博士論文

論文題目：

Constructions of amenable dynamical systems

and their applications to nuclearity of C*-algebras

(従順位相力学系の構成とC*環の核型性への応用)

氏 名 鈴木 悠平

Contents

Notation table. Here we fix notation used throughout this thesis.

- For $m \in \mathbb{N}$, denote by $\mathbb{Z}_m$ the cyclic group of order m.
- Denote by $\mathbb{F}_n$ the free group of rank n for $n \in \mathbb{N}$.
- Denote by $\mathbb{F}_\infty$ the free group of (countable) infinite rank.
- For a group Γ, we denote by $D(\Gamma)$ the commutator subgroup of Γ, i.e., the subgroup of Γ generated by commutators $sts^{-1}t^{-1}, s, t \in \Gamma$.
- For two actions σ and τ of a group Γ, denote by $\sigma \times \tau$ the diagonal action of σ and τ.
- When the action $\alpha\colon \Gamma \curvearrowright X$ is clear from the context, we denote $\alpha_s(x)$ and $\alpha_s(U)$ by $s.x$ and sU for $s \in \Gamma, x \in X$, and $U \subset X$.
- For a subset U of a topological space, its closure and interior are denoted by $\mathrm{cl}(U)$ and $\mathrm{int}(U)$ respectively.
- Denote by e the unit element of a group.
- For a subset S of a set, denote by χ_S the characteristic function of S.
- For a set X, denote by Δ_X the diagonal set $\{(x, x) : x \in X\}$ of $X \times X$.
- Denote by $\mathbb{K}(H)$ and $\mathbb{B}(H)$ the C*-algebras of all compact operators and all bounded operators on a Hilbert space H respectively. When $H = \ell^2(\mathbb{N})$, we denote them by $\mathbb{K}$ and $\mathbb{B}$ respectively.
- Denote by $\otimes$ the minimal tensor product of C*-algebras. We use the same notation for the minimal tensor product of completely positive maps.
- Let A be a C*-algebra. For a projection p in A or $A \otimes \mathbb{K}$, denote by $[p]_0$ the element of $K_0(A)$ represented by p.
- For a $*$-homomorphism α between C*-algebras, denote by $\alpha_{*,i}$ the homomorphism induced on the K_i-groups.
- For an action $\alpha\colon \Gamma \curvearrowright A$ of a group on a unital C*-algebra, let $A \rtimes_{\mathrm{alg}} \Gamma$ denote its algebraic crossed product, i.e., the $*$-subalgebra of the reduced crossed product generated by A and Γ.
- For the simplicity of notation, in the reduced crossed product $B = A \rtimes_r \Gamma$, we denote the unitary of B corresponding to $s \in \Gamma$ by the same symbol s.
- Let $E\colon A \rtimes_r \Gamma \to A$ denote the canonical conditional expectation on the reduced crossed product. That is, the unital completely positive map defined by the formula $E(as) := \delta_{e,s} a$ for $a \in A$ and $s \in \Gamma$.
- For $x \in A \rtimes_r \Gamma$ and $s \in \Gamma$, set $E_s(x) := E(xs^{-1})$. This is referred to as the sth coefficient of x.
- For a unital C*-algebra, we denote by $\mathbb{C}$ the C*-subalgebra generated by the unit.
- For $n, m \in \mathbb{N}$ and a C*-algebra A, let $\mathbb{M}_{n,m}(A)$ denote the space of n by m matrices over A. As usual, for a matrix $[a_{i,j}]_{i,j} \in \mathbb{M}_{n,m}(A)$, we set $[a_{i,j}]_{i,j}^* := [a_{j,i}^*]_{i,j} \in \mathbb{M}_{m,n}(A)$. Note that $\mathbb{M}_n(A) := \mathbb{M}_{n,n}(A)$ is a C*-algebra. We denote $\mathbb{M}_n(\mathbb{C})$ by $\mathbb{M}_n$ for short.
- For a C*-algebra A and a finite set X, denote by $\mathbb{M}_X(A)$ the C*-algebra of all A-valued X by X matrices.

CHAPTER 1

Introduction

1. Introduction

A concept of amenability is first introduced for groups by von Neumann in order to explain the Banach–Tarski paradox. Since then it plays crucial roles in many subjects, which include operator algebras, ergodic theory, and topology. Inspired by amenability of groups, this concept is introduced for many other mathematical objects. To understand dynamical systems of non-amenable groups, Zimmer introduced a notion of amenability in the measurable context ([**59**], [**60**]). In the celebrated paper [**13**], Connes, Feldman, and Weiss showed that any amenable orbit equivalence relation is hyperfinite. In particular they concluded a crucial structural result on amenable factors, namely, the unicity of Cartan subalgebras up to conjugacy. Inspired by works in measurable dynamical systems, Anantharaman-Delaroche [**1**] introduced amenability for topological dynamical systems. Nowaday, it is known that topological amenable dynamical systems have striking applications in many subjects, which include topology and theory of both C^*- and von Neumann algebras. We refer the reader to the survey paper of Ozawa [**36**] for further information.

In this thesis, we construct amenable dynamical systems with new interesting properties. We also use our examples to reveal new phenomena of nuclearity (which is equivalent to amenability) of C^*-algebras. Our results are divided to four chapters. We next introduce main results of each chapter. For the precise statements, see the introduction of the corresponding chapter.

In Chapter 2, we compute K-groups of amenable Cantor systems of free groups arising as the diagonal action of the boundary action and a profinite action. We also show that their crossed products are in a classifiable class, thus we can decide their isomorphism classes. As a result, we obtain the first continuously many examples of amenable Cantor systems of free groups whose crossed products are classified and pairwise non-isomorphic. We construct free examples, which extends a result of Elliott and Sierakowski [**17**]. The results of this chapter is based on the author's paper [**52**].

In Chapter 3, we extend the existence theorem of minimal skew product extensions of dynamical systems of amenable groups obtained by Glasner and Weiss [**20**] to general amenable dynamical systems. This provides many new examples of amenable minimal dynamical systems for arbitrary exact groups. In particular we give a generalization of a theorem of Rørdam and Sierakowski [**48**]. Roughly speaking, their result shows that the structure of a group cannot be an obstruction to form a Kirchberg algebra. Our generalization further says that the structure of neither a group nor a space can be an obstruction to form a Kirchberg algebra. The results of this chapter is based on the author's paper [**54**].

In Chapter 4, by using amenable dynamical systems, we show that the class of nuclear C^*-algebras do not form a monotone class. More strongly, we show that the decreasing intersection of nuclear C^*-algebras can lost the operator approximation property, which is a weak version of nuclearity. Note that in the von Neumann algebra case, it is well-known that the injective von Neumann algebras do form a monotone class. This reflects how C^*-algebras are sensitive compared with von Neumann algebras. The results of this chapter is based on the author's paper [**53**].

In Chapter 5, we study how typical Cantor systems behave. We show that amenability is always generic whenever a given group is exact. For infinite free product groups, we further show that a generic Cantor system has a certain extremal transitivity. In particular this shows that primeness is generic. To the best knowledge of the author, this is the first existence result of an amenable minimal prime topological dynamical system of a non-amenable group. We also use this property to show that the reduced group C*-algebra of any infinite free product group with the approximation property [**24**] (e.g., the free group $\mathbb{F}_\infty$) has an ambient nuclear C*-algebra with no proper intermediate C*-algebra. This is somehow surprising, since the developments of classification theory of nuclear C*-algebras show that nuclear C*-algebras admit mysterious isomorphisms and embeddings. We also emphasize that the existence of a minimal ambient nuclear C*-algebra of a non-nuclear C*-algebra is already new and highly nontrivial. See the introduction of Chapter 5 for the details. By using Kirchberg's $\mathcal{O}_2$-absorption theorem, we also show that the Cuntz algebra $\mathcal{O}_2$ admits non-nuclear C*-subalgebras with no intermediate C*-algebras. The results of this chapter is based on the author's paper [**55**].

2. Preliminaries

Here we collect the fundamental knowledge and notation used throughout this thesis. The basic references are the book [**6**] of Brown and Ozawa and the book [**45**] of Rørdam.

2.1. Pure infiniteness of C*-algebras and classification theorem. Recall that a unital C*-algebra A is purely infinite and simple if for any nonzero positive element $a \in A$, there is $b \in A$ with $b^*ab = 1$. This notion was introduced by Cuntz in the study of the Cuntz algebras $\mathcal{O}_n$; $2 \leq n \leq \infty$ [**10**]. Pure infiniteness plays an important role in the study of C*-algebras. See [**10**], [**28**], [**29**], [**39**], and [**44**] for example.

A C*-algebra is said to be a Kirchberg algebra if it is simple, separable, nuclear, and purely infinite. A celebrated theorem of Kirchberg [**28**] and Phillips [**39**] states that the Kirchberg algebras are classified in terms of the KK-theory. In particular, the Kirchberg algebras in the UCT class (i.e., the class of C*-algebras satisfying the universal coefficient theorem of Rosenberg–Schochet [**49**]) are classified by their K-theoretic data. More precisely, for unital cases they are classified by the triplet $(K_0, [1]_0, K_1)$ and for non-unital cases they are classified by the pair (K_0, K_1). Consequently all Kirchberg algebras in the UCT class are isomorphic to the one constructed in [**44**]. Note that all possible K-theoretic data are exhausted by a Kirchberg algebra in the UCT [**44**, Theorem 3.6]. Typical examples of Kirchberg algebras in the UCT class are the Cuntz algebras $\mathcal{O}_n$ [**9**] and the Cuntz–Krieger algebras $\mathcal{O}_A$ [**12**]. Kirchberg algebras in the UCT class also naturally arise in many constructions of C*-algebras. For example, certain graphs (see e.g., [**42**]) and certain topological dynamical systems (see e.g., [**2**], [**30**], [**43**], [**48**], and [**50**]) provide Kirchberg algebras in the UCT class.

For these reasons, it is important to know whether a given C*-algebra is purely infinite. Obviously pure infiniteness implies other infiniteness properties; e.g., tracelessness, properly infiniteness. The latter conditions are easy to check in many situations. However, even in the nuclear case, Rørdam has constructed a counterexample for the converse implications [**46**]. See [**45**] and the references therein for more information on pure infiniteness and Kirchberg algebras.

2.2. Approximation properties for C*-algebras and groups. For C*-algebras A, B and a closed subspace X of B, we define a subspace $F(A, B, X) \subset A \otimes B$ by

$$F(A, B, X) := \{a \in A \otimes B : (\varphi \otimes \mathrm{id}_B)(a) \in X \text{ for all } \varphi \in A^*\}.$$

A triplet (A, B, X) is said to have the slice map property if the equality $F(A, B, X) = A \otimes X$ holds. Here $A \otimes X$ denotes the closed subspace of $A \otimes B$ spanned by elements $a \otimes x; a \in A, x \in X$. Note that when there is a completely bounded projection from B onto X, the triplet (A, B, X) has the slice map property. Note also that a C*-algebra A is exact if and only if for any C*-algebra B and its ideal J, the triplet (A, B, J) has the slice map property. In general, deciding whether a given triplet has the slice map property is a sensitive and difficult problem. We give a definition of the SOAP (strong operator approximation property) and the OAP (operator approximation property) in terms of the slice map property. See [**6**, Section 12.4] for the detail.

Definition 2.1. A C*-algebra A is said to have the SOAP (resp. the OAP) if for any C*-algebra B (resp. for $B = \mathbb{K}$) and for any closed subspace X of B, the triplet (A, B, X) has the slice map property.

It is not hard to show the following implications

$$\text{Nuclearity} \Rightarrow \text{CBAP} \Rightarrow \text{SOAP} \Rightarrow \text{OAP, exactness.}$$

All implications are known to be proper and there are no implications in the last two properties. However, for the reduced group C*-algebras, the SOAP and the OAP are equivalent. See Chapter 12 of [**6**] for details. The SOAP and the OAP have a strong connection to the property of groups called the AP (approximation property [**24**]). Here we give the following equivalent condition as a definition of the AP.

Definition 2.2. A discrete group Γ is said to have the AP if there is a net $(\varphi_i)_{i \in I}$ of finitely supported complex valued functions on Γ such that $m_{\varphi_i} \otimes \mathrm{id}_{\mathbb{B}}$ converges to the identity map in the pointwise norm topology. Here $m_\varphi(x) := \sum_{g \in \Gamma} \varphi(g) E_g(x) g$ is the multiplier of φ for a finitely supported function φ on Γ defined on the reduced group C*-algebra $\mathrm{C}^*_r(\Gamma)$.

This property is characterized in the following way.

Proposition 2.3. *Let Γ be a discrete group. Then the following are equivalent.*

(1) *The group Γ has the AP.*
(2) *The C*-algebra $\mathrm{C}^*_r(\Gamma)$ has the SOAP.*
(3) *The C*-algebra $\mathrm{C}^*_r(\Gamma)$ has the OAP.*
(4) *There is an intermediate C*-algebra between $\mathrm{C}^*_r(\Gamma)$ and $L(\Gamma)$ which has the SOAP or the OAP.*

In particular, the AP implies exactness.

See Section 12.4 of [**6**] for the proof. Note that the implication (4)⇒(1) follows from the proofs of (2), (3)⇒ (1).

A group Γ is said to have the ITAP (invariant translation approximation property) if we have the equality

$$L(\Gamma) \cap \mathrm{C}^*_u(\Gamma) = \mathrm{C}^*_r(\Gamma) \text{ (in } \mathbb{B}(\ell^2(\Gamma))).$$

Here $\mathrm{C}^*_u(\Gamma)$ denotes the uniform Roe algebra of Γ, i.e., the C*-subalgebra of $\mathbb{B}(\ell^2(\Gamma))$ generated by $\ell^\infty(\Gamma)$ and $\mathrm{C}^*_r(\Gamma)$. Note that under the canonical isomorphism $\mathrm{C}^*_u(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_r \Gamma$, the intersection $L(\Gamma) \cap \mathrm{C}^*_u(\Gamma)$ is identified with the C*-subalgebra of $\ell^\infty(\Gamma) \rtimes_r \Gamma$ consisting of elements whose coefficients sit in $\mathbb{C}$. Zacharias [**58**] showed that the AP implies the ITAP. We do not know either the ITAP holds or not for groups without the AP.

2.3. Minimality of dynamical systems. Minimality of topological dynamical systems is an indecomposability condition of topological dynamical systems. It is regarded as a topological analogue of ergodicity. Hence it is natural and important to study minimal dynamical systems. Here we recall the definition of minimal dynamical system.

Definition 2.4. A topological dynamical system $\alpha\colon \Gamma \curvearrowright X$ is said to be minimal if every orbit of α is dense in X.

It is clear from the definition that α is minimal if and only if there is no proper Γ-invariant open/closed subset of X. Minimality has a strong relation with the simplicity of the reduced crossed product. Obviously minimality is necessary for the simplicity of the reduced crossed product. The converse is not true in general. Archbold and Spielberg [**3**, page 122, Corollary] showed the converse under the additional assumption that the action is topologically free. Recall that a topological dynamical system $\alpha\colon \Gamma \curvearrowright X$ is said to be topologically free if it acts freely on a dense subset of X.

2.4. Amenability of dynamical systems. (Topological) amenability of dynamical systems is a dynamical analogue of amenability of discrete groups. First we review the definition of topological amenability. For the definition, we need the space $\mathrm{Prob}(\Gamma)$, which is the space of all probability measures on Γ with the pointwise convergence topology. On $\mathrm{Prob}(\Gamma)$, Γ acts from the left by $s.\mu(t) := \mu(s^{-1}t)$ for $s, t \in \Gamma$ and $\mu \in \mathrm{Prob}(\Gamma)$.

Definition 2.5. A dynamical system α of a group Γ on a compact Hausdorff space X is said to be amenable if there is a sequence $(\mu_n)_n$ of continuous maps

$$\mu_n\colon x \in X \mapsto \mu_n^x \in \mathrm{Prob}(\Gamma)$$

such that for all $s \in \Gamma$, we have

$$\lim_{n\to\infty} \sup_{x\in X}(\|s.\mu_n^x - \mu_n^{s.x}\|_1) = 0.$$

Roughly speaking, what this condition means is the existence of Følner-like distributions on the orbit structure. In Chapter 3, we will see how it plays the role of Følner sets in a purely dynamical problem.

Amenable dynamical systems arise naturally in many situations. Here we review a few examples of amenable dynamical systems.

Examples 2.6 (See [**6**]).
- Any dynamical system of an amenable group is amenable.
- The Gromov boundary action of a hyperbolic group is amenable.
- For a second countable locally compact group G, a discrete subgroup Γ, and a closed cocompact amenable subgroup P, the left multiplication action of Γ on G/P is amenable.
- The left translation action of Γ on its Stone-Čech compactification $\beta\Gamma$ is amenable if and only if Γ is exact.

Next we review some basic and important properties of amenable dynamical systems. For an amenable dynamical system and for any non-amenable subgroup Λ of the acting group, there is no Λ-invariant probability measure. It is easy to check that any amenable minimal dynamical system of $\mathbb{F}_n$ must be topologically free. In fact, this holds for all C*-simple groups. See Theorem 14 in [**38**]. The most important feature of amenability for us is that it ensures that the crossed product has nice properties. For example, the reduced crossed product of an amenable dynamical system is nuclear (in fact this characterizes the amenability) [**1**], satisfies the UCT [**56**], and coincides with the full crossed product [**1**].

2.5. Extensions and factors of dynamical systems. Let $\alpha\colon \Gamma \curvearrowright X$ and $\beta\colon \Gamma \curvearrowright Y$ be actions of a group on compact Hausdorff spaces. The α is said to be an extension of β if there is a Γ-equivariant quotient map $\pi\colon X \to Y$. In this case β is said to be a factor of α. The action $\alpha\colon \Gamma \curvearrowright X$ is said to be prime if there is no nontrivial factor of α. Obviously freeness and amenability pass to extensions and minimality passes to factors.

2.6. Transformation groupoids, continuous orbit equivalence, and topological full groups. We refer the reader to Section 5.6 of [**6**] for the definition and basic facts on étale groupoids. For each topological dynamical system $\alpha\colon \Gamma \curvearrowright X$, we have an associated étale groupoid $X \rtimes_\alpha \Gamma$, called the transformation groupoid of α. As a topological space, it is usually defined to be the space

$$\{(g.x, g, x) \in X \times \Gamma \times X : x \in X, g \in \Gamma\}.$$

(Sometimes we omit the first or third coordinate from the definition, which define the isomorphic étale groupoids in the obvious way.) With this definition, the range and source map of $X \rtimes_\alpha \Gamma$ coincide with the projections onto the first and third coordinate respectively. For a composable pair $(g.x, g, x)$ and $(h.y, g, y)$, i.e., in the case $x = h.y$, their composite is defined to be $(g.x, gh, y) = (gh.y, gh, y)$.

We next recall the notion of continuous orbit equivalence, which has a strong relation with the structure of transformation groupoids. To state the definitions, first we recall the definition of orbit cocycles.

Definition 2.7. Let α_1 and α_2 be minimal topologically free dynamical systems of groups Γ_1 and Γ_2 respectively. Let $F\colon X_1 \to X_2$ be an orbit preserving homeomorphism between α_1 and α_2. A map $c\colon \Gamma_1 \times X_1 \to \Gamma_2$ is said to be an orbit cocycle of F if it satisfies the equation $F(\alpha_1(g)(x)) = \alpha_2(c(g, x))(F(x))$ for all $(g, x) \in \Gamma_1 \times X_1$.

Note that by topological freeness, the cocycle equation

$$c(g, h.x)c(h, x) = c(gh, x)$$

holds on a dense subset of $\Gamma_1 \times \Gamma_1 \times X_1$. If we further assume that either α_2 is free or c is continuous, then the cocycle equation holds on $\Gamma_1 \times \Gamma_1 \times X_1$.

Definition 2.8. Let α_1 and α_2 be as above. Two dynamical systems α_1 and α_2 are said to be continuously orbit equivalent if there is an orbit preserving homeomorphism $F\colon X_1 \to X_2$ such that both F and F^{-1} admit a continuous orbit cocycle.

It is easy to check that two minimal topologically free dynamical systems are continuously orbit equivalent if and only if their transformation groupoids are isomorphic as étale groupoids.

Next we recall the definition of the topological full group. This is the group that gathers the local behaviors of a topologically free Cantor system. Here and throughout this thesis, we call a dynamical system on the Cantor set a Cantor system.

Definition 2.9. The topological full group $[[\gamma]]$ of a topologically free Cantor system $\gamma\colon \Gamma \curvearrowright X$ is the group of all homeomorphisms F on X with the following property. For each $x \in X$, there are a neighborhood U of x and $s \in \Gamma$ satisfying $F(y) = s.y$ for all $y \in U$.

It is not hard to show that a homeomorphism F on X is contained in $[[\gamma]]$ if and only if there is a partition $(U_s)_{s\in\Gamma}$ of X by clopen sets such that $F(x) = s.x$ for any $x \in U_s$ and $s \in \Gamma$. Hence the topological full groups are countable.

It is immediate from the definition that the continuous orbit equivalence implies the isomorphism of topological full groups. With minimality assumption, Matui showed the converse [**33**]. This rigidity theorem is originally shown in [**23**] for the integer group.

2.7. Gromov boundary. For a pair of a finitely generated group Γ and a finite generating set S of Γ, we equip a geodesic left-invariant metric d_S on Γ by $d_S(g, h) := |g^{-1}h|_S$ where $|\cdot|_S$ is the length function on Γ determined by S. The quasi-isometric class of d_S is independent of the choice of S. Hence any quasi-isometric invariant property of (geodesic) metric spaces defines a property of finitely generated groups. Hyperbolicity is one such property. Recall that a discrete geodesic space is said to be hyperbolic if there is a constant $\delta > 0$ such that for any geodesic triangle, each edge is contained in the δ-neighborhood of the union of other two edges.

Basic examples of hyperbolic groups are the finitely generated free groups and the fundamental groups of closed manifolds of negative curvature. For each hyperbolic group Γ, there is a canonical boundary $\partial\Gamma$, called the Gromov boundary, which is a metrizable compact Hausdorff space. Here we do not explain the precise definitions of the Gromov boundary and boundary action, since it is technically involved. Roughly speaking, $\partial\Gamma$ is the space of all infinite geodesic rays in Γ modulo a certain equivalence relation. The left multiplication action of Γ on itself naturally induces an action on the Gromov boundary $\partial\Gamma$, called the boundary action. See Section 5.3 of [**6**] or [**19**] for details. For finitely generated free groups, we have a simple description of the Gromov boundary (which is also known as the ideal boundary). See Chapter 2 for details. The Gromov boundary actions are known to be amenable. For a proof, see Section 5.3 of [**6**] for instance.

Acknowledgement. The author would like to thank Cyril Houdayer, Masaki Izumi, David Kerr, Yoshikata Kida, Hiroki Matui, Masato Mimura, Narutaka Ozawa, Sven Raum, Mikael Rørdam, Hiroki Sako, Yasuhiko Sato, and Yoshimichi Ueda for fruitful discussions and stimulating conversations. He particularly thanks David Kerr, Hiroki Matui, and Narutaka Ozawa, whose suggestions and questions motivate some works in this thesis. He also thanks his supervisor Professor Yasuyuki Kawahigashi for encouragement and support. He also thanks Professor Narutaka Ozawa, who was his advisor during his stay in RIMS from April 2013 to March 2015, for daily discussions and advice during his stay. He was supported by Research Fellow of the Japan Society for the Promotion of Science (No.25-7810) and the Program of Leading Graduate Schools, MEXT, Japan. A part of this work was carried out while the author was visiting the University of Münster under the support of Program of Leading Graduate Schools, MEXT, Japan. He acknowledges their hospitality.

CHAPTER 2

Amenable minimal Cantor systems of free groups arising from diagonal actions

The Cantor set is characterized by the following four properties: compactness, total disconnectedness, metrizability, and not having isolated points. From this characterization, the property 'X is (homeomorphic to) the Cantor set' is preserved by many operations. Moreover, the properties which characterize the Cantor set make topological difficulties small in many situations. By these properties, the Cantor set can be considered as a topological analogue of the Lebesgue space without atoms. Moreover, in the category of minimal dynamical systems on metrizable compact spaces, the Cantor set has a "universal" property in the following sense. For any minimal dynamical system of a countable infinite group Γ on a metrizable compact space, it is realized as a factor of a minimal Cantor Γ-system. This follows from a similar proof to the case $\Gamma = \mathbb{Z}$; see Section 1 in [**22**]. Therefore the study of minimal Cantor systems is important. Furthermore, Cantor systems themselves are attractive objects. The underlying spaces of many important dynamical systems are homeomorphic to the Cantor set. This includes certain symbolic dynamical systems, the boundary actions of virtually free groups, and the odometer transformations.

The free group $\mathbb{F}_n$ is one of the most interesting and tractable non-amenable groups. Most of the known non-amenable groups contain $\mathbb{F}_n$, and it has nice properties: the universal property (namely, the freeness), exactness, the Haagerup property, weak amenability, hyperbolicity (with the nice boundary), and so on. Hence, to understand the phenomena of non-amenable groups, the free groups are suitable objects for the first study.

The aim of this chapter is to construct and study amenable minimal Cantor systems of free groups. This is motivated by the following two natural questions. The first question is finding new concrete and tractable presentations of Kirchberg algebras in the UCT class, which is asked in the book [**45**] of Rørdam. (See the last paragraph of page 85.) The second question is about how well the crossed products of amenable minimal Cantor $\mathbb{F}_n$-systems remember the information about the original systems.

Note that for the case of the group $\mathbb{Z}$, analogues of both questions have complete answers. They are the celebrated results of Giordano, Putnam, and Skau [**22**]. For the first question, they have shown that every simple unital A$\mathbb{T}$-algebra of real rank zero whose K_1-group is isomorphic to $\mathbb{Z}$ is presented as the crossed product of a minimal Cantor $\mathbb{Z}$-system and this is the only possible case. For the second question, they have shown that two minimal Cantor $\mathbb{Z}$-systems have isomorphic crossed products exactly when they are strongly orbit equivalent.

To start the study on these problems, we need as many well-understandable examples of amenable minimal Cantor $\mathbb{F}_n$-systems as possible. Until now, only a few examples have been constructed and studied. In this chapter, we construct continuously many examples of amenable minimal Cantor $\mathbb{F}_n$-systems whose crossed products are completely determined. As a consequence, for the first question, we obtain new concrete presentations for certain continuously many Kirchberg algebras in the UCT class. For the second question, our examples give a hopeful prospect. As examples, we show that the diagonal actions of the boundary actions and the products of odometer transformations are classified in terms of continuous orbit equivalence by using a C*-algebraic technique.

A recent work of Elliott and Sierakowski [**17**] gives an example of amenable minimal Cantor $\mathbb{F}_n$-systems which are distinguished by K-theory. They constructed an amenable minimal free Cantor $\mathbb{F}_n$-system whose K_0-group vanishes. In particular, it has the different K_0-group from that of the boundary action. Their construction is based on the idea developed in the paper [**48**]. Our strategy is different from them. We construct amenable minimal Cantor $\mathbb{F}_n$-systems from the diagonal actions. This construction is quite simple and gives many fruitful and concrete examples of amenable minimal Cantor $\mathbb{F}_n$-systems.

Main results. Here we collect the main results of this chapter.

Throughout this chapter, the K-theory of the reduced crossed product of a dynamical system γ is referred to as the K-theory of γ for short.

Theorem A (Theorem 3.5). *Let G be a subgroup of $\mathbb{Q}^{\oplus\infty}$ which contains $\mathbb{Z}^{\oplus\infty}$ as a subgroup of infinite index. Let $2 \leq n < \infty$ and k be an integer. Then there is an amenable minimal Cantor $\mathbb{F}_n$-system that satisfies the following properties.*

- *The pair of K_0-group and the unit $[1]_0$ is isomorphic to*
$$\left(G \oplus \Lambda_{G,n}, 0 \oplus [k(n-1)^{-1}]\right),$$
where $\Lambda_{G,n}$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of elements whose order divides the product of $(n-1)$ and the order of a finite subgroup of $G/\mathbb{Z}^{\oplus\infty}$.
- *The K_1-group is isomorphic to $\mathbb{Z}^{\oplus\infty}$.*
- *The crossed product is a Kirchberg algebra in the UCT class.*

We also show similar results for non-amenable finitely generated virtually free groups (Theorem 3.6). As a consequence of these results, we obtain the following decomposition theorem.

Corollary B (Corollary 3.7). *For a torsion free abelian group G of infinite rank, consider a Kirchberg algebra A in the UCT class satisfying $(K_0(A), [1]_0, K_1(A)) \cong (G \oplus \mathbb{Q}/\mathbb{Z}, 0, \mathbb{Z}^{\oplus\infty})$. Then for any non-amenable finitely generated virtually free group Γ, A is decomposed as the crossed product of an amenable minimal topologically free Cantor Γ-system.*

We also see that even if we restrict our attention to the free Cantor systems, we still obtain the existence of continuously many amenable minimal Cantor systems. We further work on the infinite rank free group, and finally obtain the following result.

Theorem C (Theorem 3.8). *Every non-amenable virtually free group admits continuously many amenable minimal free Cantor Γ-systems whose crossed products are mutually non-isomorphic Kirchberg algebras in the UCT class.*

In the proof of Theorem A, techniques of the computation of K-theory are developed for certain Cantor systems. In Section 4, we give computations of the K-theory for the diagonal actions of the boundary actions and the products of the odometer transformations. From our computations, their topological full groups, continuous orbit equivalence classes, and strong orbit equivalence classes (which we define later) are classified. Here we collect the classification results. First we present the Cantor systems which we will classify more precisely. For each free group $\mathbb{F}_n$, fix an enumeration $\{s_1, \ldots, s_n\}$ of the canonical generators. For $2 \leq n < \infty$, $1 \leq k \leq n$, and a sequence $N_1, \ldots, N_k$ of infinite supernatural numbers, define a Cantor $\mathbb{F}_n$-system by
$$\gamma^{(n)}_{N_1,\ldots,N_k} := \beta_n \times \left(\prod_{j=1}^{k} \alpha_{N_j} \circ \pi^{(n)}_j\right),$$

where β_n denotes the boundary action of $\mathbb{F}_n$, α_N denotes the odometer transformation of type N, and $\pi_j^{(n)}$ denotes the homomorphism $\mathbb{F}_n \to \mathbb{Z}$ given by $s_j \mapsto 1$ and $s_i \mapsto 0$ for $i \neq j$.

Theorem D (Proposition 4.2 and Theorem 4.5). *For two Cantor systems* $\gamma_1 := \gamma^{(n)}_{N_1,\ldots,N_k}$ *and* $\gamma_2 := \gamma^{(m)}_{M_1,\ldots,M_l}$ *defined as above, the following conditions are equivalent.*

(1) *They are strongly orbit equivalent.*
(2) *They are continuously orbit equivalent.*
(3) *Their topological full groups are isomorphic.*
(4) *The commutator subgroups of their topological full groups are isomorphic.*
(5) *Their crossed products are isomorphic.*
(6) *Their* K_0*-invariants* $(K_0, [1]_0)$ *are isomorphic.*
(7) *The equations* $k = l$ *and* $n = m$ *hold and there are a permutation* $\sigma \in \mathfrak{S}_k$ *and sequences* $(n_1, \ldots, n_k)$ *and* $(m_1, \ldots, m_k)$ *of natural numbers that satisfy* $\prod_{j=1}^k n_j = \prod_{j=1}^k m_j$ *and* $n_i N_i = m_i M_{\sigma(i)}$.

1. Preliminaries

1.1. Gromov boundaries of free groups. Since free groups have a combinatorial aspect (cf. [**32**]), it is not so surprising that their Gromov boundaries also have a combinatorial aspect. Here we recall an explicit description of the boundaries of free groups and their combinatorial aspect that we need in the computation of K-groups in Theorem 3.5.

Definition 1.1. Let S be the set of canonical generators of $\mathbb{F}_n$ and set $\widetilde{S} := S \sqcup S^{-1}$. Define the subspace $\partial\mathbb{F}_n$ of $\prod_{\mathbb{N}} \widetilde{S}$ by

$$\partial\mathbb{F}_n := \left\{ (s_m)_{m\in\mathbb{N}} \in \prod_{\mathbb{N}} \widetilde{S} : s_{m+1} \neq s_m^{-1} \text{ for all } m \in \mathbb{N} \right\}.$$

We equip $\partial\mathbb{F}_n$ with the topology induced from the product topology.

