# 博士論文

論文題目　Actions of locally compact abelian groups on factors with the Rohlin property

（ロホリン性をもつ局所コンパクト可換群の因子環への作用）

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# ACTIONS OF LOCALLY COMPACT ABELIAN GROUPS ON FACTORS WITH THE ROHLIN PROPERTY

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Abstract. Motivated by a classification problem of actions of locally compact abelian groups on factors, we study a property of actions which is called the Rohlin property. First of all, by generalizing a work of Masuda–Tomatsu, we establish a classification theorem of actions of locally compact abelian groups on factors with the Rohlin property. Next, we give a good sufficient condition for actions to have the Rohlin property. Namely, we show that actions of $\mathbf{R}$ with faithful Connes–Takesaki modules on AFD factors have the Rohlin property, which provides many new examples of actions with the Rohlin property. Finally, as an application of the study of the Rohlin property, we characterize an analytic property of finite index endomorphisms, approximate innerness, which is useful for classifying actions of compact groups on factors.

## 1. Introduction

An operator algebra is a *-closed algebra which consists of bounded operators on a Hilbert space. If it is closed in the operator norm topology, then it is called a $C^*$-algebra. If it is closed in the strong operator topology, then it is called a von Neumann algebra. Since the convergence in the norm topology implies the convergence in the strong operator topology, a von Neumann algebra is a $C^*$-algebra. However, if we think of a von Neumann algebra as a $C^*$-algebra, it is so large that we hardly obtain fine information. Hence they are thought to be different topics and considerable parts of their respective techniques are different. In this thesis, we mainly treat von Neumann algebras. In particular, we consider a classification problem of group actions on von Neumann algebras.

The classification problems of von Neumann algebras attract many researchers' attentions. It is a natural attempt to classify von Neumann algebras, their symmetric structures (group actions), and their subalgebras, up to appropriate isomorphisms. However, because the number of von Neumann algebras is so huge that it is hopeless to classify all of them, we often classify those with conditions which are called amenability.

The history of classifying von Neumann algebras dates back to the age of Murray and von Neumann. First, they showed that factors (von Neumann algebras with trivial centers) are devided into factors of type I, $\mathrm{II}_1$, $\mathrm{II}_\infty$ and type III and that factors of type I are isomorphic to one of the (possibly infinite dimensional) matrix algebras [45]. Then they showed that the AFD (a short expression for approximately finite dimensional, a kind of amenability) factor of type $\mathrm{II}_1$ is unique, up to *-isomorphism [46]. This result is amazing because although there are so many ways of constructing AFD factors of type $\mathrm{II}_1$ such as infinite tensor products of $n \times n$ matrices ($n \geq 2$), it turned out that all of them are mutually isomorphic. Later, Connes tried to classify all of the AFD factors, which was completely solved in 1987 by Connes [5] [6] and Haagerup [18]. In his program, classification of group actions on von Neumann algebras began to be studied. He noticed that most factors of type III are described by using von Neumann algebras of type II and actions of $\mathbf{Z}$ on them [9]. Then he classified actions of $\mathbf{Z}$ on AFD factors of type II [7]. With another crucial ingredient (a characterization of approximate finite dimensionality [5]), he succeeded in classifying most of the AFD factors. By this achievement, he was bestowed the Fields medal.

Hence the original motivation of classifying group actions is to classify factors. However, classifying group actions is itself attractive because their complete invariant is simple compared with the diversity of the ways of constructing actions. His technique for classifying group actions is also interesting. He borrowed an idea from ergodic theory. In ergodic theory, there is a classical theorem which is called the Rohlin lemma. He showed that for any outer action of $\mathbf{Z}$ on the AFD factor of type II, an analogue of the Rohlin lemma holds (the non commutative Rohlin lemma), which is one of the vital points of his proof.

Hence classifying group actions has fascinated many operator algebraists. After Connes' work, Jones [23] classified actions of finite groups on the AFD factors of type II and Ocneanu [47] classified actions of discrete amenable groups on the AFD factors of type II. At these stages, one of the difficulties was to find out the invariants. Some invariants which are needed to classify actions of these classes of groups degenerate when the group is that of the integers. After that, some researchers such as Katayama, Kawahigashi, Takesaki and Sutherland were interested in classifying actions on AFD factors of type III. When one considers factors of type III, he would face with some difficulties which do not occur when we consider only factors of type II. The lack of the trace makes analytic arguments difficult. Finding invariants is another problem. In order to construct invariants, Connes–Takesaki [11] and Kawahigashi–Sutherland–Takesaki [27], in which analytic properties of automorphisms are studied, play crucial roles. The problem was finally solved in 1998 by Sutherland–Takesaki [56], Kawahigashi–Sutherland–Takesaki [31] and Katayama–Sutherland–Takesaki [27]. We also have to say that Masuda [39] gave a simple proof of the classification theorem of actions of discrete amenable groups on AFD factors based on techniques of Evans–Kishimoto [12], in which actions on $C^*$-algebras are studied.

Anyway, classification of actions of discrete amenable groups on AFD factors has been completed. One of the next problems is to classify actions of continuous (amenable) groups. In particular, actions of $\mathbf{R}$ are important because they naturally appear in Takesaki's structural theorem of factors of type III (See Takesaki [60]). Although there are some pioneering results about actions of continuous groups due to Kawahigashi [28] [29] [30], the classification of actions of continuous groups is not completed. One of the reasons is that it is not easy to classify "outer" of actions of continuous groups. In the case of actions of discrete groups, the classification problem was separated into the outer part and the inner part and then these results were combined. However, when the group is continuous, we cannot classify outer actions by just an analogue of the discrete group case. As we have said, one of the vital points of classifying outer actions of discrete groups is the non-commutative Rohlin lemma. However, when the group is continuous, if we simply assume that an action is outer at any nontrivial point, then it may not have a similar property to the conclusion of the non-commutative Rohlin lemma. Hence in order to proceed with classification, the Rohlin property was introduced by Kishimoto [34]. Actually, he introduced the Rohlin property for actions of $\mathbf{R}$ on $C^*$-algebras and Kawamuro [33] translated it in the von Neumann setting. Roughly speaking, the Rohlin property corresponds to the conclusion of the non-commutative Rohlin lemma of the discrete group case. Later, Masuda–Tomatsu [44] established a classification theorem of actions of $\mathbf{R}$ with the Rohlin property. It is natural to try to generalize there result for actions of more general groups. For actions of locally compact abelian groups, it is not difficult to define the Rohlin property in the same way as in Kishimoto [34]. The problem is to classify actions with the Rohlin property. In this direction, Asano [3] showed a classification theorem when the group is $\mathbf{R}^d$ for some $d \in \mathbf{Z}_{>0}$. When the group is a general locally compact abelian group, the problem is that the group may not have enough compact quotients. Section 3 is devoted to considering this problem. Namely, we show the following theorem (Theorem 2 of Shimada [53]).

**Theorem 1.1.** *Let $\alpha$ and $\beta$ be actions of a locally compact abelian group $G$ on a factor $M$ with the Rohlin property. Assume that $\alpha_g \circ \beta_{-g}$ is approximately inner for any $g \in G$. Then $\alpha$ and $\beta$ are mutually cocycle conjugate.*

We also present many examples of actions with the Rohlin property.

However, there is a much more important problem. Although a classification theorem of actions with the Rohlin property is established, the definition of the Rohlin property is rather technical. Hence we have to study relation between the Rohlin property and invariants for group actions. In Section 4, we give a sufficient condition for actions of $\mathbf{R}$ to have the Rohlin property, that is, we show that an action of $\mathbf{R}$ which is "very outer" at any nontrivial point has the Rohlin property (Main Theorem of Shimada [51]).

**Theorem 1.2.** *An action of* $\mathbf{R}$ *with faithful Connes–Takesaki module on any AFD factor has the Rohlin property.*

Not only does this theorem provide many examples of actions with the Rohlin property on factors of type III but also makes a connection between pointwise outerness defined by usual invariants of automorphisms and the Rohlin property. As a corollary of this theorem, we obtain the following.

**Corollary 1.3.** *Actions of* R *with faithful Connes–Takesaki module on any AFD factor are completely classified by their Connes–Takesaki modules, up to cocycle conjugacy.*

Actually, there is a similar theorem about actions of compact groups due to Izumi [22]. He showed that actions of compact groups on any AFD factor are completely classified by their Connes–Takesaki modules, up to cocycle conjugacy. However, it is impossible to show our theorem by the same argument as his one. One evidence is the following. There is a classification theorem of actions of (any!) locally compact groups on any AFD factor due to Yamanouchi [64] based on Izumi's method. However, there is a strong restriction of Connes–Takesaki modules of the actions which are classified by his method, that is, they should be isomorphic to (an amplification of) the left translation of the group. Hence we can say that at least for actions of $\mathbf{R}$, our classification theorem covers a much wider class of actions. There is another evidence which shows the diversity of the class of actions covered by our theorem. For the class of actions which are covered by Izumi [22] and Yamanouchi [64], the coincidence of Connes–Takesaki modules of actions in fact implies conjugacy of two actions. Although this fact itself is surprising, this means that the number of actions contained in the class is small. However,by our theorem, it turns out that there are some actions of R with faithful Connes–Takesaki modules which are mutually cocycle conjugate but are not mutually conjugate.

Finally, in Section 5, we give a characterization of an analytic property of endomorphisms, approximate innerness, by using the Rohlin property for actions of R. More precisely, we have the following theorem.

**Theorem 1.4.** *Let* $\rho$, $\sigma$ *be endomorphisms of an AFD factor* $M$ *of type* III *with* $d(\rho), d(\sigma) < \infty$. *Then the following two conditions are equivalent.*

(1) *We have* $\phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}} \circ \theta_{-\log(d(\sigma))}|_{\mathcal{Z}(\tilde{M})}$.
(2) *There exists a sequence* $\{u_n\}$ *of unitaries of* $M$ *with* $\mathrm{Ad}u_n \circ \rho \to \sigma$ *as* $n \to \infty$.

Among actions of locally compact groups, actions of compact groups are special because their duals are discrete. In fact, actions of compact abelian groups on AFD factors have completely been classified by classifying their duals (See Jones–Takesaki [25] and Kawahigashi–Takesaki [32]). However, when it comes to classifying actions of non-abelian compact groups, the problem is much more difficult. One of the reasons is that the dual of an action of a non-abelian compact group is a collection of endomorphisms, not of automorphisms. Hence we need to handle endomorphisms. In classification theorems of outer actions of discrete amenable groups, approximate innerness of automorphisms is an (and the only!) obstruction for cocycle conjugacy. Hence approximate innereness is also thought to be important for the dual of an action of a compact group. This is the reason why the above characterization theorem is important. We have to mention that when the endomorphism is an automorphism, the characterization is obtained by Kawahigashi–Sutherland–Takesaki [31]. By using the Rohlin property of the trace-scaling action of $\mathbf{R}$ on the AFD factor of type $\mathrm{II}_\infty$, we have succeeded in generalizing their result for endomorphisms. By this method, it is also possible to provide a new proof of a characterization theorem of another analytic property, central triviality, of automorphisms due to Kawahigashi–Sutherland–Takesaki [31].

**Theorem 1.5.** (See Theorem 1 (2) of Kawahigashi–Sutherland–Takesaki) *For an automorphism $\alpha$ of $M$, $\alpha$ is centrally trivial if and only if its canonical extension is inner.*

Note that by our results and the result of Masuda [39], if we admit that AFD factors are completely classified by their flows of weights, it is possible to classify the actions of discrete amenable groups on AFD factors without separating cases by the types of the factors.

**Acknowledgment** The author thanks Professor Reiji Tomatsu for informing him of the topic of Section 5 and for giving him useful comments. But for his suggestion, the author would not have studied the topic of Section 5. The author also thanks Professor Toshihiko Masuda for pointing out mistakes in the first versions of Shimada [52] and [53], which are the main part of Section 5 and 3 of this thesis, respectively. He thanks Professor Masaki Izumi for useful comments on Shimada [51] and [52], which are main part of Section 3 and 4 of this thesis, respectively. Without his comment, the author would not have studied some topics of Subsection 4.3. He also thanks Professor Vaughan. F. R. Jones for useful comments on the contents of Section 5 of this thesis. He is also thankful to Professor Yasuyuki Kawahigashi, who is his adviser, not only for useful comments on examples of Section 3 and 4 but also for his constant encouragement and instruction. In particular, his suggestion that the author stay at Vanderbilt university has greatly widened his horizon. The author is supported by Research Fellowships of the Japanese Society for the Promotion of Science for Young Scientists No.26-6590. This work is also supported by the Program for Leading Graduate Schools, MEXT, Japan.

## 2. Preliminaries

2.1. **Notations.** Let $M$ be a von Neumann algebra. We denote the set of unitaries of $M$ by $U(M)$. For a weakly continuous linear functional $\phi \in M_*$ and an element $a \in M$, set $[\phi, a] := a\phi - \phi a$. For a weakly continuous positive linear functional $\phi \in M_*^+$ and an element $x \in M$, set

$$\|x\|_\phi^\sharp := \sqrt{\frac{\phi(x^*x + xx^*)}{2}}.$$

This $\|\cdot\|_\phi^\sharp$ is a seminorm on $M$. If $\phi$ is faithful, then this norm metrizes the strong* topology of the unit ball of $M$.

2.2. **A topology of groups of automorphisms.** Let $M$ be a von Neumann algebra. Let $\mathrm{Aut}(M)$ be the set of all automorphisms $\alpha$ of $M$. A topology of $\mathrm{Aut}(M)$ is defined in the following way. We have

$$\alpha_i \to \alpha$$

if, by definition, $\|\psi \circ \alpha_i^{-1} - \psi \circ \alpha^{-1}\| \to 0$ for any $\psi \in M_*$.

2.3. **Ultraproduct von Neumann Algebras.** Next, we recall ultapruduct von Neumann algebras. Basic references are Ando–Haagerup [2] and Ocneanu [47]. Let $\omega$ be a free ultrafilter on $\mathbf{N}$ and $M$ be a separable von Neumann algebra. We denote by $l^\infty(M)$ the C*-algebra consisting of all norm bounded sequences in $M$. Set

$$I_\omega := \{(x_n) \in l^\infty(M) \mid \text{strong*-}\lim_{n\to\omega} x_n = 0\},$$

$$N_\omega := \{(x_n) \in l^\infty(M) \mid \text{for all } (y_n) \in I_\omega, \text{ we have } (x_n y_n) \in I_\omega \text{ and } (y_n x_n) \in I_\omega\},$$

$$C_\omega := \{(x_n) \in l^\infty(M) \mid \text{for all } \phi \in M_*, \text{we have} \lim_{n\to\omega} \|[\phi, x_n]\| = 0\}.$$

Then we have $I_\omega \subset C_\omega \subset N_\omega$ and $I_\omega$ is a closed ideal of $N_\omega$. Hence we can take the quotient C*-algebra $M^\omega := N_\omega / I_\omega$. Denote the canonical quotient map $N_\omega \to M^\omega$ by $\pi$. Set $M_\omega := \pi(C_\omega)$. Then $M_\omega$ and $M^\omega$ are von Neumann algebras as in Proposition 5.1 of Ocneanu [47].

Let $\tau^\omega \colon M^\omega \to M$ be the map defined by $\tau^\omega(\pi((x_n))) = \lim_{n\to\omega} x_n$. Here, the limit is taken in the weak topology of $M$. This map is a faithful normal conditional expectation (see Subsection 2.4 of [44]).

Let $\alpha$ be an automorphism of $M$. We define an automorphism $\alpha^\omega$ of $M^\omega$ by $\alpha^\omega(\pi((x_n))) = \pi((\alpha(x_n)))$ for $\pi((x_n)) \in M^\omega$. Then we have $\alpha^\omega(M_\omega) = M_\omega$. By restricting $\alpha^\omega$ to $M_\omega$, we define an automorphism $\alpha_\omega$ of $M_\omega$. Hereafter we omit $\pi$ and denote $\alpha^\omega$ and $\alpha_\omega$ by $\alpha$ if no confusion arises.

2.4. **The Rohlin Property.** Next, we recall the Rohlin property. A basic reference is [44]. In the previous subsection, we have seen that it is possible to lift automorphisms of von Neumann algebras on their ultraproducts. Hence it is natural to consider lifts of actions of locally compact abelian groups on $M^\omega$ and $M_\omega$. However, lifts may not be continuous. Instead of considering $\alpha^\omega$ on whole $M^\omega$, we consider their continuous part. Let $G$ be a locally compact separable abelian group. In the rest of the paper, we always assume that groups and von Neumann algebras are separable, except for ultaproduct von Neumann algebras. We denote the group operation of $G$ by $+$. Let $d$ be a translation invariant metric on $G$ (This metric exists. See Theorem 8.3 of

[19]). Choose a normal faithful state $\varphi$ on $M$. For an action $\alpha$ of $G$ on a von Neumann algebra $M$, set

$$M^\omega_\alpha := \{(x_n) \in M^\omega \mid \text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$\{n \in \mathbf{N} \mid \|\alpha_t(x_n) - x_n\|^\sharp_\varphi < \epsilon \text{ for } t \in G \text{ with } d(0,t) < \delta\} \in \omega\},$$

$$M_{\omega,\alpha} := \{(x_n) \in M_\omega \mid \text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$\{n \in \mathbf{N} \mid \|\alpha_t(x_n) - x_n\|^\sharp_\varphi < \epsilon \text{ for } t \in G \text{ with } d(0,t) < \delta\} \in \omega\}.$$

Since all metrics on $G$ are mutually equivalent, this definition does not depend on the choice of $d$. The condition appearing in the definition of $M^\omega_\alpha$ means the $\omega$-equicontinuity of the family of maps $\{G \ni t \mapsto \alpha_t(x_n)\}$ (See Definition 3.1 and Lemma 3.2 of [44]). Now, we define the Rohlin property.

**Definition 2.1.** *An action $\theta$ of a locally compact abelian group $G$ on a von Neumann algebra $M$ is said to have the Rohlin property if for each $p \in \hat{G}$, there exists a unitary $u$ of $M_{\omega,\theta}$ satisfying $\theta_t(u) = \langle t, -p\rangle u$ for all $t \in G$.*

The Rohlin property is also defined for Borel cocycle actions (See Definition 3.4 and Definition 4.1 of [44]). For actions, by the same argument as in the proof of Proposition 3.5 of [44], it is shown that the two definitions coincide.

2.5. **Connes–Takesaki module.** First of all, we recall Connes–Takesaki module. Basic references are Connes–Takesaki [11] and Haagerup–Størmer [20].

Let $M$ be a properly infinite factor and let $\phi$ be a normal faithful semifinite weight on $M$. Set $N := M \rtimes_{\sigma^\phi} \mathbf{R}$. Then the von Neumann algebra $N$ is generated by $M$ and a one parameter unitary group $\{\lambda_s\}_{s\in\mathbf{R}}$ satisfying $\lambda_s x \lambda_{-s} = \sigma^\phi_s(x)$ for $x \in M$, $s \in \mathbf{R}$. Let $\theta^\phi$ be the dual action of $\sigma^\phi$ and let $C$ be the center of $N$. Then an automorphism $\alpha$ of $M$ extends to an automrphism $\tilde{\alpha}$ of $N$ by the following way (See Proposition 12.1 of Haagerup–Stømer [20]).

$$\tilde{\alpha}(x) = \alpha(x) \text{ for } x \in M,\ \tilde{\alpha}(\lambda_s) = [D\phi\circ\alpha^{-1} : D\phi]_s\lambda_s \text{ for } s \in \mathbf{R}.$$

This $\tilde{\alpha}$ has the following properties (See Proposition 12.2 of Haagerup–Stømer [20]).

(1) The automorphism $\tilde{\alpha}$ commutes with $\theta^\phi$.

(2) The automorphism $\tilde{\alpha}$ preserves the canonical trace on $N$.
(3) The map $\alpha \mapsto \tilde{\alpha}$ is a continuous group homomorphism.

Set $\mathrm{mod}^\phi(\alpha) := \tilde{\alpha}|_C$. This is said to be a Connes–Takesaki module of $\alpha$. Actually, this definition is different from the original definition of Connes–Takesaki [11]. However, in Proposition 13.1 of Haagerup–Størmer [20], it is shown that they are same. This Connes–Takesaki module does not depend on the choice of $\phi$, that is, if $\phi$ and $\psi$ are two normal faithful semifinite weights, then the action $\mathrm{mod}^\phi(\alpha) \circ \theta^\phi$ of $\mathbf{R} \times \mathbf{Z}$ on $C$ is conjugate to $\mathrm{mod}^\psi(\alpha) \circ \theta^\psi$. Hence, in the following, we omit $\phi$ and write $\theta_t$ and $\mathrm{mod}(\alpha)$ if there is no danger of confusion. For an automorphism of any factor of type $\mathrm{II}_\infty$, considering its Connes–Takesaki module is equivalent to considering how it scales the trace. Hence flows with faithful Connes–Takesaki modules are natural generalization of trace-scaling flows.

We explain what property of automorphisms Connes–Takesaki module indicates. By Theorem 1 of Kawahigashi–Sutherland–Takesaki [31], an automorphism of any AFD factor is approximately inner if and only if its Connes–Takesaki module is trivial. Hence Connes–Takesaki module indicates "the degree of approximate innerness".

## 3. A classification theorem of actions of locally compact abelian groups on factors with the Rohlin property

3.1. **A Classification Theorem of actions of the Rohlin property.** Let $G$ be a locally compact abelian group. Let $\alpha^1$ and $\alpha^2$ be two actions of $G$ on a von Neumann algebra $M$. Two actions $\alpha^1$ and $\alpha^2$ are said to be *cocycle conjugate* if there exist an $\alpha^2$-cocycle $u$ and an automorphism $\sigma$ of $M$ satisfying $\mathrm{Ad}u_t \circ \alpha^2_t = \sigma \circ \alpha^1_t \circ \sigma^{-1}$ for all $t \in G$. If $\sigma$ can be chosen to be approximately inner, then $\alpha^1$ is said to be *strongly cocycle conjugate* to $\alpha^2$ (see Subsection 2.1 of [44]).

Our main theorem of this section is the following.

**Theorem 3.1.** *Let $G$ be a locally compact abelian group. Let $\alpha$ and $\beta$ be actions of $G$ with the Rohlin property on a factor $M$. Then $\alpha$ and $\beta$ are strongly cocycle conjugate if and only if $\alpha_t \circ \beta_{-t} \in \overline{\mathrm{Int}}(M)$ for all $t \in G$.*

This is a generalization of the following theorem due to Masuda–Tomatsu [44].

**Theorem 3.2.** (See Theorem 5.14 of Masuda–Tomatsu [44]) *Let $\alpha^1$, $\alpha^2$ be two Rohlin flows on a separable von Neumann algebra $M$. If*

*$\alpha^1_t \circ \alpha^2_{-t}$ is approximated by inner automorphisms for each $t \in \mathbf{R}$, then they are mutually (strongly) cocycle conjugate.*

In the rest of this section, we present a proof of this theorem. The proof is modeled after that in [44]. However, at some points of the proof, we need to deal with problems different from those in their proof. One of the problems is that some locally compact abelian groups do not have enough compact quotients. Instead, we consider compact quotients of compactly generated clopen subgroups. By Theorem 9.14 of Hewitt–Ross [19], a compactly generated subgroup is isomorphic to $\mathbf{R}^n \times K \times \mathbf{Z}^m$ for some compact abelian group $K$ and non-negative integers $n$, $m$. We deal with this problem in Subsection 3.1.3.

3.1.1. *Lifts of Borel Unitary Paths.* The first step of our proof of Theorem 3.1 is to find a representing unitary sequence $\{u^\nu_t\}$ for a Borel map $U_t \colon G \to U(M^\omega_\theta)$ so that the family $\{t \mapsto u^\nu_t\}$ is "almost" $\omega$-equicontinuous. More precisely, we have the following.

**Lemma 3.3.** (See Lemma 3.24 of [44]) *Let $(\theta, c)$ be a Borel cocycle action of a locally compact abelian group $G$ on a factor $M$. Suppose that $U \colon G \to M^\omega_\theta$ is a Borel unitary map. Let $H$ be a compactly generated clopen subgroup of $G$, which is isomorphic to $\mathbf{R}^n \times K \times \mathbf{Z}^m$ for some non-negative integers $n$, $m$ and a compact abelian group $K$. Let $L$ be a subset of $H$ of the form*

$$L = [0, S_1) \times \cdots \times [0, S_n) \times K \times [0, N_1) \times \cdots [0, N_m)$$

*when we identify $H$ with $\mathbf{R}^n \times K \times \mathbf{Z}^m$. Then for any $\delta > 0$ with $0 < \delta < 1$ and a finite set $\Phi$ of $M^+_*$, there exist a compact subset $I$ of $L \times L$, a compact subset $C$ of $L$ and a lift $\{u^\nu_t\}$ of $U$ satisfying the following conditions.*

(1) *We have $\pi_\omega\big((u^\nu_t)_\nu\big) = U_t$ for almost every $t \in L$ and the equality holds for all $t \in C$.*

(2) *We have $\mu_G(L \setminus C) < \delta$, where $\mu_G$ is the Haar measure on $G$.*

(3) *For all $\nu \in \mathbf{N}$, the map $L \ni t \mapsto u^\nu_t$ is Borel and its restriction to $C$ is strongly continuous.*

(4) *The family of maps $\{C \ni t \mapsto u^\nu_t\}_\nu$ is $\omega$-equicontinuous.*

(5) *We have $(\mu_G \times \mu_G)(I) \geq (1 - \delta)(\mu_G \times \mu_G)(L \times L)$.*

(6) *The family of maps $\{I \ni (t, s) \mapsto u^\nu_t \theta_t(u^\nu_s) c(t, s)(u^\nu_{t+s})^*\}_\nu$ is $\omega$-equicontinuous.*

(7) *The following limit is the uniform convergence on $I$ for all $\phi \in \Phi$.*

$$\lim_{\nu \to \omega} \|u^\nu_t \theta_t(u^\nu_s) c(t, s)(u^\nu_{t+s})^* - 1\|^\sharp_\phi = \|U_t \theta_t(U_s) c(t, s) U^*_{t+s} - 1\|^\sharp_{\phi^\omega}.$$

The proof is similar to that of Lemma 3.24 of [44]. Here, we only prove the following lemma, which corresponds to Lemma 3.21 of [44]. The proof is a simple approximation by Borel simple step functions.

**Lemma 3.4.** (See also Lemma 3.21 of [44]) *Let $G$ be a locally compact abelian group, $\theta\colon G \to \mathrm{Aut}(M)$ be a Borel map and $U\colon G \to M_\theta^\omega$ be a Borel unitary map. Then for any Borel subset $L$ of $G$ with $0 < \mu_G(L) < \infty$ and for any $\epsilon > 0$, there exist a compact subset $C$ of $L$ and a sequence $\{u_t^\nu\}_{\nu\in\mathbf{N}}$ of unitaries of $M$ for any $t \in L$ which satisfy the following conditions.*

(1) *We have $\pi_\omega\big((u_t^\nu)_\nu\big) = U_t$ for almost every $t \in L$ and the equality holds for all $t \in C$.*

(2) *We have $\mu_G(L \setminus C) < \epsilon$.*

(3) *For all $\nu \in \mathbf{N}$, the map $L \ni t \mapsto u_t^\nu$ is Borel and its restriction to $C$ is strongly continuous.*

(4) *The family of maps $\{C \ni t \mapsto u_t^\nu\}_\nu$ is $\omega$-equicontinuous.*

*Proof.* By the same argument as in the proof of Lemma 3.21 of [44], it is shown that there exists a sequence $\{L_n\}$ of compact subsets of $L$ satisfying the following conditions.

(1) We have $L_i \cap L_j = \emptyset$ for $i \neq j$.

(2) We have $\mu_G(L \setminus \bigcup_{j=1}^\infty L_j) = 0$.

(3) The map $U|_{L_i}$ is continuous for each $i$.

Hence we may assume that $L$ is compact and that $U\mid_L$ is strongly continuous. Let $\psi \in M_*$ be a normal faithful state. For each $t \in L$, take a representing unitary $\{\tilde{U}_t^\nu\}_\nu$ of $U_t$. Note that $t \mapsto \tilde{U}_t^\nu$ may not be Borel measurable. We first show the following claim.

**Claim**. For each $k \in \mathbf{N}$, there exist $N_k \in \mathbf{N}$, $F_k \in \omega$, a finite subset $A_k$ of $L$, a finite Borel partition $P^k := \{K_l^k\}_{l=1}^{n_k}$ of $L$ and a compact subset $C_k$ of $L$ satisfying the following conditions.

(1) For $s, t \in L$ with $d(s,t) \leq 1/N_k$, we have $\|U_s - U_t\|_{\psi^\omega}^\sharp < 1/2k$.

(2) We have $N_k > N_{k-1}, 2/N_k + 1/(2N_{k-1}) < 1/N_{k-1}$ for all $k$.

(3) We have $[k, \infty) \supset F_{k-1} \supsetneq F_k$ for all $k$.

(4) We have $A_k \supset A_{k-1}$ for all $k$.

(5) We have $\bigcup_{j=1}^\infty A_j \subset C_k$, $C_{k+1} \subset C_k$, $\mu_G(L \setminus C_k) < \epsilon(1 - 2^{-k})$ for all $k$ and $C_k \cap K_l^k$'s are also compact for all $k \in \mathbf{N}$, $l = 1, \cdots n_k$.

(6) For each $k$, the partition $P^{k+1}$ is finer than $P^k$ and for each $k \in \mathbf{N}$, $l = 1, \cdots, n_k$, we have $A_k \cap K_l^k = \{t_{k,l}\}(= \{\mathrm{pt}\})$.

(7) For $s, t \in K_l^k$, we have $d(s,t) \leq 1/N_k$.

(8) For $s, t \in A_k$, $\nu \in F_k$, we have $\|\tilde{U}_s^\nu - \tilde{U}_t^\nu\|_\psi^\sharp < \|U_s - U_t\|_{\psi^\omega}^\sharp + 1/(2k)$.

*Proof of Claim.* First of all, choose a sequence $\{N_k\}_{k=1}^\infty \subset \mathbf{N}$ so that the sequence satisfies conditions (1) and (2). Next, we take $P^k$'s.

Assume that $P^1, \cdots P^k$ are chosen so that they satisfy condition (7) and that $P^{j+1}$ is a refinement of $P^j$ for $j = 1, \cdots, k-1$. By compactness of $L$, there exists a family of finite balls $\{B_f\}_{f \in F}$ of radius $1/(2N_{k+1})$ of $L$ which covers $L$. This $\{B_f\}_{f\in F}$ defines a partition $\{\tilde{B}_{f'}\}_{f' \in F'}$ of $L$. Then $P^{k+1} := \{K_k^l \cap \tilde{B}_{f'}\}_{f' \in F', l=1,\cdots,n_k}$ is a refinement of $P^k$, which satisfies condition (7). Next, we take $C_k$'s. Set $C_0 := L$ and $C_1^0 := C_0$. By Lusin's theorem, for each $l = 1, \cdots, n_k$, $k \in \mathbf{N}$, there exists a compact subset $C_l^k$ of $K_l^k$ which satisfies the following conditions.

(1) We have $C_l^{k+1} \subset C_{l'}^k$ if $K_l^{k+1} \subset K_{l'}^k$.

(2) We have $\mu_G((K_l^{k+1} \cap C_{l'}^k) \setminus C_l^{k+1}) < 2^{-(k+1)}\epsilon/n_{k+1}$ if $K_l^{k+1} \subset K_{l'}^k$.

