博士論文

論文題目　Rational singularities, ω-multiplier ideals and cores of ideals
（ 有理特異点、ω-乗数イデアルとイデアルのコア ）

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Chapter 1

Introduction

In this paper, we always assume that a ring is a domain essentially of finite type over $\mathbb{C}$ and a variety is an irreducible reduced separated scheme of finite type over $\mathbb{C}$.

Rees and Sally introduced the cores of ideals in [33]. Okuma, Watanabe and Yoshida characterized 2-dimensional local ring with a rational singularity via cores of ideals in [32]. However, in higher dimensional case we have a counterexample to the characterization. We will show another characterization of local ring with a rational singularity of arbitrary dimension via cores of ideals. We, namely, will prove the following:

Theorem 1.0.1. *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional Cohen-Macaulay local ring with an isolated singularity. Then* A *is a rational singularity if and only if* $\overline{I^n} \subset \operatorname{core}(I)$ *for any* $\mathfrak{m}$*-primary ideal* I*.*

By this Theorem, we show that a Cohen-Macaulay local ring with an isolated singularity has a rational singularity if Briançon-Skoda Theorem holds for the ring. Lipman and Teissier showed that for a local ring with rational singularities, Briançon-Skoda Theorem holds in [28]. Therefore a Cohen-Macaulay local ring with an isolated singularity has a rational singularity if and only if Briançon-Skoda Theorem holds for the ring.

The multiplier ideals are fundamental tools in birational geometry. In this paper we introduce a new notion an "ω-multiplier ideal" which has similar properties and works in a slightly different way than a multiplier ideal. The main goal of this paper is to prove the properties of ω-multiplier ideals and show some applications.

For the definition of the multiplier ideals we used the discrepancies. In order for the discrepancy to be well-defined, we need to assume that the variety is normal and $\mathbb{Q}$-Gorenstein. The advantage of ω-multiplier ideals is that they can be defined on any normal variety. If a variety X is normal Gorenstein, then the ω-multiplier ideal $\mathcal{J}^{\omega}(X, \mathfrak{a}^c)$ is equal to the usual multiplier ideal $\mathcal{J}(X, \mathfrak{a}^c)$ for any ideal $\mathfrak{a}$.

One of the most important theorem of the multiplier ideals is the Skoda's Theorem. We will prove that the Skoda's Theorem of ω-multiplier ideals of a local ring with a rational singularity.

Proposition 1.0.2. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an m-primary ideal and J be a reduction of $\mathfrak{a}$. Then for $n \in \mathbb{Z}_{\geq 2}$,*

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}).$$

Huneke and Swanson proved the many properties of cores of ideals of 2-dimensional regular local ring and the relationships between the core of an ideal and multiplier ideal of 2-dimensional regular local ring in [13]. We generalize their results to rational singularities using ω-multiplier ideals. We will prove the followings:

Proposition 1.0.3. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then*
(1) $\mathrm{core}(\mathfrak{a}) = \mathcal{J}^{\omega}(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a})$.
(2) $\mathrm{e}(\mathfrak{a}) = \ell(A/\mathrm{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^{\omega}(A, \mathfrak{a}))$.
(3) $\mathcal{J}^{\omega}(A, \mathrm{core}(\mathfrak{a})) = (\mathcal{J}^{\omega}(A, \mathfrak{a}))^2$.
(4) $\mathrm{core}(\mathfrak{a}^n) = \mathfrak{a}^{2n-1}\mathcal{J}^{\omega}(A, \mathfrak{a})$.
(5) $\mathrm{core}^n(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^{\omega}(A, \mathfrak{a}))^{2^n-1}$. *In particular,* $\mathrm{core}(\mathrm{core}(\mathfrak{a})) = \mathfrak{a}(\mathcal{J}^{\omega}(A, \mathfrak{a}))^3$.

Demailly, Ein and Lazarsfeld proved the subadditivity theorem for multiplier ideals on non-singular varieties in [4]. This theorem gives many applications of commutative algebra and algebraic geometry. Takagi and Watanabe proved that the subadditivity theorem holds for a 2-dimensional log terminal local ring in [37]. Moreover they showed the characterization of a 2-dimensional log terminal local ring via the subadditivity of multiplier ideals.

Hence it makes sense to consider the subadditivity of ω-multiplier ideals. We show the characterization of 2-dimensional local ring with a rational singularity via the subadditivity of ω-multiplier ideals.

Theorem 1.0.4. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec}A$ has a rational singularity if and only if the subadditivity theorem holds, that is, for any two ideal $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^{\omega}(X, \mathfrak{a})\mathcal{J}^{\omega}(X, \mathfrak{b}).$$

To use the subadditivity of ω-multiplier ideals, we investigate the subadditivity of cores of ideals. We show the characterization of 2-dimensional local ring with a rational singularity via the subadditivity of cores of ideals.

Corollary 1.0.5. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec}A$ is rational singularities if and only if the subadditivity theorem hold, that is, for any two $\mathfrak{m}$-primary integral closed ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathrm{core}(\mathfrak{a}\mathfrak{b}) \subset \mathrm{core}(\mathfrak{a})\mathrm{core}(\mathfrak{b}).$$

Moreover in [37] Takagi and Watanabe showed that a 2-dimensional normal ring is regular if the strong subadditivity theorem for the ring holds. We will consider the problem of a version of ω-multiplier ideals. We will prove the following:

Proposition 1.0.6. *Let (A, m) be a two-dimensional normal local ring essentially of finite type over $\mathbb{C}$. Then $X = \mathrm{Spec}A$ is regular if and only if the strong subadditivity theorem hold, that is, for any two ideal $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$ and any rational number $c, d > 0$,*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c\mathfrak{b}^d) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^c)\mathcal{J}^{\omega}(X, \mathfrak{b}^d).$$

A multiplier ideal is an integrally closed ideal. It is natural to ask that an integrally closed ideal is a multiplier ideal. In general multiplier ideals are not integrally closed ideals (see [25], [26]). Favre, Jonsson, Lipman and Watanabe gave an answer to this question when a ring is 2-dimensional regular local ring. That is, they showed that all integrally closed ideals on a regular local ring are multiplier ideals in [10] and [29]. Moreover Tucker generalized the result to a log terminal local ring in [38]. On the other hand we generalize this theorem to rational singularities by using ω-multiplier ideals. In other words, we will prove the following:

Theorem 1.0.7. *Let (A, m) be a two-dimensional local normal ring. Suppose $X = \mathrm{SpecA}$ is a rational singularity. Then every integrally closed ideal is an ω-multiplier ideal.*

Another application of ω-multiplier ideals is an upper bound of the multiplicity of a Du Bois singularity. Huneke and Watanabe gave an upper bound on the multiplicity of a rational singularity in [15]. That is, they showed the following:

Theorem 1.0.8. ([15]) *Let X be an n-dimensional variety with rational singularities. Then for a closed point $x \in X$*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x) - 1}{n - 1}.$$

In [15], Huneke and Watanabe asked the following

Question 1.0.9. Let X be an n-dimensional variety with Du Bois singularities. Is it true that for a closed point $x \in X$

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n} \ ?$$

We give the affirmative answer to the question under the condition that X is a normal Cohen-Macaulay variety.

Theorem 1.0.10. *Let X be an n-dimensional normal Cohen-Macaulay variety with Du Bois singularities. Then for a closed point $x \in X$*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n}.$$

In Chapter 2, we define rational singularities, the Mather-Jacobian discrepancy and cores of ideals and collect their results.

In Chapter 3, we define ω-multiplier ideals and prove their properties. Further we characterize local ring with a rational singularity of arbitrary dimension via cores of ideals.

In Chapter 4, we study ω-multiplier ideals of a 2-dimensional local ring with a rationals singularity. In section 4.1, we discuss the various relationships between the a core of an ideal and a ω-multiplier ideal of a 2-dimensional local ring with a rational singularity. In section 4.2, we investigate when the subadditivity theorem of ω-multiplier ideals holds in the two-dimensional case. In section 4.3, we show that all integrally closed ideals on surface with a rational singularity are ω-multiplier ideals.

In Chapter 5, we give an upper bound of the multiplicity of a Du Bois singularity.

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Chapter 2

Preliminaries

2.1 Rational singularities and Du Bois singularities

In this section, we define rational singularities and Du Bois singularities.

Definition 2.1.1. We say that a local ring A has rational singularities if A is normal and there exists a desingularizaion $Y \to \mathrm{Spec}A$ with $H^i(Y, \mathcal{O}_Y) = 0$ for every $i > 0$.

The following is well known as a characterization of rational singularities in characteristic zero (see for example [21])

Proposition 2.1.2. *Let A be a normal local Cohen-Macaulay ring essentially of finite type over a field of characteristic 0. The scheme $X = \mathrm{Spec}A$ has rational singularities if and only if there exists a desingularizaion $Y \to X$ with $f_*\omega_Y = \omega_X$, where ω_Y and ω_X are the canonical sheaves of Y and X, respectively.*

Definition 2.1.3. Suppose that X is a reduced scheme embedded as a closed subscheme of a smooth scheme Y. Let $f : \widetilde{Y} \to Y$ be a log resolution of (Y, X) that is an isomorphism outside of X. Let E denote $(f^{-1}(X))_{\mathrm{red}}$. Then X is said to have Du Bois singularities if the natural map $\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_E$ is a quasi-isomorphism.

First Du Bois singularities are introduced by Steenbrink with the different definition in [36], but Schwede ([35]) showed that it is equivalent to the condition in Definition 2.1.3.

Kovács, Schwede and Smith characterized normal Cohen-Macaulay Du Bois singularities.

Theorem 2.1.4. [22] *Suppose that X is normal and Cohen-Macaulay. Let $\pi : Y \to X$ be any log resolution and denote the reduced exceptional divisor of π by G. Then X has Du Bois singularities if and only if $\pi_*\omega_Y(G) = \omega_X$.*

Using this theorem, it is easy to see that Cohen-Macaulay log canonical singularities are Du Bois singularities and Gorenstein Du Bois singularities are log canonical singularities.

Remark 2.1.5. Kollár and Kovács showed that log canonical singularities are Du Bois singularities even if the singularities are not Cohen-Macaulay (See [20]).

2.2 Mather-Jacobian minimal log discrepancy

We start by recalling the definition and basic properties of Mather-Jacobian log discrepancy which is defined in [7], [8]. We refer to [7] for further details. Let X is a variety of dimension $\dim X = n$. The sheaf Ω_X^n is invertible over the smooth locus X_{reg} of X, hence the projection

$$\pi : \mathrm{P}(\Omega_X^n) \to X$$

is an isomorphism over X_{reg}. The Nash blow up $\widehat{X} \to X$ is defined as the closure of $\pi^{-1}(X_{\text{reg}})$ in $\mathrm{P}(\Omega_X^n)$.

If $V \supset X$ is an n-dimensional reduced, locally complete intersection scheme, then Nash blow up $\pi : \widehat{X} \to X$ is isomorphic to the blow-up of the ideal $\mathfrak{j}_V|_X$, where $\mathfrak{j}_V$ is the Jacobian ideal of V (see Proposition 2.4 in [3]).

Definition 2.2.1. Let $f : Y \to X$ be a resolution of singularities of X that factors through the Nash blow-up of X. The image of the canonical homomorphism

$$f^*(\Omega^n_X) \to \Omega^n_Y$$

is an invertible sheaf of the form $Jac_f \Omega^n_Y$, where Jac_f is the relative Jacobian which is an invertible ideal on Y and defines an effective divisor supported on the exceptional locus of f. The divisor is called the Mather discrepancy divisor and denoted by $\widehat{K}_{Y/X}$.

Remark 2.2.2. Let X be an n-dimensional normal variety and $V \supset X$ be an n-dimensional reduced, locally complete intersection scheme. If $f : Y \to X$ is a log resolution of $\mathfrak{j}_V|_X$ such that $\mathfrak{j}_V|_X \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, then we have $\widehat{K}_{Y/X} = K_Y + J_V - f^*(K_V|_X)$ (see [3]).

Definition 2.2.3. Let $f : Y \to X$ is a log resolution of $\mathfrak{j}_X$, where $\mathfrak{j}_X$ is the Jacobian ideal of a variety X. We denote by $J_{Y/X}$ the effective divisor on Y such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. This divisor is called the Jacobian discrepancy divisor.

Here, we note that every log resolution of $\mathfrak{j}_X$ factors through the Nash blow-up, see for example, Remark 2.3, in [8].

Definition 2.2.4. Let X be an n-dimensional normal variety and V be a reduced locally complete intersection n-dimensional scheme containing X. The ideal $\mathfrak{d}_{X,V}$ is the ideal such that

$$\mathrm{Im}(\omega_X \to \omega_V|_X) = \mathfrak{d}_{X,V} \otimes \omega_V|_X.$$

Remark 2.2.5. Let M be a smooth variety containing X and V. Consider the ideals I_X and I_V of X and V in M. Then, as $\mathcal{O}_V$-modules, we have

$$\omega_X \otimes \omega_V^{-1} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_X, \mathcal{O}_V) = (I_V : I_X)/I_V,$$

and therefore

$$\mathfrak{d}_{X,V} = ((I_V : I_X) + I_X)/I_X.$$

In other words, if we write $V = X \cup X'$, where X' is the residual part of V with respect to X (given by the ideal $(I_V : I_X)$), then $\mathfrak{d}_{X,V}$ is the ideal defining the intersection $X \cap X'$ in X.

Definition 2.2.6. Let X be a normal variety. The lci-defect ideal of X is defined to be

$$\mathfrak{d}_X = \sum_V \mathfrak{d}_{X,V},$$

where the sum is taken over all reduced, locally complete intersection schemes $V \supset X$ of the same dimension.

Remark 2.2.7. The support of the lci-defect ideal of X is locally a non-complete intersection locus of X. In particular $\mathfrak{d}_X = \mathcal{O}_X$ if X is locally a complete intersection.

Definition 2.2.8. A normal variety X is said to be $\mathbb{Q}$-Gorenstein if its canonical divisor K_X is $\mathbb{Q}$-Cartier.

Definition 2.2.9. Let X be an n-dimensional normal $\mathbb{Q}$-Gorenstein variety and V be a reduced locally complete intersection n-dimensional scheme containing X. Let r be a positive integer such that rK_X is Cartier. The ideal $\mathfrak{d}_{r,X,V}$ is the ideal such that

$$\mathrm{Im}(\mathcal{O}_X(rK_X) \to (\omega_V|_X)^{\otimes r}) = \mathfrak{d}_{r,X,V} \otimes (\omega_V|_X)^{\otimes r}.$$

Definition 2.2.10. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let r be a positive integer such that rK_X is Cartier. The lci-defect ideal of level r of X is defined to be

$$\mathfrak{d}_{r,X} = \sum_V \mathfrak{d}_{r,X,V},$$

where the sum is taken over all reduced, locally complete intersection schemes $V \supset X$ of the same dimension.

Proposition 2.2.11. ([3]) *Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let r be a positive integer such that rK_X is Cartier. Then $\mathfrak{d}_X^r \subset \overline{\mathfrak{d}_{r,X}}$.*

Remark 2.2.12. If X is Gorenstein, then $\mathfrak{d}_X = \mathfrak{d}_{1,X}$. In general however $\overline{\mathfrak{d}_X^r} \neq \overline{\mathfrak{d}_{r,X}}$.

Definition 2.2.13. Let X be a normal $\mathbb{Q}$-Gorestein variety. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Given a log resolution $f : Y \to X$ of $\mathfrak{a}_1 \cdots \mathfrak{a}_r$, we denote by $Z_1, \ldots, Z_r$ the effective divisors on Y such that

$\mathfrak{a}_i\mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. For a prime divisor E over X such that E appears on Y, we define the log discrepancy at E as

$$a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) := \mathrm{ord}(K_{Y/X}) - \mathrm{ord}_E(t_1 Z_1 + \cdots + t_r Z_r) + 1.$$

Definition 2.2.14. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Given a log resolution $f : Y \to X$ of $\mathfrak{j}_X \mathfrak{a}_1 \cdots \mathfrak{a}_r$, we denote by $Z_1, \ldots, Z_r$ the effective divisors on Y such that $\mathfrak{a}_i\mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. For a prime divisor E over X such that E appears on Y, we define the Mather-Jacobian-log discrepancy at E as

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X} - t_1 Z_1 - \cdots - t_r Z_r) + 1.$$

Remark 2.2.15. If X in normal and locally a complete intersection, then $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$. Indeed, in this case the image of the canonical map $\Omega_X^n \to \omega_X$ is $\mathfrak{j}_X\omega_X$, hence $\widehat{K}_{Y/X} - J_{Y/X} = K_{Y/X}$. In particular, we see that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ if X is smooth.

Note that the Mather-Jacobian log discrepancy at a prime divisor E does not depend on the choice of f. We denote $\mathrm{ord}_E \widehat{K}_{Y/X}$ by $\widehat{k}_E$.

Definition 2.2.16. Let X be a normal $\mathbb{Q}$-Gorestein variety and $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Then $(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ is canonical (resp. log canonical) if for every exceptional prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) holds.

Definition 2.2.17. Let X be a variety and $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Then $(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ is MJ-canonical (resp. MJ-log canonical) if for every exceptional prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) holds.

Remark 2.2.18. Fix a log resolution $Y \to X$ of $\mathfrak{j}_X \mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Then $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_r)$ is MJ-canonical (resp. MJ-log canonical) if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) for all exceptional prime divisor E on Y. This is proved by using the fact that

$$\widehat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + g^*(\widehat{K}_{Y/X} - J_{Y/X})$$

for a sequence $Y' \xrightarrow{g} Y \xrightarrow{f} X$ of such log resolution of $\mathfrak{j}_X \mathfrak{a}_1, \ldots, \mathfrak{a}_r$.

Definition 2.2.19. Let X be a normal $\mathbb{Q}$-Gorenstein variety and $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \dots, t_r \in \mathbb{R}$. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}_1, \dots, \mathfrak{a}_r$. Define $Z_1, \dots, Z_r$ by $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \le i \le r$. Then we can define the multiplier ideal as follows:

$$\mathcal{J}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = f_* \mathcal{O}_Y(\lceil K_{Y/X} - t_1 Z_1 - \cdots - t_r Z_r \rceil).$$

Definition 2.2.20. Let X be a normal $\mathbb{Q}$-Gorestein variety. X is said to be a log terminal singularities if $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$.