It is easy to check that $\partial\mathbb{F}_n$ is homeomorphic to the Cantor set.

Each element of $\partial\mathbb{F}_n$ is regarded as a (one-sided) infinite reduced word of the free basis S. For an element w of $\mathbb{F}_n$ or $\partial\mathbb{F}_n$ with the reduced word $w = s_1 \cdots s_k \cdots$, the elements $s_1 \cdots s_k$ and s_k are referred to as the first kth segment of w and the kth alphabet of w, respectively. For $w \in \mathbb{F}_n$, denote by $|w|$ the length of the reduced word of w. For $w \in \mathbb{F}_n$ and $k \leq |w| = m$, the element $s_{m-k+1} \cdots s_m$ is referred to as the last kth segment of w.

For $z \in \mathbb{F}_n$ and $w \in \partial\mathbb{F}_n$, we define the product $z \cdot w$ by the same rule as that of the product of two elements of $\mathbb{F}_n$. This is the boundary action of $\mathbb{F}_n$. We denote the boundary action of $\mathbb{F}_n$ by β_n, or simply by β if the rank n is obvious from the context.

Similarly to the elements of free groups, for any other free basis T of $\mathbb{F}_n$, every element w of $\partial\mathbb{F}_n$ can be expanded uniquely as an infinite reduced word of the free basis T. This enables us to identify the boundary space $\partial\mathbb{F}_n$ with the space

$$\left\{ (t_m)_{m\in\mathbb{N}} \in \prod_{\mathbb{N}} \widetilde{T} : t_{m+1} \neq t_m^{-1} \text{ for all } m \in \mathbb{N} \right\},$$

where $\widetilde{T} := T \sqcup T^{-1}$, for any free basis T of $\mathbb{F}_n$. We always identify these spaces in this way without further comments.

For a free basis T of $\mathbb{F}_n$ and $t \in \widetilde{T}$, we define the clopen subset $\Omega(t;T)$ of $\partial\mathbb{F}_n$ to be the subspace of all infinite reduced words whose first alphabet is t in the expansion with respect to the free basis T. When the free basis T is obvious from the context, we simply denote it by $\Omega(t)$.

More generally, for a free basis W of a finite index subgroup Γ of $\mathbb{F}_n$ and $w \in W \sqcup W^{-1}$, we define the clopen subset $\Theta(w; W)$ of $\partial\mathbb{F}_n$ to be the image of $\Omega(w; W)(\subset \partial\Gamma)$ under the homeomorphism $\partial\Gamma \cong \partial\mathbb{F}_n$ induced from the inclusion map. If we need to refer the entire group $\Lambda = \mathbb{F}_n$, we further denote it by $\Theta(w; W; \Lambda)$.

1.2. Supernatural numbers and associated abelian groups. To describe certain abelian groups, we recall the definition of the supernatural numbers. Denote by $\mathcal{P}$ the set of all prime numbers. A supernatural number is a map from $\mathcal{P}$ into the set $\{0, 1, \ldots, \infty\}$. A supernatural number N is formally presented as the formal infinite product $\prod_{p\in\mathcal{P}} p^{N(p)}$ of powers of prime numbers. By this presentation, supernatural numbers are naturally regarded as a generalization of natural numbers. We say a supernatural number N is infinite if it is not a natural number. Note that many operations of natural numbers are naturally extended to that of supernatural numbers. (E.g., the (possibly infinite) product, the greatest common divisor, and the least common multiple; which correspond to the summation, the infimum, and the supremum of the corresponding functions N, respectively.)

For a supernatural number N, denote by $\Lambda(N)$ the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the elements whose order divides N and denote by $\Upsilon(N)$ the inverse image of the group $\Lambda(N)$ under the quotient homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Note that for two supernatural numbers N and M, the groups $\Lambda(N)$ and $\Lambda(M)$ are isomorphic if and only if $N = M$ holds, and the groups $\Upsilon(N)$ and $\Upsilon(M)$ are isomorphic if and only if there are natural numbers n and m with $nN = mM$.

2. Elementary construction

In this section we give the construction of amenable minimal Cantor $\mathbb{F}_n$-systems which plays the fundamental role in the next two sections. The next proposition provides amenable minimal dynamical systems of hyperbolic groups.

PROPOSITION 2.1. *Let Γ be a hyperbolic group. Let σ be a transitive action of Γ on a finite set X. Then the diagonal action $\beta \times \sigma$ is amenable and minimal, and its crossed product is isomorphic to $\mathbb{M}_n(C(\partial\Gamma_0) \rtimes_r \Gamma_0)$, where Γ_0 is the stabilizer subgroup of $\sigma\colon \Gamma \curvearrowright X$ at a point, which is independent of the choice of point up to conjugacy by transitivity, and $n = \sharp X$.*

PROOF. The amenability of $\beta \times \sigma$ is clear since it has an amenable factor. Fix $x_0 \in X$ and denote by Γ_0 the stabilizer subgroup of σ at x_0. Then it is hyperbolic and the restriction of the boundary action of Γ to Γ_0 coincides with the boundary action of Γ_0. This shows that the restriction of β to Γ_0 is minimal, hence $\beta \times \sigma$ is minimal.

The last claim immediately follows from Green's imprimitivity theorem [**21**, Theorem 4.1]. However, we need a concrete isomorphism for later use, so we construct an isomorphism directly here, which is a very special case of [**21**]. To do this, first we identify X with Γ/Γ_0 by the bijective map $s \in \Gamma/\Gamma_0 \mapsto s.x_0 \in X$. Take a cross section ρ of the quotient map $\Gamma \to \Gamma/\Gamma_0$. Then define two maps π and u by

$$\pi\colon f \in C(\partial\Gamma \times X) \mapsto \bigoplus_{x\in X} \left(f \circ ((\beta \times \sigma)(\rho(x)))|_{\partial\Gamma\times\{x_0\}}\right) \in \mathbb{M}_X(C(\partial\Gamma_0) \rtimes_r \Gamma_0)$$

and

$$u\colon s \in \Gamma \mapsto \sum_{x\in X} E_{s.x,x} \otimes (\rho(s.x)^{-1} s\rho(x)) \in \mathbb{M}_X(C(\partial\Gamma_0) \rtimes_r \Gamma_0),$$

here we identify $\partial\Gamma \times \{x_0\}$ with $\partial\Gamma_0$ in the canonical way. Then the pair (π, u) is a covariant representation of $\beta \times \sigma$. This covariant representation induces a $*$-isomorphism θ between two $*$-algebras $C(\partial\Gamma \times X) \rtimes_{\mathrm{alg}} \Gamma$ and $\mathbb{M}_X(C(\partial\Gamma_0) \rtimes_{\mathrm{alg}} \Gamma_0)$. Then by the amenability of $\beta \times \sigma$, the universal C*-enveloping algebra of the $*$-algebra $C(\partial\Gamma \times X) \rtimes_{\mathrm{alg}} \Gamma$ coincides with the reduced crossed product of $\beta \times \sigma$, and similarly for the second one. This shows that the $*$-isomorphism θ extends to the desired isomorphism. $\square$

A particularly interesting case is the one that the group Γ is a free group.

REMARK 2.2. Let $2 \leq n < \infty$. We apply Proposition 2.1 to the case $\Gamma = \mathbb{F}_n$ and a transitive action $\sigma\colon \mathbb{F}_n \curvearrowright X$ on a set X with $\sharp X = k \in \mathbb{N}$. Then by Schreier's formula [**32**, Chap.1, Prop.3.9], any subgroup of $\mathbb{F}_n$ of the index k is the free group of rank $m := k(n-1)+1$. Hence the resulting crossed product is isomorphic to $\mathbb{M}_k(C(\partial\mathbb{F}_m) \rtimes_r \mathbb{F}_m)$, which only depends on the cardinality of X. However, the inclusion $C(\partial\mathbb{F}_n)\rtimes_r\mathbb{F}_n \to C(\partial\mathbb{F}_n \times X)\rtimes_r\mathbb{F}_n \cong \mathbb{M}_k(C(\partial\mathbb{F}_m)\rtimes_r\mathbb{F}_m)$ does depend on the choice of transitive action. The difference between them is crucial in the next section.

3. More general constructions of amenable minimal Cantor $\mathbb{F}_n$-systems

In this and the next sections we investigate more general constructions of amenable minimal Cantor systems for free groups. We construct continuously many amenable minimal Cantor systems and classify them in terms of the crossed products.

For computations of K-groups in Theorem 3.5, we need a few lemmas and facts about the K-theory of the boundary algebra $C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n$.

In [**50**], Spielberg showed that the boundary algebras of free groups are presented as a Cuntz–Krieger algebra. (The canonical generators are explicitly given.) This presentation and Cuntz's computation of the K-theory of Cuntz–Krieger algebras [**11**, Proposition 3.1] show that the K_0-group of the boundary algebra is equal to $\left(\bigoplus_{s\in S}\mathbb{Z}[p_s]_0\right) \oplus \mathbb{Z}_{n-1}[1]_0$, where p_s denotes the characteristic function of the clopen subset $\Omega(s)$ for each $s \in \widetilde{S}$. Here for an element x of a group G, we denote the subgroup $\langle x\rangle$ by $\mathbb{Z}x$ (resp. $\mathbb{Z}_m x$) if x is of infinite order (resp. x is of order m). Notice that for $s \in S$, the equality $s.\Omega(s^{-1}) = \partial\mathbb{F}_n \setminus \Omega(s)$ holds. This implies the equality $[p_s]_0 + [p_{s^{-1}}]_0 = [1]_0$. We also have that the K_1-group is isomorphic to $\mathbb{Z}^n$.

We also need a few notations for abelian groups. For an abelian group G, the torsion subgroup G^{tor} of G is the subgroup of G consisting of all torsion elements. For a finitely generated abelian group G, denote by G^{free} the quotient group G/G^{tor}. The subgroup G^{tor} is referred to as the torsion part of G and the quotient G^{free} is referred to as the free part of G. By the structure theorem of finitely generated abelian groups, G^{free} is indeed free abelian and G is isomorphic to $G^{\mathrm{free}} \oplus G^{\mathrm{tor}}$ (in a non-canonical way). Every homomorphism h between two finitely generated abelian groups induces a homomorphism between their free parts. We denote it by h^{free}. Similarly, for a homomorphism h between two abelian groups, we denote by h^{tor} the restriction of it to the torsion subgroup, and refer to it as the torsion part of h.

Every automorphism φ of $\mathbb{F}_n$ induces the automorphism Φ of $C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n$ by

$$s \in \mathbb{F}_n \mapsto \varphi(s) \in \mathbb{F}_n$$

and

$$f \in C(\partial\mathbb{F}_n) \mapsto f \circ (\partial\varphi)^{-1} \in C(\partial\mathbb{F}_n).$$

Here and below, for an automorphism φ of a hyperbolic group, denote by $\partial\varphi$ the homeomorphism on the Gromov boundary induced by φ. Note that the mapping $\varphi \mapsto \Phi$ preserves the composition. The next give two lemmas about $\Phi_{*,0}$.

LEMMA 3.1. *Let $2 \leq n < \infty$ and fix an enumeration $S = \{s_1, \ldots, s_n\}$ of S. Then, for any $A \in \mathrm{GL}(n, \mathbb{Z})$, there is an automorphism φ of $\mathbb{F}_n$ such that with respect to the identification*

$$K_0(C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n) = \left(\bigoplus_{i=1}^{n}\mathbb{Z}[p_{s_i}]_0\right) \oplus \mathbb{Z}_{n-1}[1]_0,$$

$\Phi_{*,0}$ *is identified with* $A \oplus \mathrm{id}$.

Proof. We claim that for the following automorphisms

(i) the automorphism induced from a permutation of S,
(ii) the automorphism given by

$$s_1 \mapsto s_2 s_1 \text{ and } s_i \mapsto s_i \text{ for } 2 \leq i \leq n,$$

the equality $\Phi_{*,0} = {}^t(\varphi^{\mathrm{ab}})^{-1} \oplus \mathrm{id}$ holds under the identification of $\bigoplus_{s\in S} \mathbb{Z}[p_s]_0$ with the abelianization $\mathbb{F}_n^{\mathrm{ab}}$ of $\mathbb{F}_n$. Then the same condition holds for any element of the subgroup of $\mathrm{Aut}(\mathbb{F}_n)$ generated by these automorphisms. This ends the proof. Case (i) is obvious, so let ψ be the automorphism of $\mathbb{F}_n$ given in case (ii). For our purpose, it is enough to expand $[\Psi(p_s)]_0$ as a linear combination of $[p_t]_0$'s. By the definition of Ψ, for each $s \in S$, the projection $\Psi(p_s)$ is the characteristic function of $\partial\psi(\Omega(s; S)) = \Omega(\psi(s); \psi(S))$. Then one can check easily that the following three equations hold.

$$\Omega(\psi(s_1); \psi(S)) = s_2.\Omega(s_1; S).$$

$$\Omega(\psi(s_2); \psi(S)) = \Omega(s_2; S) \setminus (s_2.\Omega(s_1; S)).$$

$$\Omega(\psi(s_i); \psi(S)) = \Omega(s_i; S) \text{ for } 2 < i \leq n.$$

Here we only give a proof of the inclusion $s_2.\Omega(s_1; S) \subset \Omega(\psi(s_1); \psi(S))$. The rest of the proof is done in a similar way. Let $w \in s_2.\Omega(s_1, S)$ be given. Then the reduced form of w is of the form $s_2 s_1 v_1 s_1^{k_1} s_2^{l_1} v_2 \cdots$ for some $v_i \in \langle s_3, \ldots, s_n\rangle$, $k_i, l_i \in \mathbb{Z}$. (Here we allow the possibility that $v_i = e$, $k_i = 0$, $l_i = 0$.) Then the expansion of w with respect to $\psi(S)$ is given by reducing the formal infinite product $\psi(s_1)v_1(\psi(s_2)^{-1}\psi(s_1))^{k_1}\psi(s_2)^{l_1}v_2 \cdots$ to a reduced form (with respect to $\psi(S)$). Note that since $w = s_2 s_1 v_1 s_1^{k_1} s_2^{l_1} v_2 \cdots$ is reduced with respect to S, the equality $v_1 = e$ implies $k_1 \geq 0$ and similarly for the other places. This shows that any cancellation does not remove the first $\psi(s_1)$. Thus we have $s_2.\Omega(s_1, S) \subset \Omega(\psi(s_1), \psi(S))$.

From these equations, we obtain the equations

$$\Psi([p_{s_1}]_0) = [p_{s_1}]_0,$$

$$\Psi([p_{s_2}]_0) = [p_{s_2}]_0 - [p_{s_1}]_0,$$

$$\Psi([p_{s_i}]_0) = [p_{s_i}]_0 \text{ for } 2 < i \leq n.$$

This shows that the automorphism ψ satisfies our claim. □

Lemma 3.2. *Let t and u be two distinct elements of S and let $m \in \mathbb{Z}$. Let ψ be the automorphism of $\mathbb{F}_n$ defined by*

$$t \mapsto t, u \mapsto t^m u t^{-m}, \text{ and } v \mapsto v \text{ for the other } v \in S.$$

Then $\Psi_{*,0}$ is given by

$$[1]_0 \mapsto [1]_0, [p_t]_0 \mapsto [p_t]_0 - m[1]_0, \text{ and } [p_v]_0 \mapsto [p_v]_0 \text{ for } u \in S \setminus \{t\}.$$

Proof. Since the set of integers satisfying the claim forms a group, it suffices to show it for the case $m = 1$. Let φ_1, φ_2 be the automorphisms of $\mathbb{F}_n$ defined by $\varphi_1(u) = u^{-1}, \varphi_1(v) := v$ for $v \in S \setminus \{u\}$, $\varphi_2(u) := tu$, and $\varphi_2(v) := v$ for $v \in S \setminus \{u\}$. Then a direct computation shows the equality $\psi = \varphi_2 \circ \varphi_1 \circ \varphi_2 \circ \varphi_1$. Therefore, to compute $\Psi_{*,0}$, it suffices to compute $(\Phi_1)_{*,0}$ and $(\Phi_2)_{*,0}$. The computation of $(\Phi_1)_{*,0}$ is easily derived from the equation $[p_{u^{-1}}]_0 = [1]_0 - [p_u]_0$. The $(\Phi_2)_{*,0}$ is computed in the proof of Lemma 3.1. Now the claim follows from a simple algebraic computation. □

LEMMA 3.3. *Let $\mathbb{F}_n$ be the free group. Enumerate S as $S := \{s_1, \ldots, s_n\}$. Consider the presentation*

$$K_0(C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n) = \left(\bigoplus_{i=1}^{n} \mathbb{Z}[p_{s_i}]_0\right) \oplus \mathbb{Z}_{n-1}[1]_0,$$

where $p_s := \chi_{\Omega(s;S)}$ for $s \in S \sqcup S^{-1}$. Similarly, for each enumerated free basis $W := \{w_1, \ldots, w_m\}$ of a finite index subgroup Γ of $\mathbb{F}_n$, consider the presentation

$$K_0(C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n) = \left(\bigoplus_{i=1}^{m} \mathbb{Z}[q_{(w_i;W)}]_0\right) \oplus \mathbb{Z}_{m-1}[r_W]_0$$

where $q_{(w;W)} := \chi_{\Theta(w;W)} \otimes \delta_{[e]}$ for $w \in W \sqcup W^{-1}$ and $r_W := 1 \otimes \delta_{[e]}$. (This follows from the isomorphism given in Proposition 2.1. See also the proof below.) Let

$$j\colon C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n \to C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n$$

denote the canonical inclusion. Let $A\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be an injective homomorphism and let l denote the product of all elementary divisors of A. Then for any left invertible inclusion $Q\colon \mathbb{Z}^n \to \mathbb{Z}^{l(n-1)+1}$, there is an enumerated finite subset $W = \{w_1, \ldots, w_m\}$ of $\mathbb{F}_n$ satisfying the following conditions.

- *The subset W is a free basis of a subgroup Γ of $\mathbb{F}_n$.*
- *The index $[\mathbb{F}_n : \Gamma]$ is l; hence one has $m = l(n-1)+1$.*
- *With respect to the above enumerated bases, $(j_{*,0})^{\mathrm{free}}$ is presented by QA.*
- *The torsion part of $j_{*,0}$ is injective.*
- *The image of $\bigoplus_{i=1}^{n} \mathbb{Z}[p_{s_i}]_0$ under $j_{*,0}$ is contained in the subgroup*

$$\left(\bigoplus_{i=1}^{m} \mathbb{Z}[q_{(w_i;W)}]_0\right) \oplus \Lambda$$

where Λ denotes the subgroup of $\mathbb{Z}_{m-1}[r_W]_0$ generated by elements of order 2, which must be either trivial or isomorphic to $\mathbb{Z}_2$.

PROOF. First we show that if the claim holds for a homomorphism $QA\colon \mathbb{Z}^n \to \mathbb{Z}^{l(n-1)+1}$, then it also holds for any homomorphisms of the form $BQAC$ where $B \in \mathrm{GL}(l(n-1)+1, \mathbb{Z})$ and $C \in \mathrm{GL}(n, \mathbb{Z})$. To see this, take an enumerated finite subset W which satisfies the required conditions for QA. Clearly, it suffices to show the claim for the cases $B = \mathrm{id}$ and $C = \mathrm{id}$ holds.

First we consider the case $B = \mathrm{id}$. In this case, take an automorphism φ of $\mathbb{F}_n$ satisfying $\Phi_{*,0} = C^{-1} \oplus \mathrm{id}$, which exists by Lemma 3.1. Consider the commutative diagram

$$\begin{array}{ccc} C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n & \xrightarrow{j} & C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n \\ \Phi\big\downarrow & & \widetilde{\Phi}\big\downarrow \\ C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n & \xrightarrow{j'} & C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\varphi(\Gamma))) \rtimes_r \mathbb{F}_n \end{array}$$

where each row map is the canonical inclusion and the second column map $\widetilde{\Phi}$ is the isomorphism induced from the following covariant representation

$$s \in \mathbb{F}_n \mapsto \varphi(s) \in \mathbb{F}_n,$$

$$f \in C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \mapsto (f \circ (\partial\varphi \times \tilde{\varphi})^{-1}) \in C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\varphi(\Gamma))) \rtimes_r \mathbb{F}_n.$$

Here $\tilde{\varphi}\colon \mathbb{F}_n/\Gamma \to \mathbb{F}_n/\varphi(\Gamma)$ is the bijection defined by $x\Gamma \in \mathbb{F}_n/\Gamma \mapsto \varphi(x\Gamma) \in \mathbb{F}_n/\varphi(\Gamma)$. Then on the level of K_0-groups, the above commutative diagram becomes the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}^S \oplus \mathbb{Z}_{n-1} & \xrightarrow{j_{*,0}} & \mathbb{Z}^W \oplus \mathbb{Z}_{m-1} \\ {\scriptstyle C^{-1}\oplus \mathrm{id}}\big\downarrow & & {\scriptstyle \tau_\varphi}\big\downarrow \\ \mathbb{Z}^S \oplus \mathbb{Z}_{n-1} & \xrightarrow{j'_{*,0}} & \mathbb{Z}^{\varphi(W)} \oplus \mathbb{Z}_{m-1} \end{array}$$

where τ_φ is the isomorphism given by $[q_{(w:W)}]_0 \mapsto [q_{(\varphi(w);\varphi(W))}]_0$ and $[r_W]_0 \mapsto [r_{\varphi(W)}]_0$. This shows that the enumerated finite subset $\varphi(W)$ satisfies the desired conditions for the homomorphism QAC. We remark that this operation may change the subgroup Γ.

Next we consider the case $C = \mathrm{id}$. In this case, take an automorphism ψ of Γ such that the induced automorphism Ψ satisfies $\Psi_{*,0} = B^{-1} \oplus \mathrm{id}$ with respect to the enumerated free basis W. Then from this form of $\Psi_{*,0}$, we immediately conclude that the enumerated finite subset $\psi(W)$ satisfies the desired conditions for BQA.

From this together with the elementary divisor theory, it suffices to show the assertion for the case of A being a diagonal homomorphism with respect to the standard basis. By decomposing A as a composite of finitely many homomorphisms, we only need to show the following. For each $k \in \mathbb{N} \setminus \{1\}$, there is a free basis W of a finite index subgroup Γ of $\mathbb{F}_n$ with the following properties.

- The index $[\mathbb{F}_n : \Gamma]$ is k.
- The elementary divisors of $j_{*,0}$ are given by $(1, 1, \ldots, 1, k)$.
- The $j_{*,0}$ satisfies the last two conditions in the statement.

To construct the desired W, fix $s \in S$ and set

$$W := \left\{ s^k, s^l t s^{-l} : 0 \leq l \leq k-1, t \in S \setminus \{s\} \right\}.$$

Then W is a free basis of the kernel Γ of the homomorphism $\pi_s\colon \mathbb{F}_n \to \mathbb{Z}_k$. Here π_s is given by

$$s \mapsto [1] \text{ and } t \mapsto [0] \text{ for } t \in S \setminus \{s\}.$$

In particular, the subgroup of $\mathbb{F}_n$ generated by W is of index k. Note that the group Γ coincides with the stabilizer subgroup of the action $\sigma\colon \mathbb{F}_n \curvearrowright \mathbb{F}_n/\Gamma$ of an arbitrary point.

By direct computations, we obtain the equalities

$$\Omega(s) = \bigsqcup_{w \in I} \Theta(w; W)$$

where $I := \{s^k, s^l t^{\pm 1} s^{-l} : 1 \leq l \leq k-1, t \in S \setminus \{s\}\}$, and

$$\Omega(t) = \Theta(t; W) \text{ for } t \in S \setminus \{s\}.$$

From the isomorphism $C(\partial \mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n \cong \mathbb{M}_k(C(\partial\Gamma) \rtimes_r \Gamma)$ given in Proposition 2.1, the canonical (non-unital) inclusion

$$C(\partial \mathbb{F}_n \times \{[e]\}) \rtimes_r \Gamma \to C(\partial \mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n$$

induces the isomorphism of K_0-groups. This yields the equation

$$K_0(C(\partial \mathbb{F}_n \times (\mathbb{F}_n/\Gamma)) \rtimes_r \mathbb{F}_n) = \left(\bigoplus_{w \in W} \mathbb{Z}[q_{(w;W)}]_0 \right) \oplus \mathbb{Z}_{k(n-1)}[r_W]_0.$$

Since the set $\mathrm{im}(\rho) = \{s^l : 0 \le l \le k-1\}$ is a complete system of representatives for the quotient $\mathbb{F}_n/\Gamma$, the above equations of Θ's give the equations

$$p_s = \sum_{l=0}^{k-1} \sum_{v \in I} q_{(v;W)} \circ (\beta(s^{-l}) \times \mathrm{id}) \circ (\gamma(s^l))$$

and

$$p_t = \sum_{l=0}^{k-1} q_{(t;W)} \circ (\beta(s^{-l}) \times \mathrm{id}) \circ (\gamma(s^l))$$

for $t \in S \setminus \{s\}$. (Here we use the equality $(\beta(s^{-l}) \times \mathrm{id}) \circ \gamma(s^l) = \mathrm{id} \times \sigma(s^l)$.) We also have

$$1 = \sum_{l=0}^{k-1} r_W \circ (\gamma(s^l)).$$

The last equation shows the equation $[1]_0 = k[r_W]_0$. This shows that the $j_{*,0}$ preserves the order of the unit $[1]_0$. This proves the injectivity of $(j_{*,0})^{\mathrm{tor}}$.

We observe that the homeomorphism $\beta(s^{-1})$ on $\partial\Gamma$ is induced by a group automorphism of Γ. Indeed, it is induced from the conjugating automorphism $\alpha := \mathrm{ad}(s^{-1})$. Hence it extends to the automorphism Φ of $C(\partial\Gamma) \rtimes_r \Gamma \cong C(\partial\mathbb{F}_n \times \{[e]\}) \rtimes_r \Gamma$. The proof of Lemma 3.1 and Lemma 3.2 show that $\Phi_{*,0}$ is the automorphism of $\left(\bigoplus_{w \in W} \mathbb{Z}[q_{(w;W)}]_0\right) \oplus \mathbb{Z}_{m-1}[r_W]_0$ given as follows.

$$[r_W]_0 \mapsto [r_W]_0,$$

$$[q_{(s^k;W)}]_0 \mapsto [q_{(s^k;W)}]_0 + (n-1)[r_W]_0,$$

$$[q_{(w;W)}]_0 \mapsto [q_{(\sigma(w);W)}]_0 \text{ for } w \in W \setminus \{s^k\},$$

where σ is the permutation of W given by

$$\sigma(w) := \begin{cases} w & \text{if } w = s^k, \\ s^{-1+k} w s^{1-k} & \text{if } w \in S \setminus \{s\}, \\ s^{-1} w s & \text{otherwise.} \end{cases}$$

This is because the automorphism α is equal to the composite of the automorphism induced from the permutation σ of W and $n-1$ automorphisms of the form appearing in Lemma 3.2 with $t = s^k$ and $m = -1$. (Notice that the equality $s^{-1}ts = s^{-k}(s^{k-1}ts^{1-k})s^k$ holds for $t \in S$.)

From this, the first equation is reduced to

$$\begin{aligned} [p_s]_0 &= \sum_{l=0}^{k-1} \Big([q_{(s^k;W)}]_0 + l(n-1)[r_W]_0 + (k-1)(n-1)[r_W]_0 \Big) \\ &= k[q_{(s^k;W)}]_0 + \frac{k(k-1)(n-1)}{2}[r_W]_0. \end{aligned}$$

Here we use the equation $[q_{(w;W)}]_0 + [q_{(w^{-1};W)}]_0 = [r_W]_0$ for $w \in W$. Similarly, the second equation is reduced to

$$[p_t]_0 = \sum_{v \in J_t} [q_{(v;W)}]_0,$$

where $J_t = \{s^l t s^{-l} : 0 \le l \le k-1\}$ for $t \in S \setminus \{s\}$. Since the sets $\{s^k\}, J_t; t \in S \setminus \{s\}$ are mutually disjoint subsets of W and the order of the element $\frac{k(k-1)(n-1)}{2}[r_W]_0$ is either 0 or 2, this W is what we needed. □

For a torsion group H, we define the supernatural number N_H to be the least common multiple of the orders of finite subgroups of H. For a subset X of $\mathbb{Z}^{\oplus\infty}$, define the subgroup $P(X)$ of $\mathbb{Z}^{\oplus\infty}$ to be

$$\{y \in \mathbb{Z}^{\oplus\infty} : \text{there exists } n \in \mathbb{Z} \setminus \{0\} \text{ with } ny \in \langle X \rangle\}.$$

We denote by π the quotient homomorphism $\mathbb{Q}^{\oplus\infty} \to \mathbb{Q}^{\oplus\infty}/\mathbb{Z}^{\oplus\infty}$.

LEMMA 3.4. *Let G be a subgroup of $\mathbb{Q}^{\oplus\infty}$ which contains $\mathbb{Z}^{\oplus\infty}$ as a subgroup of infinite index. Let $2 \leq n < \infty$. Then G is isomorphic to the inductive limit of an inductive system of the form $(\mathbb{Z}^{k_m}, A_m)_m$ that satisfies the following conditions.*

- *Each connecting map A_m is injective.*
- *The sequence $(k_m)_m$ satisfies $k_1 = n$ and $k_m = l_{m-1}(k_{m-1} - 1) + 1$ for $m \geq 2$, where for each m, l_m denotes the product of all elementary divisors of A_m.*
- *The formal infinite product $\prod_m l_m$ is equal to $N_{G/\mathbb{Z}^{\oplus\infty}}$.*

PROOF. Fix an enumeration $\{x_n\}_n$ of elements of $\mathbb{Z}^{\oplus\infty}$. Since the quotient $G/\mathbb{Z}^{\oplus\infty}$ is an infinite torsion abelian group, there are a sequence $(y_j)_j$ of elements of $\mathbb{Z}^{\oplus\infty}$ and a sequence $(k_j)_j$ of natural numbers greater than 1 such that the sequence $(\langle \pi(k_1^{-1}y_1), \ldots, \pi(k_j^{-1}y_j) \rangle)_j$ of subgroups of $\pi(G)$ is strictly increasing and the union of the sequence coincides with $\pi(G)$. Note that the set $\{x_j, k_j^{-1}y_j : j \in \mathbb{N}\}$ generates G. For each $j \in \mathbb{N}$, set $m_j := \sharp(\langle \pi(k_1^{-1}y_1), \ldots, \pi(k_j^{-1}y_j) \rangle)$. Take $r_1 \in \mathbb{N}$ such that the rank of $\langle y_1, x_1, \ldots, x_{r_1} \rangle$ is equal to n. Set $H_1 := P(y_1, x_1, \ldots, x_{r_1})$. Note that any subgroup of $\mathbb{Z}^{\oplus\infty}$ is free abelian [**18**, Vol.I, Theorem 14.5]. Therefore H_1 is isomorphic to $\mathbb{Z}^n$. Next take $r_2 \in \mathbb{N}$ such that $r_2 \geq r_1$ and the rank of $\langle y_1, y_2, x_1, \ldots, x_{r_2} \rangle$ is equal to $m_1(n-1)+1$. Set the subgroup H_2 of G to be $\langle k_1^{-1}y_1, P(y_1, y_2, x_1, \ldots, x_{r_2}) \rangle$. Then H_2 contains H_1, the rank of H_2 is $m_1(n-1)+1$, and H_2 is finitely generated. Hence H_2 is isomorphic to $\mathbb{Z}^{m_1(n-1)+1}$. We will determine the product l_1 of all elementary divisors of the inclusion map $\iota_1 \colon H_1 \to H_2$. This is equal to the order of $(H_2/H_1)^{\mathrm{tor}}$. By definition of H_1 and H_2, the group $(H_2/H_1)^{\mathrm{tor}}$ is generated by the image of $k_1^{-1}y_1$. Since $\langle k_1^{-1}y_1 \rangle \cap H_1 = \langle k_1^{-1}y_1 \rangle \cap \mathbb{Z}^{\oplus\infty}$, the group $(H_2/H_1)^{\mathrm{tor}}$ is isomorphic to $\langle \pi(k_1^{-1}y_1) \rangle$. Hence we have $l_1 = m_1$. Next take $r_3 \in \mathbb{N}$ such that the rank of the group $\langle y_1, y_2, y_3, x_1, \ldots, x_{r_3} \rangle$ is equal to $m_2(n-1)+1$ and set $H_3 := \langle k_1^{-1}y_1, k_2^{-1}y_2, P(y_1, y_2, y_3, x_1, \ldots, x_{r_3}) \rangle$. Note that since $m_2(n-1) - m_1(n-1) = m_1(l_1 - 1)(n-1) > 1$, we must have $r_3 > r_2$. It is clear from the definition that H_3 contains H_2. By a similar reason to above, the group H_3 is isomorphic to $\mathbb{Z}^{m_2(n-1)+1}$. We determine the product l_2 of all elementary divisors of the inclusion map $\iota_2 \colon H_2 \to H_3$. By the same reason as above, it is equal to the order of $(H_3/H_2)^{\mathrm{tor}}$. It is clear that $(H_3/H_2)^{\mathrm{tor}}$ is generated by the image of $k_2^{-1}y_2$. This, together with the equality $\langle k_2^{-1}y_2 \rangle \cap H_2 = \langle k_2^{-1}y_2 \rangle \cap \langle k_1^{-1}y_1, \mathbb{Z}^{\oplus\infty} \rangle$, shows that the group $(H_3/H_2)^{\mathrm{tor}}$ is isomorphic to $\langle \pi(k_1^{-1}y_1), \pi(k_2^{-1}y_2) \rangle / \langle \pi(k_1^{-1}y_1) \rangle$. This shows the equation $l_2 = m_2/m_1$. Continuing this process inductively, we obtain an increasing sequence $(H_j)_j$ of subgroups of G which has the following properties.