Set $C_k := \bigcup_{l=1}^{n_k} C_l^k$ for each $k \in \mathbf{N}$. Since $C_l^k$'s are compact, $C_k$ is also compact. On the other hand, we have

$$\begin{aligned}
\mu_G(C_j \setminus C_{j+1}) &= \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_j) \setminus C_{j+1}) \\
&= \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_{l'}^j) \setminus C_{j+1}) \\
&= \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_{l'}^j) \setminus C_l^{j+1}) \\
&< n_{j+1} \frac{1}{n_{j+1}} 2^{-(j+1)}\epsilon \\
&= 2^{-(j+1)}\epsilon.
\end{aligned}$$

In the above inequality, for each $l = 1, \cdots n_{j+1}$, $l' \in \{1, \cdots, n_j\}$ is the unique number with $C_l^{j+1} \subset C_{l'}^j$. Hence we have

$$\begin{aligned}
\mu_G(L \setminus C_k) &\leq \sum_{j=0}^{k-1} \mu_G(C_j \setminus C_{j+1}) \\
&< \epsilon \sum_{j=0}^{k-1} 2^{-(j+1)} \\
&= \epsilon(1 - 2^{-k}).
\end{aligned}$$

These $C_k$'s satisfy $C_{k+1} \subset C_k$ and $\mu_G(L \setminus C_k) < \epsilon(1 - 2^{-k})$, and we also have $C_k \cap K_l^k (= C_l^k)$ 's are compact. Next, we take $A_k$'s. For each $C_{l_1}^1 \supset C_{l_2}^2 \supset \cdots$, there exists $t_{l_1 l_2 \cdots} \in \bigcap_{k=1}^\infty C_{l_k}^k$ by compactness of $C_l^k$'s. By induction on $k$, it is possible to choose $A_k = \{t_{k,l}\}_{l=1}^{n_k}$ so that $A_k \subset A_{k+1}$ and that $t_{k,l} = t_{l_1 l_2 \cdots l l_{k+1} \cdots}$, i.e., $l_k = l$. These $A_k$'s satisfy conditions (4), (5) and (6). We may choose $F_k$'s so that they satisfy conditions (3) and (8). This completes the proof of Claim. □

Now, we return to the proof of Lemma 4.5. For $t \in L$, set $U_t^{k,\nu} := \tilde{U}^\nu_{t_{k,l}}$ if $t \in K_l^k$, $u_t^\nu := U_t^{k,\nu}$ for $\nu \in F_k \setminus F_{k+1}$. Set $C := \bigcap_k C_k$. Then we have $\mu_G(L \setminus C) < \epsilon$ by condition (5) of Claim. Since $U_t^{k,\nu}$'s are continuous on each $K_l^k \cap C_k (= C_l^k)$ and $C_1^k, \cdots C_{n_k}^k$ are compact, $U_t^{k,\nu}$'s are continuous on each $C_k$. Hence they are continuous on $C$. Hence by the same argument as in the proof of Lemma 3.21 of [44], the map $C \ni t \mapsto u_t^\nu$ is strongly continuous for each $\nu \in \mathbf{N}$. Then by the same argument as in Lemma 3.21 of [44], it is possible to see that $\{C \ni t \mapsto u_t^\nu\}_\nu$ is $\omega$-equicontinuous and that $\pi_\omega(u_t^\nu) = U_t$ for all $t \in C$. Now, we have chosen $\{u_t^\nu\}_\nu$ and $C$ so that they satisfy conditions (2),(3) and (4) of Lemma 4.5 and the following condition.

(1)' We have $\pi_\omega((u_t^\nu)_\nu) = U_t$ for $t \in C$.

Hence what remains to be done is to replace $\{u_t^\nu\}_\nu$ so that $\pi_\omega((u_t^\nu)_\nu) = U_t$ for almost all $t \in L$. By repeating the same process, we can find a sequence of compact subsets $\{D_n\}_{n=0}^\infty$ of $L$ and a sequence of strongly continuous maps $\{D_n \ni t \mapsto u_t^{n,\nu} \in U(M)\}_{n,\nu=0}^\infty$ which satisfy the following conditions.

(1) We have $\mu_G(L \setminus (\bigcup_{n=0}^\infty D_n)) = 0$ and $D_n$'s are mutually disjoint.

(2) We have $\pi_\omega((u_t^{n,\nu})_\nu) = U_t$ for $t \in D_n$.

(3) We have $D_0 = C$ and $u_t^{0,\nu} = u_t^\nu|_C$ for all $\nu \in \mathbf{N}$.

Set $u_t^\nu := u_t^{n,\nu}$ for $t \in D_n$. This $\{u_t^\nu\}_\nu$ satisfies all conditions of Lemma 4.5. □

3.1.2. *The Averaging Technique.* Next, we show the "averaging lemma". For the $\mathbf{R}$-action case, this means that it is possible to embed $(M \otimes L^\infty([0,S)), \theta \otimes \text{translation})$ into $(M_\theta^\omega, \theta)$ for any $S > 0$. This is a key lemma for the classification theorem. For the general case, the following lemma corresponds to this.

**Lemma 3.5.** *Let $G$ be a locally compact abelian group and $\theta$ be an action with the Rohlin property of $G$ on a factor $M$. Let $L$ be a subset of $G$ with the following properties.*

(1) *There exists a compactly generated clopen subgroup $H$ of $G$, which is isomorphic to $\mathbf{R}^n \times K \times \mathbf{Z}^m$ for some compact group $K$ and non-negative integers $n$, $m$.*

(2) *The set $L$ is a subset of $H$. When we identify $H$ with $\mathbf{R}^n \times K \times \mathbf{Z}^m$, $L$ is of the form $[0,S_1) \times \cdots \times [0,S_n) \times K \times [0,N_1) \times \cdots \times [0,N_2)$. Note that $L$ can be thought of as a quotient group of $H$.*

*Then there exist a unitary representation $\{u_k\}_{k\in\hat{L}}$ of $\hat{L}$ on $M_{\omega,\theta}$ and an injective *-homomorphism $\Theta : M\otimes L^\infty(L) \to M_\theta^\omega$ with the following properties.*

(1) *We have $\theta_t\circ\Theta = \Theta\circ(\theta_t\otimes\gamma_t)$. Here, $\gamma : H \curvearrowright L^\infty(L)$ denotes the translation.*

(2) *We have $\Theta(a\otimes\langle\cdot,k\rangle) = au_k$ for $a\in M$, $k\in\hat{L}$.*

(3) *We have $\tau^\omega\circ\Theta = \mathrm{id}_M\otimes\mu_L$, where $\mu_L$ denotes the normalized Haar measure on $L$, which is the normalization of the restriction of a Haar measure on $G$, and $\tau^\omega$ is the normal faithful conditional expectation as in Section 2.*

In order to show this lemma, by the same argument as in Lemma 5.2 of [44] (in this part, we use the fact that $M$ is a factor), it is enough to show the following proposition.

**Proposition 3.6.** *Let $\theta : G \curvearrowright M$ be an action with the Rohlin property of a locally compact abelian group $G$ on a factor $M$ and $L\subset H$ be subsets of $G$ as in the above lemma. Then there exists a family of unitaries $\{u_k\}_{k\in\hat{L}}\subset U(M_{\omega,\theta})$ with the following properties.*

(1) *We have $\theta_t(u_k) = \langle t,k\rangle u_k$ for $t\in H$.*

(2) *The map $k\mapsto u_k$ is an injective group homomorphism.*

To show the above proposition, we need to prepare some lemmas. In the rest of this subsection, $\theta$, $G$, $H$ and $L$ are as in Proposition 3.6.

**Lemma 3.7.** *Let $C$ be a subgroup of $\hat{L}$ isomorphic to $\mathbf{Z}/l\mathbf{Z}$. Then there exists a family of unitaries $\{u_k\}_{k\in C}\subset M_{\omega,\theta}$ with the following properties.*

(1) *We have $\theta_t(u_k) = \langle t,k\rangle u_k$ for $t\in H$.*

(2) *The map $C\ni k\mapsto u_k$ is an injective group homomorphism.*

*Proof.* Let $p$ be a generator of $C$. Since $\theta$ has the Rohlin property, there exists a unitary $w$ of $M_{\omega,\theta}$ satisfying $\theta_t(w) = \langle t,p\rangle w$ for $t\in H$. Since $w^l\in M_{\omega,\theta}^\theta$, there exists a unitary $v$ of $M_{\omega,\theta}^\theta\cap\{w\}'$ such that $v^{-l} = w^l$. Set $u := vw$ and $u_k := u^k$. Then the family $\{u_k\}_{k\in\mathbf{Z}/l\mathbf{Z}}$ does the job. □

By the same argument as in the proof of Lemma 3.16 of [44], we have the following lemma. See also Lemma 5.3 of Ocneanu [47], Lemma 3.16 of [44].

**Lemma 3.8.** (Fast reindexation trick.) *Let $\theta$ be an action of $G$ on a von Neumann algebra $M$ and let $F\subset M^\omega$ and $N\subset M_\theta^\omega$ be separable von Neumann subalgebras. Suppose that the subalgebra $N$ is globally* *invariant by $\theta$. Then there exists a faithful normal *-homomorphism $\Phi : N \to M_\theta^\omega$ with the following properties.*

$$\Phi = \mathrm{id} \text{ on } F \cap M,$$
$$\Phi(N \cap M_{\omega,\theta}) \subset F' \cap M_{\omega,\theta},$$
$$\tau^\omega(\Phi(a)x) = \tau^\omega(a)\tau^\omega(x) \text{ for all } a \in N,\ x \in F,$$
$$\theta_t \circ \Phi = \Phi \circ \theta_t \text{ on } N \text{ for all } t \in L.$$

**Lemma 3.9.** *Let $C$ be a subgroup of $\hat{L}$ of the form $\mathbf{Z}^n \times F$, where $F := \bigoplus_{k=1}^m \mathbf{Z}/(l_k\mathbf{Z})$ is a finite abelian group. Then there exists a family of unitaries $\{u_k\}_{k\in C} \subset M_{\omega,\theta}$ which satisfies the following conditions.*

(1) *We have $\theta_t(u_k) = \langle t, k\rangle u_k$ for $t \in H$.*

(2) *The map $k \mapsto u_k$ is an injective group homomorphism.*

*Proof.* Let $\{p_1, \cdots, p_n, q_1, \cdots, q_m\}$ be a base of $\hat{C}$. Then there exist unitaries $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ with $\theta_t(u_i) = \langle t, p_i\rangle u_i$, $\theta_t(v_j) = \langle t, q_j\rangle v_j$ for $t \in H$. By Lemma 3.7, we may assume that $v_j^{l_j} = 1$. By using the fast reindexation trick, it is possible to choose $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ so that they mutually commute. $\square$

Now, we prove Proposition 3.6.

*Proof.* Let $\psi \in M_*$ be a normal faithful state and let $\Phi = \{\phi_m\}$ be a countable dense subset of the unit ball of $M_*$. There exists an increasing sequence $\{C_\nu\}$ of finitely generated subgroups of $\hat{L}$ satisfying $\hat{L} = \bigcup_{\nu=1}^\infty C_\nu$. Then by the structure theorem of finitely generated abelian groups and the above lemma, for each $\nu$, there exists a family of unitaries $\{u_k^\nu\}_{k\in C_\nu} \subset U(M_{\omega,\theta})$ with $C_\nu \ni k \mapsto u_k^\nu$ satisfying conditions (1) and (2) of Lemma 3.9. For each $k \in \hat{L}$, set a sequence $\{k_\nu\}$ of $\hat{L}$ as follows.

$$k_\nu = \begin{cases} k & \text{if } k \in C_\nu \\ 0 & \text{if } k \notin C_\nu \ . \end{cases}$$

For each $\nu \in \mathbf{N}$, $k \in C_\nu$, take a representing sequence $\{u_k^{\nu,n}\}$ of $u_k^\nu$. Take a sequence $\{E_\nu\}$ of finite subsets of $\hat{L}$ satisfying $\bigcup E_\nu = \hat{L}$, $E_\nu \subset C_\nu$ for all $\nu \in \mathbf{N}$. By Lemma 3.3 of [**?**], the convergence

$$\lim_{n\to\omega} \|\theta_t(u_k^{\nu,n}) - \langle t, k\rangle u_k^{\nu,n}\|_\psi^\sharp = 0$$

is uniform for $t \in L$. Hence it is possible to choose $F_\nu \in \omega$ ($\nu = 1, 2, 3, \cdots$) so that

$$F_\nu \subsetneq F_{\nu-1} \subset [\nu - 1, \infty),\ \nu = 2, 3, \cdots, \tag{1}$$

$$\|u_k^{\nu,n} u_l^{\nu,n} - u_{k+l}^{\nu,n}\|_\psi^\sharp < 1/\nu,\ k, l \in E_\nu,\ n \in F_\nu, \tag{2}$$

$$\|[\phi_m, u_k^{\nu,n}]\| < 1/\nu,\ k \in E_\nu,\ m \le \nu,\ n \in F_\nu, \tag{3}$$

$$\|\theta_t(u_k^{\nu,n}) - \langle t, k\rangle u_k^{\nu,n}\|_\psi^\sharp < 1/\nu,\ k \in E_\nu,\ t \in L,\ n \in F_\nu. \tag{4}$$

Set $(u_k)_n := u_{k_\nu}^{\nu,n}$ for $n \in F_\nu \setminus F_{\nu+1}$. We show that $u_k := \{(u_k)_n\}$ is a desired family of unitaries.

We show $u_k \in M_\omega$. Fix $\mu \in \mathbf{N}$ and $k \in \hat{L}$. Then there exists $\nu \ge \mu$ with $k \in E_\nu$. Then for $n \in F_\nu$, there exists a unique $\lambda \ge \nu$ satisfying $n \in F_\lambda \setminus F_{\lambda+1}$. Then by the inequality (3), we have

$$\|[\phi_m, (u_k)_n]\| = \|[\phi_m, (u_{k_\lambda}^{\lambda,n})]\| < 1/\lambda \le 1/\mu$$

for $m \le \mu$. Thus we have $u_k \in M_\omega$.

In a similar way to the above, we obtain $\theta_t(u_k) = \langle t, k\rangle u_k$, using the inequality (4). It is also possible to show that the map $\hat{L} \ni k \mapsto u_k$ is a unitary representation by using the inequality (2). □

3.1.3. *Cohomology Vanishing.* By using Lemma 3.5, we show the following two propositions. See also Theorems 5.5 and 5.11 of [44], respectively.

**Proposition 3.10.** (2-cohomology vanishing) *Let $(\theta, c)$ be a Borel cocycle action of a locally compact abelian group $G$ on a factor $M$. Suppose that $(\theta, c)$ has the Rohlin property. Then the 2-cocycle $c$ is a coboundary, that is, there exists a Borel unitary map $v : G \to U(M)$ such that*

$$v_t \theta_t(v_s) c(t, s) v_{t+s}^* = 1$$

*for almost every $(t, s) \in G^2$.*

Furthermore, if $\|c(t, s) - 1\|_\phi^\sharp$, $\|[c(t, s), \phi]\|$ ($\phi \in M_*$) are small, then it is possible to choose $v_t$ so that $\|v_t - 1\|_\phi^\sharp$ and $\|[v_t, \phi]\|$ are small. We will explain this later.

**Proposition 3.11.** (Approximate 1-cohomology vanishing) *Let $\theta$ be an action with the Rohlin property of a locally compact abelian group $G$ on a factor $M$. Let $\epsilon$, $\delta$ be positive numbers and $\Phi$ be a compact subset of the unit ball of $M_*$. Let $H$ be a compactly generated clopen subgroup of $G$, which is isomorphic to $\mathbf{R}^n \times K \times \mathbf{Z}^m$ for some compact abelian group $K$ and non-negative integers $n$, $m$. Let $T$, $L$ be subsets of $H$ which satisfy the following conditions.*

(1) *When we identify* $H$ *with* $\mathbf{R}^n \times K \times \mathbf{Z}^m$, $L$ *is of the form*

$$[0, S_1) \times \cdots \times [0, S_n) \times K \times [0, N_1) \times \cdots \times [0, N_m),$$

*which implies that* $L$ *is a compact quotient of* $H$*.*

(2) *We have*

$$\frac{\mu_G\big(\bigcap_{t\in T}(t+L)\big)}{\mu_G(L)} > 1 - 4\epsilon^2.$$

*Then for any* $\theta$*-cocycle* $u_t$ *with*

$$\frac{1}{\mu_G(L)} \int_L \|[u_t, \phi]\| \; d\mu_G(t) < \delta$$

*for all* $\phi \in \Phi$*, there exists a unitary* $w \in M$ *such that*

$$\|[w, \phi]\| < 3\delta \text{ for all } \phi \in \Phi,$$

$$\|\phi \cdot (u_t\theta_t(w)w^* - 1)\| < \epsilon,$$

$$\|(u_t\theta_t(w)w^* - 1) \cdot \phi\| < \epsilon \text{ for all } t \in T, \phi \in \Phi.$$

By carefully examining arguments of the proofs of [44] Theorems 5.5 and 5.11, we notice that we need to choose sequences $\{L_n\}$ and $\{T_n\}$ of subsets of $G$ with the following properties.

(1) There exists an increasing sequence of compactly generated clopen subgroups $\{H_k\}$ of $G$ with $\bigcup_k H_k = G$ and $L_k$, $T_k$ are subsets of $H_k$ and $T_k$'s are compact. When we identify $H_k$ with $\mathbf{R}^{n_k} \times K_k \times \mathbf{Z}^{m_k}$ for some compact abelian group $K_k$ and non-negative integers $n_k$, $m_k$, the subset $L_k$ is of the form

$$[0, S_1) \times \cdots \times [0, S_{n_k}) \times K_k \times [0, N_1) \times \cdots \times [0, N_{m_k}).$$

(2) The translation $H_k \curvearrowright L^\infty(L_k)$ is embedded into $(\theta, M_{\omega,\theta})$ (see Proposition 3.6).

(3) The quantity

$$\frac{\mu_G\big(L_k \setminus \bigcap_{t\in T_k+T_k}(t+L_k)\big)}{\mu_G(L_k)}$$

is small.

(4) We have $L_k + T_k \subset T_{k+1}$.

(5) We have $T_k \subset T_{k+1}$ for all $k \in \mathbf{N}$ and $\bigcup_{k=1}^\infty T_k = G$.

For the $\mathbf{R}$-action case, $L_k = [0, s_k)$ and $T_k = [-t_k, t_k)$, $t_k \ll s_k \ll t_{k+1}$ do the job. In the following, we explain how to choose $L_k$'s and $T_k$'s for the general case. First, we show that there exists an increasing sequence $\{H_k\}$ of clopen subgroups of $G$ with the following conditions.

(6) For each $k$, the subgroup $H_k$ is compactly generated, which is isomorphic to $\mathbf{R}^n \times K_k \times \mathbf{Z}^{m_k}$ for some compact abelian group $K_k$. Note

that the multiplicity $n$ of $\mathbf{R}$ of $H_k$ can be chosen to be independent on $k$ by Theorem 9.14 of [19].

(7) We have $\bigcup_k H_k = G$.

This increasing sequence is chosen in the following way. There exists an increasing sequence $\{O_k\}$ of open subsets of $G$ such that $\overline{O_k}$'s are compact, $0 \in O_k$ for all $k \in \mathbf{N}$ and that $\bigcup_k \overline{O_k} = G$. For each $k \in \mathbf{N}$, let $H_k$ be the subgroup of $G$ generated by $\overline{O_k}$. We show that $H_k$ is clopen. If $t \in H_k$, then $t + O_k \subset H_k$. Hence this is open. Hence by Theorem 5.5 of [19], $H_k$ is closed. By Theorem 9.14 of [19], $H_k$ is of the form $\mathbf{R}^n \times K_k \times \mathbf{Z}^{m_k}$.

Next, take two sequences $\{L_k\}$ and $\{T_k\}$ of subsets of $G$ and a decreasing sequence $\{\epsilon_k\} \subset \mathbf{R}_{>0}$ with the following properties.

(8) The sets $L_k$, $T_k$ are subsets of $H_k$. When we identify $H_k$ with $\mathbf{R}^n \times K_k \times \mathbf{Z}^{m_k}$ for some compact abelian group $K_k$ and a non-negative integer $m_k$, the subset $L_k$ is of the form

$$[0, S_1) \times \cdots [0, S_n) \times K_k \times [0, N_1) \times \cdots \times [0, N_{m_k}).$$

Note that the way how to identify $H_k$ with $\mathbf{R}^n \times K_k \times \mathbf{Z}^{m_k}$ is not important. The point is that $L_k$ is a quotient of a clopen subgroup of $G$.

(9) We have

$$\frac{\mu_G\big(L_k \setminus \bigcap_{t \in T_k + T_k}(t + L_k)\big)}{\mu_G(L_k)} > 1 - (\frac{\epsilon_k}{6\mu_G(T_k)^2})^2.$$

(10) We have $T_k + L_k \subset T_{k+1}$, $\bigcup_k T_k = G$ and $T_k$'s are compact.

(11) We have $0 < \epsilon_k < 1/k$ and

$$\sum_{k=n+1}^{\infty} \sqrt{13\mu_G(T_k)\epsilon_k} < \epsilon_n.$$

From now on, we explain how to choose two sequences $\{L_k\}$ and $\{T_k\}$. They are chosen in the following way. For each $k \in \mathbf{N}$, set $A_k := \overline{O_k}$. Here, the set $O_k$ is chosen as in (7).

Assume that $(T_l, L_l, \epsilon_l)$, $l \leq k$ are chosen. Then since $A_{k+1} + T_k + \overline{L_k}$ is compact, it is possible to choose a subset $T_{k+1} \subset H_{k+1}$ so that when we identify $H_{k+1}$ with $\mathbf{R}^n \times K_{k+1} \times \mathbf{Z}^{m_{k+1}}$, $T_{k+1}$ is of the form

$$[-t_1, t_1] \times \cdots \times [-t_n, t_n] \times K_{k+1} \times [-M_1, M_1] \times \cdots \times [-M_{m_{k+1}}, M_{m_{k+1}}]$$

and that $A_{k+1} + T_k + L_k \subset T_{k+1}$. Since $\bigcup_k A_k = G$, we also have $\bigcup_k T_k = G$. Choose $\epsilon_{k+1} > 0$ so that

$$\epsilon_{k+1} < \epsilon_k, \ \sqrt{13\mu_G(T_{k+1})\epsilon_{k+1}} < \epsilon_k/2^k.$$

Choose $L_{k+1} \subset H_{k+1}$ so large that $L_{k+1}$ satisfies conditions (8) and (9). Thus we are done.

By using the above sequences $\{L_k\}$, $\{T_k\}$ instead of $\{S_k\}$ and $\{T_k\}$ of (5.14) of [44], Propositions 3.10 and 3.11 are shown by a similar argument to that of the proofs of Theorems 5.5 and 5.11 of [44], respectively. Furthermore, it is possible to choose $v_t$ in Proposition 3.10 so that $v_t$ satisfies the following conditions.

(1) If for some $n \geq 2$ and a finite subset $\Phi \subset (M_*)_+$, we have

$$\int_{T_{n+1}} d\mu_G(t) \int_{T_{n+1}} d\mu_G(s) \|c(t,s) - 1\|_\phi^\sharp \leq \epsilon_{n+1}$$

for all $\phi \in \Phi$, then it is possible to choose $v_t$ so that

$$\int_{T_n} \|v_t - 1\|_\phi^\sharp \, d\mu_G(t) < \epsilon_{n-1} d(\Phi)^{1/2}$$

for all $\phi \in \Phi$. Here, $d(\Phi)$ is defined in the following way.

$$d(\Phi) := \max(\{1\} \cup \{\|\phi\| \mid \phi \in \Phi\}).$$

(2) If for some $n \geq 2$ and a finite subset $\Phi \subset M_*$, we have

$$\int_{T_{n+1}} d\mu_G(t) \int_{L_{n+1}} d\mu_G(s) \|[c(t,s), \phi]\| < \epsilon$$

for all $\phi \in \Phi$, then it is possible to choose $v_t$ satisfying

$$\int_{T_n} \|[v_t, \phi]\| \, d\mu_G(t) \leq (3\epsilon_{n-1} + 3\epsilon) d(\Phi)$$

for all $\phi \in \Phi$.

In the proof, the following points are slightly different.

(1) The inequality corresponding to (5.12) of [44] is

$$\frac{2\mu_G\big(L \setminus (\bigcap_{t \in T+T} t + L)\big)^{1/2}}{\mu_G(L)^{1/2}} < \frac{\delta}{6\mu_G(T)^2}.$$

(2) We need to show a lemma which corresponds to Lemma 5.4 of [44]. In the proof, the inequality corresponding to (5.13) of [44] is the

following.

$$\begin{aligned}
&\|U_t\alpha_t(U_s)c(t,s)U_{s+t}^* - 1\|_\phi^\sharp \\
&\leq \|\chi_{\bigcap_{t\in T+T} t+L} - 1\|_{\phi\otimes\mu_L}^\sharp \\
&+ \|\chi_{L\setminus(\bigcap_{t\in T+T} t+L)}\big((\text{a unitary valued function}) - 1\big)\|_{\phi\otimes\mu_L}^\sharp \\
&\leq 0 + 2\|\chi_{L\setminus(\bigcap_{t\in T+T} t+L)}\|_{\phi\otimes\mu_L}^\sharp \\
&\leq 2\|\phi\|^{1/2}\frac{\mu_G\big(L\setminus(\bigcap_{t\in T+T} t+L)\big)^{1/2}}{\mu_G(L)^{1/2}} \\
&< \frac{\delta}{6\mu_G(T)^2}
\end{aligned}$$

for all $t, s \in T$, $\phi \in \Phi$. The other parts of of the proof are completely same.

(3) In the proof of Theorem 5.5 of [44], they show the inequality

$$\int_{T_n}^{T_n} \|W^* u_t \alpha_t^n(W) - 1\|_2^2 \, dt < 18\epsilon_n.$$

Instead, in the proof of Proposition 3.10, we show the following inequality.

$$\begin{aligned}
&\int_{T_n} \|W^* u_t \alpha_t^n(W) - 1\|_2^2 \, d\mu_G(t) \\
&\leq \frac{2}{\mu_G(L_n)}\int_{T_n} d\mu_G(t)\Big(\int_{\bigcap_{t\in T_n} t+L_n} d\mu_G(s) \;\; \|\tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1\|_2^2 \\
&+ \int_{L_n\setminus\bigcap_{t\in T_n} t+L_n} d\mu_G(s) \; \|\tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1\|_2^2\Big) \\
&\leq \frac{2}{\mu_G(L_n)}\int_{T_n} d\mu_G(t)\int_{\bigcap_{t\in T_n} t+L_n} d\mu_G(s) \; \|\tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1\|_2^2 \\
&+ \frac{8}{\mu_G(L_n)}\mu_G(T_n)\mu_G(L_n\setminus\bigcap_{t\in T_n} t+L_n) \\
&< \frac{2}{\mu_G(L_n)}\int_{T_{n+1}\times T_{n+1}} d\mu_G(t)d\mu_G(s) \; \|\tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1\|_2^2 \\
&+ \mu_G(T_n)\frac{{\epsilon_n}^2}{18\mu_G(T_n)^4} \\
&< 9\epsilon_n.
\end{aligned}$$

The other parts of the proof of Proposition 3.10 are same as corresponding parts of the proof of Theorem 5.5 of [44].

(4) In the proof of Proposition 3.11, we need to show the inequality

$$\|u_t\alpha_t(W)W^* - 1\|^\sharp_{|\phi|^\omega} \leq 2\|\chi_{L\setminus(\bigcap_{t\in T} t+L)}\|^\sharp_{|\phi|\otimes\mu_L},$$

which corresponds to the inequality

$$\|u_t\alpha_t(W)W^* - 1\|^\sharp_{|\phi|^\omega} \leq 2\frac{t^{1/2}\|\phi\|^{1/2}}{S^{1/2}}$$

in the proof of Theorem 5.11 of [44]. This is obtained by a similar computation to the above (3).

By using Proposition 3.10, it is possible to show the following lemma, which corresponds to Lemma 5.8 of [44].

**Lemma 3.12.** *Let $\alpha$, $\beta$ be actions with the Rohlin property of a locally compact abelian group $G$ on a factor $M$. Suppose that $\alpha_t\circ\beta_{-t} \in \overline{\mathrm{Int}}(M)$ for all $t \in G$. Let $H$ be a compactly generated clopen subgroup of $G$ and $T$ be a subset of $H$ such that when we identify $H$ with $\mathbf{R}^n \times K \times \mathbf{Z}^m$ for some compact abelian group $K$ and non-negative integers $n$, $m$, $T$ is of the form*

$$[-t_1, t_1] \times \cdots \times [-t_n, t_n] \times K \times [-M_1, M_1] \times \cdots \times [-M_m, M_m].$$

*Then for any $\epsilon > 0$ and a finite set $\Phi \subset M_*$, there exists an $\alpha$-cocycle $u$ such that*

$$\int_T \|\mathrm{Ad}u_t \circ \alpha_t(\phi) - \beta_t(\phi)\| \, d\mu_G(t) < \epsilon$$

*for all $\phi \in \Phi$.*

In the proof of this lemma, the set corresponding to (5.18) of [44] is obtained in the following way. For a small positive real number $\eta > 0$, take a small number $r > 0$ so that

$$\|\alpha_t(\phi) - \phi\| < \eta, \ \|\beta_t(\phi) - \phi\| < \frac{2\eta}{\mu_G(T)}$$

for $\phi \in \Phi$, $t \in G$, $d(t, 0) < r$. Choose $A(r, T) := \{t_j\}_{j=1}^N$ so that for any $t \in T$, there exists $t_j \in A(r, T)$ with $d(t, t_j) < r$. This is possible because $T$ is compact.

Now, we return to the proof of Theorem 3.1. The proof is basically the same as that of Case 2 of Lemma 5.12 of [44]. Here, we only explain the outline. By using Proposition 3.11 and Lemma 3.12 alternatively, our main theorem of this section is obtained (the Bratteli–Elliott–Evans–Kishimoto type argument). However, we need to change the following part. In the proof of Case 2 of Lemma 5.12 of [44], they take $\{M_n\} \subset \mathbf{N}$ and $\{A(M_n, T_n)\}$, which appear in conditions $(n.1)$ and $(n.8)$. Instead, in $(n.8)$, take $r_n \in \mathbf{R}_{>0}$ so that

$$\|(\hat{v^n}(t) - \hat{v^n}(s)) \cdot \phi\| < \epsilon_n,$$
$$\|\phi \cdot (\hat{v^n}(t) - \hat{v^n}(s))\| < \epsilon_n$$

for $t, s \in T_n$, $d(t,s) < r_n$, $\phi \in \hat{\Phi}_{n-1}$. Choose a finite subset $A(r_k, T_k)$ of $T_k$ so that for each $t \in T_k$, there exists $t_0 \in A(r_k, T_k)$ with $d(t, t_0) < r_k$. This is possible because $T_k$ is compact.

3.2. **Actions with the Rohlin property on the AFD factors of type II.** Here, we give some classes of actions each member of which has the Rohlin property. We separate the argument by factors on which the group acts. Namely, we will separately consider the following three classes.

(1) The factor is AFD and of type II.
(2) The factor is non-McDuff and of type II.
(3) The factor is AFD and of type III.

