of (1.24). □

We proceed to the proof of convergence results. Below, we only state the proof of Theorem 1.2.5 since that of Theorem 1.2.4 is simpler.

Proof of Theorem 1.2.5. Let $\{(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})\}_{n\geq 1}$ be the solution of the implicit scheme (1.21) with the initial condition (1.27). We note that

$$\|\boldsymbol{u}^0\| + \|\boldsymbol{\phi}^{\frac{1}{2}}\| \leq 3M.$$

Hereinafter, set $M' = 3M$. In view of Theorem 1.2.3, there exists constants $h_{M'} > 0$ and $T_{M'} > 0$, which depend only on M' and p, such that, if $h \in (0, h_{M'}]$, we have

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 2M' \quad (n \in \Lambda_{M'} = \{n \in \mathbb{N} \mid t_n \leq T_{M'}\}).$$

We set

$$\nu = \sup\{n \in \mathbb{N} \mid \|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 3M'\},$$
$$\tilde{\Lambda}_\nu = \{n \in \mathbb{N} \mid t_{n+1} \leq T,\ n \leq \nu\}.$$

The rest of the proof is divided into two steps.

Step 1. First, we show that there exist positive constants h_1 and M_0, which depend only on T and M, such that the estimate (1.28) holds for all $h \in (0, h_1]$ and $n \in \tilde{\Lambda}_\nu$.

We have for $n \in \tilde{\Lambda}_\nu$

$$e_j^n - e_j^{n-1} + \frac{\Delta t_{n-1}}{2}\left(\frac{e_j^n - e_{j-1}^n}{h} + \frac{e_j^{n-1} - e_{j-1}^{n-1}}{h}\right) = \Delta t_{n-1} E_j^{n-\frac{1}{2}}, \tag{1.40}$$

where $E_j^{n-\frac{1}{2}} = \varepsilon_j^{n-\frac{1}{2}} - E_{1j}^{n-\frac{1}{2}} - E_{2j}^{n-\frac{1}{2}}$,

$$\begin{aligned} E_{1j}^{n-\frac{1}{2}} &= u_t(t_{n-\frac{1}{2}}, x_j) - \frac{u(t_n, x_j) - u(t_{n-1}, x_j)}{\Delta t_{n-1}}, \\ E_{2j}^{n-\frac{1}{2}} &= u_x(t_{n-\frac{1}{2}}, x_j) \\ &\quad -\frac{1}{2}\left(\frac{u(t_n, x_j) - u(t_n, x_{j-1})}{h} + \frac{u(t_{n-1}, x_j) - u(t_{n-1}, x_{j-1})}{h}\right). \end{aligned}$$

Since (1.40) is equivalently written as

$$\boldsymbol{e}^n = A_{n-1}^{-1} B_{n-1} \boldsymbol{e}^{n-1} + \Delta t_{n-1} A_{n-1}^{-1} \boldsymbol{E}^{n-\frac{1}{2}},$$

where $\boldsymbol{E}^{n-\frac{1}{2}} = (E_j^{n-\frac{1}{2}})$, we have from Lemma 1.3.1

$$\begin{aligned} \|\boldsymbol{e}^n\| &\leq \|\boldsymbol{e}^{n-1}\| + \Delta t_{n-1} \|\boldsymbol{E}^{n-\frac{1}{2}}\| \\ &\leq \|\boldsymbol{e}^{n-1}\| + \Delta t_{n-1}(\|\boldsymbol{E}_1^{n-\frac{1}{2}}\| + \|\boldsymbol{E}_2^{n-\frac{1}{2}}\|) + \Delta t_{n-1}\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|. \end{aligned}$$

From the standard error estimates for the difference quotients, we obtain

$$\|\boldsymbol{E}_1^{n-\frac{1}{2}}\| \leq CM\Delta t_{n-1}, \quad \|\boldsymbol{E}_2^{n-\frac{1}{2}}\| \leq CM(\Delta t_{n-1} + h)$$

for $n \in \tilde{\Lambda}_\nu$. Consequently,

$$\|\boldsymbol{e}^n\| \leq \|\boldsymbol{e}^{n-1}\| + CM\Delta t_{n-1}(\Delta t_{n-1} + h) + \Delta t_{n-1}\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| \tag{1.41}$$

for $n \in \tilde{\Lambda}_\nu$.

Similarly, we have for $n \in \tilde{\Lambda}_\nu$

$$\varepsilon_j^{n+\frac{1}{2}} - \varepsilon_j^{n-\frac{1}{2}} - \frac{\Delta t_{n-1}}{2}\left(\frac{\varepsilon_{j+1}^{n+\frac{1}{2}} - \varepsilon_j^{n+\frac{1}{2}}}{h} + \frac{\varepsilon_{j+1}^{n-\frac{1}{2}} - \varepsilon_j^{n-\frac{1}{2}}}{h}\right) = \Delta t_{n-1}\xi_j^n,$$

or, equivalently,

$$\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = C_{n-1}^{-1} D_{n-1} \boldsymbol{\varepsilon}^{n-\frac{1}{2}} + \Delta t_{n-1} C_{n-1}^{-1} \boldsymbol{\xi}^n,$$

where $\xi_j^n = -\xi_{1j}^n + \xi_{2j}^n + \xi_{3j}^n$, $\boldsymbol{\xi}^n = (\xi_j^n)$ and

$$\begin{aligned} \xi_{1j}^n &= \phi_t(t_n, x_j) - \frac{\phi(t_{n+\frac{1}{2}}, x_j) - \phi(t_{n-\frac{1}{2}}, x_j)}{\Delta t_{n-1}}, \\ \xi_{2j}^n &= \phi_x(t_n, x_j) \\ &\quad -\frac{1}{2}\left(\frac{\phi(t_{n+\frac{1}{2}}, x_{j+1}) - \phi(t_{n+\frac{1}{2}}, x_j)}{h} + \frac{\phi(t_{n-\frac{1}{2}}, x_{j+1}) - \phi(t_{n-\frac{1}{2}}, x_j)}{h}\right), \\ \xi_{3j}^n &= |u(t_n, x_j)|^p - |u_j^n|^p. \end{aligned}$$

We know

$$\|\boldsymbol{\xi}_1^n\| \leq CM\Delta t_{n-1}, \quad \|\boldsymbol{\xi}_2^n\| \leq CM(\Delta t_{n-1} + h)$$

for $n \in \tilde{\Lambda}_\nu$. Since $|u(t_n,x_j)| \le M$ and $|u_j^n| \le 3M'$, we can estimate as

$$\Big||u(t_n,x_j)|^p - |u_j^n|^p\Big| \le C_{2p}M^{p-1}|u(t_n,x_j) - u_j^n|$$

for $n \in \tilde{\Lambda}_\nu$ and $1 \le j \le J$, where C_2p denotes a constant depending only on p. Hence, we deduce

$$\|\boldsymbol{\xi}_3^n\| \le CM^{p-1}\|\boldsymbol{e}^n\|$$

for $n \in \tilde{\Lambda}_\nu$. Thus, we obtain

$$\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1}+h) + CM^{p-1}\Delta t_{n-1}\|\boldsymbol{e}^n\|. \tag{1.42}$$

Summing up (1.41) and (1.42), we deduce

$$\begin{aligned}\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le \|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1}+h)& \\ +CM^{p-1}\Delta t_{n-1}\|\boldsymbol{e}^n\| + \Delta t_{n-1}\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|.& \end{aligned} \tag{1.43}$$

Setting $M^* = M + M^{p-1}$, we have from (1.43)

$$\begin{aligned}&(1-CM^*\Delta t_{n-1})(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|)\\ &\le \|\boldsymbol{e}^{n-1}\| + (1+\Delta t_{n-1})\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1}+h)\\ &\le (1+CM^*\Delta t_{n-1})(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + CM^*\Delta t_{n-1}(\Delta t_{n-1}+h).\end{aligned}$$

At this stage, we define

$$h_1 = \frac{1}{4\gamma CM^*}, \quad \tau_1 = 2\gamma h_1$$

and we assume that $h \in (0,h_1]$. Then, using an elementally inequality $0 \le (1-s)^{-1}(1+s) \le 1+4s$ for $s \in [0,1/2]$, we have

$$\begin{aligned}&\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|\\ &\le (1+4CM^*\Delta t_{n-1})(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + 2CM^*\Delta t_{n-1}(\Delta t_{n-1}+h)\\ &\le e^{4CM^*\Delta t_{n-1}}(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + 2CM^*\Delta t_{n-1}(\Delta t_{n-1}+h).\end{aligned}$$

Therefore

$$\begin{aligned}\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| &\le e^{4CM^*t_n}(\|\boldsymbol{e}^0\| + \|\boldsymbol{\varepsilon}^{\frac{1}{2}}\|) + 2CM^*\sum_{j=0}^{n-1}\Delta t_j(\Delta t_j+h)e^{4CM^*t_n}\\ &\le e^{4CM^*T}\|\boldsymbol{\varepsilon}^{\frac{1}{2}}\| + 2CM^*Te^{4CM^*T}(\tau+h).\end{aligned}$$

On the other hand, we have $\|\boldsymbol{\varepsilon}^{\frac{1}{2}}\| \le (\tau+h)M$, since $\varepsilon_j^{\frac{1}{2}} = \phi(t_{\frac{1}{2}},x_j) - \phi_j^{\frac{1}{2}} = u_t(t_{\frac{1}{2}},x_j) + u_x(t_{\frac{1}{2}},x_j) - u_1(x_j) - u_0'(x_j)$. Therefore, taking

$$M_0 = (Me^{4CM^*T} + 2CM^*Te^{4CM^*T}),$$

we have shown that the desired estimate (1.28) holds for all $h \in (0, h_1]$ and $n \in \tilde{\Lambda}_\nu$.

Step 2. We set

$$h_0 = \min\left\{h_1,\ \frac{M}{2M_0(1+2\gamma)},\ h_{\frac{3}{2M},1}\right\}$$

where $h_{\frac{3}{2}M,1}$ is the constant introduced in Theorem 1.2.3 with $R = \frac{3}{2}M$ and $N = 1$. Below we assume $h \in (0, h_0]$.

We prove

$$\max\{n \in \mathbb{N} \mid t_{n+1} \le T\} \le \nu \tag{1.44}$$

by showing a contradiction. Thus, we assume

$$\max\{n \in \mathbb{N} \mid t_{n+1} \le T\} > \nu.$$

Then, we have $\tilde{\Lambda}_\nu = \{1, \ldots, \nu\}$ and, since $h_0 \le h_1$ in view of Step 1,

$$\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le M_0(1+2\gamma)h$$

for all $n = 1, \ldots, \nu$. Moreover, since $t_{\nu+1} \le T$, it follows from the definition of M that

$$\max_{n=1,\ldots,\nu}(\|\boldsymbol{u}(t_n)\| + \|\boldsymbol{\phi}(t_{n+\frac{1}{2}})\|) \le M,$$

where $\boldsymbol{u}(t_n) = (u(t_n, x_j))$ and $\boldsymbol{\phi}(t_{n+\frac{1}{2}}) = (\phi(t_{n+\frac{1}{2}}, x_j))$. Combining those inequalities, we get

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \le M + M_0(1+2\gamma)h$$

for all $n = 1, \ldots, \nu$. In particular,

$$\|\boldsymbol{u}^\nu\| + \|\boldsymbol{\phi}^{\nu+\frac{1}{2}}\| \le M + M_0 h \le \frac{3}{2}M.$$

Now, we apply Theorem 1.2.3 with $\boldsymbol{a} = \boldsymbol{u}^\nu$, $\boldsymbol{b} = \boldsymbol{\phi}^{\nu+\frac{1}{2}}$, $R = \frac{3}{2}M$, and $N = 1$ to obtain

$$\|\boldsymbol{u}^{\nu+1}\| + \|\boldsymbol{\phi}^{\nu+\frac{3}{2}}\| \le 3M.$$

This contradicts the definition of ν. Therefore, (1.44) actually holds true. Hence, by the result of Step 1, we see that the desired estimate (1.28) holds for all $h \in (0, h_0]$ and $n \in \mathbb{N}$ satisfying $t_{n+1} \le T$. This completes the proof of Theorem 1.2.5. □

1.4 Proof of Theorem 1.2.8

This section is devoted to the proof of numerical blow-up result, Theorem 1.2.8. We shall deal only with the case of the explicit scheme (1.19); the case of the implicit scheme (1.21) is proved in exactly the same way.

Throughout this section, suppose that $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ denotes the solution of the explicit scheme (1.19) as in Theorem 1.2.8. Further, we suppose that all assumptions of Theorem 1.2.8 hold true. In view of (1.32), we may suppose that $\boldsymbol{\phi}^0, \boldsymbol{u}^1 \ge \mathbf{0}, \ne \mathbf{0}$ for a sufficiently small $h > 0$. Consequently, we have $\boldsymbol{u}^n, \boldsymbol{\phi}^n \ge \mathbf{0}, \ne \mathbf{0}$ for $n \ge 1$.

Before stating the proof of Theorem 1.2.8, we establish a discrete version of (1.10). To this end, we introduce the functional

$$K_h(\boldsymbol{v}) = \frac{1}{L}\sum_{j=1}^{J} v_j h \qquad (\mathbf{0} \le \boldsymbol{v} \in \mathbb{R}^J) \tag{1.45}$$

and consider the discrete version $K_h(\boldsymbol{u}^n)$ of $K(u(t))$. We note that $K_h(\boldsymbol{u}^n) \ge 0$ and $K_h(\boldsymbol{\phi}^n) \ge 0$ for $n \ge 0$. In particular,

$$K_h(\boldsymbol{\phi}^0) > 0, \quad \alpha_h = K_h(\boldsymbol{u}^0) \ge 0, \quad \beta_h = K_h(\boldsymbol{u}^1) > 0. \tag{1.46}$$

Lemma 1.4.1. *$K_h(\boldsymbol{u}^n)$ is a strictly increasing sequence in $n \ge 0$ and it satisfies*

$$\left[\frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n}\right]^2 \ge \frac{1}{p+1} K_h(\boldsymbol{u}^n)^{p+1} + M_{1h} \ge 0 \tag{1.47}$$

for $n \ge 0$, where

$$M_{1h} = \left(\frac{\beta_h - \alpha_h}{\Delta t_0}\right)^2 - \frac{1}{p+1}\alpha_h^{p+1}. \tag{1.48}$$

Proof. We have

$$\begin{aligned}
\frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n} &= \frac{1}{L}\sum_{j=1}^{J} \frac{u_j^{n+1} - u_j^n}{\Delta t_n} h \\
&= \frac{1}{L}\sum_{j=1}^{J} \left[-\frac{u_j^n - u_{j-1}^n}{h} + \phi_j^n\right] h = K_h(\boldsymbol{\phi}^n)
\end{aligned} \tag{1.49}$$

for $n \ge 0$. In particular, by (1.46)

$$\frac{K_h(\boldsymbol{u}^1) - K_h(\boldsymbol{u}^0)}{\Delta t_0} \ge K_h(\boldsymbol{\phi}^0) > 0 \tag{1.50}$$

By using Jensen's inequality, we have from (1.49)

$$\begin{aligned}
\frac{K_h(\boldsymbol{\phi}^{n+1}) - K_h(\boldsymbol{\phi}^n)}{\Delta t_n} &= \frac{1}{L}\sum_{j=1}^{J} \left[\frac{\phi_{j+1}^n - \phi_j^n}{h} + \left(u_j^{n+1}\right)^p\right] h \\
&= \frac{1}{L}\sum_{j=1}^{J} \left(u_j^{n+1}\right)^p h \ge K_h(\boldsymbol{u}^{n+1})^p.
\end{aligned}$$

Combining these, we obtain

$$\frac{K_h(\boldsymbol{u}^{n+2}) - K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}} \ge \frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n} + \Delta t_n (K_h(\boldsymbol{u}^{n+1}))^p \tag{1.51}$$

$$\ge \frac{K_h(\boldsymbol{u}^1) - K_h(\boldsymbol{u}^0)}{\Delta t_0} + \sum_{k=0}^{n} \Delta t_k (K_h(\boldsymbol{u}^{k+1}))^p > 0 \tag{1.52}$$

for $n \ge 0$. This, together with (1.50), implies that $K_h(\boldsymbol{u}^n)$ is a strictly increasing sequence in $n \ge 0$.

Again, we apply (1.51) to obtain

$$\begin{aligned}
&\left[\frac{K_h(\boldsymbol{u}^{n+2}) - K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}}\right]^2 \\
&\qquad \geq \frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n}\left[\frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n} + \Delta t_n \left(K_h(\boldsymbol{u}^{n+1})\right)^p\right] \\
&\qquad = \left[\frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n}\right]^2 + \left(K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)\right) K_h(\boldsymbol{u}^{n+1})^p.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left[\frac{K_h(\boldsymbol{u}^{n+2}) - K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}}\right]^2 \\
&\geq \sum_{k=0}^{n} \Big(K_h(\boldsymbol{u}^{k+1}) - K_h(\boldsymbol{u}^k)\Big) K_h(\boldsymbol{u}^{k+1})^p + \left[\frac{K_h(\boldsymbol{u}^1) - K_h(\boldsymbol{u}^0)}{\Delta t_0}\right]^2 \\
&\geq \int_{\alpha_h}^{K_h(\boldsymbol{u}^{k+1})} z^p \, dz + \left(\frac{\beta_h - \alpha_h}{\Delta t_0}\right)^2 \\
&= \frac{1}{p+1}\Big(K_h(\boldsymbol{u}^{n+1})^{p+1} - \alpha_h^{p+1}\Big) + \left(\frac{\beta_h - \alpha_h}{\Delta t_0}\right)^2. \qquad (1.53)
\end{aligned}$$

Since $K_h(\boldsymbol{u}^n)$ is non-decreasing in n, the right-hand side of (1.53) is non-negative. This completes the proof of Lemma 1.4.1. □

Remark 1.4.2. Under the assumptions of Theorem 1.2.4, we have $M_{1h} \to \beta^2 - \frac{1}{p+1}\alpha^{p+1}$ as $h \to 0$.

Remark 1.4.3. In view of (1.50) and (1.52),

$$\frac{K_h(\boldsymbol{u}^{n+2}) - K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}} \geq K_h(\boldsymbol{\phi}^0) \equiv \nu_h,$$

where ν_h is a positive number which is independent of n. This implies that $K_h(\boldsymbol{u}^n)$ is not a bounded sequence in n. In particular, there exists $m \in \mathbb{N}$ such that $K_h(\boldsymbol{u}^m) > 1$.

At this stage, we set

$$G(z) = \sqrt{\frac{1}{p+1}z^{p+1} + M_{1h}}.$$

Note that $G(z)$ is a strictly increasing function in $z \in [\alpha_h, \infty)$.

In view of Lemma 1.4.1, we can follow exactly the same argument of the proof of [10, Lemma 5.4] and obtain the following lemma.

Lemma 1.4.4. *There exists a positive constant C which is independent of h such that*

$$T(h) \leq 2\left(\int_{\alpha_h}^{\infty} \frac{dz}{G(z)} + C\tau\right).$$

In particular, we have $T(h) < \infty$.

Now we can state the following proof.

Proof of Theorem 1.2.8. (i) It is a direct consequence of Theorems 1.2.2 and 1.2.3.
(ii) According to Lemma 1.4.4, we have $T(h) < \infty$; $\boldsymbol{u}^n$ blows up in finite time. We prove that

$$T_\infty \leq \liminf_{h \to 0} T(h) \equiv T_* \tag{1.54}$$

by showing a contradiction. Thus, we assume that

$$T_* < T_\infty.$$

Then, there exists a subsequence $\{h_i\}_i$ such that $h_i \to 0$ as $i \to \infty$ and that

$$T(h_i) \leq T_* + \delta < T_\infty,$$

where $\delta = (T_\infty - T_*)/2$. We have

$$\max_{0 \leq t \leq T_* + \delta} \|u(t)\|_{L^\infty(S_L)} < \infty. \tag{1.55}$$

On the other hand, the solution $\boldsymbol{u}^n = \boldsymbol{u}^n(h_i)$ of the explicit scheme (1.19) corresponding to the parameter $h = h_i$ satisfies (cf. Remark 1.2.7)

$$\lim_{n \to \infty} \|\boldsymbol{u}^n(h_i)\| = \lim_{t_n \to T(h_i)} \|\boldsymbol{u}^n(h_i)\| = \infty. \tag{1.56}$$

These (1.55) and (1.56) contradict to Theorem 1.2.4. Hence, (1.54) is proved.
(iii) We assume (1.34); thus, $u(t,x)$ and $K(u(t))$ blow up in finite time $t = T_\infty$. We now prove that

$$T^* \equiv \limsup_{h \to 0} T(h) \leq T_\infty \tag{1.57}$$

by showing a contradiction. In fact, this, together with (1.54), implies $T_\infty = \lim_{h \to 0} T(h)$, which completes the proof. We assume

$$T_\infty < T^*$$

and set $\epsilon = (T^* - T_\infty)/4$. There exist $R > 0$ and $h_{**} > 0$ such that

$$2\left(\int_R^\infty \frac{dz}{G(z)} + C\gamma h_{**}\right) < \epsilon.$$

Below we fix such R and h_{**}. Further, there exists $t' = t'_R < T_\infty$ such that $K(u(t')) > 2R$. Set

$$T = t' + \frac{T_\infty - t'}{2} = \frac{t' + T_\infty}{2} < T_\infty$$

and let M and M_0 be the positive constants appearing Theorem 1.2.4 corresponding to this T. Set

$$h_* = \min\left\{h_{**},\ \frac{T_\infty - t'}{2\gamma},\ \frac{R}{M + M_0(1+\gamma)}\right\}$$

and suppose $h \in (0, h_*]$ below. Then, we have $Mh + M_0(\tau + h) \leq R$ and $\tau \leq T - t'$.

According to Theorem 1.2.4, we have

$$\begin{aligned}
&|K(u(t_n)) - K_h(\boldsymbol{u}^n)| \\
&\le \frac{1}{L}\sum_{j=1}^{J}\int_{x_{j-1}}^{x_j} |u(t_n,x) - u_j^n|\ dx \\
&\le \frac{1}{L}\sum_{j=1}^{J}\int_{x_{j-1}}^{x_j} (|u(t_n,x) - u(t_n,x_j)| + |u(t_n,x_j) - u_j^n|)\ dx \\
&\le Mh + M_0(\tau + h) \le R
\end{aligned}$$

and, therefore,

$$K_h(\boldsymbol{u}^n) \ge K(u(t_n)) - R.$$

There exists $k \in \mathbb{N}$ satisfying $t' \le t_k < T_\infty$, since $\tau \le T - t' < T_\infty - t'$. Then,

$$K_h(\boldsymbol{u}^k) \ge K(u(t_k)) - R > R. \tag{1.58}$$

At this stage, we can take a subsequence $\{h_i\}_i$ such that

$$T_\infty + \epsilon < T(h_i)$$

and $h_i \to 0$ as $i \to \infty$. However, in view of Lemma 1.4.4 and (1.58), we have

$$T(h_i) = t_k + \sum_{n=k}^{\infty} \Delta t_n < T_\infty + 2\left(\int_R^\infty \frac{dz}{G(z)} + C\tau_i\right).$$

Therefore, by the definition of R and h_{**}, we obtain $T(h_i) < T_\infty + \epsilon$, which is a contradiction. Hence, we obtain (1.57). This completes the proof of Theorem 1.2.8. □

1.5 Numerical experiments

In this section, we offer some numerical examples and examine the validity of our proposed finite-difference schemes. Suppose $L = 1$ and take

$$u_0(x) = \frac{\lambda}{2}(\sin(4\pi x) + 2), \quad u_1(x) = 2\pi\lambda + \mu$$

as initial values. Then, if $\lambda, \mu > 0$, we have $\alpha = K(u_0) = \lambda > 0$, $\beta = K(u_1) = 2\pi\lambda + \mu > 0$ and $u_0'(x) + u_1(x) \ge \mu > 0$. Below we set $\lambda = 10$ and $\mu = 5$.

1.5.1 Choice of q

We first examine the value of q in the definition of Δt_n. We consider the explicit scheme (1.19). In Fig. 1.1, we plot Δt_n as a function of t_n when $p = 2$. We see that Δt_n deceases as a linear function if $q = 0.5$ whereas it deceases very rapidly if $q = 0.25$ and very slowly if $q = 0.75, 1$. Results for the cases of $p = 3$ and 4 are reported in Fig. 1.2 and 1.3, respectively. Here, the case $p = 3$ means the nonlinearlity $f(u) = u|u|^2$; see Remark 1.2.10. For each p, there is $q = q_*$ such that Δt_n deceases linearly if $q = q_*$ and it deceases very rapidly if $q < q_*$ and very slowly if $q > q_*$.

Slowly-deceasing cases are not suitable from the viewpoint of efficiency. On the other hand, we do not prefer rapidly-deceasing cases since it is difficult to capture clearly the variation of a numerical solution near $t = T(h)$ even if Δt_n is quite small.

Consequently, as a better choice, we offer

$$q = \begin{cases} 0.5 & (p = 2) \\ 1 & (p = 3) \\ 1.5 & (p = 4). \end{cases} \tag{1.59}$$

Below we choose q as (1.59).

1.5.2 Stopping criterion

The numerical blow-up time is an infinite series defined as

$$T(h) = \sum_{n=0}^{\infty} \Delta t_n.$$

Therefore, in actual computations, we take a sufficiently large n and regard t_n as a reasonable approximation of $T(h)$. For this purpose, we introduce the *truncated numerical blow-up time* $T(h;\varepsilon)$ by setting

$$T(h;\varepsilon) = \min \left\{ t_n \mid \|\boldsymbol{u}^n\| > \varepsilon^{-1} \right\}, \tag{1.60}$$

where $\varepsilon > 0$ is the stopping criterion given below.

We still consider the explicit scheme (1.19) and plot $T(h,\varepsilon)$, $T(h;100\varepsilon)$ for several h in Fig. 1.4. For suitably small ε and h, $T(h,\varepsilon)$ and $T(h;100\varepsilon)$ are almost equal so that we can take $T(h;\varepsilon)$ as a reasonable approximation of the exact blow-up time.

1.5.3 Comparison of our schemes and Cho's scheme

We compare three finite-difference schemes; the explicit scheme (1.19), the implicit scheme (1.21) and the Cho's scheme (1.17) with obvious modification of the boundary condition.

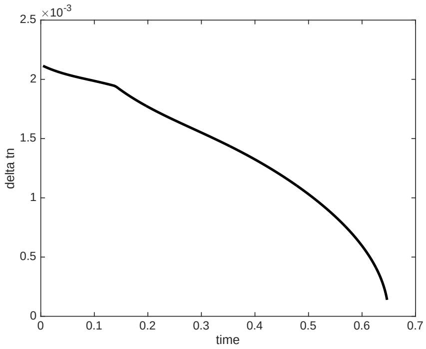
Fig. 1.4, we plot $T(h;\varepsilon)$ for several h by using those three schemes. We see that those $T(h;\varepsilon)$ converge to a certain value, say the exact blow-up time, as $h \to 0$. Thus, we can apply anyone to compute the blow-up solutions. Cho's scheme is better than ours. But, again, it should be kept in mind that our schemes and the numerical blow-up times are guaranteed to converge by the mathematical proof.

Furthermore, we conjecture form those figures that the rate of convergence of $T(h)$ is expressed as

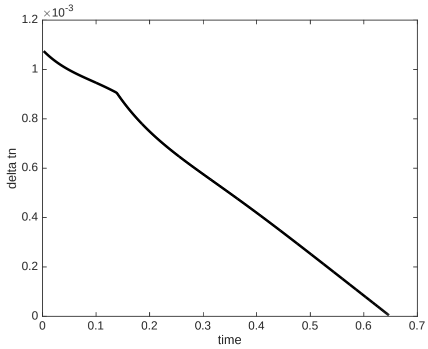
$$|T(h) - T_\infty| \le Ch = C\tau$$

if τ/h is fixed. We, however, have no mathematical proof; for similar difficulties for parabolic problems, see [13].

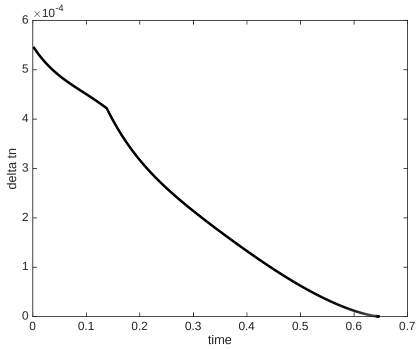
We finally give the shapes of solutions $\boldsymbol{u}^n$ of the explicit scheme (1.19) in Fig. 1.6.



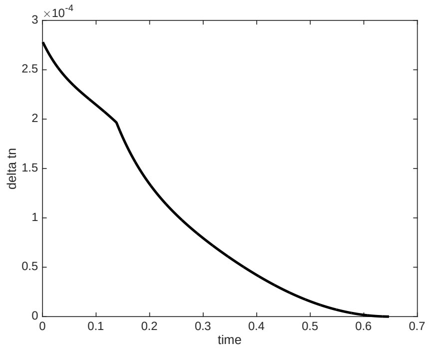
$q = 0.25$



$q = 0.5$

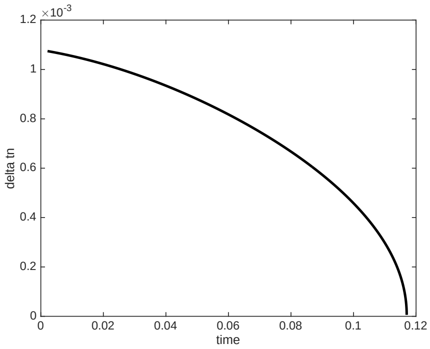


$q = 0.75$

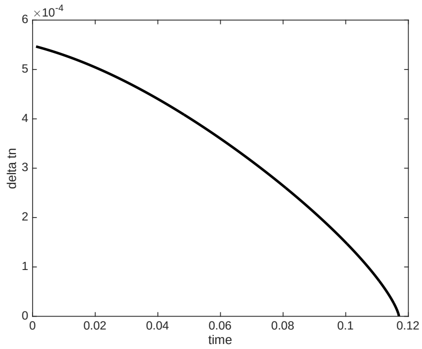


$q = 1$

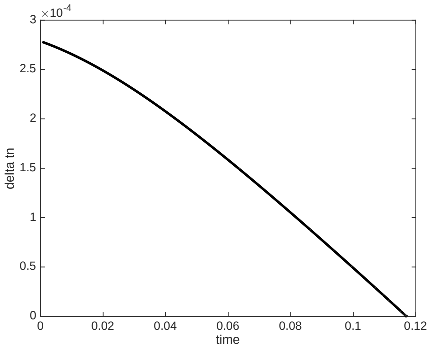
Figure 1.1: The history of Δt_n for $p = 2$.