Remark 2.2.21. Log terminal singularities are rational singularities.

Definition 2.2.22. Let X be a variety and $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \dots, t_r \in \mathbb{R}$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \mathfrak{a}_1, \dots, \mathfrak{a}_r$. Define $Z_1, \dots, Z_r$ by $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \le i \le r$. Then we can define the Mather-Jacobian multiplier ideal (or MJ-multiplier ideal for short) as follows:

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - [t_1 Z_1 - \cdots - t_r Z_r]).$$

Remark 2.2.23. Multiplier ideals and Mather-Jacobian multiplier ideals are independent of the choice of a log resolution.

Proposition 2.2.24. ([3], [8]) *If X is MJ-canonical, then it is normal and has rational singularities.*

Proposition 2.2.25. ([3]) *If X is MJ-log canonical, then it has Du Bois singularities.*

There are the relations between jet scheme and Mather-Jacobian minimal log discrepancy (see [3], [7], [18]). For the theory on jet schemes and arc space, see for example [9].

2.3 Cores of ideals

In this section, we define cores of ideals and collect their results.

Definition 2.3.1. Let A be a ring and I be an ideal of A. An ideal $J \subset I$ is called a reduction of I if there is a positive number r such that $JI^r = I^{r+1}$. An ideal $J \subset I$ is called a minimal reduction of I if J is minimal among the reductions of I.

Definition 2.3.2. Let A be a ring and I be an ideal of A. Let $f : Y \to X = \mathrm{Spec}A$ be the normalized blowing up of I such that $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. The integral closure of I is defined to be $f_*\mathcal{O}_Y(-F)$. We denote it by $\overline{I}$

Definition 2.3.3. Let X be an n-dimensional scheme. Suppose that a dualizing complex for X exists. A canonical sheaf ω_X for X is defined to be the coherent sheaf given by $(-n)$-th cohomology of a normalized dualizing complex for X.

Remark 2.3.4. Dualizing complexes exist for any equidimensional scheme essentially of finite type over an affine Gorenstein scheme (see [12]). If X is a normal algebraic variety, then the usual notion of the canonical sheaf provides the canonical sheaf of X. In the case $X = \mathrm{Spec}A$ where A is a local ring, ω_X coincides with the sheafification of the canonical module ω_A.

Let $f : Y \to X$ be a birational morphism of integral schemes. Then the trace map $\mathrm{Tr}_f : f_*\omega_Y \to \omega_X$ is injective, and it is important to observe that in this case we can consider Tr_f as an inclusion $f_*\omega_Y \subset \omega_X$.

Hyry and Villamayor proved the following lemma in [16].

Lemma 2.3.5. (Lemma 2.2 in [16]) *Let $(A, \mathfrak{m})$ be a local ring. Let $f : Y \to X = \mathrm{Spec}A$ be a proper birational morphism such that Y has rational singularities. Then $H^0(Y, \omega_Y) \subset H^0(Z, \omega_Z)$ for any proper birational morphism $g : Z \to X$. It follows, in particular, that $H^0(Y, \omega_Y) = H^0(Z, \omega_Z)$ if Z has rational singularities.*

Definition 2.3.6. Let A be a Noetherian local ring and I an ideal. The core of I, denoted core(I), is the intersection of all its reductions.

Definition 2.3.7. Let $(A, \mathfrak{m})$ be a local ring. An ideal I of A is equimultiple if a minimal reductions of I are generated by h elements, where $h = \mathrm{ht}(I)$.

Example 2.3.8. Every $\mathfrak{m}$-primary ideal in a local ring is equimultiple.

By the following theorem, we are able to compute core ideals for equimultiple ideals in Cohen-Macaulay local rings whose residue field has characteristic 0.

Theorem 2.3.9. ([14], Theorem 3.7) *Let A be a Cohen-Macaulay local ring. Let I be an equimultiple ideal of A with $h = \mathrm{ht}(I) \geq 1$, let J be a minimal reduction of I, and let r be a positive number such that $JI^r = I^{r+1}$. Then*

$$\mathrm{core}(I) = J^{r+1} : I^r.$$

Lemma 2.3.10. ([17], Lemma 3.1.5) *Let $(A, \mathfrak{m})$ be a local ring and let I be a proper ideal of A of height greater than one. Let $Y = \mathrm{Proj}A[I]$. Then $H^0(Y, I^{n+p}\omega_Y) :_{\omega_A} I^p = H^0(Y, I^n\omega_Y)$ for all $n \geq 0$ and all $p \geq 1$.*

Lemma 2.3.11. ([17], Lemma 5.1.6) *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, and I be an equimultiple ideal of height h. Then*

$$H^0(Y, I^h\omega_Y) :_A \omega_A = J^{r+1} :_A I^r$$

where $Y = \mathrm{Proj}A[I]$, $H^0(Y, I^h\omega_Y)$ is considered as a submodule of ω_A via the trace map, and J is any reduction of I with $JI^r = I^{r+1}$.

Theorem 2.3.12. ([17], Corollary 5.3.1) *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with rational singularities, and I be an equimultiple ideal of height h such that the Rees ring $A[It]$ is normal and Cohen-Macaulay. Let $Y = \mathrm{Proj}A[It]$. Then the following conditions are equivalent*
(1) *$A[It]$ has rational singularities;*
(2) *$H^0(Y, I^n\omega_Y) = \mathcal{J}(I^n)$ for all $n \geq 0$;*
(3) *$\mathrm{core}(I) = \mathcal{J}(I^h)$.*
If this is the case, then

$$\mathrm{core}(I) = I\mathcal{J}(I^{h-1}),$$

$$\mathcal{J}(I^{h-1}) = \mathrm{core}(I) : I.$$

Chapter 3

ω-multiplier ideals and cores of ideals

3.1 ω-multiplier ideals

In this section, we define ω-multiplier ideals and prove some properties of ω-multiplier ideals.

Definition 3.1.1. Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$, $c \in \mathbb{Q}_{>0}$ and let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. The ω-multiplier ideal of a pair $(X, \mathfrak{a}^c)$ is defined to be $f_*(\omega_Y(-[cF])) : \omega_X$. We will denote it by $\mathcal{J}^\omega(X, \mathfrak{a}^c)$.

Definition 3.1.2. Let X be a variety with rational singularities and $\mathfrak{a} \subsetneq \mathcal{O}_X$ be a nonzero ideal of $\mathcal{O}_X$. The rational threshold of a pair $(X, \mathfrak{a})$ is defined to be $\sup\{c > 0 | \mathcal{J}^\omega(X, \mathfrak{a}^c) = \mathcal{O}_X\}$. We will denote it by $\mathrm{rt}(X, \mathfrak{a})$.

Theorem 3.1.3. (Theorem 6.15 in [3]) *Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal and $c \in \mathbb{Q}_{>0}$. Then we have*

$$\mathcal{J}^\omega(X, \mathfrak{a}^c) = \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c \mathfrak{d}_X^{-1}).$$

De Fernex and Decampo prove the following in the proof of Theorem 6.15 in [3].

Theorem 3.1.4. ([3]) *Let X be a normal variety and $\mathfrak{a}$ be a nonzero ideal sheaf of $\mathcal{O}_X$. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $\mathfrak{d}_{V,X}$ be the ideal determined by the image of $\omega_X \to \omega_V|_X$. Let $f: Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Then*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V - [cF]).$$

Corollary 3.1.5. *Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal and $c \in \mathbb{Q}_{>0}$. Then we have*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) \supset \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c).$$

In particular, If X is locally a complete intersection, then

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c).$$

Theorem 3.1.6. (Theorem 7.1 in [3]) *Let X be a normal variety and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X. Let $f: Y \to X$ be a log resolution of $\mathfrak{d}_X$ and denote by E the reduced exceptional divisor. Then the following properties hold:*
(i) The pair $(X, \mathfrak{d}_X^{-1})$ is MJ-canonical if and only if $\mathcal{J}^{\omega}(X, \mathcal{O}_X) = \mathcal{O}_X$.
(ii) The pair $(X, \mathfrak{d}_X^{-1})$ is MJ-log canonical if and only if $f_*\omega_Y(E) = \omega_X$.

Corollary 3.1.7. (Corollary 7.2 in [3]) *Let X be a normal variety, and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X. Then the following properties hold:*
(i) If X has rational singularities, then $(X, \mathfrak{d}_X^{-1})$ is MJ-canonical.
(ii) If X has Du Bois singularities, then $(X, \mathfrak{d}_X^{-1})$ is MJ-log canonical.
Moreover, the converse holds in both cases whenever X is Cohen-Macaulay.

This corollary implies the following corollary.

Corollary 3.1.8. *Let X be a Cohen-Macaulay normal variety. Then X has rational singularities if and only if $\mathcal{J}^{\omega}(X, \mathcal{O}_X) = \mathcal{O}_X$.*

The following proposition gives the relation of Mather Jacobian discrepancies and usual multiplier discrepancies.

Proposition 3.1.9. (Proposition 3.4 in [3]) *Let X be a $\mathbb{Q}$-Gorenstein normal variety. Let r be a positive integer such that rK_X is Cartier. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot \mathfrak{d}_{r,X}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $\mathfrak{d}_{r,X} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{r,Y/X})$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$ and $D_{r,Y/X}$ on Y. Then*

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} \geq \widehat{K}_{Y/X} - J_{Y/X} + D_{r,Y/X} = K_{Y/X}.$$

In particular, if X is Gorenstein, then

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} = K_{Y/X}.$$

The following proposition gives the relation of ω-multiplier ideals and usual multiplier ideals is an immediate consequence of the above proposition.

Proposition 3.1.10. *Let X be a $\mathbb{Q}$-Gorenstein normal variety, $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$ and $c \in \mathbb{Q}_{>0}$. Then $\mathcal{J}^\omega(X, \mathfrak{a}^c) \supset \mathcal{J}(X, \mathfrak{a}^c)$. In particular, if X is Gorenstein, $\mathcal{J}^\omega(X, \mathfrak{a}^c) = \mathcal{J}(X, \mathfrak{a}^c)$.*

The assertion in the next proposition are an immediate consequence of the definition.

Proposition 3.1.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals on a normal variety X, and $c > 0$.*
(1) *If $\mathfrak{a} \subset \mathfrak{b}$, then $\mathcal{J}^\omega(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{b}^c)$.*
(2) *If $c \geq d$ are in $\mathbb{Q}_{>0}$, then $\mathcal{J}^\omega(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^d)$.*
(3) *$\mathcal{J}^\omega(X, \mathfrak{a}^c) = \mathcal{J}^\omega(X, \overline{\mathfrak{a}}^c)$, where $\overline{\mathfrak{a}}$ is integrally closure of $\mathfrak{a}$.*

Proposition 3.1.12. *Let $\mathfrak{a}$ be a nonzero ideal on a normal variety X, and $c > 0$.*
(1) *The ω-multiplier ideal $\mathcal{J}^\omega(X, \mathfrak{a}^c)$ is an integrally closed ideal of $\mathcal{O}_X$.*
(2) *Suppose that X has rational singularities. Then $\mathfrak{a} \subset \mathcal{J}^\omega(X, \mathfrak{a})$.*

Proof. Let $\mathfrak{j}_X$ be the Jacobian ideal of X and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$ and F on Y. Then we have $\mathcal{J}^\omega(X, \mathfrak{a}^c) = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - [cF])$ by Theorem 3.1.3. Therefore $\mathcal{J}^\omega(X, \mathfrak{a}^c)$ is an integrally closed ideal of $\mathcal{O}_X$. If X has rational singularities, then $\mathcal{J}^\omega(X, \mathcal{O}_X) = \mathcal{O}_X$ by Corollary 3.1.8. Therefore $\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$ is effective. Thus $\mathcal{J}^\omega(X, \mathfrak{a}) \supset f_*\mathcal{O}_Y(-F) = \overline{\mathfrak{a}} \supset \mathfrak{a}$. □

Blickle defined the multiplier module in [1].

Definition 3.1.13. Let X be a normal variety and let $\mathfrak{a}$ be a nonzero ideal on X. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then the multiplier module is defined as

$$\mathcal{J}_\omega(\mathfrak{a}^c) = f_*\mathcal{O}_Y(K_Y - [cF]) \subset \omega_X$$

for $c > 0$.

Proposition 3.1.14. *Let X be a normal variety and let $\mathfrak{a}$ be a nonzero ideal on X. Then $\mathcal{J}^\omega(\mathfrak{a}^c) = \mathcal{J}_\omega(\mathfrak{a}^c) : \omega_X$ for all $c > 0$.*

Proof. This follows immediately from the definition of ω-multiplier ideals. □

Blickle gave a formula computing the multiplier module of a monomial ideal on an arbitrary affine toric variety in [1].

Theorem 3.1.15. ([1]) *Let X_σ be an affine toric variety and* a *a monomial ideal. Then*

$$\mathcal{J}_\omega(X_\sigma, \mathfrak{a}^c) = \langle x^m | m \in interior\ of\ c\mathrm{Newt}(\mathfrak{a})\rangle \subset \omega_{X_\sigma}.$$

Proposition 3.1.16. *Let X_σ be an n-dimensional affine toric variety and $\mathfrak{m}$ be the maximal ideal. Then* $\mathrm{rt}(\mathfrak{m}) \geq 1$.

Proof. Note that $\omega_{X_\sigma} = \langle x^m | m \in \mathrm{int}(\sigma) \cap \mathbb{Z}^n\rangle \subset \mathcal{O}_{X_\sigma}$. By theorem 3.1.15, we have

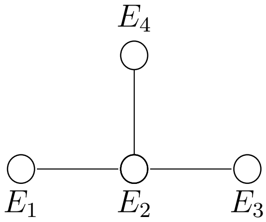
$$\mathcal{J}_\omega(X_\sigma, \mathfrak{m}^c) = \langle x^m | m \in interior\ of\ c\mathrm{Newt}(\mathfrak{m})\rangle \subset \omega_{X_\sigma}.$$

Therefore if $c < 1$, then we have $x^m \in \mathcal{J}_\omega(X_\sigma, \mathfrak{m}^c)$ for any $x^m \in \omega_{X_\sigma}$. This implies that $\mathrm{rt}(\mathfrak{m}) \geq 1$.

□

In general $\mathrm{rt}(\mathfrak{m})$ is not necessarily greater than or equal to 1.

Example 3.1.17. Let $(A = (\mathbb{C}[x, y, z]/(x^2 + y^2 z + z^3))_{(x,y,z)}, \mathfrak{m} = (x, y, z)$. Then A is a Du Val singularity of type D_4. Let Y be the minimal resolution of $X = \mathrm{Spec}A$. The dual graph of the exceptional divisor on the minimal resolution of A is as follows;



Therefore the fundamental cycle of the minimal resolution of $\mathrm{Spec}A$ is $Z = E_1 + 2E_2 + E_3 + E_4$, where $E_1, \ldots, E_4$ are exceptional divisors on the minimal resolution of $\mathrm{Spec}A$. Since A is a Gorenstein rational singularity, we have $K_{Y/X} = 0$, $\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. This implies that $\mathrm{lct}(\mathfrak{m}) = \frac{1}{2}$. Since A is Gorenstein, $\mathrm{rt}(\mathfrak{m})$ is equal to $\mathrm{lct}(\mathfrak{m})$. Thus we have $\mathrm{rt}(\mathfrak{m}) = \frac{1}{2}$.

Lemma 3.1.18. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay normal local ring and $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of A. Then $\mathcal{J}^\omega(A, \mathfrak{a}^n) \subset \mathrm{core}(\mathfrak{a})$. In particular, if $\mathrm{Proj}A[\mathfrak{a}]$ has rational singularities, then $\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathrm{core}(\mathfrak{a})$.*

Proof. Let $f : Y \to X$ be the blowing-up along $\mathfrak{a}$ and $g : Z \to X$ be a log resolution of $\mathfrak{a}$. By Theorem 2.3.9 and Lemma 2.3.11, we have

$$\mathrm{core}(\mathfrak{a}) = H^0(Y, \mathfrak{a}^n \omega_Y) :_A \omega_A.$$

Let $h : Z \to Y$ be a morphism with $g = f \circ h$. Then $h_*(\mathfrak{a}^n \omega_Z) \subset \mathfrak{a}^n \omega_Y$. Hence we have $H^0(Z, \mathfrak{a}^n \omega_Z) \subset H^0(Y, \mathfrak{a}^n \omega_Y)$. Therefore we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Z, \mathfrak{a}^n \omega_Z) :_A \omega_A \subset H^0(Y, \mathfrak{a}^n \omega_Y) :_A \omega_A = \mathrm{core}(\mathfrak{a}).$$

We assume that $Y = \mathrm{Proj}A[\mathfrak{a}]$ has rational singularities. Then $h_*(\mathfrak{a}^n \omega_Z) = \mathfrak{a}^n \omega_Y$ by the projection formula. Therefore we have $\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathrm{core}(\mathfrak{a})$. $\square$

Proposition 3.1.19. *Let X be an n-dimensional normal variety. Let $\mathfrak{j}_X$ be the Jacobian ideal of X and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$*

be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X$ *such that* $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$ *and* $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ *for some effective divisors* $J_{Y/X}$ *and* $D_{Y/X}$ *on* Y. *Then for* $i > 1$

$$R^i f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

Proof. If X is locally a complete intersection, then $\mathfrak{d}_X = \mathcal{O}_X$. Therefore $D_{Y/X} = 0$. Then by Local Vanishing Theorem (see Theorem 3.5 in [8]),

$$R^i f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

We assume that X is not locally a complete intersection. We may assume that X is affine. Note that there is a reduction of an ideal of $\mathcal{O}_X$ generated by n elements (see [2], Proposition 4.6.8). Let $I = (x_1, \dots, x_n)$ be a reduction of $\mathfrak{d}_X$. If V is the $\mathbb{C}$-vector space generated by $x_1, \dots, x_n$, then we have on Y an exact Koszul complex

$$0 \to \wedge^n V \otimes \mathcal{O}_Y(nD_{Y/X}) \to \cdots \to V \otimes \mathcal{O}_Y(D_{Y/X}) \to \mathcal{O}_Y \to 0.$$

Let $\mathcal{L}_n = \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - nD_{Y/X})$. By tensoring with $\mathcal{L}_n$ we get the exact complex

$$0 \to \wedge^n V \otimes \mathcal{L}_0 \to \cdots \to V \otimes \mathcal{L}_{n-1} \to \mathcal{L}_n \to 0.$$