The first case is deeply studied by Kawahigashi [28], [29] and [30]. We give proofs for some of his results by using Theorem 3.1. We will explain this in Subsubsection 3.2.1. The second case is studied by the author [54]. Although he handles only actions of $\mathbf{R}$ in [54], its results holds for actions of general locally compact abelian groups. We will explain this in Subsubsection 3.2.2. For the third case, in Section 4, we will give a sufficient condition for the Rohlin property.

First, we consider actions which fix Cartan subalgebras. This type of examples are classified by Kawahigashi [28]. One of the most important examples of actions of this form is an infinite tensor product action.

Let $\{p_n\}$ be a sequence of the dual group $\hat{G}$ of $G$. Set

$$M := \bigotimes_{n=1}^{\infty} (M_2(\mathbf{C}), \mathrm{tr}),$$

Then it is possible to define an action $\theta$ of $G$ by the following way.

$$\theta_t := \bigotimes \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \langle t, p_n \rangle \end{pmatrix}.$$

Then this $\theta$ has the Rohlin property if and only if the set

$$A := \{p \in \hat{G} : \text{there exists a subsequence of } \{p_n\} \text{ which converges to } p\}$$

generates a dense subgroup $\Gamma$ in $\hat{G}$. This is seen by the following way. We first show the "if" part. Here, we show this implication in the case where for each $p \in A$, a subsequence of $\{p_n\}$ which converges to $p$ can be chosen to be a constant sequence. This case is needed for the proof of Example 3.13. The general case of this implication will follow from Example 3.13.

Choose $p \in A$. By ignoring other tensor components, we may assume that $p_n = p$ for all $n$. For each $m \in \mathbf{N}$, set

$$S^m := \{\sigma : \{1, \cdots, 2m-1\} \to \{1,2\} \mid \sharp\sigma^{-1}(1) = m-1,\ \sharp\sigma^{-1}(2) = m\}.$$

For $\sigma \in S^m$, $m \in \mathbf{N}$ and $k \in \{1, \cdots, 2m-1\}$, set $\tau(k) := 3 - \sigma(k)$ and

$$v_\sigma := e_{\tau(1)\sigma(1)} \otimes \cdots \otimes e_{\tau(2m-1)\sigma(2m-1)} \otimes 1 \otimes \cdots .$$

Then we have

$$e_\sigma := v_\sigma^* v_\sigma = e_{\sigma(1)\sigma(1)} \otimes \cdots \otimes e_{\sigma(2m-1)\sigma(2m-1)} \otimes 1 \otimes \cdots ,$$
$$f_\sigma := v_\sigma v_\sigma^* = e_{\tau(1)\tau(1)} \otimes \cdots \otimes e_{\tau(2m-1)\tau(2m-1)} \otimes 1 \otimes \cdots ,$$
$$\theta_t(v_\sigma) = \langle t, p\rangle v_\sigma$$

for $t \in G$. Hence if we set

$$T := \bigcup_{m=1}^{\infty} \big\{\sigma \in S^m \mid \sharp(\sigma^{-1}(1) \cap \{1, \cdots, k\}) \geq \sharp(\sigma^{-1}(2) \cap \{1, \cdots, k\}) \text{ for } k = 1, \cdots, 2m-2\big\},$$

then the families $\{e_\sigma\}_{\sigma\in T}$ and $\{f_\sigma\}_{\sigma\in T}$ are orthogonal families, respectively. We show that $\sum_{\sigma\in T} e_\sigma = 1$, which implies that $\sum_{\sigma\in T} v_\sigma$ is a unitary. This is shown in the following way. Consider the gambler's ruin problem when one has infinite money, the other has no money and they have equal chance to win. Then $\| \sum_{\sigma\in T} e_\sigma\|_1$ is equal to the probability of the poor's ruin. This is 1. Set

$$u_n := 1 \otimes \cdots \otimes 1 \otimes \sum_{\sigma\in T} v_\sigma \in M_2(\mathbf{C})^{\otimes n} \otimes M.$$

Then we have $\{u_n\} \in M_{\omega,\theta}$ and $\theta_t((u_n)_\omega) = \langle t, p\rangle (u_n)_\omega$ for $t \in G$. By assumption, the set $A$ generates a dense subgroup of $\hat{G}$. Hence $\theta$ has the Rohlin property.

Conversely, assume that the subgroup $\Gamma$ is not dense in $\hat{G}$. Then there exists a non-empty open subset $U$ of $\hat{G}$ with $U \cap \Gamma = \emptyset$. Then by a similar argument to that of the proof of Proposition 1.2 of [28], it is shown that the Connes spectrum of $\theta$ and $U$ do not intersect, which implies that $\theta$ does not have the Rohlin property.

**Theorem 3.13.** (See also Corollary 1.9 of Kawahigashi [28]) *Let $\alpha$ be an action of a locally compact abelian group $G$ on the AFD factor $R$ of* *type* $\mathrm{II}_1$. *Assume that* $\alpha$ *fixes a Cartan subalgebra of* $R$. *Then* $\alpha$ *has the Rohlin property if and only if its Connes spectrum is* $\hat{G}$.

The proof is just a combination of an analogue of Corollary 5.17, which follows from the above example of an infinite tensor product action and Theorem 3.1, and Lemma 6.2 of [44]. In the proof, the point is that invariantly approximate innerness (see Definition 4.5 of [44]) is the dual of the Rohlin property. This fact is shown by the completely same argument as in the proof of Theorem 4.11 of [44].

By this example and the main theorem, all the actions fixing Cartan subalgebras with full Connes spectrum are cocycle conjugate to an infinite tensor product action with full Connes spectrum.

The following is a next example.

**Example 3.14.** (See Theorem 6.12 of [44]) Let $\theta$ be an almost periodic minimal action of a locally compact abelian group $G$ on the AFD factor of type $\mathrm{II}_1$. Then $\theta$ has the Rohlin property.

*Proof.* An almost periodic action is a restriction of a compact abelian group action to its dense subgroup (see Proposition 7.3 of Thomsen [61]). If $\theta$ is minimal, then the original compact group action is also minimal, which is unique up to cocycle conjugacy by Jones–Takesaki [25]. This has the Rohlin property. $\square$

3.3. **Actions with the Rohlin property on non-McDuff factors.** One of the remarkable point of Theorem 3.1 is that the theorem is also applicable to actions of locally compact abelian groups on non-McDuff factors. Hence it is natural to try to find actions with the Rohlin property on non-McDuff factors. Here we construct Rohlin flows on a non-McDuff factor.

3.3.1. *The Construction.* Although the following is written about actions of $\mathbf{R}$ for simplicity, it is also possible to construct actions of general locally compact abelian groups by completely the same argument. Let $D = L^\infty(X, \mu)$ be a diffuse separable abelian von Neumann algebra, where $\mu$ is a probability measure. Choose a free ergodic $\mu$-preserving action $\alpha : \mathbf{Z} \curvearrowright D$. Then $A := D \rtimes_\alpha \mathbf{Z} \supset D$ is a pair of the AFD type $\mathrm{II}_1$ factor and its Cartan subalgebra. There is a unique action $\alpha * \alpha : \mathbf{F}_2 \curvearrowright D$ which satisfies $\alpha * \alpha(a) = \alpha$, $\alpha * \alpha(b) = \alpha$, where $a$, $b$ are two generator of $\mathbf{F}_2$. Set $M := A *_D A$. Then $M$ is isomorphic to $D \rtimes_{\alpha * \alpha} \mathbf{F}_2$ and is a non-McDuff factor.

**Lemma 3.15.** (See also Theorem 2.6 of Ueda [62])

*Let $\theta : \boldsymbol{R} \curvearrowright D$ be a $\mu$-preserving flow commuting with $\alpha$. Let $\{u_t^i\}$ be $\theta$-cocycles $(i = 1, 2)$. Then the action $\theta$ extends to $M$ by $\theta_t(\lambda_a) = u_t^1\lambda_a$, $\theta_t(\lambda_b) = u_t^2\lambda_b$ for $t \in \boldsymbol{R}$.*

*Proof.* Fix $t \in \mathbf{R}$. Since $\theta_t$ commutes with $\alpha$, the injective homomorphisms $\pi_A : A \cong \{\{\lambda_a\} \cup D\}'' \hookrightarrow A *_D A$, $\pi_B : A \cong \{\{\lambda_b\} \cup D\}'' \hookrightarrow A *_D A$ satisfying the following are well-defined.

$$\pi_A(\lambda_a) = u_t^1\lambda_a,\ \pi_B(\lambda_b) = u_t^2\lambda_b,\ \pi_A(x) = \pi_B(x) = \theta_t(x) \text{ for } x \in D.$$

Then here exists an automorphism $\theta_t$ of $A *_D A$ such that $\theta_t|_{\{\{\lambda_a\}\cup D\}''} = \pi_A$, $\theta_t|_{\{\{\lambda_a\}\cup D\}''} = \pi_B$. It is not difficult to see that the map $t \mapsto \theta_t(y)$ is strongly continuous for $y \in M$.

□

For the above flows, we give a characterization of the Rohlin property.

In order to achieve this, we make use of the following Rohlin type theorem for $\mathbf{R} \times \mathbf{Z}$ actions on the standard probability space, which is a part of a theorem of Lind [38] or Ornstein–Weiss [48].

**Lemma 3.16.** (Theorem 1 of Lind [38]) *Let $R$ be a $\mu$ -preserving faithful ergodic action of $\boldsymbol{R} \times \boldsymbol{Z}$ on the standard probability space $(X, \mu)$. Then for any $\epsilon > 0$, for any $N \in \boldsymbol{N}$ and for any $T > 0$, there exists a Borel subset $Y \subset X$ with the following properties.*

(1) *The set $A := \bigcup_{|t|\leq T, |n|\leq N} R_{(t,n)}(Y)$ is Borel measurable and satisfies $\mu(A) > 1 - \epsilon$.*

(2) *There is a Borel isomorphism $F : A \cong Y \times [-T, T] \times \{-N, \cdots, N\}$ and a Borel measure $\nu$ on $Y$ such that*

$$\mu F^{-1} = \nu \otimes \text{Lebesgue measure} \otimes \text{counting measure}.$$

(3) *Under this identification, we have*

$$R_{(t,n)}(y, s, m) = (y, s + t, m + n)$$

*for $y \in Y$, $|s + t| \leq T$, $|s| \leq T$, $m \in \{-N, \cdots, N\}$, $|m| \leq N$, $|n + m| \leq N$.*

Now, we give the characterization of the Rohlin property for flows constructed in Lemma 3.15.

**Theorem 3.17.** *For flows constructed in Lemma 3.15, consider the following five conditions.*

(1) *The flow $\theta$ has the Rohlin property.*

(2) *The action $\{(\theta_t|_D) \circ \alpha_n\}_{(t,n)\in \boldsymbol{R}\times\boldsymbol{Z}}$ is faithful on $D$.*

(3) *The flow $\theta$ is centrally free. That is, $\theta_t$ is free on $M_\omega$ for $t \neq 0$.*

(4) *The flow $\theta$ is centrally free and has full Connes spectrum.*

(5) *We have $(M \rtimes_\theta \boldsymbol{R}) \cap M' = \boldsymbol{C}$.*

*Then we have implications (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇒ (5).*

*Proof.* The implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5) follow from Masuda–Tomatsu [44]. Hence it suffices to show the implications (3) $\Rightarrow$(2) and (2) $\Rightarrow$ (1).

First, we show the implication (3) $\Rightarrow$ (2). Assume that condition (2) does not hold. Then there exists $(t,n) \neq 0$ such that $\theta_t = \alpha_n$. We have $t \neq 0$ because $\alpha$ is ergodic. Hence for $x \in M_\omega \subset D^\omega$ (the implication $M_\omega \subset D^\omega$ is shown in Theorem 8 of Ueda [63]), we have $\theta_t(x) = \lambda_{a^n} x \lambda_{a^{-n}} = x$, which implies that condition (3) does not hold.

Next, we show the implication (2) $\Rightarrow$ (1). Suppose that the action $\{(\theta_t|_D) \circ \alpha_n\}_{(t,n)\in \mathbf{R}\times\mathbf{Z}}$ is faithful. Fix $n \in \mathbf{N}$. It is enough to construct a sequence $\{u_n\}$ of unitary elements of $D$ such that

(i) $\mu\left(|\theta_t(u_n) - e^{-ipt}u_n|^2\right) < n^{-2}$ for $|t| < n$,

(ii) $\mu\left(|\alpha(u_n) - u_n|^2\right) < n^{-2}$.

Assume that we have these $u_n$'s. Then by condition (ii), $\{u_n\}$ asymptotically commutes with $\lambda_a$ and $\lambda_b$. Hence $\{u_n\}$ is a centralizing sequence. By using condition (i), we have $\theta_t\left(\{u_n\}\right) = e^{-ipt}\{u_n\}$ for $t \in \mathbf{R}$.

Now, we show the existence of the above $\{u_n\}$. Regard $D$ as $L^\infty(X,\mu)$, where $(X,\mu)$ is a standard probability measured space and let $S : \mathbf{R} \curvearrowright (X,\mu)$ and $T : \mathbf{Z} \curvearrowright (X,\mu)$ be actions induced by $\theta$, $\alpha$, respectively. By using Lemma 3.16 for $T := 8n^3$, $N := 8n^2$, $\epsilon := 1/8n^2$, $R_{(s,m)} := S_s T_m$, there exists a Borel subset $Y \subset X$ satisfying the conditions in Lemma 3.16.

Set

$$u_n(y,s,m) := \begin{cases} e^{ips} & \text{for } (y,s,m) \in A, \\ 1 & \text{for } x \in X\backslash A. \end{cases}$$

Then by condition (3) of Lemma 3.16, we have

$$(\theta_t(u_n) - e^{-ipt}u_n)(x) = 0 \text{ for } x \in \{(y,s,m) \in A|\ |s| \leq T - n\}.$$

Hence we have

$$\begin{aligned} \mu(|\theta_t(u_n) - e^{-ipt}u_n|^2) &\leq 4\mu(X\backslash\{(y,s,m) \in A|\ |s| \leq T-n\}) \\ &= 4(\mu(X\backslash A) + \mu(\{(y,s,m) \in A|\ |s| > T-n\})) \\ &\leq 4(\epsilon + n/T) = n^{-2}. \end{aligned}$$

By similar computation to this, we have $\mu(|\alpha(u_n) - u_n|^2) < n^{-2}$. □

By this theorem, it is possible to see that there exist Rohlin flows on the factor $M$. In order to do this, first, note that if an action $\beta : \mathbf{Z} \curvearrowright D$ is free ergodic probability measure preserving, then $D \rtimes_{\beta*\beta} \mathbf{F}_2$ is isomorphic to the factor $M$, which is shown by Connes–Feldman–Weiss [10] and the uniqueness of the amalgamated free product.

**Example 3.18.** Let $(\tilde{D}, \mu)$ be a diffuse separable abelian von Neumann algebra with a normal faithful trace and let $\tilde{\theta}$ be a $\mu$-preserving faithful flow on $D$. Set $D := \otimes_{n=-\infty}^{\infty}(\tilde{D}, \mu)^n$ and $\alpha : \mathbf{Z} \curvearrowright D$ be a Bernoulli shift. Then the diagonal action $\theta : \mathbf{R} \curvearrowright D$ of $\tilde{\theta}$ extends to $M := D \rtimes_\alpha \mathbf{F}_2$ and has the Rohlin property.

Other examples are given in the following.

**Example 3.19.** Let $D = L^\infty(\mathbf{T}^2)(= L^\infty((\mathbf{R}/\mathbf{Z})^2))$ and let $\alpha : \mathbf{Z} \curvearrowright D$ be an action defined by $\alpha(f)(r, s) = f(r - 1/\sqrt{2}, s - 1/\sqrt{3})$ for $(r, s) \in \mathbf{T}^2$, $f \in D$. Then $D \rtimes_{\alpha * \alpha} \mathbf{F}_2$ is isomorphic to $M$. By Lemma 3.15, we can define a flow $\theta^{\lambda,\mu,p,q} : \mathbf{R} \curvearrowright D \rtimes_{\alpha*\alpha} \mathbf{F}_2$ by

$$\theta_t^{\lambda,\mu,p,q}(f)(r, s) = f(r - pt, s - qt)$$

for $(r, s) \in \mathbf{T}^2$, $f \in D$, $t \in \mathbf{R}$,

$$\theta_t^{\lambda,\mu,p,q}(\lambda_a) = e^{i\lambda t}\lambda_a,\ \theta_t^{\lambda,\mu,p,q}(\lambda_b) = e^{i\mu t}\lambda_b$$

for $t \in \mathbf{R}$. This $\theta^{\lambda,\mu,p,q}$ has the Rohlin property if and only if $(p, q) \neq r(n/\sqrt{2} - m, n/\sqrt{3} - l)$ for any $r \in \mathbf{R}$, $n, m, l \in \mathbf{Z}$.

*Proof.* In order to show this, by Theorem 3.17, it is enough to show that the action $\{(\theta_t^{\lambda,\mu,p,q}|_D) \circ \alpha_n\}$ is faithful if and only if the above condition holds. For $(t, n) \in \mathbf{R} \times \mathbf{Z}$, $\theta_t^{\lambda,\mu,p,q}|_D = \alpha_n$ if and only if $pt = n/\sqrt{2} + m$, $qt = n/\sqrt{3} + l$ for some $m, l \in \mathbf{Z}$. Hence $\{(\theta_t^{\lambda,\mu,p,q}|_D) \circ \alpha_n\}$ is faithful if and only if $(p, q) \neq (n/\sqrt{2} + m, n/\sqrt{3} + l)/t$ for all $t \in \mathbf{R} \setminus \{0\}, n, m, l \in \mathbf{Z}$. □

If we further assume that $(p, q) \neq r(s/\sqrt{2} - m, s/\sqrt{3} - l)$ for any $r, s \in \mathbf{R}$, $m, l \in \mathbf{Z}$, then this also gives a new example of a Rohlin flow on the $C^*$-algebra $C(\mathbf{T}^2) \rtimes_{\alpha*\alpha} \mathbf{F}_2$, which is shown by the same argument as in Proposition 2.5 of Kishimoto [34].

**Remark 3.20.** Let $\alpha : G \curvearrowright D$ be a non-singular free ergodic action of a discrete group. If the action $\alpha$ is stable (See Definition 3.1 of Jones–Schmidt [24]), then the factor $M := D \rtimes_{\alpha*\alpha} (G * G)$ admits Rohlin flows. This is shown by the argument similar to (2) ⇒ (1) of Theorem 3.17. In particular, by Corollary 5.8 of Ueda [62], for any $\lambda \in [0, 1]$, there exists a type $\mathrm{III}_\lambda$ non-McDuff factor which admits Rohlin flows.

3.3.2. *On classifications.* In this subsubsection, we discuss classification of actions on non-McDuff factors above. It is remarkable that Theorem 3.1 holds for actions on any separable factors. Hence it is natural to apply the theorem to the actions constructed above.

**Lemma 3.21.** (Lemma 2.1 of Popa [49], Theorem 5 of Ueda [63])

*Let* $M = A *_D B$, $\mu$, $E_A$, $E_B$, $E$ *be as above. Let* $x \in M^\omega$ *and let* $v$, $w$ *be unitaries of* $A$ *with* $\mu \circ E_A(u^* \cdot u) = \mu \circ E_A = \mu \circ E_A(v^* \cdot v)$. *Assume that* $E_A(v^n) = 0$, $E_A(w^n) = 0$ $(n \neq 0)$, $vDv^* = D = wDw^*$, $x = vxw^*$. *Then for* $y_1, y_2 \in \ker E_B$, *we have*

$$\| y_1x - xy_2 \|^2_{(\mu\circ E)^\omega} \geq \| y_1(x - E^\omega(x)) \|^2_{(\mu\circ E)^\omega} + \| (x - E^\omega(x))y_2 \|^2_{(\mu\circ E)^\omega},$$

*where* $(\mu \circ E)^\omega : M^\omega \to \mathbf{C}$, $E^\omega : M^\omega \to D^\omega$ *are maps induced by* $\mu \circ E$ *and* $E$, *respectively* (see subsection 2.2 of Ueda [63]) *and* $\| x \|_{(\mu\circ E)^\omega} = ((\mu \circ E)^\omega(x^*x))^{1/2}$ *for* $x \in M^\omega$.

By using this lemma, it is possible to show the following lemma, which is crucial to investigate the approximate innerness of flows. Let $M = A *_D A$ be the type $\mathrm{II}_1$ amalgamated free product factor considered in this subsection.

**Lemma 3.22.** *Let* $\theta$ *be an automorphism of* $M = A *_D A$ *which globally preserves* $D$ *and satisfies* $\theta(\lambda_a) = u^1\lambda_a$, $\theta(\lambda_b) = u^2\lambda_b$ *for some* $u^1$, $u^2 \in U(D)$. *Then the automorphism* $\theta$ *is approximately inner if and only if* $\theta|_D = \mathrm{id}$, $u^1 = u^2$.

*Proof.* This is shown in the proof of Theorem 14 of Ueda [63] in a more general setting. Here we give a proof briefly.

First, we show the "only if" part. Assume that $\theta$ is approximately inner. Then there exists a unitary $\{u_n\}$ of $M^\omega$ such that $\theta(y) = \text{strong-}\lim_{n\to\omega} u_n^* y u_n$ for $y \in M$. Then by using Lemma 3.21 for $v = \lambda_a$, $w = {u^1}^*\lambda_a$, $y_1 = \lambda_b$, $y_2 = {u^2}^*\lambda_b$, $x = \{u_n\}$, we have $\{u_n\} - E^\omega(\{u_n\}) = 0$. Hence we have $\{u_n\} \in D^\omega$, which implies that $\theta|_D = \mathrm{id}$, and we have

$$u^1 = \theta(\lambda_a)\lambda_a^* = \lim_{n\to\omega} u_n\lambda_a u_n^*\lambda_a^* = \lim_{n\to\omega} u_n\alpha(u_n^*) = \lim_{n\to\omega} u_n\lambda_b^* u_n^*\lambda_b = u^2.$$

Next, we show the "if" part. Assume that $\theta|_D = \mathrm{id}$, $u^1 = u^2$. We construct a sequence $\{u_n\}$ of unitaries of $D$ such that $u_n\alpha(u_n^*) \to u^1$. By using the Rohlin lemma for $\alpha$, there exists a partition $\{e_k\}_{k=0}^n \subset \mathrm{Proj}(D)$ of unity in $D$ such that

$$\alpha(e_k) = e_{k+1} \text{ for } k = 1, \cdots, n-1,\ \mu(e_0) < 1/(n+1).$$

Set

$$u_n = \sum_{k=0}^{n} v_k e_k, v_1 = u^1, v_{k+1} = \alpha(v_k)u^1.$$

for $k = 1, \cdots, n-1$. Note that

$$v_k e_k \alpha(e_l v_l^*) = v_k v_{l+1}^* u^1 e_k e_{l+1} = \delta_{k,l+1} u^1 e_k$$

for $k, l = 1, \cdots n-1$. Hence by a similar computation to the one in the proof of Theorem 3.17, we have $u_n \alpha(u_n^*) \to u^1$. Hence we have

$$u_n \lambda_a u_n^* \to u^1 \lambda_a = \theta(\lambda_a),$$
$$u_n \lambda_b u_n^* \to u^1 \lambda_b = u^2 \lambda_b = \theta(\lambda_b),$$

which implies that $\mathrm{Ad} u_n(y) \to \theta(y)$ strongly for $y \in M$. □

**Lemma 3.23.** *The Rohlin flows constructed in Theorem 3.17 are completely classified by* $\{\theta|_D, u_t^1 u_t^{2*}\}$*, up to strong cocycle conjugacy.*

*Proof.* This lemma immediately follows from Theorem 3.1 and Lemma 3.22. □

**Example 3.24.** The Rohlin flows considered in Example 3.19 are completely classified by $(p, q, \lambda - \mu)$, up to strong cocycle conjugacy.

However, we are mainly interested in classifying actions up to usual cocycle conjugacy. Being mutually strongly cocycle conjugate is just a sufficient condition for being mutually cocycle conjugate. When the factor is approximately finite dimensional, then the difference of these classifications does not cause any problem because we can describe how far from being approximately inner an automorphism is (See Theorem 1 of Kawahigashi–Sutherland–Takesaki [31]). However, we will see that when the factor is not approximately finite dimensional, these two classifications are completely different.

**Theorem 3.25.** *For Rohlin flows in Example 3.19, usual cocycle conjugacy and strong cocycle conjugacy are different.*

The following lemma is an essential part of Theorem 3.25. Recall that the discrete spectrum $\mathrm{Sp}_d(\theta)$ of a flow $\theta$ on a von Neumann algebra $M$ is the set

$$\mathrm{Sp}_d(\theta) := \{p \in \mathbf{R} \mid \text{there exists } x \in M \backslash \{0\} \text{ with } \theta_t(x) = e^{ipt} x \text{ for } t \in \mathbf{R}\}.$$

**Lemma 3.26.** *Let* $\theta^{\lambda_1, \mu_1, p_1, q_1}$, $\theta^{\lambda_2, \mu_2, p_2, q_2}$ *be two Rohlin flows mentioned in Example 3.19. Then they are cocycle conjugate if there exist* $r \in \boldsymbol{R}$ *and two points* $c$, $d$ *of* $\mathrm{Sp}_d(\theta^{\lambda_1, \mu_1, p_1, q_1}|_D)$ *such that one of the following conditions holds.*

(1) *We have* $(p_1, q_1) = (p_2, q_2)$ *and*

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} + \begin{pmatrix} r \\ r \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}.$$

(2) *We have* $(p_1, q_1) = -(p_2, q_2)$ *and*

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} + \begin{pmatrix} r \\ r \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}.$$

*Proof.* Assume that one of the above conditions holds. First, consider the case when condition (2) holds. Let $\sigma$ be an automorphism of $D$ defined by

$$\sigma(f)(s,t) := f(-s,-t)$$

for $f \in D$, $(s,t) \in \mathbf{T}^2$. We show the following claim.

**Claim.** The automorphism $\sigma$ extends to an automorphism of $M$ by

$$\sigma(\lambda_a) = \lambda_{b^{-1}}, \qquad \sigma(\lambda_b) = \lambda_{ab^{-2}}.$$

*Proof of Claim.* Set an automorphism $\beta$ on $D$ by $\beta := \alpha^{-1}$. Then we have $\sigma \circ \beta \circ \sigma^{-1} = \alpha$. Hence by Lemma 7.5 of Takesaki [59], there exists an isomorphism $\pi_A : D \rtimes_{\alpha * \alpha} \{a\}^{\mathbf{Z}} \cong D \rtimes_\alpha \mathbf{Z} \to D \rtimes_\beta \mathbf{Z} \cong D \rtimes_{\alpha*\alpha} \{b^{-1}\}^{\mathbf{Z}}$ satisfying

$$D \ni f \mapsto \sigma(f),\ \lambda_a \mapsto \lambda_{b^{-1}}.$$

Similarly, there exists an isomorphism $\pi_B : D \rtimes_{\alpha*\alpha} \{b\}^{\mathbf{Z}} \cong D \rtimes_\alpha \mathbf{Z} \to D \rtimes_\beta \mathbf{Z} \cong D \rtimes_{\alpha*\alpha} \{ab^{-2}\}^{\mathbf{Z}}$ satisfying

$$D \ni f \mapsto \sigma(f),\ \lambda_b \mapsto \lambda_{ab^{-2}}.$$

Note that the endomorphism $\rho$ of $\mathbf{F}_2$ defined by $a \mapsto b^{-1}$, $b \mapsto ab^{-2}$ is bijective. The inverse is given by $a \mapsto ba^{-2}$, $b \mapsto a^{-1}$. By the injectivity of $\rho$, the images of $\pi_A$ and $\pi_B$ are free over $D$. By this observation,by the uniqueness of the amalgamated free product, the automorphism $\sigma$ extends to an automorphism of $M$. $\square$

Now we continue the proof of the lemma. Since we have

$$\sigma^{-1} \circ \theta^{\lambda_1,\mu_1,p_1,q_1} \circ \sigma = \theta^{-\mu_1,\lambda_1-2\mu_1,-p_1,-q_1},$$

by replacing $\theta^{\lambda_1,\mu_1,p_1,q_1}$ by $\sigma^{-1} \circ \theta^{\lambda_1,\mu_1,p_1,q_1} \circ \sigma$, it is enough to consider the case when condition (1) holds. Assume that condition (1) holds. Since $c \in \mathrm{Sp}_d(\theta^{\lambda_1,\mu_1,p_1,q_1}\mid_D)$, there exists $u \in D$ such that $\| u \| = 1$ and $\theta_t^{\lambda_1,\mu_1,p_1,q_1}(u) = e^{ict}u$ for $t \in \mathbf{R}$. Since $u^*u (= uu^*)$ is fixed by $\theta^{\lambda_1,\mu_1,p_1,q_1}$, $u^*u = uu^* = 1$ by the ergodicity of $\theta^{\lambda_1,\mu_1,p_1,q_1}\mid_D$. Similarly, there exists a unitary $v$ of $D$ with $\theta_t^{\lambda_1,\mu_1,p_1,q_1}(v) = e^{idt}v$ for $t \in \mathbf{R}$.

Then the identity map $\sigma$ of $D$ extends to $M$ by $\sigma(\lambda_a) = u\lambda_a$, $\sigma(\lambda_b) = v\lambda_b$. By replacing $\theta^{\lambda_1,\mu_1,p_1,q_1}$ by $\sigma^{-1} \circ \theta^{\lambda_1,\mu_1,p_1,q_1} \circ \sigma$, we may assume that $c = d = 0$. Hence by using Example 3.24, $\theta^{\lambda_1,\mu_1,p_1,q_1}$ and $\theta^{\lambda_2,\mu_2,p_2,q_2}$ are cocycle conjugate. $\square$

Now, we return to the proof of Theorem 3.25.

*Proof of Theorem 3.25.* Let $\theta^{\lambda_1,\mu_1,p_1,q_1}$ and $\theta^{\lambda_2,\mu_2,p_2,q_2}$ be two Rohlin flows considered in Example 3.19. Then by Example 3.24, they are strongly cocycle conjugate if and only if $\lambda_1 - \mu_1 = \lambda_2 - \mu_2$, $p_1 = p_2$

and $q_1 = q_2$. On the other hand, by Lemma 3.26, they are cocycle conjugate if $(p_2, q_2) = (-p_1, -q_1)$ and $(\lambda_2, \mu_2) = (-\mu_1, \lambda_1 - 2\mu_1)$. $\square$

## 4. A sufficient condition of actions of $\mathbf{R}$ on AFD factors of type III for the Rohlin property

### 4.1. **The main theorems of Section 4.** The main theorem of this paper is the following.

**Theorem 4.1.** *A flow on any AFD factor with faithful Connes–Takesaki module has the Rohlin property.*

As we have explained in Subsection 2.5, Connes–Takesaki module indicates how far from being approximately iner an automorphism is. Hence this theorem means that a kind of "pointwise outerness" implies "global outerness".

As a corollary, we obtain a classification theorem up to cocycle conjugacy. For a von Neumann algebra $C$ and a flow $\beta$ of $C$, set $\mathrm{Aut}_\beta(C) := \{\sigma \in \mathrm{Aut}(C) \mid \sigma \circ \beta_t = \beta_t \circ \sigma,\ t \in \mathbf{R}\}$. By Theorem 4.1, Theorem 3.2, and the characterization of approximate innerness of automorphisms of AFD factors (Theorem 1 of Kawahigashi–Sutherland–Takesaki [31]), we have the following.