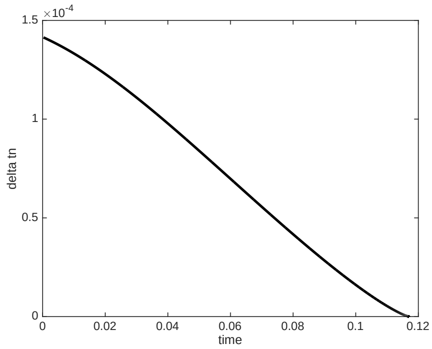
$q = 0.5$



$q = 0.75$

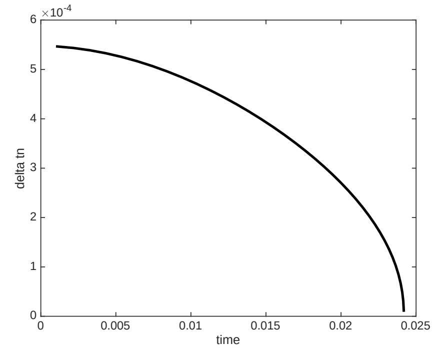


$q = 1$

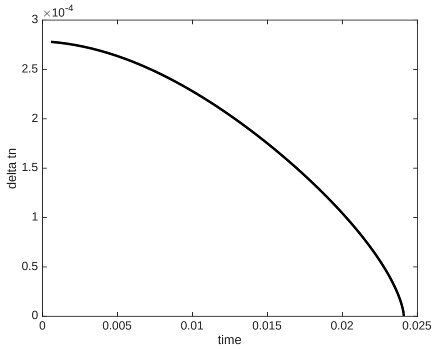


$q = 1.25$

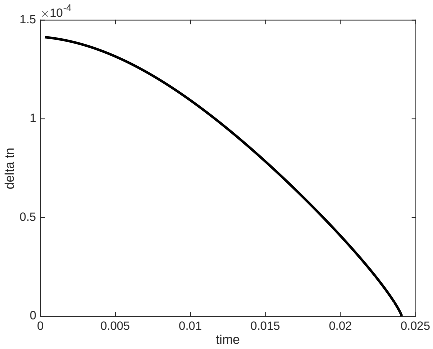
Figure 1.2: The history of Δt_n for $p = 3$.



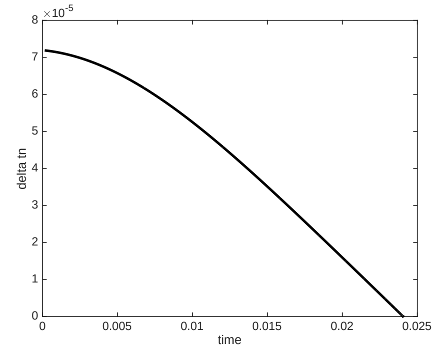
$q = 0.75$



$q = 1$

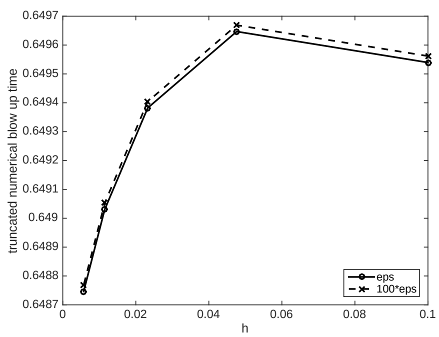


$q = 1.25$

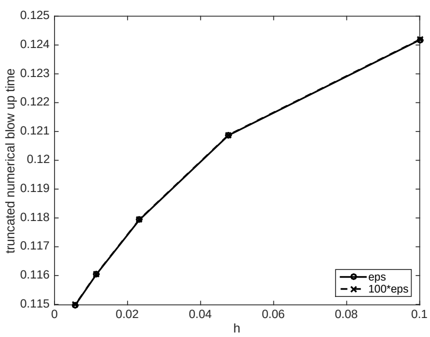


$q = 1.5$

Figure 1.3: The history of Δt_n for $p = 4$.

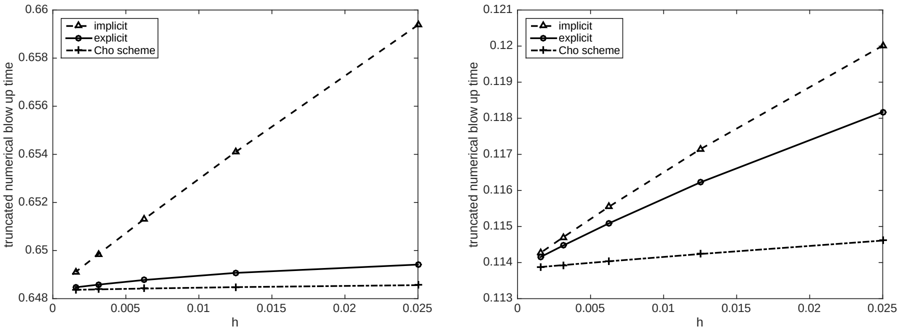


(a) $p = 2$, $\varepsilon = 10^{-12}$



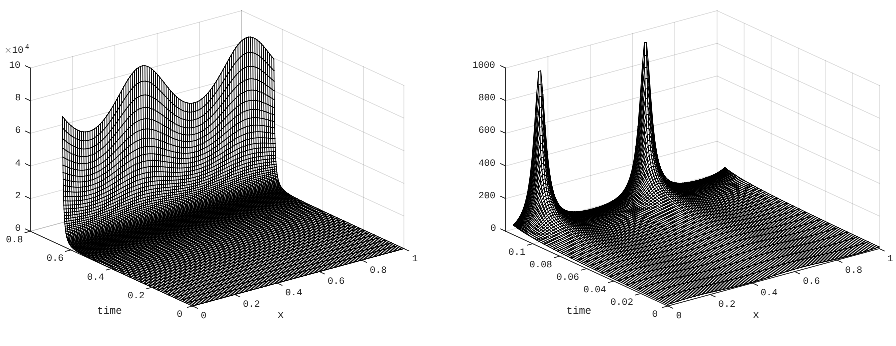
(b) $p = 3$, $\varepsilon = 10^{-7}$

Figure 1.4: Truncated numerical blow-up time $T(h; \varepsilon)$ for stopping criteria ε and 100ε



(a) $p = 2$, $\varepsilon = 10^{-10}$ (b) $p = 3$, $\varepsilon = 10^{-5}$

Figure 1.5: Truncated numerical blow-up time $T(h; \varepsilon)$ for three schemes



(a) $p = 2$ (b) $p = 3$

Figure 1.6: Shapes of finite-difference solutions $\boldsymbol{u}^n$ of the explicit scheme (1.19).

2 Error analysis of splitting methods for semilinear evolution equations

We consider a Strang-type splitting method for an abstract semilinear evolution equation $u_t = Au + F(u)$. Roughly speaking, the splitting method is a time-discretization approximation based on the decomposition of operators A and F. Particularly, the Strang method is a popular splitting method and is known to be convergent at a second order rate for some particular ODEs and PDEs. In this chapter, we propose a generalization of the Strang method and prove that our proposed method is convergent at a second order rate. Some numerical examples that confirm our theoretical result are given.

2.1 Introduction and main results

Let X be a Hilbert space equipped with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$, A be an m-dissipative linear operator in X with dence domain $D(A) \subset X$.

- For any $u \in D(A)$, $(Au, u) \leq 0$;
- For any $f \in X$ and $\lambda > 0$, there exists $u \in D(A)$ such that $u - \lambda Au = f$.

As is well-known, the operator A generates a contraction semigroup $\Phi_A(t) = e^{tA}$ if and only if A is m-dissipative with dense domain. We consider the following Cauchy problem for semilinear evolution equation:

$$\begin{cases} u_t = Au + F(u), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $F : D(A) \to D(A)$ is a nonlinear operator. Typical examples of (2.1) are nonlinear Schrödinger equations in $\Omega \in \mathbb{R}^d$

$$u_t = i\Delta u + \alpha u|u|^2, \tag{2.2}$$

$$u_t = i\Delta u + \alpha u|u|^2 + \beta u|u|^4, \tag{2.3}$$

where α and β are complex constants. Setting $D(A) = \{v \in H_0^1(\Omega) \mid \Delta v \in L^2(\Omega)\}$, $Av = i\Delta v$, and $F(v) = \alpha v|v|^2$ in (2.2), we obtain (2.1).

The main purpose of this chapter is to study the so called splitting method, which is a semi-discrete approximation of (2.1) with respect to time variable t. The idea behind the splitting method is as follows. We denote the (nonlinear) solution operator (2.1) by $S(t)$. That is, the solution of (2.1) is given as $u(t) = S(t)u_0$; see (2.9) below. Then, we consider the time-discrete approximation to (2.1) at $t = n\Delta t$ as

$$u_n = \Psi(n\Delta t)u_0,$$

where $\Delta t > 0$ denotes a time increment and n a positive integer. Typical choices of Ψ are, for example,

$$\Psi(t) = \Phi_A(t)\Phi_F(t), \tag{2.4}$$

$$\Psi(t) = \Phi_F(t)\Phi_A(t), \tag{2.5}$$

$$\Psi(t) = \Phi_A(t/2)\Phi_F(t)\Phi_A(t/2) \tag{2.6}$$

where $\Phi_F(t)$ denotes the solution operator of $w_t = F(w)$. Particularly, (2.6) is called the Strang method.

Splitting methods are useful when $S(t)u_0$ is difficult to compute, while $\Phi_A(t)u_0$ and $\Phi_F(t)u_0$ are easy to compute. In addition, if (2.8) has conservation properties, then splitting methods basically preserve its discrete version. Splitting methods are widely used numerical methods for solving ODEs and PDEs.

Analysis of splitting methods for ODEs has been presented in many studies. For example, see Hairer *et al.*[20]. Some results on error analysis are also presented for PDEs. For example, results of error analysis for nonlinear Schrödinger equations can be found in e.g., Besse *et al.* [4] and Lubich [25].

However, to our best knowledge, little is known for abstract Cauchy problem of the form (2.1). Decombes and Thalhammer[14] and Jahnke and Lubich [21] presented an error analysis for the case in which F is a linear operator. For nonlinear abstract Cauchy problems, Borgna *et al.*[5] demonstrated that various splitting methods involving Strang method have first order accuracy. Namely, if Δt is sufficiently small, we have

$$\|S(n\Delta t)u_0 - \Psi(\Delta t)^n u_0\| \leq C\Delta t.$$

However, they did not demonstrate that Strang-type splitting method is a second order scheme:

$$\|S(n\Delta t)u_0 - \Psi(\Delta t)^n u_0\| \leq C\Delta t^2. \tag{2.7}$$

It should be kept in mind that (2.7) is established for the Strang method applied to particular PDEs; see Besse *et al.*[4] and Lubich[25]. Therefore, it is worth studying the Strang method for abstract Cauchy problem of the form (2.1) and deriving the second order error estimate.

On the other hand, the majority of previous studies have considered schemes that are split into two parts; $v_t = Av$ and $w_t = F(w)$. As a matter of fact, such two-parts splitting is applied tp (2.2), then the explicit solution formula for the ordinary differential equation $w_t = \alpha w|w|^2$ is available. However, the two-parts splitting is applied to (2.3), then we have to solve the ordinary differential equation

$w_t = \alpha w|w|^2 + \beta w|w|^4$ by numerical method since the exact solution is not available in the case.

Therefore, some researchers have proposed schemes that are split into more than two parts. However, the convergence properties of such schemes are not guaranteed in the case of PDEs.

In this paper, we propose a Strang-type splitting method that is split into three parts for (2.8). Moreover, we show that it is actually convergent at a second order rate.

Let us formulate our problem. For given nonlinear operators $F_1, F_2 : D(A) \to D(A)$, we set

$$F(v) = F_1(v) + F_2(v) \qquad (v \in D(A)).$$

For $u_0 \in D(A)$, we consider the Cauchy problem

$$\begin{cases} u_t = Au + F_1(u) + F_2(u), \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.8}$$

and the corresponding integral equation:

$$u(t) = \Phi_A(t)u_0 + \int_0^t \Phi_A(t-s)F(u(s))ds, \quad t \in [0,T]. \tag{2.9}$$

We consider $D(A)$ and $D(A^2)$ as Hilbert spaces with

$$\|v\|_{D(A)} = \|v\|_X + \|Av\|_X \quad \text{for} \quad v \in D(A),$$
$$\|v\|_{D(A^2)} = \|v\|_{D(A)} + \|A^2 v\|_X \quad \text{for} \quad v \in D(A^2).$$

For $i = 1, 2$, we assume that $F_i : D(A) \to D(A)$ satisfies the following conditions:

(F0) $F_i(0) = 0$,

(F1) $\|F_i'(v)w\|_{D(A)} \le L(\|v\|_{D(A)})\|w\|_{D(A)}$ for $v, w \in D(A)$,

(F2) $F_i(v) \in D(A^2)$ and $\|F_i(v)\|_{D(A^2)} \le L_2(\|v\|_{D(A)})\|v\|_{D(A^2)}$ for $v, w \in D(A^2)$,

(F3) $F_i(v) \in D(A^2)$ and $\|F_i(v) - F_i(w)\|_{D(A^2)} \le L_3(\max\{\|v\|_{D(A^2)}, \|w\|_{D(A^2)}\})\|v - w\|_{D(A^2)}$
for $v, w \in D(A^2)$,

(F4) $\|F_i'(v)w\|_X \le L_4(\|v\|_{D(A)})\|w\|_X$ for $v, w \in D(A)$,

(F5) $\|F_i''(v)(w,w)\|_X \le L_5(\|v\|_{D(A)})\|w\|_X\|w\|_{D(A)}$ for $v, w \in D(A)$.

Herein, F_i' and F_i'' denote the first and second Fréchet derivatives, $L, L_2, \cdots, L_5 : [0,\infty) \to [0,\infty)$ are decreasing functions.

We note that it follows from (F1) and (F0) that

(F6) $\|F_i(v) - F_i(w)\|_{D(A)} \le L(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v - w\|_{D(A)}$
for $v, w \in D(A)$,

(F7) $\|F_i(v)\|_{D(A)} \leq L(\|v\|_{D(A)})\|v\|_{D(A)}$ for $v \in D(A)$.

Moreover, it follows from (F4) that

(F8) $\|F_i(v) - F_i(w)\|_X \leq L_4(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v - w\|_X$
for $v, w \in D(A)$.

For simplicity, we write $F''(v)(w, w) = F''(v)w^2$ for $v, w \in D(A)$. Before stating the schemes and main results, we recall a general result for (2.9):

Proposition 2.1.1. *Assume* (F0)–(F1). *Then, for any* $u_0 \in D(A)$*, there exist* $T_{\max}(u_0) \in (0, \infty]$ *and a unique solution*

$$u \in C([0, T_{\max}(u_0)), D(A)) \cap C^1([0, T_{\max}(u_0), X)$$

of (2.9) *such that either the following* (i) *or* (ii) *holds:*

(i) $T_{\max}(u_0) = \infty$,

(ii) $T_{\max}(u_0) < \infty$ *and* $\lim_{t \uparrow T_{\max}(u_0)} \|u(t)\|_{D(A)} = \infty$.

Moreover, if $u_0 \in D(A^2)$*, then*

$$u \in C([0, T_{\max}(u_0)), D(A^2)) \cap C^1([0, T_{\max}(u_0)), D(A)).$$

For the proof of Proposition 2.1.1, see e.g., Section 4.3 of [8].

In order to state our scheme, for $i = 1, 2$, we consider the following Cauchy problem:

$$\begin{cases} w_{i,t} = F_i(w_i), & t \in [0, T], \\ w_i(0) = w_{i,0}, \end{cases} \tag{2.10}$$

and the corresponding integral equation:

$$w_i(t) = w_{i,0} + \int_0^t F_i(w_i(s))ds, \quad t \in [0, T]. \tag{2.11}$$

We denote the solution of (2.12) by $w_i(t) = \Phi_{F_i}(t)w_{i,0}$. That is,

$$\Phi_{F_i}(t)w_{i,0} = w_{i,0} + \int_0^t F_i(w_i(s))ds, \quad t \in [0, T]. \tag{2.12}$$

Then, our scheme to find $\Psi(t)u_0 \approx S(t)u_0$, reads as

$$\Psi(t)u_0 = \Phi_A(t/2)\Phi_{F_1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0. \tag{2.13}$$

Our scheme includes the Strang method by setting $F_1 = 0$.

We are now in a position to state the main results.

Theorem 2.1.2. *Assume* (F0)–(F5). *Let* $u_0 \in D(A^2)$, $T \in (0, T_{\max}(u_0))$ *and set*

$$m_0 = 8 \max_{t \in [0,T]} \|S(t)u_0\|_{D(A)}.$$

Then, there exists a positive constant h_0*, which depends only on* T, m_0 *and* $\|u_0\|_{D(A^2)}$*, such that*

$$\|(\Psi(h))^n u_0\|_{D(A)} \le m_0, \quad \|(\Psi(h))^n u_0\|_{D(A^2)} \le e^{\gamma_1 nh}\|u_0\|_{D(A^2)}, \tag{2.14}$$

$$\|S(nh)u_0 - (\Psi(h))^n u_0\|_{D(A)} \le \kappa_1 h \|u_0\|_{D(A^2)}, \tag{2.15}$$

$$\|S(nh)u_0 - (\Psi(h))^n u_0\|_X \le \kappa_2 h^2 \|u_0\|_{D(A^2)}, \tag{2.16}$$

for all $h \in (0, h_0]$ *and* $n \in \mathbb{N}$ *satisfying* $nh \le T$*, where* γ_1 *is a positive constant depending only on* m_0*, and* κ_1, κ_2 *are positive constants depending only on* T *and* m_0.

The rest of this paper is organized as follows. In Section 2.2, we collect some lemmas that are needed to prove Theorem 2.1.2. In Section 2.3, we give local error estimates between $S(h)u_0$ and $\Psi(h)u_0$ in $D(A)$. In Section 2.4, we give local error estimates between $S(h)u_0$ and $\Psi(h)u_0$ in X. In Section 2.5, we complete the proof of Theorem 2.1.2. In Section 2.6, we present some numerical experiments that show the convergence rate of the scheme numerically.

2.2 Preliminaries

2.2.1 Estimates on the contraction semigroup $\Phi_A(t)$

Lemma 2.2.1. *Let* $k = 0, 1$. *Then,*

$$\|\Phi_A(t)v_0 - \Phi_A(s)v_0\|_{D(A^k)} \le (t - s)\|v_0\|_{D(A^{k+1})},$$

for $v_0 \in D(A^{k+1})$ *and* $0 \le s \le t$.

Proof. Set $v(t) = \Phi_A(t)v_0$. Then, we have

$$\Phi_A(t)v_0 - \Phi_A(s)v_0 = v(t) - v(s) = \int_s^t v'(\tau)d\tau = \int_s^t Av(\tau)d\tau.$$

Since

$$\|Av(\tau)\|_{D(A^k)} = \|\Phi_A(\tau)Av_0\|_{D(A^k)} \le \|Av_0\|_{D(A^k)}$$

for $\tau > 0$, we have

$$\begin{aligned}\|\Phi_A(t)v_0 - \Phi_A(s)v_0\|_{D(A^k)} &\le \int_s^t \|Av(\tau)\|_{D(A^k)}d\tau \\ &\le (t-s)\|Av_0\|_{D(A^k)} \le (t-s)\|v_0\|_{D(A^{k+1})}.\end{aligned}$$

This completes the proof. □

Lemma 2.2.2. *Let $w \in C^1([0,T], D(A)) \cap C([0,T], D(A^2))$. Then,*

$$\left\| \int_0^t [\Phi_A(t-s)w(s) - \Phi_A(t/2)w(s)]ds \right\|_X \leq t^3 \left\{ \|w\|_{C^1([0,T],D(A))} + \|w\|_{C([0,T],D(A^2))} \right\} \tag{2.17}$$

for $t \in [0,T]$.

Proof. For $0 \leq s \leq t \leq T$, by Taylor's formula, we obtain

$$\Phi_A(t-s)w(s) - \Phi_A(t/2)w(s) = (t/2-s)\Phi_A(t/2)Aw(s) + (t/2-s)^2 \int_0^1 (1-\theta)\Phi_A(\theta(t-s) + (1-\theta)t/2)A^2 w(s)d\theta.$$

Let $v(s) = \Phi_A(t/2)Aw(s)$. Then, we have

$$\|v'(s)\|_X \leq \|Aw'(s)\|_X \leq \|w'(s)\|_{D(A)},$$
$$\int_0^t (t/2-s)v(s)ds = \int_0^{t/2} (t/2-s)[v(s) - v(t-s)]ds.$$

Moreover, for $0 \leq s \leq t/2$, since

$$\|v(s) - v(t-s)\|_X = \left\| \int_0^1 \frac{d}{d\theta} v(\theta s + (1-\theta)(t-s))d\theta \right\|_X \leq (t-2s)\int_0^1 \|v'(\theta s + (1-\theta)(t-s))\|_X d\theta \leq 2(t/2-s)\|v'\|_{C([0,T],X)},$$

we have

$$\left\| \int_0^t (t/2-s)\Phi_A(t/2)Aw(s)ds \right\|_X = \left\| \int_0^{t/2} (t/2-s)[v(s) - v(t-s)]ds \right\|_X \leq 2\int_0^{t/2} (t/2-s)^2 ds \|v'\|_{C([0,T],X)} \leq t^3 \|w\|_{C^1([0,T],D(A))}. \tag{2.18}$$

Furthermore, since

$$\left\| \int_0^1 (1-\theta)\Phi_A(\theta(t-s) + (1-\theta)t/2)A^2 w(s)d\theta \right\|_X \leq \int_0^1 (1-\theta)\|A^2 w(s)\|_X d\theta \leq \|w\|_{C([0,T],D(A^2))},$$

we have

$$\left\| \int_0^t (t/2-s)^2 \int_0^1 (1-\theta)\Phi_A(\theta(t-s) + (1-\theta)t/2)A^2 w(s)d\theta ds \right\|_X \leq \|w\|_{C([0,T],D(A^2))} \int_0^t (t/2-s)^2 ds \leq t^3 \|w\|_{C([0,T],D(A^2))}. \tag{2.19}$$

Thus, by (2.18) and (2.19), we obtain (2.17). □

2.2.2 Estimates on the nonlinear flows Φ_{F_i}

Lemma 2.2.3. *Assume* (F0)–(F1)*. For any $M > 0$, there exists a positive constant $\tau(M)$ such that if $\|v_0\|_{D(A)} \le M$, then*

$$\|\Phi_{F_i}(t)v_0\|_{D(A)} \le 2M, \quad \|S(t)v_0\|_{D(A)} \le 2M$$

for all $t \in [0, \tau(M)]$ and $i = 1, 2$. Moreover, if $v_1, v_2 \in D(A)$ satisfying $\max\{\|v_1\|_{D(A)}, \|v_2\|_{D(A)}\} \le$ *M, then*

$$\begin{aligned}\|\Phi_{F_i}(t)v_1 - \Phi_{F_i}(t)v_2\|_{D(A)} &\le e^{L(2M)t}\|v_1 - v_2\|_{D(A)},\\ \|S(t)v_1 - S(t)v_2\|_{D(A)} &\le e^{L(2M)t}\|v_1 - v_2\|_{D(A)},\end{aligned}$$

for all $t \in [0, \tau(M)]$ and $i = 1, 2$.

Proof. See Proposition 4.3.3 of [8]. □

Lemma 2.2.4. *Assume* (F0)–(F3)*. Let $v_0 \in D(A^2)$ and $\|v_0\|_{D(A)} \le M$. Then,*

$$\|\Phi_{F_i}(t)v_0\|_{D(A^2)} \le e^{L_2(2M)t}\|v_0\|_{D(A^2)}, \tag{2.20}$$

for all $t \in [0, \tau(M)]$ and $i = 1, 2$, where $\tau(M)$ is previously defined in Lemma 2.2.3. Moreover, we have

$$\|\Psi(t)v_0\|_{D(A^2)} \le e^{2L_2(8M)t}\|v_0\|_{D(A^2)} \tag{2.21}$$

for all $t \in [0, \tau(4M)]$.

Proof. First, we note that it follows from (F0)–(F3) that (2.9) is local well-posed in $D(A^2)$. For $i = 1, 2$, we set $v_i(t) = \Phi_{F_i}(t)v_0$.

By (2.12) and (F2), we have

$$\begin{aligned}\|v_i(t)\|_{D(A^2)} &\le \|v_0\|_{D(A^2)} + \int_0^t \|F_i(v_i(\tau))\|_{D(A^2)}d\tau\\ &\le \|v_0\|_{D(A^2)} + \int_0^t L_2(\|v_i(\tau)\|_{D(A)})\|v_i(\tau)\|_{D(A^2)}d\tau.\end{aligned}$$

Here, it follows from Lemma 2.2.3 that

$$\|v_i(t)\|_{D(A^2)} \le \|v_0\|_{D(A^2)} + L_2(2M)\int_0^t \|v_i(\tau)\|_{D(A^2)}d\tau$$

for $t \in [0, \tau(M)]$. Thus, Gronwall's lemma implies (2.20) for $t \in [0, \tau(M)]$.

Next, since $\|\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \le 2M$ for $t \in [0, \tau(M)]$, and

$$\|\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \le 4M \tag{2.22}$$

for $t \in [0, \tau(2M)]$, it follows from (2.20) that

$$\begin{aligned}\|\Psi(t)v_0\|_{D(A^2)} &\le \|\Phi_{F1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)}\\ &\le e^{L_2(8M)t/2}\|\Phi_{F_2}(t)\Phi_{F1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)}\end{aligned}$$

for $t \in [0, \tau(4M)]$. Similarly, we have

$$\|\Phi_{F_2}(t)\Phi_{F1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)} \leq e^{L_2(4M)t+L_2(2M)t/2}\|v_0\|_{D(A^2)}$$

for $t \in [0, \tau(2M)]$. Therefore, we obtain

$$\begin{aligned}\|\Psi(t)v_0\|_{D(A^2)} &\leq e^{L_2(8M)t/2+L_2(4M)t+L_2(2M)t/2}\|v_0\|_{D(A^2)} \\ &\leq e^{2L_2(8M)t}\|v_0\|_{D(A^2)}\end{aligned}$$

for $t \in [0, \tau(4M)]$. This completes the proof. □

2.2.3 Lipschitz property of $S(t)$

Lemma 2.2.5. *Assume* (F0)–(F4). *Let* $u_0 \in D(A)$, $T \in (0, T_{\max}(u_0))$ *and set*

$$m_1 = 2 \max_{t\in[0,T]} \|S(t)u_0\|_{D(A)}, \quad \delta_0 = \min\left\{\frac{m_1}{2},\ m_1 e^{-2L(2m_1)T}\right\}.$$

If $\|v_0 - S(t_0)u_0\|_{D(A)} \leq \delta_0$, *then*

$$\|S(t)v_0\|_{D(A)} \leq 2m_1 \quad \textit{for} \quad t \in [0, T - t_0]. \tag{2.23}$$

Moreover, if $\|v_1 - S(t_0)u_0\|_{D(A)} \leq \delta_0$ *and* $\|v_2 - S(t_0)u_0\|_{D(A)} \leq \delta_0$, *then*

$$\begin{aligned}&\|S(t)v_1 - S(t)v_2\|_{D(A)} \leq e^{2L(2m_1)t}\|v_1 - v_2\|_{D(A)}, \\ &\|S(t)v_2 - S(t)v_2\|_X \leq e^{2L_4(2m_1)t}\|v_1 - v_2\|_X\end{aligned} \tag{2.24}$$

for $t \in [0, T - t_0]$.

Proof. First, we show (2.23). Since

$$\|v_0\|_{D(A)} \leq \|v_0 - S(t_0)u_0\|_{D(A)} + \|S(t_0)u_0\|_{D(A)} \leq \delta_0 + \frac{m_1}{2} \leq m_1,$$

it follows from Lemma 2.2.3 that $\|S(t)v_0\|_{D(A)} \leq 2m_1$ for $t \in [0, \tau(m_1)]$.

