博士論文 (要約)

論文題目: Rational singularities, ω-multiplier ideals and core of ideals
（ 有理特異点、ω-乗数イデアルとイデアルのコア ）

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Chapter 1

Introduction

In this paper, we always assume that a ring is a domain essentially of finite type over $\mathbb{C}$ and a variety is an irreducible reduced separated scheme of finite type over $\mathbb{C}$.

Rees and Sally introduced the cores of ideals in [10]. Okuma, Watanabe and Yoshida characterized 2-dimensional local ring with a rational singularity via cores of ideals in [9]. However, in higher dimensional case we have a counterexample to the characterization. We will show another characterization of local ring with a rational singularity of arbitrary dimension via cores of ideals. We, namely, will prove the following:

Theorem 1.0.1. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay local ring with an isolated singularity. Then A is a rational singularity if and only if $\overline{I^n} \subset \mathrm{core}(I)$ for any $\mathfrak{m}$-primary ideal I.*

By this Theorem, we show that a Cohen-Macaulay local ring with an isolated singularity has a rational singularity if Briançon-Skoda Theorem holds for the ring. Lipman and Teissier showed that for a local ring with rational singularities, Briançon-Skoda Theorem holds in [7]. Therefore a Cohen-Macaulay local ring with an isolated singularity has a rational singularity if and only if Briançon-Skoda Theorem holds for the ring.

The multiplier ideals are fundamental tools in birational geometry. In this paper we introduce a new notion an "ω-multiplier ideal" which has similar properties and works in a slightly different way than a multiplier ideal. The main goal of this paper is to prove the properties of ω-multiplier ideals and show some applications.

For the definition of the multiplier ideals we used the discrepancies. In order for the discrepancy to be well-defined, we need to assume that the variety is normal and $\mathbb{Q}$-Gorenstein. The advantage of ω-multiplier ideals is that they can be defined on any normal variety. If a variety X is normal Gorenstein, then the ω-multiplier ideal $\mathcal{J}^{\omega}(X, \mathfrak{a}^c)$ is equal to the usual multiplier ideal $\mathcal{J}(X, \mathfrak{a}^c)$ for any ideal $\mathfrak{a}$.

One of the most important theorem of the multiplier ideals is the Skoda's Theorem. We will prove that the Skoda's Theorem of ω-multiplier ideals of a local ring with a rational singularity.

Proposition 1.0.2. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an m-primary ideal and J be a reduction of $\mathfrak{a}$. Then for $n \in \mathbb{Z}_{\geq 2}$,*

$$\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^{\omega}(A, \mathfrak{a}^{n-1}).$$

Huneke and Swanson proved the many properties of cores of ideals of 2-dimensional regular local ring and the relationships between the core of an ideal and multiplier ideal of 2-dimensional regular local ring in [3]. We generalize their results to rational singularities using ω-multiplier ideals. We will prove the followings:

Proposition 1.0.3. *Let $(A, \mathfrak{m})$ be a 2-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then*
(1) $\operatorname{core}(\mathfrak{a}) = \mathcal{J}^{\omega}(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^{\omega}(A, \mathfrak{a})$.
(2) $\mathrm{e}(\mathfrak{a}) = \ell(A/\operatorname{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^{\omega}(A, \mathfrak{a}))$.
(3) $\mathcal{J}^{\omega}(A, \operatorname{core}(\mathfrak{a})) = (\mathcal{J}^{\omega}(A, \mathfrak{a}))^2$.
(4) $\operatorname{core}(\mathfrak{a}^n) = \mathfrak{a}^{2n-1}\mathcal{J}^{\omega}(A, \mathfrak{a})$.
(5) $\operatorname{core}^n(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^{\omega}(A, \mathfrak{a}))^{2^n-1}$. *In particular,* $\operatorname{core}(\operatorname{core}(\mathfrak{a})) = \mathfrak{a}(\mathcal{J}^{\omega}(A, \mathfrak{a}))^3$.

Demailly, Ein and Lazarsfeld proved the subadditivity theorem for multiplier ideals on non-singular varieties in [1]. This theorem gives many applications of commutative algebra and algebraic geometry. Takagi and Watanabe proved that the subadditivity theorem holds for a 2-dimensional log terminal local ring in [11]. Moreover they showed the characterization of a 2-dimensional log terminal local ring via the subadditivity of multiplier ideals.

Hence it makes sense to consider the subadditivity of ω-multiplier ideals. We show the characterization of 2-dimensional local ring with a rational singularity via the subadditivity of ω-multiplier ideals.

Theorem 1.0.4. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec} A$ has a rational singularity if and only if the subadditivity theorem holds, that is, for any two ideal $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b}).$$

To use the subadditivity of ω-multiplier ideals, we investigate the subadditivity of cores of ideals. We show the characterization of 2-dimensional local ring with a rational singularity via the subadditivity of cores of ideals.

