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CHAPTER 1

Introduction

One of the most fundamental problems in mathematical physics, especially in relation to statistical physics, is to relate a macroscopic system such as a system of partial differential equations with a microscopic system. A microscopic system is naturally expected to have a large number of degrees of freedom and self-interaction structures. Such a microscopic system is called a large scale interacting system in probability theory. Hydrodynamic limit, which is a limiting procedure in a space-time scaling for the large scale interacting systems, gives an answer to the above problem. For instance, by means of the hydrodynamic limit, a system such as a Stefan free boundary problem can be derived from some sort of large scale interacting systems. The theory of the hydrodynamic limit also plays an important role in the study of stationary non-equilibrium states, which is one of the main objects in thermodynamic theory. In recent years, the large deviation principle in this context has been extensively studied by both mathematicians and physicists. Especially "Macroscopic fluctuation theory" developed by Bertini et al. in [**8**] gives a unified approach to the study of the stationary non-equilibrium states from the point of view of a microscopic system. The theory of the hydrodynamic limit is used to formulate this theory in a mathematical way.

This thesis is dedicated to studying some problems related to the hydrodynamic limit for lattice-gas. In this thesis, we consider the following types of lattice-gas consisting of particles evolving in a one-dimensional discrete domain. Lattice-gas is described by a superposition of the Kawasaki dynamics and the Glauber dynamics. More precisely, for each fixed $N > 0$, let $\mathbb{T}_N$ be the one-dimensional discrete torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \cdots, N-1\}$. The state space of our process is given by $\{0,1\}^{\mathbb{T}_N}$ and denote by η an element of $\{0,1\}^{\mathbb{T}_N}$, which describes a configuration on $\mathbb{T}_N$ such that $\eta(x) = 1$ if there is a particle at $x \in \mathbb{T}_N$ and $\eta(x) = 0$, otherwise. We consider in the set $\mathbb{T}_N$ the superposition of the exclusion process with speed change (Kawasaki) with a spin-flip dynamics (Glauber). The stochastic dynamics is a Markov process on $\{0,1\}^{\mathbb{T}_N}$ whose generator, denoted by L_N in this introduction, acts on functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$L_N f \;=\; (N^2/2) L_K f + L_G f \;,$$

where L_K is the generator of the simple exclusion process with speed change (Kawasaki dynamics),

$$(L_K f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} c_{0,1}(\tau_x\eta)[f(\eta^{x,x+1}) - f(\eta)] \;,$$

and L_G is the generator of a spin flip dynamics (Glauber dynamics),

$$(L_G f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} c_0(\tau_x\eta)[f(\eta^{x}) - f(\eta)] \;.$$

In these formulas, $\eta^{x,x+1}$, η^x, $\tau_x\eta$, represents the configuration obtained from η by exchanging, flipping, translating by x, the occupation variables $\eta(x)$ and $\eta(x+1)$, $\eta(x)$, respectively. For this model, the hydrodynamic equation is formally given by

$$\partial\rho_t \;=\; \nabla(D(\rho)\nabla\rho) + F(\rho) \;,$$

where D and F are functions on $[0,1]$ and are determined only by $c_{0,1}$ and c_0 respectively.

In Chapter 2, we consider a system of particles called an exclusion process with speed change. This model is realized as a lattice-gas with the jump rates $c_0 \equiv 0$ and general $c_{0,1}$. We consider a tagged particle problem for this particle system and study a law of large numbers for a tagged particle under the diffusive scaling. Combining this with the result of the hydrodynamic limit, we derive a Stefan free boundary problem in a diffusive scaling limit. Chapter 2 is based on a work [**39**].

Several kinds of Stefan free boundary problems have been derived from particle systems by many authors. Funaki [**26**] derived a nonlinear one-phase Stefan free boundary problem from a system consisting of two types of particles called “water” and “ice” on multi-dimensional periodic lattices. Landim, Olla and Volchan [**35**] derived a Stefan free boundary problem from an infinite system of particles evolving in a one-dimensional lattice according to symmetric random walks with hard core interaction. Komoriya [**33**] derived a Stefan free boundary problem from a system of two types of particles moving on a one-dimensional lattice according to simple random walks. The methods developed in [**33, 35**] strongly depend on the analysis of the related zero-range process. We note that our proof is different from these.