Here, we define

$$\tilde{T} = \sup\left\{\tau \in (0, T_{\max}(v_0)) \mid \|S(t)v_0\|_{D(A)} \leq 2m_1, \quad \forall t \in [0, \tau]\right\},$$

and suppose that $\tilde{T} < T - t_0$. Then, we have

$$S(t)v_0 = \Phi_A(t)v_0 + \int_0^t \Phi_A(t-\tau)F(S(\tau)v_0)d\tau, \quad t \in [0, \tilde{T}].$$

Moreover, for $\tau \in [0, \tilde{T}]$, since $0 \leq \tau \leq \tilde{T}$ and $\tau + t_0 \leq T$, we have

$$\|S(\tau)v_0\|_{D(A)} \leq 2m_1, \quad \|S(\tau)(S(t_0)u_0)\|_{D(A)} = \|S(\tau + t_0)u_0\|_{D(A)} \leq m_1.$$

Thus, by (F6), for $t \in [0, \tilde{T}]$, we have

$$\begin{aligned}&\|S(t)v_0 - S(t)(S(t_0)u_0)\|_{D(A)} \\ &\leq \|v_0 - S(t_0)u_0\|_{D(A)} + \int_0^t \|F(S(\tau)v_0) - F(S(\tau)S(t_0)u_0)\|_{D(A)}d\tau \\ &\leq \delta_0 + 2L(2m_1)\int_0^t \|S(\tau)v_0 - S(\tau)S(t_0)u_0\|_{D(A)}d\tau.\end{aligned}$$

By Gronwall's lemma, for $t \in [0, \tilde{T}]$, we have

$$\|S(t)v_0 - S(t)S(t_0)u_0\|_{D(A)} \le \delta_0 e^{2L(2m_1)t} \le \delta_0 e^{2L(2m_1)T} \le m_1,$$

and

$$\begin{aligned}\|S(t)v_0\|_{D(A)} &\le \|S(t)v_0 - S(t)S(t_0)u_0\|_{D(A)} + \|S(t)S(t_0)u_0\|_{D(A)} \\ &\le m_1 + \frac{1}{2}m_1 < 2m_1.\end{aligned}$$

This contradicts the definition of $\tilde{T}$. Thus, we conclude that $T - t_0 \le \tilde{T}$, which shows (2.23).

Next, we will show (2.24). By (2.23), we have

$$\|S(t)v_1\|_{D(A)} \le 2m_1, \quad \|S(t)v_2\|_{D(A)} \le 2m_1, \quad \text{for } t \in [0, T - t_0]. \tag{2.25}$$

Thus, by (F6), for $t \in [0, T - t_0]$, we have

$$\begin{aligned}&\|S(t)v_1 - S(t)v_2\|_{D(A)} \\ &\le \|v_1 - v_2\|_{D(A)} + \int_0^t \|F(S(\tau)v_1) - F(S(\tau)v_2)\|_{D(A)} d\tau \\ &\le \|v_1 - v_2\|_{D(A)} + 2L(2m_1) \int_0^t \|S(\tau)v_1 - S(\tau)v_2\|_{D(A)} d\tau.\end{aligned}$$

and by Gronwall's lemma, we have

$$\|S(t)v_1 - S(t)v_2\|_{D(A)} \le e^{2L(2m_1)t} \|v_1 - v_2\|_{D(A)}, \quad \text{for } t \in [0, T - t_0].$$

Moreover, by (2.25) and (F8), for $t \in [0, T - t_0]$, we have

$$\begin{aligned}&\|S(t)v_1 - S(t)v_2\|_X \\ &\le \|v_1 - v_2\|_X + \int_0^t \|F(S(\tau)v_1) - F(S(\tau)v_2)\|_X d\tau \\ &\le \|v_1 - v_2\|_X + 2L_4(2m_1) \int_0^t \|S(\tau)v_1 - S(\tau)v_2\|_X d\tau.\end{aligned}$$

Hence, we obtain

$$\|S(t)v_1 - S(t)v_2\|_X \le e^{2L_4(2m_1)t} \|v_1 - v_2\|_X \quad \text{for} \quad t \in [0, T - t_0].$$

This completes the proof. $\square$

2.3 Local error estimates in $D(A)$

In this section, we will estimate local errors in $D(A)$ between the solution $u(t)$ of (2.9) and $\Psi(t)v_0$ which is defined by (2.13).

Proposition 2.3.1. *Assume* (F0)–(F3). *Let* $v_0 \in D(A^2)$ *and* $\|v_0\|_{D(A)} \leq M$ *for some* $M > 0$. *Then, there exists a positive constant* $K_1(M)$ *depending only on* M *such that*

$$\|S(t)v_0 - \Psi(t)v_0\|_{D(A)} \leq K_1(M)\|v_0\|_{(D(A^2)}t^2$$

for $t \in [0, \tau(4M)]$.

In what follows, we put

$$u(t) = S(t)v_0, \quad v(t) = \Psi(t)v_0. \tag{2.26}$$

First, $u(t)$ is expressed as

$$u(t) = \Phi_A(t)v_0 + \int_0^t \Phi_A(t-s)F(u(s))ds.$$

To derivative a useful expression for $v(t)$, we note by (2.12)

$$\begin{aligned}
\Phi_{F_1}(t/2)\underbrace{\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0}_{=w_0(t)} &= w_0(t) + \int_0^{t/2} F_1(\Phi_{F_1}(s)w_0(t))ds \\
&= w_0(t) + \frac{1}{2}\int_0^t F_1(\Phi_{F_1}(s/2)w_0(t))ds, \\
\Phi_{F_2}(t)\underbrace{\Phi_{F_1}(t/2)\Phi_A(t/2)v_0}_{=w_1(t)} &= w_1(t) + \int_0^t F_2(\Phi_{F_2(s)}w_1(t))ds,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 &= \Phi_A(t/2)v_0 + \int_0^{t/2} F_1(\Phi_{F_1}(s)\Phi_A(t/2)v_0)ds \\
&= \Phi_A(t/2)v_0 + \frac{1}{2}\int_0^t F_1(\Phi_{F_1}(s/2)\Phi_A(t/2)v_0)ds.
\end{aligned}$$

Therefore, $v(t)$ can be written as

$$\begin{aligned}
v(t) &= \Phi_A(t/2)\Phi_{F_1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0, \\
&= \Phi_A(t)v_0 + \boldsymbol{G}_1(t) + \boldsymbol{G}_2(t) + \boldsymbol{G}_3(t)
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\boldsymbol{G}_1(t) &= \frac{1}{2}\int_0^t \Phi_A(t/2)F_1(\Phi_{F_1}(s/2)\Phi_A(t/2)v_0)ds, \\
\boldsymbol{G}_2(t) &= \int_0^t \Phi_A(t/2)F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)ds, \\
\boldsymbol{G}_3(t) &= \frac{1}{2}\int_0^t \Phi_A(t/2)F_1(\Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)ds.
\end{aligned}$$

Hence, we have

$$u(t) - v(t) = \int_0^t \Phi_A(t-s)\left[F(u(s)) - F(v(s))\right]ds + R(t), \tag{2.28}$$

where

$$R(t) = \int_0^t \Phi_A(t-s)F(v(s))ds - [\boldsymbol{G}_1(t) + \boldsymbol{G}_2(t) + \boldsymbol{G}_3(t)].$$

We devide $R(t)$ as $R(t) = R_1(t) + R_2(t)$, where

$$\begin{aligned} R_1(t) &= \int_0^t \Phi_A(t-s)F_1(v(s))ds - (\boldsymbol{G}_1(t) + \boldsymbol{G}_3(t)), \\ R_2(t) &= \int_0^t \Phi_A(t-s)F_2(v(s))ds - \boldsymbol{G}_2(t). \end{aligned}$$

Moreover, we split $R_1(t)$ and $R_2(t)$ as $R_1(t) = R_{1a}(t) + R_{1b}(t)$ and $R_2(t) = R_{2a}(t) + R_{2b}(t)$, respectively. Here,

$$\begin{aligned} R_{1a}(t) = \int_0^t \Phi_A(t-s) \Big[&F_1(v(s)) - \frac{1}{2}F_1(\Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0) \\ &- \frac{1}{2}F_1(\Phi_{F_1}(s/2)\Phi_A(t/2)v_0) \Big] ds, \end{aligned} \tag{2.29}$$

$$\begin{aligned} R_{1b}(t) = \int_0^t (\Phi_A(t-s) - \Phi_A(t/2)) \Big[&\frac{1}{2}F_1(\Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0) \\ &+ \frac{1}{2}F_1(\Phi_{F_1}(s/2)\Phi_A(t/2)v_0) \Big] ds, \end{aligned} \tag{2.30}$$

$$R_{2a}(t) = \int_0^t \Phi_A(t-s) [F_2(v(s)) - F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)] ds, \tag{2.31}$$

$$R_{2b}(t) = \int_0^t (\Phi_A(t-s) - \Phi_A(t/2))F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)ds. \tag{2.32}$$

First, we prove the following lemma.

Lemma 2.3.2. *Assume* (F0)–(F3). *Let* $v_0 \in D(A^2)$ *and* $\|v_0\|_{D(A)} \le M$. *Then, there exists a positive constant* C_{12} *depending only on* M *such that*

$$\|R_2(t)\|_{D(A)} \le C_{12}\|v_0\|_{D(A^2)}t^2 \tag{2.33}$$

for $t \in [0, \tau(4M)]$.

Proof. First, we show that there exists a positive constant C_{12a} depending only on M such that

$$\|R_{2a}(t)\|_{D(A)} \le C_{12a}\|v_0\|_{D(A^2)}t^2 \tag{2.34}$$

for $t \in [0, \tau(4M)]$. For $0 \le s \le t \le \tau(4M)$, we set

$$w(s,t) = \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0.$$

Then, we have

$$R_{2a}(t) = \int_0^t \Phi_A(t-s)[F_2(v(s)) - F_2(w(s,t))]ds.$$

By Lemma 2.2.3, we have $\|w(s,t)\|_{D(A)} \le 4M$ and $\|v(s)\|_{D(A)} \le 8M$ for $0 \le s \le t \le \tau(4M)$. Thus, by (F6), we have

$$\|R_{2a}(t)\|_{D(A)} \le L(8M) \int_0^t \|v(s) - w(s,t)\|_{D(A)}\, ds \tag{2.35}$$

for $0 \le s \le t \le \tau(4M)$. Since

$$\begin{aligned} v(s) &= \Phi_A(s/2)\Phi_{F_1}(s/2)w(s,s) \\ &= \Phi_A(s/2)\left\{ w(s,s) + \int_0^{s/2} F_1(\Phi_{F_1}(\tau)w(s,s))d\tau \right\}, \end{aligned}$$

we have

$$\begin{aligned} \|v(s) - w(s,s)\|_{D(A)} \le \|\Phi_A(s/2)w(s,s) - w(s,s)\|_{D(A)} \\ + \int_0^{s/2} \|F_1(\Phi_{F_1}(\tau)w(s,s))\|_{D(A)} d\tau. \end{aligned}$$

By Lemmas 2.2.1 and 2.2.4, we have

$$\begin{aligned} &\|\Phi_A(s/2)w(s,s) - w(s,s)\|_{D(A)} \le \frac{s}{2}\|w(s,s)\|_{D(A^2)} \\ &\le \frac{s}{2} e^{2L_2(8M)s}\|v_0\|_{D(A^2)} \le \frac{s}{2} e^{2L_2(8M)\tau(4M)}\|v_0\|_{D(A^2)} \end{aligned}$$

for $0 \le s \le \tau(4M)$. Moreover, by (F7) and Lemma 2.2.3, we have

$$\begin{aligned} &\|F_1(\Phi_{F_1}(\tau)w(s,s))\|_{D(A)} \\ &\le L(\|\Phi_{F_1}(\tau)w(s,s)\|_{D(A)})\|\Phi_{F_1}(\tau)w(s,s)\|_{D(A)} \le L(8M)8M \end{aligned}$$

for $0 \le \tau \le s \le \tau(4M)$. Thus, we have

$$\|v(s) - w(s,s)\|_{D(A)} \le \frac{s}{2}\left\{ e^{2L_2(8M)\tau(4M)} + 8L(8M) \right\}\|v_0\|_{D(A^2)}$$

for $0 \le s \le \tau(4M)$, which implies (2.34).

Next, we show that there exists a positive constant C_{12b} depending only on M such that

$$\|R_{2b}(t)\|_{D(A)} \le C_{12b}\|v_0\|_{D(A^2)} t^2 \tag{2.36}$$

for $t \in [0, \tau(2M)]$.

By (F2) and Lemmas 2.2.1 and 2.2.4, we have

$$\begin{aligned} &\|R_{2b}(t)\|_{D(A)} \\ &\le \int_0^t \left| \frac{t}{2} - s \right| \|F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)\|_{D(A^2)} ds \\ &\le \int_0^t \left| \frac{t}{2} - s \right| L_2(4M)\|\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)} ds \\ &\le \int_0^t \left| \frac{t}{2} - s \right| L_2(4M)e^{L_2(4M)s + L_2(2M)t/2}\|v_0\|_{D(A^2)} ds \\ &\le \int_0^t \left| \frac{t}{2} - s \right| ds L_2(4M)e^{2L_2(4M)\tau(2M)}\|v_0\|_{D(A^2)} \end{aligned}$$

for $t \in [0, \tau(2M)]$, which implies (2.36).

Finally, (2.33) follows from (2.34) and (2.36). $\square$

Lemma 2.3.3. *Assume* (F0)–(F3). *Let* $v_0 \in D(A^2)$ *and* $\|v_0\|_{D(A)} \le M$. *Then, there exists a positive constant* C_{11} *depending only on* M *such that*

$$\|R_1(t)\|_{D(A)} \le C_{11}\|v_0\|_{D(A^2)} t^2 \tag{2.37}$$

for $t \in [0, \tau(4M)]$.

Lemma 2.3.3 can be proved in the same way as in Lemma 2.3.2, so we omit the detail. By Lemmas 2.3.2 and 2.3.3, we obtain the following lemma.

Lemma 2.3.4. *Assume* (F0)–(F3). *Let* $v_0 \in D(A^2)$ *and* $\|v_0\|_{D(A)} \le M$. *Then, there exists a positive constant* C_1 *depending* M *such that*

$$\|R(t)\|_{D(A)} \le C_1\|v_0\|_{D(A^2)} t^2 \tag{2.38}$$

for $t \in [0, \tau(4M)]$.

Now, we give the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. It follows from (F6) and Lemma 2.3.4 that there exists a positive constant C_1 depending only on M such that

$$\begin{aligned}\|u(t) - v(t)\|_{D(A)} &\le \int_0^t \|F(u(s)) - F(v(s))\|_{D(A)} ds + \|R(t)\|_{D(A)} \\ &\le \int_0^t 2L(\max\{\|u(s)\|_{D(A)}, \|v(s)\|_{D(A)}\})\|u(s) - v(s)\|_{D(A)} ds \\ &\qquad + C_1\|v_0\|_{D(A^2)} t^3\end{aligned}$$

for $t \in [0, \tau(4M)]$. Moreover, by Lemma 2.2.3, we have $\|u(s)\|_{D(A)} \le 8M$ and $\|v(s)\|_{D(A)} \le 8M$ for $s \in [0, \tau(4M)]$. Thus, we have

$$\|u(t) - v(t)\|_{D(A)} \le 2L(8M)\int_0^t \|u(s) - v(s)\|_{D(A)} ds + C_1\|v_0\|_{D(A^2)} t^2$$

for $t \in [0, \tau(4M)]$. Finally, by Gronwall's lemma, we obtain

$$\|u(t) - v(t)\|_{D(A)} \le e^{2L(8M)t} C_1\|v_0\|_{D(A^2)} t^2 \le e^{2L(8M)\tau(4M)} C_1\|v_0\|_{D(A^2)} t^2$$

for $t \in [0, \tau(4M)]$. This completes the proof. $\square$

2.4 Local error estimates in X

In this section, we prove the following local error estimates in X.

Proposition 2.4.1. *Assume* (F0)–(F5)*. Let* $v_0 \in D(A^2)$ *and* $\|v_0\|_{D(A)} \le M$*. Then, there exists a positive constant* $K_2(M)$ *depending only on* M *that*

$$\|S(t)v_0 - \Psi(t)v_0\|_X \le K_2(M)\|v_0\|_{D(A^2)}t^3$$

for $t \in [0, \tau(4M)]$.

This proposition is a readily obtainable consequence of

$$\|R_{1a}(t)\|_X \le C_{12a}\|v_0\|_{D(A^2)}t^3, \tag{2.39}$$
$$\|R_{1b}(t)\|_X \le C_{12b}\|v_0\|_{D(A^2)}t^3, \tag{2.40}$$
$$\|R_{2a}(t)\|_X \le C_{22a}\|v_0\|_{D(A^2)}t^3, \tag{2.41}$$
$$\|R_{2b}(t)\|_X \le C_{22b}\|v_0\|_{D(A^2)}t^3, \tag{2.42}$$

for $t \in [0, \tau(4M)]$, where $C_{12a}, C_{12b}, C_{22a}, C_{22b}$ are positive constants depending only on M and $R_{1a}(t), R_{1b}(t), R_{2a}(t), R_{2b}(t)$ are defined by (2.26), (2.28)–(2.32).

The proof of theses estimates are given below.

2.4.1 Proof of (2.42)

For $0 \le s \le t \le \tau(M)$, we set

$$w_0 = \Phi_{F_1}(t/2)\Phi_A(t/2)v_0, \quad w(s) = F_2(\Phi_{F_2}(s)w_0).$$

Then, it follows from Lemmas 2.2.3 and 2.2.4 that

$$\|w_0\|_{D(A)} \le 2M, \quad \|w_0\|_{D(A^2)} \le e^{L_2(2M)t}\|v_0\|_{D(A^2)}$$

for $t \in [0, \tau(M)]$. Moreover, by Lemma 2.2.2, we have

$$\begin{aligned}&\left\|\int_0^t [\Phi_A(t-s)w(s) - \Phi_A(t/2)w(s)]ds\right\|_X \\ &\quad \le t^3\left\{\|w\|_{C^1([0,\tau(M)],D(A))} + \|w\|_{C([0,\tau(M)],D(A^2))}\right\}\end{aligned}$$

for $t \in [0, \tau(M)]$.

Here, it follows from (F7) and Lemma 2.2.3 that

$$\|w(s)\|_{D(A)} \le L(\|\Phi_{F_2}(s)w_0\|_{D(A)})\|\Phi_{F_2}(s)w_0\|_{D(A)} \le 4ML(4M) \tag{2.43}$$

for $s \in [0, \tau(2M)]$. Moreover, by (F2) and Lemma 2.2.4, we see that

$$\begin{aligned}&\|w(s)\|_{D(A^2)} \le L_2(\|\Phi_{F_2}(s)w_0\|_{D(A)})\|\Phi_{F_2}(s)w_0\|_{D(A^2)} \\ &\le L_2(4M)e^{L_2(4M)s}\|w_0\|_{D(A^2)} \le L_2(4M)e^{2L_2(4M)\tau(2M)}\|v_0\|_{D(A^2)}\end{aligned}$$

for $s \in [0, \tau(2M)]$. Thus, there exists a positive constant C' depending only on M such that $\|w\|_{C([0,\tau(2M)],D(A^2))} \le C'\|v_0\|_{D(A^2)}$.

Next, since $w'(s) = F_2'(\Phi_{F_2}(s)w_0)\partial_s(\Phi_{F_2}(s)w_0) = F_2'(\Phi_{F_2}(s)w_0)F_2(\Phi_{F_2}(s)w_0)$, it follows from (F1) and (2.43) that

$$\begin{aligned}\|w'(s)\|_{D(A)} &\le L(\|\Phi_{F_2}(s)w_0\|_{D(A)})\|F_2(\Phi_{F_2}(s)w_0)\|_{D(A)} \\ &\le 4ML(4M)^2 \le 4L(4M)^2\|v_0\|_{D(A^2)}\end{aligned}$$

for $s \in [0, \tau(2M)]$. Thus, there exists a positive constant C'' depending only on M such that $\|w\|_{C^1([0,\tau(2M)],D(A))} \le C''\|v_0\|_{D(A^2)}$.

This completes the proof of (2.42).

2.4.2 Proof of (2.41)

In order to prove (2.41), we devide $R_{2a}(t)$ into several parts. By Taylor's formula, we have

$$F_2(\Psi(s)v_0) - F_2(v_0) = F_2'(v_0)[\Psi(s)v_0 - v_0] + J_1(s), \tag{2.44}$$

$$\begin{aligned}&F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0) - F_2(v_0) \\ &\quad = F_2'(v_0)[\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 - v_0] + J_2(s),\end{aligned} \tag{2.45}$$

where

$$\begin{aligned}J_1(s) &= \int_0^1 (1-\theta)F_2''(\theta\Psi(s)v_0 + (1-\theta)v_0)[\Psi(s)v_0 - v_0]^2 d\theta, \\ J_2(s) &= \int_0^1 (1-\theta)F_2''(\theta\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 + (1-\theta)v_0) \\ &\qquad\qquad \cdot [\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 - v_0]^2 d\theta.\end{aligned}$$

Moreover, it follows from (2.27) that

$$\begin{aligned}\Psi(s)v_0 = \Phi_A(s)v_0 &+ \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(\Phi_{F_1}(\tau/2)\Phi_{F_2}(s)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)d\tau \\ &+ \int_0^s \Phi_A(s/2)F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)d\tau \\ &+ \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(\Phi_{F_1}(\tau/2)\Phi_A(s/2)v_0)d\tau, \\ \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 &= \Phi_A(t/2)v_0 + \int_0^s F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)d\tau \\ &+ \frac{1}{2}\int_0^t F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0)d\tau.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\Psi(s)v_0 - \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0 \\ &= \Phi_A(s)v_0 - \Phi_A(t/2)v_0 + J_3(s,t) + J_4(s) + J_5(s,t) + J_6(s,t),\end{aligned} \tag{2.46}$$

where

$$
\begin{aligned}
J_3(s,t) &= \int_0^s \Phi_A(s/2)F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)d\tau \\
&\quad - \int_0^s F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)d\tau, \\
J_4(s) &= \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(\Phi_{F_1}(\tau/2)\Phi_{F_2}(s)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)d\tau \\
&\quad - \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(\Phi_{F_1}(\tau/2)\Phi_A(s/2)v_0)d\tau, \\
J_5(s,t) &= \int_0^s \Phi_A(s/2)F_1(\Phi_{F_1}(\tau/2)\Phi_A(s/2)v_0)d\tau \\
&\quad - \int_0^s F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0)d\tau, \\
J_6(s,t) &= \int_0^s F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0)d\tau \\
&\quad - \frac{1}{2}\int_0^t F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0)d\tau.
\end{aligned}
$$

Thus, it follows from (2.44) and (2.45) that

$$
\begin{aligned}
R_{2a}(t) &= \int_0^t \Phi_A(t-s)\Big[F_2(\Psi(s)v_0) - F_2(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)\Big]ds \\
&= \int_0^t \Phi_A(t-s)F_2'(v_0)\big[\Psi(s)v_0 - \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\big]ds \\
&\qquad + \int_0^t \Phi_A(t-s)(J_1(s) - J_2(s))ds. \qquad (2.47)
\end{aligned}
$$

By (2.46), we obtain

$$
\begin{aligned}
&\int_0^t \Phi_A(t-s)F_2'(v_0)\big[\Psi(s)v_0 - \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\big]ds \\
&= \int_0^t \Phi_A(t-s)F_2'(v_0)\big[\Phi_A(s)v_0 - \Phi_A(t/2)v_0\big]ds \\
&\qquad + \int_0^t \Phi_A(t-s)F_2'(v_0)\big\{J_3(s,t) + J_4(s) + J_5(s,t) + J_6(s,t)\big\}ds. \quad (2.48)
\end{aligned}
$$

Hence, we can express as

$$
R_{2a}(t) = \sum_{j=0}^{6} Q_j(t),
$$

where

$$
\begin{aligned}
Q_0(t) &= \int_0^t \Phi_A(t-s)F_2'(v_0)\left[\Phi_A(s)v_0 - \Phi_A(t/2)v_0\right]ds, \\
Q_j(t) &= (-1)^{j+1}\int_0^t \Phi_A(t-s)J_j(s)ds \quad (j=1,2), \\
Q_j(t) &= \int_0^t \Phi_A(t-s)F_2'(v_0)J_j(s,t)ds \quad (j=3,4,5,6).
\end{aligned}
$$

Estimation for $Q_0(t)$.

By Taylor's formula, we have

$$\Phi_A(s)v_0 - \Phi_A(t/2)v_0 = (s-t/2)\Phi_A(t/2)Av_0 + (s-t/2)^2\int_0^1 (1-\theta)\Phi_A(\theta s + (1-\theta)t/2)A^2v_0 d\theta.$$

We note, for any $\hat{w} \in X$,

$$\int_0^t (s-t/2)\Phi_A(t-s)\hat{w}ds = \int_0^{t/2}(t/2-s)(\Phi_A(s)-\Phi_A(t-s))\hat{w}ds. \tag{2.49}$$

We set $w_0 = F_2'(v_0)\Phi_A(t/2)Av_0$. By (F1), we have

$$\|w_0\|_{D(A)} \le L(\|v_0\|_{D(A)})\|\Phi_A(t/2)Av_0\|_{D(A)} \le L(M)\|v_0\|_{D(A^2)}. \tag{2.50}$$

Hence, by (F4), Lemma 2.2.1 and (2.50), we have

$$\begin{aligned}
&\|Q_0(t)\|_X \\
&\le \int_0^{t/2}(t/2-s)\|(\Phi_A(s)-\Phi_A(t-s))w_0\|_X ds \\
&\qquad + \int_0^t (s-t/2)^2 L_4(M)\left\|\int_0^1 (1-\theta)\Phi_A(\theta s+(1-\theta)t/2)A^2v_0 d\theta\right\|_X ds \\
&\le \int_0^t 2(t/2-s)^2 ds\|w_0\|_{D(A)} + \int_0^t (s-t/2)^2 ds L_4(M)\|v_0\|_{D(A^2)} \\
&\le (L(M)+L_4(M))\|v_0\|_{D(A^2)}t^3
\end{aligned}$$

for $t \ge 0$.

Estimations for $Q_1(t)$ and $Q_2(t)$

First, we consider the case $j=1$. Since $\|\Psi(s)v_0\|_{D(A)} \le 8M$ for $s \in [0,\tau(4M)]$, it follows from (F5) that

$$\begin{aligned}
&\|Q_1(t)\|_X \\
&\le \int_0^t \|J_1(s)\|_X ds \\
&\le \int_0^t\int_0^1 \left\|F_2''(\theta\Psi(s)v_0+(1-\theta)v_0)\left[\Psi(s)v_0-v_0\right]^2\right\|_X d\theta ds \\
&\le \int_0^t L_5(8M)\cdot\|\Psi(s)v_0-v_0\|_X\cdot\|\Psi(s)v_0-v_0\|_{D(A)}ds
\end{aligned}$$

for $t \in [0,\tau(4M)]$. Moreover, by (2.27) and Lemma 2.2.1, we have

$$\begin{aligned}
&\|\Psi(s)v_0-v_0\|_{D(A)} \\
&\le \|\Phi_A(s)v_0-v_0\|_{D(A)} + \|\boldsymbol{G}_1(s)\|_{D(A)} + \|\boldsymbol{G}_2(s)\|_{D(A)} + \|\boldsymbol{G}_3(s)\|_{D(A)} \\
&\le s\|v_0\|_{D(A^2)} + \|\boldsymbol{G}_1(s)\|_{D(A)} + \|\boldsymbol{G}_2(s)\|_{D(A)} + \|\boldsymbol{G}_3(s)\|_{D(A)}.
\end{aligned}$$

By (F7) and Lemma 2.2.3,

$$\begin{aligned}
\|\boldsymbol{G}_1(s)\|_{D(A)} &\le \int_0^s \|F_1(\Phi_{F_1}(\tau/2)\Phi_A(s/2)v_0)\|_{D(A)}d\tau \\
&\le 2ML(2M)s \le 2L(2M)\|v_0\|_{D(A^2)}s \\
\|\boldsymbol{G}_2(s)\|_{D(A)} &\le \int_0^s \|F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)\|_{D(A)}ds \\
&\le 4ML(4M)s \le 2L(2M)\|v_0\|_{D(A^2)}s \\
\|\boldsymbol{G}_3(s)\|_{D(A)} &\le \int_0^s \|F_1(\Phi_{F_1}(\tau/2)\Phi_{F_2}(s)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0)\|_{D(A)}ds \\
&\le 8ML(8M)s \le 8L(8M)\|v_0\|_{D(A^2)}s.
\end{aligned}$$

Thus, there exists a positive constant C'_{J_1} depending only on M such that

$$\|\Psi(s)v_0 - v_0\|_{D(A)} \le C'_{J_1}\|v_0\|_{D(A^2)}s,$$

for $s \in [0, \tau(4M)]$. Similarly, there exists a positive constant C''_{J_1} depending only on M such that $\|\Psi(s)v_0 - v_0\|_X \le C''_{J_1}s$ for $s \in [0, \tau(4M)]$.

Therefore, we have

$$\begin{aligned}
\|Q_1(t)\|_X &\le L_5(8M)\int_0^t C'_{J_1}C''_{J_1}\|v_0\|_{D(A^2)}s^2 ds \\
&\le L_5(8M)C'_{J_1}C''_{J_1}\|v_0\|_{D(A^2)}t^3. \qquad (2.51)
\end{aligned}$$

for $t \in [0, \tau(4M)]$.