**Corollary 4.2.** *Let $\alpha^1$ and $\alpha^2$ be two flows on an AFD factor $M$ with faithful Connes–Takesaki modules. Then they are cocycle conjugate if and only if there exists an automorphism $\sigma \in \mathrm{Aut}_\theta(C)$ with $\mathrm{mod}(\alpha^2_t) = \sigma \circ \mathrm{mod}(\alpha^1_t) \circ \sigma^{-1}$ for any $t \in \mathbf{R}$.*

As an obvious application, we have the following example.

**Example 4.3.** A flow on any AFD factor with faithful Connes–Takesaki module absorbs any flow on the AFD $\mathrm{II}_1$ factor, as a tensor product factor.

As we have explained in the introduction, characterization of the Rohlin property is an important problem (Conjecture 8.3 of Masuda–Tomatsu [44]). Theorem 4.1 gives a partial answer to this problem. We will proceed further to this direction in Subsection 4.3.3.

### 4.2. **The proof of the main theorem of Section 4.** In this subsection, we show Theorem 4.1. In order to achieve this, we first note that we may assume that a flow has an invariant weight. This is seen in the following way. Let $\alpha$ be a flow on an AFD factor $M$. Then by the same argument as in Lemma 5.10 of Sutherland–Takesaki [55] (or equivalently, by the combination of Lemma 5.11 and Lemma 5.12 of

[55]), there exists a flow $\beta$ and a dominant weight $\phi$ which satisfy the following conditions.

(1) We have $\phi \circ \beta_t = \phi$ for all $t \in \mathbf{R}$.
(2) The action $\beta$ is cocycle conjugate to $\alpha \otimes \mathrm{id}_{B(L^2\mathbf{R})}$.

By Lemma 2.11 of Connes [8], $(M \otimes B(L^2\mathbf{R}))_\omega = M_\omega \otimes \mathbf{C}$. Hence, by replacing $\alpha$ by $\beta$, we may assume that the action $\alpha$ has an invariant dominant weight. In the rest of the section, we denote the continuous core $M \rtimes_{\sigma^\phi} \mathbf{R}$ by $N$ and the dual action of $\sigma^\phi$ by $\theta$. Then by the same argument as in the proof of Proposition 13.1 of Haagerup–Størmer [20], the action $\tilde{\alpha}$ extends to a flow $\tilde{\tilde{\alpha}}$ of $N \rtimes_\theta \mathbf{R}$ so that if we identify $N \rtimes_\theta \mathbf{R}$ with $M \otimes B(L^2\mathbf{R})$ by Takesaki's duality, $\tilde{\tilde{\alpha}}$ corresponds to $\alpha \otimes \mathrm{id}$. By Lemma 2.11 of Connes [8] again, in order to show that $\alpha$ has the Rohlin property, it is enough to show that $\tilde{\tilde{\alpha}}$ has the Rohlin property. In order to achieve this, we need to choose $\{u_n\} \subset U(M \otimes B(L^2\mathbf{R}))_\omega$ which satisfies the conditions in the definition of the Rohlin property. Our strategy is to choose $\{u_n\}$ from $N$. Based on this strategy, it is sufficient to show the following lemma.

**Lemma 4.4.** *For each $p \in \mathbf{R}$, there exists a sequence $\{u_n\} \subset U(N)$ satisfying the following conditions.*
(1) *We have $\|[u_n, \phi]\| \to 0$ for any $\phi \in N_*$.*
(2) *We have $\theta_s(u_n) - u_n \to 0$ compact uniformly for $s \in \mathbf{R}$ in the strong* topology.*
(3) *We have $\tilde{\alpha}_t(u_n) - e^{ipt}u_n \to 0$ compact uniformly for $t \in \mathbf{R}$ in the strong* topology.*

By the first two conditions, this $\{u_n\}$ asymptotically commutes with elements in a dense subspace of $M \otimes B(L^2\mathbf{R}) \cong M$. However, in general, this does not imply that $\{u_n\}$ is centralizing (and this sometimes causes a serious problem). Hence, in order to assure that Lemma 4.4 implies Theorem 4.1, we need to show the following lemma.

**Lemma 4.5.** *Let $M$ be an AFD factor of type* III *and let $M = N \rtimes_\theta \mathbf{R}$ be the continuous decomposition. Then a sequence $\{u_n\} \subset U(N)$ with conditions* (1) *and* (2) *of the above lemma is centralizing.*

*Proof.* Let $H$ be the standard Hilbert space of $N$. Take $\xi \in H$ and $f \in L^2(\mathbf{R})$. Since

$$x(\xi \otimes f)(s) = (\theta_{-s}(x)\xi)f(s),$$

$$
\begin{aligned}
(\xi \otimes f)x(s) &= (J_M x^* J_M(\xi \otimes f))(s) \\
&= (J_N x^* J_N \xi) f(s) \\
&= (\xi x) f(s)
\end{aligned}
$$

for $s \in \mathbf{R}$, $x \in N$, we have

$$
\begin{aligned}
\|u_n(\xi \otimes f) - (\xi \otimes f)u_n\|^2 &= \int_{\mathbf{R}} \|\theta_{-s}(u_n)\xi - \xi u_n\|^2 |f(s)|^2 \, ds \\
&\leq \int_{\mathbf{R}} \|(\theta_{-s}(u_n) - u_n)\xi\|^2 |f(s)|^2 \, ds \\
&+ \int_{\mathbf{R}} \|u_n \xi - \xi u_n\|^2 |f(s)|^2 \, ds \\
&\to 0
\end{aligned}
$$

by Lebesgue's convergence theorem. Here, the convergence of the second term follows from Lemma 2.6 of Masuda–Tomatsu [44]. Any vector of $H \otimes L^2\mathbf{R}$ is approximated by finite sums of vectors of the form $\xi \otimes f$. Hence for any vector $\eta \in H \otimes L^2\mathbf{R}$, we have $\|u_n \eta - \eta u_n\| \to 0$. Hence $\{u_n\}$ is centralizing. □

By this lemma, Lemma 4.4 implies Theorem 4.1. In the following, we will show Lemma 4.4. If $M$ is of type $\mathrm{II}_\infty$, Lemma 4.4 is shown in Theorem 6.18 of Masuda–Tomatsu [44], using Connes and Haagerup's theory. If $M$ is of type $\mathrm{II}_1$ or is of type $\mathrm{III}_1$, then we need not do anything because Connes–Takesaki modules of automorphisms are always trivial. Hence we only need to consider the case when $M$ is of type $\mathrm{III}_0$ and the case when $M$ is of type $\mathrm{III}_\lambda$ $(0 < \lambda < 1)$. Actually, as we will see in Remark 4.16, if $M$ is of type $\mathrm{III}_\lambda$ $(0 < \lambda < 1)$, the Connes–Takesaki module of a flow cannot be faithful. Hence, the only problem is how to handle the case when $M$ is of type $\mathrm{III}_0$.

Let $C$ be the center of $N$. First, we list up the form of the kernel of the action $\mathrm{mod}(\alpha) \circ (\theta\ |_C)$ of $\mathbf{R}^2$ on $C$. This is a closed subgroup of $\mathbf{R}^2$. Thus the kernel must be isomorphic to one of the following groups.

$$
0, \ \mathbf{Z}, \ \mathbf{Z}^2, \ \mathbf{R}, \ \mathbf{R} \times \mathbf{Z}, \ \mathbf{R}^2.
$$

However, since $\theta\ |_C$ is faithful, the kernel cannot be isomorphic to $\mathbf{R} \times \mathbf{Z}$ or $\mathbf{R}^2$. We handle the other four cases separately.

We first consider the case when $\ker(\mathrm{mod}(\alpha) \circ (\theta\ |_C)) = 0$. In this case, by an argument similar to that of the proof of Theorem 3.3 of Shimada [54], Lemma 4.4 follows from a Rohlin type theorem due to Feldman [13]. In the following, we will explain this theorem.

**Settings.** A subset $Q$ of $\mathbf{R}^d$ is said to be a cube if $Q$ is of the form

$$[-s_1, t_1] \times \cdots \times [-s_d, t_d]$$

for some $s_1, \cdots, s_d, t_1, \cdots, t_d > 0$. Let $Q$ be a cube of $\mathbf{R}^d$ and $T$ be a non-singular action of $\mathbf{R}^d$ on a Lebesgue space $(X, \mu)$. Then a measurable subset $F$ of $X$ is said to be a $Q$-set if $F$ satisfies the following two conditions.

(1) The map $Q \times F \ni (t, x) \mapsto T_t(x) \in X$ is injective.
(2) The set $T_Q F := \{T_t(x) \mid t \in Q, x \in F\}$ is measurable and non-null.

In this setting, the following theorem holds.

**Theorem 4.6.** (p.410 of Feldman [13], Thoerem 1 of Feldman–Lind [16]) *Let $T$ be a free non-singular action of $\mathbf{R}^d$ on the standard probability space $(X, \mu)$. Then for any $\epsilon > 0$ and for any cube $P$ of $\mathbf{R}^d$, there exists a large cube $Q$ and a $Q$-set $F$ of $X$ with*

$$\mu(T_{\bigcap_{t \in P}(t+Q)} F) > 1 - \epsilon.$$

The proof is written in Feldman [14]. However, his paper is privately circulated. Hence we explain the outline of the proof in Appendix of this section (Subsection 4.4), which is based on Theorem 1 of Feldman–Lind [16] and Lind [37]. As written in the proof of Theorem 1.1 (a) of Feldman [13] (p.410 of Feldman [13]), it is possible to introduce a measure $\nu$ on $F$ so that the map $Q \times F \ni (t, x) \mapsto T_t(x) \in T_Q F$ is a non-singular isomorphism. The measure $\nu$ is defined in the following way. Set

$$\mathcal{M} := \{A \subset F \mid T_Q(A) \text{ is measurable with respect to } \mu\}.$$

Then $\mathcal{M}$ is a $\sigma$-algebra of $F$ and it is possible to define a measure $\nu$ on $F$ by

$$\nu(A) := \frac{\mu(T_Q A)}{\mu(T_Q F)}$$

for $A \in \mathcal{M}$. Then the map $(t, x) \mapsto T_t(x)$ is a non-singular isomorphism with respect to Lebesgue $\otimes\, \nu$ and $\mu|_{T_Q F}$. These things are written in p.410 of Feldman [13] and the proof may be written in Feldman–Hahn–Moore [15]. In this paper, for reader's convenience, we present a proof of what we will use (Propositions 4.30 and 4.31 of Subsection 4.4).

**Lemma 4.7.** *When* $\ker(\mathrm{mod}(\alpha) \circ (\theta|_C))$ *is zero, Lemma 4.4 holds.*

*Proof.* Think of $C$ as $L^\infty(X,\mu)$ for some probability measured space $(X,\mu)$. Let $T$ be an action of $\mathbf{R}^2$ defined by the following way.

$$f \circ T_{(s,t)} = \theta_{-s} \circ \tilde{\alpha}_{-t}(f)$$

for $f \in L^\infty(X,\mu)$, $(s,t) \in \mathbf{R}^2$. Fix a natural number $n \in \mathbf{N}$. Set $P := [-n,n]^2$. Then by Theorem 4.6, there exists a large cube $Q$ and a $Q$-set $F$ of $X$ with

$$\mu(T_{\bigcap_{t\in P}(t+Q)}F) > 1 - \frac{1}{n}.$$

Define a function $u_n$ on $X$ by the following way.

$$u_n = \begin{cases} e^{-ipt} & (x = T_{(s,t)}(y),\ (s,t) \in Q,\ y \in F) \\ 1 & (otherwise). \end{cases}$$

Then by Proposition 4.30, the function $u_n$ is Borel measurable. Then for $x \in T_{\bigcap_{t\in P}(t+Q)}F$ and $(s,t) \in P$, we have

$$\theta_s(u_n)(x) = u_n(x),$$

$$\tilde{\alpha}_t(u_n)(x) = e^{ipt}u_n(x).$$

Hence we have

$$\begin{aligned}\|\theta_s(u_n) - u_n\|_\mu^2 &\leq 4\mu(X \setminus T_{\bigcap_{t\in P}(t+Q)}F) \\ &\leq \frac{4}{n+1}\end{aligned}$$

for $s \in [-n,n]$. By the same computation, we have

$$\|\tilde{\alpha}_t(u_n) - e^{ipt}u_n\|_\mu^2 \leq \frac{4}{n+1}$$

for $t \in [-n,n]$. Hence the sequence $\{u_n\}$ of unitaries of $C$ satisfies the conditions in Lemma 4.4. □

Next, we consider the following case.

**Lemma 4.8.** *When* $\ker(\mathrm{mod}(\alpha) \circ (\theta\mid_C))$ *is isomorphic to* $\mathbf{Z}^2$*, Lemma 4.4 holds.*

In this case, there exist two pairs $(p_1,q_1)$, $(p_2,q_2)$ of non-zero real numbers with $\ker(\mathrm{mod} \circ \theta) = \mathbf{Z}(p_1,q_1) \oplus \mathbf{Z}(p_2,q_2)$. Here, we use our assumption that $\mathrm{mod}(\alpha)$ is faithful for showing $q_i \neq 0$. Set $\sigma_t := \theta_{q_1 t} \circ \tilde{\alpha}_{p_1 t}$. In order to show Lemma 4.8, it is enough to show the following lemma.

**Lemma 4.9.** *For each $r \in \mathbf{R}$, there exists a sequence of unitaries $\{u_n\}$ of $N$ which satisfies the following conditions.*

(1) *We have $\|[u_n, \phi]\| \to 0$ for any $\phi \in N_*$.*

(2) *We have $\theta_s(u_n) - u_n \to 0$ compact uniformly for $s \in \mathbf{R}$ in the strong* topology.*

(3) *We have $\sigma_t(u_n) - e^{irt}u_n \to 0$ compact uniformly for $t \in \mathbf{R}$ in the strong* topology.*

In order to show this lemma, we need to prepare some lemmas.

**Lemma 4.10.** *The action $\theta$ on $C^\sigma$ is ergodic and has a period $p \in (0, \infty)$.*

*Proof.* Ergodicity follows from the ergodicity of $\theta : \mathbf{R} \curvearrowright C$. We show that the restriction of $\theta$ on $C^\sigma$ has a period.

We first note that a Borel measurable map $T$ from $\mathbf{T}$ to itself which commutes with every translations of the torus must be a translation because we have $T(\gamma+t)-t = T(\gamma)$ for $t \in \mathbf{T}$ and for almost all $\gamma \in \mathbf{T}$. Now, we show that $C^\sigma \neq \mathbf{C}$. Assume that $C^\sigma$ were isomorphic to $\mathbf{C}$. Then since $\theta$ would commute with $\sigma$, which is a translation flow on the torus. Hence $\theta$ would be also a translation on the torus. Hence $\theta \circ \sigma$ would define a group homomorphism from $\mathbf{R}^2$ to the group of translations of the torus, which is isomorphic to $\mathbf{T}$. Hence the kernel of $\theta \circ \sigma$ would be isomorphic to $\mathbf{R} \times \mathbf{Z}$, which would contradict to the faithfulness of $\theta$. Combining this with the ergodicity of $\theta$, we have $\theta |_{C^\sigma}$ is non-trivial. Since $\mathrm{mod}(\alpha_{p_2}) = \theta_{-q_2} |_C$, we have $(\sigma_{p_2/p_1} \circ \theta_{q_2 - p_2 q_1/p_1}) |_C = \mathrm{id}_C$. Since $(p_1, q_1)$ and $(p_2, q_2)$ are independent, this $\theta |_{C^\sigma}$ has a non-trivial period. □

By this lemma, we may assume the following.

(1) We have $C^\sigma = L^\infty(\mathbf{T}_p)$, where $\mathbf{T}_p$ is the torus of length $p$, which is isomorphic to $[0, p)$ as a measured space.

(2) We have $\theta_t(f) = f(\cdot - t)$ for $f \in L^\infty(\mathbf{T}_p)$, $t \in \mathbf{R}$.

Let

$$N = \int_{[0,p)}^{\oplus} N_\gamma \, d\gamma$$

be the direct integral decomposition of $N$. For $\gamma_1, \gamma_2 \in \mathbf{R}$, $N_{\gamma_1}$ and $N_{\gamma_2}$ are mutually isomorphic by the following map.

$$\theta_{\gamma_2 - \gamma_1, \gamma_1} : N_{\gamma_1} \to N_{\gamma_2},$$

$$\theta_{\gamma_2 - \gamma_1, \gamma_1}(x_{\gamma_1}) = (\theta_{\gamma_2 - \gamma_1}(x))_{\gamma_2}$$

for $x = \int_{[0,p)}^{\oplus} x_\gamma d\gamma \in N$. These $\theta_{\gamma_1,\gamma_2}$'s satisfy the following two conditions.

**Conditions.**
(1) The equality $\theta_{0,\gamma} = \mathrm{id}_{N_\gamma}$ holds for each $\gamma \in [0,p)$.
(2) The equality $\theta_{\gamma_3-\gamma_2,\gamma_2} \circ \theta_{\gamma_2-\gamma_1,\gamma_1} = \theta_{\gamma_3-\gamma_1,\gamma_1}$ holds for each $\gamma_1, \gamma_2, \gamma_3 \in \mathbf{R}$.

By these $\theta_{\gamma_1,\gamma_2}$'s, all $N_\gamma$'s are mutually isomorphic. Thus it is possible to think of $N$ as $N_0 \otimes L^\infty([0,p))$.

Now, we need to consider the measurability of $\theta_{t,\gamma}$.

**Fact.**
If we identify $N$ with $N_0 \otimes L^\infty([0,p))$, the map $[0,p)^2 \ni (t,\gamma) \mapsto \theta_{t,\gamma} \in \mathrm{Aut}(N_0)$ is Lebesgue measurable.

Although this fact is probably well-known to specialists, for the reader's convenience, we present the proof in Appendix of this section (Subsection 4.4).

By measurability of $\theta_{t,\gamma}$, Lusin's theorem and Fubini's theorem, for almost all $\gamma \in [0,p)$, the map $t \mapsto \theta_{-t,t+\gamma}$ and $t \mapsto \theta_{t,\gamma}$ are also Lebesgue measurable. We may assume that $\gamma = 0$ and we identify $N_{\gamma_1}$ with $N_0$ by $\theta_{\gamma_1,0}$ for all $\gamma_1 \in [0,p)$, that is, if we think of $N$ as the set of all essentially bounded weak * Borel measurable maps from $[0,p)$ to $N_0$, then the set of constant functions is the following set.

$$\{\int_{[0,p)}^{\oplus} \theta_{\gamma,0}(x_0)\ d\gamma \mid x_0 \in N_0\}.$$

Take a normal faithful state $\phi_0$ of $N_0$. Then

$$\phi := \frac{1}{p} \int_{[0,p)}^{\oplus} \phi_0 \circ \theta_{-\gamma,\gamma}\ d\gamma$$

is a normal faithful state on $N$. Choose $\phi_1, \cdots, \phi_n \in N_*$, $\epsilon > 0$ and $T > 0$. Then by the above identification of $N_*$ with $L^1_{(N_0)_*}([0,p))$, each $\phi_k$ is a Lebesgue measurable map from $[0,p)$ to $(N_0)_*$. Hence it is possible to approximate each $\phi_k$ by Borel simple step functions by the following way.

$$\|\phi_k - \sum_{i=1}^{l_k} \phi_{k,i} \circ \theta_{-\gamma,\gamma} \chi_{I_i}(\gamma)\| < \epsilon.$$

for each $k \in \{1, \cdots, n\}$. Here, $\phi_{k,i} \in N_{0*}$ for $i = 1, \cdots, l_k$, $\{I_i\}_{i=1}^{l_k}$ is a Borel partition of $[0, p)$. Next, we look at actions on $N_0$. Let

$$\theta_p = \int_{[0,p)}^{\oplus} \theta_p^\gamma \, d\gamma,$$

$$\sigma_t = \int_{[0,p)}^{\oplus} \sigma_t^\gamma \, d\gamma,$$

$$\tau = \int_{[0,p)}^{\oplus} \tau_\gamma \, d\gamma$$

be the direct integral decompositions. Since $\theta$ is trace-scaling and $\tilde{\alpha}$ is trace-preserving, $\sigma$ is trace-scaling. Hence for almost all $\gamma \in [0, p)$, $\sigma^\gamma$ is $\tau_\gamma$ -scaling. Thus we may assume that $\sigma^0$ is $\tau_0$-scaling. In order to show Lemma 4.9, it is enough to show the following lemma.

**Lemma 4.11.** *In the above context, for real number $r \in \mathbf{R}$, there exists a unitary $u_0$ of $N_0$ which satisfies the following conditions.*

(1) *We have $\|[u_0, \phi_{k,i}]\| < \epsilon/(pl_k)$ for $k = 1, \cdots, n$, $i = 1, \cdots, l_k$.*

(2) *We have $\|\theta^0_{mp}(u_0) - u_0\|^\sharp_{\phi_0} < \epsilon/p$ for $m \in \mathbf{Z}$, $|m| \leq p/T + 2$.*

(3) *We have $\|\sigma^0_t(u_0) - e^{-irt}u_0\|^\sharp_{\phi_0} < \epsilon/p$ for all $t \in [-T, T]$.*

First, we show that Lemma 4.11 implies Lemma 4.9.

*Proof of Lemma 4.11 ⇒ Lemma 4.9.* Assume that there exists a unitary $u_0$ in $N_0$ which satisfies the conditions in Lemma 4.11. We set

$$u_\gamma := \theta_{\gamma,0}(u_0),$$

$$u := \int_{[0,p)}^{\oplus} u_\gamma \, d\gamma.$$

Fix $t \in [-T, T]$ and $\gamma \in [0, p)$. For each $\gamma \in [0, p)$, choose $m_\gamma \in \mathbf{Z}$ so that $-t + m_\gamma p + \gamma \in [0, p)$. Then we have

$$\begin{aligned}
(\theta_t(u))_\gamma &= \theta_{t,-t+\gamma}(u_{-t+\gamma}) \\
&= \theta_{t,-t+\gamma}(u_{-t+\gamma+m_\gamma p}) \\
&= \theta_{\gamma,0} \circ \theta^0_{m_\gamma p} \circ \theta_{t-m_\gamma p-\gamma,-t+m_\gamma p+\gamma}(u_{-t+m_\gamma p+\gamma}) \\
&= \theta_{\gamma,0} \circ \theta^0_{m_\gamma p}(u_0).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|\theta_t(u) - u\|_\phi^\sharp &= \int_{[0,p)} \|(\theta_t(u))_\gamma - u_\gamma\|_{\phi_0 \circ \theta_{-\gamma,\gamma}}^\sharp \ d\gamma \\
&= \int_{[0,p)} \|\theta_{\gamma,0} \circ \theta^0_{m_\gamma p}(u_0) - \theta_{\gamma,0}(u_0)\|_{\phi_0 \circ \theta_{-\gamma,\gamma}}^\sharp \ d\gamma \\
&= \int_{[0,p)} \|\theta^0_{m_\gamma p}(u_0) - u_0\|_{\phi_0}^\sharp \ d\gamma \\
&< \int_{[0,p)} \frac{\epsilon}{p} \ d\gamma \\
&= \epsilon.
\end{aligned}$$

Here we use that $|m_\gamma| \leq T/p + 2$ in the fourth inequality of the above estimation. By the same argument, we also have

$$\|\sigma_t(u) - e^{-irt}u\|_\phi^\sharp < \epsilon$$

for $t \in [-T, T]$. We also have

$$\begin{aligned}
\|[u, \phi_k]\| &\leq 2\|\phi_k - \sum_{i=1}^{l_k} \phi_{k,i} \otimes \chi_{I_i}\| + \|[u, \sum_{i=1}^{l_k} \phi_{k,i} \otimes \chi_{I_i}]\| \\
&< 2\epsilon + \sum_{i=1}^{l_k} \|[u, \phi_{k,i} \otimes \mathrm{id}]\| \\
&= 2\epsilon + \sum_{i=1}^{l_k} \int_{[0,p)} \|[\theta_{\gamma,0}(u_0), \phi_{k,i} \circ \theta_{-\gamma,\gamma}]\| \ d\gamma \\
&= 2\epsilon + \sum_{i=1}^{l_k} \int_{[0,p)} \|[u_0, \phi_{k,i}]\| \ d\gamma \\
&< 2\epsilon + \sum_{i=1}^{l_k} \int_{[0,p)} \frac{\epsilon}{pl_k} \ d\gamma \\
&= 3\epsilon.
\end{aligned}$$

Thus Lemma 4.9 holds. □

In order to prove Lemma 4.11, we first rewrite the lemma in a simpler form. To do this, we show that there exists a number $s \in (0, 1)$ with $(\theta^0_p \circ \sigma^0_s) |_{Z(N_0)} = \mathrm{id}$. Since the restriction of $\sigma^0$ on the center of $N_0$ has a period 1 and is ergodic, we may assume that $Z(N_0)$ is isomorphic to $L^\infty([0, 1))$, which is canonically identified with $L^\infty(\mathbf{T})$, $\sigma^0_s(f) = f(\cdot - s)$ for $s \in \mathbf{R}$, $f \in L^\infty(\mathbf{T})$. By this identification, $\theta^0_p$ commutes with all

$\sigma_s^0$'s. Hence $\theta^0$ is a translation on the torus. Thus there exists a unique $s \in (0,1)$ with $(\theta_p^0 \circ \sigma_s^0) \mid_{Z(N_0)} = \mathrm{id}$. Set $\beta^0 := \theta_p^0 \circ \sigma_s^0$. The proof of Lemma 4.11 reduces to that of the following lemma.

**Lemma 4.12.** *The action $\{\beta_m^0 \circ \sigma_t^0\}_{(m,t)}$ of $\mathbf{Z} \times \mathbf{R}$ on $N_0$ has the Rohlin property.*

*Proof of Lemma 4.12⇒ Lemma 4.11.* Assume that the action $\{\beta_m^0 \circ \sigma_t^0\}_{m,t}$ has the Rohlin property. Then there exists a unitary element $u_0$ of $N_0$ with the following conditions.

(1) We have $\|[u_0, \phi_{k,i}]\| < \epsilon/(pl_k)$ for $k = 1, \cdots, n$, $i = 1, \cdots, l_k$.
(2) We have $\|\beta_m^0(u_0) - e^{-irms}u_0\|_{\phi_0}^\sharp < \epsilon/(2p)$ for $m \in \mathbf{Z}$, $|m| \leq p/T + 2$.
(3) We have $\|\sigma_t^0(u_0) - e^{-irt}u_0\|_{\phi_0 \circ \theta_{mp}^0}^\sharp < \epsilon/(2p)$ for $t \in [-(1+s)(T+2p), (1+s)(T+2p)]$, $m \in \mathbf{Z}$, $|m| \leq p/T + 2$.

Since $\beta_m^0 = \theta_{mp}^0 \circ \sigma_{ms}^0$, we have

$$\begin{aligned}
\|\theta_{mp}^0(u_0) - u_0\| &= \|e^{-irms}\theta_{mp}^0(u_0) - e^{-irms}u_0\|_{\phi_0}^\sharp \\
&\leq \|\theta_{mp}^0(e^{-irms}u_0 - \sigma_{ms}^0(u_0))\|_{\phi_0}^\sharp + \|\beta_m^0(u_0) - e^{-irms}u_0\|_{\phi_0}^\sharp \\
&= \|e^{-irms}u_0 - \sigma_{ms}^0(u_0)\|_{\phi_0 \circ \theta_{mp}^0}^\sharp + \|\beta_m^0(u_0) - e^{-irms}u_0\|_{\phi_0}^\sharp \\
&< \frac{\epsilon}{2p} + \frac{\epsilon}{2p} \\
&= \frac{\epsilon}{p}
\end{aligned}$$

for $m \in \mathbf{Z}$, $|m| \leq p/T + 2$. Thus Lemma 4.11 holds. □

In order to show Lemma 4.12, we need further to reduce the lemma to a simpler statement. Let

$$N_0 = \int_{[0,1)}^{\oplus} (N_0)_\gamma \, d\gamma$$

be the direct integral decomposition of $N_0$ over the center of $N_0$. For each $\gamma_1, \gamma_2 \in [0,1)$, there exists an isomorphism from $(N_0)_{\gamma_1}$ to $(N_0)_{\gamma_2}$ defined by

$$\sigma_{\gamma_2-\gamma_1,\gamma_1}^0((x_0)_{\gamma_1}) = (\sigma_{\gamma_2-\gamma_1}^0(x_0))_{\gamma_2}$$

for $x_0 = \int_{[0,1)}^{\oplus}(x_0)_\gamma d\gamma \in N_0$. These $\sigma_{\gamma_2-\gamma_1,\gamma_1}^0$'s satisfy the conditions similar to conditions (1) and (2) of $\theta_{t,\gamma}$ (See Conditions between Lemma 4.10 and Lemma 4.11).

We identify $(N_0)_\gamma$'s with $(N_0)_0$ by $\sigma^0_{\gamma,0}$. Choose a normal faithful state $\psi_0$ of $(N_0)_0$. Set

$$\psi := \int_{[0,1)}^{\oplus} \psi_0 \circ \sigma^0_{-\gamma,\gamma} \, d\gamma.$$

This is a normal faithful state on $N_0$. Choose $\psi_1, \cdots \psi_n \in (N_0)_*$, $\epsilon > 0$ and $T > 0$. By the same argument as above, we may assume that $\psi_k$'s are simple step Borel functions.

$$\psi_k = \sum_{i=1}^{l_k} \psi_{k,i} \circ \sigma^0_{-\gamma,\gamma} \chi_{I_i}(\gamma)$$

for $k = 1, \cdots, n$. Here, $\psi_{k,i} \in (N_0)_*$, $\{I_i\}_{i=1}^{l_k}$ are partitions of $[0,1)$. Since $\beta^0$ and $\sigma^0$ fix the center of $N_0$, they are decomposed into the following form.

$$\beta^0 = \int_{[0,1)}^{\oplus} \beta^{0,\gamma} \, d\gamma,$$

$$\sigma^0_1 = \int_{[0,1)}^{\oplus} \sigma^{0,\gamma} \, d\gamma.$$

Then for each $\gamma \in [0,1)$, $\{\beta^{0,\gamma}_n \circ \sigma^{0,\gamma}_m\}_{(n,m)\in\mathbf{Z}^2}$ defines an action of $\mathbf{Z}^2$ on $(N_0)_\gamma$, which is isomorphic to the AFD factor of type $\mathrm{II}_\infty$. We show the following lemma, which is essentially important, that is, assumption that $\mathrm{mod}(\alpha)$ is faithful is essentially used for showing this lemma.

**Lemma 4.13.** *For almost all $\gamma \in [0,1)$, the action $\{\beta^{0,\gamma}_n \circ \sigma^{0,\gamma}_m\}$ is trace-scaling for $(n,m) \neq 0$.*

*Proof.* Take a pair $(n,m) \neq 0$. By definition of $\beta^0$ and $\sigma^0$, we have

$$\begin{aligned}\beta^0_n \circ \sigma^0_m &= (\theta_{np} \circ \sigma_{ns})^0 \circ \sigma^0_m \\ &= (\theta_{np} \circ \sigma_{ns+m})^0 \\ &= (\theta_{np} \circ \theta_{(ns+m)q_1} \circ \tilde{\alpha}_{(ns+m)p_1})^0 \\ &= (\theta_{np+(ns+m)q_1} \circ \tilde{\alpha}_{(ns+m)p_1})^0.\end{aligned}$$

If $n = 0$, we need not show anything . Assume that $n \neq 0$. Then since $\theta_{np}$ is not identity on the center of $N_0$, $\sigma_{ns+m}$ is not identity on $Z(N_0)$ by looking at the first equation. Hence $(ns+m)p_1 \neq 0$. Thus, by the faithfulness of $\mathrm{mod}(\alpha)$ and the last equation, we have $np + (ns+m)q_1 \neq 0$. Hence $\theta_{np+(ns+m)q_1}$ scales $\tau$. Besides, $\tilde{\alpha}$ preserves $\tau$. Hence we may assume that $\beta^0_n \circ \sigma^0_m$ scales $\tau_0$. Hence if we decompose $\tau^0$ by

$$\tau^0 = \int_{[0,1)}^{\oplus} \tau^{0,\gamma} \, d\gamma,$$

$(\beta_n^0 \circ \sigma_m^0)^\gamma$ scales $\tau^{0,\gamma}$ for almost all $\gamma \in [0,1)$. $\square$

From Lemma 4.13, we may assume that $\{\beta_n^{0,0} \circ \sigma_m^{0,0}\}$ is trace-scaling. Now, we will return to prove Lemma 4.12, which completes the proof of Lemma 4.9.

