# 博士論文

論文題目

Primitive ideals of Bost-Connes systems

（ボスト・コンヌ系の原始イデアルについて）

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# Primitive ideals of Bost-Connes systems

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## 1 Introduction

For an arbitrary number field $K$, a $C^*$-dynamical system $(A_K, \sigma_{t,K})$ is defined in the works of Ha-Paugam [5], Laca-Larsen-Neshveyev [6] and Yalkinoglu [20]. The $C^*$-dynamical system $(A_K, \sigma_{t,K})$ is related to class field theory. It is called the *Bost-Connes system*, after Bost and Connes [1], who defined such a system for the special case of $K = \mathbb{Q}$. It was a longstanding open problem to generalize Bost-Connes systems to arbitrary number fields, but that problem has been solved in recent years by the efforts of many researchers (especially, Yalkinoglu's work [20] was the last piece). So it is a good moment to start the investigation of those $C^*$-dynamical systems from both number theoretic and operator algebraic viewpoints. The operator algebraic viewpoint naturally asks for the classification of Bost-Connes systems. Concretely, we are interested in the following problem:

**Problem 1.1.** Does an $\mathbb{R}$-equivariant isomorphism of $(A_K, \sigma_{t,K})$ and $(A_L, \sigma_{t,L})$ imply an isomorphism of $K$ and $L$ ?

The best known result for the classification problem is the classification theorem of the KMS-states by Laca-Larsen-Neshveyev [6], obtaining the Dedekind zeta function $\zeta_K(s)$ as the partition function of $(A_K, \sigma_{t,K})$. In particular, Problem 1.1 is true if $[K : \mathbb{Q}] \leq 6$ or $[L : \mathbb{Q}] \leq 6$, thanks to the work of R. Perlis [11].

The purpose of this paper is to show that several invariants of number fields are in fact invariants for Bost-Connes systems. The main tool is the primitive ideal space, which is a common tool for the analysis of non-simple $C^*$-algebras. Using Williams' theorem [18], we describe the whole picture of the primitive ideal space of Bost-Connes $C^*$-algebras in Section 4. There is a result of Laca and Raeburn [8] determining the primitive ideal space of the original Bost-Connes $C^*$-algebra $A_{\mathbb{Q}}$. Hence Section 4 (especially, Theorem 4.5) amounts to a generalization of their work.

This paper contains two main theorems, and one of them provides an actual new invariant. For a number field $K$, $h^1_K$ denotes the narrow class number of $K$. We give a proof of the following theorem in Section 5:

**Theorem 1.2.** *Let $K, L$ be number fields and $A_K, A_L$ be associated Bost-Connes $C^*$-algebras. If $A_K$ is isomorphic to $A_L$, then we have $h^1_K = h^1_L$.*

Theorem 1.2 says that the narrow class number is an invariant of Bost-Connes $C^*$-algebras. The narrow class number measures the distance of the integer ring $\mathcal{O}_K$ from being a principal ideal domain, and some information of infinite primes is added. Hence, in principle, it is an independent invariant from the zeta function, which collects the information of finite primes. Indeed, there is an example of a pair of number fields which have the same zeta function but different narrow class numbers (Remark 5.3).

Looking at flows on the primitive ideal space, we get another invariant $(\hat{P}^1_K, \sigma_{t,K})$, which is studied in Section 5.4. This is a dynamical system on the infinite-dimensional torus (Proposition 5.5). We can also recover the norm map on $P^1_K$ from that dynamical system (Theorem 5.9). This is a sort of results like reconstructing the norm map on the whole ideal group $J_K$, which amounts to the reconstruction of the zeta function by [6], but from a different perspective.

The difference between the Dedekind zeta function and the narrow class number can be viewed from an operator algebraic perspective. The narrow class number appears as the dimension of finite dimensional irreducible representations (Theorem 5.1). As we see later, a primitive ideal of Bost-Connes $C^*$-algebra is maximal if and only if it is the kernel on an irreducible representation. Hence we can say that maximal primitive ideals have the information of the narrow class number.

On the contrary, second maximal primitive ideals contain the information of the zeta function. As we see in Section 6, those ideals are closely related to primes of $K$. Concretely, there is a one-to-one correspondence between prime ideals of the integer ring $\mathcal{O}_K$ and connected components of the space of second maximal primitive ideals (Proposition 6.4). So we can expect that the component $\mathcal{C}_\mathfrak{p}$ corresponding to a fixed prime $\mathfrak{p}$ may remember some information of $\mathfrak{p}$. Indeed, the information of $\mathfrak{p}$ can be recovered as in Theorem 6.11 by using $K$-theory. As a consequence, we prove the following second main theorem:

**Theorem 1.3.** *Let $K, L$ be number fields and $A_K, A_L$ be associated Bost-Connes $C^*$-algebras. If $A_K$ is isomorphic to $A_L$, then we have $\zeta_K = \zeta_L$.*

As mentioned, $\zeta_K$ appears as the partition function and hence an invariant of Bost-Connes systems. Theorem 1.3 says that it is in fact an invariant of Bost-Connes $C^*$-algebras. The difference of $\zeta_K$ and $h_K^1$ become clear by comparing both theorems. The space of maximal primitive ideals $\mathcal{I}_{1,K}$ is connected and contains the information of $h_K^1$ as dimensions of quotients. On the contrary, each connected component $\mathcal{C}_\mathfrak{p}$ of the space of second maximal primitive ideals $\mathcal{I}_{2,K}$ contain the information of each primes, and in summary, $\mathcal{I}_{2,K}$ contains the information of $\zeta_K$ as a $K$-theoretic invariant.

Generally speaking, in order to classify a class of non-simple $C^*$-algebras, it is effective to combine ideal structure theory and $K$-theory. Our strategy goes in a usual way in this sense, but how to use the information of ideals seems to depend on $C^*$-algebras. So we think our strategy is interesting as a concrete example of classifying a class of non-simple $C^*$-algebras.

Taking semigroup $C^*$-algebras $C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times)$ is another way to construct $C^*$-algebras related to number fields. For semigroup $C^*$-algebras, there is a work of Li [9] for the classification of such $C^*$-algebras. According to [9], the minimal primitive ideals of the semigroup $C^*$-algebras $C_r^*(R \rtimes R^\times)$ are labeled by prime ideals of $R$, and we can extract some information of original prime ideals by looking at $K$-theory of the quotient. This work is inspired by Li's work, although the proof is much different. It is interesting that there seems to exist a common philosophy behind two different constructions.

The contents of this paper is a collection of the papers "Irreducible Representations of Bost-Connes systems" ([16]) and "Primitive ideals and K-theoretic approach to Bost-Connes systems" ([15]). Several lemmas and arguments are unified and arranged.

# 2 Overview of Bost-Connes systems

## 2.1 Definition of Bost-Connes systems

In this section, we quickly review the definition of the Bost-Connes system of a number field. The reader can also consult [20, p.388] for the construction of the Bost-Connes system. Throughout this paper, $J_K$ denotes the ideal group of $K$ and $I_K$ denotes the ideal semigroup of $K$. The integer ring of $K$ is denoted by $\mathcal{O}_K$. The finite adéle ring is denoted by $\mathbb{A}_{K,f}$ and the finite idéle group is denoted by $\mathbb{A}_{K,f}^*$ (for the definition, see e.g. [7]). The Galois group $G(K^{\mathrm{ab}}/K)$ of the maximal abelian extension $K^{\mathrm{ab}}$ over $K$ is denoted by $G_K^{\mathrm{ab}}$.

Let $K$ be a number field. Put

$$Y_K = \hat{\mathcal{O}}_K \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}},$$

where $\hat{\mathcal{O}}_K$ is the profinite completion of $\mathcal{O}_K$, and $\hat{\mathcal{O}}_K^*$ acts on $\hat{\mathcal{O}}_K \times G_K^{\mathrm{ab}}$ by

$$s \cdot (\rho, \alpha) = (\rho s, [s]_K^{-1}\alpha)$$

for $\rho \in \hat{\mathcal{O}}_K, \alpha \in G_K^{\mathrm{ab}}$ and $s \in \hat{\mathcal{O}}_K^*$, where $[\cdot]_K$ is the Artin reciprocity map. Let $\mathfrak{a} \in I_K$ and take a finite idéle $a \in \mathbb{A}_{K,f}^* \cap \hat{\mathcal{O}}_K$ such that $\mathfrak{a} = (a)$. The action of $I_K$ on $Y_K$ is given by

$$\mathfrak{a} \cdot [\rho, \alpha] = [\rho a, [a]_K^{-1}\alpha].$$

Let $A_K = C(Y_K) \rtimes I_K$. Define an $\mathbb{R}$-action on $A_K$ by

$$\sigma_{t,K}(f) = f,\ \sigma_{t,K}(v_{\mathfrak{a}}) = N(\mathfrak{a})^{it} v_{\mathfrak{a}}$$

for $f \in C(Y_K), \mathfrak{a} \in I_K$ and $t \in \mathbb{R}$, where $N(\cdot)$ is the ideal norm.

**Definition 2.1.** The system $(A_K, \sigma_{t,K})$ is called the Bost-Connes system for $K$.

It is convenient to extend the Bost-Connes system to a non-unital group crossed product. Let

$$X_K = \mathbb{A}_{K,f} \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}$$

and define the action of $J_K$ on $X_K$ in the same way. Let $\tilde{A}_K = C_0(X_K) \rtimes J_K$. Then $A_K$ is a full corner of $\tilde{A}_K$. Namely, we have $A_K = 1_{Y_K} \tilde{A}_K 1_{Y_K}$. The $\mathbb{R}$-action on $\tilde{A}_K$ is defined in the same way, which is also denoted by $\sigma_{t,K}$.

For convenience, we fix notations of subspaces of $X_K$ and $Y_K$. Define four subspaces by

$$\begin{aligned}
Y_K^* &= \hat{\mathcal{O}}_K^* \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}} \cong G_K^{\mathrm{ab}}, \\
X_K^0 &= \{0\} \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}} \cong G_K^{\mathrm{ab}}/[\hat{\mathcal{O}}_K^*]_K, \\
X_K^\natural &= (\mathbb{A}_{K,f} \setminus \{0\}) \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}, \\
Y_K^\natural &= (\hat{\mathcal{O}}_K \setminus \{0\}) \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}.
\end{aligned}$$

## 2.2 KMS-states of Bost-Connes systems

Let $\Sigma_\beta$ be the set of KMS$_\beta$-states of $(A_K, \sigma_{t,K})$ and let $\mathrm{ex}(\Sigma_\beta)$ be its extremal points. The following theorem is due to Laca-Larsen-Neshveyev, which is a fundamental theorem in Bost-Connes systems.