Similarly, we can prove

$$\|Q_2(t)\|_X \le C\|v_0\|_{D(A^2)}t^3. \qquad (2.52)$$

Estimations for $Q_3(t), Q_4(t)$ and $Q_5(t)$.
First, we consider the case $j = 3$. By (F4), we have

$$\|Q_3(t)\|_X \le \int_0^t L_4(\|v_0\|_{D(A)})\|J_3(s,t)\|_X ds.$$

We set $w(\tau, s) = \Phi_{F_2}(\tau)\Phi_{F_1}(s/2)\Phi_A(s/2)v_0$. Then, we have

$$\begin{aligned}
J_3(s,t) = &\int_0^s \big[\Phi_A(s/2)F_2(w(\tau,s)) - F_2(w(\tau,s))\big]d\tau \\
&+ \int_0^s \big[F_2(w(\tau,s)) - F_2(w(\tau,t))\big]d\tau.
\end{aligned}$$

It follows from Lemma 2.2.3 that

$$\|w(\tau,s)\|_{D(A)} \le 4M \quad \text{for} \quad \tau, s \in [0, \tau(2M)].$$

Moreover, by Lemma 2.2.1 and (F1), we have

$$\begin{aligned}
&\left\|\int_0^s \big[\Phi_A(s/2)F_2(w(\tau,s)) - F_2(w(\tau,s))\big]d\tau\right\|_X \le \int_0^s s\|F_2(w(\tau,s))\|_{D(A)}d\tau \\
&\le 4ML(4M)s^2 \le 4L(4M)\|v_0\|_{D(A^2)}s^2
\end{aligned}$$

for $s \in [0, \tau(2M)]$. Furthermore, by (F8), we have

$$\left\| \int_0^s \left[F_2(w(\tau, s)) - F_2(w(\tau, t)) \right] d\tau \right\|_X \le \int_0^s L_4(4M) \|w(\tau, s) - w(\tau, t)\|_X d\tau$$
$$\le \int_0^s L_4(4M) \|w(\tau, s) - w(\tau, t)\|_{D(A)} d\tau$$

for $s, t \in [0, \tau(2M)]$. For $0 \le \tau \le s \le t \le \tau(2M)$, by Lemmas 2.2.3, 2.2.1 and (F1), we have

$$\begin{aligned}
&\|w(\tau, s) - w(\tau, t)\|_{D(A)} \\
&\le e^{L(4M)\tau} \|\Phi_{F_1}(s/2)\Phi_A(s/2)v_0 - \Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \\
&\le e^{L(4M)\tau} \|\Phi_{F_1}(s/2)\Phi_A(s/2)v_0 - \Phi_{F_1}(s/2)\Phi_A(t/2)v_0\|_{D(A)} \\
&\qquad + e^{L(4M)\tau} \|\Phi_{F_1}(s/2)\Phi_A(t/2)v_0 - \Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \\
&\le e^{L(4M)\tau + L(2M)s/2} \|\Phi_A(s/2)v_0 - \Phi_A(t/2)v_0\|_{D(A)} \\
&\qquad + \int_{s/2}^{t/2} e^{L(4M)\tau} \|F_1(\Phi_{F_1}(\tau)\Phi_A(t/2)v_0)\|_{D(A)} d\tau \\
&\le \frac{1}{2} e^{L(4M)\tau + L(2M)s/2} \|v_0\|_{D(A^2)}(t - s) + ML(2M)e^{L(4M)\tau(2M)}(t - s) \\
&\le \frac{1}{2} e^{L(4M)\tau(2M)} \left(e^{L(2M)\tau(2M)/2} + 2L(2M) \right) \|v_0\|_{D(A^2)}(t - s).
\end{aligned}$$

Thus, there exists a positive constant C'_{J_3} which depend only on M such that

$$\|J_3(s, t)\|_{D(A)} \le C'_{J_3} \|v_0\|_{D(A^2)} ts$$

for $0 \le s \le t \le \tau(2M)$. Therefore, we have

$$\begin{aligned}
\|Q_3(t)\|_X &\le \int_0^t L_4(M) C'_{J_3} \|v_0\|_{D(A^2)} ts ds \\
&\le L_4(M) C'_{J_3} \|v_0\|_{D(A^2)} t^3
\end{aligned}$$

for $t \in [0, \tau(2M)]$,

Similarly, we can prove

$$\|Q_4(t)\|_X \le C\|v_0\|_{D(A^2)} t^3, \quad \|Q_5(t)\|_X \le C\|v_0\|_{D(A^2)} t^3$$

for $t \in [0, \tau(4M)]$.

Estimation for $Q_6(t)$.

We notice that $J_6(s, t)$ can be rewritten as

$$\begin{aligned}
J_6(s, t) &= \int_0^s F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0) d\tau - \frac{1}{2} \int_0^t F_1(\Phi_{F_1}(\tau/2)\Phi_A(t/2)v_0) d\tau \\
&= 2 \int_0^{s/2} F_1(\Phi_{F_1}(\tau)\Phi_A(t/2)v_0) d\tau - \int_0^{t/2} F_1(\Phi_{F_1}(\tau)\Phi_A(t/2)v_0) d\tau
\end{aligned}$$

We set $w(\tau) = F_1(\Phi_{F_1}(\tau)\Phi_A(t/2)v_0)$. Then, we have

$$J_6(s,t) = J_{61}(s,t) + J_{62}(s,t),$$

where

$$\begin{aligned}
J_{61}(s,t) &= 2\int_0^{s/2} w(\tau)d\tau - 2\int_0^{t/4} w(\tau)d\tau, \\
J_{62}(t) &= 2\int_0^{t/4} w(\tau)d\tau - \int_0^{t/2} w(\tau)d\tau.
\end{aligned}$$

By Taylor's formula, we obtain

$$J_{61}(s,t) = J_{61a}(s,t) + \frac{1}{2}(s-t/2)^2 J_{61b}(s,t),$$

where

$$\begin{aligned}
J_{61a}(s,t) &= (s-t/2)w(t/4), \\
J_{61b}(s,t) &= \int_0^1 (1-\theta)F_1'(\Phi_{F_1}((\theta s + (1-\theta)t/2)/2)\Phi_A(t/2)v_0) \\
&\qquad\qquad \cdot F_1(\Phi_{F_1}((\theta s + (1-\theta)t/2)/2)\Phi_A(t/2)v_0))d\theta.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_0^t \Phi_A(t-s)F_2'(v_0)J_{61a}(s,t)ds \\
&= \int_0^t (s-t/2)\Phi_A(t-s)F_2'(v_0)w(t/4)ds.
\end{aligned}$$

Hence, it follows from (F1), (2.49), Lemmas 2.2.1 and 2.2.3 that

$$\begin{aligned}
&\left\|\int_0^t \Phi_A(t-s)F_2'(v_0)J_{61a}(s,t)ds\right\|_X \\
&= \left\|\int_0^{t/2} (s-t/2)(\Phi_A(t-s)-\Phi_A(s))F_2'(v_0)w(t/4)ds\right\|_X \\
&\le \int_0^{t/2} 2(t/2-s)^2 ds \|F_2'(v_0)w(t/4)\|_{D(A)} \\
&\le L(2M)^2 M t^3 \le L(2M)^2 \|v_0\|_{D(A^2)} t^3
\end{aligned}$$

for $t \in [0, \tau(M)]$.

By (F4), we have

$$\begin{aligned}
&\left\|\int_0^t (s-t/2)^2 \Phi_A(t-s)F_2'(v_0)J_{61b}(s,t)ds\right\|_X \\
&\le \int_0^t (s-t/2)^2 L_4(M)\|J_{61b}(s,t)\|_X ds.
\end{aligned}$$

By (F1) and (F7), we have

$$\|J_{61b}(s,t)\|_X \le \|J_{61b}(s,t)\|_{D(A)} \le L(2M)^2 M \le L(2M)^2 \|v_0\|_{D(A^2)}$$

for $s,t \in [0,\tau(M)]$. Thus, we obtain

$$\left\| \int_0^t (s-t/2)^2 \Phi_A(t-s) F_2'(v_0) J_{61b}(s,t) ds \right\|_X \le L(2M)^2 L_4(M) \|v_0\|_{D(A^2)} t^3 \quad (2.53)$$

for $t \in [0,\tau(M)]$.

Therefore, there exists a positive constant C_{61} depending only on M such that

$$\left\| \int_0^t \Phi_A(t-s) F_2'(v_0) J_{61}(s,t) ds \right\|_X \le C_{61} \|v_0\|_{D(A^2)} t^3 \quad (2.54)$$

for $t \in [0,\tau(M)]$.

On the other hand,

$$\begin{aligned} J_{62}(t) &= 2\int_0^{t/4} w(\tau) d\tau - \int_0^{t/2} w(\tau) d\tau \\ &= \int_0^{t/2} w(\tau/2) d\tau - \int_0^{t/2} w(\tau) d\tau. \end{aligned}$$

Hence, by (F6) and Lemma 2.2.3, we have

$$\begin{aligned} &\|J_{62}(t)\|_X \le \|J_{62}(t)\|_{D(A)} \\ &\le \int_0^{t/2} \|w(\tau) - w(\tau/2)\|_{D(A)} d\tau \\ &\le \int_0^{t/2} L(2M) \int_{\tau/2}^{\tau} \|F_1(\Phi_{F_1}(\tilde\tau)\Phi_A(t/2)v_0)\|_{D(A)} d\tilde\tau d\tau \\ &\le L(2M)^2 M t^2 \le L(2M)^2 \|v_0\|_{D(A^2)} t^2 \end{aligned}$$

for $t \in [0,\tau(M)]$.

Thus, it follows from (F1) that

$$\begin{aligned} &\left\| \int_0^t \Phi_A(t-s) F_2'(v_0) J_{62}(t) ds \right\|_X \\ &\le \left\| \int_0^t \Phi_A(t-s) F_2'(v_0) J_{62}(t) ds \right\|_{D(A)} \\ &\le \int_0^t L(M) \|J_{62}(t)\|_{D(A)} ds \le L(2M)^3 \|v_0\|_{D(A^2)} t^3 \end{aligned} \quad (2.55)$$

for $t \in [0,\tau(M)]$.

Summing up those estimates, we obtain

$$\|Q_6(t)\|_X \le C \|v_0\|_{D(A^2)} t^3. \quad (2.56)$$

for $t \in [0,\tau(M)]$.

2.4.3 Proofs of (2.39) and (2.40)

Inequality (2.40) can be proved in the same way as the proof of (2.42), so we skip the detail.

To derive the estimation for $R_{1a}(t)$, we divide $R_{1a}(t)$ as

$$R_{1a}(t) = R_{11a}(t) + R_{12a},$$

where

$$\begin{aligned}
R_{11a}(t) &= \int_0^t \Phi_A(t-s)\left\{F_1(\Psi(s)v_0) - F_1(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)\right\} ds, \\
R_{12a}(t) &= \frac{1}{2}\int_0^t \Phi_A(t-s)\left[2F_1\left(\Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\right)\right. \\
&\quad \left. - \left\{F_1(\Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0) + F_1(\Phi_{F_1}(s/2)\Phi_A(t/2)v_0)\right\}\right] ds
\end{aligned}$$

Then, in exactly the same way as the proof of (2.41), we can prove

$$\|R_{11a}(t)\|_X \leq C\|v_0\|_{D(A^2)}t^3$$

for $t \in [0, \tau(4M)]$. We proceed to shpw

$$\|R_{12a}(t)\|_X \leq C\|v_0\|_{D(A^2)}t^3 \tag{2.57}$$

for $t \in [0, \tau(4M)]$. To do this, we divide $R_{12a}(t)$ into some parts. We set

$$\left\{\begin{array}{ll}
w_1(s) = \Phi_{F_1}(s/2)\Phi_A(t/2)v_0, & w_2(s) = \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0, \\
w_3(s) = \Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0. &
\end{array}\right. \tag{2.58}$$

By Taylor's formula, we have

$$2F_1\left(w_2(s)\right) - F_1\left(w_3(s)\right) - F_1\left(w_1(s)\right) = Q(s) + 2J_7(s) - J_8(s) - J_9(s),$$

where

$$\begin{aligned}
Q(s) &= F_1'(v_0)\big[2w_2(s) - w_3(s) - w_1(s)\big], \\
J_7(s) &= \int_0^1 (1-\theta)F_1''\left(\theta w_2(s) + (1-\theta)v_0\right)\left[w_2(s) - v_0\right]^2 d\theta, \\
J_8(s) &= \int_0^1 (1-\theta)F_1''\left(\theta w_3(s) + (1-\theta)v_0\right)\left[w_3(s) - v_0\right]^2 d\theta, \\
J_9(s) &= \int_0^1 (1-\theta)F_1''\left(\theta w_1(s) + (1-\theta)v_0\right)\left[w_1(s) - v_0\right]^2 d\theta.
\end{aligned}$$

That is,

$$\begin{aligned}
R_{12a}(t) &= \frac{1}{2}\int_0^t \left[2F_1(w_2(s)) - F_1(w_3(s)) - F_1(w_1(s))\right] ds \\
&= \frac{1}{2}\int_0^t \Phi_A(t-s)\left[Q(s) + 2J_7(s) - J_8(s) - J_9(s)\right] ds.
\end{aligned}$$

We can prove

$$\left\| \int_0^t \Phi_A(t-s) J_j(s) ds \right\|_X \le C\|v_0\|_{D(A^2)} t^3 \quad (j = 7, 8, 9)$$

for $t \in [0, \tau(4M)]$ in the same way as the proofs of (2.51) akd (2.52). Hence, it remains to derive the following estimate:

$$\left\| \int_0^t \Phi_A(t-s) Q(s) ds \right\|_X \le C\|v_0\|_{D(A^2)} t^3 \tag{2.59}$$

for $t \in [0, \tau(4M)]$.

Functions $w_1(s), w_2(s), w_3(s)$ are written as

$$\begin{aligned} &w_1(s) = \Phi_A(t/2)v_0 + I_5(s), \qquad w_2(s) = \Phi_A(t/2)v_0 + I_4(s) + I_5(t), \\ &w_3(s) = \Phi_A(t/2)v_0 + I_4(t) + I_5(t) + I_6(s), \end{aligned}$$

where

$$\begin{aligned} I_4(s) &= \int_0^s F_2\left(w_2(\tau)\right) d\tau, \quad I_5(s) = \int_0^{s/2} F_1\left(w_1(2\tau)\right) d\tau, \\ I_6(s) &= \int_0^{s/2} F_1\left(w_3(2\tau)\right) d\tau. \end{aligned}$$

Thus, we obtain

$$\int_0^t \Phi_A(t-s) Q(s) ds = W_1(t) + W_2(t)$$

where,

$$\begin{aligned} W_1(t) &= \int_0^t \Phi_A(t-s) F_1'(v_0) \left[2I_4(s) - I_4(t)\right] ds, \\ W_2(t) &= \int_0^t \Phi_A(t-s) F_1'(v_0) \left[I_5(t) - 2I_5(s)\right] ds, \\ W_3(t) &= \int_0^t \Phi_A(t-s) F_1'(v_0) [I_5(s) - I_6(s)] ds. \end{aligned}$$

First, we can prove the following estimate in the same way as the proof of (2.56)

$$\|W_1(t)\|_X \le C\|v_0\|_{D(A^2)} t^3, \quad \|W_2(t)\|_X \le C\|v_0\|_{D(A^2)} t^3 \tag{2.60}$$

for $t \in [0, \tau(2M)]$. In view of (F1), (F6) and Lemma 2.2.3, we obtain

$$\begin{aligned} &\|I_5(s) - I_6(s)\|_X \\ &\le \|I_5(s) - I_6(s)\|_{D(A)} \\ &\le \int_0^{s/2} L(8M) \|\Phi_{F_1}(\tau)\Phi_A(t/2)v_0 - \Phi_{F_1}(\tau)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} d\tau \\ &\le \int_0^{s/2} L(8M) e^{L(8M)\tau(4M)} \left\|\Phi_A(t/2)v_0 - \Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\right\|_{D(A)} d\tau \end{aligned}$$

for $s, t \in [0, \tau(4M)]$. Furthermore, by (F7) and Lemma 2.2.3, we have

$$
\begin{aligned}
&\|\Phi_A(t/2)v_0 - \Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \\
&\leq \int_0^t \|F_2(\Phi_{F_2}(\tau)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0)\|_{D(A)}d\tau + \int_0^{t/2} \|F_1(\Phi_{F_1}(\tau)\Phi_A(t/2)v_0)\|_{D(A)}d\tau \\
&\leq \int_0^t L(4M)4M d\tau + \int_0^{t/2} L(2M)2M d\tau \\
&\leq 5L(4M)\|v_0\|_{D(A^2)}t
\end{aligned}
$$

for $t \in [0, \tau(2M)]$.

Thus, we have

$$
\begin{aligned}
\|I_5(s) - I_6(s)\|_X &\leq \int_0^{s/2} ds \cdot 5L(8M)^2 e^{L(8M)\tau(4M)}\|v_0\|_{D(A^2)}t \\
&\leq 5L(8M)^2 e^{L(8M)\tau(4M)}\|v_0\|_{D(A^2)}ts
\end{aligned}
$$

for $s.t \in [0, \tau(4M)]$.

Hence, there exists a positive constant C''' depending only on M such that

$$
\|W_3(t)\|_X \leq C'''\|v_0\|_{D(A^2)}t^3
$$

for $t \in [0, \tau(4M)]$.

Summing up those estimates, we obtain (2.57) and, therefore, (2.39).

2.5 Proof of Theorem 2.1.2

This section is devoted to the proof of the main result, Theorem 2.1.2. We set

$$
\gamma_1 = 2L_2(8m_0), \quad \kappa_1 = e^{\{2L(2m_0)+\gamma_1\}T}K_1(m_0)T, \quad \kappa_3 = \kappa_1\|u_0\|_{D(A^2)}, \tag{2.61}
$$

$$
\kappa_2 = e^{\{2L_4(2m_0)+\gamma_1\}T}K_2(m_0)T. \tag{2.62}
$$

We assume that $h_0 > 0$ satisfies

$$
h_0 \leq \tau(4m_0), \quad e^{2L(2m_0)h_0}\kappa_3 h_0 \leq \delta_0, \quad \kappa_3 h_0 \leq \frac{7m_0}{8}, \tag{2.63}
$$

where $m_0 = 8\max_{t\in[0,T]}\|S(t)u_0\|_{D(A)}$ and δ_0 which is previously defined in Lemma 2.2.5. We note that $\kappa_3 h \leq e^{2L(2m_0)h}\kappa_3 h \leq \delta_0$ for $h \in (0, h_0]$.

In what follows, we assume $h \in (0, h_0]$. By induction, we will show

$$
\|\Psi(h)^j u_0\|_{D(A^2)} \leq e^{\gamma_1 jh}\|u_0\|_{D(A^2)} \tag{2.64}
$$

$$
\|\Psi(h)^j u_0\|_{D(A)} \leq m_0, \tag{2.65}
$$

$$
\|S(jh)u_0 - \Psi(h)^j u_0\|_{D(A)} \leq \kappa_3 h \tag{2.66}
$$

$$
\|S(jh)u_0 - \Psi(h)^j u_0\|_X \leq \kappa_2\|u_0\|_{D(A^2)}h^2 \tag{2.67}
$$

for $j \in \mathbb{N} \cup \{0\}$ satisfying $jh \leq T$.

In the case $j = 0$, it is clear that (2.64)–(2.67) hold. We assume $nh \leq T$ and (2.64)–(2.67) holds for $j = 0, 1, \ldots, n-1$.

First, it follows from Lemma 2.2.4 and (2.64) that

$$\begin{aligned}\|\Psi(h)^n u_0\|_{D(A^2)} &= \|\Psi(h)\Psi(h)^{n-1}u_0\|_{D(A^2)} \leq e^{2L_2(8m_0)h}\|\Psi(h)^{n-1}u_0\|_{D(A^2)} \\ &\leq e^{\gamma_1 h}e^{\gamma_1(n-1)h}\|u_0\|_{D(A^2)} = e^{\gamma_1 nh}\|u_0\|_{D(A^2)}.\end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}&\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} \\ &\leq \sum_{j=0}^{n-1} \|S((n-j-1)h)S(h)\Psi(h)^j u_0 - S((n-j-1)h)\Psi(h)\Psi(h)^j u_0\|_{D(A)}.\end{aligned}$$

Moreover,

$$\|\Psi(h)^j u_0 - S(jh)u_0\|_{D(A)} \leq \kappa_3 h \leq \delta_0$$

for $j = 0, 1, \ldots, n-2$. Thus, it follows from Lemma 2.2.5 that

$$\begin{aligned}&\|S(h)\Psi(h)^j u_0 - S((j+1)h)u_0\|_{D(A)} = \|S(h)\Psi(h)^j u_0 - S(h)S(jh)u_0\|_{D(A)} \\ &\leq e^{2L(2m_0)h}\|\Psi(h)^j u_0 - S(jh)u_0\|_{D(A)} \leq e^{2L(2m_0)h}\kappa_3 h \leq \delta_0\end{aligned}$$

for $j = 0, 1, \ldots, n-2$. We see that

$$\begin{aligned}&\|\Psi(h)\Psi(h)^j u_0 - S((j+1)h)u_0\|_{D(A)} = \|\Psi(h)^{j+1}u_0 - S((j+1)h)u_0\|_{D(A)} \\ &\leq \kappa_3 h \leq \delta_0\end{aligned}$$

for $j = 0, 1, \ldots, n-2$. Hence, it follows from Lemma 2.2.5 that

$$\begin{aligned}&\|S((n-j-1)h)S(h)\Psi(h)^j u_0 - S((n-j-1)h)\Psi(h)\Psi(h)^j u_0\|_{D(A)} \\ &\leq e^{2L(2m_0)(n-j-1)h}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)} \\ &\leq e^{2L(2m_0)T}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)}.\end{aligned}$$

Hence, we have

$$\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} \leq e^{2L(2m_0)T}\sum_{j=0}^{n-1}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)}.$$

Moreover, it follows from (2.65) that $\|\Psi(h)^j u_0\|_{D(A)} \leq m_0$ for $j = 0, 1, \ldots, n-1$. By Proposition 2.3.1, we obtain

$$\begin{aligned}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)} &\leq K_1(m_0)\|\Psi(h)^j u_0\|_{D(A^2)}h^2 \\ &\leq K_1(m_0)e^{\gamma_1 T}\|u_0\|_{D(A^2)}h^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} &\leq e^{2L(2m_0)T}\sum_{j=0}^{n-1}K_1(m_0)e^{\gamma_1 T}\|u_0\|_{D(A^2)}h^2 \\ &\leq e^{\{2L(2m_0)+\gamma_1\}T}K_1(m_0)\|u_0\|_{D(A^2)}nh^2 \leq \kappa_3 h.\end{aligned}$$

Finally, it follows from (2.63) that

$$\|\Psi(h)^n u_0\|_{D(A)} \leq \|\Psi(h)^n u_0 - S(nh)u_0\|_{D(A)} + \|S(nh)u_0\|_{D(A)} \leq \kappa_3 h + m_0/8 \leq m_0.$$

We can also prove (2.67) in the same way of the proof of (2.66).

Therefore, we showed (2.65) holds for $j = n$.

This completes the proof. □

2.6 Numerical examples

In this section, we present numerical examples to confirm the validity of Theorem 2.1.2. We consider

$$\begin{cases} \partial_t u = i\Delta u - i|u|^2 u - 2|u|^4 u, & t \in [0,T],\ x \in \mathbb{R}, \\ u(0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.68}$$

Let $A = i\partial_x^2$ and $D(A) = H^2(\mathbb{R})$. Moreover, we put $F_1(v) = -i|v|^2 v$, $F_2(v) = -2|v|^4 v$.

For simplicity of computation, we consider the equation in a bounded interval $[0,1]$ and pose the Dirichlet boundary condition. We set $u_0(x) = \sin(\pi x)$. We numerically solve $\Phi_A(h)v_0$ by applying the Crank-Nicolson type finite difference method. We can obtain the following exact solutions of $\partial_t v = -i|v|^2 v$ and $\partial_t v = -|v|^4 v$, respectively:

$$\Phi_{F_1}(h)v_0 = \exp\left[-i|v_0|^2 h\right] v_0 \quad \text{and} \quad \Phi_{F_2}(h)v_0 = (1 + 8|v_0|^4 h)^{-1/4} v_0.$$

It is difficult to obtain the exact solution of (2.68). Therefore, we numerically confirm the following condition instead of (2.16). We define

$$e^{(D)}(h) = \sup_{0 \leq nh \leq T} \|\Psi(h)^n u_0 - \Psi(h/2)^{2n} u_0\|_Y.$$

In this experiment, we set $T = 1$. We confirm that there exists positive constant C such that

$$e^{(D)}(h) \leq Ch^2,$$

which is a sufficient condition for (2.16). In Figure 1, we plot $(\log h, \log e^{(D)}(h))$ with $Y = L^\infty, L^2$ and H^1. We see that $e^{(D)}(h) \approx Ch^2$ in all cases.

Moreover, (2.68) has the following property:

$$\frac{d}{dt} \int |u|^2 dx \leq 0. \tag{2.69}$$

In Figure 2, we see that the scheme (2.13) preserves the property (2.69).

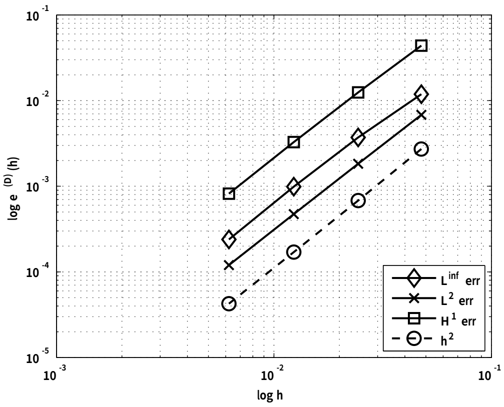


Figure 1. Convergence rate of the scheme (2.13) for (2.68).

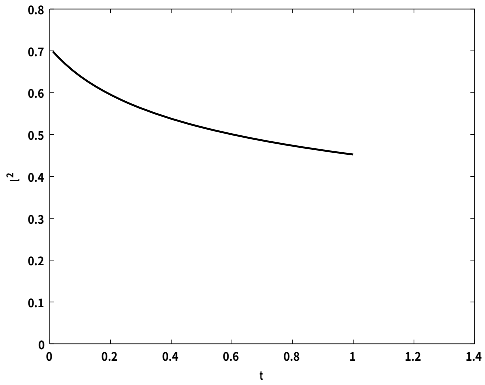


Figure 2. The dissipation property of the scheme (2.13) for (2.68).

3 Regularity and singularity of blow-up curve for $u_{tt} - u_{xx} = |u_t|^p$

We study a blow-up curve for the one dimensional wave equation $u_{tt} - u_{xx} = |u_t|^p$ with $p > 1$. The purpose of this paper is to show that the blow-up curve is a C^1 curve if the initial values are large and smooth enough. To prove the result, we convert the equation into a first order system, and then apply a modification of the method of Caffarelli and Friedman [7]. Moreover, we present some numerical investigations of the blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

3.1 Introduction

In this paper, we consider the nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = |u_t|^p, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where

$$p > 1 \text{ is a constant such that the function } |s|^p \text{ is of class } C^4. \tag{3.2}$$

Here, u is an unknown real-valued function.

Let T^* and R^* be any positive constants, and set

$$B_{R^*} = \{x \mid |x| < R^*\}, \tag{3.3}$$

$$K_-(t_0, x_0) = \{(t,x) \mid |x - x_0| < t_0 - t,\ t > 0\}, \tag{3.4}$$

$$K_{T^*,R^*} = \bigcup_{x \in B_{R^*}} K_-(T^*, x). \tag{3.5}$$

We then consider the following function

$$T(x) = \sup\{t \in (0, T^*) \mid \ |u_t(t,x)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

In this paper, we call the set $\Gamma = \{(T(x), x) \mid x \in B_{R^*}\}$ the blow-up curve. Below, we identify Γ with T itself. There are two purposes of this paper. First, we demonstrate that T is continuously differentiable for the suitable initial values. Second, we present some numerical examples of the various blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

We will state some analytical results from previous studies on the blow-up curves for nonlinear wave equations. The majority of previous studies have considered the following nonlinear wave equation:

$$u_{tt} - u_{xx} = F(u), \qquad t > 0,\ x \in \mathbb{R},$$

and corresponding blow-up curve

$$\tilde{T}(x) = \sup\{t \in (0, T^*) \mid\ |u(t,x)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

We note that the definition of the blow-up curve is different from ours. The pioneering study on this topic was done by Caffarelli and Friedman [6], [7]. They investigated the case with $F(u) = |u|^p$. They demonstrated that $\tilde{T}$ in that case is continuously differentiable under suitable initial conditions. Moreover, Godin [18] showed that the blow-up curve with $F(u) = e^u$ is also continuously differentiable under appropriate initial conditions. It was also shown that the blow-up curve can be C^∞, in the case of $F(u) = e^u$ (see Godin [19]). Furthermore, Uesaka [33] considered the blow-up curve for the system of nonlinear wave equations.

On the other hand, Merle and Zagg [27] showed that there are cases where the blow-up curve has singular points, while the above results concern the smoothness of the blow-up curve.

As mentioned above, several results have been established on the blow-up curve when there are no nonlinear terms involving the derivative of the solution. On the other hand, to the best of our knowledge only one result has been found concerning the blow-up curve with nonlinear terms involving the derivative of solution. Ohta and Takamura [30] considered the nonlinear wave equation

$$u_{tt} - u_{xx} = (u_t)^2 - (u_x)^2, \qquad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{3.6}$$

This equation can be transformed into the wave equation $\partial_t^2 v - \partial_x^2 v = 0$ by

$$v(t,x) = \exp\{-u(t,x)\}, \quad u(t,x) = -\log\{v(t,x)\}.$$

Thanks to the linearization of (3.6), we can study the blow-up curve of (3.6).

However, we cannot apply this linearization to (3.1). Therefore, we employ an alternative method, which is to rewrite to (3.1) as a system that does not include the derivative of the solution in nonlinear terms. We basically apply the method introduced by Caffarelli and Friedman [7] to this system. However, we offer an alternative proof of [7] for showing that the blow-up curve of the blow-up limits is an affine function (Section 3.5). Consequently, our proof is more elementary and easy to read. Our method would be applied to the original equation of [7].

We define ϕ and ψ as

$$\phi = u_t + u_x, \qquad \psi = u_t - u_x.$$

Then, we see that (3.1) is rewritten as

$$\begin{cases} D_-\phi = 2^{-p}|\phi+\psi|^p, & t>0,\ x\in\mathbb{R}, \\ D_+\psi = 2^{-p}|\phi+\psi|^p, & t>0,\ x\in\mathbb{R}, \\ \phi(0,x)=f(x), \quad \psi(0,x)=g(x), & x\in\mathbb{R}, \end{cases} \tag{3.7}$$

where $D_-v = v_t - v_x$, $D_+v = v_t + v_x$ and $f = u_1 + \partial_x u_0$, $g = u_1 - \partial_x u_0$. (The equivalency of between (3.1) and (3.7) will be described in Remark 3.1.2.)