Therefore we have

$$0 \to \wedge^n V \otimes f_*\mathcal{L}_0 \to \cdots \to V \otimes f_*\mathcal{L}_{n-1} \to f_*\mathcal{L}_n$$

$$\to \wedge^n V \otimes R^1 f_*\mathcal{L}_0 \to \wedge^{n-1} V \otimes R^1 f_*\mathcal{L}_1 \to \cdots \to V \otimes R^1 f_*\mathcal{L}_{n-1} \to R^1 f_*\mathcal{L}_n \to \cdots$$

$$\to \wedge^n V \otimes R^i f_*\mathcal{L}_0 \to \wedge^{n-1} V \otimes R^i f_*\mathcal{L}_1 \to \cdots \to V \otimes R^i f_*\mathcal{L}_{n-1} \to R^i f_*\mathcal{L}_n \to \cdots$$

By Local Vanishing Theorem (see Theorem 3.5 in [8]) for $j > 0$,

$$R^j f_*\mathcal{L}_1 = R^j f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X}) = 0$$

and

$$R^j f_*\mathcal{L}_n = R^j f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - (n-1)D_{Y/X}) = 0.$$

Therefore we have

$$R^{j+1} f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

$\square$

Lemma 3.1.20. *Let $(A, \mathfrak{m})$ be an n-dimensional local ring with rational singularities and I be a minimal reduction of $\mathfrak{m}$. Then $\mathfrak{m}^{n+1-\lceil \mathrm{rt}(\mathfrak{m}) \rceil} \subset I$.*

Proof. Let $X = \mathrm{Spec} A$. Let $\mathfrak{j}_X$ be the Jacobian ideal of X and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot \mathfrak{m}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $\mathfrak{m} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$ and F on Y. Since $\lceil \mathrm{rt}(\mathfrak{m}) \rceil - 1 < \mathrm{rt}(\mathfrak{m})$, we have $\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - (\lceil \mathrm{rt}(\mathfrak{m}) \rceil - 1)F \geq 0$. Therefore

$$
\begin{aligned}
I \supset \mathrm{core}(\mathfrak{m}) \supset \mathcal{J}^{\omega}(A, \mathfrak{m}^n) &= f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - nF) \\
\supset f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} &- (\lceil \mathrm{rt}(\mathfrak{m}) \rceil - 1)F - (n + 1 - \lceil \mathrm{rt}(\mathfrak{m}) \rceil)F) \\
&\supset f_* \mathcal{O}_Y(-(n + 1 - \lceil \mathrm{rt}(\mathfrak{m}) \rceil)F) \supset \mathfrak{m}^{n+1-\lceil \mathrm{rt}(\mathfrak{m}) \rceil}
\end{aligned}
$$

□

Proposition 3.1.21. *Let X be an n-dimensional variety with rational singularities. For a closed point $x \in X$,*
(1) $\mathrm{rt}(\mathfrak{m}_x) \leq n$
(2) $\mathrm{rt}(\mathfrak{m}_x) = n$ *if and only if x is a nonsingular point.*
(3) *If* $\mathrm{rt}(\mathfrak{m}_x) > n - 1$, *then x is a nonsingular point.*

Proof. For part (1), let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Let I be a minimal reduction of $\mathfrak{m}$, then $I \supset \mathfrak{m}^{n+1-\lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}$ by Lemma 3.1.20. Here, if $\mathrm{rt}(\mathfrak{m}_x) > n$, then we obtain $I \supset \mathcal{O}_{X,x}$, a contradiction.

For part (2), suppose x is a nonsingular point. Replacing X by small neighborhood of x, we may assume that X is nonsingular. Let $f : Y \to X$ be the blowup of $\mathfrak{m}_x$ and E the exceptional divisor. Then f is a log resolution of $\mathfrak{m}_x$ and the equalities $K_Y - f^* K_X = (n-1)E$, $\mathrm{val}_E(\mathfrak{m}_x) = 1$ hold. Hence $\mathrm{rt}(\mathfrak{m}_x) = n$. Conversely suppose $\mathrm{rt}(\mathfrak{m}_x) = n$, then by Lemma 3.1.20, we have $\mathfrak{m} = I$. Therefore $\mathfrak{m}$ is generated by n elements. This implies that x is a nonsingular point.

For part (3), suppose $\mathrm{rt}(\mathfrak{m}_x) > n - 1$. By the same way as above, x is a nonsingular point. □

Proposition 3.1.22. *Let X be a variety with rational singularities and $\mathfrak{a}$ a nonzero ideal of $\mathcal{O}_X$. Then* $\mathrm{rt}(\mathfrak{a}) > 1$ *if and only if for every nonzero ideal $\mathfrak{b} \subset \mathcal{O}_X$, we have $\mathcal{J}^{\omega}(X, \mathfrak{b}) \supset \mathfrak{b} : \mathfrak{a}$.*

Proof. First suppose that $\mathcal{J}^{\omega}(X, \mathfrak{b}) \supset (\mathfrak{b} : \mathfrak{a})$ for every ideal $\mathfrak{b} \subset \mathcal{O}_X$. Considering the case where $\mathfrak{a} = \mathfrak{b}$, we have $\mathcal{J}^{\omega}(X, \mathfrak{a}) = \mathcal{O}_X$, Hence $\mathrm{rt}(\mathfrak{a}) > 1$.

Conversely assume that $\mathrm{rt}(\mathfrak{a}) > 1$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot \mathfrak{a} \cdot \mathfrak{b}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_{\mathfrak{a}})$ and $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_{\mathfrak{b}})$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$, $F_{\mathfrak{a}}$ and $F_{\mathfrak{b}}$ on Y. Since $\mathrm{rt}(\mathfrak{a}) > 1$, we have $\mathcal{J}^{\omega}(X, \mathfrak{a}) = \mathcal{O}_X$. This implies that

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_{\mathfrak{a}} \geq 0.$$

We may assume that $\mathfrak{b}$ is an integrally closed ideal, that is $\mathfrak{b} = f_*\mathcal{O}_Y(-F_{\mathfrak{b}})$. Then $x \in \mathfrak{b} : \mathfrak{a} \Leftrightarrow x\mathfrak{a} \subset \mathfrak{b} \Leftrightarrow f^*x \cdot \mathcal{O}_Y(-F_{\mathfrak{a}}) \subset \mathcal{O}_Y(-F_{\mathfrak{b}}) \Leftrightarrow f^*x \in \mathcal{O}_Y(F_{\mathfrak{a}} - F_{\mathfrak{b}})$. Therefore we have $\mathrm{div} f^*x + F_{\mathfrak{a}} - F_{\mathfrak{b}} \geq 0$. Hence we have

$$\mathrm{div} f^*x + \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_{\mathfrak{b}} \geq \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_{\mathfrak{a}} \geq 0.$$

Thus $x \in \mathcal{J}^{\omega}(X, \mathfrak{b})$. □

Corollary 3.1.23. *Let X be a variety with rational singularities. Then $\mathrm{rt}(\mathfrak{m}_x) > 1$ for closed point $x \in X$ if and only if for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$, we have a strict containment $\mathcal{J}^{\omega}(X, \mathfrak{a}) \supsetneq \mathfrak{a}$.*

Proof. First suppose $\mathcal{J}^{\omega}(X, \mathfrak{a}) \supsetneq \mathfrak{a}$ for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$. Considering the case where $\mathfrak{a} = \mathfrak{m}_x$, we have $\mathcal{J}^{\omega}(X, \mathfrak{m}_x) = \mathcal{O}_X$, Hence $\mathrm{rt}(\mathfrak{m}_x) > 1$.

Conversely assume that $\mathrm{rt}(\mathfrak{m}_x) > 1$. By Proposition 3.1.22, we have $\mathcal{J}^{\omega}(X, \mathfrak{a}) \supset (\mathfrak{a} : \mathfrak{m}_x)$ for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$. If $\mathfrak{m}_x^l \subset \mathfrak{a}$, then $\mathfrak{m}_x^{l-1} \subset (\mathfrak{a} : \mathfrak{m}_x)$. Therefore we have $(\mathfrak{a} : \mathfrak{m}_x) \supsetneq \mathfrak{a}$. This implies that $\mathcal{J}^{\omega}(X, \mathfrak{a}) \supsetneq \mathfrak{a}$. □

De Fernex and Hacon defined in [5] the log canonical, log terminal singularities on an arbitrary normal variety. These singularities are generalizations of log canonical, log terminal singularities for $\mathbb{Q}$-Gorenstein variety. Moreover they defined the $\natural$-pull back of an arbitrary divisor on a normal variety in [5]. In a local situation, as we can take an effective divisor $-K_X$. Let $Y \to X$ be a log resolution of $\mathcal{O}_X(K_X)$. Define the divisor $f^{\natural}(-K_X)$ on Y by $\mathcal{O}_X(K_X)\mathcal{O}_Y = \mathcal{O}_Y(-f^{\natural}(-K_X))$.

We assume that mK_X is effective. Let $Y \to X$ be a log resolution of $\mathcal{O}_X(-mK_X)$. Define the divisor D_m on Y by $\mathcal{O}_X(-mK_X)\mathcal{O}_Y = \mathcal{O}_Y(-D_m)$.

Under this notation we define the divisor

$$K_{m,Y/X} = K_Y - \frac{1}{m} D_m$$

with the support on the exceptional divisor. In [5] De Fernex and Hacon showed that for $m, q \geq 1$,

$$K_{m,Y/X} \leq K_{qm,Y/X} \leq K_Y + f^\natural(-K_X).$$

Proposition 3.1.24. *Let $X \subset \mathbb{A}^N$ be an n-dimensional affine normal variety. Then there is a log resolution $Y \to X$ of $\mathfrak{j}_X \mathfrak{d}_X \mathcal{O}_X(K_X)$ such that*

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} = K_Y + f^\natural(-K_X).$$

Proof. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X \mathcal{O}_X(K_X)$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$. Take a reduced complete intersection scheme $M \subset \mathbb{A}^N$ of codimension $c = N - n$ such that M contains X as an irreducible component. Then we have a sequence

$$\wedge^n \Omega_X \xrightarrow{\eta} \omega_X \xrightarrow{u} \omega_M|_X.$$

By Proposition 9.1 of [9], $\mathrm{Im}(u \circ \eta) = \mathfrak{j}_M|_X \omega_M|_X$. Note that $\mathcal{O}_X(K_X)\mathcal{O}_Y = \mathcal{O}_Y(-f^\natural(-K_X))$. We have a sequence

$$f^*(\wedge^n \Omega_X) \xrightarrow{\eta'} \mathcal{O}_Y(-f^\natural(-K_X)) \xrightarrow{u'} f^*(\omega_M|_X).$$

Since $\mathcal{O}_Y(-f^\natural(-K_X))$ and $f^*(\omega_M|_X)$ are invertible, we can write

$$\mathrm{Im}\eta' = I\mathcal{O}_Y(-f^\natural(-K_X))$$

$$\mathrm{Im}u' = J_M f^*(\omega_M|_X),$$

with the ideal $I, J_M \subset \mathcal{O}_Y$. Then we obtain $IJ_M = \mathfrak{j}_M|_X \mathcal{O}_Y$. Consider all M and define $J = \sum_M J_M$, then we have $IJ = \mathfrak{j}_X \mathcal{O}_Y$. Let $g : Z \to Y$ be a log resolution of IJ such that $I\mathcal{O}_Z = \mathcal{O}_Z(-B)$ and $h : Z \to X$ be the composition of f and g. Then $B + D_{Z/X} = J_{Z/X}$ since $\mathfrak{d}_X \mathcal{O}_Y = J$.

Since h factors through the Nash blow-up, the torsion free sheaf $h^*(\wedge^n \Omega_X)/\mathrm{Tor}$ is invertible, it is written as $\mathcal{O}_Z(C)$ by a divisor C on Z. Then by the definition of $\widehat{K}_{Z/X}$, we have $\widehat{K}_{Z/X} = K_Z - C$. On the other hand we have

$C = g^*(-f^\natural(-K_X)) - B = -h^\natural(-K_X) - B$ by Lemma 2.7 in [5]. Therefore we have

$$\widehat{K}_{Z/X} - J_{Z/X} + D_{Z/X} = K_Z - C - B = K_Z + h^\natural(-K_X),$$

which completes the proof of the lemma. $\square$

In [5], De Fernex and Hacon introduced a multiplier ideal for a pair $(X, \mathfrak{a}^t)$ with normal variety X and an ideal $\mathfrak{a}$ on X. For $m \in \mathbb{N}$, they defined m-th multiplier ideal as follows:

$$\mathcal{J}_m(X, \mathfrak{a}^t) = f_* \mathcal{O}_Y(\lceil K_{m,Y/X} - tZ \rceil),$$

where $f : Y \to X$ is log resolution of $\mathfrak{a}\mathcal{O}_X(-K_X)$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. They proved that the family of ideals $\{\mathcal{J}_m(X, \mathfrak{a}^t)\}_m$ has the unique maximal element and call it the multiplier ideal of $(X, \mathfrak{a}^t)$ and denote it by $\mathcal{J}(X, \mathfrak{a}^t)$.

Corollary 3.1.25. *Let X be a normal variety and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$, Then for $c \in \mathbb{Q}_{>0}$,*

$$\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$$

Proof. Since $K_{m,Y/X} \leq K_Y + f^\natural(-K_X)$, we have $\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$. $\square$

Corollary 3.1.26. *Let X be a normal variety and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$, Then for $c \in \mathbb{Q}_{>0}$,*

$$\mathcal{J}(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$$

Proof. Since $\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$ for any m, we have $\mathcal{J}(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$. $\square$

3.2 Characterization rational singularities via cores of ideals

In this section, we characterize rational singularities via cores of ideals.

Theorem 3.2.1. ([28] Briançon-Skoda Theorem) *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional local ring with rational singularities and* I *be an ideal of* A*. Then we have*

$$\overline{I^n} \subset I$$

where $^{-}$ *denotes integral closure.*

Lemma 3.2.2. *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional Cohen-Macaulay isolated singularity local ring. Suppose that* A *is not a rational singularity. Then there exists an* $\mathfrak{m}$*-primary ideal* I *of* A *such that* $I^n \not\subset \mathrm{core}(I)$*.*

Proof. Since A is not a rational singularity, we have $H^0(Y, \omega_Y) \not\supset \omega_A$. Let I be an $\mathfrak{m}$-primary ideal such that $f : \mathrm{Proj}A[I] \to \mathrm{Spec}A$ is a desingularization. By Theorem 2.3.9 and Lemma 2.3.11, we have

$$\mathrm{core}(I) = H^0(Y, I^n\omega_Y) :_A \omega_A.$$

By Lemma 2.3.10, we have $H^0(Y, I^n\omega_Y) :_{\omega_A} I^n = H^0(Y, \omega_Y)$. This implies that $I^n\omega_A \nsubseteq H^0(Y, I^n\omega_Y)$ since $H^0(Y, \omega_Y) \not\supset \omega_A$. Therefore we have $I^n \not\subset H^0(Y, I^n\omega_Y) :_A \omega_A = \mathrm{core}(I)$. □

Theorem 3.2.3. *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional Cohen-Macaulay isolated singularity local ring . Then* A *is a rational singularity if and only if* $\overline{I^n} \subset \mathrm{core}(I)$ *for any* $\mathfrak{m}$*-primary ideal* I*.*

Proof. If A is a rational singularity, then $\overline{I^n} \subset \mathrm{core}(I)$ for any $\mathfrak{m}$-primary ideal I by Briançon-Skoda Theorem. For the converse proof, we assume that A is not a rational singularity. By Lemma 3.2.2, there is an $\mathfrak{m}$-primary ideal I of A such that $I^n \not\subset \mathrm{core}(I)$. Thus we have $\overline{I^n} \not\subset \mathrm{core}(I)$. □

The following corollary implies that a Cohen-Macaulay isolated singularity local ring is a rational singularity if Briançon-Skoda Theorem holds for the ring.

Corollary 3.2.4. *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional Cohen-Macaulay isolated singularity local ring.* A *is a rational singularity if and only if* $\overline{I^n} \subset I$ *for any* $\mathfrak{m}$*-primary ideal* I*.*

Proof. If A is a rational singularity, then $\overline{I^n} \subset I$ for any $\mathfrak{m}$-primary ideal I by Briançon-Skoda Theorem. Hence we will show the converse implication. We assume that A is not rational singularity. By Theorem 3.2.3, there are an $\mathfrak{m}$-primary ideal I and a reduction J of I such that $\overline{I^n} \not\subset J$. Therefore we have $\overline{J^n} \not\subset J$ since $\overline{I^n} = \overline{J^n}$. $\square$

Corollary 3.2.5. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay isolated singularity local ring . Then A is a rational singularity if and only if $\overline{I} \subset \mathcal{J}^\omega(I)$ for any $\mathfrak{m}$-primary ideal I.*

Proof. We assume that A is a rational singularity. Let $f : Y \to X = \mathrm{Spec} A$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X I$ such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$, $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$ and $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then Theorem 3.1.3 and Corollary 3.1.8

$$\widehat{K}_{Y/X} - J + D \geq 0, \quad \mathcal{J}^\omega(X, I) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J + D - F).$$

Therefore we have

$$\mathcal{J}^\omega(I) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J + D - F) \supset f_* \mathcal{O}_Y(-F) = \overline{I}.$$

We assume that A is not a rational singularity. Then by Theorem 3.2.3, there exists an m-primary I such that $\overline{I^n} \not\subset \mathrm{core}(I)$. Since by Lemma 3.1.18,

$$\mathcal{J}^\omega(I^n) \subset \mathrm{core}(I),$$

we have

$$\overline{I^n} \not\subset \mathcal{J}^\omega(I^n).$$

$\square$

Definition 3.2.6. Let $(A, \mathfrak{m})$ be a local domain which is a homomorphic image of a Gorenstein local ring. Suppose that $\mathrm{Spec} A \setminus \mathfrak{m}$ has rational singularities, and that there exists a proper birational morphism $f : Y \to \mathrm{Spec} A$ such that Y has rational singularities. We define the number $r(A)$ as the smallest integer r such that $\mathfrak{m}^r \omega_A \subset \Gamma(Y, \omega_Y)$.

Hyry and Villamayor gave in [16] a extension of Briançon-Skoda Theorem to normal Cohen-Macaulay local rings which have rational singularities in the punctured spectrum.