**Theorem 2.2** (KMS-classification theorem, [6]). *The following holds:*

1. *For $0 < \beta \leq 1$, there is a unique KMS$_\beta$-state of $(A_K, \sigma_{t,K})$.*
2. *For $1 < \beta \leq \infty$, there is a one-to-one correspondence between $\mathrm{ex}(\Sigma_\beta)$ and $G_K^{\mathrm{ab}}$.*
3. *The partition function coincides with $\zeta_K(\beta)$.*

In the above theorem, KMS$_\beta$-states for $1 < \beta \leq \infty$ are obtained from irreducible representations (cf. [6, Remark 2.2]). For $g \in G_K^{\mathrm{ab}}$, we have an irreducible representation $\pi_g$ on $\ell^2(I_K)$ defined by

$$\pi_g(f)\xi_{\mathfrak{b}} = f(\mathfrak{b} \cdot g)\xi_{\mathfrak{b}} \text{ for } f \in C(Y_K), \text{ and}$$
$$\pi_g(v_{\mathfrak{a}})\xi_{\mathfrak{b}} = \xi_{\mathfrak{ab}} \text{ for } \mathfrak{a} \in I_K,$$

where $g$ is identified with $[1, g] \in Y_K^*$ and $\{\xi_{\mathfrak{b}}\}_{\mathfrak{b} \in I_K}$ is the standard orthonormal basis of $\ell^2(I_K)$. Let $H$ be the positive self-adjoint operator on $\ell^2(I_K)$ defined by

$$H\xi_{\mathfrak{b}} = (\log N_K(\mathfrak{b}))\xi_{\mathfrak{b}}.$$

Then the state

$$x \mapsto \frac{\mathrm{Tr}(e^{-\beta H}\pi_g(x))}{\mathrm{Tr}(e^{-\beta H})}$$

is the KMS$_\beta$-state corresponding to $g \in G_K^{\mathrm{ab}}$. The partition function is $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$. Since the partition function is an invariant of $C^*$-dynamical systems, we obtain the following corollary:

**Corollary 2.3.** *Let $K, L$ be number fields. If $(A_K, \sigma_{t,K})$ is $\mathbb{R}$-equivariantly isomorphic to $(A_L, \sigma_{t,L})$, then we have $\zeta_K = \zeta_L$.*

One of the purposes of this paper to remove the condition of $\mathbb{R}$-equivariance. Since the above corollary crucially depends on the structure of the time evolution, the proof must be completely different if we do not consider the time evolution.

# 3 Preliminaries

## 3.1 Arithmetic Preliminary

First we fix notations (basically, we follow notations of [10]). Let $K$ be a number field. The symbol $K_+^*$ denotes the group of all totally positive nonzero elements of $K$ and let $\mathcal{O}_{K,+}^\times = \mathcal{O}_K \cap K_+^*$. The symbol $U_{K,+}$ denotes the closure of

$\mathcal{O}_{K,+}$ in $\hat{\mathcal{O}}_K^*$. The symbol $P_K^1$ denotes the subgroup of principal ideals of $J_K$ generated by totally positive elements (i.e., $P_K^1 \cong K_+^*/\mathcal{O}_{K,+}^*$). The narrow ideal class group of $K$ is denoted by $C_K^1 = J_K/P_K^1$. The order of $C_K^1$ is called the narrow class number of $K$, which is denoted by $h_K^1$. The set of all finite primes is denoted by $\mathcal{P}_K$. For any ideal $\mathfrak{m}$ of $\mathcal{O}_K$, Let

$$\begin{aligned} J_K^{\mathfrak{m}} &= \{\mathfrak{a} \in J_K \mid \mathfrak{a} \text{ is prime to } \mathfrak{m}\}, \\ P_K^{\mathfrak{m}} &= \{(k) \in J_K \mid k \in K_+^*, k \equiv 1 \mod \mathfrak{m}\}. \end{aligned}$$

Similarly, for any subset $S$ of $\mathcal{P}_K$, $J_K^S$ is the set of all fractional ideals which is prime to any $\mathfrak{p} \in S$, and $I_K^S = I_K \cap J_K^S$. For any finite prime $\mathfrak{p}$ of $K$, $K_\mathfrak{p}$ denotes the localization of $K$ at $\mathfrak{p}$, and $\mathcal{O}_\mathfrak{p}$ denotes the integer ring of $K_\mathfrak{p}$. The unit group $\mathcal{O}_\mathfrak{p}^*$ is often denoted by $U_\mathfrak{p}$. For any integer $m \geq 1$, let $U_\mathfrak{p}^{(m)} = 1 + \mathfrak{p}^m$ and $U_\mathfrak{p}^{(0)} = U_\mathfrak{p}$.

For a ring $R$, $R^\times$ denotes $R \setminus \{0\}$.

The following two lemmas are fundamental and implicitly used in this paper. They are essentially contained in [7, Proposition 1.1].

**Lemma 3.1.** *The reciprocity map $[\cdot]_K : \mathbb{A}_K^* \to G_K^{\mathrm{ab}}$ induces the isomorphism $\mathbb{A}_{K,f}^*/\overline{K_+^*} \cong G_K^{\mathrm{ab}}$, where $\overline{K_+^*}$ is the closure of $K_+^*$ in $\mathbb{A}_{K,f}^*$.*

**Lemma 3.2.** *The sequence*

$$1 \longrightarrow U_{K,+} \longrightarrow \hat{\mathcal{O}}_K^* \longrightarrow \mathbb{A}_{K,f}^*/\overline{K_+^*} \longrightarrow C_K^1 \longrightarrow 1$$

*is exact.*

Note that the homomorphism $\mathbb{A}_{K,f}^*/\overline{K_+^*} \to C_K^1$ is defined by sending the class of $a \in \mathbb{A}_{K,f}^*$ to the class of $(a)$.

## 3.2 $\mathbb{R}$-equivariant imprimitivity bimodules

**Definition 3.3.** Let $(A, \sigma_t^A)$ and $(B, \sigma_t^B)$ be $C^*$-dynamical systems. An $(A, B)$-imprimitivity bimodule $E$ is said to be an $\mathbb{R}$-*equivariant imprimitivity bimodule* if there is a one-parameter group of isometries $U_t$ on $E$ such that

- ${}_A\langle U_t\xi, U_t\eta\rangle = \sigma_t({}_A\langle \xi, \eta\rangle)$
- $\langle U_t\xi, U_t\eta\rangle_B = \sigma_t(\langle \xi, \eta\rangle_B)$

for any $\xi, \eta \in E_\mathfrak{p}$ and $t \in \mathbb{R}$.

If there exists an $\mathbb{R}$-equivariant imprimitivity bimodule, then the two $C^*$-dynamical systems are said to be $\mathbb{R}$-*equivariantly Morita equivalent.*

Note that from the above axioms we have

$$\sigma_t^A(a)U_t(\xi) = U_t(a\xi),\ U_t(\xi)\sigma_t^B(b) = U_t(\xi b)$$

for any $a \in A, b \in B$ and $\xi \in E$.

**Lemma 3.4.** *For a number field $K$, the Bost-Connes system $(A_K, \sigma_{t,K})$ is $\mathbb{R}$-equivariantly Morita equivalent to $(\tilde{A}_K, \sigma_{t,K})$.*

*Proof.* Since $A_K = 1_{Y_K}\tilde{A}_K 1_{Y_K}$ and $1_{Y_K}$ is a full projection, the $(A_K, \tilde{A}_K)$ bimodule $E = 1_{Y_K}\tilde{A}_K$ is an imprimitivity bimodule. Define a one-parameter group of isometries $U_t$ on $E$ by restricting the time-evolution of $\tilde{A}_K$. Then $U_t$ satisfies the desired property. □

If two $C^*$-algebras are Morita equivalent, then we have natural correspondences between their representations and ideals. As a consequence, their primitive ideal spaces are homeomorphic. The homeomorphism obtained in this way is called the *Rieffel homeomorphism* (cf. [12, Corollary 3.33]). We need an $\mathbb{R}$-equivariant version of this theorem. For a $C^*$-dynamical system $(A, \sigma_t)$, then we consider the $\mathbb{R}$-action on $\mathrm{Prim}A$ defined by

$$t \cdot \ker \pi = \ker(\pi \circ \sigma_t) = \sigma_{-t}(\ker \pi),$$

where $\pi$ is an irreducible representation of $A$.

**Proposition 3.5.** *Let $E$ be an $\mathbb{R}$-equivariant imprimitivity bimodule between two $C^*$-dynamical systems $(A, \sigma_t^A)$ and $(B, \sigma_t^B)$. Then the Rieffel homeomorphism $h_X : \mathrm{Prim}B \to \mathrm{Prim}A$ is $\mathbb{R}$-equivariant.*

*Proof.* Let $(\pi, \mathcal{H}_\pi)$ be a representation of $B$. We need to show that the representation $(\mathrm{id}_A \otimes 1, E \otimes_{\pi\circ\sigma_t^B} \mathcal{H}_\pi)$ is unitarily equivalent to $(\sigma_t^A \otimes 1, E \otimes_\pi \mathcal{H}_\pi)$. Let $U_t$ be a one-parameter group of isometries on $E$ which gives $\mathbb{R}$-equivariance. Then it is easy to check that the unitary

$$E \otimes_{\pi\circ\sigma_t^B} \mathcal{H}_\pi \to E \otimes_\pi \mathcal{H}_\pi,\ x \otimes_{\pi\circ\sigma_t^B} \xi \mapsto U_t(x) \otimes_\pi \xi$$

gives the unitary equivalence. □

Note that the strong continuity of the one-parameter group of isometries $U_t$ is tacitly assumed in the definition of $\mathbb{R}$-equivariant imprimitivity bimodules. However, the strong continuity is not needed for the sake of Proposition 3.5.

### 3.3 The Primitive ideal space of crossed products by abelian groups

In order to determine $\mathrm{Prim}A_K$, by Proposition 3.5, we may investigate $\mathrm{Prim}\tilde{A}_K$ instead. We have a nice structure theorem of the primitive ideal space for group crossed products. Let $G$ be a countable abelian group acting on a second countable locally compact space $X$. Define an equivalence relation on $X \times \hat{G}$ by

$$(x, \gamma) \sim (y, \delta) \text{ if } \overline{Gx} = \overline{Gy} \text{ and } \gamma\delta^{-1} \in G_x^\perp,$$

where $\hat{G}$ is the Pontrjagin dual of $G$ and $G_x$ is the isotropy group of $x$. For a representation $(\pi, \mathcal{H}_\pi)$ of $A_x = C_0(X) \rtimes G_x$, $\mathrm{Ind}_{G_x}^G \pi$ denotes the induced representation of $A = C_0(X) \rtimes G$ on the Hilbert space $A \otimes_{A_x} \mathcal{H}_\pi$.

**Theorem 3.6.** (Williams, [19, Theorem 8.39]) *We have a homeomorphism* $\Phi : X \times \hat{G}/\sim \to \mathrm{Prim} C_0(X) \rtimes G$ *defined by*

$$\Phi([x,\gamma]) = \ker(\mathrm{Ind}_{G_x}^{G}(\mathrm{ev}_x \rtimes \gamma|_{G_x})).$$

**Remark 3.7.** The quotient map $X \times \hat{G} \to X \times \hat{G}/\sim$ is an open map (cf. [19, Remark 8.40]). This fact is useful to determine the topology of the primitive ideal space.