In Chapter 3, we consider a system of particles with creation and annihilation of particles. We call this system reaction-diffusion model and the reaction-diffusion model is also realized as a lattice-gas with the jump rates $c_{0,1} \equiv 1$ and general c_0. We prove the hydrostatics and the dynamical large deviation principle for the reaction-diffusion model. Chapter 3 is based on a joint work [**36**] with Professor Claudio Landim.

The hydrostatics and the dynamical large deviations for the boundary driven exclusion processes are studied in [**20, 7, 23**]. In their models, the stationary solution of the hydrodynamic equation is always unique. Therefore the solution of the hydrodynamic equation converges to its unique stationary solution as time go to infinity. Moreover, the system conserves a total mass in the bulk. Therefore the H^{-1} norm of the solution of the hydrodynamic equation is finite. These facts allow to use usual techniques to prove the hydrostatics and the dynamical large deviation principle and are not true for our model in general. The dynamical large deviation principle for non-conservative dynamics in the bulk is studied in [**13**]. However they assumed some monotonicity for the non-linear term of the hydrodynamic equation and, under their assumption, the stationary solution of the hydrodynamic equation is always unique.

The aim of our study is to extend the previous works to the reaction-diffusion model. We also prove that the large deviations rate function is lower semicontinuous and has compact level sets. These properties play a fundamental role in the proof of the static large deviation principle, discussed in Chapter 4, for the empirical measure under the stationary state [**11, 22**]. The main difficulty in the proof of the lower bound of the large deviation principle comes from the presence of exponential terms in the rate function, denoted in this introduction by I. In contrast with conservative dynamics, for a trajectory $u(t,x)$, $I(u)$ is not expressed as a weighted H_{-1} norm. This forces the development of new tools to prove that smooth trajectories are I-dense.

In Chapter 4, we consider the stationary state of the reaction-diffusion model. We prove the static large deviation principle for the empirical measure under the stationary state. The static rate function is determined by the dynamical one considered in Chapter 3 and the structure of the set of all stationary solutions to the hydrodynamic equation. Chapter 4 is based on a joint work [**24**] with Professor Jonathan farfan and Professor Claudio Landim.

The static large deviations for the boundary driven exclusion processes are studied in [**7, 11, 22**]. As mentioned in the above, for their models, there exists a unique stationary solution of the hydrodynamic equation. In this case, the static rate function is described by the quasi potential similar to a finite-dimensional setting (c.f. [**21**]). In our case, there are several stationary solutions of the hydrodynamic equation in general. This fact makes the structure of the static rate function and the proof of the large deviation principle much complicated. To overcome this difficulty one needs to construct several paths in our infinite-dimensional setting.

This thesis is organized as follows: In Chapter 2, we prove the convergence of a tagged particle under a diffusive scaling and derive a Stefan problem from a one-dimensional exclusion process with speed change. In Chapter 3, we prove the hydrostatics and the dynamical large deviation principle for a reaction-diffusion model. In Chapter 4, we prove the static large deviation principle for a reaction-diffusion model.

CHAPTER 2

Derivation of Stefan problem from a one-dimensional exclusion process with speed change

1. Introduction to Chapter 2

Bertsch, Dal Passo and Mimura [**3**] introduced a mathematical model which describes a phenomenon of two different types of cells, called "contact inhibition of growth between two cells". They formulated this phenomenon as a one-dimensional system of partial differential equations, see Remark 2.2 below. In some cases, the system of partial differential equations can be expressed as a Stefan free boundary problem. Our goal is to relate a microscopic particle system to the macroscopic Stefan free boundary problem. We study a model of a microscopic particle system called an exclusion process with speed change to derive the Stefan free boundary problem. From the point of view of the particle system, a solution to the partial differential equation describes an evolution of the macroscopic density of particles and it is derived under a diffusive scaling limit called the hydrodynamic limit. On the other hand, the moving Stefan free boundary corresponds to the behavior of a tagged particle in the particle system. The hydrodynamic limit for the exclusion process with speed change was already studied by Funaki and Uchiyama [**27**] and Funaki, Uchiyama and Yau [**28**]. In this chapter, we study the behavior of a tagged particle for the particle system.