Theorem 3.2.7. ([16], Theorem 2.6) *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional normal Cohen-Macaulay local domain which is a homomorphic image of a Gorenstein local ring. Suppose that* $\mathrm{Spec}A \setminus \mathfrak{m}$ *has rational singularities, and that there exists a proper birational morphism* $f : Y \to \mathrm{Spec}A$ *such that* Y *has rational singularities. Set* $r = r(A)$*. Then* $\overline{I^{n+r}} \subset I$ *for all ideal* $I \subset A$*.*

Proposition 3.2.8. *Let* $(A, \mathfrak{m})$ *be an* n*-dimensional Cohen-Macaulay isolated singularity local ring. If* A *is a Du Bois singularity,* $\overline{I^{n+1}} \subset \mathrm{core}(I)$ *for all ideal* $I \subset A$*.*

Proof. Let $f : Y \to \mathrm{Spec}A$ be a resolution of $\mathrm{Spec}A$ such that f is isomorphism over $\mathrm{Spec}A \setminus \mathfrak{m}$, $f^{-1}(\mathfrak{m})$ is simple normal crossing divisor and $\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ for a divisor F on Y. Let G be the reduced exceptional divisor of f. Since A is a Du Bois singularity, we have $\Gamma(Y, \omega_Y(G)) = \omega_A$ by Theorem 2.1.4. Therefore $\mathfrak{m}\omega_A = \mathfrak{m}\Gamma(Y, \omega_Y(G)) \subset \Gamma(Y, \omega_Y(G - F)) \subset \Gamma(Y, \omega_Y)$. Thus $r(A) = 1$. By Theorem 3.2.7, we have $\overline{I^{n+1}} \subset \mathrm{core}(I)$. □

This proposition does not give a characterization of a Cohen-Macaulay Du Bois singularity. We have an example of an n-dimensional Cohen-Macaulay local ring A with non-Du Bois isolated singularity such that $\overline{I^{n+1}} \subset \mathrm{core}(I)$ for all ideal $I \subset A$.

Example 3.2.9. Let $A = (\mathbb{C}[x, y, z]/(x^3+y^3+z^4))_{(x,y,z)}$. Note Gorenstein Du Bois singularities are log canonical singularities. Then $\mathrm{Spec}A$ is Gorenstein, but not log canonical. Therefore A is not a Du Bois singularity. Let $f : Y \to \mathrm{Spec}A$ be the blowing-up at $\mathfrak{m}$. Then f is a resolution of $\mathrm{Spec}A$. Therefore we have $r(A) = 1$. By Theorem 3.2.7, $\overline{I^3} \subset \mathrm{core}(I)$ for any ideal I.

Chapter 4

Cores of ideals and ω-multiplier ideals of 2-dimensional local rings with a rational singularity

4.1 The arithmetic of cores of ideals and ω-multiplier ideals

In this section, we discuss the various relationships between the a core of an ideal and a ω-multiplier ideal of a 2-dimensional local ring with a rational singularity.

Definition 4.1.1. Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and fix a resolution of singularities $f : Y \to \mathrm{Spec}A$. For any integral divisor D on Y, f-anti-nef closure of D is defined to be a unique smallest integral f-anti-nef divisor which is bigger than or equal to D. We will denote it by $\mathrm{an}_f(D)$.

The followings are quite useful.

Theorem 4.1.2. ([27] , [11]) *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity and fix a resolution of singularities $f : X \to \mathrm{Spec}A$. Then there is a one-to-one correspondence between the set of integrally closed ideals I in A such that $I\mathcal{O}_X$ is invertible and the set of effective f-anti-*

nef cycles Z on X. The correspondence is given by $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(X, \mathcal{O}_X(-Z))$.

Lemma 4.1.3. ([29]) *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity and fix a resolution of singularities $f : Y \to \operatorname{Spec} A$. For any divisor D on Y, we have $f_*\mathcal{O}_Y(-D) = f_*\mathcal{O}_Y(-\mathrm{an}_f(D))$.*

Proposition 4.1.4. ([28]) *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed ideal of A and I be a reduction of $\mathfrak{a}$. Then $I\mathfrak{a} = \mathfrak{a}^2$.*

The following is a generalization of Lemma 5.6 in [31].

Lemma 4.1.5. *Let $(A, \mathfrak{m})$ be a 2-dimensional normal local ring, I be an m-primary ideal and J be a minimal reduction of I with $JI = I^2$. Then for $n \in \mathbb{Z}_{\geq 0}$,*

$$J^{n+1} : I = J^n(J : I) = I^n(J : I).$$

Proof. We will show that $J^{n+1} : I = J^n(J : I) = I^n(J : I)$ by induction on n. When $n = 0$, the assertion is trivial. If $n = 1$, then the equalities hold by Lemma 5.6 in [31]. Thus we may assume that $n \geq 2$. It is clear that $J^n(J : I) \subset I^n(J : I)$. Let $x \in I^n$, $y \in (J : I)$. Then $xyI \subset yI^{n+1} = yIJ^n \subset J^{n+1}$. Therefore we have $I^n(J : I) \subset J^{n+1} : I$. Hence we will show the inclusion $J^{n+1} : I \subset J^n(J : I)$. Let $J = (x_1, x_2)$. Assume that $x \in J^{n+1} : I$. Since $J^{n+1} : I \subset J^{n+1} : J \subset J^n$, there exist $a_{i_1,i_2} \in A$ such that $x = \sum_{i_1+i_2=n} a_{i_1,i_2} x_1^{i_1} x_2^{i_2}$. Since $x \in J^{n+1} : I$, for any $f \in I$ there exist $b_{i_1,i_2} \in A$ such that $xf = \sum_{j_1+j_2=n+1} b_{j_1,j_2} x_1^{j_1} x_2^{j_2}$. Then we have $a_{n,0}x_1^n f - b_{n+1,0}x_1^{n+1} \in (x_2)$, $a_{0,n}x_2^n f - b_{0,n+1}x_2^{n+1} \in (x_1)$. Since x_1, x_2 is a regular sequence, we have $a_{n,0}f - b_{n+1,0}x_1 \in (x_2)$, $a_{0,n}f - b_{0,n+1}x_2 \in (x_1)$. Thus $a_{n,0}f, a_{0,n}f \in J$. This shows that $a_{n.0}, a_{0,n} \in J : I$. We can write

$$x - a_{n.0}x_1^n - a_{0,n}x_2^n = x_1x_2 \sum_{i_1+i_2=n, i_1,i_2 \neq 0,n} a_{i_1,i_2} x_1^{i_1-1} x_2^{i_2-1}.$$

Let $y = \sum_{i_1+i_2=n, i_1,i_2 \neq 0,n} a_{i_1,i_2} x_1^{i_1-1} x_2^{i_2-1}$. Since $x \in J^{n+1} : I$ and $a_{n.0}x_1^n, a_{0,n}x_2^n \in J^n(J : I) \subset J^{n+1} : I$, we have $x_1x_2y \in J^{n+1} : I$. For any $f \in I$, we have

$$x_1x_2yf \in J^{n+1}.$$

Hence we have

$$yf \in J^{n-1}.$$

Therefore we have

$$y \in J^{n-1} : I.$$

By induction hypothesis, we have $y \in J^{n-2}(J : I)$. Thus we have $x = a_{n.0}x_1^n + a_{0,n}x_2^n + x_1x_2y \in J^n(J : I)$. □

Proposition 4.1.6. *Let $(A, \mathfrak{m})$ be a 2-dimensional normal local ring, I be an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I with $JI = I^2$. Then for $n \in \mathbb{Z}_{\geq 1}$,*

$$J^{n-1}\mathrm{core}(I) = I^{n-1}\mathrm{core}(I) = J^{n+1} : I = J^n(J : I) = I^n(J : I).$$

Proof. By Theorem 2.3.9 and Lemma 4.1.5, we have

$$\mathrm{core}(I) = J^2 : I = J(J : I) = I(J : I).$$

Thus by Lemma 4.1.5, we have

$$J^{n-1}\mathrm{core}(I) = I^{n-1}\mathrm{core}(I) = J^{n+1} : I = J^n(J : I) = I^n(J : I).$$

□

We need the following theorem to prove the properties of ω-multiplier ideals of 2-dimensional local ring with a rational singularity.

Theorem 4.1.7. ([32]) *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal and I be a minimal reduction of $\mathfrak{a}$. Let $f : Y \to X = \mathrm{Spec}A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $f : Y_0 \to X$ be the minimal resolution of singularities. Then*

$$I : \mathfrak{a} = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F)),$$

$$\mathrm{core}(\mathfrak{a}) = \mathfrak{a}H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F))$$

$$= IH^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F)) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - 2F)).$$

Hyry and Smith proved the following in the proof of Lemma 5.1.6 in [17]. We need the lemma to prove Proposition 4.1.9.

Lemma 4.1.8. ([17]) *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay local ring, $\mathfrak{a}$ be an m-primary ideal and J be a minimal reduction of $\mathfrak{a}$ with $J\mathfrak{a}^r = \mathfrak{a}^{r+1}$. Let Y be the blowing-up of $\mathfrak{a}$. Then for $m \in \mathbb{Z}_{\geq 1}$,*

$$H^0(Y, \mathfrak{a}^m \omega_Y) = J^{m+r+1-n}\omega_A :_{\omega_A} \mathfrak{a}^r$$

and

$$J^m \omega_A : \omega_A = J^m.$$

Proposition 4.1.9. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an m-primary integrally closed ideal and J be a minimal reduction of $\mathfrak{a}$. Then for $n \in \mathbb{N}$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = J^n : \mathfrak{a} = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a}).$$

Proof. Let $f : Y \to X$ be the blowing-up along $\mathfrak{a}$ and $g : Z \to X$ be a log resolution of $\mathfrak{a}$. Then Y is normal because $\mathfrak{a}^m$ is an integrally closed ideal for any $m \in \mathbb{N}$ (see Theorem 7.1 in [27]). By Proposition 1.2 in [27], Y has a rational singularity. Therefore we have by the projection formula,

$$H^0(Z, \mathfrak{a}^n \omega_Z) = H^0(Y, \mathfrak{a}^n \omega_Y).$$

Thus by Proposition 4.1.4 and Lemma 4.1.8 we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Y, \mathfrak{a}^n \omega_Y) : \omega_A = (J^n \omega_A :_{\omega_A} \mathfrak{a}) : \omega_A$$

$$= (J^n \omega_A : \omega_A) : \mathfrak{a} = J^n : \mathfrak{a}.$$

Thus by Lemma 4.1.5, we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = J^n : \mathfrak{a} = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a}).$$

□

The following proposition implies that the Skoda's Theorem of ω-multiplier ideals holds for a 2-dimensional local ring with a rational singularity.

Proposition 4.1.10. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an m-primary ideal and J be a reduction of $\mathfrak{a}$. Then for $n \in \mathbb{Z}_{\geq 2}$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}).$$

Proof. We may assume that $\mathfrak{a}$ is an integrally closed ideal and J is a minimal reduction of $\mathfrak{a}$. By Proposition 4.1.9, we have

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a})$$

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}) = J^{n-2}(J : \mathfrak{a}) = \mathfrak{a}^{n-2}(J : \mathfrak{a}).$$

Therefore we have

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}).$$

$\square$

Theorem 4.1.11. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Let $f : Y \to X$ be a log resolution of singularities of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then for $n \in \mathbb{N}$,*

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF)).$$

Proof. We may assume that $\mathfrak{a}$ is an integrally closed ideal. Let I be a minimal reduction of $\mathfrak{a}^n$. By Theorem 4.1.7 we have $I : \mathfrak{a}^n = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF))$. By Proposition 4.1.9, we have $I : \mathfrak{a}^n = \mathcal{J}^{\omega}(A, \mathfrak{a}^n)$. Therefore $\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF))$. $\square$

Corollary 4.1.12. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an m-primary integrally closed ideal and J be a minimal reduction of $\mathfrak{a}$. Then*

$$\operatorname{core}(\mathfrak{a}) = \mathcal{J}^{\omega}(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a}) = J\mathcal{J}^{\omega}(A, \mathfrak{a}).$$

Proof. By Theorem 4.1.7, Proposition 4.1.10 and Theorem 4.1.11, we have

$$\operatorname{core}(\mathfrak{a}) = \mathcal{J}^{\omega}(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a}) = J\mathcal{J}^{\omega}(A, \mathfrak{a}).$$

$\square$

Proposition 4.1.13. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then*

$$\mathrm{e}(\mathfrak{a}) = \ell(A/\operatorname{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^{\omega}(A, \mathfrak{a})).$$

Proof. Let $I = (x_1, x_2)$ be a minimal reduction of $\mathfrak{a}$. We have

$$\mathrm{e}(\mathfrak{a}) = \ell(A/I) = \ell(A/I\mathcal{J}^{\omega}(A,\mathfrak{a})) - \ell(I/I\mathcal{J}^{\omega}(A,\mathfrak{a})).$$

By Corollary 4.1.12, $\ell(A/I\mathcal{J}^{\omega}(A,\mathfrak{a})) = \ell(A/\mathrm{core}(\mathfrak{a}))$.

We will show that $I/I\mathcal{J}^{\omega}(A,\mathfrak{a})$ is isomorphic to $A/\mathcal{J}^{\omega}(A,\mathfrak{a}) \oplus A/\mathcal{J}^{\omega}(A,\mathfrak{a})$. Let $\phi : A/\mathcal{J}^{\omega}(A,\mathfrak{a}) \oplus A/\mathcal{J}^{\omega}(A,\mathfrak{a}) \to I/I\mathcal{J}^{\omega}(A,\mathfrak{a})$ be a map defined by $\phi(a + \mathcal{J}^{\omega}(A,\mathfrak{a}), b + \mathcal{J}^{\omega}(A,\mathfrak{a})) = x_1 a + x_2 b + I\mathcal{J}^{\omega}(A,\mathfrak{a})$. It is clear that ϕ is surjective. Let $(a + \mathcal{J}^{\omega}(A,\mathfrak{a}), b + \mathcal{J}^{\omega}(A,\mathfrak{a})) \in \ker\phi$. Then by Proposition 4.1.9,

$$x_1 a + x_2 b \in I\mathcal{J}^{\omega}(A,\mathfrak{a}) = I(I : \mathfrak{a}) = I^2 : \mathfrak{a}.$$

Then for any element $h \in \mathfrak{a}$, $(x_1 a + x_2 b)h \in I^2$. Therefore there are $c_1, c_2, c_3 \in A$ such that $(x_1 a + x_2 b)h = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2$. Since $x_1 a h - c_1 x_1^2 \in (x_2), x_2 b h - c_3 x_2^2 \in (x_1)$ and x_1, x_2 is a regular sequence, we have $ah - c_1 x_1 \in (x_2), bh - c_3 x_2 \in (x_1)$. Therefore we have $ah, bh \in (x_1, x_2)$. Thus we have $a, b \in I : \mathfrak{a}$. Since $\mathcal{J}^{\omega}(A,\mathfrak{a}) = I : \mathfrak{a}$, ϕ is injective. Hence ϕ is isomorphism. This implies that $\ell(I/I\mathcal{J}^{\omega}(A,\mathfrak{a})) = 2\ell(A/\mathcal{J}^{\omega}(A,\mathfrak{a}))$. Thus we have

$$\mathrm{e}(\mathfrak{a}) = \ell(A/\mathrm{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^{\omega}(A,\mathfrak{a})).$$

□

Lemma 4.1.14. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}A$ be a resolution of singularities of $\mathrm{Spec}A$. We assume that the morphism f is factorized as*

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$ and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let Z be a f-anti-nef cycle on Y and $K = K_{Y/Y_0} = \sum_{i=1}^{n} \pi_i^* E_i$. Let

$$C = \{j \in \mathbb{N} | 1 \le i \le n, Z \cdot \pi_j^* E_j < 0\}.$$

Then

$$\mathrm{an}_f(Z - K) = Z - \sum_{i \in C} \pi_i^* E_i$$

Proof. First we will show that $Z - \sum_{i\in C} \pi_i^* E_i$ is f-anti-nef. For each f_0-exceptional curve F, we have

$$(Z - \sum_{i\in C} \pi_i^* E_i) \cdot {\pi_0^{-1}}_* F \leq Z \cdot {\pi_0^{-1}}_* F \leq 0.$$

We assume that for $i \in C$ and $j \notin C$, $\pi_i^* E_i \cdot {\pi_j^{-1}}_* E_j = 1$. Then $f_i : Y_i \to Y_{i-1}$ is the blowing-up at a closed point of the strict transform of E_j on Y_{i-1}. This implies that $\pi_i^* E_i \leq \pi_j^* E_j$. Therefore $Z \cdot \pi_j^* E_j \leq Z \cdot \pi_i^* E_i < 0$ since Z is f-anti-nef. This implies that $j \in C$, which is a contradiction. Hence we have $\pi_i^* E_i \cdot {\pi_j^{-1}}_* E_j = 0$ for $i \in C$ and $j \notin C$. Thus for $j \notin C$, we have

$$(Z - \sum_{i\in C} \pi_i^* E_i) \cdot {\pi_j^{-1}}_* E_j = Z \cdot {\pi_j^{-1}}_* E_j = 0.$$

We assume that $Z \cdot {\pi_j^{-1}}_* E_j < 0$ for $j \in C$. Then we have

$$(Z - \sum_{i\in C} \pi_i^* E_i) \cdot {\pi_j^{-1}}_* E_j \leq Z \cdot {\pi_j^{-1}}_* E_j - \pi_j^* E_j \cdot {\pi_j^{-1}}_* E_j = Z \cdot {\pi_j^{-1}}_* E_j + 1 \leq 0.$$

We assume that $Z \cdot {\pi_j^{-1}}_* E_j = 0$ for $j \in C$. Then there exists $k \in C$ such that $Z \cdot \pi_k^* E_k < 0$, $\pi_k^* E_k \leq \pi_j^* E_j$ and $\pi_k^* E_k \cdot {\pi_j^{-1}}_* E_j = 1$. Therefore

$$\begin{aligned}(Z - \sum_{i\in C} \pi_i^* E_i) \cdot {\pi_j^{-1}}_* E_j &= -\sum_{i\in C} \pi_i^* E_i \cdot {\pi_j^{-1}}_* E_j \\ &\leq -\pi_j^* E_j \cdot {\pi_j^{-1}}_* E_j - \pi_k^* E_k \cdot {\pi_j^{-1}}_* E_j = 0.\end{aligned}$$

By the above discussion, $Z - \sum_{i\in C} \pi_i^* E_i$ is f-anti-nef. This implies that

$$\mathrm{an}_f(Z - K) \leq Z - \sum_{i\in C} \pi_i^* E_i.$$

Let Z' be a cycle such that $Z - K \leq Z' < Z - \sum_{i\in C} \pi_i^* E_i$. Next we will show that Z' is not f-anti-nef. Let $F = Z - \sum_{i\in C} \pi_i^* E_i - Z'$ and ${\pi_j^{-1}}_* E_j \leq F$. Then there exists $k \notin C$ such that ${\pi_j^{-1}}_* E_j \leq \pi_k^* E_k$. Thus we have $j \notin C$. Since $Z \cdot {\pi_j^{-1}}_* E_j = 0$ and $\pi_i^* E_i \cdot {\pi_j^{-1}}_* E_j = 0$ for $i \in C$ and $j \notin C$,

$$Z' \cdot F = (Z - \sum_{i\in C} \pi_i^* E_i - F) \cdot F = -F \cdot F > 0.$$

Thus Z' is not f-anti-nef. Therefore the minimal f-anti-nef cycle which are bigger than or equal to $Z - K$ is $Z - \sum_{i\in C} \pi_i^* E_i$. $\square$

Lemma 4.1.15. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec} A$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X$ such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ and $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Let Z be an exceptional f-anti-nef divisor on Y. Let $K^\omega = \widehat{K}_{Y/X} - J + D$ and $K = K_{Y/Y_0}$, where Y_0 is the minimal resolution of X. Then*

$$\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$$

for any exceptional prime divisor F with $Z \cdot F < 0$.