*Proof of Lemma 4.12.* Let $\psi_0$, $\epsilon$ and $T$ be as explained after the statement of Lemma 4.12. By Lemma 4.13, the action $\{\beta_n^{0,0} \circ \sigma_m^{0,0}\}$ is centrally outer, and hence has the Rohlin property. Hence, for $A, B \in \mathbf{N}$ with $4(T+1)^2/\epsilon^2 < B$ and $A > 1/\epsilon^2$, there exists a family of projections $\{e_{n,m}\}_{n=1,\cdots,B}^{m=1,\cdots A}$ of $N_{0,0}$ which satisfies the following conditions.

(1) The projections are mutually orthogonal.

(2) We have

$$\sum_{n,m} e_{n,m} = 1,$$

$$\| \sum_{1 \leq n \leq B, m=1,A} e_{n,m} \|_{\psi_0 + \psi_0 \circ \beta^{0,0}}^\sharp \leq 2/\sqrt{A},$$

$$\| \sum_{1 \leq m \leq A, n \geq B-(T+1), n \leq T+1} e_{n,m} \|_{\psi_0 + \sum_{l=-[T]-1}^{[T]+1} \psi_0 \circ \sigma_l^{0,0}}^\sharp \leq 2(T+1)/\sqrt{B}.$$

Here, $[T]$ is the maximal natural number which is not larger than $T$.

(3) We have $\|[e_{n,m}, \psi_{k,i}]\| < \epsilon/(ABl_k)$ for $n = 1, \cdots B$, $m = 1, \cdots, A$, $i = 1, \cdots l_k$, $k = 1, \cdots n$.

(4) We have $\|\sigma_l^{0,0}(e_{n,m}) - e_{n+1,m}\|_{\psi_0}^\sharp < \epsilon/(AB)$ for $n, m$, $l \in \mathbf{Z}$ with $|l| \leq T+1$, $n \leq B-(T+1)$.

(5) We have $\|\beta^{0,0}(e_{n,m}) - e_{n,m+1}\|_{\psi_0}^\sharp < \epsilon/(AB)$ for $n, m \in \mathbf{Z}$ with $m \neq A$.

Here, we define $e_{B+1,m} = e_{1,m}$ for $m = 1, \cdots, A$, $e_{n,A+1} = e_{n,1}$ for $n = 1, \cdots, B$. For $(s,t) \in \mathbf{T} \times \mathbf{R}$, we set

$$u_\gamma := e^{2\pi i t\gamma} \sum_{n,m} e^{2\pi i(nt+ms)} \sigma_{\gamma,0}^0(e_{n,m})$$

for $\gamma \in [0,1)$. We also set

$$u := \int_{[0,1)}^\oplus u_\gamma \, d\gamma \in U(N_0).$$

The above conditons (2) and (4) ensure that we can almost control $\sigma^{0,0}$, which is useful to show that $\sigma_q^0(u)$ is close to $e^{-2\pi i t q}u$. Conditions (2) and (5) is useful to show that $\beta^0(u)$ is close to $e^{-2\pi i s}u$. Condition (3) is useful to show that $[u, \psi_k]$ is small.

By condition (3) of the above, we have

$$
\begin{aligned}
\|[u,\psi_k]\| &\le \int_{[0,1)} \sum_{n,m}\sum_{i=1}^{l_k} \|[\sigma^0_{\gamma,0}(e_{n,m}),\psi_{k,i}\circ\sigma^0_{-\gamma,\gamma}]\|\ d\gamma\\
&= \int_{[0,1)} \sum_{n,m}\sum_{i} \|[e_{n,m},\psi_{k,i}]\|\ d\gamma\\
&< \int_{[0,1)} \sum_{n,m}\sum_{i} \frac{\epsilon}{ABl_k}\ d\gamma\\
&= \epsilon.
\end{aligned}
$$

By conditions (2) and (5), we have

$$
\begin{aligned}
&\|\beta^0(u)-e^{-2\pi is}u\|^\sharp_\psi\\
&= \int_{[0,1)} \|\beta^{0,\gamma}\circ\sigma^0_{\gamma,0}(\sum_{m,n}(e^{2\pi i((n+\gamma)t+ms)}e_{n,m}))\\
&-\sum_{m,n} e^{2\pi i((n+\gamma)t+(m-1)s)}\sigma^0_{\gamma,0}(e_{n,m})\|^\sharp_{\psi_0\circ\sigma^0_{-\gamma,\gamma}}\ d\gamma\\
&= \int_{[0,1)} \|\sigma^0_{\gamma,0}(\beta^{0,0}(\sum_{m,n} e^{2\pi i((n+\gamma)t+ms)}e_{n,m})\\
&-\sum_{m,n} e^{2\pi i((n+\gamma)t+(m-1)s)}e_{n,m})\|^\sharp_{\psi_0\circ\sigma^0_{-\gamma,\gamma}}\ d\gamma\\
&\le \sum_{m,n,m\ne A}\int_{[0,1)} \|e^{2\pi i((n+\gamma)t+ms)}\sigma^0_{\gamma,0}(\beta^{0,0}(e_{n,m})-e_{n,m+1})\|^\sharp_{\psi_0\circ\sigma^0_{-\gamma,\gamma}}\ d\gamma\\
&+\int_{[0,1)} \|\sigma^0_{\gamma,0}(\beta^{0,0}(\sum_n e_{n,A})-\sum_n e_{n,A+1})\|^\sharp_{\psi_0\circ\sigma^0_{-\gamma,\gamma}}\ d\gamma\\
&\le \sum_{m,n,m\ne A}\int_{[0,1)} \|\beta^{0,0}(e_{n,m})-e_{n,m+1}\|^\sharp_{\psi_0}\ d\gamma + 2/\sqrt{A}\\
&< AB(\frac{\epsilon}{AB})+2\epsilon\\
&= 3\epsilon.
\end{aligned}
$$

Condition (2) is used in the fourth inequality and condition (5) is used in the fifth inequality. Next, we will compute $\|\sigma^0_q(u)-e^{2\pi itq}u\|^\sharp_\psi$ for $q\in[-T,T]$. In order to do this, the following observation is useful. Let $\gamma\in[0,1)$ and let $q\in[-T,T]$. Choose $l_\gamma\in\mathbf{Z}$ so that $\gamma-q+l_\gamma\in[0,1)$.

Then we have

$$
\begin{aligned}
(\sigma_q^0(u))_\gamma &= \sigma^0_{q,\gamma-q}(u_{\gamma-q+l_\gamma}) \\
&= \sigma^0_{\gamma,0}\circ\sigma^{0,0}_{l_\gamma}\circ\sigma^0_{q-l_\gamma-\gamma,\gamma-q+l_\gamma}(u_{\gamma-q+l_\gamma}) \\
&= \sigma^0_{\gamma,0}\circ\sigma^{0,0}_{l_\gamma}(e^{2\pi it(\gamma-q+l_\gamma)}\sum_{n,m}e^{2\pi i(nt+ms)}e_{n,m}) \\
&= e^{2\pi it(\gamma-q+l_\gamma)}\sigma^0_{\gamma,0}\circ\sigma^{0,0}_{l_\gamma}(u_0).
\end{aligned}
$$

By conditions (2) and (4), we have

$$
\begin{aligned}
&\|\sigma_q^0(u)-e^{-2\pi itq}u\|^\sharp_\psi \\
&=\int_{[0,1)}\|\sigma^0_{\gamma,0}(e^{2\pi it(\gamma-q+l_\gamma)}\sigma^{0,0}_{l_\gamma}(u_0)-e^{-2\pi itq}u_\gamma)\|^\sharp_{\psi_0\circ\sigma^0_{-\gamma,\gamma}}\ d\gamma \\
&=\int_{[0,1)}\|\sum_{n,m}(e^{2\pi i((\gamma-q+l_\gamma+n)t+ms)}\sigma^{0,0}_{l_\gamma}(e_{n,m})-e^{2\pi i((\gamma+n-q)t+ms)}e_{n,m})\|^\sharp_{\psi_0}\ d\gamma \\
&\leq\int_{[0,1)}\sum_{m,l_\gamma<n\leq B-1}\|e^{2\pi i((\gamma-q+n)t+ms)}\sigma^{0,0}_{l_\gamma}(e_{n-l_\gamma,m}) \\
&-e^{2\pi i((\gamma+n-q)t+ms)}e_{n,m})\|^\sharp_{\psi_0}\ d\gamma+2\epsilon \\
&<(\frac{\epsilon}{AB})AB+2\epsilon \\
&=3\epsilon.
\end{aligned}
$$

Condition (2) is used in the third inequality and condition (4) is used in the fourth inequality. Thus $\{\sigma^0_q\circ\beta^0_m\}_{(q,m)}$ has the Rohlin property. Thus Lemma 4.12 holds. □

**Lemma 4.14.** *When* $\ker(\mathrm{mod}(\alpha_s)\circ(\theta\mid_C)_t)\cong\mathbf{R}$*, Lemma 4.4 holds.*

*Proof.* There exists $(p,q)\in(\mathbf{R}\setminus\{0\})^2$ with $\ker(\mathrm{mod}(\alpha)\circ\theta\mid_C)=(p,q)\mathbf{R}$. Set $\sigma_t:=\theta_{qt}\circ\tilde{\alpha}_{pt}$ for $t\in\mathbf{R}$. In order to show our lemma, it is enough to show that for each $r\in\mathbf{R}$, the action $\sigma$ admits a sequence of unitaries which satisfies the same conditions as in Lemma 4.9. Take a normal faithful state $\phi_0$ of $N$, $\phi_1,\cdots,\phi_n\in N_*$ with $\|\phi_k\|=1$ ($k=1,\cdots,n$), $\epsilon>0$ and $T>0$. Think of $C$ as a standard probability measured space $L^\infty(\Gamma,\mu)$. Let

$$
N=\int_\Gamma^\oplus N_\gamma\ d\mu(\gamma),
$$

$$
\phi_k=\int_\Gamma^\oplus \phi_{k,\gamma}\ d\mu(\gamma)
$$

$(k = 1, \cdots, n)$ be the direct integral decompositions. Then by Theorem 4.6 and Proposition 4.31, there exists a Borel subset $A$ of $\Gamma$ which satisfies the following three conditions.

(1) There exists a large cube $Q := [-T', T']$ and a $Q$-set $Y$ such that $A = T_Q Y$ and the map $Q \times Y \ni (t, x) \mapsto T_t(x) \in A$ is injective.
(2) We have

$$\mu(\bigcap_{t \in [-T,T]} T_t A) > 1 - \epsilon$$

and

$$\int_{\bigcap_{t\in[-T,T]} T_t A} \|\phi_{k,\gamma}\| \; d\mu(\gamma) > 1 - \epsilon$$

for $k = 1, \cdots, n$.
(3) There is a measure $\nu$ on $Y$ such that the map $Q \times Y \ni (t, x) \mapsto T_t(x) \in A$ is a non-singular isomorphism with respect to Lebesgue $\otimes\ \nu$ and $\mu$ (Note that two measures $\mu + \sum_k \int_\Gamma \|\phi_{k,\gamma}\| \; d\mu(\gamma)$ and $\mu$ are mutually equivalent).

Here, we do not assume the existence of invariant probability measures for $\theta\mid_C$. Then $N$ is isomorphic to

$$N_{\Gamma\setminus A} \oplus \int_{[-T',T']}^{\oplus} N_s \; ds.$$

Here,

$$N_s = \int_Y^{\oplus} N_{(y,s)} \; d\nu(y).$$

For $s, t \in [-T', T']$, $\theta$ defines an isomorphism $\theta_{s-t,t}$ from $N_t$ to $N_s$ by $\theta_{s-t,t}(x_t) = (\theta_{s-t}(x))_s$. As in Lemma 4.9, we identify $N_t$ with $N_0$ by this isomorphism. By this identification, we approximate $\phi_k$'s by simple step functions.

$$\|\phi_k \chi_A - \sum_{i=1}^{l_k} \phi_{k,i} \circ \theta_{-t,t} \chi_{I_i}(t)\| < \epsilon$$

for $k = 1, \cdots, n$, where $\phi_{k,i} \in (N_0)_*$ and $\{I_i\}_{i=1}^{l_k}$ are partitions of $[-T', T']$. Here, we note that it is possible to choose $\phi_{0,i}$'s so that they are positive. This is shown by the following way. Since $\phi_0 : [-T', T'] \to (N_0)_*$ is measurable, by Lusin's theorem, it is possible to choose a sufficiently large compact subset $K$ of $[-T', T']$ on which $\phi_0$ is continuous. Choose a finite partition $\{s_i\}_{i=1}^{l_0}$ of $K$ so that for every $s \in K$, there exists a number $i$ such that $\phi_0(s)$ is close to $\phi(s_i)$. It is possible to choose a partition $\{I_i\}$ of $K$ so that $\phi_0(s)$ is close to $\phi_0(s_i)$

on $I_i$. Then $\sum \phi_0(s_i)\chi_{I_i}$ well approximates $\phi_0$. Since $\sigma$ fixes the center of $N$, this is decomposed into the direct integral.

$$\sigma = \sigma_{\Gamma\setminus A} \oplus \int_{[-T',T']} \sigma^t \, dt.$$

Since $\sigma$ scales the canonical trace on $N$, for almost all $t \in \mathbf{R}$, the action $\sigma^t$ is trace-scaling, and hence has the Rohlin property by Theorem 6.18 of Masuda–Tomatsu [44]. Hence, by the same argument as in the proof of Lemma 4.12, it is possible to choose a unitary element $u_0$ of $N_0$ satisfying the following conditions.

(1) We have $\|[u_0, \phi_{k,i}]\| < \epsilon/(2l_kT')$ for $k = 1, \cdots, n$, $i = 1, \cdots, l_k$.

(2) We have $\|\sigma^0_t(u_0) - e^{-ipt}u_0\|^\sharp_{\phi_{0,i}} < \epsilon/(2l_0T')$ for $t \in [-T, T]$, $i = 1, \cdots, l_0$.

Set $u_t := \theta_{t,0}(u_0)$ for $t \in [-T', T']$ and set

$$u := \chi_{X\setminus A} \oplus \int^{\oplus}_{[-T',T']} u_t \, dt.$$

Hence by the same aregument as in the proof of Lemma 4.11 ⇒ Lemma 4.9, Lemma 4.4 holds. □

**Lemma 4.15.** *When* $\ker(\mathrm{mod}(\alpha_s) \circ (\theta\mid_C)_t) \cong \mathbf{Z}$*, Lemma 4.4 holds.*

*Proof.* Let $(p, q) \in (\mathbf{R} \setminus \{0\})^2$ be a generator of $\ker(\mathrm{mod}(\alpha_s) \circ (\theta\mid_C)_t)$. Set $\sigma_t := \theta_{qt} \circ \tilde{\alpha}_{pt}$ for $t \in \mathbf{R}$. Think of $C^\sigma$ as a standard probability space $L^\infty(\Gamma, \mu)$. We first show the following claim.

**Claim**. The action $\theta : \mathbf{R} \curvearrowright C^\sigma$ is faithful (and hence is free).

*Proof of Claim.* Assume that $\theta_t\mid_{C^\sigma} = \mathrm{id}_{C^\sigma}$. Then $\theta$ is decomposed into the direct integral over $\Gamma$.

$$\theta_t = \int^{\oplus}_{\Gamma} \theta^\gamma_t \, d\mu(\gamma),$$

$$N = \int^{\oplus}_{\Gamma} N_\gamma \, d\mu(\gamma).$$

We also decompose $\sigma$ by

$$\sigma_s = \int^{\oplus}_{\Gamma} \sigma^\gamma_s \, d\mu(\gamma).$$

Then for almost all $\gamma \in \Gamma$, $\{\sigma^\gamma_t\}_{t\in\mathbf{R}}$ defines a periodic ergodic action on the center of $N_\gamma$. Since the restriction of $\theta^\gamma_t$ on the center of $N_\gamma$ commutes with that of $\sigma^\gamma_s$'s, $\theta^\gamma_t\mid_{Z(N_\gamma)}$ is of the form $\sigma^\gamma_{s_\gamma}\mid_{Z(N_\gamma)}$. We show that there exists $s \in [0, 1)$ such that $s_\gamma = s$ for almost all $\gamma$.

Since we want to show the faithfulness of the action $\theta$, we may assume that $t \neq 0$. We think of $C$ as a probability measured space $L^\infty(X, \mu_X)$. Then there exists a projection $p$ from $X$ to $\Gamma$ induced by the inclusion $L^\infty(\Gamma) \to L^\infty(X)$. Let $T, S$ be two flows on $(X, \mu_X)$ defined by $f(T_t x) = \theta_{-t}(f)(x)$, $f(S_s x) = \sigma_{-s}(f)(x)$ for $x \in X$, $f \in L^\infty(X, \mu_X)$. We may assume that $X$ is a separable compact Hausdorff space and $T$ and $S$ are continuous. We show that the set

$$A_s = \{x \in X \mid T_t \circ S_r(x) = x \text{ for some } 0 \leq r \leq s\}$$

is Borel measurable. Let $f : \mathbf{R} \times X \to X^2$ is a map defined by $f : \mathbf{R} \times X \ni (s, x) \mapsto (T_t \circ S_s(x), x) \in X^2$. Then we have $A_s = \pi_X(f^{-1}(\Delta) \cap ([0, s] \times X))$, which is Borel measurable. Here, $\Delta$ is the diagonal set of $X \times X$ and $\pi_X : \mathbf{R} \times X \to X$ is the projection.

Next we show that there exists $s \in [0, 1)$ such that

$$B_s := \{x \in X \mid T_t \circ S_s(x) = x\}$$

has a positive measure. If not, the map $s \to \mu_X(A_s)$ would be continuous. By the first part of this proof, for each $\gamma \in \Gamma$, if $x \in X$ satisfies $p(x) = \gamma$, then we have $x \in A_{s_\gamma}$. Hence $\bigcup_{s>0} A_s$ is full measure. On the other hand, since $t \neq 0$, we have $\mu(A_0) = 0$. Thus there would exist $s \in [0, 1)$ with $\mu_X(A_s) = 1/2$. However, this would contradict to the ergodicity of $\theta$. Thus there exists $s \in [0, 1)$ with $\mu_X(B_s) > 0$.

By using the ergodicity of $\theta$ again, there exists $s \in [0, 1)$ such that $B_s$ is full measure.

Hence there exists $s \in [0, 1)$ such that $\sigma_s |_C = \theta_t |_C$. Since $\ker(\text{mod}(\alpha) \circ (\theta |_C)) = (p, q)\mathbf{Z}$, we have $s = t = 0$, which is a contradiction. Hence Claim is shown. □

Now, we return to the proof of Lemma 4.15. For almost all $\gamma \in \Gamma$, the action $\sigma^\gamma |_{Z(N_0)}$ is ergodic and has a period 1, and $\sigma^\gamma$ is trace-scaling. Hence this is the dual action of a modular automorphism of an AFD $\text{III}_\lambda$ $(0 < \lambda < 1)$ factor. Hence $\sigma^\gamma$ has the Rohlin property. Hence by the same argument as in Lemma 4.14, our lemma is shown. □

**Remark 4.16.** When $M$ is of type $\text{III}_\lambda$, $0 < \lambda < 1$, then Connes–Takesaki module of a flow on $M$ cannot be faithful. This is shown by the following way. Since $\text{mod}(\alpha)$ commutes with $\theta$, as we have seen, this is a homomorphism from $\mathbf{R}$ to $\mathbf{T}$. Hence $\text{mod}(\alpha)$ cannot be faithful.

4.3. **Remarks and Examples.** In this section, we present examples which have interesting properties.

4.3.1. *Model Actions.* In this subsection, we will construct model actions. If there were no flows with faithful Connes–Takesaki modules, then our main theorem of this section would have no value. Hence it is important to construct a flow which has a given flow as its Connes–Takesaki module.

**Proposition 4.17.** *Let $M$ be an AFD factor with its flow space $\{C, \theta\}$ and let $\sigma$ be a flow on $C$ which commutes with $\theta$ (Here, we do not assume the faithfulness of $\sigma$). Then there exists a Rohlin flow $\alpha$ on $M$ with* $\mathrm{mod}(\alpha) = \sigma$.

*Proof.* The proof is modeled after Masuda [39]. As in Corollary 1.3 of Sutherland–Takesaki [57], there exists an exact sequence

$$1 \to \overline{\mathrm{Int}}(M) \to \mathrm{Aut}(M) \to \mathrm{Aut}_\theta(C) \to 1,$$

and there exists a right inverse $s : \mathrm{Aut}_\theta(C) \to \mathrm{Aut}(M)$. The maps $p : \mathrm{Aut}(M) \to \mathrm{Aut}_\theta(C)$ and $s : \mathrm{Aut}_\theta(C) \to \mathrm{Aut}(M)$ are continuous.

Hence for a flow $\sigma$ on $C$ commuting with $\theta$, the homomorphism $\alpha := s \circ \sigma : \mathbf{R} \to \mathrm{Aut}(M)$ gives an action with its Connes–Takesaki module $\sigma$. If $\sigma$ is faithful, by our main theorem of this section, this has the Rohlin property. Assume that $\sigma$ is not faithful. Then $\mathrm{mod}(\alpha \otimes \beta) = \mathrm{mod}(\alpha)$ for a Rohlin flow $\beta$ on the AFD factor of type $\mathrm{II}_1$. Hence this $\alpha \otimes \beta$ does the job. □

For actions on the AFD factor of type $\mathrm{II}_1$, strong cocycle conjugacy is equivalent to cocycle conjugacy because every automorphism of the AFD factor of type $\mathrm{II}_1$ is approximated by its inner automorphisms. However, for flows on some AFD factor of type $\mathrm{III}_0$, cocycle conjugacy does not always imply strong cocycle conjugacy.

**Example 4.18.** Let $(X, \mu)$ be a probability measured space defined by

$$(X, \mu) := (\prod_{m \in \mathbf{Z}} (\prod_{n \in \mathbf{Z}} (\{0, 1\}, \{\frac{1}{2}, \frac{1}{2}\}))).$$

Let $S$, $T$ be two automorphisms of $X$ defined by the following way.

$$S(m \mapsto (n \mapsto {x_n}^m \in \{0, 1\})) = (m \mapsto (n \mapsto {x_{n+1}}^m)),$$

$$T(m \mapsto (n \mapsto {x_n}^m)) = (m \mapsto (n \mapsto {x_n}^{m+1})).$$

Then both $S$ and $T$ are ergodic and satisfy $S \circ T = T \circ S$. Let $\beta_1$, $\beta_2$ be two flows on $L^\infty(\mathbf{T})$ satisfying the following conditions.

(1) Two flows are faithful.
(2) The flow $\beta_1$ is not conjugate to $\beta_2$.
(3) Two flows preserve the Lebesgue measure.

As a probability measured space, we have

$$\prod_{n\in\mathbf{Z}}\{0,1\}^n = (\prod_{n:\text{odd}}\{0,1\}^n)\times(\prod_{n:\text{even}}\{0,1\}^n)\cong\mathbf{T}^2.$$

By this identification, we set

$$\beta := (\bigotimes_{m\in\mathbf{Z}}(\beta_1\otimes\beta_2))\otimes\mathrm{id} : \mathbf{R}\curvearrowright L^\infty(X\times[0,1)).$$

Let $\theta$ be a flow on $L^\infty(X\times[0,1))$ defined by $T$ and the ceiling function $r=1$. Let $\rho$ be an automorphism of $L^\infty(X\times[0,1))$ defined by $S\times\mathrm{id}$. Then we have the following.

(1) The flow $\theta$ commutes with both $\rho$ and $\beta$.
(2) The action $\rho$ does not commute with $\beta$.
(3) The flow $\theta$ is ergodic.

Now, we construct a pair of flows which are mutually cocycle conjugate but not strongly cocycle conjugate. Let $M$ be an AFD factor of type $\mathrm{III}_0$ with its flow of weights $\{\theta, X\times[0,1)\}$, $\alpha$ be a Rohlin action satisfying $\mathrm{mod}(\alpha)=\beta$ and let $\sigma$ be an an automorphism of $M$ satisfying $\mathrm{mod}(\sigma)=\rho$. Then we have

$$\mathrm{mod}(\alpha)=\beta\neq\rho\circ\beta\circ\rho^{-1}=\mathrm{mod}(\sigma\circ\alpha\circ\sigma^{-1}).$$

Hence $\alpha$ is cocycle conjugate to $\sigma\circ\alpha\circ\sigma^{-1}$ but they are not strongly cocycle conjugate.

4.3.2. *On Stability.* In Thoerem 5.9, Izumi [22] has shown that an action of a compact group on any factor of type III with faithful Connes–Takesaki module is minimal. As well as our main theorem of this section, this theorem means that actions which are "very outer" at any non-trivial point are "globally outer". He has also shown that for these actions, cocycle conjugacy coincides with conjugacy. This phenomenon also occurs for trace-scaling flows on any factor of type $\mathrm{II}_\infty$. Hence one may be tempted to think that this is true under our assumption. However, this is not the case.

**Theorem 4.19.** *Let $C$ be an abelian von Neumann algebra and $\theta$ be an ergodic flow on $C$. Let $M$ be an AFD factor with its flow of weights $(C,\theta)$. Let $\beta$ be a faithful flow on $C$ which commutes with $\theta$ and fixes a normal faithful semifinite weight $\mu$ of $C$. If the discrete spectrum of $\beta$ is not $\mathbf{R}$, then there are two flows $\alpha^1$, $\alpha^2$ which satisfies the following two conditions.*
(1) *The Connes–Takesaki modules of $\alpha^1$ and $\alpha^2$ are $\beta$.*
(2) *The flow $\alpha^1$ is not conjugate to $\alpha^2$.*

In the following, we actually construct these flows. In the following, we denote the AFD factor of type $\mathrm{II}_1$ by $R_0$ and denote the AFD factor of type $\mathrm{II}_\infty$ by $R_{0,1}$.

Let $\Lambda$ be the discrete spectrum of $\beta$ and $\mu$ be a $\beta$-invariant measure. In the rest of this subsection, we assume that $\Lambda$ is not $\mathbf{R}$. Then by the ergodicity of $\theta$ (Note that $\beta$ may not be ergodic), $\Lambda$ is a proper subgroup of $\mathbf{R}$. Hence there are at least two real numbers which do not belong to $\Lambda$. Let $\Gamma_j$ $(j = 1, 2)$ be two subgroups of $\mathbf{R}$ generated by two elements $\lambda_j, \mu_j$, respectively, satisfying the following conditions.

$$\Gamma_1 \cup \Lambda \not\subset \langle \Gamma_2, \Lambda \rangle,$$

$$\Gamma_2 \cup \Lambda \not\subset \langle \Gamma_1, \Lambda \rangle.$$

Here, $\langle \Gamma_i, \Lambda \rangle$ is the subgroup of $\mathbf{R}$ generated by $\Gamma_i$ and $\Lambda$. Let $\gamma^j$ $(j = 1, 2)$ be two ergodic flows on $R_0$ with their discrete spectrum $\Gamma_j$, respectively. Namely, we think of $R_0$ as a weak closure of an irrational rotation algebra $A_s := \mathrm{C}^*(u, v \mid u, v : \text{unitaries satisfying } vu = e^{2\pi i s} uv)$ and define flows $\gamma^j$, $j = 1, 2$, by the following way. This type of actions is considered by Kawahigashi [29].

$$\gamma^j_t(u) = e^{i\lambda_j t} u, \ \gamma^j_t(v) = e^{i\mu_j t} v$$

for $t \in \mathbf{R}$.

Set $\tau := \mu \otimes \tau_{R_{0,1}} \otimes \tau_{R_0}$. The flow $\theta$ is extended to a $\tau$-scaling flow on $N := C \otimes R_{0,1} \otimes R_0$ as in equations (1.2) of Sutherland–Takesaki [57]. Set $\overline{\alpha^j} := \beta \otimes \mathrm{id}_{R_{0,1}} \otimes \gamma^j$. Then $\overline{\alpha^j}$ commutes with $\theta$ (See the equation after equation (1.8) of Sutherland–Takesaki [57]). Hence the flow $\overline{\alpha^j}$ is extended to $M := N \rtimes_\theta \mathbf{R}$ in the following way.

$$\alpha^j_t(\lambda^\theta_s) = \lambda^\theta_s$$

for $s, t \in \mathbf{R}$. Note that the flow $\theta : \mathbf{R} \curvearrowright N$ is not so "easy". However, the flow $\alpha^j$ is very concrete. Here, we think of $M$ as a von Neumann algebra generated by $N$ and a one parameter unitary group $\{\lambda_s\}_{s \in \mathbf{R}}$.

In order to show Theorem 4.19, for these $\alpha^j$'s, it is enough to show the following lemma.

**Lemma 4.20.** *In the above context, we have the following two statements.*

(1) *The Connes–Takesaki module of $\alpha^j$ is $\beta$ for each $j = 1, 2$.*

(2) *For the discrete spectrum of $\alpha^j$, we have the following inclusion.*

$$\Gamma^j \cup \Lambda \subset \mathrm{Sp}_d(\alpha^j) \subset \langle \Gamma^j, \Lambda \rangle.$$

From statement (1) of the lemma and Corollary 4.2, it is shown that $\alpha^1$ and $\alpha^2$ are mutually cocycle conjugate. On the other hand, from statement (2) of the lemma, it is shown that the discrete spectrum of $\alpha^1$ and that of $\alpha^2$ are different. Hence they are not conjugate.

In order to show this lemma, we first show the following lemma.

**Lemma 4.21.** *The weight $\hat{\tau}$ is invariant by $\alpha^j$.*

*Proof.* Set

$$n_\tau := \{a \in N \mid \tau(a^* a) < \infty\},$$

$$K(\mathbf{R}, N) := \{x : \mathbf{R} \to N \mid \text{strongly}^* \text{ continuous map with compact support}\},$$

$$b_\tau := \operatorname{span}\{xa \mid x \in K(\mathbf{R}, N), a \in n_\tau\}.$$

For $x \in b_\tau$, set

$$\tilde{\pi}(x) := \int_{\mathbf{R}} x_t \lambda_t^\theta \, dt.$$

In order to show this lemma, it is enough to show the following two statements (For example, see Theorem X.1.17. of Takesaki [59]).

(1) For $s, t \in \mathbf{R}$, we have $\sigma_t^{\hat{\tau}} = \alpha_{-s}^j \circ \sigma_t^{\hat{\tau}} \circ \alpha_s^j$.