Let $(\tilde{\phi},\tilde{\psi})$ be the solution of

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = 2^{-p}|\tilde{\phi}+\tilde{\psi}|^p, & t>0, \\ \dfrac{d\tilde{\psi}}{dt} = 2^{-p}|\tilde{\phi}+\tilde{\psi}|^p, & t>0, \\ \tilde{\phi}(0)=\gamma_1, \quad \tilde{\psi}(0)=\gamma_2, \end{cases} \tag{3.8}$$

where γ_1 and γ_2 are some positive constants which will be fixed later. Then, we see that there exists a positive constant T_1 such that

$$\tilde{\phi}(t)+\tilde{\psi}(t) \to \infty \quad \text{as } t \to T_1.$$

We make the following assumptions.

(A1) $f \geq \gamma_1, \quad g \geq \gamma_2 \quad$ in $B_{T^*+R^*}$.

(A2) $f, g \in C^4(B_{T^*+R^*})$.

(A3) There exists a constant $\varepsilon_0 > 0$ such that

$$2^{-p}(\gamma_1+\gamma_2)^p \geq (2+\varepsilon_0)\cdot \max_{x\in B_{T^*+R^*}} \{|f_x(x)|+|g_x(x)|\}.$$

(A4) $T_1 < T^*$.

(A5.1) There exists a constant $\varepsilon_1 > \dfrac{2}{2p-3}$ such that

$$2^{-p}(\gamma_1+\gamma_2)^p \geq (2+\varepsilon_1)\cdot \max_{x\in B_{T^*+R^*}} \{|f_x(x)|+|g_x(x)|\}.$$

(We notice that it follows from (3.2) that $p > 3/2$.)

(A5.2) There exists a constant $C^{(2)} > 0$ such that

$$(f+g)^{2p-1} \geq C^{(2)}\cdot \max_{x\in B_{T^*+R^*}} \{|f_{xx}(x)|+|g_{xx}(x)|\}.$$

(A5.3) There exists a constant $C^{(3)} > 0$ such that

$$(f+g)^{3p-2} \geq C^{(3)} \cdot \max_{x \in B_{T^*+R^*}} \{|\partial_x^3 f(x)| + |\partial_x^3 g(x)|\}.$$

We now state the main results of this paper.

Theorem 3.1.1. *Let T^* and R^* be arbitrary positive numbers. Assume that (A1)-(A5.3) hold true. Then, there exists a unique $C^1(B_{R^*})$ function T such that $0 < T(x) < T^*$ $(x \in B_{R^*})$ and a unique $(C^{3,1}(\Omega))^2$ solution (ϕ, ψ) of (3.7) satisfying*

$$\phi(t,x),\ \psi(t,x) \to \infty \quad as \quad t \to T(x) \tag{3.9}$$

for any $x \in B_{R^*}$, where $\Omega = \left\{(t,x) \in \mathbb{R}^2 \mid x \in B_{R^*},\ 0 < t < T(x)\right\}$.

Remark 3.1.2. The equation (3.1) is equivalent to (3.7). We set

$$u(t,x) = u_0(x) + \frac{1}{2}\int_0^t (\phi + \psi)(s,x)ds.$$

Then, u satisfies (3.1).

Remark 3.1.3. The assertion (3.9) implies that $u_t(t,x) \to \infty$ as $t \to T(x)$ $(x \in B_{R^*})$.

Next, we will mention numerical analysis of blow-up of nonlinear partial differential equations. There are many previous works of computation of blow-up solutions of various partial differential equations; See, for example, [28], [13], [10], [34], [31], [11] and [12].

We computed blow-up curve using the method of Cho [12] and obtained the various numerical results of blow-up curves. We will show them in Section 3.7.

The remainder of this paper is organized as follows. In Section 3.2, we construct a classical solution for (3.7) in the domain Ω. In Section 3.3, we give the blow-up rates of the solutions of (3.7). Moreover, we show that the blow-up curve is Lipschitz continuous. In the course of Sections 3.4–3.6, we prove that the blow-up curve is continuously differentiable. In Section 3.7, we show some numerical examples of blow-up curves.

3.2 Existence and regularity of solutions

In this section, we will demonstrate the existence and regularity of the solutions ϕ and ψ of (3.7) by successive approximation. Let us define $\{\phi_n\}$ and $\{\psi_n\}$ by $\phi_0 \equiv \gamma_1$, $\psi_0 \equiv \gamma_2$, and

$$\begin{cases} D_-\phi_{n+1} = 2^{-p}|\phi_n + \psi_n|^p, & (t,x) \in K_{T^*,R^*}, \\ D_+\psi_{n+1} = 2^{-p}|\phi_n + \psi_n|^p, & (t,x) \in K_{T^*,R^*}, \\ \phi_{n+1}(x,0) = f(x), \quad \psi_{n+1}(x,0) = g(x), & x \in B_{T^*+R^*}, \end{cases} \tag{3.10}$$

for $n \in \mathbb{N} \cup \{0\}$. Here, γ_1 and γ_2 are initial values of (3.8). We note that (3.10) can be rewritten as

$$\begin{cases} \phi_{n+1}(t,x) = f(x+t) + \displaystyle\int_0^t 2^{-p}|\phi_n + \psi_n|^p(s, x+(t-s))ds, \\ \psi_{n+1}(t,x) = g(x-t) + \displaystyle\int_0^t 2^{-p}|\phi_n + \psi_n|^p(s, x-(t-s))ds. \end{cases} \tag{3.11}$$

Remark 3.2.1. Consider a function $F \in C^1(K_{T^*,R^*})$. We note that it follows from (3.10) and (3.11) that $F(t,x) \geq 0$ in K_{T^*,R^*} if

$$F(0,x) \geq 0 \quad \text{in } B_{T^*+R^*}, \quad \text{and} \quad \begin{cases} D_-F(t,x) \geq 0 \\ \quad\text{or} \\ D_+F(t,x) \geq 0 \end{cases} \quad \text{in } K_{T^*,R^*}.$$

3.2.1 Lemmas

Now, we introduce two important lemmas.

Lemma 3.2.2. *Assume that (A1) hold. Then, we have*

$$\begin{aligned} \phi_{n+1} \geq \phi_n \geq 0, \\ \psi_{n+1} \geq \psi_n \geq 0, \end{aligned} \quad \text{in } K_{T^*,R^*}, \tag{3.12}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. First, it follows from (A1) that

$$\phi_1(t,x) = f(x+t) + \int_0^t 2^{-p}|\phi_0 + \psi_0|^p(s, x+(t-s))ds \geq \gamma_1 = \phi_0(t,x) \geq 0$$

in K_{T^*,R^*}. Similarly, we have that $\psi_1 \geq \psi_0 \geq 0$ in K_{T^*,R^*}.

Next, we assume that

$$\phi_n \geq \phi_{n-1} \geq 0 \quad \text{and} \quad \psi_n \geq \psi_{n-1} \geq 0 \quad \text{in } K_{T^*,R^*}.$$

Then, we have

$$\begin{aligned} \phi_{n+1}(t,x) &= f(x+t) + \int_0^t 2^{-p}|\phi_n + \psi_n|^p(s, x+(t-s))ds \\ &\geq f(x+t) + \int_0^t 2^{-p}|\phi_{n-1} + \psi_{n-1}|^p(s, x+(t-s))ds \\ &= \phi_n(t,x) \geq 0 \end{aligned}$$

in K_{T^*,R^*}. Similarly, we have that $\psi_{n+1} \geq \psi_n \geq 0$ in K_{T^*,R^*}. □

Lemma 3.2.3. *Assume that (A1)–(A3) hold. Then, we have*

$$\begin{aligned} \partial_t\phi_n \geq (1+\varepsilon_0)|\partial_x\phi_n|, \\ \partial_t\psi_n \geq (1+\varepsilon_0)|\partial_x\psi_n|, \end{aligned} \quad \text{in } K_{T^*,R^*}, \tag{3.13}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. Set $\lambda = 1 + \varepsilon_0$, and

$$\begin{aligned} J_n &= \partial_t \phi_n + \lambda \partial_x \phi_n, \quad \tilde{J}_n = \partial_t \phi_n - \lambda \partial_x \phi_n, \\ L_n &= \partial_t \psi_n + \lambda \partial_x \psi_n, \quad \tilde{L}_n = \partial_t \psi_n - \lambda \partial_x \psi_n, \end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. Then, it suffices to show that J_n, $\tilde{J}_n$, L_n and $\tilde{L}_n$ are nonnegative for $n \in \mathbb{N} \cup \{0\}$, in K_{T^*,R^*}. We note that $J_0 = \tilde{J}_0 = L_0 = \tilde{L}_0 = 0$ in K_{T^*,R^*}.

First, it follows from (A3) that

$$\begin{aligned} J_1(0,x) &= \partial_t \phi_1(0,x) + \lambda \partial_x \phi_1(0,x) \\ &= (1+\lambda)\partial_x \phi_1(0,x) + 2^{-p}|\phi_0 + \psi_0|^p(0,x) \\ &= (2+\varepsilon_0) f_x + 2^{-p}(\gamma_1 + \gamma_2)^p \geq 0 \end{aligned}$$

in $B_{T^*+R^*}$, and we see that

$$\begin{aligned} D_- J_1 &= \partial_t D_- \phi_1 + \lambda \partial_x D_- \phi_1 \\ &= \partial_t 2^{-p}|\phi_0 + \psi_0|^p + \lambda \partial_x 2^{-p}|\phi_0 + \psi_0|^p \\ &= \partial_t 2^{-p}(\gamma_1 + \gamma_2)^p + \lambda \partial_x 2^{-p}(\gamma_1 + \gamma_2)^p = 0 \end{aligned}$$

in K_{T^*,R^*}. Then, we have that $J_1 \geq 0$ in K_{T^*,R^*}. Similarly, we have that $\tilde{J}_1 \geq 0$, $L_1 \geq 0$ and $\tilde{L}_1 \geq 0$ in K_{T^*,R^*}.

Next, we assume that

$$J_n \geq 0, \quad L_n \geq 0 \quad \text{in } K_{T^*,R^*}.$$

Then, it follows from (A3) that

$$\begin{aligned} J_{n+1}(0,x) &= \partial_t \phi_{n+1}(0,x) + \lambda \partial_x \phi_{n+1}(0,x) \\ &= (1+\lambda)\partial_x \phi_{n+1}(0,x) + 2^{-p}|\phi_n(0,x) + \psi_n(0,x)|^p \\ &\geq (2+\varepsilon_0) f_x + 2^{-p}(\gamma_1 + \gamma_2)^p \geq 0 \quad \text{in } B_{T^*+R^*}. \end{aligned}$$

Furthermore, it follows from Lemma 3.2.2 that

$$\begin{aligned} D_- J_{n+1} &= \partial_t(\partial_t \phi_{n+1} + \lambda \partial_x \phi_{n+1}) - \partial_x(\partial_t \phi_{n+1} + \lambda \partial_x \phi_{n+1}) \\ &= \partial_t(\partial_t \phi_{n+1} - \partial_x \phi_{n+1}) + \lambda \partial_x(\partial_t \phi_{n+1} - \partial_x \phi_{n+1}) \\ &= (\partial_t + \lambda \partial_x) 2^{-p}|\phi_n + \psi_n|^p \\ &= (\partial_t + \lambda \partial_x) 2^{-p}(\phi_n + \psi_n)^p \\ &= 2^{-p} p(\phi_n + \psi_n)^{p-1}(J_n + L_n) \geq 0 \quad \text{in } K_{T^*,R^*}. \end{aligned}$$

Therefore, we obtain $J_{n+1} \geq 0$ in K_{T^*,R^*}. Similarly, we obtain that $L_{n+1} \geq$ in K_{T^*,R^*}. In the same way of above, we can show that

$$\tilde{J}_{n+1} \geq 0, \quad \tilde{L}_{n+1} \geq 0 \quad \text{in } K_{T^*,R^*}$$

if we assume that $\tilde{J}_n \geq 0$, $\tilde{L}_n \geq 0$ in K_{T^*,R^*}. Therefore, we have obtained that $J_n, \tilde{J}_n, L_n, \tilde{L}_n \geq 0$ for $n \in \mathbb{N} \cup \{0\}$, in K_{T^*,R^*}. This completes the proof. □

3.2.2 Proof of existence and regularity of ϕ and ψ

Fix $(t,x) \in K_{T^*,R^*}$. Since $\{\phi_n(t,x)\}$ and $\{\psi_n(t,x)\}$ are increasing sequences on n, we have

$$\lim_{n\to\infty} \phi_n(t,x) = \sup_{n\in\mathbb{N}} \phi_n(t,x) \quad \text{and} \quad \lim_{n\to\infty} \psi_n(t,x) = \sup_{n\in\mathbb{N}} \psi_n(t,x). \tag{3.14}$$

We set

$$\phi(t,x) = \sup_{n\in\mathbb{N}} \phi_n(t,x) \quad \text{and} \quad \psi(t,x) = \sup_{n\in\mathbb{N}} \psi_n(t,x).$$

It follows from Lemma 3.2.3 that ϕ and ψ are monotone increasing on t. Hence, there exists a function $T(x)$ such that

$$T(x) = \sup\{t \in (0,T^*) \mid (\phi+\psi)(t,x) < \infty\} \quad \text{for} \quad x \in B_{R^*}$$

and

$$\lim_{t\uparrow T(x)} (\phi+\psi)(t,x) \to \infty \quad \text{for} \quad x \in B_{R^*}$$

if $T(x) < T^*$. We set $\Omega = \{(t,x) \mid x \in B_{R^*},\ 0 < t < T(x)\}$.

Remark 3.2.4. We will show that T is actually a blow-up curve of ϕ and ψ in Section 3.3.

We state the following local existence lemma.

Lemma 3.2.5. *Assume that (A1)–(A3) hold. Then, (ϕ,ψ) is a unique $(C^{3,1}(\Omega))^2$ solution of* (3.7)*.*

Proof. We set

$$B(t) = \left\{x \in B_{T^*+R^*} \mid |x-\tilde{x}| \le \tilde{t} - t\right\} \quad \text{for} \quad (\tilde{t},\tilde{x}) \in \Omega.$$

(Proof of regularity.)
First, we will show that (ϕ,ψ) is a $(C^{3,1}(\Omega))^2$ solution of (3.7). We split the proof into 2 steps.

(Step 1.) Fix $(\tilde{t},\tilde{x}) \in \Omega$. We will show that there exists a positive constant M_0 such that

$$\|\phi+\psi\|_{L^\infty(B(t)))} \le M_0 \quad \text{for} \quad t \in [0,\tilde{t}] \tag{3.15}$$

by showing a contradiction.

We set

$$Y_x = \left\{x \in B_{R^*} \mid |x-\tilde{x}| \le \tilde{t} - T(x)\right\}$$

and m is the 1-dimensional Lebesgue measure.

We assume that (3.15) does not hold. Then, there exists $t' \in (0,\tilde{t})$ such that there exist a', b' satisfying $a' < b'$ and

$$(a',b') \subset B(t') \quad \text{and} \quad (t',x) \notin \Omega \quad \text{for} \quad x \in (a',b').$$

By the monotonicity of $\phi + \psi$ on t, we have $T(x) \leq t'$ for $x \in (a', b')$, which implies $(a', b') \in Y_x$. Hence, we have $m(Y_x) > 0$.

It follows from the monotonicity of $\phi + \psi$ on t that

$$(t, x) \notin \Omega \qquad \text{if} \quad x \in Y_x \quad \text{and} \quad (t = x + \tilde{t} - \tilde{x} \quad \text{or} \quad t = -x + \tilde{t} + \tilde{x}).$$

Moreover, we have $m(Y_{\tilde{t},+}) > 0$ or $m(Y_{\tilde{t},-}) > 0$ if $m(Y_x) > 0$. Here,

$$\begin{aligned} Y_{\tilde{t},-} &= \left\{ s \in (0, \tilde{t}) \mid s = -x + \tilde{t} + \tilde{x}, \quad x \in Y_x \right\}, \\ Y_{\tilde{t},+} &= \left\{ s \in (0, \tilde{t}) \mid s = x + \tilde{t} - \tilde{x}, \quad x \in Y_x \right\}. \end{aligned}$$

Then, we have

$$\begin{aligned} \infty >& (\phi_{n+1} + \psi_{n+1})(\tilde{t}, \tilde{x}) \\ \geq& \int_{Y_{\tilde{t},-}} 2^{-p} |\phi_n + \psi_n|^p (s, \tilde{x} + \tilde{t} - s) ds + \int_{Y_{\tilde{t},+}} 2^{-p} |\phi_n + \psi_n|^p (s, \tilde{x} - \tilde{t} + s) ds \\ &\to \infty, \qquad \text{as} \quad n \to \infty. \end{aligned}$$

It is a contradiction. Therefore, we obtain (3.15).

(Step 2.) We will show $(\phi, \psi) \in (C^{3,1}(\Omega))^2$. Fix $(\tilde{t}, \tilde{x}) \in \Omega$. It suffices to show

$$\phi, \psi \in C^{3,1}(K_-(\tilde{t}, \tilde{x})).$$

By **(Step 1.)**, we have that there exists a positive constant C_0 depending only on $\tilde{t}$ and $\tilde{x}$ such that

$$\|\phi_n + \psi_n\|_{L^\infty(B(t)))} \leq C_0 \qquad \text{for} \quad t \in [0, \tilde{t}] \quad \text{and} \quad n \in \mathbb{N}. \tag{3.16}$$

Then, we have

$$\begin{aligned} &\|\phi_{n+1}(t, \cdot\,) - \phi_n(t, \cdot\,)\|_{L^\infty(B(t))} + \|\psi_{n+1}(t, \cdot\,) - \psi_n(t, \cdot\,)\|_{L^\infty(B(t))} \\ &\leq \int_0^t 2^{-p+1} \Big\| |\phi_n + \psi_n|^p (s_1, \cdot\,) - |\phi_{n-1} + \psi_{n-1}|^p (s_1, \cdot\,) \Big\|_{L^\infty(B(s_1))} ds_1 \end{aligned}$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N}$. By (3.16), we have that

$$\begin{aligned} &\|\phi_{n+1}(t, \cdot\,) - \phi_n(t, \cdot\,)\|_{L^\infty(B(t))} + \|\psi_{n+1}(t, \cdot\,) - \psi_n(t, \cdot\,)\|_{L^\infty(B(t))} \\ &\leq pC_0^{p-1} \int_0^t \Big(\big\|\phi_n(s_1, \cdot\,) - \phi_{n-1}(s_1, \cdot\,)\big\|_{L^\infty(B(s_1))} \\ &\qquad\qquad\qquad\qquad + \big\|\psi_n(s_1, \cdot\,) - \psi_{n-1}(s_1, \cdot)\big\|_{L^\infty(B(s_1))} \Big) ds_1 \\ &\leq \left(pC_0^{p-1}\right)^2 \int_0^t \int_0^{s_1} \Big(\big\|\phi_{n-1}(s_2, \cdot\,) - \phi_{n-2}(s_2, \cdot\,)\big\|_{L^\infty(B(s_2))} \\ &\qquad\qquad\qquad\qquad + \big\|\psi_{n-1}(s_2, \cdot\,) - \psi_{n-2}(s_2, \cdot\,)\big\|_{L^\infty(B(s_2))} \Big) ds_2 ds_1. \end{aligned}$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N}$. Repeating this argument, we obtain that

$$
\begin{aligned}
&\|\phi_{n+1}(t,\cdot) - \phi_n(t,\cdot)\|_{L^\infty(B(t))} + \|\psi_{n+1}(t,\cdot) - \psi_n(t,\cdot)\|_{L^\infty(B(t))} \\
&\vdots \\
&\leq (pC_0^{p-1})^n \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n-1}} \\
&\quad \Big(\big\|\phi_1(s_n,\cdot) - \phi_0(s_n,\cdot)\big\|_{L^\infty(B(s_n))} + \big\|\psi_1(s_n,\cdot) - \psi_0(s_n,\cdot)\big\|_{L^\infty(B(s_n))} \Big) ds_n \ldots ds_2 ds_1. \\
&\leq 4C_0 \frac{(pC_0^{p-1}T)^n}{n!} \to 0 \qquad \text{as } n \to \infty,
\end{aligned}
$$

for $t \in [0, \tilde{t}]$. Hence, it follows from (3.14) that

$$\|\phi_n - \phi\|_{L^\infty(K_-(\tilde{t},\tilde{x}))} + \|\psi_n - \psi\|_{L^\infty(K_-(\tilde{t},\tilde{x}))} \to 0 \quad \text{as } n \to \infty.$$

Next, we will show that $\phi, \psi \in W^{1,\infty}(K_-(\tilde{t},\tilde{x}))$. We see that

$$
\begin{cases}
D_- D_\theta \phi_{n+1} = D_\theta 2^{-p}(\phi_n + \psi_n)^p = p2^{-p}(\phi_n + \psi_n)^{p-1}(D_\theta \phi_n + D_\theta \psi_n), \\
D_+ D_\theta \psi_{n+1} = D_\theta 2^{-p}(\phi_n + \psi_n)^p = p2^{-p}(\phi_n + \psi_n)^{p-1}(D_\theta \phi_n + D_\theta \psi_n), \\
\begin{cases}
D_\theta \phi_1(0,x) = (\cos\theta + \sin\theta) f_x(x) + \sin\theta \cdot 2^{-p}(\gamma_1 + \gamma_2)^p, \\
D_\theta \psi_1(0,x) = (\cos\theta - \sin\theta) g_x(x) + \sin\theta \cdot 2^{-p}(\gamma_1 + \gamma_2)^p,
\end{cases} \\
\begin{cases}
D_\theta \phi_{n+1}(0,x) = (\cos\theta + \sin\theta) f_x(x) + \sin\theta \cdot 2^{-p}(f+g)^p(x), \\
D_\theta \psi_{n+1}(0,x) = (\cos\theta - \sin\theta) g_x(x) + \sin\theta \cdot 2^{-p}(f+g)^p(x),
\end{cases} \quad (n \in \mathbb{N})
\end{cases}
$$

for $n \in \mathbb{N} \cup \{0\}$. Here, $D_\theta v = \sin\theta v_t + \cos\theta v_x$.

We set $W(t) = C_0^p \exp(pC_0^{p-1}t)$. Then, we have

$$W(t) = C_0^p + \int_0^t pC_0^{p-1} W(s) ds.$$

We will show

$$\|D_\theta \phi_n(t,\cdot)\|_{L^\infty(B(t))} \leq W(t), \quad \|D_\theta \psi_n(t,\cdot)\|_{L^\infty(B(t))} \leq W(t) \tag{3.17}$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N} \cup \{0\}$.

We see

$$D_\theta \phi_0 = D_\theta \psi_0 = 0 \leq W(t)$$

for $t \geq 0$. Assume that (3.17) holds for n. Then, we have

$$\big\|p2^{-p}(\phi_n + \psi_n)^{p-1}(t,\cdot)(D_\theta \phi_n + D_\theta \psi_n)(t,\cdot)\big\|_{L^\infty(B(t))} \leq pC_0^{p-1} W(t) \tag{3.18}$$

for $t \in [0, \tilde{t}]$. It follows that (A3) that

$$
\begin{aligned}
&\|D_\theta \phi_{n+1}(t,\cdot)\|_{L^\infty(B(t))} \\
&\leq 2\|f_x\|_{L^\infty(B(0))} + 2^{-p}\|f+g\|^p_{L^\infty(B(0))} \\
&\qquad + \int_0^t \big\|p2^{-p}(\phi_n + \psi_n)^{p-1}(s,\cdot)(D_\theta \phi_n + D_\theta \psi_n)(s,\cdot)\big\|_{L^\infty(B(s))} ds \\
&\leq C_0^p + \int_0^t pC_0^{p-1} W(s) ds = W(t) \quad \text{for} \quad t \in [0, \tilde{t}].
\end{aligned} \tag{3.19}
$$

Similarly, we have that $\|D_\theta\psi_{n+1}(t,\cdot)\|_{L^\infty(B(t))} \le W(t)$ for $t \in [0,\tilde{t}]$. Thus,

$$\|D_\theta\phi_n(t,\cdot)\|_{L^\infty(B(t))} \le W(t), \quad \|D_\theta\psi_n(t,\cdot)\|_{L^\infty(B(t))} \le W(t)$$

for $t \in [0,\tilde{t}]$ and $n \in \mathbb{N}\cup\{0\}$. We set $C_1 = C_0^p \exp(pC_0^{p-1}T)$. Then, we have

$$\|D_\theta\phi_n(t,\cdot)\|_{L^\infty(B(t))} \le C_1 \quad \text{and} \quad \|D_\theta\psi_n(t,\cdot)\|_{L^\infty(B(t))} \le C_1 \tag{3.20}$$

for $t \in [0,\tilde{t}]$ and $n \in \mathbb{N}\cup\{0\}$.

We see that

$$\begin{aligned}
&\|D_\theta\phi_{n+1}(t,\cdot) - D_\theta\phi_n(t,\cdot)\|_{L^\infty(B(t))} + \|D_\theta\psi_{n+1}(t,\cdot) - D_\theta\psi_n(t,\cdot)\|_{L^\infty(B(t))} \\
&\le \int_0^t p2^{-p+1}\Big\|\Big[(\phi_n+\psi_n)^{p-1}(D_\theta\phi_n + D_\theta\psi_n) \\
&\qquad\qquad - (\phi_{n-1}+\psi_{n-1})^{p-1}(D_\theta\phi_{n-1} + D_\theta\psi_{n-1})\Big](s,\cdot)\Big\|_{L^\infty(B(s))} ds
\end{aligned}$$

It follows from (3.16) and (3.20) that

$$\begin{aligned}
&\|D_\theta\phi_{n+1}(t,\cdot) - D_\theta\phi_n(t,\cdot)\|_{L^\infty(B(t))} + \|D_\theta\psi_{n+1}(t,\cdot) - D_\theta\psi_n(t,\cdot)\|_{L^\infty(B(t))} \\
&\le \int_0^t pC_0^{p-1}\Big(\|D_\theta\phi_n(s_1,\cdot) - D_\theta\phi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_1))} \\
&\qquad\qquad + \|D_\theta\psi_n(s_1,\cdot) - D_\theta\psi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_1))}\Big)ds_1 \\
&\quad + \int_0^t 2p(p-1)C_1C_0^{p-2}\Big(\|\phi_n(s_1,\cdot) - \phi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_1))} \\
&\qquad\qquad + \|\psi_n(s_1,\cdot) - \psi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_1))}\Big)ds_1 \\
&\quad \le \left(pC_0^{p-1}\right)^2 \int_0^t\int_0^{s_1}\Big(\|D_\theta\phi_{n-1}(s_2,\cdot) - D_\theta\phi_{n-2}(s_2,\cdot)\|_{L^\infty(B(s_2))} \\
&\qquad\qquad + \|D_\theta\psi_{n-1}(s_2,\cdot) - D_\theta\psi_{n-2}(s_2,\cdot)\|_{L^\infty(B(s_2))}\Big)ds_2ds_1 \\
&\qquad + C_2^2\int_0^t\int_0^{s_1}\Big(\|\phi_{n-1}(s_2,\cdot) - \phi_{n-2}(s_2,\cdot)\|_{L^\infty(B(s_2))} \\
&\qquad\qquad + \|\psi_{n-1}(s_2,\cdot) - \psi_{n-2}(s_2,\cdot)\|_{L^\infty(B(s_2))}\Big)ds_2ds_1 \\
&\qquad + C_2\int_0^t\Big(\|\phi_n(s_1,\cdot) - \phi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_2))} \\
&\qquad\qquad + \|\psi_n(s_1,\cdot) - \psi_{n-1}(s_1,\cdot)\|_{L^\infty(B(s_2))}\Big)ds_1 \\
&\vdots \\
&\le \left(pC_0^{p-1}\right)^n\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_{n-1}} \\
&\quad \Big(\|D_\theta\phi_1(s_n,\cdot) - D_\theta\phi_0(s_n,\cdot)\|_{L^\infty(B(s_n))} \\
&\qquad\qquad + \|D_\theta\psi_1(s_n,\cdot) - D_\theta\psi_0(s_n,\cdot)\|_{L^\infty(B(s_n))}\Big)ds_1ds_2\dots ds_n \\
&\quad + \sum_{j=1}^n 4C_0\frac{T^j}{j!}\cdot\frac{(C_2T)^{n-j}}{(n-j)!}
\end{aligned}$$

$$\leq 4C_1 \frac{(pC_0^{p-1}T)^n}{n!} + \sum_{j=1}^{n} 4C_0 \frac{(C_2T)^n}{j!(n-j)!} \to 0 \qquad \text{as } n \to \infty$$

for $t \in [0, \tilde{t}]$. Here, $C_2 = \max\{pC_0^{p-1}, 2p(p-1)C_1C_0^{p-2}\}$. Thus, there exist $\phi_\theta^{(1)}, \psi_\theta^{(1)} \in L^\infty(K_-(\tilde{t}, \tilde{x}))$ such that

$$\|D_\theta \phi_n - \phi_\theta^{(1)}\|_{L^\infty(K_-(\tilde{t},\tilde{x}))} + \|D_\theta \psi_n - \psi_\theta^{(1)}\|_{L^\infty(K_-(\tilde{t},\tilde{x}))} \to 0 \qquad \text{as } n \to \infty.$$

Therefore, $(\phi, \psi) \in (W^{1,\infty}(K_-(\tilde{t}, \tilde{x})))^2$. By repeating the same arguments, we obtain that $(\phi, \psi) \in (W^{4,\infty}(K_-(\tilde{t}, \tilde{x})))^2$. That is, we have $(\phi, \psi) \in (C^{3,1}(K_-(\tilde{t}, \tilde{x})))^2$.