Proof. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$ and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let

$$C = \{j \in \mathbb{N} | 1 \leq i \leq n, Z \cdot \pi_j^* E_j < 0\}.$$

Then

$$\mathrm{an}_f(nZ - K) = nZ - \sum_{i \in C} \pi_i^* E_i$$

for any positive integer n by Lemma 4.1.14. Let $\mathfrak{a} = f_* \mathcal{O}_Y(-Z)$. Then $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal and we have $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$ by Theorem 4.1.2. Therefore $\mathcal{J}^\omega(X, \mathfrak{a}^n) = f_* \mathcal{O}_Y(K^\omega - nZ)$ by Theorem 3.1.3. By Theorem 4.1.11 and Lemma 4.1.14, we have

$$nZ - K^\omega \leq \mathrm{an}_f(nZ - K^\omega) = \mathrm{an}_f(nZ - K) = nZ - \sum_{i \in C} \pi_i^* E_i.$$

This implies that $\sum_{i \in C} \pi_i^* E_i \leq K^\omega$. Since $\mathrm{ord}_F \pi_j^* E_j = 0$ for $j \notin C$, we have

$$\mathrm{ord}_F K = \mathrm{ord}_F \sum_{i=1}^{n} \pi_i^* E_i = \mathrm{ord}_F \sum_{i \in C} \pi_i^* E_i.$$

Therefore we have $\mathrm{ord}_F K^\omega \geq \mathrm{ord}_F K$.

We assume that $\mathrm{ord}_F K^\omega > \mathrm{ord}_F K$. Then we have

$$K^\omega \geq \sum_{i \in C} \pi_i^* E_i + F.$$

Since $Z \cdot F < 0$, there exists $n \in \mathbb{N}$ such that $nZ - \sum_{i \in C} \pi_i^* E_i - F$ is f-anti-nef. Then

$$\begin{aligned}\mathrm{an}_f(nZ - K^\omega) &\leq \mathrm{an}_f\Big(nZ - (\sum_{i \in C} \pi_i^* E_i + F)\Big) \\ &\leq nZ - \sum_{i \in C} \pi_i^* E_i - F < nZ - \sum_{i \in C} \pi_i^* E_i \\ &= \mathrm{an}_f(nZ - K^\omega),\end{aligned}$$

which is a contradiction. Therefore we have $\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$. $\square$

Lemma 4.1.16. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}A$ be a resolution of singularities of X and F be a prime exceptional divisor on Y. Then there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$.*

Proof. Let Z_f be a fundamental cycle of f. Then there exists a prime exceptional divisor F_1 with $Z \cdot F_1 < 0$. Since $f^{-1}(\mathfrak{m})$ is connected, there exists a sequence $\{F_1, \ldots, F_n\}$ such that F_i is a exceptional prime divisor, $F_i \cdot F_{i+1} = 1$ for $1 \leq i \leq n-1$ and $F_n = F$.

We will make an exceptional f-anti-nef divisor Z_i such that $Z_i \cdot F_i < 0$ for i by induction on i. When $i = 1$, we can take Z_f as Z_1. By the induction hypothesis there exists an exceptional f-anti-nef divisor Z_i such that $Z_i \cdot F_i < 0$. Since $Z_i \cdot F_i < 0$, there exists a positive integer n such that $nZ_i - F_i$ is f-anti-nef divisor. Then $(nZ_i - F_i) \cdot F_{i+1} \leq -F_i \cdot F_{i+1} < 0$. Therefore we can take $nZ_i - F_i$ as Z_{i+1}.

Thus there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$. $\square$

Proposition 4.1.17. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}A$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X$ such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ and $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Let Z be an exceptional f-anti-nef divisor on Y. Let $K^\omega = \widehat{K}_{Y/X} - J + D$ and $K = K_{Y/Y_0}$, where Y_0 is the minimal resolution of X. Then*

$$K^\omega = K$$

Proof. By Lemma 4.1.16, for any prime exceptional divisor F on Y, there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$. By Lemma 4.1.15, we have $\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$. Therefore we have $K^\omega = K$. $\square$

We need the following lemma to prove Lemma 4.1.19.

Lemma 4.1.18. (Lemma 9.2.19 in [24]) *Let X be a smooth variety of dimension n, and D any $\mathbb{Q}$-divisor on X with simple normal crossing support. Suppose that $f : Y \to X$ is a log resolution of D. Then*

$$f_*\mathcal{O}_Y(K_{Y/X} - [f^*D]) = \mathcal{O}_X(-[D]).$$

Lemma 4.1.19. *Let $(A, \mathfrak{m})$ be a 2-dimensional normal local ring, $\mathfrak{a}$ be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec}A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then for $c > 0$, $f_*\mathcal{O}_Y(K_{Y/Y_0} - [cF])$ is independent of the choice of log resolutions.*

Proof. Since any two log resolutions can be dominated by a third, we consider the case of two log resolutions of $\mathfrak{a}$, $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, with a map between them:

$$\begin{tikzcd} Y_2 \arrow[r, "g"] \arrow[dr, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ & X \end{tikzcd}$$

Let $\mathfrak{a}\mathcal{O}_{Y_1} = \mathcal{O}_{Y_1}(-F_1)$, $\mathfrak{a}\mathcal{O}_{Y_2} = \mathcal{O}_{Y_2}(-F_2)$ and $g : Y_2 \to Y_1$ be the morphism with $f_2 = f_1 \circ g$. Then we have $K_{Y_2/Y_0} = K_{Y_2/Y_1} + g^*(K_{Y_1/Y_0})$ and $F_2 = g^*(F_1)$. By the projection formula and Lemma 4.1.18,

$$\begin{aligned} f_{2*}\mathcal{O}_Y(K_{Y_2/Y_0} - [cF_2]) &= f_{1*}g_*\mathcal{O}_{Y_2}(K_{Y_2/Y_1} + g^*K_{Y_1/Y_0} - [cg^*F_1]) \\ &= f_{1*}\Big(g_*\mathcal{O}_{Y_2}(K_{Y_2/Y_1} - [cg^*F_1]) \otimes \mathcal{O}_{Y_1}(K_{Y_1/Y_0})\Big) \\ &= f_{1*}\mathcal{O}_{Y_1}(K_{Y_1/Y_0} - [cF_1]). \end{aligned}$$

Therefore $f_*\mathcal{O}_Y(K_{Y/Y_0} - [cF])$ is independent of the choice of log resolutions. $\square$

Theorem 4.1.20. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec} A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then for $c > 0$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^c) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - [cZ])).$$

Proof. By Lemma 4.1.19, we may assume that f is a log resolution of $\mathfrak{j}_X \mathfrak{d}_X \mathfrak{a}$. Let J, D be divisors on Y such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ and $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Let $K^\omega = \widehat{K}_{Y/X} - J + D$ and $K = K_{Y/Y_0}$. By Proposition 4.1.17, $K^\omega = K$. This implies that

$$\mathcal{J}^\omega(A, \mathfrak{a}^c) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - [cZ])).$$

□

Proposition 4.1.21. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then*

$$\mathcal{J}^\omega(A, \mathrm{core}(\mathfrak{a})) = (\mathcal{J}^\omega(A, \mathfrak{a}))^2.$$

Proof. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ for some effective divisor Z on Y. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$ and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let $K = K_{Y/Y_0}$ and

$$C = \{j \in \mathbb{N} | 1 \leq i \leq n, Z \cdot \pi_j^* E_j < 0\}.$$

By Lemma 4.1.14 we have

$$\mathrm{an}_f(Z - K) = Z - \sum_{i \in C} \pi_i^* E_i.$$

By Theorem 4.1.7 we have

$$\mathrm{core}(\mathfrak{a}) = f_* \mathcal{O}_Y(\sum_{i \in C} \pi_i^* E_i - 2Z).$$

Let

$$C' = \{j \in \mathbb{N} | 1 \leq i \leq n, (2Z - \sum_{i \in C} \pi_i^* E_i) \cdot \pi_j^* E_j < 0\}.$$

Then by Lemma 4.1.14 we have

$$\mathrm{an}_f(2Z - \sum_{i \in C} \pi_i^* E_i - K) = 2Z - \sum_{i \in C} \pi_i^* E_i - \sum_{i \in C'} \pi_i^* E_i.$$

We will show $C = C'$. Let $j \in C$. Since $Z \cdot \pi_j^* E_j < 0$ and $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j = -1$, we have $(2Z - \sum_{i \in C} \pi_i^* E_i) \cdot \pi_j^* E_j < 0$. Therefore $C \subset C'$.

Hence we will show the opposite inclusion. We assume that we can take $j \in C' \setminus C$. Then $Z \cdot \pi_j^* E_j = 0$ and $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j > 0$ since $(2Z - \sum_{i \in C} \pi_i^* E_i) \cdot \pi_j^* E_j < 0$. On the other hand since $\pi_i^* E_i \cdot \pi_j^* E_j = 0$ for $i \neq j$, we have $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j$ is 0, which is a contradiction. Thus we have $C = C'$. This implies that

$$\mathrm{an}_f(2Z - \sum_{i \in C} \pi_i^* E_i - K) = 2(Z - \sum_{i \in C} \pi_i^* E_i).$$

Thus we have

$$\mathcal{J}^\omega(A, \mathrm{core}(\mathfrak{a})) = f_* \mathcal{O}_Y(K - (2Z - \sum_{i \in C} \pi_i^* E_i))$$

$$= f_* \mathcal{O}_Y(-2(Z - \sum_{i \in C} \pi_i^* E_i)) = (\mathcal{J}^\omega(A, \mathfrak{a}))^2.$$

□

Proposition 4.1.22. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then for $n \in \mathbb{N}$,*

$$\mathrm{core}(\mathfrak{a}^n) = \mathfrak{a}^{2n-1} \mathcal{J}^\omega(A, \mathfrak{a}).$$

Proof. We have $\mathrm{core}(\mathfrak{a}^n) = \mathfrak{a}^n \mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}^{2n-1} \mathcal{J}^\omega(A, \mathfrak{a})$ by Proposition 4.1.10 and Corollary 4.1.12. □

Now we introduce some notation: $\mathrm{core}^1(\mathfrak{a}) = \mathrm{core}(\mathfrak{a})$ and, for $n > 1$, $\mathrm{core}^n(\mathfrak{a}) = \mathrm{core}^{n-1}(\mathrm{core}(\mathfrak{a}))$.

Proposition 4.1.23. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then for $n \in \mathbb{N}$,*

$$\mathrm{core}^n(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^{2^n-1}.$$

In particular, $\mathrm{core}(\mathrm{core}(\mathfrak{a})) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^3$.

Proof. We have $\mathrm{core}(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))$ by Corollary 4.1.12. Now let $n > 1$ and assume that the proposition holds for $n-1$. Then by Proposition 4.1.21

$$\begin{aligned}\mathrm{core}^n(\mathfrak{a}) &= \mathrm{core}^{n-1}(\mathrm{core}(\mathfrak{a})) \\ &= \mathrm{core}(\mathfrak{a})\Big(\mathcal{J}^\omega(A, \mathrm{core}(\mathfrak{a}))\Big)^{2^{n-1}-1} \\ &= \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a})\Big((\mathcal{J}^\omega(A, \mathfrak{a}))^2\Big)^{2^{n-1}-1} \\ &= \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^{2^n-1}.\end{aligned}$$

□

Proposition 4.1.24. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity. Let $X = \mathrm{Spec} A$. Let $\mathfrak{j}_X$ be the Jacobian ideal of X and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$ and $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ for some effective divisors $J_{Y/X}$ and $D_{Y/X}$ on Y. Then*

$$R^1 f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

Proof. If X is locally a complete intersection, then $\mathfrak{d}_X = \mathcal{O}_X$. Therefore $D_{Y/X} = 0$. Then by Local Vanishing Theorem (see Theorem 3.5 in [8]),

$$R^1 f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

We assume that X is not locally a complete intersection. Let $I = (x_1, x_2)$ be a minimal reduction of $\mathfrak{d}_X$. If V is the $\mathbb{C}$-vector space generated by x_1, x_2, then we have on Y an exact Koszul complex

$$0 \to \wedge^2 V \otimes \mathcal{O}_Y(2D_{Y/X}) \to V \otimes \mathcal{O}_Y(D_{Y/X}) \to \mathcal{O}_Y \to 0.$$

Let $\mathcal{L}_n = \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - nD_{Y/X})$. By tensoring with $\mathcal{L}_2$ we get the exact complex

$$0 \to \wedge^2 V \otimes \mathcal{L}_0 \to V \otimes \mathcal{L}_1 \to \mathcal{L}_2 \to 0.$$

Since $\mathcal{J}^\omega(A, \mathfrak{d}^2) = I\mathcal{J}^\omega(A, \mathfrak{d}^1)$ by Proposition 4.1.10, the map $V \otimes f_*\mathcal{L}_1 \to f_*\mathcal{L}_2$ is surjective. Hence the map

$$\wedge^2 V \otimes R^1 f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) \to V \otimes R^1 f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X})$$

is injective. Since $R^1 f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X}) = 0$ by Local Vanishing Theorem (see [8]), we have

$$R^1 f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}) = 0.$$

□

4.2 Subadditivity thorem for ω-multiplier ideals of a 2-dimensional singularity

In this section, we investigate when the subadditivity theorem of ω-multiplier ideals holds in the two-dimensional case.

Demailly, Ein and Lazarsfeld proved the following theorem, which is called the subadditivity theorem.

Theorem 4.2.1. ([4]) *Let (A, m) be a regular local ring. Then for any two nonzero ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$,*

$$\mathcal{J}(X, \mathfrak{a}^c\mathfrak{b}^d) \subset \mathcal{J}(X, \mathfrak{a}^c)\mathcal{J}(X, \mathfrak{b}^d).$$

In this paper, we say that the subadditivity theorem holds if $\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b})$ for any two nonzero ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$, and the strong subadditivity theorem holds if $\mathcal{J}^\omega(X, \mathfrak{a}^c\mathfrak{b}^d) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)\mathcal{J}^\omega(X, \mathfrak{b}^d)$ for any two nonzero ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$.

The following lemma seems to be well known to the specialists, but for lack of an explicit reference we give its proof.