(2) For $x \in b_\tau$, $s \in \mathbf{R}$, we have

$$\hat{\tau} \circ \alpha_s^j(\tilde{\pi}(x)^* \tilde{\pi}(x)) = \hat{\tau}(\tilde{\pi}(x)^* \tilde{\pi}(x)).$$

Statement (1) is trivial because $\alpha^j$ commutes with $\sigma^{\hat{\tau}}$. We show statement (2). Notice that

$$\begin{aligned}
\alpha_s^j(\tilde{\pi}(x)) &= \alpha_s^j\Big(\int_{\mathbf{R}} x_t \lambda_t \, dt\Big) \\
&= \int_{\mathbf{R}} \overline{\alpha^j}_s(x_t) \lambda_t \, dt \\
&= \tilde{\pi}(\overline{\alpha^j}_s(x)).
\end{aligned}$$

Since $\tau$ is invariant by $\overline{\alpha^j}$, we have

$$\begin{aligned}
\hat{\tau} \circ \alpha_s^j(\tilde{\pi}(x)^* \tilde{\pi}(x)) &= \hat{\tau}(\tilde{\pi}(\overline{\alpha^j}_s(x))^* \tilde{\pi}(\overline{\alpha^j}_s(x))) \\
&= \tau\Big(\int_{\mathbf{R}} \overline{\alpha^j}_s(x_t^* x_t) \, dt\Big) \\
&= \tau\Big(\int_{\mathbf{R}} x_t^* x_t \, dt\Big) \\
&= \hat{\tau}(\tilde{\pi}(x)^* \tilde{\pi}(x)).
\end{aligned}$$

Thus statement (2) holds.

□

By this lemma, the canonical extention $\tilde{\alpha^j}$ of $\alpha^j$ is defined by $\tilde{\alpha^j}_t(\lambda_s^\sigma) = \lambda_s^\sigma$ if we think of $\tilde{M} := M \rtimes_{\sigma^{\hat{\tau}}} \mathbf{R}$ as a von Neumann algebra generated by $M$ and a one parameter unitary group $\{\lambda_t^\sigma\}$.

Hence by Lemma 13.3 of Haagerup–Størmer [20], if we identify $N \rtimes_\theta \mathbf{R} \rtimes_\sigma \mathbf{R}$ with $N \otimes B(H)$ by Takesaki's duality theorem, we have

$$\tilde{\alpha^j} \cong \overline{\alpha^j} \otimes \mathrm{id}.$$

Thus statement (1) of Lemma 4.20 holds.

In the following, we show statement (2) of Lemma 4.20. We need to show the following lemma.

**Lemma 4.22.** *We have* $\mathrm{Sp}_d(\overline{\alpha^j}) = \mathrm{Sp}_d(\alpha^j) = \mathrm{Sp}_d(\tilde{\alpha^j})$.

*Proof.* The action $\alpha^j$ is an extension of the action $\overline{\alpha^j}$, and the action $\tilde{\alpha^j}$ is an extension of the action $\alpha^j$. Hence we have $\mathrm{Sp}_d(\overline{\alpha^j}) \subset \mathrm{Sp}_d(\alpha^j) \subset \mathrm{Sp}_d(\tilde{\alpha^j})$. We show the implication $\mathrm{Sp}_d(\tilde{\alpha^j}) \subset \mathrm{Sp}_d(\overline{\alpha^j})$. Note that if we identify $N \rtimes_\theta \mathbf{R} \rtimes_\sigma \mathbf{R}$ with $N \otimes B(H)$ by Takesaki's duality theorem, we have $\tilde{\alpha^j} = \overline{\alpha^j} \otimes \mathrm{id}$. Choose $p \in \mathrm{Sp}_d(\tilde{\alpha^j})$. Then there exists a non-zero element $x \in N \otimes B(H)$ with $\tilde{\alpha^j}_t(x) = e^{ipt}x$ for $t \in \mathbf{R}$. If we write $x = (x_{kl})_{kl} \in N \otimes B(l^2(\mathbf{N}))$, then there exists $(k, l)$ with $x_{kl} \neq 0$. Since we have $\overline{\alpha^j}_t(x_{kl}) = e^{ipt}x_{kl}$, we have $p \in \mathrm{Sp}_d(\overline{\alpha^j})$. □

Now, we return to the proof of statement (2) of Lemma 4.20, which completes the proof of Theorem 4.19.

*Proof of Lemma 4.20.* The inclusion $\Gamma_j \cup \Lambda \subset \mathrm{Sp}_d(\overline{\alpha^j})$ is trivial. We show the inclusion $\mathrm{Sp}_d(\overline{\alpha^j}) \subset \langle \Gamma_j, \Lambda \rangle$. If we think of $N = C \otimes R_{0,1} \otimes R_0$ as a subalgebra of $C \otimes B(H) \otimes R_0$, then $\overline{\alpha^j}$ extends to $\beta \otimes \mathrm{id}_{B(H)} \otimes \gamma^j$. Hence by the same argument as in Lemma 4.22, we have $\mathrm{Sp}_d(\overline{\alpha^j}) = \mathrm{Sp}_d(\beta \otimes \gamma^j)$. Choose $p \in \mathrm{Sp}_d(\beta \otimes \gamma^j)$. Let $x \in C \otimes R_0$ be a non-zero eigenvector for $p \in \mathrm{Sp}_d(\overline{\alpha^j})$. Then $x$ is expanded as

$$x = \sum_{n,m} c_{n,m} u^n v^m$$

with $c_{n,m} \in C$ $(n, m \in \mathbf{Z})$. Hence we have

$$\begin{aligned}\sum_{n,m} e^{ipt} c_{n,m} u^n v^m &= e^{ipt} x \\ &= \beta_t \otimes \gamma_t^j(x) \\ &= \sum_{n,m} \beta_t(c_{n,m}) e^{i(n\lambda_j + m\mu_j)t} u^n v^m.\end{aligned}$$

Since $x \neq 0$, there exists $(n, m)$ with $c_{n,m} \neq 0$. Hence by the uniqueness of the Fourier expansion, we have

$$\beta_t(c_{n,m}) = e^{i(p - n\lambda_j - m\mu_j)t} c_{n,m}.$$

Thus $p \in \langle \Gamma^j, \Lambda \rangle$. □

**Remark 4.23.** (1) As shown in Corollary 8.2 of Yamanouchi [64], if we further assume that $\alpha^1$, $\alpha^2$ and $\beta$ are integrable, then $\alpha^1$ is conjugate to $\alpha^2$. In this case, $\beta$ contains the translation of $\mathbf{R}$ as a direct product component.

(2) Another important difference between flows and actions of compact groups is about extended modular actions. The duals of extended modular flows are important examples of flows with faithful Connes–Takesaki modules (See Theorem 4.20 of Masuda–Tomatsu [44]). Actions of compact groups with faithful Connes–Takesaki modules are duals of skew products (See Definition 5.6 and Theorem 5.9 of Izumi [22]). However, this is not true for flows by subsection 4.3.1 of this paper and Theorem 4.20 of Masuda–Tomatsu [44].

4.3.3. *On a Characterization of the Rohlin Property.* One of the ultimate goals of the study of flows is to completely classify all flows on AFD von Neumann algebras. In order to achieve this, it is important to characterize the Rohlin property by using invariants for flows. A candidate for this characterization is the following conjecture.

**Conjecture 4.24.** (See Section 8 of Masuda–Tomatsu [44]) *Let $M$ be an AFD von Neumann algebra and let $\alpha$ be a flow on $M$. Let $\tilde{\alpha} : \mathbf{R} \curvearrowright \tilde{M}$ be a canonical extension of $\alpha$. Then the following three conditions are equivalent.*

(1) *The action $\alpha$ has the Rohlin property.*
(2) *We have $\pi_{\tilde{\alpha}}(\tilde{M})' \cap \tilde{M} \rtimes_{\tilde{\alpha}} \mathbf{R} = \pi_{\tilde{\alpha}}(Z(\tilde{M}))$.*
(3) *The action $\alpha$ has full Connes spectrum and is centrally free.*

We will give a partial answer for this conjecture by generalizing Theorem 4.1. We start off by showing the following lemma.

**Lemma 4.25.** *Let $M$ be an AFD factor of type III. Let $\alpha$ be an automorphism of $M$ with trivial Connes–Takesaki module. Then $\alpha$ is centrally outer if and only if $\tilde{\alpha}^\gamma$ is outer for almost every $\gamma \in \Gamma$. Here, $C = L^\infty(\Gamma, \mu)$ is the center of $\tilde{M}$ and $\tilde{\alpha} = \int_\Gamma^\oplus \tilde{\alpha}^\gamma d\mu(\gamma)$ is the direct integral decomposition.*

*Proof.* This is shown by Proposition 5.4 of Haagerup–Størmer [21] and Theorem 3.4 of Lance [36]. □

In order to state our theorem, we define the following notion.

**Definition 4.26.** *Let $C$ be an abelian von Neumann algebra and let $\beta$ be a flow on $C$. Then $\beta$ is said to be nowhere trivial if for any $e \in \mathrm{Proj}(C^\beta)$, $\beta|_{C_e}$ is not $\mathrm{id}_{C_e}$ as a flow.*

The following theorem means that we need not consider Conjecture 4.24 for flows on AFD von Neumann algebras of type $\mathrm{III}_0$ anymore.

**Theorem 4.27.** (a) *Let $M$ be a von Neumann algebra of type $\mathrm{III}_0$ and $\alpha$ be a flow on $M$. Assume that $\mathrm{mod}(\alpha)$ is nowhere trivial, then conditions (1)–(3) in the above conjecture are all equivalent to the following condition.*

(4) *The action $\alpha$ is centrally free.*

(b) *If conditions (1)–(3) are equivalent for flows on the AFD factor of type $\mathrm{II}_\infty$, then these conditions are also equivalent for flows on AFD von Neumann algebras of type $\mathrm{III}_0$.*

*Proof.* **Step 0.** The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are shown in Lemma 3.17 and Corollary 4.13 of Masuda–Tomatsu [44]. The implication (3) $\Rightarrow$ (4) is trivial.

**Step 1.** First, we show (a) and (b) when $M$ is a factor.

(a) We show the implication (4) $\Rightarrow$ (1). If $\mathrm{mod}(\alpha) : \mathbf{R} \curvearrowright Z(M)$ is faithful, then $\alpha$ satisfies condition (1) by Theorem 4.1. In the following, we assume that $\mathrm{mod}(\alpha)$ is not faithful. By the ergodicity of $\theta$, $\mathrm{mod}(\alpha)$ has a non-trivial period $p \in (0, \infty)$. Since $\theta$ is faithful and commutes with $\mathrm{mod}(\alpha)$, $C^{\mathrm{mod}(\alpha)}$ is not trivial. Hence, the restriction of $\theta$ to $C^{\mathrm{mod}(\alpha)}$ is either free or periodic.

When the restriction of $\theta$ to $C^{\mathrm{mod}(\alpha)}$ is free, then the proof goes parallel to Lemma 4.15, using Lemma 4.25.

When the restriction of $\theta$ to $C^{\mathrm{mod}(\alpha)}$ is periodic, then the proof goes parallel to Lemma 4.8.

(b) What remains to do is to reduce the case when $\mathrm{mod}(\alpha)$ is trivial to Conjecture 4.24 for flows on the AFD factor of type$\mathrm{II}_\infty$. This goes parallel to the proof of Lemma 4.14.

**Step 2.** Next, we consider the proof of this theorem for the case when $M$ is not a factor. Decomposing into a direct integral, we may assume that $\alpha$ is centrally ergodic. We need to consider the case when $\alpha|_{Z(M)}$ is faithful, the case when $\alpha|_{Z(M)}$ has a non-trivial period and the case when $\alpha|_{Z(M)}$ is trivial separately. When $\alpha|_{Z(M)}$ is faithful, the implication (4) $\Rightarrow$ (1) follows from Theorem 4.6 and Proposition 4.30. When $\alpha|_{Z(M)}$ has non-trivial period, then the proof is similar to that of Lemma 4.15. When $\alpha|_{Z(M)}$ is trivial, then the implication follows from the case when $M$ is a factor. □

**Remark 4.28.** By the same argument, it is possible to reduce Conjecture 4.24 for flows on the AFD factor of type $\mathrm{III}_\lambda$ $(0 < \lambda < 1)$, $\mathrm{III}_1$ to Conjecture 4.24 for actions of $\mathbf{R} \times \mathbf{Z}$, actions of $\mathbf{R}^2$ on the AFD factor of type $\mathrm{II}_\infty$, respectively.

4.4. **Appendix of Section 4.** In this section, we explain the proof of two statements which are used in the proof of the main theorem of Section 4.

4.4.1. *Proof of Theorem 4.6.* For readers who do not have any access to Feldman [14], we will explain the outline of the proof of Theorem 4.6.

*Proof of Theorem 4.6.*
The proof consists of two parts. The first is, for any cube $Q$ of $\mathbf{R}^d$, constructing a $Q$-set $F$ with $\mu(QF) > 0$. This part is shown by the same argument as in the proof of Lemma of Lind [37] (Note that Wiener's ergodic theorem holds for actions without invariant measures). The second is to show this theorem by using the first part. This is achieved by the same argument as in the proof of Theorem 1 of Feldman–Lind [16]. In the proof, they show two key statements (Statements (i) and (ii) in p.341 of Feldman–Lind [16]). We need statements corresponding to them. Let $L$, $N$, $P$ be positive natural numbers. Assume that $P$ is a multiple of $L$. Set

$$Q_P := [0, P)^d,$$

$$S_L(Q_P) := \{t = (t_1, \cdots, t_d) \in \mathbf{R}^d \mid \frac{P}{L} \leq t_j < P - \frac{P}{L} \text{ for all } j\},$$

$$B_N(Q_P) := \{t = (t_1, \cdots, t_d) \in \mathbf{R}^d \mid -N \leq t_j < P + N \text{ for all } j\} \setminus Q_P,$$

$$C_{P/L} := \{n = (n_1, \cdots, n_d) \in \mathbf{Z}^d \mid 0 \leq n_j < \frac{P}{L} \text{ for all } j\}.$$

The corresponding statements are the following.

(i)' Let $\eta > 0$ be a positive number. Then for any sufficiently large even integer $M$, any integer $L$, any multiple $P = NLM$ of $LM$ and any $Q_P$-set $F$, we have

$$\mu(B_{2N}(Q_P)(tF)) < \eta$$

for over 9/10 of the elements $t$ of $C_{P/L}$.

(ii)' Let $\xi > 0$ be a positive number. Then for any sufficiently large integer $L$, any integer $M$, any multiple $P = NLM$ of $LM$ by a multiple $N$ of $L$ and any $Q_P$-set $F$, we have

$$\mu(S_L(Q_N)(NC_{P/L})(tF)) > \mu(Q_P(tF)) - \xi$$

for over 9/10 of the elements $t$ of $C_{P/L}$.

The other parts of the proof is the same as that of Theorem 1 of Feldman–Lind [16]. □

We may assume that $X$ is a compact metric space and the map $T : \mathbf{R}^d \times X \to X$ is continuous.

**Lemma 4.29.** *In the context of Theorem 4.6, the set $F$ can be chosen to be a Borel subset of $X$.*

*Proof.* This follows from the proof of Lemma of Lind [37]. By removing a null set, we may assume that the set $D$ in p.181 of Lind [37] is a Borel subset of $X$. Then the set

$$\{(t, x) \in \mathbf{R}^n \times X \mid T_t(x) \in D\}$$

is a Borel subset of $Q \times X$. Hence by Fubini's theorem, the map $\psi_j^{\pm}(x)$ in p.181 of Lind is Borel measurable. Thus the set $F$ can be chosen to be a Borel subset. □

**Proposition 4.30.** *In the context of Theorem 4.6, the map*

$$Q \times F \ni (t, x) \mapsto T_t(x) \in T_Q F$$

*is a Borel isomorphism.*

*Proof.* By Lemma 4.29, if $C \subset T_Q F$ is a Borel subset, then $C$ is also Borel in $X$. Hence the map $Q \times F \ni (t, x) \mapsto T_t(x) \in T_Q F$ is a Borel bijection. Hence by Corollary A.10 of Takesaki [58], this map is a Borel isomorphism. □

**Proposition 4.31.** *In the context of Theorem 4.6, if $\mathbf{R}^d = \mathbf{R}$, then the map*

$$Q \times F \ni (t, x) \mapsto T_t(x) \in T_Q F$$

*is non-singular.*

*Proof.* This is based on Lemma 3.1 of Kubo [35]. The action $T$ of $\mathbf{R}$ on $X$ induces an action $\tilde{T}$ of $\mathbf{R}$ on $T_Q(F)$. Then $\tilde{T}$ defines an action $S$ of $\mathbf{Z}$ on $F$. Then $(F, \nu)$, $S$ and $(T_Q F, \mu)$ satisfy the assumptions of Lemma 3.1 of Kubo [35]. □

4.4.2. *On a Measurability of a Certain Map.* In the proof of Lemma 4.9, we use the fact that a map from a measured space to the automorphism group of a von Neumann algebra is measurable (See Fact between Lemma 4.10 and Lemma 4.11). Probably it is well-known to specialists. However, we could not find appropriate references. Hence, we present the proof here.

**Proposition 4.32.** *If we identify $N$ with $N_0 \otimes L^\infty([0,p))$, the map $[0,p)^2 \ni (t,\gamma) \mapsto \theta_{t,\gamma} \in \mathrm{Aut}(N_0)$ is Lebesgue measurable.*

*Proof.* By Lusin's theorem, it is enough to show that the map $[0,p)^2 \ni (t,\gamma) \mapsto \phi_0 \circ \theta_{t,\gamma} \in (N_0)_*$ is Lebesgue measurable for $\phi_0 \in (N_0)_*$. We identify $N_*$ with $L^1_{(N_0)_*}([0,p))$ and set $\phi := \phi_0 \otimes \mathrm{id}$. Since the map $s \mapsto \phi \circ \theta_s \in L^1_{(N_0)_*}([0,p))$ is continuous, for any $\epsilon > 0$, there exists a positive number $\delta$ such that

$$\|\phi \circ \theta_s - \phi\| < \epsilon^2 \tag{5}$$

for $|s| < \delta$. Take a partition $0 = s_0 < s_1 < \cdots < s_n = p$ so that $|s_i - s_{i+1}| < \delta$. For each $i = 0, \cdots, n$, the map $[0,p) \ni \gamma \mapsto (\phi \circ \theta_{s_i})_\gamma$ is Lebesgue measurable and integrable. Hence it is possible to approximate $\phi \circ \theta_{s_i}$ by Borel simple step functions, that is, for each $i$, there exists a compact subset $K_i$ of $[0,p)$ which satisfies the following conditions.

(2) We have $\mu(K_i) > p - \epsilon$.

(3) There exist a Borel partition $\{I_j\}$ of $K_i$ and $\phi_{i,j} \in (N_0)_*$ such that

$$\|(\phi \circ \theta_{s_i})_\gamma - \sum_j \phi_{i,j} \chi_{I_j}(\gamma)\| < \epsilon$$

for $\gamma \in K_i$.

Set

$$\psi_{t,\gamma} := \sum_{i,j} \phi_{i,j} \chi_{[s_i, s_{i+1})}(t) \chi_{I_j}(\gamma).$$

for each $(t,\gamma) \in [0,p)^2$. For each $s \in [s_i, s_{i+1})$, set

$$K_s := \{\gamma \in [0,p) \mid \|(\phi \circ \theta_s)_\gamma - (\phi \circ \theta_{s_i})_\gamma\| < \epsilon\}.$$

Then by the above inequality (1), we have $\mu(K_s) > p - \epsilon$. For $\gamma \in K_s \cap K_i$, we have

$$\|(\phi \circ \theta_s)_\gamma - \psi_{s,\gamma}\| < 2\epsilon.$$

Set

$$K := \{(s,\gamma) \in [0,p)^2 \mid \|(\phi \circ \theta_s)_\gamma - \psi_{s,\gamma}\| < 2\epsilon\}.$$

Then we have $\mu(K) > p(p-2\epsilon)$. Hence $(s,\gamma) \mapsto (\phi \circ \theta_s)_\gamma$ is well-approximated by simple step Borel functions in measure convergence. Hence this is Lebesgue measurable. $\square$

## 5. Characterization of approximate innerness of finite index endomorphisms of AFD factors

In this subsection, as an application of the study of the Rohlin property, we present a characterization of approximate innerness of endomorphisms.

5.1. **Preliminaries of Section 5.** First, we explain some notions necessary to understand our main theorem of this section. In order to understand endomorphisms, some notions of automorphisms were generalized to that of endomorphisms by Izumi [22] and Masuda–Tomatsu [40].

5.1.1. *A topology of semigroups of endomorphisms.* Let $M$ be a factor of type III. Let $\mathrm{End}(M)_0$ be the set of all finite index endomorphisms $\rho$ of $M$. Let $d(\rho)$ be the square root of the minimal index of $M \supset \rho(M)$ and $E_\rho$ be the minimal expectation from $M$ to $\rho(M)$. Set $\phi_\rho := \rho^{-1} \circ E_\rho$. In Masuda–Tomatsu [40], a topology of $\mathrm{End}(M)_0$ is introduced in the following way. We have

$$\rho_i \to \rho$$

if, by definition, $\|\psi \circ \phi_{\rho_i} - \psi \circ \phi_\rho\| \to 0$ for any $\psi \in M_*$.

5.1.2. *Canonical extension of endomorphisms.* Let $\varphi$ be a normal faithful semifinite weight of $M$ and $\sigma^\varphi$ be the group of modular automorphisms of $\varphi$. In Izumi [22], an extension $\tilde{\rho}$ of $\rho \in \mathrm{End}(M)_0$ on the continuous core $\tilde{M} := M \rtimes_{\sigma^\varphi} \mathbf{R}$ is introduced in the following way. We have

$$\tilde{\rho}(x\lambda_t^{\sigma^\varphi}) = d(\rho)^{it}\rho(x)[D\varphi \circ \phi_\rho : D\varphi]_t \lambda_t^{\sigma^\varphi}$$

for $t \in \mathbf{R}$, $x \in M$, where $[D\varphi \circ \phi_\rho : D\varphi]_t$ is the Connes cocycle between $\varphi \circ \phi_\rho$ and $\varphi$. This extension does not depend on the choice of $\varphi$ under a specific identification (See Theorem 2.4 of Izumi [22]). The extension $\tilde{\rho}$ is said to be the canonical extension of $\rho$.

In Lemma 3.5 of Masuda–Tomatsu [40], it is shown that there exists a left inverse $\phi_{\tilde{\rho}}$ of $\tilde{\rho}$ satisfying

$$\phi_{\tilde{\rho}}(x\lambda_t^\varphi) = d(\rho)^{-it}\phi_\rho(x[D\phi : D\phi \circ \phi_\rho]_t)\lambda_t^\varphi$$

for $x \in M$, $t \in \mathbf{R}$.

5.2. **The main theorem of Section 5.** The main theorem of this paper is the following.

**Theorem 5.1.** *Let $\rho$, $\sigma$ be endomorphisms of an AFD factor $M$ of type* III *with $d(\rho), d(\sigma) < \infty$. Then the following two conditions are equivalent.*

(1) *We have $\phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}} \circ \theta_{-\log(d(\sigma))}|_{\mathcal{Z}(\tilde{M})}$.*

(2) *There exists a sequence $\{u_n\}$ of unitaries of $M$ with $\mathrm{Ad}u_n \circ \rho \to \sigma$ as $n \to \infty$.*

As a corollary, we have the following result.

**Corollary 5.2.** *Let $M$ be an AFD factor and $R_0$ be the AFD factor of type* $\mathrm{II}_1$. *Take endomorphisms $\rho_1$, $\rho_2 \in \mathrm{End}(M)_0$. Then the following two conditions are equivalent.*

(1) *There exists a sequence of unitaries $\{u_n\}$ of $M \otimes R_0$ with $\mathrm{Ad}u_n \circ (\rho_1 \otimes \mathrm{id}_{R_0}) \to \rho_2 \otimes \mathrm{id}_{R_0}$ as $n \to \infty$.*

(2) *There exists a sequence of unitaries $\{v_n\}$ of $M$ with $\mathrm{Ad}v_n \circ \rho_1 \to \rho_2$ as $n \to \infty$.*

*Proof.* By the identification $\mathcal{Z}((M \otimes R_0) \rtimes_{\sigma^\varphi \otimes \mathrm{id}_{R_0}} \mathbf{R}) \cong \mathcal{Z}((M \rtimes_{\sigma^\varphi} \mathbf{R}) \otimes R_0) \cong \mathcal{Z}(M \rtimes_{\sigma^\varphi} \mathbf{R})$ by

$$(x \otimes y)\lambda_t^{\sigma^\varphi \otimes \mathrm{id}_{R_0}} \mapsto (x\lambda_t^{\sigma^\varphi}) \otimes y,$$

we have $\phi_{\rho_i \otimes \mathrm{id}_{R_0}} = \phi_{\rho_i}$ on the center of the continuous core for $i = 1, 2$. We also have $d(\rho_i \otimes \mathrm{id}_{R_0}) = d(\rho_i)$. Hence by Theorem 5.1, conditions (1) and (2) are equivalent. □

Note that this corollary would be quite difficult to show without Theorem 5.1 (See also Section 3 of Connes [5]).

As we will explain later, this is a generalization of a work of Kawahigashi–Sutherland–Takesaki [31], in which our main theorem of this section is shown when $\rho$ and $\sigma$ are automorphisms. We briefly explain this. For an automorphism $\alpha$ of a factor $M$, we have $\phi_{\tilde{\alpha}} = \tilde{\alpha}^{-1}$. Hence in this case, considerling $\phi_{\tilde{\alpha}}$ is equivalent to considering $\tilde{\alpha}$. Set $\mathrm{mod}(\alpha) := \tilde{\alpha}|_{\mathcal{Z}(\tilde{M})}$. The following is a special case of Theorem 5.1.

**Corollary 5.3.** (Theorem 1(1) of Kawahigashi–Sutherland–Takesaki [31]) *Let $M$ be an AFD factor of type* III *and $\alpha$ be an automorphism of $M$. Then the following two conditions are equivalent.*

(1) *The automorphism $\alpha$ is approximately inner.*

(2) *The automorphism $\mathrm{mod}(\alpha) \in \mathrm{Aut}(\mathcal{Z}(\tilde{M}))$ is trivial.*

Theorem 5.1 should also be useful for classifying actions of compact groups on AFD factors of type III. Popa–Wassermann [50] and Masuda–Tomatsu [42] showed that any compact group has only one minimal action on the AFD factor of type $\mathrm{II}_1$, up to conjugacy. One of the next problems is to classify actions of compact groups on AFD factors of type III. In Masuda–Tomatsu [41] and [43], they are trying to solve this problem, and some partial answers to this problem are obtained (Theorems A, B of [41] and Theorem 2.4 of [43]). However, still the problem has not been solved completely. In Masuda–Tomatsu [41], a conjecture about this classification problem is proposed (Conjecture 8.2). Our main theorem of this section implies that if two actions of discrete Kac algebras on AFD factors of type III have the same invariants, the difference of these two actions is approximately inner (See Problem 8.3 and the preceding argument to that problem of Masuda–Tomatsu [41]). In order to classify group actions, whether the difference of two actions is approximately inner or not is very important. Kawahigashi–Sutherland–Takesaki [31] and Masuda–Tomatsu [40] characterize the approximate innerness of endomorphisms under such a motivation. Theorem 5.1 is a generalization of their results.

In the following, we will show Theorem 5.1. Implication (2)$\Rightarrow$ (1) is shown easily by using known results.

*Proof of implication* (2) $\Rightarrow$ (1) *of Theorem 5.1.* This is shown by the same argument as that of the proof of implication (1) $\Rightarrow$ (2) of Theorem 3.15 of [40]. Assume that we have $\mathrm{Ad}u_n \circ \rho \to \sigma$ as $n \to \infty$. Then by the continuity of normalized canonical extension (Theorem 3.8 of Masuda–Tomatsu [40]), we have

$$\phi_{\tilde{\rho}} \circ \theta_{-\log d(\rho)} \circ \mathrm{Ad}u_n^*(x) \to \phi_{\tilde{\sigma}} \circ \theta_{-\log d(\sigma)}(x)$$

in the strong* topology for any $x \in \tilde{M}$. Hence we have

$$\phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho)/d(\sigma))}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}.$$

□

In the following, we will show the reverse implication. Our strategy is to reduce the problem to that of endomorphisms on semifinite von Neumann algebras. In order to achieve this, in Kawahigashi–Sutherland–Takesaki [31] and Masuda–Tomatsu [40], they have used discrete decomposition theorems (See Connes [9]). However, in our situation, the centers of the images of canonical extensions may not coincide with that of $\tilde{M}$. This makes the problem difficult. It seems that Corollary 4.4 of Izumi [22] means that it is difficult to show Theorem 5.1 by the same strategy as those in them. Instead, we will use continuous decomposition. We also note that our method gives a proof of Theorem (1) of Kawahigashi–Sutherland–Takesaki [31] which does not depend on the types of AFD factors.

5.3. **Approximation on the continuous core.** In order to prove implication (1) $\Rightarrow$ (2) of Theorem 5.1, we need to prepare some lemmas. We first show the implication when $\phi_{\tilde{\rho}} = \phi_{\tilde{\sigma}}$ on the center of the continuous core. Until the end of the proof of Lemma 5.22, we always assume that $d(\rho) = d(\sigma)$ and $\phi_{\tilde{\rho}} = \phi_{\tilde{\sigma}}$ on the center of the continuous core. Choose a dominant weight $\varphi$ of $M$ (For the definition of dominant weights, see Definition II.1.2. and Theorem II.1.3. of Connes–Takesaki [11]). Then by Lemma 2.3 (3) of Izumi [22], it is possible to choose unitaries $u$ and $v$ of $M$ so that $(\varphi, \mathrm{Ad}u \circ \rho)$ and $(\varphi, \mathrm{Ad}v \circ \sigma)$ are invariant pairs (See Definition 2.2 of Izumi [22]). More precisely, we have

$$\varphi \circ \mathrm{Ad}u \circ \rho = d(\rho)\varphi, \ \varphi \circ E_{\mathrm{Ad}u\circ\rho} = \varphi,$$

$$\varphi \circ \mathrm{Ad}v \circ \sigma = d(\sigma)\varphi, \ \varphi \circ E_{\mathrm{Ad}v\circ\sigma} = \varphi.$$

By replacing $\rho$ by $\mathrm{Ad}u\circ\rho$ and $\sigma$ by $\mathrm{Ad}v\circ\sigma$ respectively, we may assume that $(\varphi, \rho)$ and $(\varphi, \sigma)$ are invariant pairs. In the rest of this paper, we identify $\tilde{M}$ with $M \rtimes_{\sigma^\varphi} \mathbf{R}$. Let $h$ be a positive self-adjoint operator affiliated to $\tilde{M}$ satisfying $h^{-it} = \lambda_t^\varphi$ and $\hat{\varphi}$ be the dual weight of $\varphi$. Let $\tau$ be a trace of $\tilde{M}$ defined by $\hat{\varphi}(h\cdot)$.