In this section, we look into the dynamics of the primitive ideal space in a general setting. Let $N : G \to \mathbb{R}_+$ be a group homomorphism and define the time evolution on $A$ by

$$\sigma_t(fu_s) = N(s)^{it} fu_s$$

for any $f \in C_0(X), s \in G$ and $t \in \mathbb{R}$. Take $x \in X, \gamma \in \hat{G}$ and let $\pi = \mathrm{ev}_x \rtimes \gamma|_{G_x}$. Then $\pi_x$ defines a character of $A_x$. By [18, Proposition 8.24], $\mathrm{Ind}_{G_x}^{G}\pi$ is unitarily equivalent to the representation $\pi_{x,\gamma}$ on $\mathcal{H}_{x,\gamma} = C^*(G) \otimes_{C^*(G_x)} \mathbb{C}$ defined by

$$\pi_{x,\gamma}(f)\xi_s = f(sx)\xi_s,\ \ \pi_{x,\gamma}(u_t)\xi_s = \xi_{ts}$$

for $f \in C_0(X)$ and $s, t \in G$. The inner product of $\mathcal{H}_{x,\gamma}$ is defined by

$$\langle \xi_s, \xi_t \rangle = \begin{cases} \gamma(s^{-1}t) & \text{if } s^{-1}t \in G_x, \\ 0 & \text{if } s^{-1}t \notin G_x, \end{cases}$$

for any $s, t \in G$. We would like to determine the representation $\pi_{x,\gamma} \circ \sigma_t$. We have $\pi_{x,\gamma} \circ \sigma_t(u_s)\xi_r = N(s)^{it}\xi_{sr}$. Let $\tilde{\mathcal{H}} = \mathcal{H}_{x,\gamma}$ as a linear space. Define a linear map $U : \mathcal{H}_{x,\gamma} \to \tilde{\mathcal{H}}$ by

$$U(N(s)^{it}\xi_s) = \tilde{\xi}_s$$

for $s \in G$. To make $U$ a unitary, the inner product on $\tilde{\mathcal{H}}$ needs to be defined by

$$\langle \tilde{\xi}_s, \tilde{\xi}_r \rangle = \begin{cases} N(s^{-1}r)^{it}\gamma(s^{-1}r) & \text{if } s^{-1}r \in G_x, \\ 0 & \text{if } s^{-1}r \notin G_x. \end{cases}$$

Then we can see that $U\pi_{x,\gamma} \circ \sigma_t U^* = \pi_{x,\tilde{\gamma}}$, where $\tilde{\gamma} = N(\cdot)^{it}\gamma$. Thus we have the following proposition:

**Proposition 3.8.** *Let* $A = C_0(X) \rtimes G$ *and consider the* $\mathbb{R}$*-action on* $\mathrm{Prim}A = X \times \hat{G}/\sim$ *defined in* Section 3.2 *(this action is also denoted by* $\sigma$*). Then we have*

$$\sigma_t([x,\gamma]) = [x, N(\cdot)^{it}\gamma]$$

*for* $[x,\gamma] \in X \times \hat{G}/\sim$*.*

The Bost-Connes systems for number fields are not Type I $C^*$-algebras, because it is known that they have type $\mathrm{III}_1$ representations. So we cannot expect that Williams' theorem gives complete classification of irreducible representations. However, we can still get some information about irreducible representations, such as their dimensions. The following lemma will be used:

**Lemma 3.9.** *For $(x, \gamma) \in X \times \hat{G}$, let $(\pi_{x,\gamma}, \mathcal{H}_{x,\gamma})$ be the representation of $A = C_0(X) \rtimes G$ defined as above. Then $\dim \mathcal{H}_{x,\gamma} = [G : G_x]$. In particular, $\pi_{x,\gamma}$ is finite-dimensional if and only if $G_x$ has a finite index in $G$.*

*Proof.* Let $\{s_i\}$ be a complete representative of $G/G_x$. Then the family $\{\xi_{s_i}\}$ is orthogonal in $\mathcal{H}_{x,\gamma}$. We can see that $\{\xi_{s_i}\}$ is an orthogonal basis. In fact, we have $\xi_{s_i t} = \gamma(t)\xi_{s_i}$ for $t \in G_x$ because

$$\langle \gamma(t)\xi_{s_i}, \xi_{s_i r} \rangle = \gamma(t^{-1}r) = \langle \xi_{s_i t}, \xi_{s_i r} \rangle,$$
$$\langle \gamma(t)\xi_{s_i}, \xi_{s_j r} \rangle = 0 = \langle \xi_{s_i t}, \xi_{s_j r} \rangle,$$

for $t, r \in G_x$ and $j \neq i$. □

**Remark 3.10.** In fact, there is a canonical orthonormal basis of $\mathcal{H}_{x,\gamma}$. If $\{s_i\}$ is a complete set of representatives of $G/G_x$, then the family $\{\gamma(s_i^{-1})\xi_{s_i}\}$ is an orthonormal basis and independent of the choice of $\{s_i\}$.

We need to study the dimensions of irreducible representations. Clearly, if $E$ is an $(A, B)$-imprimitivity bimodule and $\pi$ is a finite-dimensional representation of $B$, $E-\mathrm{Ind}\pi$ may be infinite-dimensional (e.g., $A = \mathbb{K}(\mathcal{H})$ and $B = \mathbb{C}$). However, we have the following criterion in our case.

**Lemma 3.11.** *Let $A$ be a $C^*$-algebra and $e \in A$ be a full projection and Let $E = eA$ be the natural $(eAe, A)$-imprimitivity bimodule. Let $\pi$ be a non-degenerate representation of $A$. Then $E-\mathrm{ind}\pi$ is unitarily equivalent to $(\pi|_{eAe}, \pi(e)\mathcal{H})$. In particular, $\dim(E-\mathrm{ind}\pi) = \dim \pi(e)\mathcal{H}$.*

*Proof.* The unitary

$$eA \otimes_A \mathcal{H}_\pi \to \pi(e)\mathcal{H}_\pi,\ ea \otimes \xi \mapsto \pi(ea)\xi$$

gives the desired unitary equivalence. □

# 4 Picture of the primitive ideal space

## 4.1 The formal description of the primitive ideal space

In this section, we study the equivalence relation that appeared in Section 3.3 in our case, and determine the structure of $\mathrm{Prim}A_K$ in a formal way. This section amounts to an actual generalization of the work of Laca and Raeburn [8].

The first step is to determine the closure of the orbit $\overline{J_K x}$ for $x \in Y_K$.

**Lemma 4.1.** (cf. [8, Lemma 2.3]) *For $\rho \in \mathbb{A}_{K,f}$, we have*

$$\overline{K_+^* \rho} = \{\sigma \in \mathbb{A}_{K,f} \mid \rho_\mathfrak{p} = 0 \text{ implies } \sigma_\mathfrak{p} = 0\}.$$

*Proof.* We may assume $\rho \in \hat{\mathcal{O}}_K$ because $\overline{K_+^* a\rho} = \overline{K_+^* \rho}$ for any $a \in \mathcal{O}_{K,+}$ and the right hand side is invariant under multiplication by an element of $\mathbb{A}_{K,f}^*$. Take

$\sigma$ from the right hand side. Enumerate the primes of $K$ as $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$. Define $\tau \in \mathbb{A}_{K,f}$ by

$$\tau_{\mathfrak{p}} = \begin{cases} \rho_{\mathfrak{p}}^{-1}\sigma_{\mathfrak{p}} & \text{if } \rho_{\mathfrak{p}} \neq 0, \\ 0 & \text{if } \rho_{\mathfrak{p}} = 0. \end{cases}$$

Take $a \in \mathcal{O}_{K,+}$ satisfying $a\tau \in \hat{\mathcal{O}}_K$. For each $n$, take $k_n \in \mathcal{O}_{K,+}$ such that $k_n \equiv a\tau_{\mathfrak{p}} \bmod \mathfrak{p}^n$ for $\mathfrak{p} = \mathfrak{p}_k$ with $1 \leq k \leq n$. Then we have $a\sigma \in \hat{\mathcal{O}}_K$ and $k_n\rho_{\mathfrak{p}} \equiv a\sigma_{\mathfrak{p}} \bmod \mathfrak{p}^n$ for such $\mathfrak{p}$. This implies that $k_n\rho$ converges to $a\sigma$ in $\mathbb{A}_{K,f}$, so $a^{-1}k_n\rho$ converges to $\sigma$. The other inclusion is obvious. □

**Lemma 4.2.** *For $x = [\rho, \alpha] \in X_K$, we have*

$$\overline{J_K x} = \{y = [\sigma, \beta] \in X_K \mid \rho_{\mathfrak{p}} = 0 \text{ implies } \sigma_{\mathfrak{p}} = 0\}.$$

*Proof.* Take $y = [\sigma, \beta]$ from the right hand side. Take a finite idéle $a \in \mathbb{A}_{K,f}^*$ such that $\alpha[a]_K^{-1} = \beta$ and let $\mathfrak{a}$ be the ideal generated by $a$. Then $\mathfrak{a}[\rho, \alpha] = [\rho a, \beta]$. By Lemma 4.1, there exists a sequence $k_n \in K_+^*$ such that $k_n\rho a$ converges to $\sigma$. Since $[k_n]_K = 1$, the sequence $(k_n)\mathfrak{a}x$ converges to $y$. □

The next step is to determine what the isotropy group is.

**Definition 4.3.** For any subset $S$ of $\mathcal{P}_K$, define the subgroup $\Gamma_S$ of $J_K$ by

$$\Gamma_S = \{(a) \mid a \in \overline{K_+^*} \subset \mathbb{A}_{K,f}^*, a_{\mathfrak{p}} = 1 \text{ for } \mathfrak{p} \notin S\}.$$

Note that $\Gamma_S$ is a subgroup of $P_K^1$, because $\overline{K_+^*}$ is contained in $K_+^*\hat{O}_K^*$. We can see that $\Gamma_\emptyset = 1$ and $\Gamma_{\mathcal{P}_K} = P_K^1$. For two subsets $S, T \subset \mathcal{P}_K$, we have $\Gamma_S \subset \Gamma_T$ if and only if $S \subset T$.

For $x = [\rho, \alpha] \in X_K$, let $S_x = \{\mathfrak{p} \in \mathcal{P}_K \mid \rho_{\mathfrak{p}} = 0\}$. By Lemma 4.2, for $x, y \in X_K$, $\overline{J_K x} = \overline{J_K y}$ if and only if $S_x = S_y$.

**Lemma 4.4.** (cf. [8, Lemma 2.1]) *For $x \in X_K$, the isotropy group $J_{K,x}$ coincides with $\Gamma_{S_x}$.*

*Proof.* Let $\mathfrak{a} \in J_{K,x}$. Take $\rho \in \mathbb{A}_{K,f}$ and $\alpha \in G_K^{\mathrm{ab}}$ such that $x = [\rho, \alpha]$. Then we can choose a finite idéle $a \in \mathbb{A}_{K,f}$ generating $\mathfrak{a}$ and satisfies $[a]_K = 1$ and $\rho a = \rho$. Hence $a$ belongs to $\overline{K_+^*}$ and $a_{\mathfrak{p}} = 1$ for $\mathfrak{p}$ satisfying $\rho_{\mathfrak{p}} \neq 0$. This implies that $\mathfrak{a} \in \Gamma_{S_x}$. The converse inclusion can be shown in a similar way. □

Combining Lemma 4.2, Lemma 4.4 and Theorem 3.6, we get the following conclusion.

**Theorem 4.5.** *We have* $\mathrm{Prim} A_K = \bigcup_{S \subset \mathcal{P}_K} \hat{\Gamma}_S$*, where $S$ runs through all subsets of $\mathcal{P}_K$. Let $P_{S,\gamma}$ be the ideal which corresponds to $\gamma \in \hat{\Gamma}_S$. Then we have*

$$P_{S,\gamma} = \ker((\mathrm{Ind}_{\Gamma_S}^{J_K}(\mathrm{ev}_x \rtimes \gamma))|_{A_K}),$$

*where $x = [\rho, \alpha] \in X_K$ which satisfies that $\rho_{\mathfrak{p}} = 0$ if and only if $\mathfrak{p} \in S$.*

**Remark 4.6.** Using Remark 3.10 and Lemma 3.11, the explicit form of the representation $(\mathrm{Ind}_{\Gamma_S}^{J_K}(\mathrm{ev}_x \rtimes \gamma))|_{A_K}$ can be determined. Let $\tilde{\mathcal{H}}_{S,\gamma} = \ell^2(J_K/\Gamma_S)$. Take a lift of $\gamma$ on $\hat{J}_K$, which is still denoted by $\gamma$. Define a representation $\tilde{\pi}_{S,\gamma}$ of $\tilde{A}_K$ by

$$\tilde{\pi}_{S,\gamma}(f)\xi_{\bar{t}} = f(tx)\xi_{\bar{t}},\ \tilde{\pi}_{S,\gamma}(u_s)\xi_{\bar{t}} = \gamma(s)\xi_{\bar{st}},$$

where $s, t \in J_K$ and $f \in C_0(X_K)$ (note that the unitary $u_s$ corresponding to $s \in J_K$ actually lies in the multiplier algebra of $\tilde{A}_K$). Then $(\pi_{S,\gamma}, \mathcal{H}_{S,\gamma}) = (\tilde{\pi}_{S,\gamma}|_{A_K}, \tilde{\pi}_{S,\gamma}(1_{Y_K})\tilde{\mathcal{H}}_{S,\gamma})$ is exactly the above irreducible representation. Up to unitary equivalence, $\pi_{S,\gamma}$ is independent of the choice of a lift of $\gamma$. Indeed, for any element $\omega \in \Gamma_S^{\perp} = \widehat{J_K/\Gamma_S} \subset \hat{J}_K$, we have a unitary $U_\omega$ on $\ell^2(J_K/\Gamma_S)$ defined by

$$U_\omega \xi_{\bar{t}} = \omega(t)\xi_{\bar{t}}$$

for $\bar{t} \in J_K/\Gamma_S$, which gives the equivalence between $\pi_{S,1}$ and $\pi_{S,\omega}$. Moreover, we have $\tilde{\pi}_{S,\gamma}(1_{Y_K})\tilde{\mathcal{H}}_{S,\gamma} = \ell^2(J_K^{S^c}/\Gamma_S \times I_K^S)$. Note that $\pi_{S,\gamma}(v_{\mathfrak{p}})$ is a unitary if $\mathfrak{p} \notin S$.