Lemma 4.2.2. *Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and fix a resolution of singularities $f : Y \to \operatorname{Spec} A$. Let Z_1, Z_2 be two effective f-anti-nef divisors on Y. Then $f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2)$ if and only if $Z_1 \geq Z_2$.*

Proof. If $Z_1 \geq Z_2$, then $f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2)$. Hence we will show the converse implication. Suppose, by way of contradiction, $f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2)$ and $Z_1 \not\geq Z_2$. Note that $f_*\mathcal{O}_Y(-Z_1)\cdot\mathcal{O}_Y = \mathcal{O}_Y(-Z_1)$ by Theorem 4.1.2. Then

$$\begin{gathered} x \in f_*\mathcal{O}_Y(-Z_2) : f_*\mathcal{O}_Y(-Z_1) \Leftrightarrow x f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2) \\ \Leftrightarrow f^*x \cdot \mathcal{O}_Y(-Z_1) \subset \mathcal{O}_Y(-Z_2) \Leftrightarrow f^*x \in \mathcal{O}_Y(Z_1 - Z_2) \\ \Leftrightarrow x \in f_*\mathcal{O}_Y(Z_1 - Z_2). \end{gathered}$$

Therefore we have $f_*\mathcal{O}_Y(-Z_2) : f_*\mathcal{O}_Y(-Z_1) = f_*\mathcal{O}_Y(Z_1 - Z_2)$. Since

$$f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2),$$

we have $f_*\mathcal{O}_Y(-Z_2) : f_*\mathcal{O}_Y(-Z_1) = A$. On the other hand we have $f_*\mathcal{O}_Y(Z_1 - Z_2) \neq A$ since $Z_1 \not\geq Z_2$. Thus if $f_*\mathcal{O}_Y(-Z_1) \subset f_*\mathcal{O}_Y(-Z_2)$, then $Z_1 \geq Z_2$. $\square$

Theorem 4.2.3. *Let (A, m) be a two-dimensional normal local ring. Then $X = \operatorname{Spec} A$ has a rational singularity if and only if the subadditivity theorem of ω-multiplier ideals holds, that is, for any two nonzero ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b}).$$

Proof. If the subadditivity theorem holds, then $\mathcal{J}^\omega(X, \mathcal{O}_X) \subset \mathcal{J}^\omega(X, \mathcal{O}_X)^2$. Thus $\mathcal{J}^\omega(X, \mathcal{O}_X) = \mathcal{O}_X$, namely X has a rational singularity. Hence we will show the converse implication, that is, we will prove that for any two ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$, $\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b})$, when X has a rational singularity. Let $f : Y \to X$ be a resolution of singularities such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{a})$ and $\mathfrak{b}\mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{b})$ are invertible and $\operatorname{Exc}(f) \cup \operatorname{Supp} F_\mathfrak{a} \cup \operatorname{Supp} F_\mathfrak{b}$ is a simple normal crossing divisor. Denote by K the relative canonical divisor K_{Y/Y_0}, where Y_0 is the minimal resolution of X. By Theorem 4.1.20, we have

$$\mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b}) = H^0(Y, \mathcal{O}_Y(K - F_\mathfrak{a}))H^0(Y, \mathcal{O}_Y(K - F_\mathfrak{b})),$$

$$\mathcal{J}^{\omega}(X, \mathfrak{ab}) = H^0(Y, \mathcal{O}_Y(K - F_{\mathfrak{a}} - F_{\mathfrak{b}})).$$

Since X has a rational singularity, the product of integrally closed ideals of X is also integrally closed (see [27]). Hence $\mathcal{J}^{\omega}(X, \mathfrak{a})\mathcal{J}^{\omega}(X, \mathfrak{b})$ and $\mathcal{J}^{\omega}(X, \mathfrak{ab})$ are integrally closed, and $\mathcal{J}^{\omega}(X, \mathfrak{a})\mathcal{J}^{\omega}(X, \mathfrak{b})$ and $\mathcal{J}^{\omega}(X, \mathfrak{ab})$ correspond to the cycles $\mathrm{an}_f(F_{\mathfrak{a}} - K) + \mathrm{an}_f(F_{\mathfrak{b}} - K)$ and $\mathrm{an}_f(F_{\mathfrak{a}} + F_{\mathfrak{b}} - K)$, respectively. Therefore, it suffices to show that

$$\mathrm{an}_f(F_{\mathfrak{a}} - K) + \mathrm{an}_f(F_{\mathfrak{b}} - K) \leq \mathrm{an}_f(F_{\mathfrak{a}} + F_{\mathfrak{b}} - K).$$

In order to prove this, we prepare some notation. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$ and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Using Lemma 4.1.14, we will prove

$$\mathrm{an}_f(F_{\mathfrak{a}} - K) + \mathrm{an}_f(F_{\mathfrak{b}} - K) \leq \mathrm{an}_f(F_{\mathfrak{a}} + F_{\mathfrak{b}} - K).$$

Let

$$C_{\mathfrak{a}} = \{j \in \mathbb{N} | 1 \leq i \leq n, F_{\mathfrak{a}} \cdot \pi_j^* E_j < 0\},$$
$$C_{\mathfrak{b}} = \{j \in \mathbb{N} | 1 \leq i \leq n, F_{\mathfrak{b}} \cdot \pi_j^* E_j < 0\}$$

and

$$C_{\mathfrak{ab}} = \{j \in \mathbb{N} | 1 \leq i \leq n, (F_{\mathfrak{a}} + F_{\mathfrak{b}}) \cdot \pi_j^* E_j < 0\}.$$

Then we have $C_{\mathfrak{ab}} \subset C_{\mathfrak{a}} \cup C_{\mathfrak{b}}$. Therefore by Lemma 4.1.14,

$$\begin{aligned} \mathrm{an}_f(F_{\mathfrak{a}} - K) + \mathrm{an}_f(F_{\mathfrak{b}} - K) &= F_{\mathfrak{a}} - \sum_{i \in C_{\mathfrak{a}}} \pi_i^* E_i + F_{\mathfrak{b}} - \sum_{i \in C_{\mathfrak{b}}} \pi_i^* E_i \\ &\leq F_{\mathfrak{a}} + F_{\mathfrak{b}} - \sum_{i \in C_{\mathfrak{ab}}} \pi_i^* E_i = \mathrm{an}_f(F_{\mathfrak{a}} + F_{\mathfrak{b}} - K). \end{aligned}$$

□

Lemma 4.2.4. *Let $(A, \mathfrak{m})$ be an n-dimensional local ring and I be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec} A$ be a log resolution of I such that $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then for any divisor K on Y,*

$$f_* \mathcal{O}_Y(K) : I = f_* \mathcal{O}_Y(K + F).$$

Proof. Then

$$x \in f_*\mathcal{O}_Y(K) : I \Leftrightarrow xI \subset f_*\mathcal{O}_Y(K)$$
$$\Leftrightarrow f^*x \cdot \mathcal{O}_Y(-F) \subset \mathcal{O}_Y(K) \Leftrightarrow f^*x \in \mathcal{O}_Y(K+F)$$
$$\Leftrightarrow x \in f_*\mathcal{O}_Y(K+F).$$

Therefore we have $f_*\mathcal{O}_Y(K) : I = f_*\mathcal{O}_Y(K+F)$. □

Corollary 4.2.5. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec}A$ has a rational singularity if and only if the subadditivity theorem of cores of ideals holds, that is, for any two $\mathfrak{m}$-primary integral closed ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathrm{core}(\mathfrak{ab}) \subset \mathrm{core}(\mathfrak{a})\mathrm{core}(\mathfrak{b}).$$

Proof. If A has a rational singularity, then

$$\mathrm{core}(\mathfrak{ab}) = \mathcal{J}^\omega(X, \mathfrak{a}^2\mathfrak{b}^2) \subset \mathcal{J}^\omega(X, \mathfrak{a}^2)\mathcal{J}^\omega(X, \mathfrak{b}^2) = \mathrm{core}(\mathfrak{a})\mathrm{core}(\mathfrak{b})$$

by Corollary 4.1.12 and Theorem 4.2.3. Hence we will show the converse implication. Let I be an $\mathfrak{m}$-primary integral closed ideal such that $g : Z = \mathrm{Proj}A[I] \to X = \mathrm{Spec}A$ is a resolution of singularities. Let F' be an effective divisor on Z such that $I\mathcal{O}_Z = \mathcal{O}_Z(-F')$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot I$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, and $I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$ and F on Y. Let $K = \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$. Then

$$\mathrm{core}(I) = g_*\mathcal{O}_Z(K_Z - 2F') : \omega_X = f_*\mathcal{O}_Y(K_Y - 2F) : \omega_X$$
$$= \mathcal{J}^\omega(X, I^2) = f_*\mathcal{O}_Y(K - 2F)$$

by Lemma 2.3.9, Lemma 2.3.11 and Theorem 3.1.3. In the same manner, we have

$$\mathrm{core}(I^2) = f_*\mathcal{O}_Y(K - 4F).$$

Next we will show that

$$f_*\mathcal{O}_Y(K - 2F) \subset f_*\mathcal{O}_Y(2^{n-1}K - 2F)$$

for any $n \in \mathbb{N}$ by induction on n. When $n = 1$, the assertion is trivial. By the induction hypothesis and subadditivity of cores of ideals, we have

$$f_*\mathcal{O}_Y(K - 4F) = \mathrm{core}(I^2) \subset (\mathrm{core}(I))^2 = (f_*\mathcal{O}_Y(K - 2F))^2$$

$$\subset (f_*\mathcal{O}_Y(2^{n-1}K - 2F))^2 \subset f_*\mathcal{O}_Y(2^n K - 4F).$$

Therefore we have

$$f_*\mathcal{O}_Y(K - 2F) = f_*\mathcal{O}_Y(K - 4F) : I^2$$

$$\subset f_*\mathcal{O}_Y(2^n K - 4F) : I^2 = f_*\mathcal{O}_Y(2^n K - 2F)$$

by Lemma 4.2.4. By the above discussion, we have

$$f_*\mathcal{O}_Y(K - 2F) \subset f_*\mathcal{O}_Y(2^{n-1}K - 2F)$$

for any $n \in \mathbb{N}$. By Lemma 4.2.4 we have for any $n \in \mathbb{N}$,

$$f_*\mathcal{O}_Y(K) = f_*\mathcal{O}_Y(K - 2F) : I^2 \subset f_*\mathcal{O}_Y(2^{n-1}K - 2F) : I^2 = f_*\mathcal{O}_Y(2^{n-1}K).$$

This implies that K is effective. Since $\mathcal{J}^\omega(A) = f_*\mathcal{O}_Y(K) = A$, A has a rational singularity.

□

In order that the strong subadditivity theorem of ω-multiplier ideal holds, non-singularness is necessary.

Proposition 4.2.6. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec}A$ is regular if and only if the strong subadditivity theorem of ω-multiplier ideals holds, that is, for any two nonzero ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$,*

$$\mathcal{J}^\omega(X, \mathfrak{a}^c\mathfrak{b}^d) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)\mathcal{J}^\omega(X, \mathfrak{b}^d).$$

Proof. If A is regular, then the strong subadditivity theorem holds (see [4]). Hence we will show the converse implication. In order that the strong subadditivity theorem holds, by Theorem 4.2.3, it is necessary that A is a rational singularity. Assume that A is not regular. Let $f : Y \to X$ be the minimal resolution and F be the fundamental cycle of f.

We assume that the exceptional locus of f is irreducible. Then F is the f-exceptional prime divisor. Let $g : Z \to Y$ be the blowing-up at closed point of F and $h : Z \to X$ be the composite morphism of f and g. We denote by E_1 the strict transform of F and by E_2 the exceptional divisor of g. Let $n = -E_1 \cdot E_1$, $C = (n-1)E_1 + 2nE_2$ and $K = K_{Z/Y} = E_2$. Then C and $(n-1)E_1 + (2n-1)E_2$ are h-anti-nef since $n = -E_1 \cdot E_1 = -F \cdot F + 1 \geq 3$. Since $E_1 + E_2$ is the fundamental cycle of h, we have

$$\mathrm{an}_h(\lfloor \frac{1}{n} C - K \rfloor) = E_1 + E_2$$

$$\mathrm{an}_h(C - K) = (n-1)E_1 + (2n-1)E_2.$$

These imply that

$$h_*\mathcal{O}_Z(-\mathrm{an}_h(C-K)) \not\subset \Big(h_*\mathcal{O}_Z(-\mathrm{an}_h(\lfloor \frac{1}{n} C - K \rfloor))\Big)^n$$

by Lemma 4.2.2. Therefore, denoting the ideal $I = h_*\mathcal{O}_Z(-C)$, we have $\mathcal{J}^\omega(X, I) \not\subset \mathcal{J}^\omega(X, I^{\frac{1}{n}})^n$ by Theorem 4.1.20. Thus the strong subadditivity theorem does not hold on A.

We assume that the exceptional locus of f is reducible. Let E be a f-exceptional prime divisor such that $F \cdot E < 0$. Then there exists $n \in \mathbb{N}$ such that $nF - E$ is f-anti-nef. Since F is the fundamental cycle of f, we have

$$\mathrm{an}_f(\lfloor \frac{1}{n}(nF - E) \rfloor)) = F,$$

$$\mathrm{an}_f(nF - E) = nF - E.$$

These imply that

$$f_*\mathcal{O}_Y(-\mathrm{an}_f(nF-E)) \not\subset \Big(f_*\mathcal{O}_Y(-\mathrm{an}_f(\lfloor \frac{1}{n}(nF - E) \rfloor))\Big)^n$$

by Lemma 4.2.2. Therefore, denoting the ideal $I = f_*\mathcal{O}_Y(-nF+E)$, we have $\mathcal{J}^\omega(X, I) \not\subset \mathcal{J}^\omega(X, I^{\frac{1}{n}})^n$ by Theorem 4.1.20. Thus the strong subadditivity theorem does not hold on A.

According to the above discussion, if A is not regular, then the strong subadditivity theorem does not hold on A.

$\square$

Remark 4.2.7. In higher dimensional case, we have a counterexample to Theorem 4.2.3.

Takagi and Watanabe gave the following counterexample to the subadditivity of multiplier ideals in a 3-dimensional hypersurface local ring in [37]. Since the ring is Gorenstein, the multiplier ideals are ω-multiplier ideals by Proposition 3.1.10.

Example 4.2.8. Let $A = (\mathbb{C}[X,Y,Z,W]/(X^2 + Y^4 + Z^4 + W^5))_{(X,Y,Z,W)}$ and $\mathfrak{m} = (x, y, z, w)$, where x, y, z, w are the images of X, Y, Z, W in A. Then A is a Gorenstein canonical singularity, but not a terminal singularity. Therefore A is a rational singularity, $\mathcal{J}^\omega(\mathfrak{m}) = \mathfrak{m}$ and $\overline{\mathfrak{m}^2} \subset \mathcal{J}^\omega(\mathfrak{m}^2)$. Since $x^2 \in \mathfrak{m}^4$, we have $x \in \overline{\mathfrak{m}^2}$. Hence $x \in \mathcal{J}^\omega(\mathfrak{m}^2)$ and $x \notin \mathcal{J}^\omega(\mathfrak{m})\mathcal{J}^\omega(\mathfrak{m})$. Thus $\mathcal{J}^\omega(\mathfrak{m}^2) \not\subset \mathcal{J}^\omega(\mathfrak{m})\mathcal{J}^\omega(\mathfrak{m})$.

4.3 Integrally closed ideals on surface with a rational singularity

In this section, we show that all integrally closed ideals on surface with a rational singularity are ω-multiplier ideals.

Theorem 4.3.1. *Let (A, m) be a two-dimensional normal local ring. Suppose $X = \mathrm{SpecA}$ has a rational singularity. Then every integrally closed ideal is an ω-multiplier ideal.*

Favre, Jonsson, Lipman and Watanabe showed that all integrally closed ideals on regular surfaces are multiplier ideals (see [10] and [29]). Our result is a generalization of this theorem since ω-multiplier ideals of regular scheme are multiplier ideals.

Definition 4.3.2. Let (A, m) be a two-dimensional normal local ring. Let $f : Y \to X$ be a resolution of singularities such that $f^{-1}(\mathfrak{m})$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(\mathfrak{m})$. $\check{E}_i$ is defined to be a effective exceptional $\mathbb{Q}$-divisor such that

$$\check{E}_i \cdot E_j = \begin{cases} -1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{4.1}$$

Definition 4.3.3. Let Y be a 2-dimensional regular scheme and $x^{(i)}$ be a closed point of Y. A generic sequence of n-blowing-ups over $x^{(i)}$ is:

$$Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 = Y$$

where f_i is the blowing-up of $Y_0 = Y$ at $x_1 := x^{(i)}$, and $f_k : Y_k \to Y_{k-1}$ is the blowing-up of Y_{k-1} at general closed point x_k of $(f_{k-1})^{-1}(x_{k-1})$ for $k = 2, \ldots, n$. Let $f : Y_n \to Y$ be the composition $f_1 \circ \cdots \circ f_n$. Let $E(x^{(i)}, k), k = 1, \ldots, n$ be the strict transforms of the n new f-exceptional divisors created by blowing-ups $f_1, \ldots, f_n$ respectively.

Lipman and Watanabe stated the following in [29].

Remark 4.3.4. f^{-1} is a chain of n integral curve $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ such that for $0 < k < n$,

$$E(x^{(i)}, k) \cdot E(x^{(i)}, k+1) = 1,$$

$$E(x^{(i)}, k) \cdot E(x^{(i)}, k) = -2$$

while

$$E(x^{(i)}, n) \cdot E(x^{(i)}, n) = -1;$$

and if $|k' - k| > 1$ then

$$E(x^{(i)}, k') \cdot E(x^{(i)}, k) = 0.$$

Lemma 4.3.5. *Let Y be a 2-dimensional regular scheme and $x^{(i)}$ be a closed point of Y. Let $f : Y_n \to Y$ be a generic sequence of n-blowing-ups over $x^{(i)}$. As in Definition 4.3.3 denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$. Then*

$$K_f := K_{Y_n} - f^*(K_Y) = \sum_{k=1}^{n} kE(x^{(i)}, k).$$

Proof. We will show the lemma by induction of n. When $n = 1$, we have $K_f := K_{Y_1} - f^*(K_Y) = E(x^{(i)}, k)$. By in the induction hypothesis, $K_{Y_{n-1}/K_Y} = \sum_{k=1}^{n-1} kE(x^{(i)}, k)$. Therefore

$$K_{Y_n} - f^*(K_Y) = K_{Y_n/Y_{n-1}} + f_n^* K_{Y_{n-1}/K_Y}$$

$$= nE(x^{(i)}, n) + \sum_{k=1}^{n-1} kE(x^{(i)}, k) = \sum_{k=1}^{n} kE(x^{(i)}, k).$$

□

Lemma 4.3.6. *Let Y be a 2-dimensional regular scheme and $x^{(i)}$ be a closed point of Y. Let $f : Y_n \to Y$ be a generic sequence of n-blowing-ups over $x^{(i)}$. As in Definition 4.3.3 denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$. Let $K_f = K_{Y_n} - f^*(K_Y)$. Then*

$$K_f \cdot E(x^{(i)}, k) = \begin{cases} -1 & (k = n) \\ 0 & (k \neq n) \end{cases} \tag{4.2}$$

Proof. By Lemma 4.3.5, $K_f := K_{Y_n} - f^*(K_Y) = \sum_{k=1}^{n} kE(x^{(i)}, k)$. For $k \neq n$, by Remark 4.3.4

$$\begin{aligned} &K_f \cdot E(x^{(i)}, k) \\ =&\Big((k-1)E(x^{(i)}, k-1) + kE(x^{(i)}, k) + (k+1)E(x^{(i)}, k+1)\Big) \cdot E(x^{(i)}, k) \\ &= (k-1) - 2k + (k+1) = 0. \end{aligned}$$

By Remark 4.3.4

$$\begin{aligned} &K_f \cdot E(x^{(i)}, n) \\ =&\Big((n-1)E(x^{(i)}, n-1) + nE(x^{(i)}, n)\Big) \cdot E(x^{(i)}, n) \\ &= (n-1) - n = -1. \end{aligned}$$

□

Tucker showed the following in [38].

Lemma 4.3.7. ([38]) *Let $(A, \mathfrak{m})$ be a 2-dimensional normal local ring. Let $f : Y \to X = \mathrm{Spec}A$ be a resolution of singularities such that $f^{-1}(\mathfrak{m})$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(\mathfrak{m})$. Suppose $x^{(i)}$ be a closed point of E_i with $x^{(i)} \notin E_j$ for $j \neq i$. Let $g : Y_n \to Y$ be a generic sequence of n-blowing-ups over $x^{(i)}$. As in Definition 4.3.3 denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$ and $E(i)$ the strict transforms of $E_1, \ldots, E_u$ on Y_n. Then*
(1) $\check{E}(i) \leq \check{E}(x^{(i)}, 1) \leq \cdots \leq \check{E}(x^{(i)}, n)$.