(Proof of uniqueness.)
Next, we will show that (ϕ, ψ) is a unique solution of (3.7). We suppose (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are solutions of (3.7) and T_1 and T_2 are corresponding blow-up curves. Let

$$\Omega_j = \{(t, x) \mid x \in B_{R^*},\ 0 < t < T_j(x)\} \quad \text{for} \quad j = 1, 2.$$

Take $(\tilde{t}, \tilde{x}) \in \Omega_1 \cap \Omega_2$ arbitrarily. In the same way of proof of **(Step 2.)**, we have

$$\begin{aligned}
&\sup_{0 \leq t' \leq t} \Big(\|\phi_1(t', \cdot) - \phi_2(t', \cdot)\|_{L^\infty(B(t'))} + \|\psi_1(t', \cdot) - \psi_2(t', \cdot)\|_{L^\infty(B(t'))} \Big) \\
&\leq \sup_{0 \leq t' \leq t} \Big(\int_0^{t'} 2^{-p+1} \big\| |\phi_1 + \psi_1|^p(s, \cdot) - |\phi_2 + \psi_2|^p(s, \cdot) \big\|_{L^\infty(B(s))} ds \Big) \\
&\leq tpC_0^{p-1} \sup_{0 \leq t' \leq t} \Big(\|\phi_1(t', \cdot) - \phi_2(t', \cdot)\|_{L^\infty(B(t'))} \\
&\qquad\qquad\qquad + \|\psi_1(t', \cdot) - \psi_2(t', \cdot)\|_{L^\infty(B(t'))} \Big)
\end{aligned}$$

for t satisfying $0 \leq t \leq \tilde{t}$. Thus,

$$\sup_{0 \leq t' \leq t} \Big(\|\phi_1(t', \cdot) - \phi_2(t', \cdot)\|_{L^\infty(B(t'))} + \|\psi_1(t', \cdot) - \psi_2(t,' \cdot)\|_{L^\infty(B(t'))} \Big) = 0$$

if t is small enough. Since C_0 does not depend on t, by repeating this argument, we obtain

$$\sup_{0 \leq t' \leq \tilde{t}} \Big(\|\phi_1(t', \cdot) - \phi_2(t', \cdot)\|_{L^\infty(B(t'))} + \|\psi_1(t', \cdot) - \psi_2(t', \cdot)\|_{L^\infty(B(t'))} \Big) = 0.$$

Therefore, we have

$$(\phi_1, \psi_1) = (\phi_2, \psi_2) \quad \text{in} \quad \Omega_1 \cap \Omega_2$$

and

$$T_1(x) = T_2(x) \quad \text{for} \quad x \in B_{R^*}.$$

This completes the proof. □

Lemma 3.2.6. *Assume that (A1)–(A4) hold. Then, we have*

$$T(x) < T^* \quad \textit{for } x \in B_{R^*}.$$

Proof. Let us define $\{\tilde{\phi}_n\}$ and $\{\tilde{\psi}_n\}$ by $\tilde{\phi}_0 = \gamma_1$, $\tilde{\psi}_0 = \gamma_2$ and

$$\begin{cases} \dfrac{d}{dt}\tilde{\phi}_{n+1} = 2^{-p}|\tilde{\phi}_n + \tilde{\psi}_n|^p, \quad t > 0, \\ \dfrac{d}{dt}\tilde{\psi}_{n+1} = 2^{-p}|\tilde{\phi}_n + \tilde{\psi}_n|^p, \quad t > 0, \\ \tilde{\phi}_{n+1}(0) = \gamma_1, \quad \tilde{\psi}_{n+1}(0) = \gamma_2. \end{cases}$$

It suffices to show that $\phi_n(t,x) \geq \tilde{\phi}_n(t)$ and $\psi_n(t,x) \geq \tilde{\psi}_n(t)$ in K_{T^*,R^*}, for $n \in \mathbb{N}$. First, we see that

$$\begin{aligned} \phi_1(t,x) - \tilde{\phi}_1(t) &= f(x+t) - \gamma_1 + \int_0^t 2^{-p}|\phi_0 + \psi_0|^p(s, x+(t-s))ds \\ &\quad - \int_0^t 2^{-p}|\tilde{\phi}_0 + \tilde{\psi}_0|^p(s)ds \\ &= f(x+t) - \gamma_1 \geq 0, \end{aligned}$$

in K_{T^*,R^*}. Similarly, we have that $\psi_1(t,x) \geq \tilde{\psi}_1(t)$ in K_{T^*,R^*}.

Next, we assume that $\phi_n(t,x) \geq \tilde{\phi}_n(t)$ and $\psi_n(t,x) \geq \tilde{\psi}_n(t)$ in K_{T^*,R^*}. Then, we have that

$$\begin{aligned} \phi_{n+1}(t,x) - \tilde{\phi}_{n+1}(t) &= f(x+t) - \gamma_1 + \int_0^t 2^{-p}|\phi_n + \psi_n|^p(s, x+(t-s))ds \\ &\quad - \int_0^t 2^{-p}|\tilde{\phi}_n + \tilde{\psi}_n|^p(s)ds \\ &\geq 0, \end{aligned}$$

in K_{T^*,R^*}. Similarly, we obtain that $\psi_{n+1}(t,x) \geq \tilde{\psi}_{n+1}(t)$ in K_{T^*,R^*}. Therefore, we have

$$\phi_n(t,x) \geq \tilde{\phi}_n(t), \qquad \psi_n(t,x) \geq \tilde{\psi}_n(t) \quad \text{in } K_{T^*,R^*}$$

for $n \in \mathbb{N}$.

This completes the proof. □

3.3 Blow-up rates of solutions and Lipschitz continuity of T

Now, we will show that T is Lipschitz continuous in B_{R^*}. To prove this fact, we first introduce the following proposition.

Proposition 3.3.1. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending only on p and ε_0 such that*

$$C_1(\phi+\psi)^p \leq \phi_t \leq C_2(\phi+\psi)^p, \tag{3.21}$$

$$C_1(T(x)-t)^{-q-1} \leq \phi_t(t,x) \leq C_2(T(x)-t)^{-q-1}, \tag{3.22}$$

$$C_1(\phi+\psi)^p \leq \psi_t \leq C_2(\phi+\psi)^p, \tag{3.23}$$

$$C_1(T(x)-t)^{-q-1} \leq \psi_t(t,x) \leq C_2(T(x)-t)^{-q-1}, \tag{3.24}$$

$$C_1(T(x)-t)^{-q} \leq (\phi+\psi)(t,x) \leq C_2(T(x)-t)^{-q}, \tag{3.25}$$

in Ω. Here, $q = 1/(p-1)$.

Proof. First, we will show that (3.21) holds. We see that

$$\begin{aligned} D_-\partial_t\phi_{n+1} &= \partial_t D_-\phi_{n+1} = \partial_t 2^{-p}|\phi_n+\psi_n|^p = \partial_t 2^{-p}(\phi_n+\psi_n)^p \\ &= 2^{-p}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \quad \text{in } K_{T^*,R^*}, \end{aligned} \tag{3.26}$$

for $n \in \mathbb{N}\cup\{0\}$. From Lemma 3.2.3, we obtain that

$$\begin{aligned} D_-2^{-p}(\phi_n+\psi_n)^p &= 2^{-p}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n-\partial_x\phi_n+\partial_t\psi_n-\partial_x\psi_n) \\ &\le 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \quad \text{in } K_{T^*,R^*}, \end{aligned} \tag{3.27}$$

for $n \in \mathbb{N}\cup\{0\}$. We set $J_{\phi,n+1} = 2\partial_t\phi_{n+1} - 2^{-p}(\phi_n+\psi_n)^p$. Then, by (3.26) and (3.27), we have

$$\begin{aligned} &D_-J_{\phi,n+1} \\ &\ge 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) - 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \\ &= 0 \quad \text{in } K_{T^*,R^*}, \end{aligned} \tag{3.28}$$

for $n \in \mathbb{N}\cup\{0\}$. It follows from (A3) that

$$\begin{aligned} J_{\phi,n+1}(0,x) &= 2\partial_t\phi_{n+1}(0,x) - 2^{-p}(\phi_n+\psi_n)^p(0,x) \\ &= 2\partial_x\phi_{n+1}(0,x) + 2^{-p}(\phi_n+\psi_n)^p(0,x) \\ &\ge 2f_x + 2^{-p}(\gamma_1+\gamma_2)^p \ge 0 \quad \text{in } B_{T^*+R^*} \end{aligned} \tag{3.29}$$

for $n \in \mathbb{N}\cup\{0\}$. Then, by (3.29) and (3.28), we obtain that $J_{\phi,n} \ge 0$ in K_{T^*,R^*}, for $n \in \mathbb{N}$.

On the other hand, it follows from Lemma 3.2.3 that

$$\partial_t\phi_{n+1} = \partial_x\phi_{n+1} + 2^{-p}(\phi_n+\psi_n)^p \le \frac{1}{1+\varepsilon_0}\partial_t\phi_{n+1} + 2^{-p}(\phi_n+\psi_n)^p$$

in K_{R^*,T^*}, for $n \in \mathbb{N}\cup\{0\}$. Hence,

$$\partial_t\phi_{n+1} \le \frac{1+\varepsilon_0}{\varepsilon_0}2^{-p}(\phi_n+\psi_n)^p \quad \text{in } K_{T^*,R^*}, \tag{3.30}$$

for $n \in \mathbb{N}\cup\{0\}$. It follows from the fact that $J_{\phi,n} \ge 0$ and (3.30) that

$$2^{-p-1}(\phi_n+\psi_n)^p \le \partial_t\phi_{n+1} \le \frac{1+\varepsilon_0}{\varepsilon_0}\cdot 2^{-p}(\phi_{n+1}+\psi_{n+1})^p \text{ in } K_{T^*,R^*}, \tag{3.31}$$

for $n \in \mathbb{N}\cup\{0\}$, which implies (3.21) holds. Similarly, we can prove that (3.23) holds.

Next, we will show that (3.25) holds. By considering (3.21), we see that

$$\frac{\partial(\phi+\psi)}{\partial t} \le 2^{-p+1}(1+\varepsilon_0)\varepsilon_0^{-1}(\phi+\psi)^p \quad \text{in} \quad \Omega.$$

Thus, we have

$$\frac{\partial t}{\partial(\phi+\psi)} \ge 2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0(\phi+\psi)^{-p} \quad \text{in} \quad \Omega. \tag{3.32}$$

Fix $x_0 \in B_{R^*}$. By (3.32), we have

$$T(x_0) - \varepsilon - \tau \geq \int_{(\phi+\psi)(\tau,x_0)}^{(\phi+\psi)(T(x_0)-\varepsilon,x_0)} 2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-p} dz$$
$$= \left[-(p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-(p-1)}\right]_{(\phi+\psi)(\tau,x_0)}^{(\phi+\psi)(T(x_0)-\varepsilon,x_0)}.$$

for $\tau > 0$ and $\varepsilon > 0$ satisfying $T(x_0) - \varepsilon - \tau > 0$. Hence, by letting $\varepsilon \to 0$, we obtain

$$T(x_0) - \tau \geq \left[-(p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-(p-1)}\right]_{(\phi+\psi)(\tau,x_0)}^{\infty}$$
$$= (p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0(\phi+\psi)^{-(p-1)}(\tau, x_0).$$

Thus, we have that

$$(\phi+\psi)(\tau,x_0) \geq 2\left((p-1)\varepsilon_0^{-1}(1+\varepsilon_0)\right)^{-1/(p-1)}(T(x_0)-\tau)^{-1/(p-1)} \tag{3.33}$$

for $\tau \in [0, T(x_0))$. Similarly, we obtain that

$$(2^p(p-1)^{-1})^{1/(p-1)}(T(x_0)-\tau)^{-1/(p-1)} \geq (\phi+\psi)(\tau,x_0) \tag{3.34}$$

for $\tau \in [0, T(x_0))$. It follows from (3.33) and (3.34) that (3.25) holds. Moreover, it follows from (3.21) and (3.25) that (3.22) holds. Similarly, we have that (3.24) also holds. This completes the proof. $\square$

By combining the above Lemma 3.2.3 with Proposition 3.3.1, we obtain that the blow-up curve T is Lipschitz continuous. That is, the following lemma holds.

Lemma 3.3.2. *Suppose that (A1)–(A4) hold. Then, we have that*

$$|T(x') - T(x'')| \leq \frac{1}{1+\varepsilon_0}|x'-x''| \quad \textit{for} \quad x', x'' \in B_{R^*}. \tag{3.35}$$

Proof. This proof is based on the Implicit Function Theorem. Let $\varepsilon > 0$ be arbitrary. By (3.25), we see that there exists a positive constant C_1 depending p and ε_0 such that

$$C_1\varepsilon^{-q} \leq (\phi+\psi)(t,x) \quad \text{for} \quad x \in B_{R^*} \quad \text{and} \quad t \in [T(x)-\varepsilon, T(x)).$$

Thus, there exists a positive constant M satisfying $M \geq C_1\varepsilon^{-q}$, and a function $E(x)$ ($x \in B_{R^*}$) such that

$$(\phi+\psi)(E(x),x) = M \quad \text{and} \quad T(x) - E(x) \leq \varepsilon \quad \text{for } x \in B_{R^*}.$$

First, we will demonstrate continuity of E in B_{R^*}. That is, for $x' \in B_{R^*}$, we will show that $t_n \to E(x')$ if $x_n \to x'$, where $t_n = E(x_n)$.

We take an arbitrary converging subsequence $\{t_{nk}\} \subset \{t_n\}$, and denote its limit by η. Following from the definition of E, we have that $(\phi+\psi)(t_{nk}, x_{nk}) = (\phi+\psi)(E(x_{nk}), x_{nk}) = M$. Thus, it follows from continuity of ϕ and ψ that $(\phi+\psi)(\eta, x') = M$. Since $\partial_t(\phi+\psi) > 0$ in Ω, we have that $\eta = E(x')$. Therefore,

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = E(x').$$

Thus, we have demonstrated the continuity of E at x'.

Next, we will prove Lipschitz continuity of E. We see that there exists a positive constant h' for $x' \in B_{R^*}$ such that

$$\boldsymbol{B}(x', h') \subset \Omega,$$

where $\boldsymbol{B}(x', h') = \left\{(t, x) \mid \sqrt{(t - E(x'))^2 + (x - x')^2} < h'\right\}$. Following from continuity of E, there exits a positive constant h'' such that $0 < h'' \leq h'$ satisfying

$$(E(x_1), x_1), (E(x_2), x_2) \in \boldsymbol{B}(x', h') \quad \text{for } x_1, x_2 \in (x' - h'', x' + h'').$$

Let $k = E(x_2) - E(x_1)$ and

$$H(\xi) = (\phi + \psi)(t + \xi k, x_1 + \xi(x_2 - x_1)),$$

where ξ is a constant satisfying $0 \leq \xi \leq 1$. Then, we have

$$\begin{aligned}
&H(0) = (\phi + \psi)(t, x_1), \\
&H(1) = (\phi + \psi)(t + k, x_2) = (\phi + \psi)(t + E(x_2) - E(x_1), x_2).
\end{aligned}$$

Take t as $t = E(x_1)$. Then, we have $H(0) = H(1) = M$. By Rolle's Theorem, there exists $\xi' \in (0, 1)$ such that

$$\begin{aligned}
H'(\xi') =& (x_2 - x_1)\partial_x(\phi + \psi)(E(x_1) + \xi' k, x_1 + \xi'(x_2 - x_1)) \\
&+ k\partial_t(\phi + \psi)(E(x_1) + \xi' k, x_1 + \xi'(x_2 - x_1)) = 0. \qquad (3.36)
\end{aligned}$$

Hence, it follows from Lemma 3.2.3 and (3.36) that

$$\begin{aligned}
|E(x_1) - E(x_2)| = |k| &= \left| \frac{-\partial_x(\phi + \psi)(E(x_1) + \xi' k, x_1 + \xi'(x_2 - x_1))}{\partial_t(\phi + \psi)(E(x_1) + \xi' k, x_1 + \xi'(x_2 - x_1))} \right| |x_1 - x_2| \\
&\leq \frac{1}{1 + \varepsilon_0} |x_1 - x_2|.
\end{aligned}$$

Thus, E is Lipschitz continuous in $(x' - h'', x' + h'')$. Moreover, it follows from the continuity of E that

$$\begin{aligned}
\frac{E(x + h) - E(x)}{h} &= \frac{-h\partial_x(\phi + \psi)(E(x) + \xi(E(x + h) - E(x)), x + \xi h)}{h\partial_t(\phi + \psi)(E(x) + \xi(E(x + h) - E(x)), x + \xi h)} \\
&\to \frac{-\partial_x(\phi + \psi)(E(x), x)}{\partial_t(\phi + \psi)(E(x), x)} \quad \text{as } h \to 0 \quad \text{for} \quad x \in B_{R^*}.
\end{aligned}$$

Hence, we have that

$$\frac{\partial}{\partial x} E(x) = \frac{-\partial_x(\phi + \psi)(E(x), x)}{\partial_t(\phi + \psi)(E(x), x)} \quad \text{for} \quad x \in B_{R^*}.$$

By continuity of $\partial_x(\phi + \psi)$, $\partial_t(\phi + \psi)$ and E, we see that $E \in C^1(B_{R^*})$. Hence, we have that

$$|E(x') - E(x'')| \leq \left(\sup_{x \in B_{R^*}} |E'(x)| \right) |x' - x''| \leq \frac{1}{1 + \varepsilon_0} |x' - x''| \qquad (3.37)$$

for $x', x'' \in B_{R^*}$. Therefore, E is Lipschitz continuous in B_{R^*}.

Finally, we will prove Lipschitz continuity of T in B_{R^*}. It follows from (3.37) that

$$\begin{aligned} |T(x') - T(x'')| &\le |T(x') - E(x')| + |E(x') - E(x'')| + |E(x'') - T(x'')| \\ &\le 2\varepsilon + \frac{1}{1+\varepsilon_0}|x' - x''| \quad \text{for} \quad x', x'' \in B_{R^*}. \end{aligned}$$

Since we let $\varepsilon > 0$ take an arbitrary value, this completes the proof. $\square$

By applying Lemma 3.3.2, we obtain the following results.

Definition 2. By $d(t,x)$, we denote the distance from a point (t,x) in Ω to $\Gamma = \{(T(x), x) \mid x \in B_{R^*}\}$.

Remark 3.3.3. It follows from Lemma 3.3.2 that

$$\frac{T(x) - t}{\sqrt{2}} \le d(t,x) \le T(x) - t.$$

By replacing $T(x) - t$ by $d(x,t)$ in Proposition 3.3.1, we obtain the following Corollary.

Corollary 3.3.4. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending only on p and ε_0 such that*

$$C_1 d^{-q}(t,x) \le (\phi + \psi)(t,x) \le C_2 d^{-q}(t,x), \tag{3.38}$$

$$C_1 d^{-q-1}(t,x) \le \phi_t(t,x) \le C_2 d^{-q-1}(t,x), \tag{3.39}$$

$$C_1 d^{-q-1}(t,x) \le \psi_t(t,x) \le C_2 d^{-q-1}(t,x), \tag{3.40}$$

where $q = 1/(p-1)$, in Ω.

From Corollary 3.3.4, we obtain the following lemma, which states that T is the blow-up curve of both ϕ and ψ:

Lemma 3.3.5. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending on p and ε_0 such that*

$$C_1 (T(x) - t)^{-q} \le \phi(t,x) \le C_2 (T(x) - t)^{-q}, \tag{3.41}$$

$$C_1 (T(x) - t)^{-q} \le \psi(t,x) \le C_2 (T(x) - t)^{-q}, \tag{3.42}$$

where $q = 1/(p-1)$, in Ω.

Proof. We will only show that (3.41) holds. By Corollary 3.3.4 and Lemma 3.3.2, there exist positive constants c_1 and c_2 depending p and ε_0 such that

$$\begin{aligned} \phi(T(x) - \varepsilon, x) &= f(x + T(x) - \varepsilon) \\ &\quad + \int_0^{T(x)-\varepsilon} 2^{-p} (\phi + \psi)^p (s, x + (T(x) - \varepsilon) - s) ds \\ &\ge \int_{T(x)-2\varepsilon}^{T(x)-\varepsilon} 2^{-p} (\phi + \psi)^p (s, x + (T(x) - \varepsilon) - s) ds \\ &\ge c_1 \varepsilon \inf_{T(x)-2\varepsilon \le s \le T(x)-\varepsilon} d(s, x + (T(x) - \varepsilon) - s)^{-qp} \\ &\ge c_2 \varepsilon \cdot \varepsilon^{-q-1} = c_2 \varepsilon^{-q}. \end{aligned}$$

On the other hand, it follows from Proposition 3.3.1 that there exists a positive constant C_2 depending only on p and ε_0 such that $\phi(T(x)-\varepsilon, x) \leq C_2\varepsilon^{-q}$. This completes the proof. $\square$

3.4 Blow-up limits of solutions

In the following, we will show that $T \in C^1(B_{R^*})$. In order to achieve this, we will consider limits of the scaled functions T_λ, ϕ_λ, and ψ_λ (we will define these later) and their properties.

3.4.1 Estimates of blow-up limits

We set D_θ as

$$D_\theta = \sin\theta\partial_t + \cos\theta\partial_x, \quad \text{where } 0 \leq \theta < 2\pi.$$

First, we introduce the following lemma.

Lemma 3.4.1. *Assume that (A1)–(A5.3) hold. Then, there exist positive constants C_α and C_α^* depending only on p and ε_1 such that*

$$\max\{|D_\theta^\alpha\phi(t,x)|, |D_\theta^\alpha\psi(t,x)|\} \leq C_\alpha(\phi+\psi)^{p+(\alpha-1)/q}(t,x) \tag{3.43}$$
$$\leq C_\alpha^* d(t,x)^{-(pq+(\alpha-1))} \tag{3.44}$$

for $(t,x) \in \Omega$, where $q = 1/(p-1)$ and $\alpha = 0,1,2,3$.

Proof. We can easily obtain that (3.44) holds by Corollary 3.3.4 if we prove (3.43). So, we will only prove (3.43).

We also obtain that (3.43) holds in the case of $\alpha = 0, 1$, by Lemmas 3.2.2, 3.2.3 and Proposition 3.3.1.

First, we will show that (3.43) holds in the case of $\alpha = 2$. It suffices to show that there exists a positive constant C_2 depending only on p and ε_1 such that

$$\begin{aligned}&\max\{|D_\theta^2\phi_n(t,x)|, |D_\theta^2\psi_n(t,x)|\}\\ &\qquad \leq C_2(\phi_n+\psi_n)^{2p-1}(t,x) \quad \text{for } n \in \mathbb{N}\cup\{0\},\end{aligned} \tag{3.45}$$

in K_{T^*,R^*}. We see that $D_\theta\phi_0 = D_\theta\psi_0 = 0$ in K_{T^*,R^*}. Hence, (3.43) holds for $n = 0$.

Assume

$$\max\big\{|D_\theta^2\phi_n(t,x)|, |D_\theta^2\psi_n(t,x)|\big\} \leq C_2(\phi_n+\psi_n)^{2p-1}(t,x) \quad \text{in} \quad K_{T^*,R^*}.$$

Then, it follows from (3.43) in the case $\alpha = 1$ and Proposition 3.3.1 that

$$\begin{aligned}&|D_-(D_\theta^2\phi_{n+1})(t,x)|\\ &= 2^{-p}|D_\theta^2(\phi_n+\psi_n)^p(t,x)|\\ &\leq 2^{-p}p(p-1)(\phi_n+\psi_n)^{p-2}(t,x)(D_\theta\phi_n+D_\theta\psi_n)^2(t,x)\\ &\qquad + 2^{-p}p(\phi_n+\psi_n)^{p-1}(t,x)|(D_\theta^2\phi_n+D_\theta^2\psi_n)(t,x)|\\ &\leq 2^{-p+1}p\Big(2(p-1)C_1^2+C_2\Big)|(\phi_n+\psi_n)^{3p-2}(t,x)| \quad \text{in } K_{T^*,R^*},\end{aligned} \tag{3.46}$$

where C_α is the constant in the case of $\alpha = 1, 2$ of (3.43). Moreover, it follows from Lemma 3.2.3 and Proposition 3.3.1 that

$$\begin{aligned} &D_- C_2(\phi_{n+1} + \psi_{n+1})^{2p-1}(t,x) \\ &= C_2(2p-1)(\phi_{n+1} + \psi_{n+1})^{2p-2}(t,x) D_-(\phi_{n+1} + \psi_{n+1})(t,x) \\ &\geq 2^{-p} C_2(2p-1)\Big(1 + \frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big)(\phi_n + \psi_n)^{3p-2}(t,x) \quad \text{in } K_{T^*,R^*}. \end{aligned} \tag{3.47}$$

Let

$$M_n(t,x) = C_2(\phi_n + \psi_n)^{2p-1}(t,x) - D_\theta^2 \phi_n(t,x).$$

Then, it follows from (A3) and (A5.2) that

$$M_{n+1}(0,x) \geq \Big(C_2 - 4C^{(2)^{-1}} - p2^{-2p+3}\Big)(f+g)^{2p-1}(x), \tag{3.48}$$

in $B_{T^*+R^*}$. On the other hand, it follows from (3.46) and (3.47) that

$$\begin{aligned} D_- M_{n+1}(t,x) \geq 2^{-p} C_2 \Big\{(2p-1)\Big(1 + \frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big) - 2p\Big\}(\phi_n + \psi_n)^{3p-2}(t,x) \\ - 2^{-p} 4p(p-1) C_1^2 (\phi_n + \psi_n)^{3p-2}(t,x) \quad \text{in } K_{T^*,R^*}. \end{aligned} \tag{3.49}$$

By (A5.1), we have

$$(2p-1)\Big(1 + \frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big) - 2p > 0.$$

We take C_2 as

$$\begin{aligned} C_2 > \max \Big\{ & 4C^{(2)^{-1}} + p2^{-2p+3}, \\ & \Big\{(2p-1)\Big(1 + \frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big) - 2p\Big\}^{-1} 4p(p-1)C_1^2 \Big\}. \end{aligned}$$

Then, it follows from (3.48) and (3.49) that $M_{n+1} \geq 0$ in K_{T^*,R^*}. Consequently, we obtain that $M_n \geq 0$ in K_{T^*,R^*}, for $n \in \mathbb{N} \cup \{0\}$. That is, there exists a positive constant C_2 depending p and ε_1 such that

$$C_2(\phi_n + \psi_n)^{2p-1} \geq D_\theta^2 \phi_n \quad \text{in } K_{T^*,R^*}$$

for $n \in \mathbb{N} \cup \{0\}$. Similarly, we have the following inequality by retaking C_2 if necessary.

$$\left\{ \begin{array}{ll} C_2(\phi_n + \psi_n)^{2p-1} \geq -D_\theta^2 \phi_n, & \\ C_2(\phi_n + \psi_n)^{2p-1} \geq D_\theta^2 \psi_n, & \text{in } K_{T^*,R^*}, \\ C_2(\phi_n + \psi_n)^{2p-1} \geq -D_\theta^2 \psi_n, & \end{array} \right.$$

for $n \in \mathbb{N} \cup \{0\}$. This means (3.45) holds. In the same way, we can prove (3.43) in the case of $\alpha = 3$. □

Let $x_0 \in B_{R^*}$. Then, we introduce the following scaled functions:

$$\phi_\lambda(s,y) = \lambda^q \phi(T(x_0) + \lambda s, x_0 + \lambda y), \tag{3.50}$$

$$\psi_\lambda(s,y) = \lambda^q \psi(T(x_0) + \lambda s, x_0 + \lambda y), \tag{3.51}$$

where $\lambda > 0$ and $q = 1/(p-1)$. Any sequences $\{\phi_{\lambda_n}\}$ and $\{\psi_{\lambda_n}\}$ with $\lambda_n \downarrow 0$ are called blow-up sequences (see. [7]). Now, we see that

$$\begin{cases} D_-\phi_\lambda = 2^{-p}(\phi_\lambda + \psi_\lambda)^p, \\ D_+\psi_\lambda = 2^{-p}(\phi_\lambda + \psi_\lambda)^p \end{cases} \tag{3.52}$$

for $(s, y) \in \Omega_\lambda$, where $\Omega_\lambda = \big\{(s, y) \in \mathbb{R}^2 \mid (T(x_0) + \lambda s, x_0 + \lambda y) \in \Omega\big\}$. By $d_\lambda(s, y)$, we denote the distance from a point $(s, y) \in \Omega_\lambda$ to $\Gamma_\lambda = \{(s, y) \mid s = T_\lambda(y)\}$. Here, T_λ is a blow-up curve of ϕ_λ.

Lemma 3.4.2. *For each fixed $\lambda > 0$,*

$$T_\lambda(y) = \frac{T(x_0 + \lambda y) - T(x_0)}{\lambda}. \tag{3.53}$$

Proof. By Lemma 3.3.5, there exist positive constants C_1 and C_2 depending on p and ε_1 such that

$$\begin{aligned} &\lambda^q C_1 \left(T(x_0 + \lambda y) - (T(x_0) + \lambda s)\right)^{-q} \\ &\quad \le \lambda^q \phi(T(x_0) + \lambda s, x_0 + \lambda y) \le \lambda^q C_2 \left(T(x_0 + \lambda y) - (T(x_0) + \lambda s)\right)^{-q}. \end{aligned}$$

We see that

$$\lambda^q \left(T(x_0 + \lambda y) - (T(x_0) + \lambda s)\right)^{-q} = \left(\frac{T(x_0 + \lambda y) - T(x_0)}{\lambda} - s\right)^{-q}. \tag{3.54}$$

Therefore, we obtain (3.53). □

Similarly, we can show that the blow-up curve of $\psi_\lambda(s, y)$ is $T_\lambda(y)$.