Note that the unitary equivalence class of the representation $\pi_{S,\gamma}$ may change if we change $x$ to another one — only the kernel of $\pi_{S,\gamma}$ is independent of the choice of $x$. However, for our purpose, this difference does not matter in most of the cases, so we abuse the notation. In the case that this difference actually matters (as in Section 4.2), we prepare another notation.

We can say that $\mathrm{Prim}A_K$ is a bundle over $2^{\mathcal{P}_K}$ with fibers $\hat{\Gamma}_S$. In other words, $\mathrm{Prim}A_K$ is considered as an increasing net of compact groups indexed by subsets of $\mathcal{P}_K$. In fact, the lattice structure of primitive ideals reflects this net structure in the following sense:

**Proposition 4.7.** *Let $S, T$ be two subsets of $\mathcal{P}_K$ and let $\gamma \in \hat{J}_K$. Then we have $P_{S,\gamma} \subset P_{T,\gamma}$ if and only if $S \subset T$.*

*Proof.* Take $x = [\rho, \alpha], y = [\sigma, \beta] \in Y_K$ satisfying that $\rho_{\mathfrak{p}} = 0$ if and only if $\mathfrak{p} \in S$, and $\sigma_{\mathfrak{p}} = 0$ if and only if $\mathfrak{p} \in T$.

Suppose $P_{S,\gamma} \subset P_{T,\gamma}$. Since $C_0((\overline{J_K x})^c \cap Y_K) \subset P_{S,\gamma}$, any function of $C(Y_K)$ which vanishes on $\overline{J_K x} \cap Y_K$ also vanishes on $\overline{J_K y} \cap Y_K$. Hence $\overline{J_K x} \cap Y_K \supset \overline{J_K y} \cap Y_K$, which is equivalent to $S \subset T$ by Lemma 4.2.

Suppose $S \subset T$. By Lemma 4.2, there exists a sequence $\{w_n\}_{n \in \mathbb{N}} \subset J_K$ such that $w_n x$ converges to $y$ in $X_K$. For any $n \in \mathbb{N}$, let $\tilde{\pi}_n$ be the representation of $\tilde{A}_K$ on $\ell^2(J_K/\Gamma_S)$ determined by

$$\tilde{\pi}_n(f)\xi_{\bar{t}} = f(tw_n x)\xi_{\bar{t}},\ \tilde{\pi}_n(u_s)\xi_{\bar{t}} = \gamma(s)\xi_{\bar{st}},$$

where $s, t \in J_K$ and $f \in C_0(X_K)$. Then we can see that $\tilde{\pi}_n$ is unitarily equivalent to $\tilde{\pi}_{S,\gamma}$. For any $a \in \tilde{A}_K$, we can see that there exists a limit of $\tilde{\pi}_n(a)$ with respect to the strong operator topology, which is denoted by $\pi(a)$. The representation $\pi$ of $\tilde{A}_K$ is determined by

$$\pi(f)\xi_{\bar{t}} = f(ty)\xi_{\bar{t}},\ \pi(u_s)\xi_{\bar{t}} = \gamma(s)\xi_{\bar{st}},$$

for $s, t \in J_K$ and $f \in C_0(X_K)$. If $a \in P_{S,\gamma}$, then we have $\pi(a) = 0$ by definition. Hence it suffices to show that $\ker \pi \cap A_K$ is contained in $P_{T,\gamma}$.

Let $\rho = \tilde{\pi}_{T,\gamma}$. First, we consider the case of $\gamma = 1$. By the universality of group crossed products, we have a quotient map $\phi : C_0(X_K) \rtimes J_K \to C_0(\overline{J_K x}) \rtimes (J_K/\Gamma_S)$. Then the representations $\pi, \rho$ factors through $\phi$, i.e., there exist representations $\pi', \rho'$ of $C_0(\overline{J_K x}) \rtimes (J_K/\Gamma_S)$ on $\mathbb{B}(\ell^2(J_K/\Gamma_S))$ such that $\pi = \pi' \circ \phi$ and $\rho = \rho' \circ \phi$. We can see that $\pi'$ is in fact a faithful representation (cf. [3, Lemma 2.5.1]). Hence $\ker \pi = \ker \phi \subset \ker \rho$.

Next, we consider general cases. In fact, the $C^*$-algebras $\pi(\tilde{A}_K), \rho(\tilde{A}_K)$ are independent of the choice of $\gamma$. Hence we have a quotient map $\psi : \pi(\tilde{A}_K) \to \rho(\tilde{A}_K)$ obtained from the case of $\gamma = 1$. We can directly check that $\psi \circ \pi = \rho$ for general $\gamma$. Therefore, $\ker \pi \subset \ker \rho$ holds for general $\gamma$. □

Theorem 4.5 does not say anything about the topology of $\mathrm{Prim} A_K$. The most important fact is that the inclusion $\hat{\Gamma}_S \hookrightarrow \mathrm{Prim} A_K$ is a homeomorphism onto its range. However, we describe the topology of $\mathrm{Prim} A_K$ in detail because it is needed in Section 6.

**Definition 4.8.** (cf. [8, pp.437]) Let $2^{\mathcal{P}_K}$ be the power set of $\mathcal{P}_K$. The *power-cofinite topology* of $2^{\mathcal{P}_K}$ is the topology generated by

$$U_F = \{S \in 2^{\mathcal{P}_K} \mid S \cap F = \emptyset\},$$

where $F$ is a finite subset of $\mathcal{P}_K$.

Note that $\{U_F\}_F$ is a basis of the topology since we have $U_{F_1} \cap U_{F_2} = U_{F_1 \cup F_2}$.

**Proposition 4.9.** (cf. [8, Proposition 2.4]) *The canonical surjection*

$$Q : 2^{\mathcal{P}_K} \times \hat{J}_K \to \bigcup_{S \subset \mathcal{P}_K} \hat{\Gamma}_S = \mathrm{Prim} A_K, \ (S, \gamma) \mapsto \gamma|_{\Gamma_S} \in \hat{\Gamma}_S$$

*is an open continuous surjection.*

*Proof.* Define $Q_1 : X_K \times \hat{J}_K \to 2^{\mathcal{P}_K} \times \hat{J}_K$ by sending $(x, \gamma)$ to $(S_x, \gamma)$. Let $Q_2 : X_K \times \hat{J}_K \to \mathrm{Prim} A_K = X_K \times \hat{J}_K / \sim$ be the natural quotient map. Then we have $Q_2 = Q \circ Q_1$. The quotient map $\mathbb{A}_{K,f} \times G_K^{\mathrm{ab}} \to \mathbb{A}_{K,f} \times_{\hat{O}_K^*} G_K^{\mathrm{ab}} = X_K$ is denoted by $R$. Then we can show in the same way as in [8, Proposition 2.4] that

$$Q_1 \left( R \left( \prod_{\mathfrak{p} \in F} V_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin F} \mathcal{O}_{K,\mathfrak{p}} \times V \right) \times W \right) = U_G \times W, \text{ and}$$

$$Q_1^{-1}(U_F \times W) = R \left( \prod_{\mathfrak{p} \in F} K_{\mathfrak{p}}^* \times \prod_{\mathfrak{p} \notin F} (K_{\mathfrak{p}}, \hat{O}_{K_{\mathfrak{p}}}) \times G_K^{\mathrm{ab}} \right) \times W$$

for a finite set $F$ of $\mathcal{P}_K$, non-empty open sets $V_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$, $V$ of $G_K^{\mathrm{ab}}$ and $W$ of $\hat{J}_K$, where $G = \{\mathfrak{p} \in F \mid 0 \notin V_{\mathfrak{p}}\}$. This means that $Q_1$ is open and continuous. Since $Q_1$ is surjective and $Q_2 = Q \circ Q_1$ is open and continuous by Remark 3.7, $Q$ is also an open and continuous surjection. □

Let us briefly view when two points in $\mathrm{Prim} A_K$ can be separated by open sets. Take two distinct subsets $S_1, S_2$ of $\mathcal{P}_K$. If $S_1 \not\subset S_2$, then $Q(U_G \times \hat{J}_K) \cap \hat{\Gamma}_{S_1} = \emptyset$ and $Q(U_G \times \hat{J}_K) \supset \hat{\Gamma}_{S_2}$ for any finite subset $G$ of $S_1 \setminus S_2$. Hence, if $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$, then $\hat{\Gamma}_{S_1} \cup \hat{\Gamma}_{S_2}$ is Hausdorff with respect to the relative topology. If $S_1 \subset S_2$, then any open set which contains $\hat{\Gamma}_{S_2}$ also contains $\hat{\Gamma}_{S_1}$.

**Remark 4.10.** As mentioned, $\mathrm{Prim} A_K$ is a bundle over $2^{\mathcal{P}_K}$ with fibers $\hat{\Gamma}_S$. By Proposition 4.7, maximal primitive ideals sit in the fiber on $\mathcal{P}_K$, and second maximal primitive ideals sit in the fiber of $\mathcal{P}_K \setminus \{\mathfrak{p}\}$ for some $\mathfrak{p}$.

We study those ideals in Section 5 and Section 6 respectively. If $K = \mathbb{Q}$ or $K$ is imaginary quadratic, then $\Gamma_S$ is trivial for $S \neq \mathcal{P}_K$ because $K_+^*$ is closed in $\mathbb{A}_{K,f}^*$. In such cases, we have

$$\mathrm{Prim} A_K = 2^{\mathcal{P}_K} \setminus \{\mathcal{P}_K\} \cup \hat{P}_K^1.$$

In general cases, the concrete form of $\hat{\Gamma}_S$ is not known.

## 4.2 Faithful irreducible representations

By the KMS-classification theorem in [6], extremal KMS$_\beta$-states for $\beta > 1$ are obtained from irreducible representations $\pi_g$'s as in Section 2.2. We can check that $\pi_g$ is unitarily equivalent to $\pi_{\emptyset,1}$ if we choose $x$ in Remark 4.6 to $[g, 1] \in Y_K$.

We can see directly that these representations are not mutually unitarily equivalent.