(2) *Suppose D is an integral $f \circ g$-anti-nef divisor on Y_n such that E_i is the unique component of g_*D containing $x^{(i)}$. Then*

$$\mathrm{ord}_{E(i)}D \le \mathrm{ord}_{E(x^{(i)},1)}D \le \cdots \le \mathrm{ord}_{E(x^{(i)},n)}D.$$

Further $\mathrm{ord}_{E(i)}D < \mathrm{ord}_{E(x^{(i)},n)}D$ *if and only if*

$$\sum_{k=1}^{n}(-D \cdot E(x^{(i)},k))\check{E}(x^{(i)},k) \ge \check{E}(i).$$

Tuker showed that all integrally closed ideals on log terminal surfaces are multiplier ideals (see [38]). Our proof is just an imitation of the proof of the Theorem 1.1 of [38].

We will begin the proof of Theorem 4.3.1.

Proof. Let $I \subset \mathcal{O}_X$ be an integrally closed ideal. We will construct an ideal $\mathfrak{a}$ and $c \in \mathbb{Q}_{>0}$ such that $I = \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot I$ with exceptional divisors $E_1, \ldots, E_u$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_0)$. Let $K = \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$. Write

$$K = \sum_{i=1}^{u} b_i E_i,$$

$$F_0 = (f^{-1})_* f_*(F_0) + \sum_{i=1}^{u} a_i E_i.$$

Note that $b_i \ge 0$ since X has a rational singularity. Let $0 < \epsilon < 1/2$ such that $\lfloor \epsilon (f^{-1})_* f_*(F_0) \rfloor = 0$ and $\epsilon(a_i + 1) < 1 + b_i$ for $i = 1, \ldots, u$. Let $n_i := \lfloor \frac{1+b_i}{\epsilon} - (a_i + 1) \rfloor > 0$ and $e_i := (-F_0 \cdot E_i)$. Choose e_i distinct closed points $x_1^{(i)}, \ldots, x_{e_i}^{(i)}$ on E_i such that $x_j^{(i)} \notin \mathrm{Supp}((f^{-1})_* f_*(F_0))$ and $x_j^{(i)} \notin E_l$ for $l \neq i$. Denote by $g : Z \to Y$ the composition of generic sequence of n_i-blowing-ups over each of the points $x_j^{(i)}$ for $j = 1, \ldots, e_i$ and $i = 1, \ldots, u$. As in Definition 4.3.3 denote by $E(x_j^{(i)}, 1), \ldots, E(x_j^{(i)}, n_i)$ the strict transforms of the n_i exceptional divisors over $x_j^{(i)}$ and $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$.

Let $h := f \circ g$ and $F = g^*(F_0)$. By Lemma 4.3.5 and Lemma 4.3.6,

$$K_g := K_Z - g^*(K_Y) = \sum_{i=1}^{u} \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} k E(x_j^{(i)}, k)$$

and

$$K_g \cdot E(i) = e_i$$

$$K_g \cdot E(x_j^{(i)}, k) = \begin{cases} -1 & (k = n_i) \\ 0 & (k \neq n_i) \end{cases} \tag{4.3}$$

Then $F + K_g$ is h-anti-nef since

$$F \cdot E(i) = F_0 \cdot E_i = -e_i, \quad F \cdot E(x_j^{(i)}, k) = 0.$$

Let $K' = K_g + g^*(K)$, $\mathfrak{a} = h_* \mathcal{O}_Z(-(F + K_g))$ and $c = 1 + \epsilon$. Then by Theorem 4.1.2, we have $\mathfrak{a}\mathcal{O}_Z = \mathcal{O}_Z(-(F + K_g))$.

We will show $I = \mathcal{J}^\omega(X, \mathfrak{a}^c) = h_* \mathcal{O}_Z(-F)$. By Theorem 3.1.3,

$$\mathcal{J}^\omega(X, \mathfrak{a}^c) = h_* \mathcal{O}_Z(-\lfloor c(F + K_g) - K' \rfloor).$$

Therefore it suffices to show that

$$F' := \mathrm{an}_h(\lfloor c(F + K_g) - K' \rfloor) = F,$$

by Lemma 4.1.3.

Claim 1 We have $F' \leq F$ and $h_* F' = h_* F$. In addition, for $i = 1, \ldots, u$ and $j = 1, \ldots, e_i$,

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(F') = \mathrm{ord}_{E(x_j^{(i)}, n_i)}(F) = \mathrm{ord}_{E(i)}(F).$$

proof of Claim 1. By the definition of a generic sequence of blowing-up, we have

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(F) = \mathrm{ord}_{E(i)}(F).$$

Since $F' = \mathrm{an}_h(\lfloor c(F + K_g) - K' \rfloor)$ and F are h-anti-nef, it suffices to show that

$$\lfloor c(F + K_g) - K' \rfloor \leq F,$$

$$h_* \lfloor c(F + K_g) - K' \rfloor = h_* F,$$

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(\lfloor c(F + K_g) - K' \rfloor) = \mathrm{ord}_{E(x_j^{(i)}, n_i)}(F).$$

We have

$$\lfloor c(F + K_g) - K' \rfloor = F + \lfloor \epsilon(F + K_g) - g^* K \rfloor.$$

Since $\lfloor \epsilon(f^{-1})_* f_*(F_0) \rfloor = 0$, it follows that $h_* \lfloor c(F+K_g) - K' \rfloor = h_* F$. Consider the coefficients of $\epsilon(F+K_g) - g^*K$. We have

$$\mathrm{ord}_{E(i)}(\epsilon(F+K_g) - g^*K) = \epsilon a_i - b_i < 1,$$

$$\mathrm{ord}_{E(x_j^{(i)},k)}(\epsilon(F+K_g) - g^*K) = \epsilon(a_i + k) - b_i.$$

Since $0 < \epsilon < 1/2$ and $\frac{1+b_i}{\epsilon} - (a_i+1) - 1 < n_i \leq \frac{1+b_i}{\epsilon} - (a_i+1)$, we have

$$0 < 1 - 2\epsilon < \epsilon(a_i + n_i) - b_i \leq 1 - \epsilon < 1.$$

Therefore we have

$$\mathrm{ord}_{E(x_j^{(i)},k)} \lfloor \epsilon(F+K_g) - g^*K \rfloor \leq 0$$

$$\mathrm{ord}_{E(x_j^{(i)},n_i)} \lfloor \epsilon(F+K_g) - g^*K \rfloor = 0.$$

Thus we have $F' \leq F$ and

$$\mathrm{ord}_{E(x_j^{(i)},n_i)}(F') = \mathrm{ord}_{E(x_j^{(i)},n_i)}(F).$$

$\square$

Claim 2 For each $i = 1, \dots, u$,

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)},k))\check{E}(x_j^{(i)},k) \geq (-F \cdot E(i))\check{E}(i).$$

proof of Claim 2.

1. We assume that $\mathrm{ord}_{E(i)} F' = \mathrm{ord}_{E(i)} F$.
   We have $F' \cdot E(i) \leq F \cdot E(i)$ since we have $F' \leq F$ by Claim 1. Since $\check{E}(i)$ and $\check{E}(x_j^{(i)},k)$ are effective and F' is h-anti-nef, we have

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)},k))\check{E}(x_j^{(i)},k) \geq (-F \cdot E(i))\check{E}(i).$$

2. We assume that $\mathrm{ord}_{E(i)} F' < \mathrm{ord}_{E(i)} F = \mathrm{ord}_{E(x_j^{(i)},n_i)} F'$.
   Then for each $j = 1, \dots, e_i$ we have

$$\sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)},k))\check{E}(x_j^{(i)},k) \geq \check{E}(i)$$

by Lemma 4.3.7. Therefore we have

$$
\begin{aligned}
(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i}\sum_{k=1}^{n_i}(-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k) \\
\geq \sum_{j=1}^{e_i}\sum_{k=1}^{n_i}(-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k) \\
\geq e_i\check{E}(i) = (-F \cdot E(i))\check{E}(i)
\end{aligned}
$$

□

Next we will prove that $F' \geq F$. By the two claims, we have

$$
\begin{aligned}
F' &= h^*h_*F' + \sum_{i=1}^{u}\left((-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i}\sum_{k=1}^{n_i}(-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k)\right) \\
&\geq h^*h_*F + \sum_{i=1}^{u}(-F \cdot E(i))\check{E}(i) = F.
\end{aligned}
$$

Therefore we have $F = F'$ by Claim 1. Thus $I = \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$.

□

Remark 4.3.8. In higher dimensional case, we have counterexamples to Theorem 4.3.1 (see [25] and [26]).

Chapter 5

Upper bound of the multiplicity

5.1 Upper bound of the multiplicity of a Du Bois singularity

In this section, we show bounds of the multiplicity of a Du Bois singularity. Recall that for an n-dimensional variety X and an n-dimensional locally complete intersection variety $V \supset X$, the ideal $\mathfrak{d}_{X,V}$ is the ideal such that

$$\mathrm{Im}(\omega_X \to \omega_V|_X) = \mathfrak{d}_{X,V} \otimes \omega_V|_X.$$

The following is a generalization of Theorem 3.1 in [15]. But our proof is just an imitation of the proof of Theorem 3.1 in [15].

Proposition 5.1.1. *Let X be an n-dimensional variety with rational singularities. Then for a closed point $x \in X$,*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X,x) - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}{n - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and I be a minimal reduction of $\mathfrak{m}$. Since $\mathcal{O}_{X.x}$ is Cohen-Macaulay, we have $\mathrm{e}(\mathfrak{m}) = \ell(\mathcal{O}_{X,x}/I)$. Let $v = \mathrm{emb}(X,x)$. we may assume that $\{x_1, \ldots, x_n, y_1, \ldots, y_{v-n}\}$ is a minimal generators of $\mathfrak{m}$ with $I = (x_1, \ldots, x_n)$. Then $\mathcal{O}_{X,x}/I$ is generated as a $\mathbb{C}$-vector space by 1 and the monomials of $y_1, \ldots, y_{v-n}$. Here, we can take generators as monomials of degree $\leq n - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil$, since $I \supset \mathfrak{m}^{n+1-\lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}$

by Lemma 3.1.20. Therefore we obtain $\ell(\mathcal{O}_{X,x}/I) \leq \binom{v - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}{n - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}$. Then we obtain

$$\mathrm{e}(\mathfrak{m}) \leq \binom{\mathrm{emb}(X,x) - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}{n - \lceil \mathrm{rt}(\mathfrak{m}_x) \rceil}.$$

□

Corollary 5.1.2. *Let X be an n-dimensional variety with rational singularities. If $\lceil \mathrm{rt}(\mathfrak{m}_x) \rceil = n-1$ for a closed point $x \in X$, then*

$$\mathrm{e}(\mathfrak{m}_x) + n - 1 = \mathrm{emb}(X,x).$$

Proof. Since X is Cohen-Macaulay, $\mathrm{e}(\mathfrak{m}_x) + n - 1 \geq \mathrm{emb}(X,x)$. By Proposition 5.1.1, $\mathrm{e}(\mathfrak{m}_x) + n - 1 \leq \mathrm{emb}(X,x)$. □

Lemma 5.1.3. *Let X be a normal variety, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Let $C = \sum F_i$, where F_i is exceptional prime divisor on Y which center is not x. Then for any integer l,*

$$f_*\omega_Y(C - lF) : \omega_X = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF).$$

Proof. We have $\omega_X = \mathfrak{d}_{X,V}\omega_V|_X$ and $\widehat{K}_{Y/X} = K_Y + J_V - f^*K_V|_X$ by the definition of $\mathfrak{d}_{X,V}$ and Remark 2.2.2. Hence

$$\begin{aligned} f_*\omega_Y(C - lF) : \omega_X &= f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + f^*K_V|_X + C - lF) : \mathfrak{d}_{X,V}\omega_V|_X \\ &= f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + C - lF) : \mathfrak{d}_{X,V}. \end{aligned}$$

Next we will prove

$$f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + C - lF) : \mathfrak{d}_{X,V} = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF).$$

$$\begin{aligned} & x \in f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + C - lF) : \mathfrak{d}_{X,V} \\ \Leftrightarrow\ & x\mathfrak{d}_{X,V} \subset f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + C - lF) \\ \Leftrightarrow\ & f^*x \cdot \mathcal{O}_Y(-D_V) \subset f\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + C - lF) \end{aligned}$$

$$\Leftrightarrow f^*x \in \mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF)$$
$$\Leftrightarrow x \in f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF)$$

Therefore we have

$$f_*\omega_Y(C - lF) : \omega_X = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF).$$

□

Lemma 5.1.4. *Let X be a normal variety with Du Bois singularities, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Let $C = \sum F_i$, where F_i is exceptional prime divisor on Y which center is not x. Then*

$$\widehat{K}_{Y/X} - J_V + D_V + C + F \geq 0.$$

Proof. By Lemma 5.1.3, we have

$$f_*\omega_Y(C + F) : \omega_X = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C + F).$$

By Theorem 2.1.4, we have $f_*\omega(C + F) : \omega_X = \mathcal{O}_X$. Therefore

$$\widehat{K}_{Y/X} - J_V + D_V + C + F \geq 0.$$

□

Lemma 5.1.5. *Let X be an n-dimensional normal Cohen-Macaulay variety with Du Bois singularities and x be a closed point of X. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and I be a minimal reduction of $\mathfrak{m}$. Then $\mathfrak{m}^{n+1} \subset I$*

Proof. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{m}_x$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{m}_x \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Let $C = \sum F_i$, where F_i is exceptional prime divisor on Y which center is not x. Since X has Du Bois singularities, we have $\widehat{K}_{Y/X} - J_V + D_V + C + F \geq 0$ by Lemma

5.1.4. Therefore we have $\widehat{K}_{Y/X} - J_V + D_V + C - nF \geq -(n+1)F$. Thus we have

$$\mathfrak{m}_x^{n+1} \subset f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - nF) = f_*\omega_Y(C - nF) : \omega_X.$$

Let $g : Z \to X$ be the blow-up at x such that $\mathfrak{m}_x\mathcal{O}_Z = \mathcal{O}_Z(-F')$ and $h : Y \to Z$ be the morphism such that $f = g \circ h$. Then we have $h_*\omega_Y(C - nF) \subset \omega_Z(-nF')$. Hence by Theorem 2.3.9 and Lemma 2.3.11, we have

$$(f_*\omega_Y(C - nF) : \omega_X)_x \subset (g_*\omega_Z(-nF') : \omega_X)_x$$

$$\subset g_*\omega_Z(-nF')_x : \omega_{X,x} \subset \mathrm{core}(\mathfrak{m}).$$

Therefore

$$\mathfrak{m}^{n+1} \subset (f_*\omega_Y(C - nF) : \omega_X)_x \subset \mathrm{core}(\mathfrak{m}) \subset I.$$

$\square$

In [15], Huneke and Watanabe asked the following

Question 5.1.6. Let X be an n-dimensional variety with Du Bois singularities. Is it true that for a closed point $x \in X$,

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n} \ ?$$

The following Theorem gives an answer to the above question in the case where X is normal Cohen-Macaulay.

Theorem 5.1.7. *Let X be an n-dimensional normal Cohen-Macaulay variety with Du Bois singularities. Then for a closed point $x \in X$,*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n}.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and I be a minimal reduction of $\mathfrak{m}$. Since $\mathcal{O}_{X.x}$ is Cohen-Macaulay, we have $\mathrm{e}(\mathfrak{m}) = \ell(\mathcal{O}_{X,x}/I)$. Let $v = \mathrm{emb}(X, x)$. we may assume that $\{x_1, \dots, x_n, y_1, \dots, y_{v-n}\}$ is a minimal generators of $\mathfrak{m}$ with $I = (x_1, \dots, x_n)$. Then $\mathcal{O}_{X,x}/I$ is generated as a $\mathbb{C}$-vector space by 1 and the monomials of $y_1, \dots, y_{v-n}$. Here, we can take generators as monomials of degree $\leq n$, since $I \supset \mathfrak{m}^{n+1}$ by Lemma 5.1.5. Therefore we obtain $\ell(\mathcal{O}_{X,x}/I) \leq \binom{v}{n}$. Then we obtain

$$\mathrm{e}(\mathfrak{m}) \leq \binom{\mathrm{emb}(X,x)}{n}.$$

□

Corollary 5.1.8. *Let X be an n-dimensional Cohen-Macaulay variety with log canonical singularities. Then for a closed point $x \in X$,*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X,x)}{n}.$$

Proof. Since log canonical singularities are Du Bois singularities, the statement follows by Theorem 5.1.7. □

Huneke and Watanabe proved the following using Matlis duality in the proof of Theorem 5.1 in [15].

Lemma 5.1.9. ([15]) *If $(A, \mathfrak{m})$ is a Gorenstein Artin local ring with $\mathfrak{m}^s = 0$, then $\ell(\mathfrak{m}^t) \leq \ell(A/\mathfrak{m}^{s-t})$ for each $0 \leq t \leq s$.*

If X is a Gorenstein variety, then the upper bound is largely reduced by the lemma. Our proof is just an imitation of the proof of Theorem 5.1 in [15].

Proposition 5.1.10. *Let X be an n-dimensional normal Gorenstein variety with Du Bois singularities and let $x \in X$ be a closed point. Let* $\mathrm{emb}(X,x) = v$.