- The union of H_j's is equal to G.
- The group H_j is isomorphic to $\mathbb{Z}^{m_{j-1}(n-1)+1}$ for each $j \in \mathbb{N}$. Here we put $m_0 = 1$ for convenience.
- The product l_j of all elementary divisors of the inclusion map $\iota_j \colon H_j \to H_{j+1}$ is equal to m_j/m_{j-1} for each $j \in \mathbb{N}$.

By the last property, we have $\prod_j l_j = N_{G/\mathbb{Z}^{\oplus\infty}}$. Therefore the inductive system $(H_m, \iota_m)_m$ satisfies the desired properties. □

Now we prove the main theorem.

THEOREM 3.5. *Let G be a subgroup of $\mathbb{Q}^{\oplus\infty}$ which contains $\mathbb{Z}^{\oplus\infty}$ as a subgroup of infinite index. Set $\tilde{G} := G/\mathbb{Z}^{\oplus\infty}$. Let $2 \leq n < \infty$ and k be an integer. Then there is an amenable minimal Cantor $\mathbb{F}_n$-system with the following properties.*

- *The pair of K_0-group and the unit $[1]_0$ is isomorphic to*
$$\left(G \oplus \Lambda((n-1)N_{\tilde{G}}), 0 \oplus [k(n-1)^{-1}]\right).$$
- *The K_1-group is isomorphic to $\mathbb{Z}^{\oplus\infty}$.*
- *The crossed product is a Kirchberg algebra in the UCT class.*

PROOF. First we consider the case $k = 1$. Let G be a group as stated. Take an inductive system $(\mathbb{Z}^{k_m}, A_m)_m$ as in Lemma 3.4 for the case of n and G. Fix an enumeration of S. By Lemma 3.3, there is an enumerated subset W_1 of $\mathbb{F}_n$ such that the equation $\sharp W_1 = (n-1)l_1+1 = k_2$ holds, the subset W_1 is a free basis of a finite index subgroup Γ_1 of $\mathbb{F}_n$, and the K_0-map $h_1 := (j_1)_{*,0}$ of the canonical inclusion
$$j_1 \colon C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n \to C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_1)) \rtimes_r \mathbb{F}_n$$
satisfies the following conditions.

- With respect to the bases $([\chi_{\Omega(s)}]_0 : s \in S)$ and $([\chi_{\Theta(w_1;W_1;\mathbb{F}_n)} \otimes \delta_e]_0 : w_1 \in W_1)$, the free part of h_1 is presented by A_1.
- The image of $\bigoplus_{s\in S} \mathbb{Z}[\chi_{\Omega(s)}]_0$ under h_1 is contained in
$$\left(\bigoplus_{w_1\in W_1} \mathbb{Z}[\chi_{\Theta(w_1;W_1;\mathbb{F}_n)}]_0\right) \oplus \Lambda_1$$
where Λ_1 is the subgroup of the torsion part generated by elements of order 2.
- The torsion part of h_1 is injective.

From Lemma 3.3 and the proof of Proposition 2.1, we can further take an enumerated subset W_2 of Γ_1 such that the equation $\sharp W_2 = (\sharp W_1 - 1)l_2 + 1 = k_3$ holds, the subset W_2 is a free basis of a finite index subgroup Γ_2 of Γ_1, and the K_0-map $h_2 := (j_2)_{*,0}$ of the canonical inclusion
$$j_2 \colon C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_1)) \rtimes_r \mathbb{F}_n \to C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_2)) \rtimes_r \mathbb{F}_n$$
satisfies the following conditions.

- With respect to the bases $([\chi_{\Theta(w_1;W_1;\mathbb{F}_n)} \otimes \delta_e]_0 : w_1 \in W_1)$ and $([\chi_{\Theta(w_2;W_2;\mathbb{F}_n)} \otimes \delta_e]_0 : w_2 \in W_2)$, the free part of h_2 is presented by A_2.
- The image of $\bigoplus_{w_1\in W_1} \mathbb{Z}[\chi_{\Theta(w_1;W_1;\mathbb{F}_n)} \otimes \delta_e]_0$ under h_2 is contained in
$$\left(\bigoplus_{w_2\in W_2} \mathbb{Z}[\chi_{\Theta(w_2;W_2;\mathbb{F}_n)} \otimes \delta_e]_0\right) \oplus \Lambda_2,$$
where Λ_2 is the subgroup of the torsion part generated by elements of order 2.
- The torsion part of h_2 is injective.

Note that we have $h_1(\Lambda_1) \subset \Lambda_2$ and each Λ_i is either trivial or isomorphic to $\mathbb{Z}_2$.

Continuing this process inductively, we finally obtain a sequence $(W_m)_m$ of enumerated subsets of $\mathbb{F}_n$ such that each W_m is a free basis of a finite index subgroup of $\Gamma_{m-1} := \langle W_{m-1}\rangle$ (here we set $W_0 := S$ for convenience), the equation $\sharp W_m = k_{m+1}$ holds, and for each m, the K_0-map $h_m := (j_m)_{*,0}$ of the canonical inclusion
$$j_m \colon C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_{m-1})) \rtimes_r \mathbb{F}_n \to C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_m)) \rtimes_r \mathbb{F}_n$$
satisfies the analogues of the above three conditions.

For each m we denote by α_m the canonical action of $\mathbb{F}_n$ on $Y_m := \mathbb{F}_n/\Gamma_m$. We denote by $\alpha\colon \mathbb{F}_n \curvearrowright Y$ the projective limit of the projective system $(\alpha_m\colon \mathbb{F}_n \curvearrowright Y_m)_m$ and set

$$X_m := \partial\mathbb{F}_n \times Y_m, X := \partial\mathbb{F}_n \times Y, \gamma_m := \beta \times \alpha_m, \gamma := \beta \times \alpha.$$

Note that by definition γ is the projective limit of $(\gamma_m)_m$. Then since both amenability and minimality pass to projective limits, the γ is an amenable minimal Cantor system. Since the UCT class and the class of all Kirchberg algebras are closed under inductive limits, the crossed product of γ is a Kirchberg algebra in the UCT class.

We next show that the Cantor $\mathbb{F}_n$-system γ has the desired K-theory. First we determine the pair $(K_0(C(X) \rtimes_r \mathbb{F}_n), [1]_0)$. From our construction, we have

$$K_0(C(X) \rtimes_r \mathbb{F}_n) \cong \varinjlim(\mathbb{Z}^{k_m} \oplus \mathbb{Z}_{k_m - 1}, h_m).$$

This shows that the group $K_0(C(X) \rtimes_r \mathbb{F}_n)/K_0(C(X) \rtimes_r \mathbb{F}_n)^{\mathrm{tor}}$ is isomorphic to G and that the group $K_0(C(X) \rtimes_r \mathbb{F}_n)^{\mathrm{tor}}$ is isomorphic to $\Lambda((n-1)N_{\tilde{G}})$. (Note that $k_{m+1} - 1 = (n-1)l_1 \cdots l_m$ for $m \in \mathbb{N}$.) We show that the torsion subgroup $K_0(C(X) \rtimes_r \mathbb{F}_n)^{\mathrm{tor}}$ is a direct summand of $K_0(C(X) \rtimes_r \mathbb{F}_n)$. This proves that the group $K_0(C(X) \rtimes_r \mathbb{F}_n)$ is isomorphic to the expected group. To see this, consider the subgroup H of $K_0(C(X) \rtimes_r \mathbb{F}_n)$ generated by elements of order 2 (at most one such element exists) and the images of $[\chi_{\Theta(w_m;W_m;\mathbb{F}_n)} \otimes \delta_e]_0$ for all $m \in \mathbb{N}$ and $w_m \in W_m$. Then the torsion part of H is either trivial or of order 2. In both cases, Szele's Theorem [**18**, Vol. I. Prop. 27.1] shows that H^{tor} is a direct summand of H. This shows that the torsion free quotient $K_0(C(X) \rtimes_r \mathbb{F}_n)/K_0(C(X) \rtimes_r \mathbb{F}_n)^{\mathrm{tor}}$ is lifted to $H \subset K_0(C(X) \rtimes_r \mathbb{F}_n)$ by homomorphism, as desired. Furthermore, by the construction, the above isomorphism maps the unit $[1]_0$ to $0 \oplus [(n-1)^{-1}]$. This shows that γ has the desired K_0-group.

Next we determine the K_1-group. By the Pimsner–Voiculescu exact sequence for free groups [**40**], for any Cantor $\mathbb{F}_n$-system $\tau\colon \mathbb{F}_n \curvearrowright Z$, we have $K_1(C(Z) \rtimes_r \mathbb{F}_n) \cong \ker(\eta_\tau)$, where $\eta_\tau\colon C(Z,\mathbb{Z})^{\oplus S} \to C(Z,\mathbb{Z})$ is the group homomorphism given by

$$(f_s)_{s\in S} \in C(Z,\mathbb{Z})^{\oplus S} \mapsto \sum_{s\in S}(f_s - f_s \circ \tau(s^{-1})) \in C(Z,\mathbb{Z}).$$

From the above isomorphism and the functoriality of the Pimsner–Voiculescu exact sequence, the canonical map $K_1(C(X_m) \rtimes_r \mathbb{F}_n) \to K_1(C(X) \rtimes_r \mathbb{F}_n)$ is injective for each m (since η_{γ_m} is identified with the restriction of η_γ). The isomorphism

$$C(X_m) \rtimes_r \mathbb{F}_n \cong \mathbb{M}_{L_m}(C(\partial\mathbb{F}_{k_m}) \rtimes_r \mathbb{F}_{k_m}),$$

shows that the rank of $K_1(C(X_m) \rtimes_r \mathbb{F}_n)$ is k_m. Here $L_m := l_1 \cdots l_m$. This shows that the rank of $K_1(C(X) \rtimes_r \mathbb{F}_n)$ must be infinite. Since the group $K_1(C(X) \rtimes_r \mathbb{F}_n)$ is a subgroup of the free abelian group $C(X,\mathbb{Z})^{\oplus S}$, it is free abelian [**18**, Vol.I, Theorem 14.5]. This shows that the K_1-group is isomorphic to $\mathbb{Z}^{\oplus\infty}$.

To end the proof for general case, we need the skyscraper construction. For G as above, let $\gamma_G\colon \mathbb{F}_n \curvearrowright X$ be the Cantor $\mathbb{F}_n$-system constructed above for the case of G. Then for each natural number k, we define the new Cantor $\mathbb{F}_n$-system $\widetilde{\gamma_G}^{(k)}$ as follows. Set $\widetilde{X}^{(k)} := X \times \{1,\ldots,k\}$ and fix $s \in S$. Then define a dynamical system $\widetilde{\gamma_G}^{(k)}$ of $\mathbb{F}_n$ on $\widetilde{X}^{(k)}$ by

$$\widetilde{\gamma_G}^{(k)}(s)(x,j) := \begin{cases} (x, j+1) & \text{if } j \neq k \\ (\gamma_G(s)(x), 1) & \text{if } j = k \end{cases}$$

and $\widetilde{\gamma_G}^{(k)}(t)(x,j) := (\gamma_G(t)(x), j)$ for the other $t \in S$. Then by definition, $\widetilde{\gamma_G}^{(k)}$ is an amenable minimal Cantor system and its crossed product is isomorphic to the tensor product of $\mathbb{M}_k$ and the crossed product of γ_G. □

Similar results also hold for virtually free groups.

THEOREM 3.6. *Let Γ be a group. Supposes we have a subgroup Λ of index k isomorphic to $\mathbb{F}_n$. Then for each triplet (G_0, u, G_1) given in Theorem 3.5 for the case n, there is an amenable minimal topologically free Cantor Γ-system with the following properties.*

- *The K-theory $(K_0, [1]_0, K_1)$ is isomorphic to (G_0, ku, G_1).*
- *The crossed product is a Kirchberg algebra in the UCT class.*

PROOF. The claim follows from the induced construction of dynamical systems with Green's imprimitivity theorem. For the convenience of the reader, we give the precise construction. Let Γ and Λ be as above. Take an amenable minimal Cantor Λ-system $\gamma\colon \Lambda \curvearrowright X$ such that its crossed product is a Kirchberg algebra in the UCT class and its K-theory $(K_0, [1]_0, K_1)$ is isomorphic to (G_0, u, G_1). On the space $\Gamma \times X$, define the equivalence relation $\sim_\Lambda$ by

$$(g, x) \sim_\Lambda (h, y) \Longleftrightarrow \exists k \in \Lambda, (h, y) = (gk^{-1}, \gamma(k)(x)).$$

Define the space $\Gamma \times_\Lambda X$ to be the quotient space of $\Gamma \times X$ by the equivalence relation $\sim_\Lambda$. It is easy to check that $\Gamma \times_\Lambda X$ is the Cantor set. Define an action $\widetilde{\gamma}$ of Γ on $\Gamma \times_\Lambda X$ by $\widetilde{\gamma}(g)([h, x]) := [gh, x]$. Here $[h, x]$ denotes the equivalence class of (h, x) under $\sim_\Lambda$. By the definition of $\widetilde{\gamma}$ (with the corresponding properties of γ), we can check easily that $\widetilde{\gamma}$ is minimal, amenable, and topologically free.

Define $\pi\colon \Gamma \times_\Lambda X \to \Gamma/\Lambda$ by $[h, x] \mapsto h\Lambda$. Then by the definition of $\sim_\Lambda$, π is a (well-defined) Γ-equivariant quotient map. Notice that $\pi^{-1}(e\Lambda)$ is Λ-equivariantly homeomorphic to X. Now applying Green's imprimitivity theorem to π, we get the isomorphism

$$C(\Gamma \times_\Lambda X) \rtimes_{\widetilde{\gamma}, r} \Gamma \cong \mathbb{M}_k(C(X) \rtimes_{\gamma, r} \Lambda).$$

(An isomorphism can be given by a similar way to that in Proposition 2.1.) This shows that the Cantor Γ-system $\widetilde{\gamma}$ has the desired properties. □

Combining Theorems 3.5 and 3.6, we obtain the following decomposition theorem.

COROLLARY 3.7. *For a torsion free abelian group G of infinite rank, consider a Kirchberg algebra A in the UCT class satisfying $(K_0(A), [1]_0, K_1(A)) \cong (G \oplus \mathbb{Q}/\mathbb{Z}, 0, \mathbb{Z}^{\oplus\infty})$. Then for any finitely generated non-amenable virtually free group Γ, A is decomposed as the crossed product of an amenable minimal topologically free Cantor Γ-system.*

PROOF. Let G be as stated. Thanks to Theorems 3.5 and 3.6, it suffices to show that there is an embedding $\iota\colon G \hookrightarrow \mathbb{Q}^{\oplus\infty}$ such that its image G' contains $\mathbb{Z}^{\oplus\infty}$ and satisfies $N_{G'/\mathbb{Z}^{\oplus\infty}} = \prod_{p \in \mathcal{P}} p^\infty$. To see this, take a maximal linear independent sequence $(x_n)_n$ of G. Then the mapping $x_n \mapsto n^{-1}e_n \in \mathbb{Q}^{\oplus\infty}, n \in \mathbb{N}$ extends to the desired inclusion. Here $(e_n)_n$ denotes the canonical basis of $\mathbb{Z}^{\oplus\infty}$. □

We say a Cantor system is profinite if it is of the form $\varprojlim(\Gamma \curvearrowright \Gamma/\Gamma_m)_m$ for some (strictly) decreasing sequence $(\Gamma_m)_m$ of finite index subgroups of Γ. When each Γ_m is normal in Γ, the corresponding profinite Cantor system is free if and only if the intersection $\bigcap_m \Gamma_m$ only consists of the unit element. Our constructions in Theorems 3.5 and 3.6 also provide continuously many amenable minimal free Cantor systems for every virtually free group. We also show the same result for non-finitely generated virtually free groups, by using a restriction of the boundary action. Consequently, we obtain the following result.

THEOREM 3.8. *Let Γ be a non-amenable virtually free group. Then there are continuously many amenable minimal free Cantor Γ-systems whose crossed products are mutually non-isomorphic Kirchberg algebras in the UCT class.*

PROOF OF THEOREM 3.8:FINITELY GENERATED CASE. We first show the assertion for the case $\Gamma = \mathbb{F}_n$. For each nonempty subset Q of $\mathcal{P}$, take a decreasing sequence of finite index subgroups of $\mathbb{F}_n$ as follows. First take an increasing sequence $(F_m)_m$ of finite subsets of Q whose union is Q and set $q_m := \prod_{p \in F_m} p$ for each m. Let $\pi_1 \colon \mathbb{F}_n \to G_1$ be the quotient homomorphism where G_1 is the quotient $\mathbb{F}_n^{\mathrm{ab}}/q_1\mathbb{F}_n^{\mathrm{ab}}$ of $\mathbb{F}_n$. Then $\Gamma_1 := \ker \pi_1$ is a proper characteristic subgroup of $\mathbb{F}_n$ whose index is a power of q_1. Next consider the quotient homomorphism $\pi_2 \colon \Gamma_1 \to G_2$ where G_2 is the quotient $\Gamma_1^{\mathrm{ab}}/q_2\Gamma_1^{\mathrm{ab}}$ of Γ_1. Then $\Gamma_2 := \ker \pi_2$ is a proper characteristic subgroup of Γ_1 whose index is a power of q_2. Continuing this process inductively, we get a decreasing sequence $(\Gamma_m)_m$ of subgroups of $\mathbb{F}_n$ that satisfies the following conditions.

- Each Γ_m is a proper characteristic subgroup of Γ_{m-1}.
- Each index $[\Gamma_m : \Gamma_{m-1}]$ is a power of q_m.

From the first condition, Levi's theorem [**32**, Chap.1, Prop.3.3] implies that the intersection $\bigcap_m \Gamma_m$ only consists of the unit element. Denote by α_Q the profinite Cantor system defined by the sequence $(\Gamma_m)_m$. Then the proof of Theorem 3.5 shows that the Cantor system $\gamma_Q := \beta \times \alpha_Q$ is amenable and minimal, the crossed product is a Kirchberg algebra in the UCT class, and the torsion subgroup of the K_0-group is isomorphic to $\Lambda((n-1)\prod_{q \in Q} q^\infty)$. This completes the proof. The case of virtually free groups is derived from the case of free groups by the same method as that in the proof of Theorem 3.6. □

LEMMA 3.9. *The restriction of the boundary action to the commutator subgroup $D(\mathbb{F}_2)$ of $\mathbb{F}_2$ is amenable, minimal, and its crossed product satisfies the following properties.*

- *It is a Kirchberg algebra in the UCT class.*
- *The unit $[1]_0$ generates a subgroup isomorphic to $\mathbb{Z}$.*
- *For any $n \geq 2, [1]_0 \notin nK_0(A)$.*

The same statement also holds for any finite index subgroup of $D(\mathbb{F}_2)$.

PROOF. The amenability of $\beta|_{D(\mathbb{F}_2)}$ is clear. We observe that for any nontrivial element t of $\mathbb{F}_2$, there are two points t^∞ and $t^{-\infty}$ in $\partial\mathbb{F}_2$ with the following property. For any compact subset K of $\partial\mathbb{F}_2 \setminus \{t^{-\infty}\}$, the sequence $(t^m.x)_m$ converges to t^∞ uniformly on K. (To see this, note that for any $n \in \mathbb{N}$, the first n segments of the sequence $(t^m)_m$ is eventually constant. This defines an element, say t^∞, in $\partial\mathbb{F}_2$. We define $t^{-\infty}$ in an analogous way. Then it is not hard to check that these points satisfy the above condition. Note that this is a general result for hyperbolic groups. See Chapter 8 of [**19**] for details.) This property with the normality of $D(\mathbb{F}_2)$ in $\mathbb{F}_2$ shows the minimality of $\beta|_{D(\mathbb{F}_2)}$. Then thanks to the above property with minimality, we can apply Theorem 5 of [**30**] to conclude the pure infiniteness of the crossed product of $\beta|_{D(\mathbb{F}_2)}$.

Now we consider the reminded two conditions. For any $n \geq 3$, there is a finite index subgroup Λ_n of $\mathbb{F}_2$ which contains $D(\mathbb{F}_2)$ and is isomorphic to $\mathbb{F}_n$. For such Λ_n, the restriction $\beta_2|_{\Lambda_n}$ is isomorphic to the boundary action of $\mathbb{F}_n$. Hence for any $n \geq 2$, there is a unital embedding $C(\partial\mathbb{F}_2) \rtimes_r D(\mathbb{F}_2) \to C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n$. This shows that for any $n \geq 2$, there is a group homomorphism $K_0(C(\partial\mathbb{F}_2) \rtimes_r D(\mathbb{F}_2)) \to \mathbb{Z}^n \oplus \mathbb{Z}_{n-1}$ that maps the unit $[1]_0$ to the canonical generator of $\mathbb{Z}_{n-1}$. This shows the claim for $D(\mathbb{F}_2)$.

Now let a finite index subgroup Λ of $D(\mathbb{F}_2)$ be given. The conditions on K-groups follows from the above proof. To show the minimality of $\beta_2|_\Lambda$ and pure infiniteness of the crossed product, by the proof in the case $\Lambda = D(\mathbb{F}_2)$, it suffices to show that Λ contains a nontrivial normal subgroup of $\mathbb{F}_2$. To see this, consider the group action $D(\mathbb{F}_2) \curvearrowright \bigsqcup_{g \in \mathbb{F}_2} D(\mathbb{F}_2)/g\Lambda g^{-1}$ given by the left multiplication action on each component. Then the kernel of the action is a normal subgroup of $\mathbb{F}_2$ contained in Λ. Moreover, it must be nontrivial because $D(\mathbb{F}_2)$ contains a torsion free element. □

PROOF OF THEOREM 3.8: NON-FINITELY GENERATED CASE. By the induced dynamical system construction, it suffices to show the claim for $\Gamma = \mathbb{F}_\infty$. For any nonempty set Q of prime numbers, take a sequence $(q_n)_n$ in Q satisfying $\{q_n\}_n = Q$. Then take a decreasing sequence $(\Lambda_n)_n$ of finite index normal subgroups of $\mathbb{F}_\infty$ that satisfies $\Lambda_1 = \mathbb{F}_\infty$ and $[\Lambda_n : \Lambda_{n+1}] = q_n$ for $n \in \mathbb{N}$. We identify $\mathbb{F}_\infty$ with the commutator subgroup of $\mathbb{F}_2$ by a fixed isomorphism [**32**, Chap.1 Prop.3.12]. Take a decreasing sequence $(\Gamma_n)_n$ of finite index normal subgroups of $\mathbb{F}_2$ such that the index $[\Gamma_n : \Gamma_{n+1}]$ is a power of q_n and $\bigcap_n \Gamma_n = \{1\}$ holds. (Such a sequence can be constructed in the same way as the proof for the finitely generated case.) Now set $\Upsilon_n := \Gamma_n \cap \Lambda_n$. Then $(\Upsilon_n)_n$ is a decreasing sequence of finite index normal subgroups of $\mathbb{F}_\infty$ such that the index $[\mathbb{F}_\infty : \Upsilon_n]$ divides a power of $q_1 \cdots q_{n-1}$ and is divisible by $q_1 \cdots q_{n-1}$ for each n, and $\bigcap_n \Upsilon_n = \{1\}$ holds. Now consider the Cantor system $\gamma_Q := \varprojlim((\beta_2|_{\mathbb{F}_\infty}) \times \alpha_n \colon \mathbb{F}_\infty \curvearrowright \partial\mathbb{F}_2 \times (\mathbb{F}_\infty/\Upsilon_n))$. Then Lemma 3.9 shows that the Cantor system γ_Q is amenable, minimal, free, and its crossed product is a Kirchberg algebra in the UCT class. Furthermore, a similar argument to that in the proof of Theorem 3.5 shows the equality

$$\{p \in \mathcal{P} : [1]_0 \in pK_0(C(X) \rtimes_{\gamma_Q, r} \mathbb{F}_\infty)\} = Q.$$

This shows that the crossed products of γ_Q's are mutually non-isomorphic. $\square$

4. Classification of diagonal actions of boundary actions and products of odometer transformations

In this section, using the technique of computation of K-groups developed in Section 3, we classify the amenable minimal Cantor $\mathbb{F}_n$-systems given by the diagonal actions of the boundary actions and the products of the odometer transformations.

First we recall the definition of the odometer transformation. For an infinite supernatural number N, take a sequence $(k_m)_m$ of natural numbers whose least common multiple is equal to N with the condition $k_m | k_{m+1}$ for all m. The odometer transformation of type N is then defined as the projective limit of the projective system $(\mathbb{Z} \curvearrowright \mathbb{Z}_{k_m})_m$. We denote it by α_N. (Obviously, the definition of α_N only depends on N.)

Let $2 \leq n < \infty$, let $1 \leq k \leq n$, and let $N_1, \ldots, N_k$ be a sequence of infinite supernatural numbers. Fix an enumeration $\{s_1, \ldots, s_n\}$ of $S (\subset \mathbb{F}_n)$. Then define a Cantor $\mathbb{F}_n$-system by

$$\gamma^{(n)}_{N_1, \ldots, N_k} := \beta_n \times \left(\prod_{j=1}^{k} \alpha_{N_j} \circ \pi^{(n)}_j \right),$$

where for each j, $\pi^{(n)}_j$ denotes the homomorphism $\pi^{(n)}_j \colon \mathbb{F}_n \to \mathbb{Z}$ given by

$$s_i \mapsto \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By the result of the previous section, each $\gamma^{(n)}_{N_1, \ldots, N_k}$ is an amenable minimal Cantor $\mathbb{F}_n$-system and similar computations to those in Lemmas 3.1 and 3.3 and Theorem 3.5 show the following theorem.

THEOREM 4.1. *Let $\gamma^{(n)}_{N_1, \ldots, N_k}$ be as above. Then the crossed product of $\gamma^{(n)}_{N_1, \ldots, N_k}$ satisfies the following conditions.*

- *The pair of K_0-group and the unit $[1]_0$ is isomorphic to*

$$\left(\left(\bigoplus_{i=1}^{k} \Upsilon(N_i) \right) \oplus \mathbb{Z}^{\oplus \infty} \oplus \Lambda((n-1)N_1 \cdots N_k), 0 \oplus 0 \oplus [(n-1)^{-1}] \right).$$

- *The K_1-group is isomorphic to $\mathbb{Z}^{\oplus\infty}$.*
- *It is a Kirchberg algebra in the UCT class.*

PROOF. For each $j = 1, \ldots, k$, take a sequence $(n(m,j))_m$ of natural numbers that satisfies the equation $\prod_m n(m,j) = N_j$. We further assume that for each m, only one j, say j_m, satisfies $n(m,j) \neq 1$. Put $N(m,j) := n(1,j) \cdots n(m,j)$ and $M(m) := N(m-1, j_m)$. Then for each m, consider the surjective homomorphism $q_m \colon \mathbb{F}_n \to \bigoplus_{j=1}^{k} \mathbb{Z}/N(m,j)$ defined by mapping s_j to the canonical generator of the jth direct summand for $j = 1, \ldots, k$ and the other $s \in S$ to 0. Set $\Gamma_m := \ker(q_m)$. Then the sequence $(\Gamma_m)_m$ defines a profinite Cantor system α. By definition we have $\gamma^{(n)}_{N_1,\ldots,N_k} = \beta \times \alpha$.

Next we inductively choose suitable free bases of Γ_m's as follows. First set $W_0 := S$ and $N(0,j) = 1$ for convenience. Then define W_m by

$$W_m := \left(W_{m-1} \setminus \{s_{j_m}^{M(m)}\}\right) \cup \{s_{j_m}^{N(m,j_m)}\} \cup Z_m,$$

where

$$Z_m := \left\{ w^{-1} s_{j_m}^{lM(m)} w s_{j_m}^{-lM(m)} : w \in W_{m-1} \setminus \{s_{j_m}^{M(m)}\}, 1 \leq l < n(j_m, m) \right\}.$$

It is easy to check that for each m, the set W_m is a free basis of Γ_m.

Combining the computations used in the proofs of Lemmas 3.1 and 3.3, we can show that the free part of the K_0-map induced from the canonical inclusion

$$C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_{m-1})) \rtimes_r \mathbb{F}_n \to C(\partial\mathbb{F}_n \times (\mathbb{F}_n/\Gamma_m)) \rtimes_r \mathbb{F}_n$$

is given by

$$[q_{(s_{j_m}^{N(m-1,j_m)}; W_{m-1})}]_0 \mapsto n(m, j_m)[q_{(s_{j_m}^{N(m,j_m)}; W_m)}]_0$$

and

$$[q_{(t, W_{m-1})}]_0 \mapsto [q_{(t; W_m)}]_0 \text{ for } t \in W_{m-1} \setminus \{s_{j_m}^{N(m-1,j_m)}\}.$$

Now the proof of Theorem 3.5 completes the computation. □

The invariants appearing in Theorem 4.1 are completely classified in terms of $(n; N_1, \ldots, N_k)$ as follows. A supernatural number N is recovered from the group $\Lambda(N)$ as the least common multiple of the orders of finite subgroups of $\Lambda(N)$. On the other hand, from the group $G = \left(\bigoplus_{i=1}^{k} \Upsilon(N_i)\right) \oplus \mathbb{Z}^{\oplus\infty} (\cong K_0/K_0^{\mathrm{tor}})$, we can recover the subgroup $\bigoplus_{i=1}^{k} \Upsilon(N_i)$ as the subgroup generated by the subset of all elements x such that the set $\{n \in \mathbb{N} : \text{there exists } y \in G \text{ with } ny = x\}$ is infinite. Then it is known that the two groups $\bigoplus_{i=1}^{k} \Upsilon(N_i)$ and $\bigoplus_{i=1}^{m} \Upsilon(M_i)$ are isomorphic if and only if $k = m$ and there are a permutation $\sigma \in \mathfrak{S}_k$ and natural numbers $n_1, \ldots, n_k$, $m_1, \ldots, m_k$ such that $n_i N_i = m_i M_{\sigma(i)}$ holds for all i. This follows from Baer's theorem [**18**, Vol.II, Prop.86.1] and the isomorphism condition of groups $\Upsilon(M)$.