**Lemma 5.4.** *For $\rho \in \mathrm{End}(M)_0$, we have $\phi_{\tilde{\rho}} = \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}$, where $E_{\tilde{\rho}}$ is the conditional expectation with respect to $\tau$.*

*Proof.* For $x \in M$ and $t \in \mathbf{R}$, we have

$$\begin{aligned}\tilde{\rho} \circ \phi_{\tilde{\rho}}(x\lambda_t^\varphi) &= \tilde{\rho}(d(\rho)^{-it}\phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t)\lambda_t^\varphi) \\ &= d(\rho)^{it}d(\rho)^{-it}\rho(\phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t))[D\varphi \circ \phi_\rho : D\varphi]_t\lambda_t^\varphi \\ &= E_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t)[D\varphi \circ \phi_\rho : D\varphi]_t\lambda_t^\varphi\end{aligned}$$

Since $(\varphi, \rho)$ is an invariant pair, we have

$$[D\varphi \circ \phi_\rho : D\phi]_t = d(\rho)^{-it}.$$

Hence we have

$$E_\rho(x[D\varphi : D\varphi\circ\phi_\rho]_t)[D\varphi\circ\circ\phi_\rho : D\varphi]_t\lambda_t^\varphi = E_\rho(x)d(\rho)^{it}d(\rho)^{-it}\lambda_t^\varphi = E_\rho(x)\lambda_t^\varphi.$$

Hence by an argument of p.226 of Longo [38], it is shown that $\tilde{\rho} \circ \phi_{\tilde{\rho}}$ is the expectation with respect to $\tau$. ☐

**Lemma 5.5.** *For $\rho \in \mathrm{End}(M)_0$, we have $\tau \circ \phi_{\tilde{\rho}} = d(\rho)^{-1}\tau$.*

*Proof.* By Lemma 5.4, we have $\phi_{\tilde{\rho}} = \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}$. On the other hand, by Proposition 2.5 (4) of Izumi [22], we have $\tau \circ \tilde{\rho} = d(\rho)\tau$. Hence we have

$$\begin{aligned}
\tau \circ \phi_{\tilde{\rho}} &= d(\rho)^{-1}\tau \circ \tilde{\rho} \circ \phi_{\tilde{\rho}} \\
&= d(\rho)^{-1}\tau \circ \tilde{\rho} \circ \tilde{\rho}^{-1} \circ E_{\tilde{\rho}} \\
&= d(\rho)^{-1}\tau \circ E_{\tilde{\rho}} \\
&= d(\rho)^{-1}\tau.
\end{aligned}$$

□

In the following, we identify $\mathcal{Z}(\tilde{M})$ with $L^\infty(X,\mu)$. Let

$$\tau = \int_X^\oplus \tau_x \, d\mu(x)$$

be the direct integral decomposition of $\tau$.

**Lemma 5.6.** *Let $\rho, \sigma$ be elements of* $\mathrm{End}(M)_0$. *Assume that $\phi_{\tilde{\rho}}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}$ and $d(\rho) = d(\sigma)$. For $a \in \tilde{M}_+$ with $\tau(a) < \infty$, set*

$$b := \tilde{\rho}(a) = \int_X^\oplus b_x \, d\mu(x),$$

$$c := \tilde{\sigma}(a) = \int_X^\oplus c_x \, d\mu(x).$$

*Then we have*

$$\tau_x(b_x) = \tau_x(c_x)$$

*for almost every $x \in X$.*

*Proof.* Take an arbitrary positive element $z$ of $\mathcal{Z}(\tilde{M})_+$. Then we have

$$\begin{aligned}
\tau(bz) &= \int_X \tau_x(b_x z_x) \, d\mu(x) \\
&= \int_X \tau_x(b_x) z_x \, d\mu(x).
\end{aligned}$$

Similarly, we have

$$\tau(cz) = \int_X \tau_x(c_x) z_x \, d\mu(x).$$

On the other hand, by Lemma 5.5, we have

$$\begin{aligned}\tau(bz) &= d(\rho)\tau \circ \phi_{\tilde{\rho}}(bz)\\ &= d(\rho)\tau \circ \phi_{\tilde{\rho}}(\tilde{\rho}(a)z)\\ &= d(\rho)\tau \circ \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}(\tilde{\rho}(a)z)\\ &= d(\rho)\tau \circ \tilde{\rho}^{-1}(\tilde{\rho}(a)E_{\tilde{\rho}}(z))\\ &= d(\rho)\tau(a\phi_{\tilde{\rho}}(z)).\end{aligned}$$

Since we assume $d(\rho) = d(\sigma)$ and $\phi_{\tilde{\rho}}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}$, the last number of the above equality is $d(\sigma)\tau(a\phi_{\tilde{\sigma}}(z))$, which is shown to be $\tau(cz)$ in a similar way. Hence we have

$$\int_X \tau_x(b_x)z_x \, d\mu(x) = \int_X \tau_x(c_x)z_x \, d\mu(x).$$

Since the maps $x \mapsto \tau_x(b_x)$ and $x \mapsto \tau_x(c_x)$ are integrable functions and $z \in L^\infty(X, \mu) = L^1(X, \mu)^*$ is arbitrary, we have $\tau_x(b_x) = \tau_x(c_x)$ for almost every $x \in X$. □

Note that we have never used the assumption that $M$ is approximately finite dimensional up to this point. However, in order to show the following lemma, we need to assume that $M$ is approximately finite dimensional. Let

$$\tilde{M} = \int_X^\oplus \tilde{M}_x \, d\mu(x)$$

be the direct integral decomposition.

**Lemma 5.7.** *Let $M$ be an AFD factor of type* III *and $\rho$, $\sigma$ be as in Lemma 5.6. Then for almost every $x \in X$, there exist a factor $B_x$ of type $I_\infty$, a unitary $u$ of $\tilde{M}_x$ and a sequence $\{u_n\}$ of unitaries of $\tilde{M}_x$ with the following properties.*

(1) *The relative commutant $B_x' \cap \tilde{M}_x$ is finite.*
(2) *There exists a sequence of unitaries $\{v_n\}$ of $B_x' \cap \tilde{M}_x$ with $u_n = (v_n \otimes 1)u$, where we identify $\tilde{M}_x$ with $(B_x' \cap \tilde{M}_x) \otimes B_x$.*
(3) *For almost every $x \in X$ and for any $a \in \tilde{M}$, we have $\mathrm{Ad}u_n((\tilde{\rho}(a))_x) \to (\tilde{\sigma}(a))_x$ in the strong * topology.*
(4) *We have $B_x \subset u(\tilde{\rho}(\tilde{M}))_x u^* \cap (\tilde{\sigma}(\tilde{M}))_x$.*

*Proof.* Let $B_0 \subset \tilde{\rho}(\tilde{M})$ be a factor of type $\mathrm{I}_\infty$ with $Q := \tilde{\rho}(\tilde{M}) \cap B_0'$ finite. Let $\{f_{ij}^0\}$ be a matrix unit generating $B_0$. We may assume that $\tau(f_{ii}^0) < \infty$. Then since $(\tau \circ E_{\tilde{\rho}})_x((f_{11}^0)_x) < \infty$ for almost every $x \in X$,

$P := \tilde{M} \cap B_0'$ is also finite. Then by Lemma 5.6, there exists a partial isometry $v$ of $\tilde{M}$ with $v^*v = \tilde{\rho}(f_{11}^0)$, $vv^* = \tilde{\sigma}(f_{11}^0)$. Set

$$u := \sum_{j=1}^{\infty} \tilde{\sigma}(f_{j1}^0) v \tilde{\rho}(f_{1j}^0).$$

Then $u$ is a unitary of $\tilde{M}$ with $u\tilde{\sigma}(f_{ij}^0)u^* = \tilde{\rho}(f_{ij}^0)$. Set

$$B := \tilde{\sigma}(B_0)(= u\tilde{\rho}(B_0)u^*),$$

$$f_{ij} := \tilde{\sigma}(f_{ij}^0)(= u\tilde{\rho}(f_{ij}^0)u^*).$$

By replacing $\tilde{\rho}$ by $\mathrm{Ad}u \circ \tilde{\rho}$, we may assume that $\tilde{\rho}(f_{ij}) = \tilde{\sigma}(f_{ij})$. In the following, we identify $\tilde{M}$ with $P \otimes B$ and $P$ with $R \otimes \mathcal{Z}(\tilde{M})$, where $R$ is the AFD factor of type $\mathrm{II}_1$. By the approximate finite dimensionality of $R$ and $\mathcal{Z}(\tilde{M})$, there exists a sequence $\{\{e_{ij}^n \otimes a_k^n\}_{i,j,k}\}_{n=1}^{\infty}$ of systems of partial isometries of $P$ with the following properties.

(1) For each $n$, the system $\{e_{ij}^n\}_{i,j}$ is a matrix unit of $R$.
(2) For each $n$, the system $\{a_k^n\}_k$ is a partition of unity in $\mathcal{Z}(\tilde{M})$.
(3) For each $n$, $\{e_{ij}^{n+1}\}_{i,j}$ is a refinement of $\{e_{ij}^n\}_{i,j}$.
(4) For each $n$, $\{a_k^{n+1}\}_k$ is a refinement of $\{a_k^n\}_k$.
(5) We have $\bigvee_{n=1}^{\infty}\{e_{ij}^n \otimes a_k^n\}_{i,j,k}'' = P$.

Fix a natural number $n$. Then by Lemma 5.6, we have

$$\tau_x((\tilde{\rho}(e_{11}^n \otimes a_k^n \otimes f_{11}))_x) = \tau_x((\tilde{\sigma}(e_{11}^n \otimes a_k^n \otimes f_{11}))_x)$$

for almost every $x \in X$. Here, we should notice that we have $\tau(e_{11}^n \otimes a_k^n \otimes f_{11}) \leq \tau(1 \otimes 1 \otimes f_{11}) < \infty$. Hence the assumption of Lemma 5.6 is satisfied. Hence for almost every $x \in X$, there exists a partial isometry $v_k^n$ of $P_x = (\tilde{\rho}(f_{11})\tilde{M}\tilde{\rho}(f_{11}))_x$ with

$$v_k^{n*}v_k^n = \tilde{\rho}(e_{11}^n \otimes a_k^n \otimes f_{11})_x,\ v_k^n v_k^{n*} = \tilde{\sigma}(e_{11}^n \otimes a_k^n \otimes f_{11})_x.$$

Set

$$v_n := \sum_{k,j} \tilde{\sigma}(e_{j1} \otimes a_k^n \otimes f_{11})_x v_k^n \tilde{\rho}(e_{1j} \otimes a_k^n \otimes f_{11})_x.$$

Then $v_n$ is a unitary of $\tilde{\rho}(f_{11})_x \tilde{M}_x \tilde{\rho}(f_{11})_x$ with

$$v_n \tilde{\rho}(e_{ij}^n \otimes a_k^n \otimes f_{11})_x v_n^* = \tilde{\sigma}(e_{ij}^n \otimes a_k^n \otimes f_{11})_x.$$

Hence for almost every $x \in X$, there exists a sequence $\{v_n\}$ of unitaries of $P_x$ with

$$\mathrm{Ad}(v_n \otimes 1)(\tilde{\rho}(a)_x) \to \tilde{\sigma}(a)_x$$

for any $a \in \tilde{M}$. $\square$

**Lemma 5.8.** *Let $M$, $\rho$ and $\sigma$ be as in Lemma 5.7. Then there exist a unital subfactor $B$ of $\tilde{M}$, a unitary $u$ of $\tilde{M}$ and a sequence $\{u_n\}$ of unitaries of $\tilde{M}$ with the following properties.*

(1) *The factor $B$ is of type $I_\infty$.*
(2) *The relative commutant $B' \cap \tilde{M}$ is finite.*
(3) *There exists a sequence of unitaries $\{v_n\}$ of $B' \cap \tilde{M}$ with $u_n = (v_n \otimes 1)u$, where we identify $\tilde{M}$ with $(B' \cap \tilde{M}) \otimes B$.*
(4) *For any $a \in \tilde{M}$, we have $\mathrm{Ad}u_n \circ \tilde{\rho}(a) \to \tilde{\sigma}(a)$ in the strong * topology.*
(5) *We have $B \subset u\tilde{\rho}(\tilde{M})u^* \cap \tilde{\sigma}(\tilde{M})$.*

*Proof.* This is shown by "directly integrating" the above lemma. □

The conclusion of Lemma 5.8 means that $\mathrm{Ad}u_n \circ \tilde{\rho}$ converges to $\tilde{\sigma}$ point *strongly. However, this convergence is slightly weaker than that of the topology we consider. We need to fill this gap. In order to achieve this, the following criterion is very useful.

**Lemma 5.9.** (Lemma 3.8 of Masuda–Tomatsu [42]). *Let $\rho$ and $\rho_n$, $n \in \mathbf{N}$ be endomorphisms of a von Neumann algebra $N$ with left inverses $\Phi$ and $\Phi_n$, $n \in \mathbf{N}$, respectively. Fix a normal faithful state $\phi$ of $N$. Then the following two conditions are equivalent.*
(1) *We have $\lim_{n\to\infty} \|\psi \circ \Phi_n - \psi \circ \Phi\| = 0$ for all $\psi \in N_*$.*
(2) *We have $\lim_{n\to\infty} \|\phi \circ \Phi_n - \phi \circ \Phi\| = 0$ and $\lim_{n\to\infty} \rho_n(a) = \rho(a)$ for all $a \in N$.*

Hence what we need to do is to find a normal faithful state of $\tilde{M}$ satisfying condition (2) of Lemma 5.9.

**Lemma 5.10.** *Let $M$, $\rho$, $\sigma$ be as in Lemma 5.7. Then there exists a sequence of unitaries $u_n$ of $\tilde{M}$ with $\mathrm{Ad}u_n \circ \tilde{\rho} \to \tilde{\sigma}$.*

*Proof.* Take a subfactor $B$ of $\tilde{M}$, a unitary $u$ of $\tilde{M}$ and a sequence $\{v_n\}$ of unitaries of $\tilde{M}$ as in Lemma 5.8. By condition (5) in Lemma 5.8, we have $u^*Bu \subset \tilde{\rho}(\tilde{M})$. Set

$$F := \tilde{\rho}^{-1}(u^*Bu).$$

Then we have

$$\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*(B) = F,$$
$$\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*(B' \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M})) = F' \cap \tilde{M}.$$

We also have

$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*|_B = \mathrm{id}_B,$$
$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(B' \cap \tilde{M}) = B' \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M}).$$

Let $\{f_{ij}\}$ be a matrix unit generating $B$. Set

$$\overline{\tau}(a) := \tau(a\tilde{\rho}^{-1}(u^* f_{11} u))$$

for $a \in F' \cap \tilde{M}$, which is a faithful normal finite trace of $F' \cap \tilde{M}$. Let $\psi_F$ be a normal faithful state of $F$. Let $\Psi_F : \tilde{M} \to (F' \cap \tilde{M}) \otimes F$ is the natural identification map. Then by the above observation, for $a \in B' \cap \tilde{M}$ and $i$, $j$, we have

$$\begin{aligned}
&(\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*(af_{ij}) \\
&= (\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ (\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*) \circ (\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(af_{ij}) \\
&= (\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ (\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*)((\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*|_{B' \cap \tilde{M}})(a) f_{ij}) \\
&= (\overline{\tau} \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(a)(\psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(f_{ij}).
\end{aligned}$$

Since $B \subset \tilde{\sigma}(\tilde{M}) \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M})$, we have

$$E_{\tilde{\sigma}}(af_{ij}) = E_{\tilde{\sigma}}(a) f_{ij},$$

$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(af_{ij}) = \mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(a) f_{ij}$$

for $a \in B' \cap \tilde{M}$. Notice that $\tilde{\sigma}^{-1}(f_{ij}) = \tilde{\rho}^{-1}(u^* f_{ij} u)$ by conditions (3) and (4)of Lemma 5.8. Then for any $a \in B' \cap \tilde{M}$, we have

$$\begin{aligned}
&(\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*(v_n^* \otimes 1)(af_{ij}) \\
&= (\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\rho}}((u^*(v_n^* a v_n) u)(u^* f_{ij} u)) \\
&= \overline{\tau} \circ \phi_{\tilde{\rho}}(u^*(v_n^* a v_n) u) \psi_F(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= \tau(\phi_{\tilde{\rho}}(u^*(v_n^* a v_n) u) \tilde{\rho}^{-1}(u^* f_{11} u)) \psi_F(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= \tau \circ \phi_{\tilde{\rho}}(u^*(v_n^* a v_n) f_{11} u) \psi_F(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= d(\rho) \tau(u^*(v_n^* a v_n) f_{11} u) \psi_F(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= d(\sigma) \tau(af_{11}) \psi_F(\tilde{\sigma}^{-1}(f_{ij})) \\
&= \tau(\phi_{\tilde{\sigma}}(a) \tilde{\sigma}^{-1}(f_{11})) \psi_F(\tilde{\sigma}^{-1}(f_{ij})) \\
&= \tau(\phi_{\tilde{\sigma}}(a) \tilde{\rho}^{-1}(u^* f_{11} u)) \psi_F(\tilde{\sigma}^{-1}(f_{ij})) \\
&= (\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\sigma}}(af_{ij}).
\end{aligned}$$

Hence we have $(\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}(u^*(v_n \otimes 1)^*) = (\overline{\tau} \otimes \psi_F) \circ \Psi_F \circ \phi_{\tilde{\sigma}}$ for any $n$. Hence by Lemma 5.8 and Lemma 5.9, we have $\mathrm{Ad}((v_n \otimes 1)u) \circ \tilde{\rho} \to \tilde{\sigma}$. □

5.4. **Averaging by the trace-scaling action.** In this subsection, we always assume that $M$ is an AFD factor of type III. Let $\varphi$ be a dominant weight of $M$ and $\rho, \sigma \in \mathrm{End}(M)_0$ be finite index endomorphisms with $(\varphi, \rho)$ and $(\varphi, \sigma)$ invariant pairs. Set

$$\tilde{M} := M \rtimes_{\sigma^\varphi} \mathbf{R}.$$

Let $\psi_0$ be a normal faithful state of $\tilde{M}$ and $\{\psi_i\}_{i=1}^{\infty}$ be a norm dense sequence of the unit ball of $\tilde{M}_*$. Let $\theta$ be the dual action on $\tilde{M}$ of $\sigma^\varphi$. We will replace the sequence $\{u_n\}$ chosen in the previous section so that it is almost invariant by $\theta$. In order to achieve this, we use a property of $\theta$ which is said to be the Rohlin property. In order to explain this property, we first need to explain related things. Let $\omega$ be a free ultrafilter of $\mathbf{N}$. A sequence $\{[-1,1] \ni t \mapsto x_{n,t} \in \tilde{M}\}_{n=1}^{\infty}$ of maps from $[-1,1]$ to $\tilde{M}$ is said to be $\omega$-equicontinuous if for any $\epsilon > 0$, there exist an element $U \subset \mathbf{N}$ of $\omega$ and $\delta > 0$ with $\|x_{n,t} - x_{n,s}\| < \epsilon$ for any $s, t \in [-1,1]$ with $|s-t| < \delta$, $n \in U$. Set

$$\mathcal{C}_\omega := \{(x_n) \in l^\infty(\tilde{M}) \mid \|x_n\psi - \psi x_n\| \to 0 \text{ as } n \to \omega \text{ for any } \psi \in \tilde{M}.\},$$

$$\mathcal{C}_{\theta,\omega} := \{(x_n) \in \mathcal{C}_\omega \mid \text{the maps } \{t \mapsto \theta_t(x_n)\}_{n=1}^{\infty} \text{ are } \omega \text{ equicontinuous.}\},$$

$$\mathcal{I}_\omega := \{(x_n) \in l^\infty(\tilde{M}) \mid x_n \to 0 \text{ in the } *\text{strong topology as } n \to \omega.\}.$$

Then $\mathcal{I}_\omega$ is a (norm) closed ideal of $\mathcal{C}_{\theta,\omega}$, and the quotient $\tilde{M}_{\theta,\omega} := \mathcal{C}_{\theta,\omega}/\mathcal{I}_\omega$ is a von Neumann algebra. As mentioned in Masuda–Tomatsu [44], the action $\theta$ has the Rohlin property, that is, for any $R > 0$, there exists a unitary $v$ of $\tilde{M}_{\theta,\omega}$ with

$$\theta_t(v) = e^{-iRt}v$$

for any $t \in \mathbf{R}$ (See Section 4 of Masuda–Tomatsu [44]). Choose arbitrary numbers $r > 0$ and $0 < \epsilon < 1$. Then since $M$ is of type III, there exists a real number $R$ such that any times of which is not of the discrete spectrum of $\theta|_{\mathcal{Z}(\tilde{M})}$ and which satisfies $r/R < \epsilon^2$. Then as shown in Theorem 5.2 of Masuda–Tomatsu [44], there exists a normal injective *-homomorphism $\Theta$ from $\tilde{M} \otimes L^\infty([-R,R])$ to $\tilde{M}^\omega$ satisfying $x \otimes f \mapsto xf(v)$ for any $x \in \tilde{M}$, $f \in L^\infty([-R,R])$. For each $t \in \mathbf{R}$, set

$$\gamma_t : L^\infty([-R,R]) \ni f \mapsto f(\cdot - t) \in L^\infty([-R,R]),$$

where we identify $[-R,R]$ with $\mathbf{R}/2R\mathbf{Z}$ as measured spaces. Then the *-homomorphisms $\Theta$ and $\gamma_t$ satisfy

$$\Theta \circ (\theta_t \otimes \gamma_t) = \theta_t \circ \Theta$$

(See Theorem 5.2 of Masuda–Tomatsu [44]).

**Lemma 5.11.** *For $\psi \in \tilde{M}_*$, we have*

$$\psi^\omega \circ \Theta = \psi \otimes \tau_{L^\infty},$$

*where $\tau_{L^\infty}$ is the trace coming from the normalized Haar measure of $L^\infty([-R,R])$.*

*Proof.* Let $\{v_n\}$ be a representing sequence of $v$. For $x \otimes f \in \tilde{M} \otimes L^\infty([-R,R])$, we have

$$\begin{aligned}\psi^\omega \circ \Theta(x \otimes f) &= \psi^\omega(xf(v)) \\ &= \lim_{n\to\omega} \psi(xf(v_n)) \\ &= \psi(x) \lim_{n\to\omega} f(v_n) \\ &= \psi(x)\tau_{L^\infty}(f) \\ &= (\psi \otimes \tau_{L^\infty})(x \otimes f).\end{aligned}$$

□

Since the maps

$$[-R,R] \ni t \mapsto \psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t \in (\tilde{M})_*,$$

$$[-R,R] \ni t \mapsto \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t \in (\tilde{M})_*$$

are norm continuous, the union of their images

$$\{\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t \mid t \in [-R,R]\} \cup \{\psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t \mid t \in [-R,R]\}$$

is compact. Hence there exists a finite set $-R = t_0 < \cdots < t_J = R$ of $[-R,R]$ such that

$$\|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_{t_j} - \psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t\| < \epsilon,$$

$$\|\psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j} - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t\| < \epsilon$$

for any $i = 1, \cdots, n$, $j = 0, \cdots, J-1$ and $t \in [t_j, t_{j+1}]$. We may assume that $t_j = 0$ for some $j$. Then by Lemma 5.10, there exists a unitary $u$ of $\tilde{M}$ with

$$\|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_{t_j} \circ \mathrm{Ad}u - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j}\| < \epsilon$$

for any $j = 0, \cdots, J-1$, $i = 1, \cdots, n$ (Notice that we used the fact that we have $\phi_{\tilde{\rho}} \circ \theta_{t_j} = \theta_{t_j} \circ \phi_{\tilde{\rho}}$ and that we have $\phi_{\tilde{\sigma}} \circ \theta_{t_j} = \theta_{t_j} \circ \phi_{\tilde{\sigma}}$ for any $j = 0, \cdots, J-1$). Hence we have

$$\|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t \circ \mathrm{Ad}u - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t\| < 3\epsilon$$

for any $t \in [-R,R]$. Set

$$U : [-R,R] \ni t \mapsto \theta_t(u) \in \tilde{M},$$

which is a unitary of $\tilde{M} \otimes L^\infty([-R,R])$.

**Lemma 5.12.** *We have*

$$\|\theta_s(\Theta(U)) - \Theta(U)\|^\sharp_{\psi_0^\omega} < 2\epsilon$$

*for* $|s| \leq r$.

*Proof.* Notice that we have

$$(\theta_s \otimes \gamma_s)(U) : t \mapsto \theta_s(U_{t-s}),$$

where $U_t$ denotes the evaluation of the function $U$ at the point $t$. Hence by the definition of $U$, we have

$$(\theta_s \otimes \gamma_s)(U)_t = \theta_t(u)$$

for any $t \in [-R+r, R-r]$, where the left hand side is the evaluation of the function $(\theta_s \otimes \gamma_s)(U)$ at the point $t$. Hence by Lemma 5.11, we have

$$\begin{aligned}
&\|\theta_s(\Theta(U)) - \Theta(U)\|^\sharp_{\psi_0^\omega} \\
&= \|(\theta_s \otimes \gamma_s)(U) - U\|^\sharp_{\psi_0 \otimes \tau_{L^\infty}} \\
&= (\int_{[-R,R]} (\|((\theta_s \otimes \gamma_s)(U))_t - U_t\|^\sharp_{\psi_0})^2 \ dm(t))^{1/2} \\
&\leq (\int_{[-R,-R+r]\cup[R-r,R]} 4 \ dm(t))^{1/2} \\
&\leq (4\epsilon^2)^{1/2} \\
&= 2\epsilon.
\end{aligned}$$

□

**Lemma 5.13.** *There exists a finite subset $-R = s_0 < \cdots < s_K = R$ of $[-R, R]$ with*

$$\|U - \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k\|^\sharp_{(\psi_0\circ\theta_{t_j})\otimes\tau_{L^\infty}} < \epsilon$$

*for any $j = 0, \cdots, J-1$, where $e_k := \chi_{[s_k, s_{k+1}]} \in L^\infty([-R, R])$.*

*Proof.* Since the map $t \mapsto \theta_t(u)$ is continuous in the strong * topology, there exists a finite set $-R = s_0 < \cdots < s_K = R$ of $[-R, R]$ with

$$\|\theta_t(u) - \theta_{s_k}(u)\|^\sharp_{\psi_0\circ\theta_{t_j}} < \epsilon$$

for $j = 0, \cdots, J-1$, $k = 0, \cdots, K-1$ and $t \in [s_k, s_{k+1}]$. Then we have

$$\begin{aligned}
&\|U - \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k\|^\sharp_{\psi_0\circ\theta_{t_j}\otimes\tau_{L^\infty}} \\
&= (\sum_{k=0}^{K-1} \int_{[s_k,s_{k+1})} (\|\theta_t(u) - \theta_{s_k}(u)\|^\sharp_{\psi_0\circ\theta_{t_j}})^2 \ dm(t))^{1/2} \\
&< (\sum_{k=0}^{K-1} \int_{[s_k,s_{k+1})} \epsilon^2 \ dm(t))^{1/2} \\
&= \epsilon.
\end{aligned}$$

□

Set

$$V := \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k.$$

Take a representing sequence $\{e_k^n\}_{n=1}^\infty$ of $\Theta(e_k)$ so that $\{e_k^n\}_{k=0}^{K-1}$ is a partition of unity in $\tilde{M}$ by projections for each $n$. Set

$$v_n := \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k^n.$$

The sequence $\{v_n\}_{n=1}^\infty$ represents the unitary $\Theta(V)$. Let $\{u_n\}_{n=1}^\infty$ be a representing sequence of $\Theta(U)$.

**Lemma 5.14.** *We have*

$$\lim_{n\to\omega} \|\theta_t(v_n) - v_n\|^\sharp_{\psi_0} < 6\sqrt{\epsilon}.$$

*for* $t \in [-r, r]$.

*Proof.* Note that we have

$$\begin{aligned}
&(\|\theta_t(a)\|^\sharp_{\psi_0})^2 \\
&= \frac{1}{2}\psi_0 \circ \theta_t(a^*a + aa^*) \\
&= \frac{1}{2}(\psi_0 \circ \theta_{t_j}(a^*a + aa^*)) - \frac{1}{2}((\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t)(a^*a + aa^*)) \\
&\leq (\|a\|^\sharp_{\psi_0\circ\theta_{t_j}})^2 + \|a\|^2\|\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t\|
\end{aligned}$$

for any $a \in \tilde{M}$. Hence for $t \in [t_j, t_{j+1}] \cap [-r, r]$, we have

$$\begin{aligned}
&\|\theta_t(v_n) - v_n\|_{\psi_0}^\sharp \\
&\leq \|\theta_t(v_n - u_n)\|_{\psi_0}^\sharp + \|\theta_t(u_n) - u_n\|_{\psi_0}^\sharp + \|u_n - v_n\|_{\psi_0}^\sharp \\
&\leq (4\|\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t\| + (\|v_n - u_n\|_{\psi_0 \circ \theta_{t_j}}^\sharp)^2)^{1/2} \\
&+ \|\theta_t(u_n) - u_n\|_{\psi_0}^\sharp + \|u_n - v_n\|_{\psi_0}^\sharp \\
&< (4\epsilon + (\|v_n - u_n\|_{\psi_0 \circ \theta_{t_j}}^\sharp)^2)^{1/2} \\
&+ \|\theta_t(u_n) - u_n\|_{\psi_0}^\sharp + \|u_n - v_n\|_{\psi_0}^\sharp.
\end{aligned}$$

Hence by Lemmas 5.12 and 4.11, we have

$$\begin{aligned}
&\lim_{n\to\omega} \|\theta_t(v_n) - v_n\|_{\psi_0}^\sharp \\
&\leq (4\epsilon + (\|V - U\|_{(\psi_0 \circ \theta_{t_j}) \otimes \tau_{L^\infty}}^\sharp)^2)^{1/2} \\
&+ \|\theta_t(U) - U\|_{\psi_0 \otimes \tau_{L^\infty}}^\sharp + \|U - V\|_{\psi_0 \otimes \tau_{L^\infty}}^\sharp \\
&< (4\epsilon + \epsilon^2)^{1/2} + 2\epsilon + \epsilon \\
&< 6\sqrt{\epsilon}.
\end{aligned}$$

□

**Lemma 5.15.** *We have*

$$\lim_{n\to\omega} \|v_n^* \psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} v_n^*\| \leq 3\epsilon$$

*for any* $i = 1, \cdots, n$.

*Proof.* Fix $i$ and $k$. Set $\lambda := \theta_{s_k}(u)^* \psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} \theta_{s_k}(u)^*$. Notice that we have seen that $\|\lambda\| < 3\epsilon$ in the argument preceding to Lemma 5.12. We have

$$\|e_k^n \theta_{s_k}(u)^* \psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} e_k^n \theta_{s_k}(u)^*\| \leq \|e_k^n \lambda\| + \|(e_k^n \psi_i \circ \psi_{\tilde{\sigma}} - \psi_i \circ \phi_{\tilde{\sigma}} e_k^n) \theta_{s_k}(u)^*\|.$$

Let $\lambda = |\lambda| v_\lambda$ be the polar decomposition of $\lambda$. For $x \in \tilde{M}$ with $\|x\| \leq 1$, we have

$$\begin{aligned}
|e_k^n \lambda(x)| &= |e_k^n |\lambda| (v_\lambda x e_k^n)| \\
&\leq |(e_k^n |\lambda| - |\lambda| e_k^n)(v_\lambda x e_k^n)| + ||\lambda|(e_k^n v_\lambda x e_k^n)| \\
&\leq \|[e_k^n, |\lambda|]\| + ||\lambda|(e_k v_\lambda x e_k)| \\
&\leq \|[e_k^n, |\lambda|]\| + |\lambda|(e_k^n) \\
&\to \|\lambda\| \tau_{L^\infty}(e_k) \\
&< 3\epsilon \tau_{L^\infty}(e_k).
\end{aligned}$$

Hence we have

$$\lim_{n\to\omega} \|e_k^n\theta_{s_k}(u)^*\psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} e_k^n \theta_{s_k}(u)^*\| < 3\epsilon\tau_{L^\infty}(e_k)$$

for $k = 1, \cdots, K$. Summing up these inequalities, we have

$$\lim_{n\to\omega} \|v_n^*\psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} v_n^*\| \leq 3\epsilon.$$

□

By Lemmas 5.14 and 5.15, we have the following proposition.