(1) *If $n = 2r + 1$, then*

$$\mathrm{e}(\mathfrak{m}_x) \leq 2\binom{v-r-1}{r}.$$

(2) *If $n = 2r$, then*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{v-r}{r} + \binom{v-r-1}{r-1}.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Let I be a minimal reduction of $\mathfrak{m}$. Let $A = \mathcal{O}_{X,x}/I$ and $\mathfrak{n} = \mathfrak{m}/I$. Then $(A, \mathfrak{n})$ is a Gorenstein Artin local ring, as $\mathcal{O}_{X,x}$ is Gorenstein and I is generated by a system of parameters. Now, by Lemma 5.1.5, we have $I \supset \mathfrak{m}^{n+1}$, which yields $\mathfrak{n}^{n+1} = 0$. We need to evaluate $\ell(A)$ in order to get bounded of $\mathrm{e}(\mathfrak{m}_x)$. We may assume that $\{y_1, \ldots, y_n, z_1, \ldots, z_{v-n}\}$ is minimal generators of $\mathfrak{m}$ with $I = (y_1, \ldots, y_n)$. Then $A/\mathfrak{n}^{l+1}$ is generated as $\mathbb{C}$-vector space by the monomials of $z_1, \ldots, z_{v-n}$ of degree $\leq l$. Therefore $\ell(A/\mathfrak{n}^{l+1}) \leq \binom{v-n+l}{l}$. If $n = 2r+1$, then by Lemma 5.1.9 we obtain

$$\begin{aligned}\ell(A) = \ell(A/\mathfrak{n}^{r+1}) + \ell(\mathfrak{n}^{r+1}) &\leq \ell(A/\mathfrak{n}^{r+1}) + \ell(A/\mathfrak{n}^{r+1}) \\ &\leq 2\binom{v-r-1}{r}.\end{aligned}$$

If $n = 2r$, then by Lemma 5.1.9 we obtain

$$\ell(A) = \ell(A/\mathfrak{n}^{r}) + \ell(\mathfrak{n}^{r}) \leq \ell(A/\mathfrak{n}^{r}) + \ell(A/\mathfrak{n}^{r+1}) \leq \binom{v-r-1}{r-1} + \binom{v-r}{r}.$$

□

Laufer proved the relation between the multiplicity and embedding dimension of the minimal elliptic singularity in [23].

Corollary 5.1.11. ([23]) *Let X be a 2-dimensional normal Gorenstein variety and let $x \in X$ be a closed point. Suppose that X is Du Bois singularity but not regular. Then*

$$\mathrm{e}(\mathfrak{m}_x) = \mathrm{emb}(X, x) - 1 = 2$$

or

$$\mathrm{e}(\mathfrak{m}_x) = \mathrm{emb}(X, x) \geq 3.$$

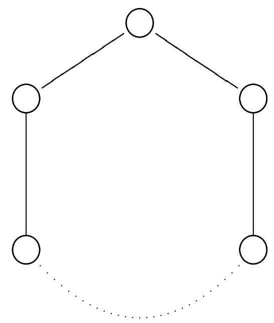
Proof. By Proposition 5.1.10, we have $\mathrm{e}(\mathfrak{m}_x) \leq \mathrm{emb}(X, x)$. On the other hand, since X is Cohen-Macaulay, we have $\mathrm{emb}(X, x) \leq \mathrm{e}(\mathfrak{m}_x)+1$. Therefore $\mathrm{emb}(X, x)$ is $\mathrm{e}(\mathfrak{m}_x)$ or $\mathrm{e}(\mathfrak{m}_x) + 1$. Let I be a minimal reduction of maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$, $A = \mathcal{O}_{X,x}/I$ and $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of A. We assume that $\mathrm{e}(\mathfrak{m}_x) = 2$. Since $\ell(\mathcal{O}_{X,x}/I) = 2$, $\mathrm{emb}(X, x) = 3$.

We assume that $\mathrm{emb}(X,x) = \mathrm{e}(\mathfrak{m}_x) + 1$. Then $I\mathfrak{m} = \mathfrak{m}^2$ (see [34]). Therefore $\mathrm{e}(\mathfrak{m}_x) = \ell(A) = \ell(A/\mathfrak{n}) + \ell(\mathfrak{n}) \leq \ell(A/\mathfrak{n}) + \ell(A/\mathfrak{n}) = 2$ by Lemma 5.1.9. By the above discussion, if $\mathrm{e}(\mathfrak{m}_x) = \mathrm{emb}(X,x)$, then $\mathrm{e}(\mathfrak{m}_x) \geq 3$. $\square$

Definition 5.1.12. A two-dimensional normal singularity (X,x) is called a simple elliptic singularity if the exceptional curve E of the minimal resolution $f : Y \to X$ is an irreducible nonsingular elliptic curve.

Definition 5.1.13. If the exceptional divisor $E = \sum_{i=1}^{r} E_i$ of the minimal resolution $f : Y \to X$ of two-dimensional normal singularity (X,x) satisfies the following, then we call (X,x) a cusp singularity.

The total exceptional divisor E is an irreducible rational curve with an ordinary node or the equalities $E_i \cong \mathbb{P}^1$ $({}^\forall i = 1, \ldots, r)$ hold and E is of normal crossings with the dual graph as the following cyclic form (ignoring the weight):



We know that a 2-dimensional normal singularity (X,x) is Gorenstein Du Bois singularity if and only if (X,x) is a rational double point, a simple elliptic singularity or a cusp singularity (see [19]).

Example 5.1.14. Suppose X is a simple elliptic singularity or a cusp singularity. Since simple elliptic singularity and cusp singularity are Gorenstein Du Bois singularities, we have $\mathrm{e}(\mathfrak{m}_x) = \mathrm{emb}(X,x) - 1 = 2$ or $\mathrm{e}(\mathfrak{m}_x) = \mathrm{emb}(X,x) \geq 3$.

5.2 Upper bound of the multiplicity of $\mathfrak{m}_x$-primary ideal

In this section, we show bounds of the multiplicity by functions of birational invariants for singularities.

Definition 5.2.1. Let X be a variety and Z be a proper reduced subscheme of X defined by an ideal sheaf $\mathfrak{q}$. The i-th symbolic power $\mathfrak{q}^t$ is then defined on any affine open set U by $\mathfrak{q}^{(t)}(U) = \{f \in \mathcal{O}_X(U) | f \in \mathfrak{m}_\eta^t$, for all generic point η of $Z\}$, where $\mathfrak{m}_\eta$ means the maximal ideal in the local ring $\mathcal{O}_{X,\eta}$.

Definition 5.2.2. Let X be a variety and $\mathfrak{a}$ be a non-zero ideal of $\mathcal{O}_X$. Let $\nu : W \to X$ be the normalization of the blowing-up of X along $\mathfrak{a}$ so that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$ where E is an effective Cartier divisor on W. We can write $E = \sum_{i=1}^t r_i E_i$ as a sum of distinct prime divisor E_i's with some positive integer coefficients r_i. Write $Z_i = \nu(E_i)$ to be the image E_i on X with the reduced scheme structure. Then Z_i's are called the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i.

Niu showed the relation between the Mather-Jacobian ideals and symbolic powers in [30]. Our proof of the following lemma is just an imitation of the proof of Claim 3.1.1 in [30].

Lemma 5.2.3. *Let X be a variety with rational singularities and $\mathfrak{a}$ be a non-zero ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. For $l \geq \lceil \mathrm{rt}(\mathfrak{a}) \rceil - 1$, we have the inclusion*

$$\mathfrak{q}_{Z_1}^{(r_1(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \subset \mathcal{J}^\omega(X, \mathfrak{a}^l).$$

Proof. Since the inclusion is local, we can assume that X is affine. Let $\mathfrak{j}_X$ be the Jacobian ideal of X and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot \mathfrak{a}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$ and F on Y. Then by Theorem 3.1.3 we have

$$\mathcal{J}^\omega(X, \mathfrak{a}^l) = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - lF).$$

Therefore, it suffices to show that for any element $h \in \mathfrak{q}_{Z_1}^{(r_1(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))}$,

$$\mathrm{div} f^* h \geq -\widehat{K}_{Y/X} + J_{Y/X} - D_{Y/X} + lF,$$

where $\mathrm{div} f^* h$ means the effective divisor defined by $f^* h$ on Y. To see this let $\nu : W \to X$ be the normalization of the blowing-up of X along $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$ where E is an effective Cartier divisor on W. Write E as a sum of prime divisors $E = \sum_{i=1}^t r_i E_i$. Note that $Z_i = \nu(E_i)$ and f factors through ν via a morphism $g : Y \to W$ such that $F = g^* E$. For any element $h \in \mathfrak{q}_{Z_1}^{(r_1(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))}$, we have $\mathrm{ord}_{E_i} \nu^* h \geq r_i(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)$. Therefore we have $\mathrm{div} \nu^* h \geq (l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)E$. Thus we have $\mathrm{div} f^* h = \mathrm{div} g^*(\nu^* h) \geq g^*((l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)E) = (l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)F$. Since $\lceil \mathrm{rt}(\mathfrak{a})\rceil - 1 < \mathrm{rt}(\mathfrak{a})$, $\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - (\lceil \mathrm{rt}(\mathfrak{a})\rceil - 1)F \geq 0$. Thus we have $\mathrm{div} f^* h \geq (l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)F \geq -\widehat{K}_{Y/X} + J_{Y/X} - D_{Y/X} + lF$. Therefore the lemma is proved. □

Lemma 5.2.4. *Let X be a variety with rational singularities and $\mathfrak{a}$ be a non-zero ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. Let $r = \max_i\{r_i\}$. For $l \geq \lceil \mathrm{rt}(\mathfrak{a})\rceil - 1$, we have the inclusion*

$$(\sqrt{\mathfrak{a}})^{r(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)} \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^l).$$

Proof. By Lemma 5.2.3, we have

$$\mathfrak{q}_{Z_1}^{(r_1(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^l).$$

Since $r \geq r_i$, $(\sqrt{\mathfrak{a}})^{r(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)} \subset \mathfrak{q}_{Z_1}^{(r_1(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil))}$. Therefore we have

$$(\sqrt{\mathfrak{a}})^{r(l+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)} \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^l).$$

□

Theorem 5.2.5. *Let X be an n-dimensional variety with rational singularities, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. Let $r = \max_i\{r_i\}$. Then*

$$\mathrm{e}(\mathfrak{a}) \leq \binom{\mathrm{emb}(X,x) + r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil) - 1}{\mathrm{emb}(X,x)}.$$

Proof. By Lemma 5.2.4, we have

$$\mathfrak{m}_x^{r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)} \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^n).$$

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and I be a minimal reduction of $\mathfrak{a}\mathcal{O}_{X,x}$. Let $g : Z \to X$ be the blow-up of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_Z = \mathcal{O}_Z(-F')$ and $h : Y \to Z$ be the morphism such that $f = g \circ h$. Then we have $h_*\omega_Y(-nF) \subset \omega_Z(-nF')$. Hence by Theorem 2.3.9 and Lemma 2.3.11, we have

$$\mathfrak{m}^{r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil)} \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^n)_x = (f_*\omega_Y(-nF) : \omega_X)_x$$

$$\subset (g_*\omega_Z(-nF') : \omega_X)_x \subset g_*\omega_Z(-nF')_x : \omega_{X,x} \subset \mathrm{core}(\mathfrak{a}\mathcal{O}_{X.x}) \subset I.$$

Since $\mathcal{O}_{X.x}$ is Cohen-Macaulay, we have $\mathrm{e}(\mathfrak{a}) = \ell(\mathcal{O}_{X,x}/I)$.

Let $v = \mathrm{emb}(X, x)$. we may assume that $\{y_1, \ldots, y_v\}$ is a minimal generators of $\mathfrak{m}$. Then $\mathcal{O}_{X,x}/I$ is generated as a $\mathbb{C}$-vector space by 1 and the monomials of $y_1, \ldots, y_v$. Here, we can take generators as monomials of degree $\leq d := r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil) - 1$, since $I \supset \mathfrak{m}^{r(n+1-\lceil \mathrm{rt}(\mathfrak{m}_x)\rceil)}$. Therefore we obtain $\ell(\mathcal{O}_{X,x}/I) \leq \binom{v+d}{d}$. Then we obtain

$$\mathrm{e}(\mathfrak{a}) \leq \binom{\mathrm{emb}(X,x) + r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil) - 1}{\mathrm{emb}(X,x)}.$$

□

Lemma 5.2.6. *Let X be a normal Cohen-Macaulay variety with Du Bois singularities, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Let $C = \sum F_i$, where F_i is exceptional prime divisor on Y which center is not x. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. For $l \geq 0$, we have the inclusion*

$$\mathfrak{q}_{Z_1}^{(r_1(l+1))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1))} \subset f_*\omega_Y(C - lF) : \omega_X.$$

Proof. Since the inclusion is local, we can assume that X is affine. By Lemma 5.1.3, we have

$$f_*\omega_Y(C - lF) : \omega_X = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V + C - lF).$$

Therefore it suffices to show that for any element $h \in \mathfrak{q}_{Z_1}^{(r_1(l+1))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1))}$,

$$\mathrm{div} f^*h \geq -\widehat{K}_{Y/X} + J_V - D_V - C + lF,$$

where $\mathrm{div} f^*h$ means the effective divisor defined by f^*h on Y. To see this let $\nu : W \to X$ be the normalization of the blowing-up of X along $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$ where E is an effective Cartier divisor on W. Write E as a sum of prime divisors $E = \sum_{i=1}^t r_i E_i$. Note that $Z_i = \nu(E_i)$ and f factors through ν via a morphism $g : Y \to W$ such that $F = g^*E$. For any element $h \in \mathfrak{q}_{Z_1}^{(r_1(l+1))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1))}$, we have $\mathrm{ord}_{E_i} \nu^*h \geq r_i(l+1)$. Therefore we have $\mathrm{div}\nu^*h \geq (l+1)E$. Thus we have $\mathrm{div} f^*h = \mathrm{div} g^*(\nu^*h) \geq g^*((l+1)E) = (l+1)F$. Since X has Du Bois singularities, $\widehat{K}_{Y/X} - J_V + D_V + C + F \geq 0$ by Lemma 5.1.4. Thus we have $\mathrm{div} f^*h \geq (l+1)F \geq -\widehat{K}_{Y/X} + J_V - D_V - C + lF$. Therefore the lemma is proved. $\square$

Theorem 5.2.7. *Let X be an n-dimensional normal Cohen-Macaulay variety with Du Bois singularities, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. Let $r = \max_i\{r_i\}$. Then*

$$\mathrm{e}(\mathfrak{a}) \leq \binom{\mathrm{emb}(X,x) + r(n+1) - 1}{\mathrm{emb}(X,x)}.$$

Proof. Let V be a reduced locally complete intersection scheme containing $X$of the same dimension. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V and F on Y. Let $C = \sum F_i$, where F_i is exceptional prime divisor on Y which center is not x. By Lemma 5.2.6, we have

$$\mathfrak{m}_x^{r(n+1)} \subset \mathfrak{q}_{Z_1}^{(r_1(l+1))} \cap \cdots \cap \mathfrak{q}_{Z_t}^{(r_t(l+1))} \subset f_*\omega_Y(C - lF) : \omega_X.$$

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and I be a minimal reduction of $\mathfrak{a}\mathcal{O}_{X,x}$.

Let $g : Z \to X$ be the blow-up of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_Z = \mathcal{O}_Z(-F')$ and $h : Y \to Z$ be the morphism such that $f = g \circ h$. Then we have $h_*\omega_Y(C - nF) \subset \omega_Z(-nF')$. Hence by Theorem 2.3.9 and Lemma 2.3.11, we have

$$\mathfrak{m}^{r(n+1)} \subset (f_*\omega_Y(C - nF) : \omega_X)_x \subset (g_*\omega_Z(-nF') : \omega_X)_x$$

$$\subset g_*\omega_Z(-nF')_x : \omega_{X,x} \subset \mathrm{core}(\mathfrak{a}\mathcal{O}_{X.x}) \subset I.$$

Since $\mathcal{O}_{X.x}$ is Cohen-Macaulay, we have $\mathrm{e}(\mathfrak{a}) = \ell(\mathcal{O}_{X,x}/I)$.

Let $v = \mathrm{emb}(X, x)$. we may assume that $\{y_1, \ldots, y_v\}$ is a minimal generators of $\mathfrak{m}$. Then $\mathcal{O}_{X,x}/I$ is generated as a $\mathbb{C}$-vector space by 1 and the monomials of $y_1, \ldots, y_v$. Here, we can take generators as monomials of degree $\leq d := r(n + 1) - 1$, since $I \supset \mathfrak{m}^{r(n+1)}$. Therefore we obtain $\ell(\mathcal{O}_{X,x}/I) \leq \binom{v+d}{d}$. Then we obtain

$$\mathrm{e}(\mathfrak{a}) \leq \binom{\mathrm{emb}(X, x) + r(n + 1) - 1}{\mathrm{emb}(X, x)}.$$

□

Corollary 5.2.8. *Let X be an n-dimensional Cohen-Macaulay variety with log canonical singularities, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. Let $r = \max_i\{r_i\}$. Then*

$$\mathrm{e}(\mathfrak{a}) \leq \binom{\mathrm{emb}(X, x) + r(n + 1) - 1}{\mathrm{emb}(X, x)}$$

Proof. Since log canonical singularities are Du Bois singularities, the statement follows by Theorem 5.2.7. □

Definition 5.2.9. Let $(X, \mathfrak{a})$ be a pair consisting of a variety X and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$. The Mather log- canonical threshold of $(X, \mathfrak{a})$ is defined as follows:

$$\widehat{\mathrm{lct}}(\mathfrak{a}) = \sup\{c | \widehat{k}_E - c\mathrm{ord}_E(\mathfrak{a}) + 1 \geq 0, E \text{ divisor over } X\}.$$

De Fernex and Mustaţă showed the relationship between the Mather log canonical threshold and the multiplicity of an $\mathfrak{m}_x$-primary ideal.

Theorem 5.2.10. ([6]) *Let X be an n-dimensional Cohen-Macaulay variety, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$.*

$$\left(\frac{n}{\widehat{\mathrm{lct}}(\mathfrak{a})}\right)^n \leq \mathrm{e}(\mathfrak{a}).$$

Corollary 5.2.11. *Let X be an n-dimensional normal Cohen-Macaulay variety, x be a closed point of X and $\mathfrak{a}$ be an $\mathfrak{m}_x$-primary ideal sheaf of $\mathcal{O}_X$. Let Z_i, $i = 1, \ldots, t$, be the distinguished subvarieties of $\mathfrak{a}$ with the coefficient r_i defined by the ideal $\mathfrak{q}_{Z_i}$. Let $r = \max_i\{r_i\}$.*
(1) *If X has rational singularities, then*

$$\left(\frac{n}{\widehat{\mathrm{lct}}(\mathfrak{a})}\right)^n \leq \binom{\mathrm{emb}(X,x) + r(n+1-\lceil \mathrm{rt}(\mathfrak{a})\rceil) - 1}{\mathrm{emb}(X,x)}.$$

(2) *If X has Du Bois singularities, then*

$$\left(\frac{n}{\widehat{\mathrm{lct}}(\mathfrak{a})}\right)^n \leq \binom{\mathrm{emb}(X,x) + r(n+1) - 1}{\mathrm{emb}(X,x)}.$$

Proof. The statements follow by Theorem 5.2.5, Theorem 5.2.7 and Theorem 5.2.10. □

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