On the set of all finite sequences of infinite supernatural numbers, we define the equivalence relation $\sim$ as follows. For two finite sequences $(N_1, \ldots, N_k)$ and $(M_1, \ldots, M_l)$, we say the relation $\sim$ holds if $k = l$ and there are a permutation $\sigma \in \mathfrak{S}_k$ and natural numbers $n_1, \ldots, n_k$, $m_1, \ldots, m_k$ such that $\prod_{i=1}^{k} n_i = \prod_{i=1}^{k} m_i$ and $n_i N_i = m_i M_{\sigma(i)}$ hold for all i. Denote by $[N_1, \ldots, N_k]$ the equivalence class of $(N_1, \ldots, N_k)$ under $\sim$. From the above observations, the equivalence class $[N_1, \ldots, N_k]$ is a complete invariant of the group $\left(\bigoplus_{i=1}^{k} \Upsilon(N_i)\right) \oplus \mathbb{Z}^{\oplus\infty} \oplus \Lambda(N_1 \cdots N_k)$. Here we record it as a proposition.

PROPOSITION 4.2. *For a sequence $N_1, \dots, N_k$ of infinite supernatural numbers, define the group $G(N_1, \dots, N_k)$ by*

$$\left(\bigoplus_{i=1}^{k} \Upsilon(N_i)\right) \oplus \mathbb{Z}^{\oplus\infty} \oplus \Lambda(N_1 \cdots N_k).$$

Then two groups $G(N_1, \dots, N_k)$ and $G(M_1, \dots, M_l)$ are isomorphic if and only if $[N_1, \dots, N_k] = [M_1, \dots, M_m]$, where $[\cdot]$ denotes the equivalence class of the equivalence relation $\sim$ defined above. In particular, for two free groups $\mathbb{F}_n$, $\mathbb{F}_m$ and for two finite sequences of infinite supernatural numbers $N_1, \dots, N_k$, $M_1, \dots, M_l$ with $k \le n$ and $l \le m$, the pairs $(K_0, [1]_0)$ of the corresponding two γ are isomorphic if and only if $n = m$ and $[N_1, \dots, N_k] = [M_1, \dots, M_l]$ hold.

We next introduce a notion of strong orbit equivalence of Cantor systems for general groups.

DEFINITION 4.3. We define the relation R on the class of Cantor systems as follows. For two Cantor systems $\alpha_i \colon \Gamma_i \curvearrowright X_i$, $i = 1, 2$, we declare the relation $R(\alpha_1, \alpha_2)$ holds if the following conditions hold. There is an orbit preserving homeomorphism $F \colon X_1 \to X_2$ and a generating set S_i of Γ_i for $i = 1, 2$ that admit an orbit cocycle c of F with the property that for each $s \in S_1$, the restriction of c on $\{s\} \times X_1$ has at most one point of discontinuity, and the same condition also holds when we replace X_1 by X_2, F by F^{-1}, and S_1 by S_2. Unfortunately, the relation R seems not to satisfy the transitivity. (This is in fact an equivalence relation if we only consider the minimal Cantor $\mathbb{Z}$-systems. This is already highly nontrivial; this is a consequence of a classification result in [**22**].) For this reason, we define the equivalence relation $\sim$ to be the one generated by R, and say α_1 is strongly orbit equivalent to α_2 if $\alpha_1 \sim \alpha_2$ holds.

From Proposition 4.2 and Matui's theorem [**33**] with a little extra effort, we can classify the strong orbit equivalence classes, the topological full groups, the crossed products, and the continuous orbit equivalence classes of $\gamma^{(n)}_{N_1,\dots,N_k}$'s.

Before completing the classification, we need a lemma about the strong orbit equivalence. This claims that for two Cantor systems of free groups, the isomorphism of K_0-invariants is a necessary condition for strong orbit equivalence. The idea of the proof comes from [**22**].

LEMMA 4.4. *Let $\gamma_i \colon \mathbb{F}_{n_i} \curvearrowright X_i$ be topologically free Cantor systems with $2 \le n_i < \infty$ for $i = 1, 2$. Assume that γ_1 is strongly orbit equivalent to γ_2. Then their K_0-invariants $(K_0(C(X_i) \rtimes_{\gamma_i, r} \mathbb{F}_{n_i}), [1]_0); i = 1, 2$ are isomorphic.*

PROOF. We may assume that the equalities $X_1 = X_2 = X$ hold and that the identity map is an orbit preserving homeomorphism that has orbit cocycles each of which has discontinuous points at most one on each element of some generating sets S_i of Γ_i. By the Pimsner–Voiculescu six term exact sequence for free groups [**40**], we obtain the isomorphism

$$K_0(C(X) \rtimes_{\gamma_i, r} \mathbb{F}_{n_i}) \cong C(X, \mathbb{Z})/N_i,$$

where N_i is the subgroup of $C(X, \mathbb{Z})$ generated by elements of the form $\chi_E - \chi_{\gamma_i(s)(E)}$ for clopen subsets E of X and $s \in \mathbb{F}_{n_i}$. Note that under the isomorphism, the unit $[1]_0$ is mapped to $1_X + N_i$.

From the above isomorphism, it suffices to show $N_1 = N_2$. To see this, let E be a clopen subset and $s \in S_1$. Replacing E be $X \setminus E$ if necessary, which does not change the difference $\chi_E - \chi_{\gamma_1(s)(E)}$ up to sign, we may assume that there is an orbit cocycle c that is continuous on $\{s\} \times E$. Define $c_s(x) := c(s, x)$ for $s \in \Gamma_1$ and $x \in X$. Set $\mathfrak{F} := c_s(E)$, which is finite by the continuity assumption. Then we have

$$\gamma_1(s)(E) = \bigsqcup_{g \in \mathfrak{F}} \gamma_2(g)(c_s^{-1}(\{g\}) \cap E).$$

This shows

$$\chi_E - \chi_{\gamma_1(s)(E)} = \sum_{g \in \mathfrak{F}} (\chi_{c_s^{-1}(\{g\}) \cap E} - \chi_{\gamma_2(g)(c_s^{-1}(\{g\}) \cap E)}) \in N_2.$$

Since S_1 generates Γ_1, we obtain $N_1 \subset N_2$. The reverse inclusion is shown in a similar way. □

Now we give the classification results for γ's.

THEOREM 4.5. *Let* $\gamma_1 = \gamma^{(n)}_{N_1,\ldots,N_k}$ *and* $\gamma_2 = \gamma^{(m)}_{M_1,\ldots,M_l}$ *be as before. Then the following conditions are equivalent.*

(1) *They are strongly orbit equivalent.*
(2) *They are continuously orbit equivalent.*
(3) *Their topological full groups are isomorphic.*
(4) *The commutator subgroups of their topological full groups are isomorphic.*
(5) *Their crossed products are isomorphic.*
(6) *Their* K_0*-invariants* $(K_0, [1]_0)$ *are isomorphic.*

PROOF. The implications (2) ⇒ (3) ⇒ (4), (2) ⇒ (1) and (2) ⇒ (5) ⇒ (6) are clear. The implication (4) ⇒ (2) follows from Theorem 3.10 of [**33**] and the implication (1) ⇒ (6) follows from Lemma 4.4. Now it is left to prove the implication (6) ⇒ (2).

Assume condition (6) holds. Then by Proposition 4.2, the equalities $n = m$ and $k = l$ hold and there are a permutation $\sigma \in \mathfrak{S}_k$ and two sequences $n_1, \ldots, n_k$ and $m_1, \ldots, m_k$ of natural numbers that satisfy $\prod_j n_j = \prod_j m_j$ and $n_j N_j = m_j M_{\sigma(j)}$ for all j. By conjugating an automorphism of the free group, we may assume σ is trivial. Since the continuous orbit equivalence is an equivalence relation, we further assume that there are a sequence of infinite supernatural numbers $L_1, \ldots, L_k$ and a natural number l satisfying $N_1 = lL_1$, $N_j = L_j$ for $j \neq 1$, $M_2 = lL_2$, and $M_j = L_j$ for $j \neq 2$. Denote by Λ_i the kernel of the surjection $\rho_i := q \circ \pi_i$ for $i = 1, 2$. Here q denotes the quotient homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_l$. By the definition of γ's, for $i = 1, 2$, we have an $\mathbb{F}_n$-equivariant quotient map $p_i \colon X_i \to \mathbb{F}_n/\Lambda_i$. Here X_i denotes the underlying space of γ_i for $i = 1, 2$. Then, with the notion $Y_i := p_i^{-1}(\Lambda_i)$, the homeomorphism F_i from $X_i = \bigsqcup_{j=0}^{l-1} s_i^j Y_i$ onto $Y_i \times \mathbb{Z}_l$ given by $x = s_i^j y \in s_i^j Y_i \mapsto (y, [j])$ shows that γ_i is continuously orbit equivalent to the Cantor system $\tilde{\gamma}_i \boxtimes \lambda \colon \Lambda_i \times \mathbb{Z}_l \curvearrowright Y_i \times \mathbb{Z}_l$. Here $\tilde{\gamma}_i$ denotes the restriction of $\gamma|_{\Lambda_i}$ to the Λ_i-invariant subspace Y_i of X_i, λ denotes the left translation action of $\mathbb{Z}_l$ on itself, and the symbol '⊠' stands for the product action. From this, it suffices to show that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are continuously orbit equivalent. Notice that for $i = 1, 2$, the set

$$T_i := \left\{ s_i^l, t, s_i^j t s_i^{-j} t^{-1} : t \in S \setminus \{s_i\}, 1 \leq j \leq l-1 \right\}$$

is a free basis of Λ_i. Set $r := \sharp T_1 = \sharp T_2 = l(n-1) + 1$. Then the isomorphism $\Lambda \cong \mathbb{F}_r$ given by the free basis T_i shows that the dynamical system $\tilde{\gamma}_i$ is conjugate to $\gamma^{(r)}_{L_1,\ldots,L_k}$. Thus $\tilde{\gamma}_1$ is continuously orbit equivalent to $\tilde{\gamma}_2$. □

REMARK 4.6. It is not hard to check that the transformation groupoid of $\gamma^{(n)}_{N_1,\ldots,N_k}$ is purely infinite in Matui's sense [**33**]. Hence, by Theorem 4.16 in [**33**], the commutator groups appeared in Theorem 4.5 are simple.

CHAPTER 3

Construction of minimal skew products of amenable minimal dynamical systems

It is an interesting question to ask that for a given group Γ which space admits a minimal (topologically) free dynamical system of Γ. Certainly a space admitting a minimal Γ-dynamical system must have a nice homogeneity. However, this is not sufficient even for the simplest case, that is, the case $\Gamma = \mathbb{Z}$. For example, an obstruction from homological algebra shows the non-existence of a minimal homeomorphism on even dimensional spheres S^{2n} (see Chapter I.6 of [**5**] for instance).

In [**20**], Glasner and Weiss have shown the existence of minimal skew product extensions of a minimal homeomorphism under mild conditions. Their result in particular shows that many spaces admit a minimal homeomorphism. For example, it follows that there is a minimal homeomorphism on the product of the Hilbert cube and S^1. This solved a question asked by Chapman [**8**]. For certain amenable groups, their result is generalized in [**34**]. In this chapter, following the argument of Glasner and Weiss in [**20**], we construct minimal skew products of amenable minimal topologically free dynamical systems (Theorem 1.1). This provides many new examples of (amenable) minimal topologically free dynamical systems of exact groups.

We also study the reduced crossed product of these minimal skew products. In Section 2, under certain assumptions on Y and $\alpha\colon \Gamma \curvearrowright Z$, we show that the crossed products of many of dynamical systems obtained in our result are Kirchberg algebras in the UCT class (Proposition 2.11). For this purpose, we generalize the notion of the finite filling property, which is introduced in [**25**]. It turns out that the generalized version is useful to construct minimal skew products with the purely infinite crossed products. This result is applied particularly to the case that Y is a connected closed topological manifold and that α is a Cantor system constructed in [**48**]. As a consequence, we generalize a result of Rørdam and Sierakowski [**48**], which is a result for the Cantor set, to the products of connected closed topological manifolds and the Cantor set (Theorem 2.12). This is the first generalization of their result, and shows that for topological dynamical systems, not only the structure of groups but also the structure of spaces is not an obstruction to form a Kirchberg algebra.

In Section 3, we study the K-theory of the crossed products of these minimal skew products in the free group case. Using the Pimsner–Voiculescu six-term exact sequence, we prove a Künneth-type formula for them. As an application, for any connected closed topological manifold M and for any (non-amenable, countable) virtually free group Γ, we show that there are continuously many amenable minimal free dynamical systems of Γ on the product of M and the Cantor set whose crossed products are mutually non-isomorphic Kirchberg algebras. This generalizes a result in Chapter 2.

Spaces of dynamical systems. For a compact metrizable space X, let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms on X. We equip the metric d on $\mathrm{Homeo}(X)$ as follows. First let us fix a metric d_X on X. Then define

$$d(\varphi, \psi) := \max_{x\in X}(d_X(\varphi(x), \psi(x))) + \max_{x\in X}(d_X(\varphi^{-1}(x), \psi^{-1}(x)))$$

for $\varphi, \psi \in \mathrm{Homeo}(X)$. It is not hard to check that the metric d is complete and it defines a topology that makes $\mathrm{Homeo}(X)$ a topological group. Note that the sequence $(\varphi_n)_n$ in $\mathrm{Homeo}(X)$ converges to φ in this topology if and only if φ_n uniformly converges to φ. For a countable group Γ, let $\mathcal{S}(\Gamma, X)$ denote the set of dynamical systems of Γ on X, i.e., $\mathcal{S}(\Gamma, X) = \mathrm{Hom}(\Gamma, \mathrm{Homeo}(X))$. This set is naturally regarded as a closed subset of $\prod_\Gamma \mathrm{Homeo}(X)$. Since Γ is countable, this makes $\mathcal{S}(\Gamma, X)$ to be a complete metric space.

Next let Y be a compact metrizable space and let $\mathcal{G} \curvearrowright Y$ be a continuous action of a topological group $\mathcal{G}$ on Y. Let $\alpha\colon \Gamma \curvearrowright Z$ be a topological dynamical system of a group Γ on a compact metrizable space Z. Put $X = Z \times Y$. Recall that a continuous map $c\colon \Gamma \times Z \to \mathcal{G}$ is said to be a cocycle if it satisfies the equation $c(s, t.z)c(t, z) = c(st, z)$ for all $s, t \in \Gamma$ and $z \in Z$. When there is a continuous map $h\colon Z \to \mathcal{G}$ satisfying $c(s, z) = h(s.z)^{-1}h(z)$ for all $s \in \Gamma$ and $z \in Z$, the cocycle c is said to be a coboundary. Each cocycle $c\colon \Gamma \times Z \to \mathcal{G}$ defines an extension of α on X by the following equation.

$$s.(z, y) = (s.z, c(s, z)y) \text{ for } s \in \Gamma \text{ and } (z, y) \in X.$$

Such extension is called a skew product extension. Note that when c is a coboundary, the associated skew product extension is conjugate to $\bar{\alpha}$. Here and throughout this chapter, for a dynamical system $\alpha\colon \Gamma \curvearrowright Z$ and a compact space Y, we denote by $\bar{\alpha}$ the diagonal action of α and the trivial action on Y. Since the space Y is always clear from the context, we omit Y in our notation.

For a continuous map h from Z into $\mathcal{G}$, we have an associated homeomorphism H on X defined by the formula $H(z, y) := (z, h_z(y))$ for $(z, y) \in X$. We denote by $\mathcal{G}_s$ the set of homeomorphisms given in the above way. Obviously, $\mathcal{G}_s$ is a subgroup of $\mathrm{Homeo}(X)$. For a topological dynamical system $\alpha\colon \Gamma \curvearrowright Z$, we define a subset $\mathcal{S}_{\mathcal{G}}(\alpha)$ of $\mathcal{S}(\Gamma, X)$ to be

$$\mathcal{S}_{\mathcal{G}}(\alpha) := \{H^{-1} \circ \bar{\alpha} \circ H : H \in \mathcal{G}_s\}.$$

We note that the set $\mathcal{S}_{\mathcal{G}}(\alpha)$ consists of skew product extensions of α by coboundaries. We denote by $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ the closure of $\mathcal{S}_{\mathcal{G}}(\alpha)$ in $\mathcal{S}(\Gamma, X)$. Note that any $\beta \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ is a skew product extension of α on X whose associated cocycle takes the value in $\overline{\mathcal{G}}$. Here $\overline{\mathcal{G}}$ denotes the closure of the image of $\mathcal{G}$ in $\mathrm{Homeo}(X)$. In particular, when α is amenable, every dynamical system contained in $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ is amenable. Throughout this chapter, we always fix metrics d_Y and d_Z on Y and Z respectively and consider the metric on $X = Z \times Y$ defined by $d_X((z_1, y_1), (z_2, y_2)) = d_Y(y_1, y_2) + d_Z(z_1, z_2)$, and use these metrics to define metrics on the homeomorphism groups.

1. Construction of minimal skew product

The goal of this section is to prove the following theorem. The proof is done by following the same line as that of Theorem 1 in [**20**].

In the proof of the following theorem, we use amenability of dynamical systems to construct suitable continuous functions. In other word, amenability of dynamical systems plays the role of the Følner sets in the proof of Theorem 1 of [**20**].

Theorem 1.1. *Let $\mathcal{G} \curvearrowright Y$ be a minimal action of a path connected group $\mathcal{G}$ on a compact metrizable space Y. Let $\alpha\colon \Gamma \curvearrowright Z$ be an amenable minimal topologically free dynamical system of a countable group Γ on a compact metrizable space Z. Then the set*

$$\{\beta \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha) : \beta \text{ is minimal}\}$$

is a G_δ-dense subset of $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$.

Proof. Let $\mathcal{G} \curvearrowright Y$ and $\alpha\colon \Gamma \curvearrowright Z$ be as in the statement. For an open set U of $X = Z \times Y$, we define the subset $\mathcal{E}_U$ of $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ to be

$$\mathcal{E}_U := \{\beta \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha) : \bigcup_{g\in\Gamma} \beta_g(U) = X\}.$$

Since X is compact, it is not hard to check that the set $\mathcal{E}_U$ is open in $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$.

Let $(U_n)_n$ be a countable basis of X. We observe that an element in $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ is minimal if and only if it is contained in $\bigcap_n \mathcal{E}_{U_n}$. Therefore, thanks to the Baire category theorem, our claim follows once we show the density of $\mathcal{E}_U$ in $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ for each non-empty open set U in X. To see this, it is enough to show the following claim. For any $H \in \mathcal{G}_s$ and any non-empty open set $U \subset X$, $H^{-1} \circ \bar{\alpha} \circ H \in \mathrm{cl}(\mathcal{E}_U)$. This is equivalent to the condition $\bar{\alpha} \in \mathrm{cl}(H\mathcal{E}_U H^{-1})$. A direct computation shows that $H\mathcal{E}_U H^{-1} = \mathcal{E}_{H(U)}$. Since $H(U)$ is again a non-empty open set, now it is enough to show the following statement. For any non-empty open set $U \subset X$, we have $\bar{\alpha} \in \mathrm{cl}(\mathcal{E}_U)$. Now let U be a non-empty set. Let S be a finite subset of Γ and let $\epsilon > 0$. Take non-empty open sets $V \subset Y$ and $W \subset Z$ with $W \times V \subset U$. By assumption, there are $\tilde{h}_0, \ldots, \tilde{h}_n \in \mathcal{G}$ satisfying $\bigcup_{0\leq i\leq n} \tilde{h}_i(V) = Y$. Since $\mathcal{G}$ is path-connected, there is a continuous map $h\colon [0,1] \to \mathcal{G}$ satisfying $h_{i/n} = \tilde{h}_i$ for $0 \leq i \leq n$. By the continuity of h, there is $\delta > 0$ such that the condition $|t_1 - t_2| < \delta$ implies $d(h_{t_1}^{-1} h_{t_2}, \mathrm{id}_Y) < \epsilon$. Now we use the amenability of α to choose a continuous map $\mu\colon Z \to \mathrm{Prob}(\Gamma)$ satisfying $\sup_{z\in Z} \|s.\mu^z - \mu^{s.z}\|_1 < \delta$ for all $s \in S$. By perturbing μ within a small error and replacing W by a smaller one, we may assume that there is a finite set $F \subset \Gamma$ such that $\mathrm{supp}(\mu^w) \subset F$ for all $w \in W$. (Cf. Lemma 4.3.8 of [**6**].) Since α is topologically free, by replacing W by a smaller one further, we may assume that the open sets $(g.W)_{g\in F^{-1}}$ are mutually disjoint. Since W is a locally compact metrizable space without isolated points, we can choose a compact subset K of W homeomorphic to the Cantor set.

Next take a continuous surjection $\theta_0\colon K \to [0,1]$. Extend θ_0 to a map $\bigsqcup_{g\in F^{-1}} gK \to [0,1]$ by the formula $\theta_0(g.z) := \theta_0(z)$ for $g \in F^{-1}$ and $z \in K$. Then take a continuous extension $\tilde{\theta}\colon Z \to [0,1]$ of θ_0. Using $\tilde{\theta}$ and μ, we define $\theta\colon Z \to [0,1]$ by

$$\theta(z) := \sum_{g\in\Gamma} \mu^z(g^{-1})\tilde{\theta}(g.z).$$

Note that the continuity of $\tilde{\theta}$ and μ implies that of θ. For $z \in K$, since $\mathrm{supp}(\mu^z) \subset F$, we have $\theta(z) = \theta_0(z)$. In particular, $\theta(K) = [0,1]$. Moreover, for $z \in Z$ and $s \in S$, we have

$$\begin{aligned}
|\theta(s.z) - \theta(z)| &= |\sum_{g\in\Gamma} (\mu^{s.z}(g^{-1})\tilde{\theta}(gs.z) - \mu^z(g^{-1})\tilde{\theta}(g.z))| \\
&= |\sum_{g\in\Gamma} (\mu^{s.z}(g^{-1})\tilde{\theta}(gs.z) - \mu^z(s^{-1}g^{-1})\tilde{\theta}(gs.z))| \\
&\leq \|\mu^{s.z} - s.\mu^z\|_1 \\
&< \delta.
\end{aligned}$$

Now define the map $g\colon Z \to \mathcal{G}$ by $g_z := h_{\theta(z)}$ for $z \in Z$. We will show that the corresponding homeomorphism $G \in \mathcal{G}_s$ satisfies the following conditions.

(1) $d(\bar{\alpha}_s, G^{-1} \circ \bar{\alpha}_s \circ G) < \epsilon$ for $s \in S$.
(2) $G^{-1} \circ \bar{\alpha} \circ G \in \mathcal{E}_U$.

Since U, ϵ, and S are arbitrarily, this ends the proof. Let $s \in S$ and $(z, y) \in X$. Then a direct computation shows that

$$(G^{-1} \circ \bar{\alpha}_s \circ G)(z, y) = (\alpha_s(z), g_{s.z}^{-1} g_z(y)).$$

Since $d(g_{s.z}^{-1}g_z, \mathrm{id}_Y) < \epsilon$ for all $z \in Z$, we obtain the first condition.

For the second condition, note that $G^{-1} \circ \bar{\alpha} \circ G \in \mathcal{E}_U$ if and only if $\bigcup_{g\in\Gamma} \bar{\alpha}_g(G(U)) = X$ holds. By the choice of G, for any $0 \leq i \leq n$, there is $w \in W$ satisfying $g_w = \tilde{h}_i$. It follows that for any $0 \leq i \leq n$, there is $w \in W$ with $\{w\} \times \tilde{h}_i(V) \subset G(U)$. Since $\bigcup_i \tilde{h}_i(V) = Y$, this shows that for any $y \in Y$, the intersection $(Z \times \{y\}) \cap G(U)$ is non-empty (which is open in $Z \times \{y\}$). This with the minimality of α shows that $\bigcup_{g\in\Gamma} \bar{\alpha}_g(G(U)) = X$. □

2. Pure infiniteness of crossed products of minimal skew products

In this section, we discuss pure infiniteness of reduced groupoid C*-algebras. Throughout this chapter, we always assume that étale groupoids are locally compact Hausdorff and their unit spaces are compact and infinite (as a set). For an étale groupoid G, we denote by r and s the range and source map unless they are specified.

2.1. Finite filling property for étale groupoids. To study the pure infiniteness of crossed products of dynamical systems arising from Theorem 1.1, we introduce a notion of the finite filling property for étale groupoids. First recall from [**25**] the finite filling property for dynamical systems. Although their definition and result also cover noncommutative C*-dynamical systems, in this thesis, we concentrate on the commutative case. We remark that, although the following formulation is slightly different from the original one, it is easily checked that they are equivalent.

Definition 2.1. A dynamical system $\Gamma \curvearrowright X$ is said to have the n-filling property if for any non-empty open set U of X, there are n elements $g_1, \ldots, g_n \in \Gamma$ with $\bigcup_{i=1}^n g_i(U) = X$. We say that a dynamical system has the finite filling property if it has the n-filling property for some $n \in \mathbb{N}$.

Note that the finite filling property implies minimality. In [**25**], it is shown that the finite filling property of a topological dynamical system implies the pure infiniteness of the reduced crossed product by a similar way to the one in [**30**]. However, as shown in [**25**], the n-filling property is inherited to factors. This makes the usage of the n-filling property restrictive in our application. To avoid this difficulty, we introduce a notion of the finite filling property for étale groupoids, which can be regarded as a localized version of [**25**]. This helps to construct minimal skew products with purely infinite reduced crossed products.

Next we recall a few terminologies of groupoids. A subset U of an étale groupoid G is said to be a G-set if both the range and source map are injective on U. For two G-sets U and V, we set $UV := \{uv \in G : u \in U, v \in V, s(u) = r(v)\}$. Obviously it is again a G-set. Furthermore, if both U and V are open, then UV is again open. An étale groupoid is said to be minimal if for any $x \in G^{(0)}$, the set $\{r(u) : u \in G, s(u) = x\}$ is dense in $G^{(0)}$. Note that the unit space $G^{(0)}$ has no isolated points whenever G is minimal. (Recall that $G^{(0)}$ is always assumed to be infinite.)

Definition 2.2. Let G be an étale groupoid. For a natural number n, we say that G has the n-filling property if every non-empty open set W of $G^{(0)}$ satisfies the following conditon. There are n open G-sets $U_1, \ldots, U_n$ satisfying

$$\bigcup_{i=1}^n r(U_i W) = G^{(0)}.$$

For short, we say that a dynamical system has the weak n-filling (resp. weak finite filling) property if its transformation groupoid has the n-filling (resp. finite filling) property.

Obviously, for dynamical systems, the n-filling (resp. finite filling) property implies the weak n-filling (resp. weak finite filling) property. However, the converses are not true.

We also remark that it is possible to define the weak finite filling property without going through the transformation groupoid. However, this specialization does not make the arguments below easier and this generality makes notation simpler. Considering applications elsewhere also, we study the property under this generality.

When the unit space $G^{(0)}$ has finite covering dimension, we have a useful criteria for the finite filling property. The following definition is inspired from [**33**] and [**48**].

Definition 2.3. We say that an étale groupoid G is purely infinite if for any non-empty open set U of $G^{(0)}$, there is a non-empty open subset V of U with the following condition. There are open G-sets U_1 and U_2 such that $r(U_i) \subset V \subset s(U_i)$ for $i = 1, 2$ and $r(U_1)$ and $r(U_2)$ are disjoint. We say that a dynamical system is purely infinite if its transformation groupoid is purely infinite.

We remark that Matui [**33**] has introduced pure infiniteness for totally disconnected étale groupoids for the study of the topological full groups. Clearly, our definition is weaker than Matui's one. We will see later that our definition of pure infiniteness coincides with Matui's one for minimal totally disconnected étale groupoids.

Proposition 2.4. *Let G be a minimal purely infinite étale groupoid and assume that* $\dim(G^{(0)}) = n < \infty$*. Then G has the $(n+1)$-filling property.*

Proof. Let U be a non-empty open subset of $G^{(0)}$. Replacing U by a smaller one, we may assume that there are open G-sets U_1 and U_2 such that $r(U_i) \subset U \subset s(U_i)$ for $i = 1, 2$ and $r(U_1)$ and $r(U_2)$ are disjoint. We first show that for any $N \in \mathbb{N}$, there are N open G-sets $V_1, \ldots, V_N$ satisfying $r(V_i) \subset U \subset s(V_i)$ for $i = 1, \ldots, N$ and the ranges $r(V_1), \ldots, r(V_N)$ are mutually disjoint. To see this, first take $M \in \mathbb{N}$ with $2^M \geq N$ and then take N mutually distinct elements from the set

$$\{U_{i_1} U_{i_2} \cdots U_{i_M} : i_k = 1 \text{ or } 2 \text{ for each } k\}.$$

Then it gives the desired sequence.

By the compactness of $G^{(0)}$ and the minimality of G, for some natural number N, there are N open G-sets $W_1, \ldots, W_N$ with $\bigcup_{i=1}^N r(W_i U) = G^{(0)}$. Take N open G-sets $V_1, \ldots, V_N$ as in the previous paragraph and put $Z_i := W_i V_i^{-1}$ for each i. Then we have

$$\bigcup_{i=1}^{N} r(Z_i U) \supset \bigcup_{i=1}^{N} r(W_i U) = G^{(0)}.$$

Note that since $s(Z_i) \subset r(V_i)$, the sources of Z_i's are mutually disjoint. Since $\dim(G^{(0)}) = n$, we can choose a refinement $(Y_j)_{j \in J}$ of $(r(Z_i U))_{i=1}^N$ with the decomposition $J = J_0 \sqcup J_1 \sqcup \cdots \sqcup J_n$ such that the members of the family $(Y_j)_{j \in J_k}$ are mutually disjoint for each k. Choose a map $\varphi\colon J \to \{1, \ldots, N\}$ satisfying $Y_j \subset r(Z_{\varphi(j)} U)$ for each $j \in J$. Set $X_k := \bigcup_{j \in J_k} Y_j Z_{\varphi(j)}$ for each k. Then it is not hard to check that each X_k is an open G-set and that $r(X_k U) = \bigcup_{j \in J_k} Y_j$. This shows $\bigcup_{k=0}^n r(X_k U) = G^{(0)}$. $\square$

Remark 2.5. The argument in Remark 4.12 of [**33**] shows that for totally disconnected étale groupoids, the finite filling property implies pure infiniteness in Matui's sense. Thus for a minimal totally disconnected étale groupoid G, pure infiniteness in Matui's sense [**33**], that in our sense, the finite filling property, and the 1-filling property are equivalent. (Here total disconnectedness is used to replace open G-sets by clopen ones.)

Next we see a few examples of dynamical systems with the weak finite filling property. The following three examples are particularly important for us. See [**25**] for more examples of dynamical systems with the finite filling property.

EXAMPLE 2.6. It follows from the proof of Theorem 6.11 of [**48**] that every countable non-amenable exact group admits an amenable minimal free purely infinite dynamical system on the Cantor set. (To see this, use the equivalence of conditions (i) and (iii) in Proposition 5.5 in the proof of Proposition 6.8.) By Proposition 2.4, it has the weak 1-filling property. We remark that these dynamical systems almost never have the finite filling property.

Recall that a manifold is said to be closed if it is compact and has no boundaries.

LEMMA 2.7. *Let M be a connected closed topological manifold. Let* $\mathrm{Homeo}(M)_0$ *denote the path connected component of* $\mathrm{Homeo}(M)$ *containing the identity. Then the action* $\mathrm{Homeo}(M)_0 \curvearrowright M$ *has the finite filling property.*

PROOF. It is not hard to show that the above action is transitive by using the connectedness of M with the fact that M is locally homeomorphic to $\mathbb{R}^n$.

Take an open cover $U_1, \ldots, U_N$ of M each of which is homeomorphic to $\mathbb{R}^n$. We show that for any non-empty open set V in M, for any i, and for any compact subset K of U_i, there is an element $g \in \mathrm{Homeo}(M)_0$ with $g(V) \supset K$. Since M is compact, the claim with a standard argument for compactness shows the N-filling property of the action in the question. Since the action is transitive, replacing V by $g(V)$ for a suitable $g \in \mathrm{Homeo}(M)_0$ and replacing it by a smaller one further, we may assume that V is contained in U_i. Take a homeomorphism $\varphi \colon U_i \to \mathbb{R}^n$ satisfying $0 \in \varphi(V)$. Take a sufficiently large positive number $\lambda > 0$ with $\varphi(K) \subset \lambda\varphi(V)$. Then choose a continuous function $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying the following conditions.

(1) For $t \leq \mathrm{diam}(\varphi(V))$, we have $f(t) = \lambda$.
(2) For all sufficiently large t, we have $f(t) = 1$.
(3) The function $t \mapsto tf(t)$ is strictly monotone increasing.