**Proposition 5.16.** *There exists a sequence $\{v_n\}_{n=1}^\infty$ of unitaries of $\tilde{M}$ with*

$$\lim_{n\to\infty} \|\theta_t(v_n) - v_n\|_{\psi_0}^\sharp = 0,$$
$$\lim_{n\to\infty} \|v_n^*\psi_i \circ \phi_{\tilde{\rho}} - \psi_i \circ \phi_{\tilde{\sigma}} v_n^*\| = 0$$

*for any $i = 1, 2, \cdots$.*

*Proof.* Take a representing sequence $\{v_n'\}$ of $\Theta(V)$ consisting of unitaries. Then by Lemmas 5.14 and 5.15, we have

$$\lim_{n\to\omega} \|\theta_t(v_n') - v_n'\|_{\psi_0}^\sharp < 6\sqrt{\epsilon}$$

for $t \in [-r, r]$,

$$\lim_{n\to\omega} \|{v_n'}^*(\psi_i \circ \phi_{\tilde{\rho}}) - (\psi_i \circ \phi_{\tilde{\sigma}}){v_n'}^*\| < 3\epsilon$$

for finitely many $i$. Hence by the usual diagonal argument, it is possible to take the sequence. □

5.5. **Approximation on $\tilde{M} \rtimes_\theta \mathbf{R}$.** Set

$$n_\tau := \{x \in \tilde{M} \mid \tau(x^*x) < \infty\}.$$

**Lemma 5.17.** (*See also Appendix A. of Guido–Longo* [17]) *Let $L^2(\tilde{M})$ be the standard Hilbert space of $\tilde{M}$ and $\Lambda : n_\tau \to L^2(\tilde{M})$ be the canonical injection. For each $x \in n_\tau$, set $V_{\tilde{\rho}}(\Lambda(x)) := \sqrt{d(\rho)}^{-1}\Lambda(\tilde{\rho}(x))$. Then $V_{\tilde{\rho}}$ defines an isometry of $L^2(\tilde{M})$ satisfying*

$$V_{\tilde{\rho}}^* x V_{\tilde{\rho}} = \phi_{\tilde{\rho}}(x)$$

*for any $x \in \tilde{M}$. Moreover, the isometry $V_{\tilde{\rho}}$ is the canonical implementation in the sense of Guido–Longo* [17].

*Proof.* Take $x \in n_\tau$. Then by Lemma 2.5 (4) of Izumi [22], we have

$$\begin{aligned}\|V_{\tilde{\rho}}\Lambda(x)\|^2 &= d(\rho)^{-1}\tau(\tilde{\rho}(x^*x)) \\ &= \tau(x^*x) = \|\Lambda(x)\|^2.\end{aligned}$$

Hence $V_{\tilde{\rho}}$ defines an isometry of $L^2(\tilde{M})$. Next, we show the latter statement. We have $V_{\tilde{\rho}}^*(\Lambda(x)) = \sqrt{d(\rho)}\Lambda(\phi_{\tilde{\rho}}(x))$ because

$$\begin{aligned}\langle V_{\tilde{\rho}}^*\Lambda(x), \Lambda(y)\rangle &= \langle \Lambda(x), \sqrt{d(\rho)}^{-1}\Lambda(\tilde{\rho}(y))\rangle \\ &= \sqrt{d(\rho)}^{-1}\tau(\tilde{\rho}(y)^*x) \\ &= \sqrt{d(\rho)}\tau(y^*\phi_{\tilde{\rho}}(x)) \\ &= \langle \sqrt{d(\rho)}\Lambda(\phi_{\tilde{\rho}}(x)), \Lambda(y)\rangle\end{aligned}$$

for any $x, y \in n_\tau$. In order to show the third equality of the above, we used Lemma 5.5. Hence for any $x \in \tilde{M}$ and $y \in n_\tau$, we have

$$\begin{aligned}V_{\tilde{\rho}}^* x V_{\tilde{\rho}}\Lambda(y) &= \sqrt{d(\rho)}^{-1} V_{\tilde{\rho}}^*\Lambda(x\tilde{\rho}(y)) \\ &= \Lambda(\phi_{\tilde{\rho}}(x\tilde{\rho}(y))) \\ &= \phi_{\tilde{\rho}}(x)\Lambda(y).\end{aligned}$$

Next, we show that $V_{\tilde{\rho}}$ is the standard implementing. Let $\xi$ be a cyclic separating unit vector of $L^2(\tilde{M})_+$. By (b) of Proposition A.2 of Guido–Longo [17], it is enough to show that $V_{\tilde{\rho}}\xi \in L^2(\tilde{M}, \xi)_+$. In order to achieve this, by the self duality of $L^2(\tilde{M}, \xi)$, it is enough to show that

$$\langle V_{\tilde{\rho}}\xi, J_\xi a J_\xi a\xi\rangle \geq 0$$

for $a \in \tilde{M}$. However, by a characterization of the modular conjugations (Theorem 1 of Araki [1]), $J_\xi$ is the modular conjugation of $L^2(\tilde{M})$. Hence it is enough to show that

$$\langle V_{\tilde{\rho}}\xi, \Lambda_\tau(a^*a)\rangle \geq 0$$

for $a \in n_\tau$. This is trivial because we have $V_{\tilde{\rho}}^*(\Lambda(x)) = \sqrt{d(\rho)}\Lambda(\phi_{\tilde{\rho}}(x))$. Hence $V_{\tilde{\rho}}$ is the standard implementing. □

Let $\rho$ be an endomorphism of a von Neumann algebra $M$. Then since its canonical extension $\tilde{\rho}$ commutes with $\theta$, the endomorphism $\tilde{\rho}$ extends to $\tilde{M} \rtimes_\theta \mathbf{R}$ by $\lambda_t^\theta \mapsto \lambda_t^\theta$ for any $t \in \mathbf{R}$. We denote this extension by $\tilde{\tilde{\rho}}$.

**Lemma 5.18.** *Let $\rho$ and $\sigma$ be finite index endomorphisms of a separable infinite factor $M$ and $\varphi$ be a dominant weight of $M$. Assume that there exists a sequence $\{u_n\}$ of unitaries of $\tilde{M} \rtimes_\theta \mathbf{R}$ with $\mathrm{Ad}u_n \circ \tilde{\tilde{\rho}} \to \tilde{\tilde{\sigma}}$ as $n \to \infty$. Then there exists a sequence $\{v_n\}$ of unitaries of $M$ with $\mathrm{Ad}v_n \circ \rho \to \sigma$.*

*Proof.* Since $(\varphi, \rho)$ and $(\varphi, \sigma)$ are invariant pairs, it is possible to identify $\tilde{\tilde{\rho}}$ with $\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})}$ and $\tilde{\tilde{\sigma}}$ with $\sigma \otimes \mathrm{id}_{B(L^2\mathbf{R})}$ through Takesaki duality, respectively (It is possible to choose the same identification between $M \otimes B(L^2\mathbf{R})$ and $\tilde{M} \rtimes_\theta \mathbf{R}$ for $\tilde{\tilde{\rho}}$ and $\tilde{\tilde{\sigma}}$. See the argument preceding to Lemma 3.10 of Masuda–Tomatsu [40]). Then by (the proof of) Lemma 3.11 of Masuda–Tomatsu [40], there exist an isomorphism $\pi$ from $M \otimes B(L^2\mathbf{R})$ to $M$ and unitaries $u_\rho$, $u_\sigma$ of $M$ satisfying

$$\pi \circ (\rho \otimes \mathrm{id}) \circ \pi^{-1} = \mathrm{Ad}u_\rho \circ \rho,$$
$$\pi \circ (\sigma \otimes \mathrm{id}) \circ \pi^{-1} = \mathrm{Ad}u_\sigma \circ \sigma$$

(Although in the statement of Lemma 3.11 of Masuda–Tomatsu [40], the isomorphism $\pi$ depends on the choice of $\rho$, $\pi$ turns out to be independent of $\rho$ by its proof). Then we have

$$\begin{aligned}
&\mathrm{Ad}(u_\sigma^*\pi(u_n)u_\rho) \circ \rho \\
&= \mathrm{Ad}(u_\sigma^*\pi(u_n)) \circ \pi \circ (\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})}) \circ \pi^{-1} \\
&= \mathrm{Ad}u_\sigma^* \circ \pi \circ (\mathrm{Ad}u_n \circ (\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})})) \circ \pi^{-1} \\
&\to \mathrm{Ad}u_\sigma^* \circ \pi \circ (\sigma \otimes \mathrm{id}_{B(L^2\mathbf{R})}) \circ \pi^{-1} \\
&= \mathrm{Ad}u_\sigma^* \circ (\mathrm{Ad}u_\sigma \circ \sigma) \\
&= \sigma.
\end{aligned}$$

□

**Lemma 5.19.** *Let $\rho$ be an endomorphism with finite index and with $(\varphi, \rho)$ an invariant pair. Let $E_{\tilde{\tilde{\rho}}}$ be the minimal expectation from $\tilde{\tilde{M}}$ to $\tilde{\tilde{\rho}}(\tilde{\tilde{M}})$. Then we have the following.*

(1) *For each $x \in \tilde{M}$, we have $E_{\tilde{\tilde{\rho}}}(x) = E_{\tilde{\rho}}(x)$.*

(2) *For any $t \in \mathbf{R}$, we have $E_{\tilde{\tilde{\rho}}}(\lambda_t^\theta) = \lambda_t^\theta$.*

*Proof.* This is shown in the proof of Theorem 4.1 of Longo [38]. □

**Lemma 5.20.** *For $\xi \in L^2(\mathbf{R}, \tilde{M})$, set*

$$V_{\tilde{\tilde{\rho}}}(\xi)(s) := V_{\tilde{\rho}}(\xi(s)).$$

*Then $V_{\tilde{\tilde{\rho}}}$ is an isometry of $L^2(\mathbf{R}, \tilde{M})$ satisfying*

$$V_{\tilde{\tilde{\rho}}}^* x V_{\tilde{\tilde{\rho}}} = \phi_{\tilde{\tilde{\rho}}}(x)$$

*for any $x \in M$, where $\phi_{\tilde{\tilde{\rho}}} = \tilde{\tilde{\rho}}^{-1} \circ E_{\tilde{\tilde{\rho}}}$.*

*Proof.* The first statement is shown by the following computation.

$$\begin{aligned}
\|V_{\tilde{\tilde{\rho}}}(\xi)\|^2 &= \int_{\mathbf{R}} \|V_{\tilde{\rho}}(\xi(s))\|^2 \, d\mu(s) \\
&= \int_{\mathbf{R}} \|\xi(s)\|^2 \, d\mu(s) \\
&= \|\xi\|^2
\end{aligned}$$

for $\xi \in L^2(\mathbf{R}, \tilde{M})$. Next, we show the latter statement. Choose $x \in M$ and $\xi \in L^2(\mathbf{R}, \tilde{M})$. Then we have

$$\begin{aligned} V_{\tilde{\tilde{\rho}}}^* \circ \pi_\theta(x) \circ V_{\tilde{\tilde{\rho}}}(\xi) &= V_{\tilde{\tilde{\rho}}}^* \pi_\theta(x)(s \mapsto V_{\tilde{\rho}}(\xi(s))) \\ &= V_{\tilde{\tilde{\rho}}}^*(s \mapsto \theta_{-s}(x) \circ V_{\tilde{\rho}}(\xi(s))) \\ &= (s \mapsto V_{\tilde{\rho}}^* \circ \theta_{-s}(x) \circ V_{\tilde{\rho}}(\xi(s))) \\ &= (s \mapsto \phi_{\tilde{\rho}}(\theta_{-s}(x))(\xi(s))) \\ &= (s \mapsto \theta_{-s}(\phi_{\tilde{\rho}}(x))(\xi(s))) \\ &= \pi_\theta(\phi_{\tilde{\rho}}(x))(\xi) \\ &= \phi_{\tilde{\tilde{\rho}}}(\pi_\theta(x))(\xi). \end{aligned}$$

In order to show the fourth equality of the above, we used Lemma 5.17. The last equality of the above follows from Lemma 5.19. For $t \in \mathbf{R}$ and $\xi \in L^2(\mathbf{R}, \tilde{M})$, we have

$$\begin{aligned} V_{\tilde{\tilde{\rho}}}^* \lambda_t^\theta V_{\tilde{\tilde{\rho}}} \xi &= V_{\tilde{\tilde{\rho}}}^*(s \mapsto V_{\tilde{\rho}}(\xi(s-t)) \\ &= s \mapsto V_{\tilde{\rho}}^* V_{\tilde{\rho}}(\xi(s-t)) \\ &= \lambda_t^\theta(\xi). \end{aligned}$$

Thus we are done. □

**Lemma 5.21.** *Let $N$ be a von Neumann algebra and $\{V_n\}_{n=0}^\infty$ be a sequence of isometries on the standard Hilbert space $L^2(N)$ such that for each $n$, the map $\Phi_n : N \ni x \mapsto V_n^* x V_n$ is a left inverse of an endomorphism $\rho_n$ of $N$. Consider the following two conditions.*

(1) *The sequence of operators $\{V_n\}_{n=1}^\infty$ converges to $V_0$ strongly.*
(2) *We have $\|\psi \circ \Phi_n - \psi \circ \Phi_0\| \to 0$ for any $\psi \in N_*$.*

*Then we have implication (1) $\Rightarrow$ (2). If each isometry $V_n$ is the standard implementing in the sense of Guido–Longo* [17] *(See Appendix A. of* [17]*), then we have (2) $\Rightarrow$ (1).*

*Proof.* Implication (1) $\Rightarrow$ (2) is shown by just using the Cauchy–Schwartz inequality. When $V_n$ is the standard implementing, implication (2) $\Rightarrow$ (1) is implication (1) $\Rightarrow$ (3) of Lemma 3.3 of Masuda–Tomatsu [40]. □

Note that the isometries $V_{\tilde{\rho}} u_n^* J_{\tilde{M}} u_n^* J_{\tilde{M}}$ and $V_{\tilde{\sigma}}$ are examples of the standard implementing, where $J_{\tilde{M}}$ is the modular conjugation of $\tilde{M}$.

**Lemma 5.22.** *Let $\{u_n\}$ be a sequence of unitaries of $\tilde{M}$ satisfying the following conditions.*

(1) *We have $\mathrm{Ad} u_n \circ \tilde{\rho} \to \tilde{\sigma}$ as $n \to \infty$.*

(2) *For any compact subset $F$ of $\mathbf{R}$, we have $\theta_t(u_n) - u_n \to 0$ uniformly for $t \in F$.*

*Then there exists a sequence of unitaries $\{v_n\}$ with $\mathrm{Ad}v_n \circ \rho \to \sigma$.*

*Proof.* By Lemma 5.18, it is enough to show that $\mathrm{Ad}u_n \circ \tilde{\tilde{\rho}} \to \tilde{\tilde{\sigma}}$. By Lemma 5.20 and implication (1) $\Rightarrow$ (2) of Lemma 5.21, it is enough to show that $V_{\tilde{\tilde{\rho}}} u_n^* J u_n^* J \to V_{\tilde{\tilde{\sigma}}}$, where $J$ is the modular conjugation of $\tilde{M} \rtimes_\theta \mathbf{R}$. Recall that $J : L^2(\tilde{M}) \otimes L^2\mathbf{R} \to L^2(\tilde{M}) \otimes L^2\mathbf{R}$ is given by the following.

$$J : \xi \mapsto (s \mapsto J_{\tilde{M}} \theta_{-s}(\xi(-s))),$$

where $J_{\tilde{M}}$ is the modular conjugation of $\tilde{M}$. Hence we have

$$\begin{aligned} & V_{\tilde{\tilde{\rho}}} u_n^* J u_n^* J(\xi \otimes f) \\ & = (s \mapsto V_{\tilde{\rho}}(\theta_{-s}(u_n^*)\xi u_n) f(s)) \end{aligned}$$

for any $\xi \in L^2(\tilde{M})$ and $f \in L^2\mathbf{R}$. Hence we have

$$\begin{aligned} & \|V_{\tilde{\tilde{\rho}}} u_n^* J u_n^* J(\xi \otimes f) - V_{\tilde{\tilde{\sigma}}}(\xi \otimes f)\|^2 \\ & = \int_{\mathbf{R}} \|V_{\tilde{\rho}}(\theta_{-s}(u_n^*)\xi u_n) - V_{\tilde{\sigma}}(\xi)\|^2 |f(s)|^2 \, ds \\ & \leq \int_{\mathbf{R}} \|(V_{\tilde{\rho}}((\theta_{-s}(u_n^*) - u_n^*)\xi u_n)\|^2 |f(s)|^2 \, ds + \int_{\mathbf{R}} \|V_{\tilde{\rho}}(u_n^* \xi u_n) - V_{\tilde{\sigma}}(\xi)\|^2 |f(s)|^2 \, ds \\ & \to 0 \end{aligned}$$

by the Lebesgue dominant convergence theorem. Note that in order to show the last convergence, we use Lemmas 5.16, 5.17 and implication (2) $\Rightarrow$ (1) of Lemma 5.21. $\square$

### 5.6. **The proof of the main theorem of Section 5.**

**Lemma 5.23.** *Let $M$ be an AFD factor and $\sigma$ be a finite index endomorphism of $M$ with $d(\sigma) = d$. Then there exists an endomorphism $\lambda$ with the following properties.*

(1) *The endomorphism $\lambda$ is approximately inner.*
(2) *We have $d(\lambda) = d$.*
(3) *The endomorphism $\lambda$ has Connes–Takesaki module and it is $\theta_{-\log d}|_{\mathcal{Z}(\tilde{M})}$.*

*Proof.* By the proof of Theorem 3 of Kosaki–Longo [26], there exists an endomorphism $\lambda_0$ of the AFD factor of type $\mathrm{II}_1$ with $d(\lambda_0) = d$. Then $\mathrm{id}_M \otimes \lambda_0$ is an endomorphism of $M$ with $d(\mathrm{id} \otimes \lambda_0) = d$ and with $\mathrm{mod}(\mathrm{id} \otimes \lambda_0)$ trivial. Hence by the existence of a right inverse of the Connes–Takesaki module of automorphisms (See Sutherland–Takesaki

[57]), there exists an automorphism $\alpha$ of $M$ with $\mathrm{mod}(\alpha\circ\lambda_0) = \theta_{-\log(d)}$. By Theorem 3.15 of Masuda–Tomatsu (or by the same argument of our paper), it is shown that $\lambda := \alpha \circ \lambda_0$ is approximately inner. $\square$

Now, we return to the proof of the main theorem.

*Proof of implication* (1) $\Rightarrow$ (2) *of Theorem 5.1.* Let $\rho, \sigma$ be endomorphisms of $\mathrm{End}(M)_0$ with the first condition of Theorem 5.1. Then by Lemma 5.23, there exist endomorphisms $\lambda, \mu \in \mathrm{End}(M)_0$ with the following properties.

(1) We have $d(\lambda) = d(\sigma)$, $d(\mu) = d(\rho)$.
(2) We have $\tilde{\lambda}|_{\mathcal{Z}(\tilde{M})} = \theta_{-\log(d(\sigma))}|_{\mathcal{Z}(\tilde{M})}$ and $\tilde{\mu}|_{\mathcal{Z}(\tilde{M})} = \theta_{-\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})}$.
(3) The endomorphisms $\lambda$ and $\mu$ are approximately inner.

By the second condition, we have

$$\begin{aligned}\phi_{\tilde{\rho}} \circ \phi_{\tilde{\lambda}}|_{\mathcal{Z}(\tilde{M})} &= \phi_{\tilde{\rho}} \circ \theta_{\log d(\sigma)}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \theta_{-\log(d(\sigma)/d(\rho))} \circ \theta_{\log d(\sigma)}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \theta_{\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \phi_{\tilde{\mu}}|_{\mathcal{Z}(\tilde{M})}.\end{aligned}$$

Hence by replacing $\rho$ by $\lambda \circ \rho$ and $\sigma$ by $\mu \circ \sigma$ respectively, we may assume that $d(\rho) = d(\lambda)$ and $\phi_{\tilde{\rho}}|_{\mathcal{Z}(M)} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(M)}$. By Proposition 5.16, there exists a sequence $\{u_n\}$ of unitaries of $\tilde{M}$ satisfying the assumptions of Lemma 5.22. Hence by Lemma 5.22, we have $\mathrm{Ad}u_n \circ \rho \to \sigma$. $\square$

5.7. **Appendix of Section 5 (A proof of the characterization of central triviality of automorphisms of AFD factors).** In this subsection, we will see that it is possible to give a proof of a characterization theorem of central triviality of automorphisms of AFD factors by a similar strategy to the proof of Theorem 5.1, which is independent of the types of the AFD factors.

Let $M$ be an AFD factor of type III. Let $\alpha$ be an automorphism of $M$ and $\tilde{\alpha}$ be its canonical extension. Set

$$p := \min\{q \in \mathbf{N} \mid \tilde{\alpha}^q \text{ is centrally trivial}\},$$

$$G := \mathbf{Z}/p\mathbf{Z}.$$

Note that when $\tilde{\alpha}^n$ is not centrally trivial for any $n \neq 0$, we set $G := \mathbf{Z}$.

**Lemma 5.24.** *The action* $\{\tilde{\alpha}_n \circ \theta_t\}_{(n,t)\in G\times\mathbf{R}}$ *of* $G \times \mathbf{R}$ *on* $\tilde{M}_{\omega,\theta}$ *is faithful.*

*Proof.* We will show this lemma by contradiction. Let $\varphi$ be a normal faithful state of $\tilde{M}$ and $\{\psi_j\}_{j=1}^\infty$ be a norm dense sequence of the unit ball of $\tilde{M}_*$. Assume that there existed a pair $(n,t) \in (G\times \mathbf{R})\setminus\{(0,0)\}$ satisfying $\tilde{\alpha}_n \circ \theta_{-t}(a) = a$ for any $a \in \tilde{M}_{\omega,\theta}$. Then the automorphism $\tilde{\alpha}_n \circ \theta_{-t}$ would be centrally non-trivial because $\tilde{\alpha}_n \circ \theta_{-t}$ is trace-scaling if $t \neq 0$. Hence there would exist an element $x$ of $\tilde{M}_\omega$, which can never be of $\tilde{M}_{\omega,\theta}$, with $\tilde{\alpha}_n(x) \neq \theta_t(x)$. We may assume that $x$ is a unitary because any element of a von Neumann algebra is a linear combination of four unitaries. Take a representing sequence $\{x_k\}$ of $x$ consisting of unitaries. Then we would have

$$\begin{aligned}&\lim_{k\to\omega} \|\tilde{\alpha}_n(x_k) - \theta_t(x_k)\|^\sharp_{\varphi\circ\theta_s}\\ &= \text{weak} \lim_{k\to\omega} \frac{1}{2}(|\tilde{\alpha}_n(x_k) - \theta_t(x_k)|^2 + |(\tilde{\alpha}_n(x_k) - \theta_t(x_k))^*|^2)\\ &= 2\delta > 0\end{aligned}$$

for some $\delta > 0$. Then for each natural number $L$, there would exist $k \in \mathbf{N}$ satisfying the following two conditions.

(1) We have

$$\|\theta_s(x_k)\psi_j - \psi_j\theta_s(x_k)\| (= \|x_k(\psi_j\circ\theta_s) - (\psi_j\circ\theta_s)x_k\|) < \frac{1}{L}$$

for $j = 1, \cdots, L$, $|s| \leq L$ (Use the compactness of $\{\psi_j \circ \theta_s \mid |s| \leq L\}$. See also the argument just after Lemma 5.11).

(2) We have

$$\|\tilde{\alpha}_n(x_k) - \theta_t(x_k)\|^\sharp_\varphi > \delta.$$

Let $\Theta : L^\infty([-L,L], dm(s)) \otimes (\tilde{M}, \varphi) \to (\tilde{M}_{\omega,\theta}, \varphi^\omega)$ be the inclusion mentioned in Section 5 (an inclusion coming from the Rohlin property of $\theta$), where $dm(s)$ is the normalized Haar measure of $[-L,L]$. Set

$$\tilde{y} := ([-L,L] \ni s \mapsto \theta_s(x_k)) \in L^\infty([-L,L], dm(s)) \otimes \tilde{M},$$

$$y := \Theta(\tilde{y}).$$

Since we would have $\tilde{\alpha}_n \circ \theta_{-t}$ is trivial on $\tilde{M}_{\omega,\theta}$, we would have

$$\begin{aligned}\tilde{\alpha}_n(\Theta(f\otimes b)) &= (\tilde{\alpha}_n(b\Theta(f))\\ &= \tilde{\alpha}_n(b)\theta_t(\Theta(f))\\ &= \tilde{\alpha}_n(b)\Theta(f(\cdot - t))\\ &= \Theta(\tilde{\alpha}_n(b)f(\cdot - t))\end{aligned}$$

for $f \in L^\infty([-L,L])$ and $b \in \tilde{M}$. Hence we would have

$$\tilde{\alpha}_n(y) = \Theta(z),$$

where $z$ is an element of $L^\infty([-L,L], dm(s)) \otimes \tilde{M}$ satisfying $z(s) = \tilde{\alpha}_n(\theta_{s-t}(x_k))$ for $s \in [-L+t, L-t]$ (Note that the value $z(s)$ is a unitary element of $\tilde{M}$ for each $s \in [-L,L] \setminus [-L+t, L-t]$ which is completely determined by $\tilde{y}$. However, it is not important what it is). Hence we would have

$$\begin{aligned}
\|\tilde{\alpha}_n(y) - y\|_{\varphi^\omega}^\sharp &\geq (\int_{[-L+t,L-t]} (\|\tilde{\alpha}_n(\theta_{s-t}(x_k)) - \theta_s(x_k)\|_\varphi^\sharp)^2 \, dm(s) \\
&\quad - \int_{[-L,-L+t]\cup[L-t,L]} 2^2 \, dm(s))^{1/2} \\
&\geq (\int_{[-L,L]} \delta^2 \, dm(s) - \frac{4t}{L})^{1/2} \\
&= (\delta^2 - \frac{4t}{L})^{1/2}.
\end{aligned}$$

Since we have

$$(\theta_r(y))_s = \theta_s(y)$$

for any $0 < r < 1$, $s \in [-L+r, L-r]$, we have

$$\begin{aligned}
\|\theta_r(y) - y\|_{\varphi^\omega}^\sharp &= (\int_{[-L,L]} (\|(\theta_r(y))_s - y_s\|_\varphi^\sharp)^2 \, dm(s))^{1/2} \\
&\leq (\int_{[-L,-L+1]\cup[L-1,L]} 2^2 \, dm(s))^{1/2} \\
&= \frac{2}{\sqrt{L}}
\end{aligned}$$

for $|r| \leq 1$. We also have

$$\begin{aligned}
\|[y, \psi_i]|_{\Theta(L^\infty([-L,L])\otimes\tilde{M})}\| &= \int_{[-L,L]} \|[\tilde{y}_s, \psi_i]\| \, dm(s) \\
&= \int_{[-L,L]} \|[\theta_s(x_k), \psi_j]\| \, dm(s) \\
&< \int_{[-L,L]} \frac{1}{L} \, dm(s) \\
&= \frac{1}{L}
\end{aligned}$$

for $j = 1, \cdots, L$. Hence by Lemma 5.3 of Masuda–Tomatsu [44] (or by the same argument as that of Lemmas 4.11 and 5.15), there exists a representing sequence $\{y_l'\}$ of $y$ with

$$\lim_{n\to\omega} \|[y_l', \psi_j\}\| < \frac{1}{L} + 2\epsilon.$$

Hence there would exist a sequence $\{y_l\}$ of $\tilde{M}$ with the following properties.

(1) We have $\|y_l\| \leq 1$.
(2) We have $\|[y_l, \psi_j]\| \to 0$ for any $j = 1, 2, \cdots$.
(3) For any $j = 1, 2, \cdots$, we have $\|\theta_r(y_l) - y_l\|_\varphi^\sharp \to 0$ uniformly for $r \in [-1, 1]$.
(4) We have $\|\tilde{\alpha}_n(y_l) - \theta_t(y_l)\|_\varphi^\sharp \geq \delta/2$ for any $l$.

This would contradict the assumption that $\tilde{\alpha}_n \circ \theta_{-t}$ were trivial on $\tilde{M}_{\omega,\theta}$. □

**Lemma 5.25.** *For each $\gamma \in \widehat{G \times \mathbf{R}} = \hat{G} \times \mathbf{R}$, there exists a unitary $u$ of $\tilde{M}_{\omega,\theta}$ with $\tilde{\alpha}_m \circ \theta_s(u) = \langle (m, s), \gamma \rangle u$ for any $(m, s) \in G \times \mathbf{R}$.*

*Proof.* The proofs of Theorems 4.10 and 7.7 of Masuda–Tomatsu [44] work in our case. □

**Lemma 5.26.** *There exist a non-zero projection $e$ of $(\tilde{M}_{\omega,\theta})^\theta$ with $\tilde{\alpha}(e)$ orthogonal to $e$.*

*Proof.* By the previous lemma, when $p \neq 0$, for each natural number $l$, there exists a unitary $u$ of $\tilde{M}_{\omega,\theta}$ with $\tilde{\alpha}(u) = e^{2\pi i/p}u$ and with $\theta_s(u) = e^{-is/l}u$ for any $s$. When $p = 0$, there exists a unitary $u$ of $\tilde{M}_{\omega,\theta}$ with $\tilde{\alpha}(u) = -u$ and with $\theta_s(u) = e^{-is/l}u$ for any $s$. Hence when $p \neq 0$, there exists a spectral projection $e$ of $u$ with $\tilde{\alpha}(e) \leq 1 - e$, $\tau^\omega(e) = 1/p$ and with $\tau^\omega(|e - \theta_s(e)|^2) \leq 1/(2l)$ for $|s| \leq 1$. When $p = 0$, it is possible to choose a spectral projection $e$ of $u$ with $\tilde{\alpha}(e) = 1 - e$, $\tau^\omega(e) = 1/2$ and with $\tau^\omega(|e - \theta_s(e)|^2) \leq 1/(2l)$ for $|s| \leq 1$. By the usual diagonal argument, it is possible to choose a desired projection. □

**Theorem 5.27.** (See Theorem 1 (2) of Kawahigashi–Sutherland–Takesaki) *For an automorphism $\alpha$ of $M$, $\alpha$ is centrally trivial if and only if its canonical extension is inner.*

*Proof.* First, assume that $\tilde{\alpha}$ is not centrally trivial. Then by the previous lemma, neither is $\tilde{\tilde{\alpha}}$. Hence neither is $\alpha$ centrally trivial (See, for example, Lemmas 5.11 and 5.12 of Sutherland–Takesaki [56]). The above argument means that if $\alpha$ is centrally trivial, then $\tilde{\alpha}$ is centrally trivial. Since $\tilde{M}$ is of type II, any centrally trivial automorphism of $\tilde{M}$ is inner. The reverse direction is trivial by the central triviality of a modular automorphism group. □

**Remark 5.28.** Finally, we remark that by our results and the result of Masuda [39], if we admit that AFD factors are completely classified by their flows of weights, it is possible to classify the actions of discrete amenable groups on AFD factors without separating cases by the types of the factors.

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