From Proposition 3.3.1 and Lemmas 3.2.3, 3.3.2 and 3.4.1, there exist positive constants C_1, C_2, $C_{3,\alpha}$, and $C_{4,\alpha}$, depending only on p and ε_1 such that

$$C_1(\phi_\lambda + \psi_\lambda)^p \le \partial_s \phi_\lambda \le C_2(\phi_\lambda + \psi_\lambda)^p, \tag{3.55}$$

$$C_1(\phi_\lambda + \psi_\lambda)^p \le \partial_s \psi_\lambda \le C_2(\phi_\lambda + \psi_\lambda)^p, \tag{3.56}$$

$$C_1(T_\lambda(y) - s)^{-q} \le \phi_\lambda(s, y) \le C_2(T_\lambda(y) - s)^{-q}, \tag{3.57}$$

$$C_1(T_\lambda(y) - s)^{-q} \le \psi_\lambda(s, y) \le C_2(T_\lambda(y) - s)^{-q}, \tag{3.58}$$

$$|\partial_y \phi_\lambda| \le \frac{1}{1+\varepsilon_1}\partial_s \phi_\lambda, \quad |\partial_y \psi_\lambda| \le \frac{1}{1+\varepsilon_1}\partial_s \psi_\lambda, \tag{3.59}$$

$$|T_\lambda(y) - T_\lambda(y')| \le \frac{1}{1+\varepsilon_1}|y - y'| \quad \text{for} \quad y, y' \in \left(\frac{-R - x_0}{\lambda}, \frac{R - x_0}{\lambda}\right), \tag{3.60}$$

$$\frac{T_\lambda(y) - s}{\sqrt{2}} \le d_\lambda(y, s) \le T_\lambda(y) - s, \tag{3.61}$$

$$\begin{aligned} &\max\left\{|D_\theta^\alpha \phi_\lambda(s, y)|, |D_\theta^\alpha \psi_\lambda(s, y)|\right\} \\ &\quad \le C_{3,\alpha}(\phi_\lambda(s, y) + \psi_\lambda(s, y))^{p + (\alpha - 1)/q} \le C_{4,\alpha} d_\lambda(s, y)^{-(pq + \alpha - 1)}. \end{aligned} \tag{3.62}$$

where $(s, y) \in \Omega_\lambda$. Here $\alpha = 0, 1, 2, 3$.

3.4.2 Strategy of proof of the differentiability of T

We will consider the limits of the functions T_{λ_n}, ϕ_{λ_n}, and ψ_{λ_n}. It follows from (3.60) that T_{λ_n} is equicontinuous.

We define I_n by a closed interval satisfying

- $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$,
- $\bigcup_{n=1}^{\infty} . \ I \subset I_{n_0}$.

By (3.60), there exists a positive constant M_1 such that

$$|T_{\lambda_n}(y)| \leq M_1 \qquad \text{for} \quad y \in I_1.$$

By the Ascoli and Arzela theorem, there exist a sequence $\{\lambda_n^{(1)}\} \subset \{\lambda_n\}$ and $T_0^{(1)} \in C(I_1)$ such that $T_{\lambda_n^{(1)}}$ converges to $T_0^{(1)}$ uniformly in I_1.

In the same manner as above, we can see that there exist a sequence $\{\lambda_n^{(2)}\} \subset \{\lambda_n^{(1)}\}$ and $T_0^{(2)} \in C(I_2)$ such that $T_{\lambda_n^{(2)}}$ converges to $T_0^{(2)}$ uniformly in I_2. By repeating the same arguments, there exists $T_0 \in C(\mathbb{R})$ such that T_{Λ_n} converges to T_0 locally uniformly in $\mathbb{R}$, where $\Lambda_n = \lambda_n^{(n)}$.

In the remainder of this paper, we will show that $T \in C^1(B_R)$. We demonstrate this proof through the following two steps.

(Step 1.) First (in Section 3.5), we will show that T_0, which is defined as above, is an affine function. That is, there exists a constant α_{x_0} such that $T_0(y) = \alpha_{x_0} y$ for $y \in \mathbb{R}$.

(Step 2.) Next (in Section 3.6), we will demonstrate that a contradiction arises if we assume that there exists $x_0 \in B_{R^*}$ such that T is not differentiable at $x_0 \in B_{R^*}$.

We start by assuming that T is not differentiable at $x_0 \in B_{R^*}$. On the other hand, by (Step 1), we have that for all $y \in \mathbb{R}$,

$$\frac{T_{\Lambda_n}(y)}{y} = \frac{T(x_0 + \Lambda_n y) - T(x_0)}{\Lambda_n y} \to \alpha_{x_0} \quad \text{as} \quad \Lambda_n \to 0,$$

where $\{\Lambda_n\} \subset \{\lambda_n\}$ is the sequence appeared in (Step 1). This means that there exist $\{\lambda_{n'}\} \subset \{\lambda_n\}$ and $y' \in \mathbb{R}$ such that

$$\limsup_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y') > \liminf_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y'). \tag{3.63}$$

On the other hand, there exist $\{\lambda_{n'}^{(1)}\} \subset \{\lambda_{n'}\}$ and $\{\lambda_{n'}^{(2)}\} \subset \{\lambda_{n'}\}$ such that

$$\lim_{\lambda_{n'}^{(1)} \to 0} T_{\lambda_{n'}^{(1)}}(y') = \limsup_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y'),$$

$$\lim_{\lambda_{n'}^{(2)} \to 0} T_{\lambda_{n'}^{(2)}}(y') = \liminf_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y').$$

By repeating the above arguments, there exist $\{\lambda^{(1)}_{n'_k}\} \subset \{\lambda^{(1)}_{n'}\}$ and $\{\lambda^{(2)}_{n'_k}\} \subset \{\lambda^{(2)}_{n'}\}$, and corresponding functions $T_0^{(1)}$, $T_0^{(2)} \in C(\mathbb{R})$, such that

$$T_{\lambda^{(1)}_{n'_k}} \to T_0^{(1)}, \quad T_{\lambda^{(2)}_{n'_k}} \to T_0^{(2)} \quad \text{locally uniformly in } \mathbb{R}.$$

It follows from **(Step 1)** that there exist constants $\alpha^{(1)}_{x_0}$ and $\alpha^{(2)}_{x_0}$ such that $T_0^{(1)}(y) = \alpha^{(1)}_{x_0} y$ and $T_0^{(2)}(y) = \alpha^{(2)}_{x_0} y$, respectively. By (3.63), we see that $\alpha^{(1)}_{x_0} \neq \alpha^{(2)}_{x_0}$.

In Section 3.6, we will demonstrate that a contradiction arises if there exist $\alpha^{(1)}_{x_0}$ and $\alpha^{(2)}_{x_0}$ such that $\alpha^{(1)}_{x_0} \neq \alpha^{(2)}_{x_0}$ and

$$T_0^{(1)}(y) = \alpha^{(1)}_{x_0} y, \quad T_0^{(2)}(y) = \alpha^{(2)}_{x_0} y \qquad \text{for } y \in \mathbb{R}.$$

That is, we obtain that T is differentiable in B_{R^*}. Moreover, we can show that a contradiction arises if we assume that the derivative T' is not continuous in B_{R^*}.

In the remainder of this section, we prepare for our proof of **(Step 1.)**. We consider the limits of blow-up sequences ϕ_{λ_n} and ψ_{λ_n}. We set $\Omega_0 = \{(s, y) \mid y \in \mathbb{R},\ s < T_0(y)\}$. Then, we set J_n as a closed subset of Ω_0 satisfying

- $J_n \subset J_{n+1}$ for $n \in \mathbb{N}$,
- $\bigcup_{n=1}^{\infty} J_n = \Omega_0$.

It follows from the Ascoli and Arzela theorem that there exists a subsequence $\{\tilde\lambda_n\} \subset \{\Lambda_n\}$, such that there exist

$$v_\phi, v_\psi, v_\phi^{1,\theta}, v_\psi^{1,\theta}, v_\phi^{2,\theta}, v_\psi^{2,\theta}, v_\phi^{3,\theta}, v_\psi^{3,\theta} \in C(\Omega_0)$$

satisfying

$$\begin{cases} \phi_{\tilde\lambda_n} \to v_\phi, \quad \psi_{\tilde\lambda_n} \to v_\psi, \\ D_\theta \phi_{\tilde\lambda_n} \to v_\phi^{1,\theta}, \quad D_\theta \psi_{\tilde\lambda_n} \to v_\psi^{1,\theta}, \\ D_\theta^2 \phi_{\tilde\lambda_n} \to v_\phi^{2,\theta}, \quad D_\theta^2 \psi_{\tilde\lambda_n} \to v_\psi^{2,\theta}, \\ D_\theta^3 \phi_{\tilde\lambda_n} \to v_\phi^{3,\theta}, \quad D_\theta^3 \psi_{\tilde\lambda_n} \to v_\psi^{3,\theta}, \end{cases} \text{locally uniformly in } \Omega_0 \tag{3.64}$$

for $\theta \in [0, 2\pi)$. Thus, we have that $v_\phi, v_\psi \in C^3(\Omega_0)$. The functions v_ϕ and v_ψ are called blow-up limits of ϕ and ψ (see [7]). By (3.52), (3.55)–(3.62), we have that

$$\begin{cases} D_- v_\phi = 2^{-p}(v_\phi + v_\psi)^p, \\ D_+ v_\psi = 2^{-p}(v_\phi + v_\psi)^p, \end{cases} \tag{3.65}$$

and there exist positive constants $C_1, C_2, C_{3,\alpha}$ and $C_{4,\alpha}$, depending only on p and ε_1, such that

$$C_1(v_\phi + v_\psi)^p \le \partial_s v_\phi \le C_2(v_\phi + v_\psi)^p, \tag{3.66}$$

$$C_1(v_\phi + v_\psi)^p \le \partial_s v_\psi \le C_2(v_\phi + v_\psi)^p, \tag{3.67}$$

$$C_1(T_0(y) - s)^{-q} \le v_\phi(s, y) \le C_2(T_0(y) - s)^{-q}, \tag{3.68}$$

$$C_1(T_0(y) - s)^{-q} \le v_\psi(s, y) \le C_2(T_0(y) - s)^{-q}, \tag{3.69}$$

$$|\partial_y v_\phi| \le \frac{1}{1+\varepsilon_1} \partial_s v_\phi, \quad |\partial_y v_\psi| \le \frac{1}{1+\varepsilon_1} \partial_s v_\psi, \tag{3.70}$$

$$|T_0(y) - T_0(y')| \le \frac{1}{1+\varepsilon_1}|y - y'| \quad \text{for} \quad y, y' \in \mathbb{R}, \tag{3.71}$$

$$\frac{T_0(y) - s}{\sqrt{2}} \le d_0(s,y) \le T_0(y) - s, \tag{3.72}$$

$$\begin{aligned} &\max\{|D_\theta^\alpha v_\phi(s,y)|, |D_\theta^\alpha v_\psi(s,y)|\} \\ &\quad \le C_{3,\alpha}(v_\phi(s,y) + v_\psi(s,y))^{p+(\alpha-1)/q} \le C_{4,\alpha} d_0(s,y)^{-(pq+\alpha-1)}, \end{aligned} \tag{3.73}$$

where $(s,y) \in \Omega_0$. Here, $d_0(s,y)$ is the distance from a point $(s,y) \in \Omega_0$ to $\Gamma_0 = \{(s,y) \mid s = T_0(y), \quad y \in \mathbb{R}\}$ and $\alpha = 0, 1, 2, 3$.

3.4.3 Convexity of blow-up limits

In order to demonstrate that T_0 is an affine function, we will prove the following lemma.

Lemma 3.4.3. *Assume that (A1)–(A5.3) hold. Then, we have that*

$$D_\theta^2 v_\phi \ge 0, \quad D_\theta^2 v_\psi \ge 0 \quad \textit{in} \quad \Omega_0 \tag{3.74}$$

for $0 \le \theta < 2\pi$.

Proof. We fix a point $(\tilde{s}, \tilde{y}) \in \Omega_0$. Let $\boldsymbol{K}_-(\tilde{s}, \tilde{y}) = \{(s,y) \in \Omega_0 \mid |\tilde{y} - y| < \tilde{s} - s\}$. Then, it suffices to show that $D_\theta^2 v_\phi, D_\theta^2 v_\psi \ge 0$ in $\boldsymbol{K}_-(\tilde{s}, \tilde{y})$.

Let

$$J_\phi = D_\theta^2 v_\phi + \eta \partial_s v_\phi, \qquad J_\psi = D_\theta^2 v_\psi + \eta \partial_s v_\psi,$$

where η is a positive constant.

In what follows, we will show that

$$J_\phi > 0 \quad \text{and} \quad J_\psi > 0 \quad \text{in } \boldsymbol{K}_-(\tilde{s}, \tilde{y}). \tag{3.75}$$

We see that

$$\begin{aligned} D_- J_\phi &= D_+ J_\psi \\ &= 2^{-p} p(p-1)(v_\phi + v_\psi)^{p-2}(D_\theta v_\phi + D_\theta v_\psi)^2 \\ &\qquad + 2^{-p} p(v_\phi + v_\psi)^{p-1}(J_\phi + J_\psi). \end{aligned} \tag{3.76}$$

We consider J_ϕ and J_ψ in $\boldsymbol{K}_-(\tilde{s}, \tilde{y})$. By (3.72), we have

$$\frac{1}{\sqrt{2}}\left(\frac{T_0(y) - s}{|s|}\right) \le \frac{d_0(s,y)}{|s|} \le \frac{T_0(y) - s}{|s|}.$$

Thus, we obtain that

$$\frac{1}{\sqrt{2}} \le \frac{d_0(s,y)}{|s|} \le 1 \quad \text{for} \quad (s,y) \in \boldsymbol{K}_-(\tilde{s}, \tilde{y}), \quad \text{as } s \to -\infty. \tag{3.77}$$

By (3.73), (3.66), (3.68), (3.69) and (3.72), we have that there exist positive constants c_1 and c_2, depending only on p and ε_1, such that

$$\begin{aligned} \max\{|D_\theta^2 v_\phi(s,y)|, |D_\theta^2 v_\psi(s,y)|\} &\le c_1(v_\phi + v_\psi)^p(s,y)(v_\phi+v_\psi)^{1/q}(s,y) \\ &\le c_2 \partial_s v_\phi(s,y) d_0(y,s)^{-1}. \end{aligned} \tag{3.78}$$

Hence, it follows from (3.77) and (3.78) that

$$J_\phi = \eta \partial_s v_\phi(1 + O(1/|s|)), \quad J_\psi = \eta \partial_s v_\psi(1 + O(1/|s|)), \quad \text{as } s \to -\infty \tag{3.79}$$

in $\boldsymbol{K}_-(\tilde{s}, \tilde{y})$. Since $\partial_s v_\phi, \partial_s v_\phi > 0 \quad$ in Ω_0, we have that $J_\phi, J_\psi > 0$ in $\boldsymbol{K}_-(\tilde{s}, \tilde{y}) \cap \{(s,y) \mid s < -\sigma\}$ if σ is large enough.

We assume that (3.75) does not hold. Then, there exists $(s', y') \in \boldsymbol{K}_-(\tilde{s}, \tilde{y})$ such that

$$J_\phi(s', y') = 0 \quad \text{or} \quad J_\psi(s', y') = 0$$

and

$$J_\phi(s,y) > 0 \quad \text{and} \quad J_\psi(s,y) > 0 \quad \text{for } (s,y) \in \boldsymbol{K}_-(\tilde{s}, \tilde{y}) \cap \{(s,y) \mid y \in \mathbb{R},\ s < s'\}.$$

We assume $J_\phi(s', y') = 0$. Then, it follows from (3.76) that

$$\begin{aligned} 0 &= J_\phi(s', y') \\ &= J_\phi(s' - M, y' + M) \\ &\quad + \int_0^M 2^{-p} p(p-1)(v_\phi + v_\psi)^{p-2}(D_\theta v_\phi + D_\theta v_\psi)^2(s, y' + M - s) ds \\ &\quad + \int_0^M 2^{-p} p(v_\phi + v_\psi)^{p-1}(J_\phi + J_\psi)(s, y' + M - s) ds \\ &> 0 \quad \text{for} \quad M > 0. \end{aligned}$$

This is a contradiction. In the same manner as above, we can show that a contradiction arises if we assume that $J_\psi(s', y') = 0$. Therefore, we obtain that (3.75) holds.

By taking $\eta \to 0$, we have

$$D_\theta^2 v_\phi \ge 0 \quad \text{and} \quad D_\theta^2 v_\psi \ge 0 \quad \text{in } \boldsymbol{K}_-(\tilde{s}, \tilde{y}).$$

This completes the proof. □

3.5 Linearity of the blow-up curve of blow-up limits

In this section, we will prove **(Step 1.)** as stated in Section 3.4.2. In order to prove this, we will consider

$$\begin{cases} D_- V_\phi = 2^{-p}(V_\phi + V_\psi)^p, \\ D_+ V_\psi = 2^{-p}(V_\phi + V_\psi)^p, \end{cases} \tag{3.80}$$

with some constant $\alpha \in \mathbb{R}$ and the corresponding blow-up curve

$$\{(s, y) \mid s = \alpha y, \quad y \in \mathbb{R}\}. \tag{3.81}$$

We know that (3.80)–(3.81) yield the following special solution:

$$(V_{\phi,\alpha}(s, y), V_{\psi,\alpha}(s, y)) = (C_{\phi,\alpha}(\alpha y - s)^{-q}, C_{\psi,\alpha}(\alpha y - s)^{-q}), \tag{3.82}$$

where

$$C_{\phi,\alpha} = (q(1 + \alpha)(1 - \alpha)^p)^q, \quad C_{\psi,\alpha} = (q(1 + \alpha)^p(1 - \alpha))^q.$$

In this section, we will prove the following lemma.

Lemma 3.5.1. *Assume that (A1)–(A5.3) hold. Then, there exists a positive constant $\alpha \in \mathbb{R}$ such that*

$$T_0(y) = \alpha y \quad \textit{for } y \in \mathbb{R}. \tag{3.83}$$

Moreover, the constant α satisfies $-1 < \alpha < 1$ and

$$v_\phi = V_{\phi,\alpha} \quad \textit{and} \quad v_\psi = V_{\psi,\alpha}. \tag{3.84}$$

In order to prove Lemma 3.5.1, we will first introduce some lemmas.

Lemma 3.5.2. *Assume that (A1)–(A5.3) hold. Then, T_0 is concave.*

Proof. Let $\varepsilon > 0$ be arbitrary. Then, by (3.68) we see that there exists a positive constant c_1, depending only on p and ε_1, such that

$$c_1\varepsilon^{-q} \leq v_\phi(s, y) \quad \text{for} \quad y \in \mathbb{R} \quad \text{and} \quad s \in [T_0(y) - \varepsilon, T_0(y)).$$

Thus, there exist $M \geq c_1\varepsilon^{-q}$ and $E_0(y)$ such that

$$v_\phi(E_0(y), y) = M \quad \text{and} \quad T_0(y) - E_0(y) \leq \varepsilon \quad \text{for} \quad y \in \mathbb{R}.$$

We set $H_M = \{(s, y) \mid \ s \leq E_0(y), \quad y \in \mathbb{R}\}$.

We will show that E_0 is concave. It suffices to show that H_M is convex. We assume that H_M is not convex. Then, there exist $(s_1, y_1), (s_2, y_2) \in H_M$ and $\xi' \in (0, 1)$ such that $\xi'(s_1, y_1) + (1 - \xi')(s_2, y_2) \notin H_M$ and $\xi'(s_1, y_1) + (1 - \xi')(s_2, y_2) \in \Omega_0$. We notice that $\partial_s v_\phi > 0$ in Ω_0. Then, we have

$$\begin{aligned} M = \xi' M + (1 - \xi')M &\geq \xi' v_\phi(s_1, y_1) + (1 - \xi')v_\phi(s_2, y_2) \\ &\geq v_\phi(\xi'(s_1, y_1) + (1 - \xi')(s_2, y_2)) \\ &> M. \end{aligned}$$

This is a contradiction. Hence, H_M is convex. Therefore, E_0 is concave. Thus, we have

$$\begin{aligned} &\xi T_0(y) + (1 - \xi)T_0(y') \\ &= \xi(T_0(y) - E_0(y)) + (\xi E_0(y) + (1 - \xi)E_0(y')) + (1 - \xi)(T_0(y') - E_0(y')) \\ &\leq \xi(T_0(y) - E_0(y)) + E_0(\xi y + (1 - \xi)y') + (1 - \xi)(T_0(y') - E_0(y')) \\ &\leq \varepsilon + E_0(\xi y + (1 - \xi)y') < \varepsilon + T_0(\xi y + (1 - \xi)y'), \end{aligned}$$

for $y, y' \in \mathbb{R}$ and $\xi \in (0, 1)$. Since we let $\varepsilon > 0$ take an arbitrary value, this completes the proof. $\square$

We set

$$v_{\phi,\lambda}(s,y) = \lambda^q v_\phi(\lambda s, \lambda y), \qquad v_{\psi,\lambda}(s,y) = \lambda^q v_\psi(\lambda s, \lambda y),$$

with $\lambda \to \infty$. Then, we can easily see that the blow-up curve of $v_{\phi,\lambda}$ and $v_{\psi,\lambda}$ is

$$T_{0,\lambda}(y) = \frac{T_0(\lambda y)}{\lambda}.$$

Lemma 3.5.3. *Assume that (A1)–(A5.3) hold. Then, we have*

$$T_{0,\lambda_n}(y) \to \begin{cases} \alpha y & (y \geq 0) \\ \beta y & (y < 0) \end{cases} \qquad \textit{as} \quad \lambda_n \to \infty$$

where α and β are constants satisfying $-1 < \alpha \leq \beta < 1$.

Proof. First, we see that $T_{0,\lambda_n}(0) = 0$.

Next, since T_0 is concave, we see that $\dfrac{T_{0,\lambda_n}(y)}{y} = \dfrac{T_0(\lambda_n y) - T_0(0)}{\lambda_n y}$ is monotone decreasing on n, for $y > 0$. Here, $\{\lambda_n\}$ is a monotone increasing sequence satisfying $\lambda_n \to \infty$. Thus, we have that

$$\lim_{\lambda_n \to \infty} \frac{T_{0,\lambda_n}(y)}{y} = \inf_{\lambda_n} \frac{T_{0,\lambda_n}(y)}{y} = \inf_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} \quad \text{for } y > 0.$$

Let $\alpha = \inf\limits_{\lambda_n} \dfrac{T_0(\lambda_n y)}{\lambda_n y}$. Then, we have that

$$T_{0,\lambda_n}(y) \to \alpha y \quad \text{as } \lambda_n \to \infty,$$

for all $y > 0$ and monotone increasing sequences $\{\lambda_n\}$ satisfying $\lambda_n \to \infty$. By (3.71), we have $-1 < \alpha < 1$. We notice that α does not depend on y and λ_n.

Finally, we can prove

$$\lim_{\lambda_n \to \infty} \frac{T_{0,\lambda_n}(y)}{y} = \sup_{\lambda_n} \frac{T_{0,\lambda_n}(y)}{y} = \sup_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} \quad \text{for } y < 0,$$

in the same way of above. We set $\beta = \sup\limits_{\lambda_n} \dfrac{T_0(\lambda_n y)}{\lambda_n y}$. We notice that $-1 < \alpha \leq \beta < 1$. Then, it follows that

$$T_{0,\lambda_n}(y) \to \beta y \quad \text{as } \lambda_n \to \infty,$$

for all $y < 0$ and monotone increasing sequences $\{\lambda_n\}$ satisfying $\lambda_n \to \infty$. This completes the proof. □

Now, we set

$$\tilde{T}_0(y) = \begin{cases} \alpha y & (y \geq 0) \\ \beta y & (y < 0) \end{cases} \quad , \quad \tilde{\Omega}_0 = \left\{ (s,y) \in \mathbb{R}^2 \mid s < \tilde{T}_0(y), \quad y \in \mathbb{R} \right\}.$$

Remark 3.5.4. In the same way of proof of Lemma 3.5.2, we obtain that $\tilde{T}_0$ is concave. That is, α and β have the same sign.

Lemma 3.5.5. *Assume that (A1)–(A5.3) hold. Then, we have that $\alpha = \beta$. Here, α and β are constants as defined in Lemma 3.5.3.*

Proof. There exists a sequence $\{\lambda_n\}$ such that

$$v_{\phi,\lambda_n} \to w_\phi, \quad v_{\psi,\lambda_n} \to w_\psi, \quad \text{as } \lambda_n \to \infty, \quad \text{locally uniformly in } \tilde{\Omega}_0.$$

In the same arguments for Lemma 3.4.3, we see that $D_\theta^2 w_\phi \geq 0$ and $D_\theta^2 w_\psi \geq 0$ in $\tilde{\Omega}_0$, for $0 \leq \theta < 2\pi$. Thus, $D_\theta w_\phi$ and $D_\theta w_\psi$ are monotone increasing along the direction θ. We also have that it follows from the estimates $|D_\theta w_\phi|$ and $|D_\theta w_\psi|$, corresponding (3.73) that $|D_\theta w_\phi(s,y)|, |D_\theta w_\psi(s,y)| \to 0$ as $\tilde{d}_0(s,y) \to \infty$, where $\tilde{d}_0(s,y)$ is the distance from a point $(s,y) \in \tilde{\Omega}_0$ to $\tilde{\Gamma}_0 = \{(\tilde{T}_0(y), y) \mid y \in \mathbb{R}\}$. Therefore, $D_\theta w_\phi$ and $D_\theta w_\psi$ do not occur sign changes in $\tilde{\Omega}_0$.

By Remark 3.5.4, we see that α and β have the same sign.

We assume that $0 < \alpha < \beta$. We set θ_α and θ_β as $\theta_\alpha = \arctan\alpha$ and $\theta_\beta = \arctan\beta$, respectively. Let us assume that $0 \leq \theta_\alpha < \theta_\beta < \pi/2$ without loss of generality.

If we take $\theta \in S$ where $S = \{\theta \in [0, 3\pi/2) \mid \theta_\alpha < \theta < \theta_\beta + \pi\}$, then $D_\theta w_\phi > 0$, since the closer w_ϕ gets to the blow-up curve $s = \beta y$ $(y < 0)$ or $s = \alpha y$ $(y \geq 0)$, the bigger w_ϕ becomes.

We take $\tilde{\theta}$ as $\theta_\alpha < \tilde{\theta} < \theta_\beta$. Then, we have that $D_{\tilde{\theta}} w_\phi > 0$, since $\tilde{\theta} \in S$. On the other hand, $D_{\tilde{\theta}+\pi} w_\phi > 0$, since $\tilde{\theta} + \pi \in S$. This contradicts the fact that

$$D_{\tilde{\theta}} w_\phi = -D_{\tilde{\theta}+\pi} w_\phi \quad \tilde{\Omega}_0.$$

In the same manner, we can prove that a contradiction arises if we assume that $\alpha < \beta < 0$. Therefore, we have that $\alpha = \beta$. This completes the proof. ☐

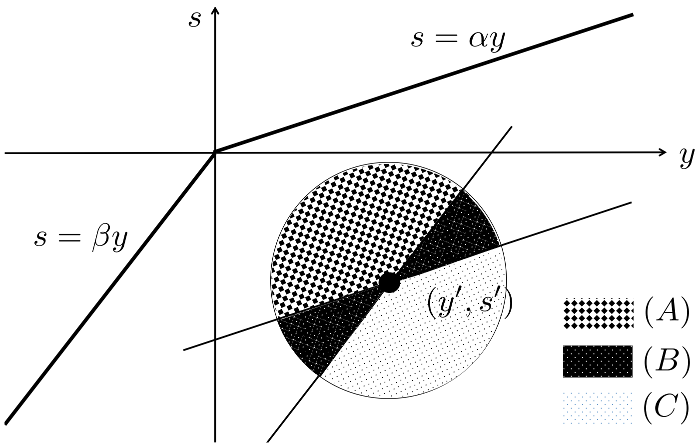


Figure A. The sign of the directional derivative at (s', y').

- (A) and (B) areas: The sign of the directional derivative is positive.

- (C) area : The sign of the directional derivative is negative.
  $\to$ If (B) area exists, we can show that a contradiction arises.

Proof of Lemma 3.5.1. First, we will show that $T_0(y) = \alpha y$. It follows from Lemma 3.5.5 that

$$\sup_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} = \inf_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} = \alpha \qquad \text{for } y \in \mathbb{R}.$$

Thus, $T_0(\lambda_n y) = \alpha \lambda_n y$ for $\lambda_n > 0$ and $y \in \mathbb{R}$. Therefore, we obtain that $T_0(y) = \alpha y$ for $y \in \mathbb{R}$.

Next, we will show that $v_\phi = V_{\phi,\alpha}$ and $v_\psi = V_{\psi,\alpha}$. By applying the proof of Lemma 3.5.5, we obtain that

$$(\alpha \partial_s + \partial_y) v_\phi = 0, \qquad \text{and} \qquad (\alpha \partial_s + \partial_y) v_\psi = 0 \tag{3.85}$$

in Ω_0. By substituting (3.85) for (3.65), we obtain the following system of equations:

$$\begin{cases} (1+\alpha)\partial_s v_\phi = 2^{-p}(v_\phi + v_\psi) \\ (1-\alpha)\partial_s v_\phi = 2^{-p}(v_\phi + v_\psi), \end{cases}$$

with the blow-up curve $T_0(y) = \alpha y$. Therefore, we obtain that $v_\phi = V_{\phi,\alpha}$ and $v_\psi = V_{\psi,\alpha}$ in Ω_0. This completes the proof. $\square$

3.6 Continuous differentiability of the blow-up curve

In this section, we complete the proof of Theorem 3.1.1.