Now set $\varphi_f(x) := \varphi^{-1}(f(\|\varphi(x)\|)\varphi(x))$ for $x \in U_i$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. From the assumptions on f, the map φ_f is a homeomorphism on U_i satisfying $K \subset \varphi_f(V)$. We extend φ_f to a homeomorphism ψ_f on M as follows.

$$\psi_f(x) := \begin{cases} \varphi_f(x) & \text{if } x \in U_i, \\ x & \text{if } x \in M \setminus U_i. \end{cases}$$

It is clear from the properties of f that ψ_f is indeed a homeomorphism on M. Clearly we have $K \subset \psi_f(V)$. Moreover, the map $t \in [0,1] \mapsto \psi_{(1-t)f+tk}$ defines a continuous path in $\mathrm{Homeo}(M)$ from ψ_f to the identity. Here k denotes the constant function of value 1 defined on $\mathbb{R}_{\geq 0}$. Thus we have $\psi_f \in \mathrm{Homeo}(M)_0$. □

Next we see examples of finite filling actions of path-connected groups on infinite dimensional spaces. Let $Q := \prod_{\mathbb{N}}[0,1]$ be the Hilbert cube. Recall that a topological space is said to be a Hilbert cube manifold if there is an open cover each of the member is homeomorphic to an open subset of Q. It is not hard to show that open subsets of Q in the definition can be taken to be $[0,1) \times Q$. (See Theorem 12.1 of [**8**] for instance.) Obvious examples are Q itself and the product of Q and a topological manifold (possible with boundary). We refer the reader to [**8**] for more information of Hilbert cube manifolds.

LEMMA 2.8. *Let M be a connected compact Hilbert cube manifold. Then the action* $\mathrm{Homeo}(M)_0 \curvearrowright M$ *has the finite filling property.*

Proof. We first show the following claim. For any open subset U of $[0,1) \times [0,1]^n$ of the form $(a,b)^n \times [0,1]$ $(0 < a < b < 1)$ and for any compact subset K of $[0,1) \times [0,1]^n$, there is a homeomorphism $h \in \mathrm{Homeo}([0,1) \times [0,1]^n)_{c,0}$ satisfying $K \subset h(U)$. Here, for a locally compact metrizable space Y, $\mathrm{Homeo}(Y)_{c,0}$ denotes the subgroup of homeomorphisms on Y defined as follows. First we define $\mathrm{Homeo}(Y)_c$ to be the group of homeomorphisms on Y which coincide with the identity off a compact subset. Then we identify $\mathrm{Homeo}(Y)_c$ with the inductive limit of subgroups of homeomorphism groups of compact subsets of Y in the natural way. Then we topologize $\mathrm{Homeo}(Y)_c$ with the inductive topology. Now we define $\mathrm{Homeo}(Y)_{c,0}$ to be the path-connected component of $\mathrm{Homeo}(Y)_c$ containing the identity with respect to this topology. To show the claim, we first construct a homeomorphism $h_1 \in \mathrm{Homeo}([0,1) \times [0,1]^n)_{c,0}$ satisfying $h_1(\{0\} \times [0,1]^n) \subset (a,b)^n \times [0,1]$ in a similar way to the proof of Lemma 2.7. Then, since h_1 is a homeomorphism, there is a positive number $\delta > 0$ satisfying $h_1([0,\delta) \times [0,1]^n) \subset (a,b)^n \times [0,1]$. Also, it is easy to find $h_2 \in \mathrm{Homeo}([0,1) \times [0,1]^n)_{c,0}$ satisfying $K \subset h_2([0,\delta) \times [0,1]^n)$. Now the homeomorphism $h := h_2 \circ h_1^{-1}$ satisfies the required condition.

Next we observe that for any compact metrizable space X and its open subset U, any $h \in \mathrm{Homeo}(U)_{c,0}$ extends to a homeomorphism $\tilde{h}$ in $\mathrm{Homeo}(X)_0$ by defining $\tilde{h}(x) = x$ off U. Now thanks to the claim in the previous paragraph with this observation, the rest of the proof can be completed by a similar way to that of Lemma 2.7. □

We next show that the finite filling property gives a sufficient condition for the pure infiniteness of the reduced groupoid C*-algebra. Recall from [**33**] that an étale groupoid G is said to be essentially principal if the interior of the set $\{g \in G : r(g) = s(g)\}$ coincides with $G^{(0)}$. Note that for transformation groupoids, this condition is equivalent to the topological freeness of the original dynamical system.

Proposition 2.9. *Let G be an étale groupoid with the finite filling property. Assume further that G is essentially principal. Then the reduced groupoid* C*-*algebra* $\mathrm{C}^*_r(G)$ *is purely infinite and simple. In particular, if G is additionally assumed to be second countable and amenable, then* $\mathrm{C}^*_r(G)$ *is a Kirchberg algebra in the UCT class.*

To show the main statement, we need the following lemma, which is an analogue of Lemma 1.5 of [**25**].

Lemma 2.10. *Let G be an étale groupoid with the n-filling property. Let b be a positive element in $C(G^{(0)})$ with norm one. Then for any $\epsilon > 0$, there is $c \in \mathrm{C}^*_r(G)$ such that $\|c\| \leq \sqrt{n}$ and $c^*bc \geq 1 - \epsilon$.*

Proof. Set $U := \{x \in G^{(0)} : b(x) > 1 - \epsilon\}$. Take n mutually disjoint non-empty open subsets $U_1, \ldots, U_n$ of U. Since G is minimal, there are n open G-sets $V_1, \ldots, V_n$ with the property that the intersection $\bigcap_i r(V_i U_i)$ is non-empty. Using the n-filling property of G with this observation, we can find n open G-sets $W_1, \ldots, W_n$ satisfying

$$\bigcup_{i=1}^{n} r(W_i U_i) = G^{(0)}.$$

By replacing W_i by $W_i U_i$, we may assume $s(W_i) \subset U_i$. Since G is locally compact and $G^{(0)}$ is compact, replacing each W_i by a smaller one if necessary, we may assume further that each W_i is relatively compact in G. Since G is locally compact, for each i, it is not hard to find an increasing net $(W_{i,\lambda})_{\lambda \in \Lambda}$ of open subsets of W_i that satisfies the following conditions. The closure of $W_{i,\lambda}$ in G is contained in W_i for each λ, and the union $\bigcup_\lambda W_{i,\lambda}$ is equal to W_i. Since the unit space $G^{(0)}$ is compact, there is $\lambda \in \Lambda$ satisfying $\bigcup_{i=1}^n r(W_{i,\lambda}) = G^{(0)}$. Now fix such

λ and put $Z_i := \mathrm{cl}(W_{i,\lambda})$ for each i. Then, by the choice of $W_{i,\lambda}$, the Z_i is a compact G-set. Moreover we have

$$G^{(0)} = \bigcup_{i=1}^{n} r(W_{i,\lambda}) \subset \bigcup_{i=1}^{n} r(Z_i).$$

Now for each i, take a continuous function $f_i \in C_c(G)$ satisfying the following conditions.

(1) $0 \leq f_i \leq 1$.
(2) $\mathrm{supp}(f_i) \subset W_i$.
(3) $f_i \equiv 1$ on Z_i.

(Since Z_i and the closure of W_i in G are compact, such function exists.) Since W_i is a G-set, these conditions imply that $f_i * f_i^* \in C(G^{(0)})$ and that $f_i * f_i^* \leq 1$. Since the sets $s(W_1), \ldots, s(W_n)$ are mutually disjoint, we have $f_i * f_j^* = 0$ for two distinct i and j. Now put $c := \sum_{i=1}^{n} f_i^*$. The above observations show that $c^* * c \in C(G^{(0)})$ and that $c^* * c \leq n$. Thus $\|c\| \leq \sqrt{n}$. Since the G-sets $W_1, \ldots, W_n$ have mutually disjoint sources, we also get $c^* * b * c \in C(G^{(0)})$. Since $s(W_i) \subset U$ for each i and $\bigcup_{i=1}^{n} r(Z_i) = G^{(0)}$, we further obtain $c^* * b * c \geq 1 - \epsilon$. ☐

Proof of Proposition 2.9. The rest of the proof is basically the same as that in [**25**]. We first observe that since G is essentially principal, it is not hard to show that for any $b \in C_c(G)$ and $\epsilon > 0$, there is a positive element $y \in C(G^{(0)})$ with norm one satisfying $yby = yE(b)y$ and $\|yby\| > \|E(b)\| - \epsilon$, where E denotes the restriction map $C_c(G) \to C(G^{(0)})$. Note that the map E extends to a faithful conditional expectation on $\mathrm{C}^*_r(G)$. From this with Lemma 2.10, for any positive element $b \in C_c(G)$ with $\|E(b)\| = 1$, there is an element $c \in C_c(G^{(0)})$ satisfying $\|c\| \leq \sqrt{n}$ and $c^*yc \geq 1/2$. Since the norm of c is bounded by the fixed constant $\sqrt{n}$, now a standard argument completes the proof. ☐

2.2. Minimal skew products with purely infinite crossed products. Now using the finite and weak finite filling property, we construct minimal skew products whose crossed products are purely infinite.

Proposition 2.11. *Let $\alpha\colon \Gamma \curvearrowright Z$ be an amenable topologically free dynamical system with the weak n-filling property. Let $\mathcal{G} \curvearrowright Y$ be a minimal dynamical system of a path connected group $\mathcal{G}$ with the m-filling property. Then the set*

$$\left\{\beta \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha) : \beta \text{ has the weak } (nm)\text{-filling property}\right\}$$

is a G_δ-dense subset of $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$.

Proof. For an open set U of $X = Z \times Y$, let $\mathcal{F}_U$ denote the set of elements β of $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ satisfying the following condition. There are nm open G_β-sets $V_1, \ldots, V_{nm}$ with $\bigcup_i r(V_i U) = X$. Here G_β denotes the transformation groupoid $X \rtimes_\beta \Gamma$ of β. Then for a countable basis $(U_n)_n$ of X, the set in the question coincides with the intersection $\bigcap_n \mathcal{F}_{U_n}$. Hence it suffices to show that each $\mathcal{F}_U$ is open and dense in $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$.

We first show the openness of $\mathcal{F}_U$. Let $\beta \in \mathcal{F}_U$. Let $V_1, \ldots, V_{nm}$ be open G_β-sets as above. Replacing V_i's by smaller ones, we may assume that they are relatively compact in G_β and that the sources $s(V_i)$ are contained in U. Set $F := \pi(\bigcup_i V_i)$, where $\pi\colon X \rtimes_\beta \Gamma \to \Gamma$ denotes the projection onto the second coordinate. Since each V_i is relatively compact in G_β, the set F is a finite subset of Γ. Now we apply the argument in the proof of Lemma 2.10 to $(V_i)_i$ to choose compact G_β-sets $W_1, \ldots, W_{nm}$ with the following properties. The W_i is contained in V_i for each i and the union $\bigcup_i r(\mathrm{int}(W_i))$ is equal to X. Now for a G_β-set W and $g \in \Gamma$, define the subset $W_g \subset X$ to be $r(W \cap \pi^{-1}(\{g\}))$. Then, for each i, the sets $(W_{i,g})_{g \in F}$ are mutually disjoint compact sets in X. Moreover, the union $\bigcup_{i,g} \mathrm{int}(W_{i,g})$ is equal to X.

For $W \subset X$ and $\delta > 0$, we define the (open) subsets $\mathcal{N}_\delta(W)$ and $\mathcal{I}_\delta(W)$ of X as follows.

$$\mathcal{N}_\delta(W) := \bigcup_{x \in W} B(x, \delta),$$

$$\mathcal{I}_\delta(V) := \{x \in G : \text{there is } \eta > \delta \text{ with } B(x, \eta) \subset V\}.$$

Here for $x \in X$ and $\eta > 0$, $B(x, \eta)$ denotes the open ball of center x and radius η. Then, from the properties of W_i's and the compactness of X, for a sufficiently small positive number $\delta > 0$, the following conditions hold. The sets $(\mathcal{N}_\delta(W_{i,g}))_g$ are mutually disjoint for each i and the sets $(\mathcal{I}_\delta(W_{i,g}))_{i,g}$ cover X. We fix such positive number δ. From the first condition, for any $\gamma \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$ satisfying $d(\gamma_s, \beta_s) < \delta$ for all $s \in F$, each W_i is a G_γ-set. Here W_i is regarded as a subset of G_γ by identifying the transformation groupoids with the set $\Gamma \times X$ by ignoring the first coordinates. Let r_β and r_γ denote the range map of G_β and G_γ respectively. Then we have

$$\bigcup_i r_\gamma(\mathrm{int}(W_i)) \supset \bigcup_i \mathcal{I}_\delta(r_\beta(\mathrm{int}(W_i))) = \bigcup_{i,g} \mathcal{I}_\delta(W_{i,g}) = X.$$

Therefore we have $\gamma \in \mathcal{F}_U$, which proves the openness of $\mathcal{F}_U$.

To show the density of $\mathcal{F}_U$, by the similar reason to that in the proof of Theorem 1.1, it suffices to show the following statement. For any $\epsilon > 0$ and any finite subset $S \subset \Gamma$, there is a homeomorphism $H \in \mathcal{G}_s$ satisfying the following conditions.

(1) $d(\bar{\alpha}_s, H^{-1} \circ \bar{\alpha}_s \circ H) < \epsilon$ for $s \in S$.
(2) $H^{-1} \circ \bar{\alpha} \circ H \in \mathcal{F}_U$.

Replacing U by a smaller open set, we may assume $U = W \times V$ for some $W \subset Z$ and $V \subset Y$. By the m-filling property of $\mathcal{G} \curvearrowright Y$, we can choose m elements $\tilde{h}_1, \ldots, \tilde{h}_m$ of $\mathcal{G}$ with $\bigcup_i \tilde{h}_i(V) = Y$. Now proceeding the same argument as in the proof of Theorem 1.1, we get a continuous map $g \colon Z \to \mathcal{G}$ with the following conditions.

(1) $d(g_{s.z}^{-1} g_z, \mathrm{id}_Y) < \epsilon$ for all $z \in Z$ and $s \in S$.
(2) There are m elements $w_1, \ldots, w_m$ in W with the condition $\bigcup_i g_{w_i}(V) = Y$.

Let $H \in \mathcal{G}_s$ be the element corresponding to g. Then from the first condition, we conclude $d(\bar{\alpha}_s, H^{-1} \circ \bar{\alpha}_s \circ H) < \epsilon$ for $s \in S$. To show $\beta := H^{-1} \circ \bar{\alpha} \circ H \in \mathcal{F}_U$, it suffices to show the following claim. There are nm open $G_{\bar{\alpha}}$-sets $W_1, \ldots, W_{nm}$ with $\bigcup_i r(W_i H(U)) = X$. Indeed the sets

$$\{(H^{-1}(z), s, H^{-1}(w)) \in X \times \Gamma \times X : (z, s, w) \in W_i\} \ (i = 1, \ldots, nm)$$

then define the desired open G_β-sets. To show the claim, first note that since g is continuous, there are an open subset U_i of U containing w_i for $i = 1, \ldots, m$ and an open covering $(V_i)_{i=1}^m$ of Y satisfying the following condition. For any $z \in U_i$, we have $V_i \subset g_z(V)$. From these conditions, we have $H(U) \supset \bigcup_{i=1}^m (U_i \times V_i)$. Now for each $1 \le i \le m$, take n open G_α-sets $W_{i,1}, \ldots, W_{i,n}$ with $\bigcup_{j=1}^n r(W_{i,j} U_i) = Z$. For each $1 \le i \le m$ and $1 \le j \le n$, set $Z_{i,j} := \varphi^{-1}(W_{i,j})$, where $\varphi \colon G_{\bar{\alpha}} \to G_\alpha$ denotes the canonical quotient map. Then each $Z_{i,j}$ is an open $G_{\bar{\alpha}}$-set and we further get

$$\bigcup_{i,j} r_{\bar{\alpha}}(Z_{i,j} H(U)) \supset \bigcup_{i,j} r_{\bar{\alpha}}(Z_{i,j}(U_i \times V_i)) = \bigcup_{i,j} (r_\alpha(W_{i,j} U_i) \times V_i) = X.$$

□

In [**48**], Rørdam and Sierakowski have shown that every countable non-amenable exact group admits an amenable minimal free dynamical system on the Cantor set whose crossed product is a Kirchberg algebra in the UCT class. Proposition 2.11 particularly gives an extension of their result to more general spaces.

THEOREM 2.12. *Let M be a connected closed topological manifold, a connected compact Hilbert cube manifold, or a countable direct product of these manifolds. Let X be the Cantor set. Then every countable non-amenable exact group admits an amenable minimal free dynamical system on $M \times X$ whose crossed product is a Kirchberg algebra in the UCT class.*

PROOF. For the first two cases, the statement immediately follows from Example 2.6, Lemmas 2.7 and 2.8, and Propositions 2.9 and 2.11.

For the last case, let $M_1, M_2, \ldots$ be a sequence of spaces each of which is either connected closed topological manifold or connected compact Hilbert cube manifold. Set $N_n := M_1 \times \cdots \times M_n \times X$ for each n. We put $\alpha_0 := \alpha$ and $N_0 := X$ for convenience. We inductively apply Proposition 2.11 to $\alpha_n \colon \Gamma \curvearrowright N_n$ and M_{n+1} to get a minimal skew product extension $\alpha_{n+1} \colon \Gamma \curvearrowright N_{n+1}$ of α_n with the weak finite filling property. Then we get the projective system $(\alpha_n)_{n=1}^{\infty}$ of dynamical systems of Γ. Now it is not hard to show that the projective limit $\varprojlim \alpha_n$ possesses the desired properties. □

3. Minimal dynamical systems of free groups on products of Cantor set and closed manifolds

In this section, we investigate the K-groups of the crossed products of minimal dynamical systems obtained in Theorem 1.1 for the free group case. By using the Pimsner–Voiculescu exact sequence [**40**], we give a Künneth-type formula for K-groups of their crossed products. As an application, we give the following generalization of Theorem 3.8 in Chapter 2.

THEOREM 3.1. *Let Γ be a countable non-amenable virtually free group. Let M be either connected closed topological manifold or connected compact Hilbert cube manifold. Then there are continuously many amenable minimal free dynamical systems of Γ on the product of M and the Cantor set whose crossed products are mutually non-isomorphic Kirchberg algebras.*

In the below, we regard abelian groups as $\mathbb{Z}$-modules. We simply denote the tensor product '$\otimes_{\mathbb{Z}}$' by '$\otimes$' for short. Recall that for two abelian groups G, H, the group $\mathrm{Tor}_1^{\mathbb{Z}}(G, H)$ is defined as follows. First take a projective resolution of G.

$$\cdots \to P_2 \to P_1 \to P_0 \to G \to 0.$$

Then by tensoring H with the above resolution, we obtain a complex

$$\cdots \to P_2 \otimes H \to P_1 \otimes H \to P_0 \otimes H \to 0.$$

The group $\mathrm{Tor}_1^{\mathbb{Z}}(G, H)$ is then defined as the first homology of the above complex. Note that the definition does not depend on the choice of the projective resolution. We remark that when we have a projective resolution of length one

$$0 \to P_1 \to P_0 \to G \to 0,$$

then $\mathrm{Tor}_1^{\mathbb{Z}}(G, H)$ is computed as the kernel of the homomorphism $P_1 \otimes H \to P_0 \otimes H$. See [**5**] for the detail.

For a compact space X, we denote $K_i(C(X))$ by $K^i(X)$ for short. Note that this coincides with the usual definition of K^i-group.

PROPOSITION 3.2. *Let $\alpha \colon \mathbb{F}_d \curvearrowright X$ be an amenable minimal topologically free dynamical system of the free group $\mathbb{F}_d$ on the Cantor set X $(d \in \mathbb{N} \cup \{\infty\})$. Let $\mathcal{G} \curvearrowright Y$ be a minimal action of a path-connected group $\mathcal{G}$ on a compact metrizable space Y. Let $\beta \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$. Let A and B denote the crossed product of α and β respectively. Then for $i = 0, 1$, we have the following short exact sequence.*

$$0 \to K_0(A) \otimes K^i(Y) \to K_i(B) \to (K_1(A) \otimes K^{1-i}(Y)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K^{1-i}(Y)) \to 0.$$

Moreover, the first map maps $[1_A]_0 \otimes [1_Y]_0$ *to* $[1_B]_0$ *when* $i = 0$.

PROOF. Since $C(X)$ is an AF-algebra, we have a canonical isomorphism

$$K^i(X \times Y) \to C(X, K^i(Y)) (\cong K^0(X) \otimes K^i(Y))$$

for $i = 0, 1$. Here $C(X, K^i(Y))$ denotes the group of continuous maps from X into $\mathrm{K}^i(Y)$ and $\mathrm{K}^i(Y)$ is regarded as a discrete group. For $i = 0$, the isomorphism is given by mapping the element $[p]_0$ where p is a projection in $\mathbb{K} \otimes C(X) \otimes C(Y)$ to the map $x \in X \mapsto [p(x, \cdot)]_0 \in K^0(Y)$ and similarly for the case $i = 1$.

From this isomorphism and the fact that $\mathcal{G}$ is path-connected, for any $\gamma \in \mathcal{S}_{\mathcal{G}}(\alpha)$ and $g \in \mathbb{F}_d$, we have $(\gamma_g)_{*,i} = (\alpha_g)_{*,0} \otimes \mathrm{id}_{K^i(Y)}$ for $i = 0, 1$. Here we identify $K^i(X \times Y)$ with $K^0(X) \otimes K^i(Y)$ under the above isomorphism. By continuity of the K-theory, the above equality holds for all $\gamma \in \overline{\mathcal{S}}_{\mathcal{G}}(\alpha)$. Now let S be a free basis of $\mathbb{F}_d$. Then by the Pimsner–Voiculescu six term exact sequence [**40**], we have the following short exact sequence.

$$0 \to \mathrm{coker}(\varphi \otimes \mathrm{id}_{K^i(Y)}) \to K_i(B) \to \ker(\varphi \otimes \mathrm{id}_{K^{1-i}(Y)}) \to 0.$$

Here φ denotes the homomorphism

$$\varphi \colon K^0(X)^{\oplus S} \to K^0(X)$$

which maps $(f_s)_{s \in S}$ to $\sum_{s \in S}(f_s - (\alpha_s)_{*,0}(f_s))$. Since $K^0(X)$ is a free abelian group, the exact sequence

$$0 \to K_1(A) \to K^0(X)^{\oplus S} \to K^0(X) \to K_0(A) \to 0$$

obtained by the Pimsner–Voiculescu six-term exact sequence is a free resolution of $K_0(A)$. This also gives the free resolution

$$0 \to \mathrm{im}(\varphi) \to K^0(X) \to K_0(A) \to 0$$

of $K_0(A)$. Here the first map is given by the inclusion map, say ι.

Let $\psi \colon K^0(X)^{\oplus S} \to \mathrm{im}(\varphi)$ be the surjective homomorphism obtained by restricting the range of φ. By tensoring $K^i(Y)$ with the second free resolution, we obtain the following exact sequence.

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K^i(Y)) \to \mathrm{im}(\varphi) \otimes K^i(Y) \to K^0(X) \otimes K^i(Y) \to K_0(A) \otimes K^i(Y) \to 0.$$

This shows that

$$\ker(\iota \otimes \mathrm{id}_{K^i(Y)}) \cong \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K^i(Y)).$$

Since the second map surjects onto $\mathrm{im}(\varphi \otimes \mathrm{id}_{K^i(Y)})$, we also obtain the isomorphism

$$\mathrm{coker}(\varphi \otimes \mathrm{id}_{K^i(Y)}) \cong K_0(A) \otimes K^i(Y).$$

Since $\varphi = \iota \circ \psi$ and ψ is surjective, we have the following exact sequence.

$$0 \to \ker(\psi \otimes \mathrm{id}_{K^i(Y)}) \to \ker(\varphi \otimes \mathrm{id}_{K^i(Y)}) \to \ker(\iota \otimes \mathrm{id}_{K^i(Y)}) \to 0. \tag{1}$$

Here the first map is the canonical inclusion and the second map is the restriction of $\psi \otimes \mathrm{id}_{K^i(Y)}$. Since $\mathrm{im}(\varphi)$ is free abelian, there is a direct complement K of $\ker(\varphi)$ in $K^0(X)^{\oplus S}$. Note that the restriction of ψ on K is an isomorphism. Hence we have the isomorphism

$$\ker(\psi \otimes \mathrm{id}_{K^i(Y)}) = \ker(\psi) \otimes K^i(Y) \cong K_1(A) \otimes K^i(Y).$$

Again by the freeness of $\mathrm{im}(\varphi)$, we have a right inverse σ of ψ. Then the homomorphism $\sigma \otimes \mathrm{id}_{K^i(Y)}$ gives a splitting of the short exact sequence (1). Combining these observations, we obtain the isomorphism

$$\ker(\varphi \otimes \mathrm{id}_{K^i(Y)}) \cong (K_1(A) \otimes K^i(Y)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K^i(Y)).$$

Now the first exact sequence completes the proof. □

Remark 3.3. Certainly, when $\mathrm{K}^*(Y)$ has a good property, the short exact sequence in Proposition 3.2 is spilitting. However, we do not know whether it is splitting in general. Recall that a splitting of the Künneth tensor product theorem is obtained by replacing considered C*-algebras by easier ones by using suitable elements of the KK-groups (see Remark 7.11 of [**49**]). However, in our setting, this argument does not work. Such replacement does not respect the relation among $C(X), C(Y), A, B$, and $\mathbb{F}_d$.

Proof of Theorem 3.1. We first prove the claim for free groups. Theorem 4.3 of Chapter 4 shows that for any finite d, there is an amenable minimal topologically free dynamical system γ of $\mathbb{F}_d$ on the Cantor set whose crossed product A satisfies the following condition. The unit $[1]_0 \in K_0(A)$ generates a direct summand of $K_0(A)$ isomorphic to $\mathbb{Z}$. Note that this property passes to unital C*-subalgebras of A. Moreover, since γ is found as a factor of the ideal boundary action, its restriction to any finite index subgroup of $\mathbb{F}_d$ is minimal. It is also not hard to show that the restriction of γ to any finite index subgroup of $\mathbb{F}_d$ is purely infinite. Applying the argument in the proof of Theorem 3.8 for non-finitely genereted cases in Chapter 2 to γ instead of the action used there, we obtain the following consequence. For any non-empty set $\mathcal{Q}$ of prime numbers, there is an amenable minimal free purely infinite dynamical system $\alpha_{\mathcal{Q}}$ of $\mathbb{F}_d$ on the Cantor set whose K_0-group G satisfies the following condition.

$$\{p \in \mathcal{P} : [1]_0 \in pG\} = \mathcal{Q}.$$

Here $\mathcal{P}$ denotes the set of all prime numbers. The similar statement for $\mathbb{F}_\infty$ is also shown in the proof of Theorem 3.8 in Chapter 2. We also denote by $\alpha_{\mathcal{Q}}$ a dynamical system of $\mathbb{F}_\infty$ satisfying the above conditions.

Now let M be as in the statement. Put

$$\mathcal{R} := \{p \in \mathcal{P} : K^1(M) \text{ contains an element of order } p\}.$$

Then by [**7**], $\mathcal{R}$ is finite. (Indeed, in either case, $M \times [0,1]^{\mathbb{N}}$ is a compact Hilbert cube manifold. Now the main theorem of [**7**] shows that $\mathrm{K}^1(M)$ is in fact finitely generated.)

Let $\mathcal{G}$ denote the path-connected component of Homeo(M) containing the identity. For each non-empty subset $\mathcal{Q}$ of $\mathcal{P} \setminus \mathcal{R}$, we apply Proposition 2.11 to $\alpha_{\mathcal{Q}}$ to choose β from $\overline{\mathcal{S}}_{\mathcal{G}}(\alpha_{\mathcal{Q}})$ whose crossed product is a Kirchberg algebra. For $i = 0, 1$, denote by G_i and H_i the K_i-group of the crossed products of $\alpha_{\mathcal{Q}}$ and β respectively. We claim that

$$\tilde{\mathcal{Q}} := \{p \in \mathcal{P} \setminus \mathcal{R} : [1]_0 \in pH_0\} = \mathcal{Q}.$$

Since the cardinal of the power set of $\mathcal{P} \setminus \mathcal{R}$ is continuum, this ends the proof. The inclusion $\mathcal{Q} \subset \tilde{\mathcal{Q}}$ is obvious. To see the converse, let $p \in \tilde{\mathcal{Q}}$ and take $h \in H_0$ with $ph = [1]_0$. Denote by ∂_i the third map of the short exact sequence in Proposition 3.2. Then since $\partial_0([1]_0) = 0$, we have $p\partial_0(h) = 0$. On the other hand, by the definition of $\mathcal{R}$ and the fact that G_1 is torsion free, there is no element of order p in the third term of the short exact sequence. Thus $p\partial_0(h) = 0$ implies $\partial_0(h) = 0$. Hence there is an element y in the first term of the short exact sequence with $\sigma_0(y) = h$. Here σ_i denotes the second map in the short exact sequence. Then from the injectivity of σ_0 and the equality $ph = [1]_0$, we must have $py = [1]_0 \otimes [1_M]_0$. Now let $\tau \colon K^0(M) \to \mathbb{Z}$ be the homomorphism induced from a character on $C(M)$. Put $w := (\mathrm{id} \otimes \tau)(y) \in G_0$. (We identify G_0 with $G_0 \otimes \mathbb{Z}$ in the obvious way.) Then we have $pw = (\mathrm{id} \otimes \tau)([1]_0 \otimes [1_M]_0) = [1]_0$. Thus we get $p \in \mathcal{Q}$ as desired.

The proof for general case is done by taking the induced dynamical systems of the actions obtained in above. See the proof of Theorem 3.6 in Chapter 2 for the detail. □

CHAPTER 4

Group C*-algebras as decreasing intersection of nuclear C*-algebras

It is well-known that every exact discrete group admits an amenable action on a compact space [**35**], and each such action gives rise to an ambient nuclear C*-algebra of the reduced group C*-algebra via the crossed product construction [**1**]. More generally, it is known that every separable exact C*-algebra is embeddable into the Cuntz algebra $\mathcal{O}_2$ [**29**]. Motivated by these phenomena, we are interested in the following question. How small can we take an ambient nuclear C*-algebra/ Cuntz algebra $\mathcal{O}_2$ for a given exact C*-algebra? In this chapter, we give an answer to the question for the reduced group C*-algebras of discrete groups with the AP. The next theorem states that ambient nuclear C*-algebras of the reduced group C*-algebras with the AP can be arbitrarily small in a certain sense. This in particular shows that, unlike injective von Neumann algebras, nuclear C*-algebras do not form a monotone class.

Main Theorem . *Let Γ be a countable discrete exact group. Then there is an intermediate* C*-*algebra A between the reduced group* C*-*algebra $\mathrm{C}^*_r(\Gamma)$ and $L(\Gamma) \cap \mathrm{C}^*_u(\Gamma)$ satisfying the following properties.*

- *There is a decreasing sequence of isomorphs of the Cuntz algebra $\mathcal{O}_2$ whose intersection is isomorphic to A.*
- *There is a decreasing sequence $(A_n)_{n=1}^{\infty}$ of separable nuclear* C*-*algebras whose intersection is isomorphic to A and the sequence admits compatible multiplicative conditional expectations $(E_n \colon A_1 \to A_n)_{n=1}^{\infty}$. Here the compatibility means that the equality $E_n \circ E_m = E_n$ holds for all $n \geq m$.*

In particular, when the group Γ has the AP, the statements hold for the reduced group C*-*algebra $\mathrm{C}^*_r(\Gamma)$.*

As a consequence of Main Theorem, we obtain the following result.

Corollary A. *The decreasing intersection of nuclear* C*-*algebras need not have the following properties.*

(1) *The OAP, hence nuclearity, the CBAP, the WEP, and the SOAP.*
(2) *The local lifting property.*

They can happen simultaneously. The statements are true even when the decreasing sequence admits a compatible family of multiplicative conditional expectations.

Thus the decreasing intersection of nuclear C*-algebras can lost most of good properties. Since the decreasing intersection of injective von Neumann algebras is injective, the analogous results for von Neumann algebras can never be true.