**Proposition 4.11.** *The representations $\{\pi_g\}_g$ are not unitarily equivalent.*

*Proof.* We have the tensor product decomposition of the Hilbert space as follows:

$$\ell^2(I_K) \cong \bigotimes_{\mathfrak{p}} \ell^2(\mathbb{N}_\mathfrak{p}), \ \xi_{\prod_{\mathfrak{p} \in F} \mathfrak{p}^{k_\mathfrak{p}}} \mapsto \bigotimes_{\mathfrak{p} \in F} \xi_{k_\mathfrak{p}} \otimes \bigotimes_{\mathfrak{p} \notin F} 1,$$

where $\mathbb{N}_\mathfrak{p}$ is a copy of $\mathbb{N}$ and $F$ is a finite set of primes of $K$. In this decomposition, the $C^*$-subalgebra $C^*(I_K)$ of $\mathbb{B}(\ell^2(I_K))$ moves to $\bigotimes_\mathfrak{p} T_\mathfrak{p}$, where $T_\mathfrak{p}$ is a copy of the Toeplitz algebra ($T_\mathfrak{p}$ is generated by the unilateral shift on $\ell^2(\mathbb{N}_\mathfrak{p})$). Since $T_\mathfrak{p}$ contains $\mathbb{K}(\ell^2(\mathbb{N}_\mathfrak{p}))$, its commutant is trivial. Hence the commutant of $C^*(I_K)$ is trivial.

Suppose that $\pi_g$ and $\pi_h$ are unitarily equivalent. Then the implementing unitary $U$ commutes with $C^*(I_K)$. The above argument implies $U = 1$, so we have $\pi_g = \pi_h$. Hence $g = h$. $\square$

The representations $\pi_g$'s have the same kernel by Theorem 3.6. In fact, we have the following proposition:

**Proposition 4.12.** (cf. [8, Proposition 2.10]) *The representations $\pi_g$'s are faithful.*

*Proof.* It suffices to see that the conditional expectation $E : C(Y_K) \rtimes I_K \to C(Y_K)$ is recovered by $\pi_g$. From Lemma 4.2, we have $\overline{I_K g} = Y_K$. Indeed, if the sequence $\mathfrak{a}_n g$ for $\mathfrak{a}_n \in J_K$ converges to some $x \in Y_K$, then $\mathfrak{a}_n g \in Y_K$ for large $n$, which implies $\mathfrak{a}_n \in I_K$ for large $n$. Hence $C(Y_K)$ can be embedded into $\prod_{\mathfrak{a}\in I_K} \mathbb{C}$ by $f \mapsto \prod_{\mathfrak{a}\in I_K} f(\mathfrak{a}g)$. For $\mathfrak{a} \in I_K$, let $\varphi_{\mathfrak{a}}$ be the vector state $\langle \cdot \xi_{\mathfrak{a}}, \xi_{\mathfrak{a}} \rangle$ on $\mathbb{B}(\ell^2(I_K))$. Define a unital completely positive map $E'$ by

$$E' = \prod_{\mathfrak{a}\in I_K} \varphi_{\mathfrak{a}} : \mathbb{B}(\ell^2(I_K)) \to \prod_{\mathfrak{a}\in I_K} \mathbb{C}.$$

Then $E = E' \circ \pi_g$, which completes the proof. $\square$

Since $\Gamma_\emptyset = 1$, the fiber on $\emptyset \subset \mathcal{P}_K$ inside $\mathrm{Prim}A_K$ is just one point. The above proposition implies that the point is in fact the zero ideal.

# 5 Maximal primitive ideals

## 5.1 Dynamics on $\hat{P}_K^1$

Since we use the dynamics on $\hat{P}_K^1$ later, we prepare it in advance. We fix a notation of a dynamical system on a torus. For a (finite or infinite) sequence of positive numbers $\{r_j\}$, $(\prod_j \mathbb{T}_j, \prod_j r_j^{it})$ denotes the dynamical system determined by

$$\sigma_t((x_j)_j) = (r_j^{it} x_j)_j$$

for $x_j \in \mathbb{T}$ and $t \in \mathbb{R}$.

Let $K$ be a number field. We consider an action of $\mathbb{R}$ on $\hat{P}_K^1$ (as a topological space) defined by

$$\langle x, \sigma_t(\gamma) \rangle = N(x)^{it} \langle x, \gamma \rangle$$

for any $x \in P_K^1$, $\gamma \in \hat{P}_K^1$ and $t \in \mathbb{R}$, where $\hat{P}_K^1$ is the Pontrjagin dual of $P_K^1$. Note that $P_K^1$ is a free abelian group, since it is a subgroup of the free abelian group $J_K$. Hence $\hat{P}_K^1$ is isomorphic to the infinite product of circles. If $\{a_j\}$ is a basis of $P_K^1$, then the dynamical system $(\hat{P}_K^1, \sigma)$ is conjugate to $(\prod_j \mathbb{T}_j, \prod_j N(a_j)^{it})$.

## 5.2 Extraction of the narrow class number

The purpose of this section is to study quotients by maximal primitive ideals. As a consequence, maximal primitive ideals have information of the narrow class number. In a representation theoretic language, it is described as follows:

**Theorem 5.1.** *Let $(A_K, \sigma_t)$ be the Bost-Connes system for a number field $K$ and let $h_K^1$ be the narrow class number of $K$. Then $A_K$ has $h_K^1$-dimensional irreducible representations, and does not have $n$-dimensional irreducible representations for $n \neq h_K^1$ and $n < \infty$.*

**Lemma 5.2.** *The statement of* Theorem 5.1 *holds for $\tilde{A}_K$.*

*Proof.* Let $S \subset \mathcal{P}_K$ and let $\gamma \in \hat{J}_K$. By Lemma 3.9, the dimension of $\tilde{\pi}_{S,\gamma}$ in Remark 4.6 equals $[J_K : \Gamma_S]$. In general, if $\ker \pi = \ker \rho$ holds for irreducible representations $\pi, \rho$ of a $C^*$-algebra $A$, then we have $\dim \pi = \dim \rho$ because if either $\rho$ or $\pi$ is finite dimensional, then $A/\ker \pi \cong M_{\dim \pi}(\mathbb{C})$ is isomorphic to $A/\ker \rho \cong M_{\dim \rho}(\mathbb{C})$. Hence it suffices to show the following:

1. If $S \neq \mathcal{P}_K$, then $[J_K : \Gamma_S] = \infty$.
2. If $S = \mathcal{P}_K$, then $[J_K : \Gamma_S] = h^1_K$.

Since $\Gamma_{\mathcal{P}_K} = P^1_K$, the second statement is obvious. Suppose $S \neq \mathcal{P}_K$ and let $\mathfrak{p}$ be a prime of $K$ which does not belong to $S$. If $\mathfrak{a} \in \Gamma_S$, then we can see that $\mathfrak{p}^n\mathfrak{a}$'s are distinct elements in $J_K/\Gamma_S$. Therefore the index of $\Gamma_S$ is infinite. □

*Proof of Theorem 5.1.* Let $S \subset \mathcal{P}_K$ and $\gamma \in \hat{J}_K$. We need to show that $\dim \tilde{\pi}_{S,\gamma} = \dim \pi_{S,\gamma}$. If $S = \mathcal{P}_K$, then we have $\pi_{S,\gamma}(1_{Y_K}) = 1$ by definition of $\pi_{S,\gamma}$. Hence $\dim \tilde{\pi}_{S,\gamma} = \dim \pi_{S,\gamma}$ holds by Lemma 3.11. So it suffices to show that $\pi_{S,\gamma}$ is infinite dimensional if $S \neq \mathcal{P}_K$.

Take $x = [\rho, \alpha] \in X_K$ as in Remark 4.6 and take an integral ideal $\mathfrak{a} \in I_K$ such that $\mathfrak{a}x \in Y_K$ (we can always take such $\mathfrak{a}$ because $\rho_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$). Let $\mathfrak{p}$ be a prime of $K$ which does not belong to $S$. Then we have seen in the proof of Lemma 5.2 that the classes of $\mathfrak{p}^n$'s are distinct in $J_K/\Gamma_S$. Hence so are for $\mathfrak{p}^n\mathfrak{a}$'s. Since $\mathfrak{p}^n\mathfrak{a}x \in Y_K$ for $n \geq 0$, $\{\xi_{\mathfrak{p}^n\mathfrak{a}}\}_{n\in\mathbb{Z}}$ is an orthogonal family in $\pi_{S,\gamma}(1_{Y_K})\ell^2(J_K/\Gamma_S)$. Therefore $\pi_{S,\gamma}(1_{Y_K})\ell^2(J_K/\Gamma_S)$ is infinite dimensional. □

Theorem 1.2 is obtained as a corollary of Theorem 5.1.

**Example 5.3.** From the classification theorem of the KMS-states by Laca-Larsen-Neshveyev [6], we know that the Dedekind zeta function is an invariant of Bost-Connes systems. From Theorem 5.1, we know that the narrow class number is also an invariant. We can see that this is actually a new invariant. Indeed, there exist two fields which have the same Dedekind zeta function but different narrow class numbers. For example, let $K = \mathbb{Q}(\sqrt[8]{a}), L = \mathbb{Q}(\sqrt[8]{16a})$ for $a = -15$. Then $K$ and $L$ are totally imaginary fields, so their narrow class numbers $h^1_K, h^1_L$ are equal to their class numbers $h_K, h_L$. By the result of de Smit and Perlis [4], we have $\zeta_K = \zeta_L$ and $h^1_K/h^1_L = h_K/h_L = 2$.

**Definition 5.4.** Let $\mathcal{I}_{1,K}$ be the set of all maximal primitive ideals of $A_K$. We consider $\mathcal{I}_{1,K}$ as topological spaces with the relative topology of $\mathrm{Prim}A_K$.

For any maximal primitive ideal $P$ of $A_K$, Theorem 5.1 tells us that $A_K/P$ is isomorphic to $M_{h^1_K}(\mathbb{C})$. The space $\mathcal{I}_{1,K}$ is identified with $\hat{\Gamma}_{\mathcal{P}_K} = \hat{P}^1_K$, which is homeomorphic to $\mathbb{T}^\infty$. By Proposition 3.8, $\mathbb{R}$ acts on $\hat{P}^1_K$ as in Section 5.1. Hence we can get another invariant by restricting our attention to dynamics on $\hat{P}^1_K$.

**Proposition 5.5.** *Let $K, L$ be two number fields. If their Bost-Connes systems $(A_K, \sigma_{t,K})$ and $(A_L, \sigma_{t,L})$ are $\mathbb{R}$-equivariantly Morita equivalent, then $\hat{P}^1_K$ and $\hat{P}^1_L$ are $\mathbb{R}$-equivariantly homeomorphic.*

*Proof.* Let $\Phi : \mathrm{Prim}A_K \to \mathrm{Prim}A_L$ be the Rieffel homeomorphism induced from an $\mathbb{R}$-equivariant imprimitivity bimodule between Bost-Connes systems of $K$ and $L$. This is $\mathbb{R}$-equivariant by Proposition 3.5. Note that the Rieffel homeomorphism preserves the inclusion of ideals (cf. [12, Section 3.3]). Hence we have $\Phi(\hat{P}^1_K) = \hat{P}^1_L$, since those subspaces are characterized by the maximality of ideals. □

We study the dynamics $\hat{P}^1_K$ in Section 5.4.

## 5.3 Finite dimensional irreducible representations

We can determine finite dimensional irreducible representations explicitly by using Remark 4.6.