Corollary 1.0.5. *Let (A, m) be a two-dimensional normal local ring. Then $X = \mathrm{Spec} A$ is rational singularities if and only if the subadditivity theorem hold, that is, for any two $\mathfrak{m}$-primary integral closed ideals $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathrm{core}(\mathfrak{a}\mathfrak{b}) \subset \mathrm{core}(\mathfrak{a})\mathrm{core}(\mathfrak{b}).$$

Moreover in [11] Takagi and Watanabe showed that a 2-dimensional normal ring is regular if the strong subadditivity theorem for the ring holds. We will consider the problem of a version of ω-multiplier ideals. We will prove the following:

Proposition 1.0.6. *Let (A, m) be a two-dimensional normal local ring essentially of finite type over $\mathbb{C}$. Then $X = \mathrm{Spec} A$ is regular if and only if the strong subadditivity theorem hold, that is, for any two ideal $\mathfrak{a}$, $\mathfrak{b} \subset \mathcal{O}_X$ and any rational number $c, d > 0$,*

$$\mathcal{J}^\omega(X, \mathfrak{a}^c\mathfrak{b}^d) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)\mathcal{J}^\omega(X, \mathfrak{b}^d).$$

A multiplier ideal is an integrally closed ideal. It is natural to ask that an integrally closed ideal is a multiplier ideal. In general multiplier ideals are not integrally closed ideals (see [5], [6]). Favre, Jonsson, Lipman and Watanabe gave an answer to this question when a ring is 2-dimensional regular local ring. That is, they showed that all integrally closed ideals on a regular local ring are multiplier ideals in [2] and [8]. Moreover Tucker generalized the result to a log terminal local ring in [12]. On the other hand we generalize this theorem to rational singularities by using ω-multiplier ideals. In other words, we will prove the following:

Theorem 1.0.7. *Let (A, m) be a two-dimensional local normal ring. Suppose $X = \mathrm{SpecA}$ is a rational singularity. Then every integrally closed ideal is an ω-multiplier ideal.*

Another application of ω-multiplier ideals is an upper bound of the multiplicity of a Du Bois singularity. Huneke and Watanabe gave an upper bound on the multiplicity of a rational singularity in [4]. That is, they showed the following:

Theorem 1.0.8. ([4]) *Let X be an n-dimensional variety with rational singularities. Then for a closed point $x \in X$*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x) - 1}{n - 1}.$$

In [4], Huneke and Watanabe asked the following

Question 1.0.9. Let X be an n-dimensional variety with Du Bois singularities. Is it true that for a closed point $x \in X$

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n} \ ?$$

We give the affirmative answer to the question under the condition that X is a normal Cohen-Macaulay variety.

Theorem 1.0.10. *Let X be an n-dimensional normal Cohen-Macaulay variety with Du Bois singularities. Then for a closed point $x \in X$*

$$\mathrm{e}(\mathfrak{m}_x) \leq \binom{\mathrm{emb}(X, x)}{n}.$$

In the sequel, we will outline the contents of each chapter of this thesis.

In Chapter 2, we define rational singularities, the Mather-Jacobian discrepancy and cores of ideals and collect their results.

In Chapter 3, we define ω-multiplier ideals and prove their properties. Further we characterize local ring with a rational singularity of arbitrary dimension via cores of ideals. Let A be an n-dimensional Cohen-Macaulay isolated singularity local ring. Assume that A is not a rational singularity. Then we will find the ideal I of local ring such that $I^n \not\subset \mathrm{core}(I)$ to prove Theorem 1.0.1.

In Chapter 4, we study ω-multiplier ideals of a 2-dimensional local ring with a rationals singularity.

In section 4.1, we discuss the various relationships between the a core of an ideal and a ω-multiplier ideal of a 2-dimensional local ring with a rational singularity. We show that we can compute ω-multiplier ideals of 2-dimensional local rings with rational singularities using the minimal resolutions. Using this result, we prove Theorem 1.0.2 and Proposition 1.0.3.

In section 4.2, we investigate when the subadditivity theorem of ω-multiplier ideals holds in the two-dimensional case. Let A be a 2-dimensional local ring with a rational singularity and $f : X \to \mathrm{Spec} A$ be a resolution of singularities. There is a one-to-one correspondence between the set of integrally closed ideals I in A such that $I\mathcal{O}_X$ is invertible and the set of effective f-anti-nef divisors on X. Therefore we investigate f-anti-nef divisor on X to prove Theorem 1.0.4.

In section 4.3, we show that all integrally closed ideals on surface with a rational singularity are ω-multiplier ideals. Let A be a 2-dimensional local ring with a rational singularity and I be an integrally closed ideal of A. We will construct an ideal $\mathfrak{a}$ and $c \in \mathbb{Q}_{>0}$ such that $I = \mathcal{J}^{\omega}(A, \mathfrak{a}^c)$ to prove Theorem 1.0.7. Our proof is just an imitation of the proof of the Theorem 1,1 of [12].

In Chapter 5, we give an upper bound of the multiplicity of a Du Bois singularity. Let X be a n-dimensional normal Cohen-Macaulay variety and x be a closed point of X. We show that $\mathfrak{m}_x^{n+1}$ is contained in a minimal reduction of $\mathfrak{m}_x$ to prove Theorem 1.0.10.

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