We also give a geometric construction of a decreasing sequence of Kirchberg algebras whose intersection is isomorphic to the hyperbolic group C*-algebra. Although the result follows from Main Theorem, this approach has good points. Our decreasing sequence is taken inside the boundary algebra $C(\partial\Gamma) \rtimes_r \Gamma$. Moreover, the proof does not depend on Kirchberg's $\mathcal{O}_2$-absorption theorem and the theory of reduced free products, both of which are used in the proof of Main Theorem. Using the sequence constructed by this method, we also study absorbing extensions of the reduced free group C*-algebras by stable separable nuclear C*-algebras, and prove the following theorem.

Theorem B. *Let A be a stable separable nuclear* C*-*algebra and let*

$$0 \to A \to B \to \mathrm{C}^*_r(\mathbb{F}_d) \to 0$$

be an extension of $\mathrm{C}^*_r(\mathbb{F}_d)$ by A $(2 \leq d \leq \infty)$. Assume B is exact and the extension is either absorbing or unital absorbing. Then B is realized as a decreasing intersection of isomorphs of the Cuntz algebra $\mathcal{O}_2$. In particular, any exact extension of $\mathrm{C}^*_r(\mathbb{F}_d)$ by $\mathbb{K}$ is realized in this way.

The proof of Theorem B is based on the KK-theory.

Organization of this chapter. In Section 1, we review some notions and facts used in this chapter. In Section 2, we prove Main Theorem. We also give few more examples satisfying the conditions in Main Theorem. In Section 3, we deal with the hyperbolic groups. Based on the study of the boundary action, we construct a decreasing sequence of nuclear C*-algebras inside the boundary algebra $C(\partial\Gamma) \rtimes_r \Gamma$ whose intersection is the reduced group C*-algebra $\mathrm{C}^*_r(\Gamma)$. In Section 4, using the decreasing sequence constructed in Section 3, we prove Theorem B.

1. Preliminaries

1.1. Reduced free product. We refer the reader to [**6**, Section 4.7] for the definition of the reduced free product. First we recall a few terminology related to theorems we will use. Let A be a C*-algebra and φ be a state on A. Recall that φ is said to be non-degenerate if its GNS-representation is faithful. Recall that the centralizer of φ is the set of all elements $b \in A$ satisfying the equality $\varphi(ba) = \varphi(ab)$ for all $a \in A$. An abelian C*-subalgebra D of A is said to be diffuse with respect to φ if $\varphi|_D$ is a diffuse measure on the spectrum of D.

In the proofs of Main Theorem and Theorem B, we use the reduced free product to make C*-algebras simple. The following two theorems are important in our proof. The first theorem guarantees the nuclearity of the reduced free product under certain conditions. The second one gives a sufficient condition for the simplicity of the reduced free product.

Theorem 1.1 (Dykema–Smith [**6**, Exercise 4.8.2]). *Let (A, φ) be a pair of a unital nuclear* C*-*algebra and a non-degenerate state on A. Let ψ be a pure state on the matrix algebra $\mathbb{M}_n$ $(n \geq 2)$. Then the reduced free product $(A, \varphi) * (\mathbb{M}_n, \psi)$ is nuclear.*

Theorem 1.2 (Dykema [**15**, Theorem 2]). *Let (A, φ) and (B, ψ) be pairs of a unital* C*-*algebra and a non-degenerate state on it. Assume that $B \neq \mathbb{C}$ and the centralizer of φ contains a diffuse abelian* C*-*subalgebra D containing the unit of A. Then the reduced free product $(A, \varphi) * (B, \psi)$ is simple.*

A good aspect of these theorems is that we only need to force a condition on one of the states. Thus we can apply these theorems at the same time in many situations.

1.2. Extensions of C*-algebras. Here we recall basic facts and terminologies related to the extensions of C*-algebras. We refer the reader to [**4**, Sections 15, 17] for the details. Let A be a unital separable C*-algebra, B be a stable (i.e., $B \cong B \otimes \mathbb{K}$) nuclear C*-algebra. Let

$$0 \to B \to C \to A \to 0$$

be an essential extension of A by B. Here essential means that the ideal B of C is essential (i.e., $cB = 0$ implies $c = 0$ for $c \in C$).

Let $\sigma\colon A \to Q(B) := M(B)/B$ be the Busby invariant of the above extension. Here $M(B)$ denotes the multiplier algebra of B. As usual, we identify an extension with its Busby invariant.

To define the addition of two extensions, we fix an isomorphism $B \cong B \otimes \mathbb{K}$. (Note that up to canonical identifications, the choice of the isomorphism does not affect to the following definitions.)

An extension σ is said to be trivial (resp. strongly unital trivial) if it has a $*$-homomorphism (resp. unital $*$-homomorphism) lifting $\tilde{\sigma}\colon A \to M(B)$. Two extensions σ_1 and σ_2 are said to be strongly equivalent if there is a unitary element u in $M(B)$ satisfying $\mathrm{ad}(\pi(u)) \circ \sigma_1 = \sigma_2$. An extension σ is said to be absorbing (resp. unital absorbing) if for any trivial extension (resp. strongly unital trivial extension) τ, $\sigma \oplus \tau$ is strongly equivalent to σ. On the class of extensions of A by B, we define an equivalence relation as follows. Two extensions σ_1 and σ_2 are equivalent if there are trivial representations τ_1 and τ_2 such that the direct sums $\sigma_i \oplus \tau_i$ are strongly equivalent. The quotient $\mathrm{Ext}(A, B)$ of the class of all extensions by this equivalence relation naturally becomes an abelian semigroup.

Kasparov showed that there is a unital absorbing strongly unital trivial extension τ of A by B [**27**, Theorem 6]. Therefore any $[\sigma] \in \mathrm{Ext}(A, B)$ has a unital absorbing representative. Moreover, if $[\sigma]$ contains a unital extension, then $[\sigma]$ has a unital absorbing unital representative. Note that an element $[\sigma] \in \mathrm{Ext}(A, B)$ contains a unital extension if and only if $[\sigma(1)]_0 = 0$ in $K_0(Q(B))$.

A theorem of Kasparov [**27**, Theorem 2] shows that for a unital absorbing extension σ, the direct sum $\sigma \oplus 0$ is an absorbing extension. Thus, by the same reason as above, any element of $\mathrm{Ext}(A, B)$ has an absorbing representative. By definition, such a representative is unique up to strongly equivalence.

It follows from [**27**, Theorem 6] that for any unital C*-subalgebra $C \subset A$, the restriction of the absorbing (resp. unital absorbing) extension to C again has the same property.

Let $\mathrm{Ext}(A, B)^{-1}$ be the subsemigroup of $\mathrm{Ext}(A, B)$ consisting of invertible elements. Then there is a natural group isomorphism between $\mathrm{Ext}(A, B)^{-1}$ and $KK^1(A, B)$ [**4**, Corollary 18.5.4].

2. Proof of Main Theorem

Let Γ be an exact group. Take an amenable action $\Gamma \curvearrowright X$ on a compact metrizable space. Define $A_n := C(\prod_{k=n}^{\infty} X) \rtimes_r \Gamma$ for each $n \in \mathbb{N}$. Here the action $\Gamma \curvearrowright \prod_{k=n}^{\infty} X$ is given by the diagonal action. We regard A_{n+1} as a C*-subalgebra of A_n in the canonical way. Since the Γ-space $\prod_{k=n}^{\infty} X$ is metrizable and amenable, each A_n is separable and nuclear. Put $A := \bigcap_{n=1}^{\infty} A_n$. We will show that A is isomorphic to an intermediate C*-algebra between $\mathrm{C}^*_r(\Gamma)$ and $\mathrm{C}^*_u(\Gamma) \cap L(\Gamma)$, To see this, take an arbitrary point $x \in \prod_{k=1}^{\infty} X$ and define $\rho\colon C(\prod_{k=1}^{\infty} X) \to \ell^\infty(\Gamma)$ by $\rho(f)(s) := f(s.x)$ for $f \in C(\prod_{k=1}^{\infty} X)$ and $s \in \Gamma$. Then ρ is a Γ-equivariant $*$-homomorphism. Hence it induces a $*$-homomorphism $\tilde{\rho}\colon A_1 \to \ell^\infty(\Gamma) \rtimes_r \Gamma$. Note that for all $a \in A$ and $g \in \Gamma$, we have $E_g(a) \in \bigcap_{n=1}^{\infty} C(\prod_{k=n}^{\infty} X) = \mathbb{C}$. This shows that $\tilde{\rho}$ is injective on A and $\tilde{\rho}(A)$ is contained in $\mathrm{C}^*_u(\Gamma) \cap L(\Gamma)$. Thus A is isomorphic to the desired C*-algebra.

Next we show that there is a compatible family of multiplicative conditional expectations $(E_n\colon A_1 \to A_n)_{n=1}^{\infty}$. Let E_n be the $*$-homomorphism induced from the Γ-equivariant $*$-homomorphism

$$E_n\colon C(\prod_{k=1}^{\infty} X) \to C(\prod_{k=n}^{\infty} X)$$

defined by

$$E_n(f)(x_n, x_{n+1}, x_{n+2}, \ldots) := f(x_n, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots),$$

where, in the right hand side, x_n is iterated n times. Then it is not difficult to check that they satisfy the desired conditions.

To make terms isomorphic to the Cuntz algebra $\mathcal{O}_2$, we first make terms simple. To do this, take a faithful state ν on A_1. Take a compact metric space Y consisting at least two points and a faithful measure μ on Y. On $(\bigotimes_{k=1}^{\infty} C(Y)) \otimes A_1$, define a faithful state φ by $\varphi := (\bigotimes_{k=1}^{\infty} \mu) \otimes \nu$. Then define a faithful state φ_n on $B_n := (\bigotimes_{k=n}^{\infty} C(Y)) \otimes A_n$ to be the restriction of φ. Now take a pure state ψ on $\mathbb{M}_2$ and put $C_n := (B_n, \varphi_n) * (*_{k=n}^{\infty}(\mathbb{M}_2, \psi))$. Then by Theorem 1.2, each C_n is simple. Moreover, since C_n is the increasing union of finite free products $((B_n, \varphi_n) * (*_{k=n}^{m}(\mathbb{M}_2, \psi)))_{m=n}^{\infty}$, each C_n is nuclear by Theorem 1.1. By Theorem 4.8.5 of [**6**], for each $n \in \mathbb{N}$, we have a conditional expectation from C_1 onto $(B_1, \varphi) * (*_{k=1}^{n}(\mathbb{M}_2, \psi))$ which maps C_{n+1} onto B_{n+1}. This proves

$$\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} A_n = A.$$

Finally, to make terms isomorphic to $\mathcal{O}_2$, we apply Kirchberg–Phillips's $\mathcal{O}_2$-absorption theorem [**29**]. We define a new sequence $(D_n)_{n=1}^{\infty}$ by $D_n := C_n \otimes (\bigotimes_{k=n}^{\infty} \mathcal{O}_2)$. Then each D_n is isomorphic to $\mathcal{O}_2$ and we have

$$\bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} C_n = A.$$

□

REMARK 2.1. There is an isomorphism between the decreasing intersection $A = \bigcap_{n \in \mathbb{N}} (C(\prod_{k=n}^{\infty} X) \rtimes_r \Gamma)$ and the C*-algebra

$$B = \{b \in C(X) \rtimes_r \Gamma : E_g(b) \in \mathbb{C} \text{ for all } g \in \Gamma\}$$

that preserves the reduced group C*-algebra. To see this, consider the quotient map $\pi \colon C(\prod_{k=1}^{\infty} X) \rtimes_r \Gamma \to C(X) \rtimes_r \Gamma$ induced from the diagonal embedding $X \to \prod_{k \in \mathbb{N}} X$. Then π is injective on A. To see the equality $\pi(A) = B$, consider the embedding of $C(X)$ into $C(\prod_{k=1}^{\infty} X)$ induced from the quotient map from $\prod_{k=1}^{\infty} X$ onto the nth product component for each $n \in \mathbb{N}$.

Therefore, the question either the equation

$$\bigcap_{n \in \mathbb{N}} \left(C(\prod_{k=n}^{\infty} X) \rtimes_r \Gamma \right) = \mathrm{C}_r^*(\Gamma)$$

holds or not seems difficult when the group Γ does not have the AP. Indeed, if the equation holds for every compact metrizable Γ-space X (when Γ is exact, we only need to consider the amenable one), then Γ has the ITAP. However, we do not know either a given group has the ITAP or not for groups without the AP.

Now we can prove Corollary A.

PROOF OF COROLLARY A. We apply Main Theorem to $\Gamma := \mathrm{SL}(3, \mathbb{Z})$. (See [**6**, Section 5.4] for the exactness of Γ.) This gives an intermediate C*-algebra A between $\mathrm{C}_r^*(\Gamma)$ and $L(\Gamma) \cap \mathrm{C}_u^*(\Gamma)$ satisfying the conditions in Main Theorem. We show that A does not have the OAP and the local lifting property. Since Γ does not have the AP [**31**], Proposition 2.3 in Chapter 1 yields that A does not have the OAP.

Next take a subgroup Λ of Γ isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. Denote by $p \in \mathbb{B}(\ell^2(\Gamma))$ the projection onto the subspace $\ell^2(\Lambda)$. Then the compression by p gives a conditional expectation

$$E_\Lambda^\Gamma \colon \mathrm{C}_u^*(\Gamma) \to \mathrm{C}_u^*(\Lambda).$$

It is clear from the definition that E_Λ^Γ maps $L(\Gamma) \cap \mathrm{C}_u^*(\Gamma)$ onto $L(\Lambda) \cap \mathrm{C}_u^*(\Lambda)$. Since Λ has the AP [**6**, Corollary 12.3.5], we obtain the conditional expectation

$$\Phi \colon A \to \mathrm{C}_r^*(\Lambda).$$

Since $\mathrm{C}_r^*(\Lambda)$ does not have the local lifting property [**6**, Corollary 3.7.12], neither does A. □

Other examples. We end this section by giving few more examples satisfying the conditions in Main Theorem.

PROPOSITION 2.2. *Let A be a unital separable nuclear* C*-*algebra,* Γ *be a group with the AP. Then for any action of* Γ *on* A*, the reduced crossed product* $A \rtimes_r \Gamma$ *satisfies the conditions mentioned in Main Theorem.*

Let A be a unital C*-algebra. Let Γ be a group and S be a Γ-set. Consider the reduced crossed product $A^{\otimes S} \rtimes_r \Gamma$, where Γ acts on $A^{\otimes S}$ by the shift of tensor components. We say it the generalized wreath product of A with respect to S and denote it by $A \wr_S \Gamma$.

PROPOSITION 2.3. *The class of unital* C*-*algebras with the SOAP satisfying the conditions in Main Theorem is closed under taking the following operations.*

(1) *Countable minimal tensor products.*
(2) *The generalized wreath product with respect to any* Γ-*set with* Γ *the AP.*

To prove Propositions 2.2 and 2.3, we need the following proposition. The idea of the proof is essentially contained in [**58**].

PROPOSITION 2.4. *Let* Γ *be a group with the AP. Let* A *be a* Γ-C*-*algebra and let* X *be a closed subspace of* A*. Assume that an element* $x \in A \rtimes_r \Gamma$ *satisfies* $E_g(x) \in X$ *for all* $g \in \Gamma$*. Then* x *is contained in the closed subspace*

$$X \rtimes_r \Gamma := \overline{\mathrm{span}}\{xg : x \in X, g \in \Gamma\}.$$

Conversely, if the above implication always holds for any Γ-C*-*algebra and its closed subspace, then the group* Γ *has the AP.*

PROOF. Since Γ has the AP, there is a net $(\varphi_i)_{i\in I}$ of finitely supported functions on Γ satisfying the condition in Definition 2.2. For $i \in I$, define the linear map $\Phi_i \colon A \rtimes_r \Gamma \to A \rtimes_r \Gamma$ by $\Phi_i(y) := \sum_{g\in\Gamma} \varphi_i(g) E_g(y) g$. We claim that the net $(\Phi_i)_{i\in I}$ converges to the identity map in the pointwise norm topology. To show this, consider the embedding $\iota \colon A \rtimes_r \Gamma \to (A \rtimes_r \Gamma) \otimes \mathrm{C}^*_r(\Gamma)$ induced from the maps $a \in A \mapsto a \otimes 1$ and $g \in \Gamma \mapsto g \otimes g$. (This indeed defines an embedding by Fell's absorption principle [**6**, Prop.4.1.7].) Then the composite $\iota \circ \Phi_i$ coincides with the composite $(\mathrm{id}_{A\rtimes_r\Gamma} \otimes m_{\varphi_i}) \circ \iota$. This proves the convergence condition. Now let x be as stated. Then for any $i \in I$, we have $\Phi_i(x) \in X \rtimes_r \Gamma$. Since the net $(\Phi_i(x))_{i\in I}$ converges in norm to x, we have $x \in X \rtimes_r \Gamma$.

To show the converse, apply the above condition to the case Γ-action is trivial. □

As a consequence, we obtain a permanence property of the SOAP and the OAP.

COROLLARY 2.5. *The SOAP and the OAP are preserved under taking the reduced crossed product of a group with the AP.*

PROOF. We only give a proof for the SOAP. Let A be a Γ-C*-algebra with the SOAP. Let B be a C*-algebra and X be its closed subspace. To show the SOAP of $A \rtimes_r \Gamma$, it suffices to prove the inclusion $F(A \rtimes_r \Gamma, B, X) \subset (A \rtimes_r \Gamma) \otimes X$. Let $x \in F(A \rtimes_r \Gamma, B, X)$. Then $(E_g \otimes \mathrm{id}_B)(x) \in F(A, B, X)$ for all $g \in \Gamma$. Since A has the SOAP, we have $F(A, B, X) \subset A \otimes X$. Then from Proposition 2.4, we conclude $x \in (A \rtimes_r \Gamma) \otimes X$. Here we use the canonical identification of $(A \rtimes_r \Gamma) \otimes B$ with $(A \otimes B) \rtimes_r \Gamma$. □

REMARK 2.6. The similar proofs also show the W*-analogues of Proposition 2.4 and Corollary 2.5. We note that the W*-analogue of Corollary 2.5 is shown by Haagerup and Kraus for locally compact groups with the AP [**24**, Theorem 3.2].

Proof of Proposition 2.2. Replace $C(\prod_{k=n}^{\infty} X)$ by $C(\prod_{k=n}^{\infty} X) \otimes A$ with the diagonal Γ-action in the proof of Main Theorem. □

Proof of Proposition 2.3. We only prove the second claim.

First take a decreasing sequence $(A_n)_{n=1}^{\infty}$ of separable nuclear C*-algebras whose intersection is isomorphic to A and that admits a compatible family of multiplicative conditional expectations. We will use $C(\prod_{k=n}^{\infty} X) \otimes A_n^{\otimes S}$ instead of $C(\prod_{k=n}^{\infty} X)$ in the proof of Main Theorem. To do this, we remark that the equality

$$\bigcap_{n=1}^{\infty} \left(C(\prod_{k=n}^{\infty} X) \otimes \left(A_n^{\otimes S}\right) \right) = A^{\otimes S}$$

holds since C*-algebras A and A_n have the SOAP. □

3. Hyperbolic group case

In this section, we give a geometric construction of a decreasing sequence of Kirchberg algebras whose decreasing intersection is isomorphic to the hyperbolic group C*-algebra. We construct such a sequence inside the boundary algebra $C(\partial\Gamma) \rtimes_r \Gamma$. To find such a sequence, we construct amenable factors of the boundary space. The proof does not depend on both reduced free product theory and Kirchberg's $\mathcal{O}_2$-absorption theorem. We will use the sequence constructed in this section for the free group case in the next two sections.

For the reader who is only interested in the free group case, we recommend to concentrate on that case. In this case, some arguments related to geodesic paths become much simpler. Next we recall a few facts on hyperbolic groups. (See 8.16, 8.21, 8.28, and 8.29 in [**19**].) For a torsion free element t of a hyperbolic group Γ, the sequence $(t^n)_{n=1}^{\infty}$ is quasi-geodesic. The boundary action of t has exactly two fixed points. They are the points represented by the quasi-geodesic paths $(t^n)_{n=1}^{\infty}$ and $(t^{-n})_{n=1}^{\infty}$. We denote them by $t^{+\infty}$ and $t^{-\infty}$ respectively. For any neighborhoods $U_{\pm}$ of $t^{\pm\infty}$, there is $n \in \mathbb{N}$ such that for any $m \geq n$, $t^m(\partial\Gamma \setminus U_-) \subset U_+$ holds.

For a metric space (X, d) and its points $x, y, z \in X$, denote by $\langle y, z\rangle_x$ the Gromov product $(d(y, x) + d(z, x) - d(y, z))/2$ of y, z with respect to x.

We recall the following criteria for the Hausdorffness of a quotient space. We left the proof to the reader.

Proposition 3.1. *Let X be a compact Hausdorff space. Let $\mathcal{R}$ be an equivalence relation on X. Assume that the quotient map $\pi\colon X \to X/\mathcal{R}$ is closed. Then the quotient space $X/\mathcal{R}$ is Hausdorff.*

The next lemma guarantees the amenability of certain factors of amenable dynamical systems. We are grateful to Narutaka Ozawa for letting us know Lusin's theorem.

Lemma 3.2. *Let Γ be a group, X be an amenable compact metrizable Γ-space. Let $\mathcal{R}$ be a Γ-invariant equivalence relation on X such that the quotient space $X/\mathcal{R}$ is Hausdorff. Assume that each equivalence class of $\mathcal{R}$ is finite. Then $X/\mathcal{R}$ is again an amenable compact Γ-space.*

To prove Lemma 3.2, we need the following characterization of amenability due to Anantharaman-Delaroche [**1**, Theorem 4.5]. See also [**6**, Prop.5.2.1] for a generalized version.

Proposition 3.3. *Let $\alpha\colon \Gamma \curvearrowright X$ be an action of Γ on a compact metrizable space X. Then α is amenable if and only if there is a net $(\zeta_i\colon X \to \mathrm{Prob}(\Gamma))_{i\in I}$ of Borel maps satisfying the following condition.*

$$\lim_{i\in I} \int_X \|g.\zeta_i(x) - \zeta_i(g.x)\|_1 \, d\mu = 0 \text{ for all } \mu \in \mathrm{Prob}(X) \text{ and } g \in \Gamma.$$

Here $\mathrm{Prob}(X)$ *denotes the set of all Borel probability measures on* X.

Proof of Lemma 3.2. Since $\mathcal{R}$ is closed in $X \times X$ and each equivalence class is finite, Lusin's theorem [**51**, Theorem 5.8.11] tells us that $\mathcal{R}$ is presented as a countable disjoint union of graphs of Borel maps between Borel subsets of X. Then it is not hard to check that for each $f \in C(X)$, the function $\tilde{f}$ on $X/\mathcal{R}$ defined by

$$\tilde{f}([x]) := \frac{1}{\sharp[x]} \sum_{y \in [x]} f(y)$$

is Borel. By the same reason, the similar formula also defines the map Φ from $C(X, \mathrm{Prob}(\Gamma))$ to $\mathcal{B}(X/\mathcal{R}, \mathrm{Prob}(\Gamma))$. Here $C(X, \mathrm{Prob}(\Gamma))$ denotes the set of all continuous maps from X into $\mathrm{Prob}(\Gamma)$ and $\mathcal{B}(X/\mathcal{R}, \mathrm{Prob}(\Gamma))$ denotes the set of all Borel maps from $X/\mathcal{R}$ into $\mathrm{Prob}(\Gamma)$.

Let $(\zeta_i \colon X \to \mathrm{Prob}(\Gamma))_{i \in I}$ be a net of continuous maps that satisfies the condition in the definition of amenability for $\Gamma \curvearrowright X$. Consider the net $(\Phi(\zeta_i))_{i \in I}$. Then for any $g \in \Gamma$, $x \in X$, and $i \in I$, we have

$$\|(g.\Phi(\zeta_i))([x]) - \Phi(\zeta_i)(g.[x])\|_1 \leq \frac{1}{\sharp[x]} \sum_{y \in [x]} \|g.\zeta_i(y) - \zeta_i(g.y)\|_1.$$

Thus, for each $g \in \Gamma$, the norms $\|(g.\Phi(\zeta_i))([x]) - \Phi(\zeta_i)(g.[x])\|_1$ converge to 0 uniformly on $X/\mathcal{R}$ as i tends to ∞. In particular, the net $(\Phi(\zeta_i))_{i \in I}$ satisfies the condition in Proposition 3.3. $\square$

Lemma 3.4. *Let* Γ *be a hyperbolic group. Let* T *be a finite set of torsion free elements of* Γ. *Then the set*

$$\mathcal{R}_T := \Delta_{\partial\Gamma} \cup \left\{(g.t^{+\infty}, g.t^{-\infty}) : g \in \Gamma, t \in T \cup T^{-1}\right\}$$

is a Γ*-invariant equivalence relation on* $\partial\Gamma$. *Moreover, the quotient space* $\partial\Gamma/\mathcal{R}_T$ *is a Hausdorff space.*

Proof. Clearly $\mathcal{R}_T$ is Γ-invariant. Let s and t be torsion free elements of Γ. Then the two sets $\{s^{\pm\infty}\}$ and $\{t^{\pm\infty}\}$ are either disjoint or the same [**19**, 8.30]. Therefore the set $\mathcal{R}_T$ is an equivalence relation. Note that this shows that each equivalence class of $\mathcal{R}_T$ contains at most two points.

For the Hausdorffness of the quotient space, it suffices to show that the quotient map $\pi \colon \partial\Gamma \to \partial\Gamma/\mathcal{R}_T$ is closed. Let A be a closed subset of $\partial\Gamma$. Then $\pi^{-1}(\pi(A)) = A \cup B$, where

$$B := \left\{g.t^{-\infty} \in \partial\Gamma : g \in \Gamma, t \in T \cup T^{-1}, g.t^{+\infty} \in A\right\}.$$

To show the closedness of $\pi(A)$, which is equivalent to that of $\pi^{-1}(\pi(A))$, it suffices to show that $\mathrm{cl}(B) \subset A \cup B$. Fix a finite generating set S of Γ and denote by $|\cdot|$ and $d(\cdot, \cdot)$ the length function and the left invariant metric on Γ determined by S respectively. Take $\delta > 0$ with the property that every geodesic triangle in (Γ, d) is δ-thin [**6**, Proposition 5.3.4]. Let $x \in \mathrm{cl}(B)$ and take a sequence $(g_n.t_n^{-\infty})_{n=1}^{\infty}$ in B which converges to x. By passing to a subsequence, we may assume that there is $t \in T \cup T^{-1}$ with $t_n = t$ for all $n \in \mathbb{N}$. Replace g_n by $g_n t^{l(n)}$ for some $l(n) \in \mathbb{Z}$ for each $n \in \mathbb{N}$, we may further assume $|g_n| \leq |g_n t^k|$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. If the sequence $(g_n)_{n=1}^{\infty}$ has a bounded subsequence, then it has a constant subsequence. Hence we have $x \in B$. Assume $|g_n| \to \infty$. For each $k \in \mathbb{Z}$, take a geodesic path $[e, t^k]$ from e to t^k. Since t is torsion free, the sequences $(t^n)_{n=1}^{\infty}$ and $(t^{-n})_{n=1}^{\infty}$ are quasi-geodesic. Therefore, by [**6**, Prop.5.3.5], there is $D > 0$ such that the Hausdorff distance between $[e, t^k]$ and $(t^n)_{n=0}^{k}$ is less than D for all $k \in \mathbb{Z}$. This shows the inequality $\mathrm{dist}(g_n^{-1}, [e, t^k]) \geq |g_n| - D$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Now consider a geodesic triangle Δ with the vertices $\{e, g_n^{-1}, t^k\}$. Let f denote the comparison tripod of Δ (see Section 5.3 of [**6**] for the definition.) Let u, v, w be (unique) points in Δ lying on the geodesic paths $[e, g_n^{-1}], [g_n^{-1}, t^k], [t^k, e] \subset \Delta$ respectively that satisfy $f(u) = f(v) = f(w)$. Put

$l_1 := d(e,u) = d(w,e), l_2 := d(u, g_n^{-1}) = d(g_n^{-1}, v)$, and $l_3 := d(v, t^k) = d(t^k, w)$. Then, since Δ is δ-thin, we have $l_2 + \delta \geq \mathrm{dist}(g_n^{-1}, [e, t^k]) \geq |g_n| - D$. Since $l_1 + l_2 = |g_n|$, this implies $l_1 \leq D + \delta$. Then since $l_1 + l_3 = |t^k|$, we further obtain $l_3 \geq |t^k| - D - \delta$. Combining these inequalities, we have $|g_n t^k| = l_2 + l_3 \geq |t^k| + |g_n| - 2(D+\delta)$. This yields

$$\langle g_n t^k, g_n t^{-l}\rangle_e \geq |g_n| - 2(D+\delta) \text{ for all } n, k, l \in \mathbb{N}.$$

Since both $(t^k)_{k=1}^\infty$ and $(t^{-k})_{k=1}^\infty$ are quasi-geodesic and the left multiplication action of Γ on itself is isometric, the paths $\{(g_n t^k)_{k=1}^\infty, (g_n t^{-k})_{k=1}^\infty : n \in \mathbb{N}\}$ are uniformly quasi-geodesic (i.e., there are constants $C \geq 1$ and $r > 0$ such that all paths in the set are (C, r)-quasi-geodesic). This with the above inequality shows that the limits of $(g_n.t^{+\infty})_{n=1}^\infty$ and $(g_n.t^{-\infty})_{n=1}^\infty$ coincide. (Cf. Lemmas 5.3.5, 5.3.8 in [**6**] and the definition of the topology on $\partial\Gamma$.) Since A is closed, we have $x \in A$ as required. □

For a subgroup Λ of a hyperbolic group Γ, we define the limit set L_Λ of Λ to be the closure of the set $\{t^{+\infty} \in \partial\Gamma : t \in \Lambda \text{ torsion free}\}$ in $\partial\Gamma$. Recall that every hyperbolic group does not contain an infinite torsion subgroup [**19**, 8.36]. Therefore the limit set L_Λ is nonempty when Λ is infinite. Since we have $(sts^{-1})^{+\infty} = s.t^{+\infty}$ for any torsion free element t of Γ and any element s of Γ, the limit set L_Λ is Λ-invariant. Hence Λ acts on L_Λ in the canonical way. Next we give two lemmas on the action on the limit set, which are familiar to specialists.

LEMMA 3.5. *Let Λ be an ICC subgroup of a hyperbolic group Γ. Then the action φ_Λ of Λ on its limit set L_Λ is amenable, minimal, and topologically free.*

PROOF. The amenability of φ_Λ is clear since boundary actions are amenable. Since Λ is ICC, it is neither finite nor virtually cyclic. Hence Λ contains a free group of rank 2 [**19**, Theorem 8.37]. Hence there are two torsion free elements s and t of Λ which do not have a common fixed point. This shows the minimality of φ_Λ.