By Lemma 3.2, we have $X^0_K = C^1_K$ ($X^0_K$ is defined in Section 2.1). Since $X^0_K$ is a closed invariant set of $J_K$, we have a canonical quotient map $q_K : C(Y_K) \rtimes I_K \to C(C^1_K) \rtimes J_K$. Take a character $\gamma \in \hat{J}_K$. Then we have the $*$-homomorphism $\varphi_\gamma : C(C^1_K) \rtimes J_K \to C(C^1_K) \rtimes C^1_K$ defined by

$$\varphi_\gamma(f) = f, \text{ and } \varphi_\gamma(u_s) = \gamma(s)u_{\bar{s}},$$

for $f \in C(C^1_K)$ and $s \in J_K$, where $\bar{s}$ denotes the class of $s$ in $C^1_K$. Since $C(C^1_K) \rtimes C^1_K \cong M_n(\mathbb{C})$ for $n = |C^1_K| = h^1_K$, we obtain the surjection $\varphi_\gamma \circ q_K : A_K \to M_n(\mathbb{C})$. As usual, the $C^*$-algebra $C(C^1_K) \rtimes C^1_K$ acts on $\ell^2(C^1_K)$ by

$$(f\xi)(s) = f(s)\xi(s), \text{ and } (u_t\xi)(s) = \xi(t^{-1}s).$$

for $f \in C(C^1_K)$, $s, t \in C^1_K$ and $\xi \in \ell^2(C^1_K)$. So $\rho_\gamma = \varphi_\gamma \circ q_K$ defines an irreducible representation. If two elements $\gamma, \delta \in \hat{J}_K$ satisfy $\gamma\delta^{-1} \in \hat{P}^{1,\perp}_K$, then $\rho_\gamma$ is unitarily equivalent to $\rho_\delta$. Identifying those representations, the family $\{\rho_\gamma\}_{\gamma \in \hat{P}^1_K}$ is the complete representative of unitary equivalence classes of finite dimensional irreducible representations of $A_K$.

Benefiting from writing down representations associated to $\hat{P}^1_K$ in this form, we can prove the following proposition:

**Proposition 5.6.** *We have* $\ker q_K = \displaystyle\bigcap_{\gamma \in \hat{P}^1_K} \ker \rho_\gamma$.

The right hand side is equal to the intersection of all ideals in $\mathcal{I}_{1,K}$. Hence the above proposition follows from the following lemma, which is also used in Section 6.

**Lemma 5.7.** *Let $S$ be a subset of $\mathcal{P}_K$, and let*

$$Y^S_K = \{x = [\rho, \alpha] \in Y_K \mid \rho_\mathfrak{p} = 0 \textit{ for any } \mathfrak{p} \in S\},$$
$$P_S = C_0((Y^S_K)^c) \rtimes I_K.$$

*Then we have*

$$\bigcap_{\gamma \in \hat{\Gamma}_S} P_{S,\gamma} = P_S.$$

*In particular, $P_S$ is a primitive ideal if and only if $\Gamma_S = 1$.*

*Proof.* Here, we consider that representations $\pi_{S,\gamma}$ in Remark 4.6 are defined for any $\gamma \in \hat{J}_K$ (some of them are mutually unitarily equivalent, but we distinguish them).

By Lemma 4.2, $P_S$ is contained in $\ker \pi_{S,\gamma} = P_{S,\gamma}$ for any $\gamma$. Let $B = \mathbb{B}(\pi_{S,\gamma}(1_{Y_K})\ell^2(J_K/\Gamma_S))$. Then the image of the homomorphism

$$\prod_{\gamma \in \hat{J}_K} \pi_{S,\gamma} : A_K/P_S \to \prod_{\gamma \in \hat{J}_K} B.$$

is actually contained in $C(\hat{J}_K, B)$. Let $\Phi : A_K/P_S \to C(\hat{J}_K, B)$ be the restriction of that map. Since $\ker \Phi = \bigcap_\gamma \ker \pi_{S,\gamma}/P_S$, it suffices to show that $\Phi$ is injective.

We have $A_K/P_S \cong C(Y_K^S) \rtimes (J_K^{S^c} \times I_K^S)$, and $\Phi(f v_{\mathfrak{a}}) = \chi_{\mathfrak{a}} \otimes \pi_{S,1}(f v_{\mathfrak{a}})$ for $f \in C(Y_K^S), \mathfrak{a} \in J_K^{S^c} \times I_K^S$, where $\chi_{\mathfrak{a}}$ is the character on $\hat{J}_K$ corresponding to $\mathfrak{a}$. So we have the following commutative diagram:

$$\begin{tikzcd}
C(Y_K^S) \rtimes (J_K^{S^c} \times I_K^S) \arrow[r, "\Phi"] \arrow[d, "E"] & C(\hat{J}_K) \otimes B \arrow[d, "\mu \otimes \mathrm{id}_B"] \\
C(Y_K^S) \arrow[r] & B,
\end{tikzcd}$$

where $\mu$ is the Haar measure of $\hat{J}_K$. The homomorphism of the bottom line is injective by Lemma 4.2. This implies that $\Phi$ is injective. $\square$

**Corollary 5.8.** *Let $K, L$ be number fields. Then any isomorphism from $A_K$ to $A_L$ carries $\ker q_K = C_0(Y_K^\natural) \rtimes I_K$ to $\ker q_L = C_0(Y_L^\natural) \rtimes I_L$.*

## 5.4 Norm preserving map on $P_K^1$

As we have seen, the dynamics on $\mathcal{I}_{1,K} = \hat{P}_K^1$ is an invariant of Bost-Connes systems. The purpose of this section is to study the meaning of this invariant. In short, we have the following theorem:

**Theorem 5.9.** *Let $K, L$ be number fields. If their Bost-Connes systems $(A_K, \sigma_{t,K})$ and $(A_L, \sigma_{t,L})$ are $\mathbb{R}$-equivariantly Morita equivalent, then we have a group isomorphism $P_K^1 \to P_L^1$ which preserves the norm map.*

By definition in Section 5.1, the dynamics on $P_K^1$ is determined from the norm map $N : P_K^1 \to \mathbb{Q}$. Theorem 5.9 says the converse.

By Proposition 5.5, the above theorem is reduced to the following proposition:

**Proposition 5.10.** *Let $K, L$ be number fields. If $\hat{P}_K^1$ and $\hat{P}_L^1$ are $\mathbb{R}$-equivariantly homeomorphic, then there exists an $\mathbb{R}$-equivariant isomorphism between them.*

**Remark 5.11.** If $\hat{\varphi} : \hat{P}^1_L \to \hat{P}^1_K$ is an $\mathbb{R}$-equivariant isomorphism, then the isomorphism $\varphi : P^1_K \to P^1_L$ induced by $\hat{\varphi}$ preserves the norm. Indeed, let $a \in P^1_K$ and $b = \varphi(a) \in P^1_L$. Then, by taking the Pontrjagin duals, we have the following commutative diagram:

$$\begin{tikzcd} \hat{P}^1_L \arrow[r, "\sim", "\hat{\varphi}"'] \arrow[d] & \hat{P}^1_K \arrow[d] \\ (\hat{b^{\mathbb{Z}}}, N(b)^{it}) \arrow[r, "\sim"] & (\hat{a^{\mathbb{Z}}}, N(a)^{it}). \end{tikzcd}$$

The isomorphism $\hat{\varphi}$ is $\mathbb{R}$-equivariant by assumption, and it is easy to show that the vertical maps are $\mathbb{R}$-equivariant. Using these facts, we can show that the isomorphism $\hat{b^{\mathbb{Z}}} \to \hat{a^{\mathbb{Z}}}$ is $\mathbb{R}$-equivariant. This implies that $N(a) = N(b)$.

Note that the isomorphism in Proposition 5.10 is not canonical. The key observation is that the space $\hat{P}^1_K$ has a nice orbit decomposition.

**Lemma 5.12.** *Let $K$ be a number field. The compact group $\hat{P}^1_K$ is $\mathbb{R}$-equivariantly isomorphic to $(\prod_{j=1}^{\infty} \mathbb{T}_j \times \mathbb{T}^{\infty}, \prod_{j=1}^{\infty} n_j^{it} \times 1)$, where $n_j > 1$ and $\{n_j\}$ is linearly independent over $\mathbb{Z}$ in the free abelian group $\mathbb{Q}^*_+$.*

*Proof.* Let $N : P^1_K \to \mathbb{Q}^*_+$ be the ideal norm and let $A = N(P^1_K)$. Then the exact sequence

$$0 \longrightarrow \ker N \longrightarrow P^1_K \xrightarrow{N} A \longrightarrow 0$$

splits, because $\ker N, P^1_K$ and $A$ are all free abelian groups. Let $s : A \to P^1_K$ be the splitting of $N$, and take a basis $\{a_j\}_j$ of $s(A)$. Then we have the decomposition

$$P^1_K = \bigoplus_j a_j^{\mathbb{Z}} \oplus \ker N.$$

Taking the Pontrjagin duals, we have the desired decomposition. $\square$

**Remark 5.13.** The condition that $\{n_j\}$ is linearly independent in $\mathbb{Q}^*_+$ means that the homeomorphism on $\prod_j \mathbb{T}_j$ by multiplying $\prod_j n_j^{it}$ is minimal for appropriate $t \in \mathbb{R}$. Indeed, the family $\{1, \frac{t}{2\pi} \log n_j\}$ is linearly independent over $\mathbb{Q}$ if we choose $t = 2\pi$.

*Proof of Proposition 5.10.* Let $\varphi : \hat{P}^1_K \to \hat{P}^1_L$ be an $\mathbb{R}$-equivariant homeomorphism. Take the decomposition

$$P^1_K = \bigoplus a_j^{\mathbb{Z}} \oplus \ker N_K, \hat{P}^1_K = (\prod_j \mathbb{T}_j \times \mathbb{T}^{\infty}, \prod_j N(a_j)^{it} \times 1),$$
$$P^1_L = \bigoplus b_k^{\mathbb{Z}} \oplus \ker N_L, \hat{P}^1_L = (\prod_k \mathbb{T}_k \times \mathbb{T}^{\infty}, \prod_k N(b_k)^{it} \times 1)$$

as in Lemma 5.12. By Remark 5.13, We have the closed orbit decomposition

$$\hat{P}_K^1 = \coprod_{x\in\mathbb{T}^\infty} \prod_j \mathbb{T}_j \times \{x\}, \hat{P}_L^1 = \coprod_{y\in\mathbb{T}^\infty} \prod_k \mathbb{T}_k \times \{y\}.$$

Hence we have $\varphi(\prod_j \mathbb{T}_j \times \{1\}) = \prod_k \mathbb{T}_k \times \{y\}$ for some $y \in \mathbb{T}^\infty$, so $\varphi$ induces an $\mathbb{R}$-equivariant homeomorphism

$$\bar{\varphi} : (\prod_j \mathbb{T}_j, \prod_j N(a_j)^{it}) \to (\prod_k \mathbb{T}_k, \prod_k N(b_k)^{it}).$$

Let $\psi = \bar{\varphi}(1)^{-1}\bar{\varphi}$ and $x = \prod_j N(a_j)^{2\pi i}, y = \prod_k N(b_k)^{2\pi i}$. Then we have $\psi(a^l) = b^l$ for any $l \in \mathbb{Z}$. Hence $\psi$ is an $\mathbb{R}$-equivariant group isomorphism, since $a$ and $b$ generates dense subgroups in $\prod_j \mathbb{T}_j$ and $\prod_k \mathbb{T}_k$ respectively. Taking any group isomorphism $\tau$ of $\mathbb{T}^\infty$, we obtain an $\mathbb{R}$-equivariant group isomorphism $\psi \times \tau : \hat{P}_K^1 \to \hat{P}_L^1$. □

**Remark 5.14.** By the classification theorem of the KMS-states in [6], we know that if the Bost-Connes systems of two number fields $K, L$ are isomorphic then their Dedekind zeta functions are the same, which implies that there exists a group isomorphism $J_K \to J_L$ which preserves the norm.