First, we will show that T is differentiable in B_{R^*}. We start by assuming that there exists $x_0 \in B_{R^*}$ such that T is not differentiable at $x_0 \in B_{R^*}$. Then, it follows from the arguments of **(Step 2.)** of Section 3.4.2 that there exist sequences $\{\lambda_n^{(1)}\}, \{\lambda_n^{(2)}\}$ such that there exist constants α_1 and α_2 satisfying

$$\begin{aligned} &\alpha_1, \alpha_2 \in (-1,1), \quad \alpha_1 \neq \alpha_2, \\ &\phi_{\lambda_n^{(j)}} \to V_{\phi,\alpha_j} \quad \text{as} \quad \lambda_n^{(j)} \to 0, \qquad \text{locally uniformly in } \Omega_{j,0}, \end{aligned}$$

where

$$\Omega_{j,0} = \big\{(s,y) \in \mathbb{R}^2 \mid s < \alpha_j y, \quad y \in \mathbb{R}\big\}$$

for $j = 1, 2$.

Let θ_{α_1} and θ_{α_2} be defined such that $\theta_{\alpha_1} = \arctan \alpha_1$ and $\theta_{\alpha_2} = \arctan \alpha_2$. Let us suppose that

$$0 \le \theta_{\alpha_j} < \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4} < \theta_{\alpha_j} < \pi \quad (j = 1, 2)$$
$$\text{and}$$
$$\theta_{\alpha_1} < \theta_{\alpha_2}$$

without loss of generality.

We assume that $0 \leq \theta_{\alpha_1} < \theta_{\alpha_2} < \pi/4$. We take $0 < \varepsilon < \pi/2$ as

$$0 < \theta_{\alpha_1} + \varepsilon < \theta_{\alpha_2} - \varepsilon < \frac{\pi}{4}.$$

Then, for $j = 1, 2$, we have that there exist θ_j such that

$$0 < \theta_{\alpha_j} + \varepsilon < \theta_j < \theta_{\alpha_j} + \pi - \varepsilon < \frac{5\pi}{4}.$$

We define

$$S_{\varepsilon}^{(j)} = \left\{\theta_j \mid \theta_{\alpha_j} + \varepsilon < \theta_j < \theta_{\alpha_j} + \pi - \varepsilon\right\} \quad \text{for} \quad j = 1, 2.$$

We see that there exists $\varepsilon' > 0$ such that

$$D_{\theta'} V_{\phi,\alpha_j} > 2\varepsilon' \quad \text{in} \quad \Omega_{j,0} \cap B_1(0,0),$$

where $B_\rho(s', y') = \left\{(s, y) \in \mathbb{R}^2 \mid \sqrt{(s - s')^2 + (y - y')^2} < \rho\right\}$. Here, ρ is a positive constant.

For $j = 1, 2$, let $(s_j^{\pm}, y_j^{\pm})$ and $(s_j^{\delta_0,\pm}, y_j^{\delta_0,\pm})$ be the intersections of $s^2 + y^2 = 1$ and

$$s = \alpha_{x_j} y \quad \text{and} \quad s = \alpha_{x_j} y - \delta_0,$$

respectively. Here, δ_0 is a positive constant.

We see that there exist $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that for $j = 1, 2$,

$$\begin{aligned}
&\Omega_{1,0}^{\delta_0} \cap \Omega_{2,0}^{\delta_0} \cap B_1(0,0), \\
&\Omega_{\lambda_{n_0}^{(j)}} \subset \Omega_{j,0}^{-\delta_0}, \quad \Omega_{j,0}^{\delta_0} \subset \Omega_{\lambda_{n_0}^{(j)}} \\
&s_j^{\delta_0,-} < s_j^{-}, \\
&\text{For } \theta_j \in S_{\varepsilon'}^{(j)}, \quad |D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} - D_{\theta_j} V_{\phi,\alpha_j}| \leq \varepsilon' \quad \text{in} \quad \Omega_{j,0}^{\delta_0} \cap B_1(0,0).
\end{aligned}$$

Here, $\Omega_{j,0}^{\delta_0} = \left\{(s, y) \mid s < \alpha_{x_j} y - \delta_0, \quad y \in \mathbb{R}\right\}$. This means that

$$D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad \Omega_{j,0}^{\delta_0} \cap B_1(0,0)$$

for $\theta_j \in S_{\varepsilon'}^{(j)}$ and $j = 1, 2$. By (3.55), we can prove

$$D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad K_j^{\delta_0} \tag{3.86}$$

where

$$K_j^{\delta_0} = \left\{(s, y) \in \Omega_{\lambda_{n_0}^{(j)}} \cap B_1(0,0) \mid y < \min\{|y_j^{\delta_0,-}|, |y_j^{\delta_0,+}|\}\right\}$$

for $\theta_j \in S_{\varepsilon'}^{(j)}$ and $j = 1, 2$. (3.86) means that there exists there exists a positive constant ρ such that

$$0 < \rho \leq 1 \quad \text{and} \quad D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad \Omega_{\lambda_{n_0}^{(j)}} \cap B_\rho(0,0) \tag{3.87}$$

for $\theta_j \in S^{(j)}_{\varepsilon'}$ and $j = 1, 2$.

Let $\lambda_{n_1} = \min\{\lambda^{(1)}_{n_0}, \lambda^{(2)}_{n_0}\}$. It follows from (3.87) that

$$D_\theta \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(T(x_0), x_0).$$

for $\theta \in S^{(1)}_{\varepsilon'} \cup S^{(2)}_{\varepsilon'}$.

In particular,

$$D_{\theta^*} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(T(x_0), x_0) \tag{3.88}$$

for $\theta^* \in (\theta_{\alpha_1} + \varepsilon, \theta_{\alpha_2} - \varepsilon)$, since $(\theta_{\alpha_1} + \varepsilon, \theta_{\alpha_2} - \varepsilon) \subset S^{(1)}_{\varepsilon}$. Moreover, we have

$$D_{\theta^*+\pi} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(T(x_0), x_0) \tag{3.89}$$

since $\theta^* + \pi \in (\theta_{\alpha_1} + \pi + \varepsilon, \theta_{\alpha_2} + \pi - \varepsilon) \subset S^{(2)}_{\varepsilon}$. Then, (3.88) and (3.89) contradict the fact

$$D_{\theta^*} \phi = -D_{\theta^*+\pi} \phi \quad \text{in } \Omega.$$

We can show contradictions in the other cases, that is, in the cases

$$\begin{aligned} &0 \leq \theta_{\alpha_1} < \pi/4, \ 3\pi/4 < \theta_{\alpha_2} < \pi, \\ &3\pi/4 < \theta_{\alpha_1} < \theta_{\alpha_2} < \pi. \end{aligned}$$

Therefore, T is differentiable in B_{R^*}.

Next, we will show that the derivative T' is continuous in B_{R^*}. We start by assuming that there exists $x_0 \in B_{R^*}$ such that T' is discontinuous at $x_0 \in B_{R^*}$. Set $\alpha_{x_0} = T'(x_0)$. Let us suppose that $0 \leq \theta_{\alpha_{x_0}} < \pi/4$ or $3\pi/4 \leq \theta_{\alpha_{x_0}} < 5\pi/4$ without loss of generality.

Since T' is discontinuous at $x_0 \in B_{R^*}$, there exists $0 < \varepsilon' < \pi/2$ such that there exists $\{x_j\} \subset B_{R*}$ satisfying

$$|x_j - x_0| \to 0 \quad \text{as} \quad j \to \infty \quad \text{and} \quad |\theta_{\alpha_{x_j}} - \theta_{\alpha_{x_0}}| > 2\varepsilon' \quad \text{for all } j \in \mathbb{N}. \tag{3.90}$$

By the above argument, there exists $n_0 \in \mathbb{N}$ and $\rho \in \mathbb{R}$ such that

$$D_{\theta_0} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0}\rho}(T(x_0), x_0)$$

for $\theta_0 \in S_{\varepsilon', x_0} = \{\theta_0 \mid \theta_{\alpha_{x_0}} + \varepsilon' < \theta_0 < \theta_{\alpha_{x_0}} + \pi - \varepsilon'\}$.

Moreover, by the continuity of T and (3.90), there exists $j_0 \in \mathbb{N}$ such that

$$(T(x_{j_0}), x_{j_0}) \in B_{\lambda_{n_0}\rho}(T(x_0), x_0).$$

We see that there exists $n_{j_0} \in \mathbb{N}$ such that

$$D_{\theta_{j_0}} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_{j_0}}\rho}(T(x_{j_0}), x_{j_0})$$

for $\theta_{j_0} \in S_{\varepsilon', x_{j_0}} = \{\theta_{j_0} \mid \theta_{\alpha_{x_{j_0}}} + \varepsilon' < \theta_{j_0} < \theta_{\alpha_{x_{j_0}}} + \pi - \varepsilon'\}$.

Then, we have

$$D_\theta \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0}\rho}(T(x_0), x_0) \cap B_{\lambda_{n_{j_0}}\rho}(T(x_{j_0}), x_{j_0})$$

for $\theta \in S_{\varepsilon', x_0} \cup S_{\varepsilon', x_{j_0}}$.

Assume $0 < \theta_{x_0} < \theta_{x_{j_0}} < \pi/4$. By (3.90),

$$\theta_{\alpha_{x_0}} + \varepsilon' < \theta_{\alpha_{x_{j_0}}} - \varepsilon'.$$

Take $\tilde{\theta}$ as $\theta_{\alpha_{x_0}} + \varepsilon' < \tilde{\theta} < \theta_{\alpha_{x_{j_0}}} - \varepsilon'$.

Then,

$$D_{\tilde{\theta}} \phi > 0 \quad \text{and} \quad D_{\tilde{\theta}+\pi} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0}\rho}(T(x_0), x_0) \cap B_{\lambda_{n_{j_0}}\rho}(T(x_{j_0}), x_{j_0}),$$

since $\tilde{\theta}, \tilde{\theta}+\pi \in S_{\varepsilon', x_0} \cup S_{\varepsilon', x_{j_0}}$. This contradicts the fact that

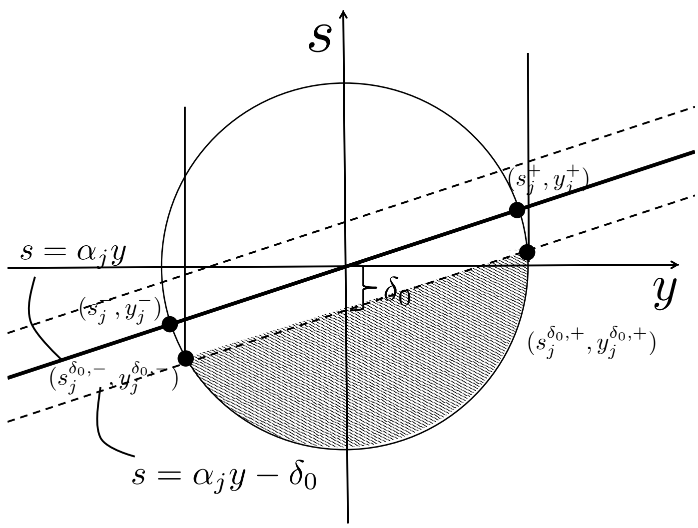
$$D_{\tilde{\theta}+\pi} \phi = -D_{\tilde{\theta}} \phi \quad \text{in} \quad \Omega.$$

In the the other cases, that is, in the cases,

$$\begin{aligned}
&0 \leq \theta_{\alpha_{x_0}} < \pi/4,\ 3\pi/4 < \theta_{\alpha_{x_{j_0}}} < \pi, \\
&3\pi/4 < \theta_{\alpha_{x_0}} < \theta_{\alpha_{x_{j_0}}} < \pi, \\
&0 \leq \theta_{\alpha_{x_{j_0}}} < \theta_{\alpha_{x_0}} < \pi/4, \\
&0 \leq \theta_{\alpha_{x_{j_0}}} < \pi/4,\ 3\pi/4 < \theta_{\alpha_{x_0}} < \pi, \\
&3\pi/4 < \theta_{\alpha_{x_{j_0}}} < \theta_{\alpha_{x_0}} < \pi,
\end{aligned}$$

we can show that contradictions arise in the same way.

This completes the proof.



$\tau = h = 1/500.$

3.7 Numerical examples

In this section, we will show some numerical examples of the blow-up curves for (3.7). For simplicity of computation, we consider the equations in a bounded interval $(0, 1)$ and pose the periodic boundary condition. We follow the method proposed by Cho [12] for computing the numerical blow-up curve.

For discretization, we employ the finite difference scheme for (3.7). Take a positive integer J and set $x_j = jh$ with $h = 1/J$. As a time variable, we take a positive constant τ as $\tau = h$ and set $t_n = \tau \cdot n$. Then, we consider the following scheme for (3.7):

$$\phi_j^n \approx \phi(t_n, x_j), \qquad \psi_j^n \approx \psi(t_n, x_j) \qquad (1 \le j \le J,\ n \ge 0),$$

$$\begin{cases} \dfrac{\phi_j^{n+1} - \phi_j^n}{\tau} - \dfrac{\phi_{j+1}^n - \phi_j^n}{h} = 2^{-p}\left|\phi_j^n + \psi_j^n\right|^p, \\ \dfrac{\psi_j^{n+1} - \psi_j^n}{\tau} + \dfrac{\psi_j^n - \psi_{j-1}^n}{h} = 2^{-p}\left|\phi_j^n + \psi_j^n\right|^p, \\ \phi_j^0 = f(x_j), \qquad \psi_j^0 = g(x_j), \end{cases}$$
$$(1 \le j \le J,\ n \ge 0),$$

where ϕ_{J+1} and ψ_0^n are set as $\phi_{J+1}^n = \phi_1^n$ and $\psi_0^n = \psi_J^n$.

We define the numerical blow-up curve T_j approximated to $T(x_j)$ by

$$T_j = \tau \cdot n_j(\tau).$$

Here, $n_j(\tau)$ is the smallest positive integer such that

$$\tau \cdot \left(\phi_j^{n_j(\tau)-1} + \psi_j^{n_j(\tau)-1}\right) \ge 1/\mathrm{eps} \quad \text{and} \quad \tau \cdot \left(\phi_j^{n_j(\tau)} + \psi_j^{n_j(\tau)}\right) < 1/\mathrm{eps},$$

where $\mathrm{eps} > 0$ is a stopping criterion given below. We set $\boldsymbol{T} = (T_j)$.

We plot two numerical blow-up curves $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ with two stopping criterion eps1 and eps2, respectively, for several τ in Figure 1–3. We see that $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are almost equal under suitable eps1, eps2 and τ. Therefore, we can regard $\boldsymbol{T}$ is a reasonable approximation of the exact blow-up curve T for (3.7).

First, we examine the shape of blow-up curve T for $p = 2$ and $f(x) = (1+\sqrt{2.3}) + \frac{1}{2\pi}\sin(2\pi x)$, $g(x) = (1+\sqrt{2.3}) - \frac{1}{2\pi}\sin(2\pi x)$. In Figure 1, we see that the numerical blow-up curve $\boldsymbol{T}$ converges to a smooth function as $\tau \to 0$. Therefore, we numerically obtain that the blow-up curve T is continuously differentiable if initial values f and g are smooth and large enough. In Figure 2, we also obtain the same result for $p = 3$.

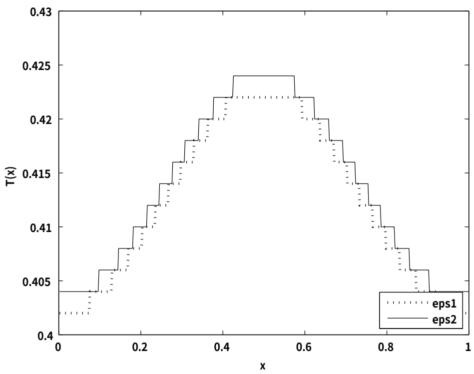
On the other hand, we obtain different results of regularity of the blow-up curve in Figure 3. We see that there is a case where the blow-up curve has the singular points. We notice that all the initial values are smooth in Figures 1–3. However, the initial values f and g occur the sign changes in Figure 3, while the initial values f and g are positive for $x \in (0, 1)$ in the case of Figures 1 and 2.

Consequently, we see that we have to impose not only regularity but also largeness on the initial values.

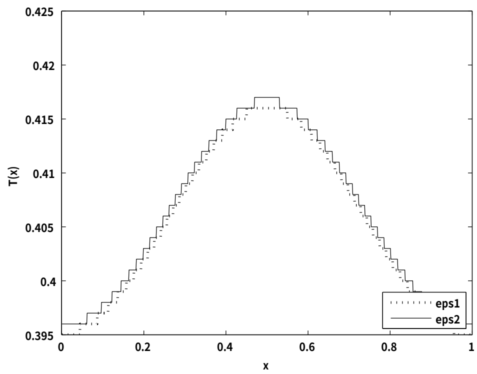
Remark 3.7.1. Merle and Zagg [27] considered

$$\partial_t^2 u - \partial_x^2 u = u^p.$$

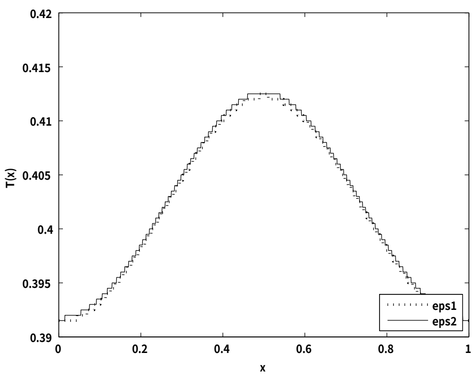
They analytically showed that there are cases where the blow-up curve T has the singular points. However, we do not know the relationship between the our numerical results and the results of [27]



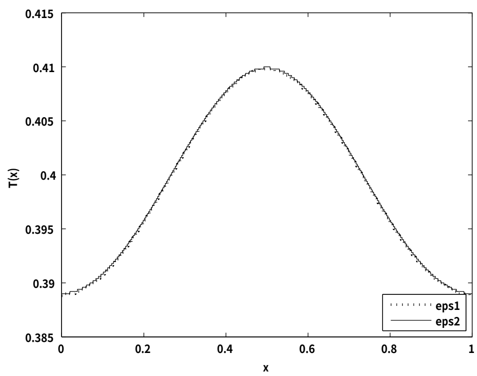
$\tau = h = 1/500.$



$\tau = h = 1/1000.$

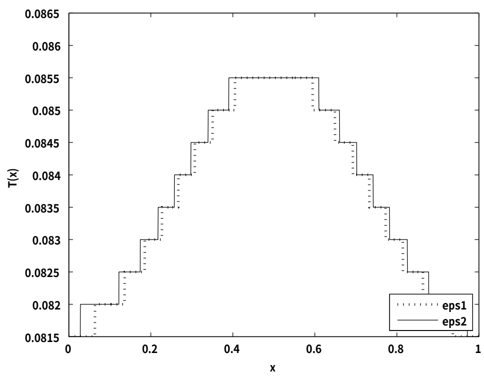


$\tau = h = 1/2000.$

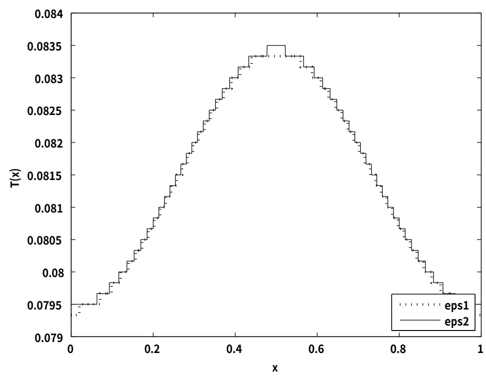


$\tau = h = 1/5000.$

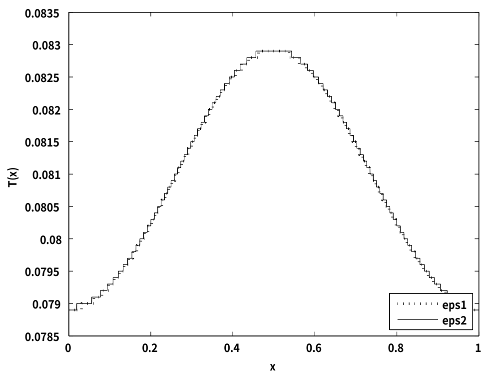
Figure 3.1: The history of (T_j) for $p = 2$, $f(x) = (1 + \sqrt{2.3}) + \frac{1}{2\pi}\sin(2\pi x)$ and $g(x) = (1 + \sqrt{2.3}) - \frac{1}{2\pi}\sin(2\pi x)$ and stopping criteria $\mathrm{eps1} = 1e - 2$ and $\mathrm{eps2} = 1e - 3$.



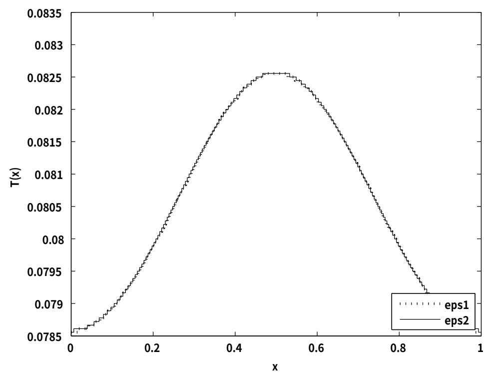
$\tau = h = 1/2000.$



$\tau = h = 1/6000.$

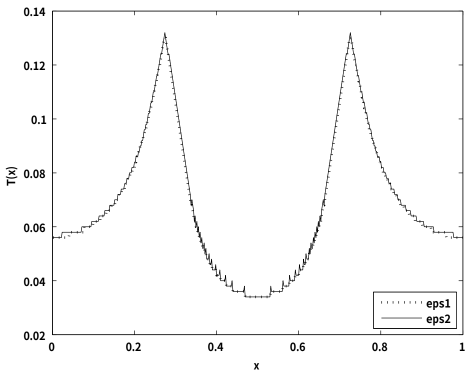


$\tau = h = 1/10000.$

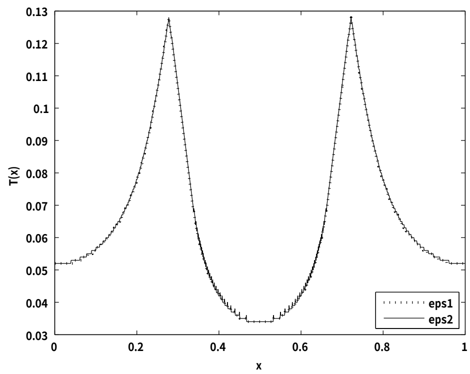


$\tau = h = 1/18000.$

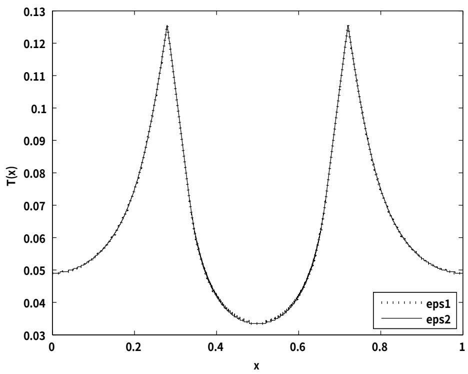
Figure 3.2: The history of (T_j) for $p = 3$, $f(x) = 2.5 + \frac{1}{2\pi}\sin(2\pi x)$, $g(x) = 2.5 - \frac{1}{2\pi}\sin(2\pi x)$ and stopping criteria eps1 $= 1e-2$ and eps2 $= 1e-3$.



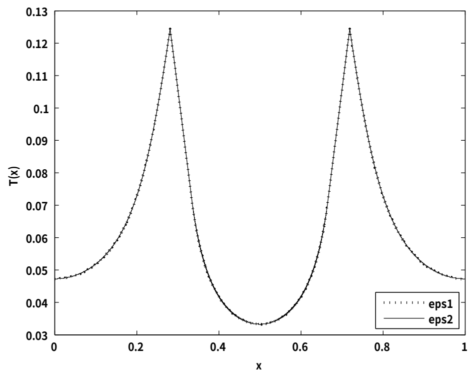
$\tau = h = 1/500.$



$\tau = h = 1/1000.$



$\tau = h = 1/2000.$



$\tau = h = 1/5000.$

Figure 3.3: The history of (T_j) for $p = 3$, $f(x) = 2 + 10\sin(2\pi x)$, $g(x) = 2 - 10\sin(2\pi x)$ and stopping criteria eps1 $= 1e-2$, and eps2 $= 1e-3$.

Bibliography

[1] L. M. Abia, J. C. López-Marcos and J. Martínez: *On the blow-up time convergence of semidiscretizations of reaction-diffusion equations*, Appl. Numer. Math. **26** (1998) 399–414.

[2] S. Alinhac: *Blowup for Nonlinear Hyperbolic Equations,* Birkhäuser, 1995.

[3] C. Besse, R. Carles, N. J. Mauser and H. P. Stimming: *Monotonicity properties of the blow-up time for nonlinear Schrödinger equations : numerical evidence*, Discrete Contin. Dyn. Syst. Ser. B **9** (2008) 11–36.

[4] C. Besse, B. Bidegaray and S. Descombes: *Order estimates in the time of splitting methods for the nonlinear Schrödinger equation*, SIAM J. Numer. Anal. **40** (2002) 26-40.

[5] J. P. Borgna, M. De Leo, D. Rial and C. Sánchez de la Vega: *General splitting methods for abstract semilinear evolution equations*, Commun. Math. Sci. **13** (2015) 83–101.

[6] L. A. Caffarelli and A. Friedman: *Differentiability of the blow-up curve for one-dimensional nonlinear wave equations*, Arch. Rational Mech. Anal. **91** (1985) 83–98. a

[7] L. A. Caffarelli and A. Friedman: *The blow-up boundary for nonlinear wave equations*, Trans. Amer. Math. Soc. **297** (1986) 223–241.

[8] T. Cazenave and A. Haraux: *An introduction to semilinear evolution equations*, The Clarendon Press, Oxford University Press, (1998).

[9] Y. G. Chen: *Asymptotic behaviours of blowing-up solutions for finite difference analogue of* $u_t = u_{xx} + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986) 541–574.

[10] C. H. Cho: *A finite difference scheme for blow-up solutions of nonlinear wave equations*, Numer. Math. Theory Methods Appl. **3** (2010) 475–498.

[11] C. H. Cho: *On the computation of the numerical blow-up time,* Jpn. J. Ind. Appl. Math. **30** (2013) 331–349.

[12] C. H. Cho: *On the computation for blow-up solutions of the nonlinear wave equation,* SIAM J. Numer. Anal. (submitted).

[13] C. H. Cho, S. Hamada and H. Okamoto: *On the finite difference approximation for a parabolic blow-up problem*, Japan J. Indust. Appl. Math. **24** (2007) 131–160.

[14] S. Descombes and M. Thalhammer: *An exact local error representation of exponential operator splitting methods for evolutionary problems and applications to linear Schrödinger equations in the semi-classical regime*, BIT **50** (2010) 729–749.

[15] L. C. Evans: Partial Differential Equations, American Mathematical Society, 1998.

[16] R. T. Glassey: *Blow-up theorems for nonlinear wave equations*, Math. Z., **132** (1973) 183–203.

[17] R. T. Glassey: *Finite-time blow-up for solutions of nonlinear wave equations*, Math. Z. **177** (1981) 323–340.

[18] P. Godin: *The blow-up curve of solutions of mixed problems for semilinear wave equations with exponential nonlinearities in one space dimension. I*, Calc. Var. Partial Differential Equations **13** (2001) 69–95.

[19] P. Godin: *A discrete phenomenon in the C^∞ and analytic regularity of the blow-up curve of solutions to the Liouville equation in one space dinemsion,* J. Differential Equations **183** (2002) 224–238.

[20] E. Hairer, C. Lubich and G. Wanner: *Geometric numerical integration (2nd ed.)*, Springer-Verlag, Berlin Heidelberg, 2006.

[21] T. Jahnke and C. Lubich: *Error bounds for exponential operator splittings*, BIT. **40** (2000) 735–744.

[22] F. John: *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979) 235–268.

[23] T. Kato: *Blow-up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **33** (1980) 501–505.

[24] H. Kawarada: *On solutions of nonlinear wave equations*, J. Phys. Soc. Japan **31** (1971) 280–282.

[25] C. Lubich: *On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations*, Math. Comput. **77** (2008) 2141-2153.

[26] K. Matsuya: *A blow-up theorem of a discrete semilinear wave equation*, J. Difference Eq. Appl. **19** (2013) 457–465.

[27] F. Merle and H. Zaag: *Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension,* Amer. J. Math., **134** (2012), no. 3, 581–648.

[28] T. Nakagawa: *Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$*, Appl. Math. Optim. **2** (1976) 337–350.

[29] T. Nakagawa and T. Ushijima: *Finite element analysis of the semi-linear heat equation of blow-up type*, Topics Numer. Anal. III, Academic Press, (1977) 275–291.

[30] M. Ohta and H. Takamura: *Remarks on the blow-up boundaries and rates for nonlinear wave equations,* Nonlinear Anal. **33** (1998) 693–698.

[31] N. Saito and T. Sasaki: *Blow-up of finite-difference solutions to nonlinear wave equations*, to appear in J. Math. Sci. Univ. Tokyo.

[32] T. Sasaki: *A second-order time-discretization scheme for a system of nonlinear Schrödinger equations*, Proc. Japan. Acad. Ser. A Math. Sci. **90** (2014) 15–20.

[33] H. Uesaka: *The blow-up boundary for a system of semilinear wave equations,* Further progress in analysis, World Sci. Publ., Hackensack, NJ, (2009) 845–853.

[34] T. K. Ushijima: *On the approximation of blow-up time for solutions of nonlinear parabolic equations*, Publ. Res. Inst. Math. Sci. **36** (2000) 613–640.

[35] S. Wang and L. Zhang: *An efficient split-step compact finite difference method for cubic-quintic complex Ginzburg-Landau equations,* Comput. Phys. Commun. **184** (2013) 1511–1521.