Assume now that φ_Λ is not topologically free. Take an element $g_1 \in \Lambda \setminus \{e\}$ such that the set $F_{g_1} := \{x \in L_\Lambda : g_1.x = x\}$ has a nontrivial interior. Since L_Λ does not have an isolated point, the order of g_1 must be finite. Assume $F_{g_1} = L_\Lambda$. This means that the kernel of φ_Λ is nontrivial. Since it cannot contain a torsion free element, it is a nontrivial torsion subgroup. Therefore it must be finite. This contradicts to the ICC condition. For a subgroup G of Λ, we set $F_G := \bigcap_{g \in G} F_g$. Note that for a subgroup G of Λ and $g \in \Lambda$, we have $F_{gGg^{-1}} = gF_G$. Set $G_1 := \langle g_1 \rangle$. Then $\mathrm{int}(F_{G_1}) = \mathrm{int}(F_{g_1}) \neq \emptyset$. We will show that there is $g_2 \in \Lambda$ satisfying

$$\emptyset \neq g_2(\mathrm{int}(F_{G_1})) \cap \mathrm{int}(F_{G_1}) \subsetneq \mathrm{int}(F_{G_1}).$$

Indeed, if such g_2 does not exist, then the family $\{g(\mathrm{int}(F_{G_1})) : g \in \Lambda\}$ makes an open covering of L_Λ whose members are mutually disjoint. (Note that if $g \in \Lambda$ satisfies $\mathrm{int}(F_{G_1}) \subsetneq g(\mathrm{int}(F_{G_1}))$, then g^{-1} satisfies the required condition.) This forces that the subgroup

$$\Lambda_0 := \{g \in \Lambda : g(\mathrm{int}(F_{G_1})) = \mathrm{int}(F_{G_1})\}$$

has finite index in Λ. Since Λ is ICC, the subgroup $G := \langle gG_1g^{-1} : g \in \Lambda_0\rangle$ must be infinite. Moreover, by definition, we have $\mathrm{int}(F_G) = \mathrm{int}(F_{G_1}) \neq 0$. Hence G must be an infinite torsion subgroup, a contradiction. Thus we can take $g_2 \in \Lambda$ as above. Set $G_2 = \langle G_1, g_2G_1{g_2}^{-1}\rangle$. Then we have $\emptyset \neq \mathrm{int}(F_{G_2}) \subsetneq \mathrm{int}(F_{G_1})$. This shows that G_2 is still finite and is larger than G_1. Continuing this argument inductively, we obtain a strictly increasing sequence $(G_n)_{n=1}^\infty$ of finite subgroups of Λ. Then the union $\bigcup_{n=1}^\infty G_n$ is an infinite torsion subgroup of Λ, again a contradiction. □

REMARK 3.6. Conversely, if Λ is not ICC, then the action on the limit set L_Λ is not faithful. In this case, Λ contains a finite index subgroup Λ_0 with the nontrivial center. Since $L_{\Lambda_0} = L_\Lambda$, the center of Λ_0 acts on L_Λ trivially.

LEMMA 3.7. *For* Λ *as in Lemma* 3.5, *the equivalence relation*

$$\mathcal{R} := \left(\bigcup_{t \in \Lambda, \text{ torsion free}} \mathcal{R}_{\{t\}} \right) \cap (L_\Lambda \times L_\Lambda)$$

on L_Λ *is dense in* $L_\Lambda \times L_\Lambda$.

PROOF. Let s and t be two torsion free elements in Λ which do not have a common fixed point. For any neighborhoods $U_\pm$ of $s^{\pm\infty}$ and neighborhoods $V_\pm$ of $t^{\pm\infty}$ with the properties $U_+ \cap V_- = \emptyset$ and $U_- \cap V_+ = \emptyset$, take a natural number N satisfying $s^N(\partial\Gamma \setminus U_-) \subsetneq U_+$ and $t^N(\partial\Gamma \setminus V_-) \subsetneq V_+$. Then, for any $m \in \mathbb{N}$, we have $(s^N t^N)^m(\partial\Gamma \setminus V_-) \subsetneq U_+$ and $(s^N t^N)^{-m}(\partial\Gamma \setminus U_+) \subsetneq V_-$. This shows that the element $s^N t^N$ is torsion free, $(s^N t^N)^{+\infty} \in \mathrm{cl}(U_+)$, and $(s^N t^N)^{-\infty} \in \mathrm{cl}(V_-)$. Thus the product $\mathrm{cl}(U_+) \times \mathrm{cl}(V_-)$ intersects with $\mathcal{R}$. This proves the density of $\mathcal{R}$. □

Recall that an action $\Gamma \curvearrowright X$ of a group on a compact Hausdorff space is called a locally boundary action if for any nonempty open set $U \subset X$, there is an open set $V \subset U$ and an element $t \in \Gamma$ such that $\mathrm{cl}(t.V) \subsetneq V$ holds [**30**, Definition 6].

LEMMA 3.8. *Let* Λ *and* Γ *be as in Lemma 3.5. Let* T *be a finite set of torsion free elements of* Λ. *Then* $\Lambda \curvearrowright L_\Lambda/(\mathcal{R}_T \cap (L_\Lambda \times L_\Lambda))$ *is a locally boundary action.*

PROOF. Let s be a torsion free element of Λ whose fixed points are not equal to $g.t^{\pm\infty}$ for any $g \in \Lambda$ and $t \in T$. Then $\pi(s^{+\infty}) \neq \pi(s^{-\infty})$. Hence, on the set $\pi(L_\Lambda \setminus \{s^{+\infty}\})$, the sequence $(s^n.x)_{n=1}^\infty$ converges to $\pi(s^{+\infty})$ uniformly on compact subsets. Thus for any neighborhood U of $\pi(s^{+\infty})$ whose closure does not contain $\pi(s^{-\infty})$, there is $n \in \mathbb{N}$ such that $s^n(\mathrm{cl}(U)) \subsetneq U$. From the minimality of $\Lambda \curvearrowright L_\Lambda$, now it is easy to conclude that the action is a locally boundary action. □

THEOREM 3.9. *Let* Λ *be a subgroup of a hyperbolic group* Γ. *Then there is a decreasing sequence of nuclear* C*-*subalgebras of* $C(L_\Lambda) \rtimes_r \Lambda$ *whose intersection is equal to* $C^*_r(\Lambda)$. *Moreover, if* Λ *is ICC, then we can find such a sequence with the terms Kirchberg algebras in the UCT class.*

PROOF. Let $(\mathfrak{F}_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of torsion free elements of Λ whose union contains all torsion free elements. Define $\mathcal{R}_n := \mathcal{R}_{\mathfrak{F}_n} \cap (L_\Lambda \times L_\Lambda)$ for each n. Note that by Lemma 3.4, each quotient space $L_\Lambda/\mathcal{R}_n$ is Hausdorff. Put $A_n := C(L_\Lambda/\mathcal{R}_n) \rtimes_r \Lambda$. Then by Lemma 3.2, each A_n is nuclear. Moreover, by Lemma 3.7, we have $\bigcap_{n=1}^\infty C(L_\Lambda/\mathcal{R}_n) = \mathbb{C}$. Since every hyperbolic group is weakly amenable [**37**], we have the equality

$$\bigcap_{n=1}^{\infty} A_n = C^*_r(\Lambda).$$

When Λ is ICC, a similar proof to that of Lemma 3.5 shows the topological freeness of $\Lambda \curvearrowright L_\Lambda/\mathcal{R}_n$. Since each action $\Lambda \curvearrowright L_\Lambda/\mathcal{R}_n$ is a locally boundary action, Theorem 9 of [**30**] yields that each A_n is a Kirchberg algebra. □

4. Extensions of free group C*-algebras by nuclear C*-algebras

In this section, we prove Theorem B. We first consider the case d is finite. We deal the case $d = \infty$ in the end of this section. Denote by S the set of all canonical generators of $\mathbb{F}_d$. We denote by $|\cdot|$ the length function on $\mathbb{F}_d$ determined by S. To prove Theorem B, first we compute the K-groups of the crossed product $C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d$.

We always use the following standard picture of the Gromov boundary $\partial\mathbb{F}_d$.

$$\partial\mathbb{F}_d := \left\{ (x_n)_{n=1}^{\infty} \in \prod_{n\in\mathbb{N}} S \sqcup S^{-1} : x_n \neq x_{n+1}^{-1} \text{ for all } n \in \mathbb{N} \right\}$$

equipped with the relative product topology. For $w \in \mathbb{F}_d$, we denote by $p[w]$ the characteristic function of the clopen set

$$\left\{ (x_n)_{n=1}^{\infty} \in \partial\mathbb{F}_d : x_1 \cdots x_{|w|} = w \right\}$$

and set $q[w] := p[w] + p[w^{-1}]$. Throughout this section, we identify $C(\partial\mathbb{F}_d/\mathcal{R}_S)$ with the C*-subalgebra of $C(\partial\mathbb{F}_d)$ in the canonical way. Under this identification, it is not difficult to check that for $s \in S$, $q[s]$ is contained in $C(\partial\mathbb{F}_d/\mathcal{R}_S)$. We denote the action $\mathbb{F}_d \curvearrowright C(\partial\mathbb{F}_d)$ by $w.f$ for $w \in \mathbb{F}_d$ and $f \in C(\partial\mathbb{F}_d)$.

LEMMA 4.1. *The* C*-*algebra* $C(\partial\mathbb{F}_d/\mathcal{R}_S)$ *is generated by the set*

$$\mathcal{P} := \{w.q[s] : w \in \mathbb{F}_d, s \in S\}.$$

In particular, the space $\partial\mathbb{F}_d/\mathcal{R}_S$ *is homeomorphic to the Cantor set.*

PROOF. By the Stone–Weierstrass theorem, it suffices to show that the set $\mathcal{P}$ separates the points of $\partial\mathbb{F}_d/\mathcal{R}_S$. Let $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ be two elements in $\partial\mathbb{F}_d$ satisfying $(x, y) \notin \mathcal{R}_S$. If $x \notin \{ws^{+\infty} : w \in \mathbb{F}_d, s \in S \sqcup S^{-1}\}$, then take $n \in \mathbb{N}$ with $x_n \neq y_n$. Let m be the smallest integer greater than n satisfying $x_m \neq x_n$ (which exists by assumption). Then the projection $(x_1 \cdots x_{m-1}).(q[x_m])$ separates x and y. Next consider the case $x = zs^{+\infty}, y = wt^{+\infty}$, where $s, t \in S \sqcup S^{-1}$ and z, w are elements of $\mathbb{F}_d$ whose last alphabets are not equal to $s^{\pm 1}$, $t^{\pm 1}$, respectively. Assume $|z| \geq |w|$. Note that the equality $z = w$ implies $s \neq t^{\pm 1}$ by assumption. Hence the projection $z.q[s]$ separates x and y. Thus $\mathcal{P}$ satisfies the required condition.

The last assertion now follows from the following fact. A topological space is homeomorphic to the Cantor set if and only if it is compact, metrizable, totally disconnected, and does not have an isolated point. □

LEMMA 4.2. *The* K_0*-group of* $C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d$ *is generated by* $\{[q[s]]_0 : s \in S\}$.

PROOF. By Lemma 4.1 and the Pimsner–Voiculescu exact sequence [**40**], the K_0-group is generated by the elements represented by a projection in $C(\partial\mathbb{F}_d/\mathcal{R}_S)$. Let r be a projection in $C(\partial\mathbb{F}_d/\mathcal{R}_S)$. Then r can be presented as a sum $\sum_{w\in F} p[w]$, where F is a subset of $\mathbb{F}_d \setminus \{e\}$ whose elements have the same lengths. Let w be an element of $\mathbb{F}_d$ whose reduced form is $s_1^{n(1)} \cdots s_k^{n(k)}$, where $s_i \in S \sqcup S^{-1}$, $n(i) \in \mathbb{N}$, and $s_i \neq s_{i+1}$ for all i. We define $\hat{w} \in \mathbb{F}_d$ by $s_1^{n(1)} \cdots s_{k-1}^{n(k-1)} s_k^{-n(k)}$. We will show that $w \in F$ implies $\hat{w} \in F$. Indeed, if $w \in F$, then $r(ws_k^{+\infty}) = 1$. Hence we must have $r(ws_k^{-\infty}) = 1$. This implies $\hat{w} \in F$ as desired. Since $w \neq \hat{w}$ and $[p[w] + p[\hat{w}]]_0 = [q[s_k^{n(k)}]]_0$, it suffices to show that for $s \in S$ and $n \in \mathbb{N}$, the element $[q[s^n]]_0$ is contained in the subgroup generated by $[q[s]]_0, s \in S$. This follows from the equations

$$q[s^2] = s.q[s] + s^{-1}.q[s] + q[s] - 2$$

and

$$q[s^k] = s.q[s^{k-1}] + s^{-1}.q[s^{k-1}] - q[s^{k-2}]$$

for $s \in S$ and $k > 2$. □

We denote the triplet $(K_0, [1]_0, K_1)$ by K_*.

THEOREM 4.3. *The* $K_*(C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d)$ *is isomorphic to* $(\mathbb{Z}^d, (1, 1, \ldots, 1), \mathbb{Z}^d)$.

PROOF. We first compute the pair $(K_0, [1]_0)$. By Lemma 4.2, it suffices to show the linear independence of the family $([q[s]]_0)_{s\in S}$. Let

$$\eta\colon C(\partial\mathbb{F}_d, \mathbb{Z})^{\oplus S} \to C(\partial\mathbb{F}_d, \mathbb{Z})$$

be the additive map defined by $(f_s)_{s\in S} \mapsto \sum_{s\in S}(f_s - s.f_s)$ and denote by τ the restriction of η to $C(\partial\mathbb{F}_d/\mathcal{R}_S, \mathbb{Z})^{\oplus S}$. Then the Pimsner–Voiculescu exact sequence [**40**] shows that the canonical map

$$C(\partial\mathbb{F}_d/\mathcal{R}_S, \mathbb{Z}) \to K_0(C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d)$$

is surjective and its kernel is equal to $\mathrm{im}(\tau)$. Hence it suffices to show that $\mathrm{im}(\tau)$ does not contain a nontrivial linear combination of the projections $q[s], s \in S$. The isomorphisms $\ker(\eta) \cong K_1(C(\partial\mathbb{F}_d) \rtimes_r \mathbb{F}_d) \cong \mathbb{Z}^d$ (see [**11, 40, 50**]) show that $\ker(\eta) = \{(f_s)_{s\in S} : \text{each } f_s \text{ is constant}\}$. Now let $r = \sum_{s\in S} n(s)q[s]$ be a nontrivial linear combination of $q[s]$'s. If $\sum_{s\in S} n(s) \not\equiv 0 \mod (d-1)$, then $r \notin \mathrm{im}(\eta)$ by [**11, 50**]. If $\sum_{s\in S} n(s) = (d-1)m$ for some $m \in \mathbb{Z}$, then $\sum_{s\in S} n(s)q[s] = \eta((g_s)_{s\in S})$, where $g_s := (n(s) - m)p[s^{-1}]$ for $s \in S$. Hence $\eta^{-1}(\{r\}) = (g_s)_{s\in S} + \ker(\eta)$, which does not intersect with $C(\partial\mathbb{F}_d/\mathcal{R}_S, \mathbb{Z})^{\oplus S}$. Thus we have $r \notin \mathrm{im}(\tau)$ in either case.

The isomorphism of the K_1-group follows from the Pimsner–Voiculescu exact sequence [**40**] and the equality $\ker(\tau) = \ker(\eta)$. □

PROOF OF THEOREM B:THE CASE d IS FINITE. Let A be a stable separable nuclear C*-algebra. Let $\iota\colon C_r^*(\mathbb{F}_d) \to C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d$ be the inclusion map. Then the above computation yields that the homomorphism $\iota_{*,0}$ has a left inverse and the homomorphism $\iota_{*,1}$ is an isomorphism. Consequently, the homomorphism

$$\mathrm{Hom}(K_i(C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d), K_{1-i}(A)) \to \mathrm{Hom}(K_i(C_r^*(\mathbb{F}_d)), K_{1-i}(A))$$

induced from ι is surjective for $i = 0, 1$. Recall that both $C_r^*(\mathbb{F}_d)$ and $C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d$ satisfy the universal coefficient theorem [**48**, Corollary 7.2]. Since $K_i(C_r^*(\mathbb{F}_d))$ is a free $\mathbb{Z}$-module for $i = 0, 1$, the universal coefficient theorem [**48**] yields that the canonical homomorphism

$$\mathrm{Ext}(C_r^*(\mathbb{F}_d), A)^{-1} \to \bigoplus_{i=0,1} \mathrm{Hom}(K_i(C_r^*(\mathbb{F}_d)), K_{1-i}(A))$$

is an isomorphism. Combining these facts, we see that the homomorphism

$$\iota^*\colon \mathrm{Ext}(C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d, A) \to \mathrm{Ext}(C_r^*(\mathbb{F}_d), A)^{-1}$$

induced from ι is surjective.

Now let B be the exact C*-algebra obtained by an extension σ of $C_r^*(\mathbb{F}_d)$ by A which is either absorbing or unital absorbing. Since A is nuclear and $C_r^*(\mathbb{F}_d)$ is exact, the Effros–Haagerup lifting theorem [**16**, Theorem B and Prop. 5.5] shows that $[\sigma] \in \mathrm{Ext}(C_r^*(\mathbb{F}_d), A)$ is invertible in the semigroup $\mathrm{Ext}(C_r^*(\mathbb{F}_d), A)$. Note that in either case, the direct sum $\sigma \oplus 0$ is absorbing. Thus, by the surjectivity of ι^*, the direct sum $\sigma \oplus 0$ extends to a $*$-homomorphism $\varphi\colon C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d \to \mathbb{M}_2(Q(A))$. Then, since $\varphi(1) = \sigma(1) \oplus 0 \leq 1 \oplus 0$, the map

$$\tilde{\sigma}\colon C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d \ni x \mapsto \varphi(x)_{1,1} \in Q(A)$$

defines a $*$-homomorphism which extends σ.

We next show that B is realized as a decreasing intersection of separable nuclear C*-algebras. Take a decreasing sequence $(A_n)_{n=1}^\infty$ of nuclear C*-subalgebras of $C(\partial\mathbb{F}_d/\mathcal{R}_S) \rtimes_r \mathbb{F}_d$ whose decreasing intersection is equal to $C_r^*(\mathbb{F}_d)$. Put $B_n := \tilde{\sigma}^{-1}(\tilde{\sigma}(A_n))$ for each n. Then, since nuclearity is preserved under taking the extension, each B_n is nuclear. Moreover, we have the equality

$$\bigcap_{n=1}^\infty B_n = \bigcap_{n=1}^\infty \tilde{\sigma}^{-1}(\tilde{\sigma}(A_n)) = B.$$

For the unital case, the rest of the proof is similarly done to the proof of Main Theorem. For the non-unital case, let $(B_n)_{n=1}^{\infty}$ be a decreasing sequence of separable nuclear C*-algebras whose intersection is B. Denote by 1 the unit of the unitization $\widetilde{B_1}$ of B_1. Define C*-subalgebras C_n of $\widetilde{B_1} \oplus \ell^{\infty}(\mathbb{N})$ by $C_n := \mathrm{C}^*(B_n, \{1 \oplus p_k : k \in \mathbb{N}\})$, where p_k is the characteristic function of the set $\{l \in \mathbb{N} : l \geq k\}$. Set $D_n := C_n \otimes \bigotimes_{k=n}^{\infty} C(X)$ for each n, where X is a compact metrizable space consisting at least two points. Take a faithful state ϕ on C_1 and a faithful measure μ on X. Then define a state φ on D_1 by $\varphi := \phi \otimes \bigotimes_{k=1}^{\infty} \mu$. Now take a pure state ψ on $\mathbb{M}_2$ and define $E_n := q_n \left((D_n, \varphi|_{D_n}) * (\ast_{k=n}^{\infty}(\mathbb{M}_2, \psi))\right) q_n$, where $q_n := (1 \oplus p_n) \in D_n$. Then, being as a corner of a simple unital separable nuclear C*-algebra, each E_n also has these properties. Now put $F_n := E_n \otimes \bigotimes_{k=n}^{\infty} \mathcal{O}_2$. Then each F_n is isomorphic to $\mathcal{O}_2$ [**29**]. Now it is easy to see that the intersection of the decreasing sequence $(F_n)_{n=1}^{\infty}$ is isomorphic to B.

Finally, when $A = \mathbb{K}$, by Voiculescu's theorem [**57**], any essential unital extension is unital absorbing and any essential non-unital extension is absorbing. Moreover, since $\mathrm{C}^*_r(\mathbb{F}_d)$ is simple [**41**], the only non-essential extension is the zero extension $\mathrm{C}^*_r(\mathbb{F}_d) \oplus \mathbb{K}$. In this case, the claim follows from the above argument. □

We remark that in the proof of Main Theorem and the above argument, the following is implicitly proved.

PROPOSITION 4.4. *Let A be a (possibly non-unital) C*-algebra which is realized as a decreasing intersection of separable nuclear C*-algebras. Then it is realized as a decreasing intersection of isomorphs of the Cuntz algebra $\mathcal{O}_2$.*

PROOF OF THEOREM B: THE CASE $d = \infty$. Let Λ be the commutator subgroup of $\mathbb{F}_2$. Then Λ is isomorphic to $\mathbb{F}_{\infty}$. Therefore we only need to show the claim for Λ. Let S be the canonical generator of $\mathbb{F}_2$ and consider the restriction α of the action $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2/\mathcal{R}_S$ to Λ. Let

$$\iota \colon \mathrm{C}^*_r(\Lambda) \to C(\partial\mathbb{F}_2/\mathcal{R}_S) \rtimes_r \Lambda$$

denote the inclusion. We will show that the induced homomorphism ι_* on the K-theory is left invertible. To show the claim for the K_0-group, consider the following inclusion map

$$\tilde{\iota} \colon \mathrm{C}^*_r(\Lambda) \to C(\partial\mathbb{F}_2/\mathcal{R}_S) \rtimes_r \mathbb{F}_2.$$

Then by Theorem 4.3, the homomorphism $\tilde{\iota}_{*,0}$ is left invertible. This proves the left invertibility of $\iota_{*,0}$.

To show the claim for the K_1-group, first take a free basis A of $\Lambda \cong \mathbb{F}_{\infty}$. Define the homomorphism

$$\eta \colon C(\partial\mathbb{F}_2/\mathcal{R}_S, \mathbb{Z})^{\oplus A} \to C(\partial\mathbb{F}_2/\mathcal{R}_S, \mathbb{Z})$$

by $\eta((f_a)_{a \in A}) := \sum_{a \in A}(f_a - a(f_a))$. Then by the Pimsner–Voiculescu six term exact sequence, we obtain an isomorphism

$$K_1(C(\partial\mathbb{F}_2/\mathcal{R}_S) \rtimes_r \Lambda) \cong \ker(\eta)$$

which maps $[u_a]_1$ to $(\delta_{a,b}1)_{b \in A}$ for each $a \in A$. Since the subgroup generated by 1 is a direct summand of the group $C(\partial\mathbb{F}_2/\mathcal{R}_S, \mathbb{Z})$, the homomorphism $\mathbb{Z}^{\oplus A} \to \ker(\eta)$ given by $\delta_a \mapsto (\delta_{a,b}1)_{b \in A}$ is left invertible. Consequently, the homomorphism $\iota_{*,1}$ is left invertible. Now the rest of the proof is similarly done to the case d is finite. □

By Theorem 4.1 of [**39**], for unital Kirchberg algebras in the UCT class, every homomorphism between the triplets K_* is implemented by a unital $*$-homomorphism. Combining this fact with our results in this section, we obtain the following consequence.

COROLLARY 4.5. *For any countable free group $\mathbb{F}$, there is a unital embedding of $\mathrm{C}^*_r(\mathbb{F})$ into a Kirchberg algebra which implements the KK-equivalence.*

CHAPTER 5

Minimal ambient nuclear C*-algebras

A deep theorem of Kirchberg–Phillips [**29**] states that every separable exact C*-algebra has an ambient nuclear C*-algebra. (In fact, one can choose it to be isomorphic to the Cuntz algebra $\mathcal{O}_2$.) When we consider reduced group C*-algebras, thanks to Ozawa's result [**35**], we have more natural ambient nuclear C*-algebras, namely, the reduced crossed products of amenable dynamical systems. Nuclear ambient C*-algebras play important roles in theory of both C*- and von Neumann algebras. We refer the reader to the books [**6**] and [**45**] for details. In this chapter, based on (new) results on topological dynamical systems, we give the first example of a minimal ambient nuclear C*-algebra of a non-nuclear C*-algebra. In fact, we have a stronger result: our examples of minimal ambient nuclear C*-algebras have no proper intermediate C*-algebras.

Note that as we have seen in Chapter 4, in contrast to injectivity of von Neumann algebras, nuclearity of C*-algebras is not preserved under taking the decreasing intersection. We also note that the increasing union of non-nuclear C*-algebras can be nuclear. See Remark 2.10 for the detail. Thus there is no obvious way to provide a minimal ambient nuclear C*-algebra.

In 1975, Powers [**41**] invented a celebrated method to study structures of the reduced group C*-algebras. His idea has been applied to more general situations, particularly for reduced crossed products, and to more general groups, by many hands. See [**14**] for instance. We use his technique with certain properties of dynamical systems to obtain the following main theorem of this chapter.

We say that a group is an infinite free product group if it is a free product of infinitely many nontrivial groups. Throughout this chapter, groups are supposed to be countable.

Main Theorem (Corollary 1.4, Theorem 2.9). *Let Γ be an infinite free product group with the AP ([**24**]) (or equivalently, each free product component has the AP). Then there is an amenable action of Γ on the Cantor set X with the following property. There is no proper intermediate C*-algebra of the inclusion $\mathrm{C}^*_r(\Gamma) \subset C(X) \rtimes_r \Gamma$. In particular $C(X) \rtimes_r \Gamma$ is a minimal ambient nuclear C*-algebra of the non-nuclear C*-algebra $\mathrm{C}^*_r(\Gamma)$.*

In Main Theorem, we need the AP to determine when a given element of the reduced crossed product sits in the reduced group C*-algebra. Cf. [**58**] and Proposition 2.4 in Chapter 4. By modifying Main Theorem, we also provide examples of non-nuclear C*-subalgebras of $\mathcal{O}_2$ with no proper intermediate C*-algebras (Theorem 3.1).

In theory of both measurable and topological dynamical systems, the Baire category theorem is a powerful tool to produce an example with a nice property. We follow this strategy to construct dynamical systems as in Main Theorem. To apply the Baire category theorem, we again work on the space $\mathcal{S}(\Gamma, X)$ of all actions $\Gamma \curvearrowright X$ introduced in Chapter 3.

For the simplicity of notion, here we introduce a few terminology. We say that a property of topological dynamical systems is open, G_δ, dense, G_δ-dense, respectively when the subset of $\mathcal{S}(\Gamma, X)$ consisting of actions with this property has the corresponding property. We say that a property is generic when the corresponding set contains a G_δ-dense subset of $\mathcal{S}(\Gamma, X)$. Note that thanks to the Baire category theorem, the intersection of countably many G_δ-dense properties is again G_δ-dense, and similarly for genericity. Although some results (e.g., genericity of amenability, minimality, primeness, for infinite free product groups) can be extended to more general spaces by minor modifications, we concentrate on the Cantor set. This is enough for Main Theorem.

1. Some generic properties of Cantor systems

In this section, we summarize generic properties of Cantor systems. From now on we denote by X the Cantor set. We recall that the Cantor set is the topological space characterized (up to homeomorphism) by the following four properties: compactness, total disconnectedness, metrizability, and perfectness (i.e., no isolated points).

LEMMA 1.1. *For any group Γ, the following properties are G_δ in $\mathcal{S}(\Gamma, X)$.*
(1) *Freeness.*
(2) *Amenability.*

PROOF. The first claim is well-known. For completeness, we include a proof.
(1): For $s \in \Gamma$, set $V_s := \{\alpha \in \mathcal{S}(\Gamma, X) : \alpha_s(x) \neq x \text{ for all } x \in X\}$. By the compactness of X, each V_s is open. The G_δ-set $\bigcap_{s\in\Gamma\setminus\{e\}} V_s$ consists of all free Cantor systems.
(2): For each finite subset S of Γ, we say that an action $\alpha\colon \Gamma \curvearrowright X$ has property $\mathcal{A}_S$ if it admits a continuous map $\mu\colon X \to \mathrm{Prob}(\Gamma)$ satisfying

$$\|s.\mu^x - \mu^{s.x}\|_1 < \frac{1}{|S|}$$

for all $s \in S$ and $x \in X$. Let $\alpha \in \mathcal{S}(\Gamma, X)$ be given and suppose we have a continuous map μ that witnesses $\mathcal{A}_S$ of α. Then, by the continuity of μ, it guarantees $\mathcal{A}_S$ for any β sufficiently close to α. This shows that $\mathcal{A}_S$ is open. Now obviously, the intersection $\bigwedge_S \mathcal{A}_S$ is equivalent to amenability, where S runs over finite subsets of Γ. □

The following simple lemma is crucial to show the genericity of some properties.

LEMMA 1.2. *Let $\alpha\colon \Gamma \curvearrowright X$ be a given Cantor system. Then the set of extensions of α is dense in $\mathcal{S}(\Gamma, X)$.*

PROOF. Let us regard the Cantor set X as the direct product of infinitely many copies Y of the Cantor set: $X = Y^{\mathbb{N}}$. We regard α as a dynamical system on Y via a homeomorphism $X \cong Y$. For each $N \in \mathbb{N}$, define a map $\sigma_N : \mathbb{N} \to \mathbb{N}$ by

$$\sigma_N(n) := \begin{cases} n & \text{when } n < N, \\ n+1 & \text{when } n \geq N. \end{cases}$$

Now let $\beta \in \mathcal{S}(\Gamma, X)$ be given. Let $\gamma\colon \Gamma \curvearrowright Y \times X$ be the diagonal action of α and β. For each $N \in \mathbb{N}$, define a homeomorphism $\varphi_N\colon X \to Y \times X$ by $\varphi_N(x) := (x_N, (x_{\sigma_N(n)})_{n\in\mathbb{N}})$. Put $\beta^{(N)} := \varphi_N^{-1} \circ \gamma \circ \varphi_N \in \mathcal{S}(\Gamma, X)$. Then for each $N \in \mathbb{N}$, the projection from X onto the Nth coordinate gives a factor map of $\beta^{(N)}$ onto α. Moreover the sequence $(\beta^{(n)})_{n=1}^\infty$ converges to β. Since β is arbitrary, this proves the claim. □

The next lemma is well-known. For completeness, we give a proof.

LEMMA 1.3. *Every group admits a free Cantor system. Also, every exact group admits an amenable Cantor system.*

PROOF. Let Γ be a group. We first show that the left translation action of Γ on its Stone–Čech compactification $\beta\Gamma$ is free. Let $s \in \Gamma \setminus \{e\}$ be given. Put $\Lambda := \langle s \rangle$. Take a Λ-equivariant map $\Gamma \to \Lambda$ where Λ acts on both groups by the left multiplication. This extends to the Λ-equivariant quotient map $\beta\Gamma \to \beta\Lambda$. By universality, $\beta\Lambda$ factors onto every minimal dynamical system of Λ (on a compact space). Since any cyclic group admits a minimal free action on a compact space, this shows that s has no fixed points in $\beta\Gamma$.

Let $(A_\mu)_{\mu\in M}$ be the increasing net of Γ-invariant unital C*-subalgebras of $\ell^\infty(\Gamma) = C(\beta\Gamma)$ generated by countably many projections. Note that $\bigcup_{\mu\in M} A_\mu = \ell^\infty(\Gamma)$. Let X_μ denote the spectrum of A_μ. Obviously, each X_μ is totally disconnected and metrizable. Let $\alpha_\mu\colon \Gamma \curvearrowright X_\mu$ be the action induced from the action $\Gamma \curvearrowright A_\mu$. By the freeness of $\Gamma \curvearrowright \beta\Gamma$, for sufficiently large μ, the α_μ must be free. When Γ is exact, then as stated in Theorem 5.1.7 of [**6**], for sufficiently large μ, the α_μ must be amenable. Hence for sufficiently large μ, the diagonal action of α_μ and the trivial Cantor system gives the desired action. □

We now summarize the results of this section.

Corollary 1.4. *For any group Γ, freeness is a G_δ-dense property in $\mathcal{S}(\Gamma, X)$. Moreover, when Γ is exact, then amenability is also a G_δ-dense property in $\mathcal{S}(\Gamma, X)$.*

Proof. Since both freeness and amenability are inherited to extensions, it follows from Lemmas 1.1 through 1.3. □

2. Construction of dynamical systems and proof of Main Theorem

In this section, we prove Main Theorem. Let $(\Gamma_i)_{i=1}^\infty$ be a sequence of nontrivial groups and let $\Gamma := \ast_{i=1}^\infty \Gamma_i$ be their free product. By replacing Γ_i by $\Gamma_{2i-1} \ast \Gamma_{2i}$ for all i if necessary, in the rest of this chapter, we assume that each free product component Γ_i contains a torsion-free element. We start with the following elementary lemmas. We remark that in the case that Γ is the free group $\mathbb{F}_\infty$, we do not need these lemmas.