By Theorem 5.9, the pair $(P_K^1, N : P_K^1 \to \mathbb{Q}_+^*)$ is an invariant of Bost-Connes systems. The difference between $(P_K^1, N : P_K^1 \to \mathbb{Q}_+^*)$ and $(J_K, N : J_K \to \mathbb{Q}_+^*)$ is thought to be very subtle because $P_K^1$ is of finite index in $J_K$. We do not know what difference exists between the two invariants. Instead, we can see that large information which is obtained by $(J_K, N : J_K \to \mathbb{Q}_+^*)$ can also be obtained by $(P_K^1, N : P_K^1 \to \mathbb{Q}_+^*)$. Here is an example:

**Proposition 5.15.** *Let $K, L$ be number fields with $n = [K : \mathbb{Q}] = [L : \mathbb{Q}]$. Suppose that there exists a group isomorphism $P_K^1 \to P_L^1$ which preserves the norm. Then for rational prime $p$, $p$ is non-split in $K$ if and only if $p$ is non-split in $L$.*

*Proof.* It suffices to show the equivalence of the following conditions:

1. $p$ is non-split in $K$.
2. There does not exist an element $a$ in $K_+^*$ satisfying $1 \le v_p(N(a)) < n$, where $v_p$ denotes the valuation of $\mathbb{Q}$ at $p$.

Suppose that $p$ is non-split in $K$. Then any element $a \in K_+^*$ satisfying $1 \le v_p(N(a))$ is a multiple of $p$ in $K$. Hence $n \le v_p(N(a))$ holds for such $a$.

Suppose that $p$ splits in $K$ and let $(p) = \prod \mathfrak{p}_i^{e_i}$ be the prime decomposition of $p$. Put $\mathfrak{p} = \mathfrak{p}_1$. By assumption, we have $1 \le v_p(N(\mathfrak{p})) < n$. Let $\mathfrak{m} = \prod \mathfrak{p}_i$ and let $J_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}$ be the ray class group modulo $\mathfrak{m}$. Since the natural map $J_K^{\mathfrak{m}}/P_K^{\mathfrak{m}} \to J_K/P_K^1$ is surjective, we can choose a fractional ideal $\mathfrak{b}$ that is prime to $(p)$ and satisfies $\mathfrak{b}\mathfrak{p} \in P_K^1$. Then $a = \mathfrak{b}\mathfrak{p}$ satisfies $1 \le v_p(N(a)) < n$. □

**Example 5.16.** Two quadratic fields $K, L$ can be distinguished by primes which are non-split in $K$ and $L$, because non-splitness of primes can be known by the Legendre symbol (cf. [10, Chapter I, Proposition 8.5], [13, Chapter VI, Proposition 14]). Hence, all Bost-Connes systems for quadratic fields are mutually non-isomorphic. This fact can also be obtained by the KMS classification theorem. So Theorem 5.9 gives another proof of this fact.

# 6 Second maximal primitive ideals

## 6.1 Structure of second maximal primitive ideals

Our next step is to give a proof of Theorem 1.3. The proof is based on the analysis of second maximal primitive ideals.

**Definition 6.1.** Let $A$ be a $C^*$-algebra and let $P$ be a primitive ideal of $A$. We say that $P$ is second maximal if the following holds:

1. There exists a primitive ideal $Q$ of $A$ such that $P \subsetneq Q$.
2. There does not exist a pair of primitive ideals $Q_1, Q_2$ of $A$ such that $P \subsetneq Q_1 \subsetneq Q_2$.

Note that a maximal primitive ideal $Q$ in the condition (1) may not be unique. For Bost-Connes $C^*$-algebras, second maximal primitive ideals are exactly of the form of $P_{\{\mathfrak{p}\}^c,\gamma}$ for some prime $\mathfrak{p}$ and $\gamma \in \hat{\Gamma}_{\{\mathfrak{p}\}^c}$ by Proposition 4.7. In the case of $K = \mathbb{Q}$ or imaginary quadratic fields, $P_{\{\mathfrak{p}\}^c}$ in Lemma 5.7 is a second maximal primitive ideal.

**Definition 6.2.** Let $\mathcal{I}_{2,K}$ be the set of all second maximal primitive ideals of $A_K$. We consider $\mathcal{I}_{2,K}$ as topological spaces with the relative topology of $\mathrm{Prim}A_K$.

**Lemma 6.3.** *The space $\mathcal{I}_{2,K}$ is equal to the direct sum of $\hat{\Gamma}_{\{\mathfrak{p}\}^c}$ for all $\mathfrak{p}$ as a topological space. In particular, $\mathcal{I}_{2,K}$ is Hausdorff.*

*Proof.* Let $Q_2 = \{\{\mathfrak{p}\}^c \in 2^{\mathcal{P}_K} \mid \mathfrak{p} \text{ is a prime}\}$. Then we can check that $Q_2$ is Hausdorff with respect to the relative topology of the power-cofinite topology. Let $\pi : 2^{\mathcal{P}_K} \times \hat{J}_K \to \mathrm{Prim}A_K$ be the canonical map. Then we can see that the restriction $\pi : Q_2 \times \hat{J}_K \to \mathcal{I}_{2,K}$ is an open continuous surjection. This means that each $\hat{\Gamma}_{\{\mathfrak{p}\}^c}$ is compact open inside $\mathcal{I}_{2,K}$. □

In summary, we have the following proposition:

**Proposition 6.4.** *There is one-to-one correspondence between $\mathcal{P}_K$ and connected components of $\mathcal{I}_{2,K}$. The connected component $\mathcal{C}_\mathfrak{p}$ corresponding to $\mathfrak{p}$ is equal to $\hat{\Gamma}_{\{\mathfrak{p}\}^c}$, and we have $\bigcap \mathcal{C}_\mathfrak{p} = P_{\{\mathfrak{p}\}^c}$.*

Our strategy is to extract information of a prime $\mathfrak{p}$ by looking at the corresponding connected component $\mathcal{C}_\mathfrak{p}$.

## 6.2 Arithmetic Observations

Fix a finite prime $\mathfrak{p}$ of $K$ which is above a rational prime $p \in \mathbb{Q}$. Let

$$U = \{1\} \times \prod_{\mathfrak{q} \neq \mathfrak{p}} \mathcal{O}_{\mathfrak{q}}^* \subset \mathbb{A}_{K,f}^*,$$
$$G_{\mathfrak{p}} = \mathbb{A}_{K,f}^* / \overline{K_+^*} U.$$

The group $G_{\mathfrak{p}}$ plays an important role later. We can see that the group $J_K^{\mathfrak{p}}$ can be considered as a subgroup of $G_{\mathfrak{p}}$ (for notations, see Section 3.1). The important point is that this is a dense subgroup. This follows from the following proposition:

**Proposition 6.5.** *We have*

$$G_{\mathfrak{p}} = \varprojlim_m J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}.$$

*Proof.* First, we define a homomorphism $\varphi_m : G_{\mathfrak{p}} \to J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}$. For $a \in \mathbb{A}_{K,f}^*$, take $k \in K_+^*$ such that $a_{\mathfrak{p}} k \equiv 1 \mod \mathfrak{p}^m$, then define $\tilde{\varphi}_m(a) = (ak)$. Then it is independent of the choice of $k$, and $\tilde{\varphi}_m$ is a group homomorphism from $\mathbb{A}_{K,f}^*$ to $J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}$. The homomorphism $\tilde{\varphi}_m$ is trivial on $\overline{K_+^*} U$ because $\overline{K_+^*}$ is contained in the open subgroup $K_+^* U_{\mathfrak{p}}^{(m)} U$. Hence, it induces a group homomorphism $\varphi_m : \mathbb{A}_{K,f}^* / \overline{K_+^*} U \to J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}$. One can see that this homomorphism is open and continuous.

Let us determine the kernel of $\varphi_m$. Clearly, $\ker \varphi_m$ contains the open subgroup $K_+^* U_{\mathfrak{p}}^{(m)} U / \overline{K_+^*} U$. For the reverse inclusion, let $a \in \mathbb{A}_{K,f}^*$ such that $\varphi_m(\bar{a}) = 1$. Here, $\bar{a}$ means the image of $a$ in $G$. Then we can take $k, l \in K_+^*$ such that $(ak) = (l)$ and $ak \equiv 1, l \equiv 1 \mod \mathfrak{p}^m$. This means $akl^{-1} \in U_p^{(m)} U$. So we have $\ker \varphi_m = K_+^* U_{\mathfrak{p}}^{(m)} U / \overline{K_+^*} U$.

The homomorphisms $\varphi_m$ commutes with the projective system $J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}$, so it induces a homomorphism $G_{\mathfrak{p}} \to \varprojlim_m J_K^{\mathfrak{p}} / P_K^{\mathfrak{p}^m}$. It is automatically surjective. To see that it is injective, it suffices to check $\bigcap_m \ker \varphi_m = 1$. Take $a \in \mathbb{A}_{K,f}^*$ such that $\bar{a} \in \bigcap_m \ker \varphi_m$, and we show that $\bar{a} = 1$. By multiplying an element of $\mathcal{O}_{K,+}^\times$, we may assume that $v_{\mathfrak{q}}(a) \geq 0$ for any $\mathfrak{q}$. Let

$$C = \{x \in \mathbb{A}_{K,f}^* \mid v_{\mathfrak{q}}(a) \geq v_{\mathfrak{q}}(x) \geq 0 \text{ for any } \mathfrak{q}\}.$$

Then $C$ is a compact open subset of $\mathbb{A}_{K,f}^*$ containing $\hat{\mathcal{O}}_K^*$. By assumption, for any $m$ there exist $k_m \in K_+^*$ and $a_m \in U_{\mathfrak{p}}^{(m)} U$ such that $a = k_m a_m$. Since we have $k_m = a a_m^{-1} \in C$, we can take an accumulation point $k \in \overline{K_+^*} \cap C$ and a subsequence $\{k_{m_j}\}$ of $\{k_m\}$ converging to $k$. Then the sequence $\{a_{m_j}\}$ converges to $ak^{-1}$. This implies $ak^{-1} \in \bigcap_m U_{\mathfrak{p}}^{(m)} U = U$, so $a \in \overline{K_+^*} U$, which completes the proof. $\square$

Proposition 6.5 gives an inductive limit structure of the $C^*$-algebra $C(G_{\mathfrak{p}}) \rtimes J_K^{\mathfrak{p}}$, which is useful to look into $K$-theory. The next lemma is used to examine the connecting maps of the inductive limit.

**Lemma 6.6.** *Let $f$ be the inertia degree of $\mathfrak{p}$ over $p$. For any $m \geq 1$, We have $[P_K^{\mathfrak{p}^m} : P_K^{\mathfrak{p}^{m+1}}] = p^f$.*

*Proof.* The group $P_K^{\mathfrak{p}^m}/P_K^{\mathfrak{p}^{m+1}}$ can be embedded into $U_{\mathfrak{p}}^{(m)}/U_{\mathfrak{p}}^{(m+1)}$. We show that this map is an isomorphism. Let $a \in \mathfrak{p}^m$, and take $k \in \mathcal{O}_{K,+}^{\times}$ such that $a \equiv k \mod \mathfrak{p}^{m+1}$. Then $l = 1 + k$ is in $P_K^{\mathfrak{p}^m}$, and $l(1+a)^{-1} \in U_{\mathfrak{p}}^{(m+1)}$. This implies the surjectivity, and the injectivity is clear.

The group $U_{\mathfrak{p}}^{(m)}/U_{\mathfrak{p}}^{(m+1)}$ is isomorphic to the additive group $\kappa_{\mathfrak{p}}$, where $\kappa_{\mathfrak{p}}$ is the residual field $\mathcal{O}_K/\mathfrak{p}$ (see [10, Chapter II, Proposition 3.10]). The order of $\kappa_{\mathfrak{p}}$ equals to $p^f$, so we have $[P_K^{\mathfrak{p}^m} : P_K^{\mathfrak{p}^{m+1}}] = p^f$. □

## 6.3 Structure of quotients

We focus on the $C^*$-algebra $P_{\mathcal{P}_K}/P_{\{\mathfrak{p}\}^c}$. By definition, we have

$$\begin{aligned}
P_{\mathcal{P}_K} &= \ker(C(Y_K) \rtimes I_K \to C(Y_K^{\mathcal{P}_K}) \rtimes J_K), \\
P_{\{\mathfrak{p}\}^c} &= \ker C(Y_K) \rtimes I_K \to C(Y_K^{\{\mathfrak{p}\}^c}) \rtimes (J_K^{\mathfrak{p}} \times \mathfrak{p}^{\mathbb{N}}).
\end{aligned}$$

and by Lemma 5.7, $P_{\mathcal{P}_K} = \bigcap \mathcal{I}_{1,K}, P_{\{p\}^c} = \bigcap \mathcal{C}_{\mathfrak{p}}$.

**Lemma 6.7.** *We have*

$$P_{\mathcal{P}_K}/P_{\{\mathfrak{p}\}^c} \cong \mathbb{K} \otimes C(G_{\mathfrak{p}}) \rtimes J_K^{\mathfrak{p}},$$

*where $G_{\mathfrak{p}}$ is the profinite group in Section 6.2.*

*Proof.* We have

$$\begin{aligned}
P_{\mathcal{P}_K}/P_{\{\mathfrak{p}\}^c} &= \ker(C(Y_K^S) \rtimes (J_K^{\mathfrak{p}} \times \mathfrak{p}^{\mathbb{N}}) \to C(Y_K^{\mathcal{P}_K}) \rtimes J_K) \\
&= C_0(\mathcal{O}_{\mathfrak{p}}^{\times} \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}) \rtimes (J_K^{\mathfrak{p}} \times \mathfrak{p}^{\mathbb{N}}),
\end{aligned}$$

and we focus on the dynamical system $J_K^{\mathfrak{p}} \times \mathfrak{p}^{\mathbb{N}} \curvearrowright \mathcal{O}_{\mathfrak{p}}^{\times} \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}$. First, $\mathcal{O}_{\mathfrak{p}}^{\times} \times_{\hat{\mathcal{O}}_K^*} G_K^{\mathrm{ab}}$ is naturally identified with

$$\mathcal{O}_{\mathfrak{p}}^{\times} \times_{\mathcal{O}_{\mathfrak{p}}^*} (G_K^{\mathrm{ab}}/U) = \mathcal{O}_{\mathfrak{p}}^{\times} \times_{\mathcal{O}_{\mathfrak{p}}^*} G_{\mathfrak{p}},$$

where $G_{\mathfrak{p}}$ and $U$ are as in Section 6.2. Fix a prime element $\pi_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$. Then $\mathcal{O}_{\mathfrak{p}}^{\times} \times_{\mathcal{O}_{\mathfrak{p}}^*} G_{\mathfrak{p}}$ is homeomorphic to $\mathbb{N} \times G_{\mathfrak{p}}$ by sending $[\pi_{\mathfrak{p}}^n, \alpha]$ to $(n, \alpha)$ for $n \in \mathbb{N}$ and $\alpha \in G_{\mathfrak{p}}$. Under this identification, the action is identified with the following action:

$$\mathfrak{q}(n, \alpha) = (n, \mathfrak{q}^{-1}\alpha), \ \mathfrak{p}(n, \alpha) = (n+1, \pi_{\mathfrak{p}}^{-1}\alpha).$$

Moreover, by the homeomorphism of $\mathbb{N} \times G_{\mathfrak{p}} \to \mathbb{N} \times G_{\mathfrak{p}}$ defined by $(n, \alpha) \mapsto (n, \pi_{\mathfrak{p}}^n \alpha)$, this action is conjugate to the following action:

$$\mathfrak{q}(n, \alpha) = (n, \mathfrak{q}^{-1}\alpha), \ \mathfrak{p}(n, \alpha) = (n+1, \alpha).$$

Hence, we have

$$P_{\mathcal{P}_K}/P_{\{\mathfrak{p}\}^c} \cong C_0(\mathbb{N} \times G_{\mathfrak{p}}) \rtimes_{\alpha \otimes \beta} \mathbb{N} \times J_K^{\mathfrak{p}} \cong (C_0(\mathbb{N}) \rtimes_\alpha \mathbb{N}) \otimes (C(G_{\mathfrak{p}}) \rtimes_\beta J_K^{\mathfrak{p}}),$$

where $\alpha$ is the action by addition and $\beta$ is the action by multiplication ($J_K^{\mathfrak{p}}$ is naturally identified with a subgroup of $G_{\mathfrak{p}}$). The second isomorphism is established by writing both algebras as corners of group crossed products and applying the well-known decomposition theorem for tensor product actions. The $C^*$-algebra $C_0(\mathbb{N}) \rtimes_\alpha \mathbb{N}$ is isomorphic to $\mathbb{K}$, because it is written as a full corner of $C_0(\mathbb{Z}) \rtimes \mathbb{Z}$. $\square$

**Remark 6.8.** Brownlowe-Larsen-Putnam-Raeburn give a similar presentation in the case of $K = \mathbb{Q}$. Lemma 6.7 is a kind of generalization of [2, Theorem 4.1 (2)] in the case of $\mathcal{P} \setminus S$ is one point.

## 6.4 $K_0$-groups of profinite actions

In this section, we prepare a general machinery which is used in the next section. Let $\Gamma$ be a discrete amenable torsion-free group. In our case, we only need the case of $\Gamma = \mathbb{Z}^\infty$, but here we treat the general case. Let $P_m$ be a decreasing sequence of finite index normal subgroups of $\Gamma$ such that $\bigcap_m P_m = 1$. Define a profinite group $G$ by $G = \varprojlim_m \Gamma/P_m$. Then, by assumption, $\Gamma$ is a dense subgroup of $G$ and $G/\overline{P_m} = \Gamma/P_m$. Let $h_m = [\Gamma : P_m]$. The $C^*$-algebra $C(G) \rtimes \Gamma$ is simple and has a unique tracial state.

**Proposition 6.9.** *Let $\tau$ be the unique tracial state of $C(G) \rtimes \Gamma$. Then we have*

$$\tau_*(K_0(C(G) \rtimes \Gamma)) = \bigcup_m h_m^{-1}\mathbb{Z}.$$

*Proof.* By assumption, we have

$$C(G) \rtimes \Gamma \cong \varinjlim C(\Gamma/P_m) \rtimes \Gamma,$$

and by the imprimitivity theorem, $C(\Gamma/P_m) \rtimes \Gamma$ is Morita equivalent to $C_r^*(P_m)$. In this case, $1_{P_m} C_r^*(P_m) = 1_{P_m}(C(\Gamma/P_m) \rtimes \Gamma)1_{P_m}$ and $1_{P_m}$ is a full projection of $C(\Gamma/P_m) \rtimes \Gamma$. The inclusion $1_{P_m} C_r^*(P_m) \hookrightarrow C(\Gamma/P_m) \rtimes \Gamma$ gives the isomorphism of $K$-groups.

Let $\tau$ be the unique tracial state of $C(G) \rtimes \Gamma$. Then it is equal to $\mu \circ E$, where $\mu$ is the Haar measure of $G$ and $E$ is the canonical conditional expectation $C(G) \rtimes \Gamma \to C(G)$. Then the restriction of $\tau$ onto $C(\Gamma/P_m) \rtimes \Gamma$ is equal to $\mu_m \circ E$, where $\mu_m$ is the normalized counting measure of $\Gamma/P_m$. Hence the restriction of $\tau$ onto $1_{P_m} C_r^*(P_m)$ is equal to $h_m^{-1}\tau^{P_m}$, where $\tau^{P_m}$ is the canonical trace of $C_r^*(P_m)$.

Since $P_m$ is torsion-free and satisfies the Baum-Connes conjecture (cf. [17, Proposition 6.3.1]), we have

$$\tau_*^{P_m}(K_0(C_r^*(P_m))) = \mathbb{Z}$$

as a subgroup of $\mathbb{R}$. Hence

$$\begin{aligned}\tau_*(K_0(C(G) \rtimes \Gamma)) &= \bigcup_m \tau_*(C(\Gamma/P_m) \rtimes \Gamma) \\ &= \bigcup_m h_m^{-1}\tau_*^{P_m}(C_r^*(P_m)) \\ &= \bigcup_m h_m^{-1}\mathbb{Z}.\end{aligned}$$

□

In our case we actually have to treat with unbounded traces. In this paper, we tacitly assume unbounded traces have finite values on finite projections. The following lemma is proved in usual way:

**Lemma 6.10.** *Let $A$ be a simple $C^*$-algebra with a unique tracial state $\tau$. Then $\mathbb{K} \otimes A$ has a unique unbounded trace $\mathrm{Tr} \otimes \tau$ up to scalar multiplication.*

## 6.5 Reconstruction of the zeta function

In summary, we have the following theorem:

**Theorem 6.11.** *Let $K$ be a number field and let $\mathfrak{p}$ be a finite prime of $K$, and let $\mathfrak{p} \cap \mathbb{Z} = (p)$. Then $\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}_\mathfrak{p}$ is a simple $C^*$-algebra with a unique unbounded trace $T$ up to scalar multiplication, and we have*

$$T_*(K_0(\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}_\mathfrak{p})) \cong \mathbb{Z}[1/p].$$

*Proof.* By Proposition 6.4 and Lemma 6.7, the $C^*$-algebra $\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}_\mathfrak{p}$ is isomorphic to $\mathbb{K} \otimes C(G_\mathfrak{p}) \rtimes J_K^\mathfrak{p}$. Let $\tau$ be the unique trace of $C(G_\mathfrak{p}) \rtimes J_K^\mathfrak{p}$. By Lemma 6.10, we may assume $T = \mathrm{Tr} \otimes \tau$ because the isomorphism class of $T_*(K_0(\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}_\mathfrak{p}))$ is independent of the choice of $T$. By Proposition 6.5, we have

$$C(G_\mathfrak{p}) \rtimes J_K^\mathfrak{p} \cong \varinjlim_m C(J_K^\mathfrak{p}/P_K^{\mathfrak{p}^m}) \rtimes J_K^\mathfrak{p}.$$

So by applying Proposition 6.9, we have

$$T_*(K_0(\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}_\mathfrak{p})) = \tau_*(K_0(C(G_\mathfrak{p}) \rtimes J_K^\mathfrak{p})) = \bigcup_m h_m^{-1}\mathbb{Z},$$

where $h_m = [J_K^\mathfrak{p} : P_K^{\mathfrak{p}^m}]$. By Lemma 6.6, we have $h_{m+1}/h_m = p^f$, so $\bigcup_m h_m^{-1}\mathbb{Z} \cong \mathbb{Z}[1/p]$. □

The proof of Theorem 1.3 is obtained by applying the theorem of Stuart-Perlis [14]. For a rational prime $p$, $g_K(p)$ denotes the splitting number of $p$, i.e., the number of primes of $K$ which is above $p$.

*Proof of Theorem 1.3.* By Theorem 6.11 and Proposition 6.4, $g_K(p)$ is equal to the number of connected components $\mathcal{C}$ of $\mathcal{I}_{2,K}$ which satisfy

$$T_*(K_0(\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C})) = \mathbb{Z}[1/p]$$

for some unbounded trace $T$ of $\bigcap \mathcal{I}_{1,K}/\bigcap \mathcal{C}$. Since $\mathbb{Z}[1/p] \cong \mathbb{Z}[1/q]$ if and only if $p = q$ for any rational prime $p, q$, this number is preserved under the isomorphism. By [14, Main Theorem], the equality of splitting numbers for all rational primes implies the equality of zeta functions. $\square$

Zeta functions consist of the information of the rational prime which is below a prime and the inertia degree. Since we applied a number theoretic theorem, the inertia degree is not naturally obtained. It may be interesting to ask how to get the inertia degree in an operator algebraic way from the $C^*$-algebra $A_K$.

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