Lemma 2.1. *Let Λ be a group and Υ be its subgroup. Then for any minimal dynamical system α of Υ on a compact metrizable space, there is a Cantor system of Λ whose restriction on Υ is an extension of α.*

Proof. Let $\alpha\colon \Upsilon \curvearrowright Y$ be an action as in the statement. Fix an element $y \in Y$. Then the map $\Upsilon \to Y$ defined by $s \mapsto s.y$ extends to a factor map $\beta\Upsilon \to Y$. This induces an Υ-equivariant unital embedding of $C(Y)$ into $\ell^\infty(\Upsilon)$. By the right coset decomposition of Λ with respect to Υ, we have an Υ-equivariant unital embedding of $\ell^\infty(\Upsilon)$ into $\ell^\infty(\Lambda)$. We identify $C(Y)$ with a unital Υ-invariant C*-subalgebra of $\ell^\infty(\Lambda)$ via the composite of these two embeddings. Take a Λ-invariant C*-subalgebra A of $\ell^\infty(\Lambda)$ which contains $C(Y)$ and is generated by countably many projections. Let Z be the spectrum of A. Note that Z is metrizable and totally disconnected. Let $\beta\colon \Lambda \curvearrowright Z$ be the action induced from the action $\Lambda \curvearrowright A$. Since A contains $C(Y)$ as a unital C*-subalgebra, the restriction of β on Υ is an extension of α. Now the diagonal action of β with the trivial Cantor system gives the desired Cantor system. □

Lemma 2.2. *Let Λ be a group. Let s be a torsion-free element of Λ. Then for any finite family $\mathcal{U} = \{U_1, \ldots, U_n\}$ of pairwise disjoint proper clopen subsets of X, there is a Cantor system $\alpha\colon \Lambda \curvearrowright X$ with $sU_i = U_{i+1}$ for all i. Here and below, we put $U_{n+1} := U_1$ for convenience.*

Proof. By Lemma 2.1, there is a Cantor system $\alpha\colon \Lambda \curvearrowright X$ whose restriction on $\langle s\rangle$ factors a transitive action on the set $\{1, \ldots, n\}$. For such α, there is a partition $\{V_1, \ldots, V_n\}$ of X by clopen subsets satisfying $sV_i = V_{i+1}$ for all i. Set $I := \{0, 1\}$ if $\bigcup_{i=1}^n U_i \neq X$. Otherwise we set $I := \{0\}$. Then define a new action $\beta\colon \Lambda \curvearrowright X \times I$ by

$$\beta_t(x, j) := \begin{cases} (\alpha_t(x), 0) & \text{when } j = 0, \\ (x, 1) & \text{otherwise.} \end{cases}$$

Since nonempty clopen subsets of the Cantor set are mutually homeomorphic, there is a homeomorphism $\varphi\colon X \times I \to X$ which maps $V_i \times \{0\}$ onto U_i for each i. For such φ, the conjugate $\varphi \circ \beta \circ \varphi^{-1}$ gives the desired Cantor system. □

We next introduce a property of Cantor systems which is one of the key of the proof of Main Theorem and show that this property is G_δ-dense for infinite free product groups.

PROPOSITION 2.3. *Let $\Gamma = *_{i=1}^{\infty}\Gamma_i$ be an infinite free product group. Then the following property $\mathcal{R}$ of Cantor systems is G_δ-dense in $\mathcal{S}(\Gamma, X)$.*

($\mathcal{R}$): *For any finite family $\mathcal{U} = \{U_1, \ldots, U_n\}$ of mutually disjoint proper clopen subsets of X, there are infinitely many $i \in \mathbb{N}$ satisfying the following condition. The group Γ_i contains a torsion-free element s satisfying $sU_j = U_{j+1}$ for all j.*

Here we put $U_{n+1} := U_1$ as before.

PROOF. For any $i \in \mathbb{N}$ and a family $\mathcal{U}$ as stated, we say that an element $\alpha \in \mathcal{S}(\Gamma, X)$ has property $\mathcal{R}(i,\mathcal{U})$ if it satisfies the following condition. There are $k \geq i$ and a torsion-free element $s \in \Gamma_k$ satisfying $sU_j = U_{j+1}$ for all j. Then observe that for any two clopen subsets U and V of X, the set

$$\{\varphi \in \mathrm{Homeo}(X) : \varphi(U) = V\}$$

is clopen in $\mathrm{Homeo}(X)$. This shows that property $\mathcal{R}(i,\mathcal{U})$ is open in $\mathcal{S}(\Gamma, X)$.

To show the density of $\mathcal{R}(i,\mathcal{U})$, for each $m \in \mathbb{N}$, take a Cantor system $\varphi_m\colon \Gamma_m \curvearrowright X$ as in Lemma 2.2. Let $\alpha \in \mathcal{S}(\Gamma, X)$ be given. Then, for each $m \in \mathbb{N}$, we define $\alpha^{(m)} \in \mathcal{S}(\Gamma, X)$ as follows.

$$\alpha^{(m)}|_{\Gamma_k} := \begin{cases} \alpha|_{\Gamma_k} & \text{for } k < m, \\ \varphi_k & \text{for } k \geq m. \end{cases}$$

Then each $\alpha^{(m)}$ satisfies property $\mathcal{R}(i,\mathcal{U})$ and the sequence $(\alpha^{(m)})_{m=1}^{\infty}$ converges to α. This proves the density of $\mathcal{R}(i,\mathcal{U})$.

Now observe that property $\mathcal{R}$ is equivalent to the intersection $\bigwedge_{i,\mathcal{U}} \mathcal{R}(i,\mathcal{U})$. Since there are only countably many clopen subsets in X, the intersection is taken over a countable family. Now the Baire category theorem completes the proof. □

REMARK 2.4. It is not hard to check that an action with $\mathcal{R}$ is a boundary in Furstenberg's sense (see Definition 3.8 of [**26**] for the definition). Also, by Theorem 5 of [**30**], property $\mathcal{R}$ with topological freeness implies the pure infiniteness of the reduced crossed product.

REMARK 2.5. Since every infinite group admits a weak mixing Cantor system of all orders (e.g., the Bernoulli shift), in a similar way to the proof of Proposition 2.3, it can be shown that weak mixing of all orders is G_δ-dense for infinite free product groups. Here recall that a topological dynamical system α is said to be weak mixing of all orders if for any $n \in \mathbb{N}$, the diagonal action of n copies of α has a dense orbit. Similarly, it can also be shown that the set of disjoint pairs $(\alpha, \beta) \in \mathcal{S}(\Gamma, X)^2$ is generic in $\mathcal{S}(\Gamma, X)^2$. Here recall that two minimal dynamical systems are disjoint if and only if their diagonal action is minimal.

REMARK 2.6. Consider the case $\Gamma = \mathbb{F}_\infty$. Then by the Pimsner–Voiculescu exact sequence [**40**], property $\mathcal{R}$ implies $K_0(C(X) \rtimes_r \mathbb{F}_\infty) = 0$. We also have $K_1(C(X) \rtimes_r \mathbb{F}_\infty) \cong \mathbb{Z}^{\oplus\infty}$ for any Cantor system of $\mathbb{F}_\infty$. This with the classification theorem of Kirchberg–Phillips [**28**], [**39**] shows that generically the crossed products give only a single C*-algebra. However, as we have seen in Chapter 2, there are continuously many Kirchberg algebras which are realized as the reduced crossed product of an amenable minimal free Cantor system of $\mathbb{F}_\infty$.

The next proposition says that property $\mathcal{R}$ implies the non-existence of nontrivial Γ-invariant closed subspace of $C(X)$. This result may be of independent interest.

Proposition 2.7. *Assume $\alpha \in \mathcal{S}(\Gamma, X)$ satisfies $\mathcal{R}$. Then there is no Γ-invariant closed subspace of $C(X)$ other than 0, $\mathbb{C}$, or $C(X)$. In particular $\mathcal{R}$ implies primeness.*

Proof. Let V be a closed Γ-invariant subspace of $C(X)$ other than 0 or $\mathbb{C}$. We first show that V contains $\mathbb{C}$. Take a nonzero function $f \in V$. Then for any $\epsilon > 0$, there is a partition $\mathcal{U} := \{U_1, \ldots, U_n\}$ of X by proper clopen sets and complex numbers $c_1, \ldots, c_n$ with $|c_1| = \|f\|$ such that with $g := \sum_{i=1}^n c_i \chi_{U_i}$, we have $\|f - g\| < \epsilon$. Put $c := \frac{1}{n}\sum_{i=1}^n c_i$. By replacing $\mathcal{U}$ by dividing U_1 into sufficiently many clopen subsets and replacing the sequence $(c_i)_i$ suitably, we may assume $|c| \geq \|f\|/2$. By property $\mathcal{R}$, we can take $s \in \Gamma$ with $sU_i = U_{i+1}$ for all i. We then have $\sum_{i=1}^n s^i g s^{-i} = \sum_{i=1}^n c_i$. This yields the inequality

$$\|\frac{1}{n}\sum_{i=1}^n s^i f s^{-i} - c\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary and $|c| \geq \|f\|/2$, we obtain $\mathbb{C} \subset V$.

From this, we can choose a nonzero function $f \in V$ with $0 \in f(X)$. For any $\epsilon > 0$, take a partition $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ of X by proper clopen sets and complex numbers $c_1, \ldots, c_n$ such that with $g := \sum_{i=1}^n c_i \chi_{U_i}$, we have $\|f - g\| < \epsilon$. Put $c := \frac{1}{n}\sum_{i=1}^n c_i$. As before, we may assume $|c| \geq \|f\|/2$. By using property $\mathcal{R}$ to the family $\{U_1, \ldots, U_n\}$, we can take $s \in \Gamma$ satisfying $sU_0 = U_0$ and $sU_i = U_{i+1}$ for $1 \leq i < n$. Then we have $\frac{1}{n}\sum_{i=1}^n s^i g s^{-i} = c\chi_{X \setminus U_0}$. Now let U be any proper clopen subset of X. Take $t \in \Gamma$ with $t(X \setminus U_0) = U$. (To find such t, use property $\mathcal{R}$ twice.) We then have

$$t(\frac{1}{n}\sum_{i=1}^n s^i g s^{-i})t^{-1} = ct(\chi_{X\setminus U_0})t^{-1} = c\chi_U.$$

This shows the inequality

$$\|(\frac{1}{n}\sum_{i=1}^n ts^i f s^{-i} t^{-1}) - c\chi_U\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves $\chi_U \in V$. Since U is arbitrary, we obtain $V = C(X)$. □

We need the following restricted version of the Powers property for free product groups. Although the proof is essentially contained in [**41**], for completeness, we include a proof.

Lemma 2.8 (Compare with Lemma 5 of [**41**] and Lemma 5 of [**14**]). *Let Λ_1, Λ_2 be groups and set $\Lambda := \Lambda_1 * \Lambda_2$. Let $s \in \Lambda_1$, $t \in \Lambda_2$ be torsion-free elements. Then for any finite subset F of $\Lambda \setminus \{e\}$, there are a partition $\Lambda = D \sqcup E$ of Λ and elements $u_1, u_2, u_3 \in \langle s, t\rangle$ with the following properties.*

(1) *$fD \cap D = \emptyset$ for all $f \in F$.*
(2) *$u_j E \cap u_k E = \emptyset$ for any two distinct $j, k \in \{1, 2, 3\}$.*

Proof. Let $F \subset \Lambda \setminus \{e\}$ be given. Then for sufficiently large $n \in \mathbb{N}$, with $z := ts^n$, any element of zFz^{-1} is started with t and ended with t^{-1}. Here for $u \in \Lambda_i \setminus \{e\}$, we say an element w of Λ is started with u if $w = uw_1 \ldots w_n$ for some (possibly empty) sequence $w_1, \ldots, w_n$ with $w_j \in \Lambda_{k_j} \setminus \{e\}$ and $i \neq k_1 \neq k_2 \neq \cdots \neq k_n$. The word "ended with u" is similarly defined. (Thus, in our terminology, the element u^2 is not started with u.)

Let E' be the subset of Λ consisting of all elements started with t. Put $E := z^{-1}E'$, $D := \Lambda \setminus E$, and $D' := \Lambda \setminus E'$. Then note that $fD \cap D = \emptyset$ for all $f \in F$ if and only if $f'D' \cap D' = \emptyset$ for all $f' \in zFz^{-1}$. Since elements $f' \in zFz^{-1}$ are started with t and ended with

t^{-1} but D' consists of elements not started with t, we have $f'D' \cap D' = \emptyset$. Now for $j \in \{1, 2, 3\}$, put $u_j := s^j z$. Obviously each u_j is contained in $\langle s, t\rangle$. By definition, we have $u_j E = s^j E'$. This shows that $u_j E$ consists of only elements started with s^j. Therefore u_1E, u_2E, and u_3E are pairwise disjoint. □

Now we prove Main Theorem. Before the proof, we remark that the AP is preserved under taking free products. Hence Γ has the AP if and only if each free product component Γ_i has it. See Section 12.4 of [**6**] for the detail.

THEOREM 2.9. *Let Γ be an infinite free product group with the AP. Then, for $\alpha \in \mathcal{S}(\Gamma, X)$ with property $\mathcal{R}$, there is no proper intermediate* C*-*algebra of the inclusion* $C^*_r(\Gamma) \subset C(X) \rtimes_r \Gamma$. *In particular, when additionally α is amenable, then $C(X) \rtimes_r \Gamma$ is a minimal ambient nuclear* C*-*algebra of the non-nuclear* C*-*algebra* $C^*_r(\Gamma)$

PROOF. Let A be an intermediate C*-algebra of the inclusion $C^*_r(\Gamma) \subset C(X) \rtimes_r \Gamma$. We first consider the case $E(A) = \mathbb{C}$. In this case, thanks to Theorem 3.2 of [**58**] (see also Proposition 2.4 in Chapter 4), we have the equality $A = C^*_r(\Gamma)$.

We next consider the case $E(A) \neq \mathbb{C}$. In this case, by Proposition 2.7, $E(A)$ is dense in $C(X)$. Let U be a proper clopen subset of X. Let $\epsilon > 0$ be given. Then take a self-adjoint element $x \in A$ with $\|E(x) - \chi_U\| < \epsilon$. By property $\mathcal{R}$, there are torsion-free elements $s_1 \in \Gamma_i$ and $s_2 \in \Gamma_j$ with $i \neq j$ which fix χ_U. Put $\Lambda := \langle s_1, s_2\rangle$. Take $y \in C(X) \rtimes_{\rm alg} \Gamma$ satisfying $E(y) = \chi_U$ and $\|y - x\| < \epsilon$. By Lemma 2.8, we can apply the Powers argument, Lemma 5 of [**14**], by elements of Λ. Iterating the Powers argument sufficiently many times, we obtain a sequence $t_1, \ldots, t_n \in \Lambda$ satisfying the inequality

$$\|\frac{1}{n}\sum_{i=1}^{n} t_i(y - \chi_U)t_i^{-1}\| < \epsilon.$$

Since χ_U is Λ-invariant, we have

$$\|\frac{1}{n}\sum_{i=1}^{n} t_i x t_i^{-1} - \chi_U\| < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows $\chi_U \in A$. Therefore $A = C(X) \rtimes_r \Gamma$. □

REMARK 2.10. It is impossible to find a minimal ambient nuclear C*-algebra of a non-nuclear C*-algebra by maximality arguments. From the outside, it is shown in Chapter 4 that the decreasing sequence of nuclear C*-algebras need not be nuclear. From the inside, it can be shown that the increasing union of non-nuclear C*-algebras can be nuclear. Here we give an example. Let A be a unital nuclear C*-algebra and let B be a non-nuclear C*-subalgebra of A containing the unit of A. Put $A_n := A^{\otimes n} \otimes B$ for each $n \in \mathbb{N}$. Then they are canonically identified with C*-subalgebras of the infinite tensor power $A^{\otimes\infty}$ of A. Then each A_n is not nuclear but their increasing union is the nuclear C*-algebra $A^{\otimes\infty}$.

REMARK 2.11. When Γ is exact without the AP (e.g., $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ [**31**]), the proof of 2.9 shows that for an amenable Cantor system $\Gamma \curvearrowright X$ with $\mathcal{R}$, any proper intermediate C*-algebra of the inclusion $C^*_r(\Gamma) \subset C(X) \rtimes_r \Gamma$ is contained in the following C*-algebra.

$$A := \{x \in C(X) \rtimes_r \Gamma : E(xs) \in \mathbb{C} \text{ for all } s \in \Gamma\}.$$

By Proposition 2.3 in Chapter 1, any intermediate C*-algebra of the inclusion $C^*_r(\Gamma) \subset A$ does not have the OAP. Therefore $C(X) \rtimes_r \Gamma$ gives a minimal ambient nuclear C*-algebra of the reduced group C*-algebra $C^*_r(\Gamma)$. Also, the inclusion $A \subset C(X) \rtimes_r \Gamma$ gives an example of an ambient nuclear C*-algebra of a C*-algebra without the OAP with no proper intermediate C*-algebra.

REMARK 2.12. Let Γ be an exact infinite free product group. We show that amenable Cantor systems of Γ with property $\mathcal{R}$ are not unique at the level of continuous orbit equivalence. When additionally Γ has the AP, we also show that minimal ambient nuclear C*-algebras of $C^*_r(\Gamma)$ are not unique in the following sense: there is no isomorphism between them that is identity on $C^*_r(\Gamma)$. We say that two ambient C*-algebras are conjugate if such an isomorphism exists. For a topologically free Cantor system α, set

$$E[[\alpha]] := \{\theta \in \mathbb{R}/\mathbb{Z} : \text{some } \varphi \in [[\alpha]] \text{ factors the rotation } R_\theta \colon \mathbb{T} \to \mathbb{T}\}.$$

It is clear that the set is invariant under continuous orbit equivalence. Note also that $E[[\alpha]]$ is countable by the metrizability of X. For any amenable Cantor system $\beta\colon \Gamma \curvearrowright Y$ and any subset $I \subset \mathbb{N}$ with the infinite complement, by working on the closed subset

$$\{\alpha \in \mathcal{S}(\Gamma, X) : \alpha|_{\Lambda_i} = (\beta^{\otimes \mathbb{N}})|_{\Lambda_i} \text{ for all } i \in I\}$$

instead of $\mathcal{S}(\Gamma, X)$, we can find an amenable Cantor system α with $\mathcal{R}$ in this set. Here we identify X with $Y^{\mathbb{N}}$ and we denote by $\beta^{\otimes\mathbb{N}}$ the diagonal action of infinitely many copies of β. Hence, with the aid of (a modification of) Lemme 2.1, for any irrational number θ, we can find an amenable free Cantor system α of Γ with $\mathcal{R}$ satisfying $\theta \in E[[\alpha]]$. Thus there is a family of continuously many amenable free Cantor systems with $\mathcal{R}$ whose members are pairwise not continuously orbit equivalent. We show that their crossed products give pairwise non-conjugacy ambient C*-algebras. Suppose two of them are conjugate. Then the composite of a conjugating isomorphism with the canonical conditional expectation gives a Γ-equivariant unital completely positive map between two $C(X)$. This is impossible by Lemma 3.10 of [**26**] (with Remark 2.4) and Proposition 2.7.

3. Further examples

We close this chapter with the following result on minimal tensor products. Recall that a C*-algebra A is of real rank zero if every self-adjoint element of A is a norm limit of self-adjoint elements of A with finite spectrum.

THEOREM 3.1. *Let A be a simple C*-algebra of real rank zero. Let Γ be an infinite free product group with the AP. Let $\alpha\colon \Gamma \curvearrowright X$ be a Cantor system with property $\mathcal{R}$. Then the inclusion $A \otimes C^*_r(\Gamma) \subset A \otimes (C(X) \rtimes_r \Gamma)$ has no proper intermediate C*-algebra.*

Before the proof, we give a few remarks. Since purely infinite simple C*-algebras are of real rank zero (Proposition 4.1.1 of [**45**]), Kirchberg's $\mathcal{O}_2$-absorption theorem (Theorem 3.8 of [**29**]) with Theorem 3.1 provides maximal non-nuclear C*-subalgebras of $\mathcal{O}_2$. We also obtain examples of minimal ambient nuclear C*-algebras of non-unital C*-algebras.

PROOF OF THEOREM 3.1. Let B be an intermediate C*-algebra of the inclusion $A\otimes C^*_r(\Gamma) \subset A \otimes (C(X) \rtimes_r \Gamma)$. Put $\Phi := \mathrm{id}_A \otimes E$. Throughout the proof, we identify A with a C*-subalgebra of $A \otimes C(X)$ in the canonical way. Note that the image $\Phi(B)$ contains A. When the equality $\Phi(B) = A$ holds, by Proposition 2.4 in Chapter 4 (with Exercise 4.1.3 of [**6**]), we have the equality $B = A \otimes C^*_r(\Gamma)$.

Suppose $\Phi(B) \neq A$. We observe first that for an element $x \in A \otimes C(X)$ satisfying $(\varphi \otimes \mathrm{id}_{C(X)})(x) \in \mathbb{C}$ for all pure states φ on A, we have $x = (\mathrm{id}_A \otimes \psi)(x) \in A$ for any state ψ on $C(X)$. Hence we can choose a pure state φ on A and an element $b \in B$ satisfying $f := (\varphi \otimes \mathrm{id}_{C(X)})(\Phi(b)) \in C(X) \setminus \mathbb{C}$. Now let $\epsilon > 0$ be given. Since A is of real rank zero, the Akemann–Anderson–Pedersen excision theorem (Theorem 1.4.10 of [**6**]) shows that there is a nonzero projection $p \in A$ with $\|\Phi(pbp) - p \otimes f\| < \epsilon$. (Cf. Lemma 1.1 of [**29**].) By the simplicity of A, for any nonzero projection $q \in A$, there are $n \in \mathbb{N}$ and a sequence $v_1, \ldots, v_n \in \mathbb{M}_{1,n}(A)$ satisfying the following conditions.

- $v_i v_i^* = p$ for each i.
- The projections $p_i := v_i^* v_i \in \mathbb{M}_n(A)$ are pairwise orthogonal.
- $r \otimes q \leq \sum_{i=1}^{n} p_i$ for some minimal projection r of $\mathbb{M}_n(\mathbb{C})$.

(See Exercise 4.8 of [**47**].) For such a sequence, with $\Phi^{(n)} := \mathrm{id}_{\mathbb{M}_n(A)} \otimes E$, we have

$$\|\Phi^{(n)}(\sum_{i=1}^{n} v_i^* b v_i) - (\sum_{i=1}^{n} p_i) \otimes f)\| < \epsilon.$$

By cutting off the difference above by the projection $r \otimes q$ and identifying $\mathbb{C}r \otimes A$ with A in the canonical way, we obtain a sequence $x_1, \ldots, x_n \in A$ with

$$\|\Phi(\sum_{i=1}^{n} x_i^* b x_i) - q \otimes f\| < \epsilon.$$

This shows that the closure of $\Phi(B)$ contains $q \otimes f$. Proposition 2.7 then shows that the closure of $\Phi(B)$ contains the subspace $\mathbb{C}q \otimes C(X)$. From this with the proof of Theorem 2.9, we have $\mathbb{C}q \otimes C(X) \subset B$. Since A is simple, we obtain the equality $B = A \otimes (C(X) \rtimes_r \Gamma)$. □

REMARK 3.2. Let $(A_i)_{i \in I}$ be a family of C*-algebras. For each $i \in I$, let B_i be a minimal ambient nuclear C*-algebra of A_i. Then it is not hard to check that the direct sum $\bigoplus_{i \in I} B_i$ is a minimal ambient nuclear C*-algebra of the C*-algebra $\bigoplus_{i \in I} A_i$. In particular this gives examples of minimal ambient nuclear C*-algebras of non-simple C*-algebras.

Bibliography

[1] C. Anantharaman-Delaroche, *Systèmes dynamiques non commutatifs et moyennabilité.* Math. Ann. **279** (1987). 297–315.

[2] C. Anantharaman-Delaroche, *Purely infinite* C**-algebras arising from dynamical systems.* Bull. Soc. Math. France **125** (1997), 199–225.

[3] R. J. Archbold, J.S. Spielberg, *Topologically free actions and ideals in discrete* C**-dynamical systems.* Proc. Edinburgh Math. Soc. (2) **37** (1994), no. 1, 119–124.

[4] B. Blackadar, *K-theory for operator algebras.* Cambridge University Press, Cambridge (1998).

[5] K. S. Brown, *Cohomology of Groups.* Graduate Texts in Mathematics 87 (1982), Springer-Verlag, New York-Berlin.

[6] N. P. Brown, N. Ozawa, C**-algebras and finite-dimensional approximations.* Graduate Studies in Mathematics **88**. American Mathematical Society, Providence, RI, 2008. xvi+509 pp.

[7] T. A. Chapman, *Compact Hilbert cube manifolds and the invariance of Whitehead torsion.* Bull. Amer. Math. Soc. **79** (1973), 52–56.

[8] T. A. Chapman, *Lectures on Hilbert cube manifolds.* Vol. 28 (1975), Amer. Math. Soc.

[9] J. Cuntz, *Simple* C**-algebras generated by isometries.* Comm. Math. Phys. **57** (1977), 173–185.

[10] J. Cuntz, *K-theory for certain* C**-algebras.* Ann. of Math. **113** (1981), no. 1, 181–197.

[11] J. Cuntz, *A class of* C**-algebras and topological Markov chains II: reducible chains and the Ext-functor for* C**-algebras.* Invent. Math. **63** (1981), 25–40.

[12] J. Cuntz, W. Krieger, *A class of* C**-algebras and topological Markov chains.* Invent. Math. **56** (1980), no. 3, 251–268.

[13] A. Connes, J. Feldman, B. Weiss, *An amenable equivalence relation is generated by a single transformation.* Ergodic Theory Dynamical Systems **1** (1981), no. 4, 431–450.

[14] P. de la Harpe, G. Skandalis, *Powers' property and simple* C**-Algebras.* Math. Ann. **273** (1986), 241–250.

[15] K. Dykema, *Simplicity and the stable rank of some free product C*-algebras.* Trans. Amer. Math. Soc. **351** (1999), 1–40.

[16] E. G. Effros, U. Haagerup, *Lifling problems and local reflexivity for* C**-algebras.* Duke Math. J. **52** (1985), 103–128.

[17] G. A. Elliott, A. Sierakowski, *K-theory of certain purely infinite crossed products.* Preprint, arXiv:1110.6614.

[18] L. Fuchs, *Infinite Abelian Groups volume I, II.* Academic Press (1970), (1973).

[19] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'aprés Mikhael Gromov.* Progress in Math. 83, (1990), Birkhäuser.

[20] E. Glasner, B. Weiss, *On the construction of minimal skew products.* Israel J. Math. **34** (1979), 321–336.

[21] P. Green, *The structure of imprimitivity algebras.* J. Funct. Anal. **36** (1980), 88–104.

[22] T, Giordano, I. F. Putnam, C. F. Skau, *Topological orbit equivalence and* C**-crossed products.* J. reine angew. Math. **469** (1995), 51–112.

[23] T, Giordano, I. F. Putnam, C. F. Skau, *Full groups of Cantor minimal systems.* Israel J. Math. **111** (1999), 285–320.

[24] U. Haagerup, J. Kraus, *Approximation properties for group* C**-algebras and group von Neumann algebras.* Trans. Amer. Math. Soc. **344** (1994), 667–699.

[25] P. Jolissaint and G. Robertson, *Simple purely infinite* C**-algebras and n-filling actions.* J. Funct. Anal. **175** (2000), no. 1, 197-213.

[26] M. Kalantar, M. Kennedy, *Boundaries of reduced* C**-algebras of discrete groups.* To appear in J. reine angew. Math., arXiv:1405.4359.

[27] G. Kasparov, *Hilbert* C**-modules: Theorems of Stinespring and Voiculescu.* J. Operator Theory **4** (1980), 133–150.

[28] E. Kirchberg, *The classification of purely infinite* C**-algebras using Kasparov's theory.* Preprint.

[29] E. Kirchberg, N. C. Phillips, *Embedding of exact* C**-algebras in the Cuntz algebra* $\mathcal{O}_2$. J. reine angew. Math. **525** (2000), 17-53.

[30] M. Laca, J. Spielberg, *Purely infinite* C**-algebras from boundary actions of discrete groups.* J. reine angew. Math. **480** (1996), 125–139.

[31] V. Lafforgue, M. de la Salle, Noncommutative L^p-spaces without the completely bounded approximation property. Duke Math. J. **160** (2011), 71–116.

[32] R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory.* Springer-Verlag, New York (1977).

[33] H. Matui, *Topological full groups of one-sided shifts of finite type.* J. reine angew. Math. **705** (2015), 35–84.

[34] M. Nerurkar, *Ergodic continuous skew product actions of amenable groups.* Pacific J. Math. **119** (1985), 343–363.

[35] N. Ozawa, *Amenable actions and exactness for discrete groups.* C. R. Acad. Sci. Paris Ser. I Math. **330** (2000), 691–695.

[36] N. Ozawa, *Amenable actions and applications.* International Congress of Mathematicians Vol II, Eur. Math. Soc., 2006, 1563–1580.

[37] N. Ozawa, *Weak amenability of hyperbolic groups.* Groups Geom. Dyn. **2** (2008), 271–280.

[38] N. Ozawa, *Lecture on the Furstenberg boundary and* C**-simplicity.* available on the website http://www.kurims.kyotou.ac.jp/narutaka/notes/yokou2014.pdf.

[39] N. C. Phillips. *A classification theorem for nuclear purely infinite simple* C**-algebras.* Doc. Math **5** (2000), 49–114.

[40] M. Pimsner, D. Voiculescu, *K-groups of reduced crossed products by free groups.* J. Operator. Theory **8** (1982), 131–156.

[41] R. T. Powers, *Simplicity of the* C**-algebra associated with the free group on two generators.* Duke Math. J. **42** (1975), 151–156.

[42] I. Raeburn, *Graph algebras.* CBMS Regional Conference Series in Mathematics **103** (2005).

[43] G. Robertson, T. Steger, *Asymptotic K-theory for groups acting on* $\tilde{A}_2$ *buildings.* Canad. J. Math. **53** (2001), 809–833.

[44] M. Rørdam, *Classification of certain infinite simple* C**-algebras.* J. Funct. Anal. **131** (1995), 415–458.

[45] M. Rørdam, *Classification of nuclear* C**-algebras.* Encyclopaedia of Mathematical Sciences **126** (2002), 1–145.

[46] M. Rørdam, *A simple C*-algebra with a finite and an infinite projection.* Acta Math. **191** (2003), 109–142.

[47] M. Rørdam, F. Larsen, N. Laustsen, *An introduction to K-theory for* C**-algebras.* London Mathematical Society Student Texts 49, Cambridge University Press, Cambridge, 2000. xii+242 pp.

[48] M. Rørdam, A. Sierakowski, *Purely infinite* C**-algebras arising from crossed products.* Ergodic Theory Dynam. Systems **32** (2012), 273–293.

[49] J. Rosenberg, C. Schochet, *The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor.* Duke Math. J. **55** (1987), no. 2, 431–474.

[50] J. Spielberg, *Free product groups, Cuntz–Krieger algebras, and covariant maps.* Internat. J. Math. **2** (1991), 457–476.

[51] S. M. Srivastava, *A course on Borel Sets.* Grad. Texts in Math. 180 (1998), Springer.

[52] Y. Suzuki, *Amenable minimal Cantor systems of free groups arising from diagonal actions.* To appear in J. reine angew. Math., doi:10.1515/crelle-2014-0071, arXiv:1312.7098.

[53] Y. Suzuki, *Group* C**-algebras as decreasing intersection of nuclear* C**-algebras.* To appear in Amer. J. Math., arXiv:1410.8347

[54] Y. Suzuki, *Construction of minimal skew products of amenable minimal dynamical systems.* Preprint, arXiv:1503.01317.

[55] Y. Suzuki, *Minimal ambient nuclear* C**-algebras.* Preprint, arXiv:1510.05592.

[56] J.-L. Tu, *La conjecture de Baum–Connes pour les feuilletages moyennables.* K-theory **17** (1999), 215–264.

[57] D. Voiculescu, *A non-commutative Weyl–von Neumann theorem.* Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.

[58] J. Zacharias, *On the invariant translation approximation property for discrete groups.* Proc. Amer. Math. Soc. **134** (2006), 1909–1916.

[59] R. Zimmer, *Amenable ergodic actions, hyperfinite factors, and Poincar flows.* Bull. Amer. Math. Soc. **83** (1977), no. 5, 1078–1080.

[60] R. Zimmer, *Hyperfinite factors and amenable ergodic actions.* Invent. Math. **41** (1977), no. 1, 23-31.