Let us explain the relationship between the Stefan problem introduced in [**3**] and our particle system more precisely. Let $\mathbb{T}$ be the one-dimensional torus $\mathbb{R}/\mathbb{Z} = [0, 1)$ under the identification of 0 and 1. Consider two intervals I_1 and I_2 which satisfy $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \mathbb{T}$. Assume that two different types of cells are initially distributed on I_1 and I_2 respectively. For each time $t \geq 0$, the moving Stefan free boundaries describes the time evolution of intervals, say $I_1(t)$ and $I_2(t)$. Since $I_1(t) \cap I_2(t) = \emptyset$ and $I_1(t) \cup I_2(t) = \mathbb{T}$, the moving boundary actually consists of two points. We denote them by $\tilde{u}_t^1$ and $\tilde{u}_t^2$. We now consider two types of particles, say red particles and blue particles. We interpret a system of these red and blue particles as a microscopic model of two different types of cells. If red particles and blue particles have same velocity, the Stefan free boundary problem considered in [**3**] reduces to the following system:

$$\partial_t \rho = \partial_u (D(\rho)\partial_u \rho), \tag{1.1}$$

$$\frac{d}{dt}\tilde{u}_t^i = -\frac{(D(\rho)\partial_u \rho)(t, \tilde{u}_t^i)}{\rho(t, \tilde{u}_t^i)}, \quad i = 1, 2\ , \tag{1.2}$$

where $\rho(t,u) : [0,\infty) \times \mathbb{T} \to [0,\infty)$, $\tilde{u}_t^i : [0,\infty) \to \mathbb{T}$ and $D : [0,1] \to [0,\infty)$. Though we consider the special situation such that red particles and blue particles have same velocity, Theorem 1.2 in [**3**] actually discusses such situation. Under this situation, the derivation of the Stefan problem can be reduced to solving a tagged particle problem for one type of particle systems. We will describe the relationship in Remark 2.2 much clearly.

We emphasize two facts. One is that, if one of the moving boundaries $\tilde{u}_t^1$ is determined, then the other moving boundary $\tilde{u}_t^2$ is automatically determined by the conservation law of the total number of red and blue particles. Moreover the other moving boundary automatically satisfies the same equation (1.2), see Remark 2.3. The other is that, once (1.1) is derived, the derivation of (1.2) under Neumann boundary conditions is not difficult. However, on the periodic domain $\mathbb{T}$, it is not clear that one of the moving boundaries $\tilde{u}_t^1$ satisfies (1.2), see Remark 2.4. Hence we consider a tagged particle problem to derive the equation (1.2) on the periodic domain $\mathbb{T}$ and we derive it as a scaling limit for a tagged particle.

This chapter is organized as follows: In Section 2, we introduce our model and state our main result. We relate our results and the Stefan free boundary problem considered in [**3**] in Remark 2.2. In Section 3, we show the law of large numbers for the diffusively scaled current across the bond. In Section 4, we prove our main theorem, Theorem 2.2, using the results established in Section 3.

2. Model and main result

In this section, we precisely formulate our particle system and state a main result. We consider an exclusion process with speed change. Before defining the process, we introduce some notation. Let $\mathbb{T}_N$ be the one dimensional discrete torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \cdots, N-1\}$. The state space of our process is given by $\{0,1\}^{\mathbb{T}_N}$ and denote by η an element of $\{0,1\}^{\mathbb{T}_N}$, which describes a configuration on $\mathbb{T}_N$ such that $\eta(x) = 1$ if there is a particle at $x \in \mathbb{T}_N$ and $\eta(x) = 0$, otherwise. The time evolution η_t^N of the exclusion process with speed change is determined as a Markov process whose generator acting on local functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ is given by

$$L_N f(\eta) = N^2 \sum_{x \in \mathbb{T}_N} c_{x,x+1}(\eta)\{f(\eta^{x,x+1}) - f(\eta)\},$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging configurations on x and $x+1$:

$$\eta^{x,x+1}(z) = \begin{cases} \eta(x+1) & \text{if } z = x, \\ \eta(x) & \text{if } z = x+1, \\ \eta(z) & \text{otherwise.} \end{cases}$$

Note that we have already put the time change factor N^2 in L_N. We first consider the jump rates $c_{x,x+1}$ as functions on $\{0,1\}^{\mathbb{T}_N}$. The fact of the matter, under the condition (2.2) below, the jump rates $c_{x,x+1}$ can be regarded as local functions on $\{0,1\}^{\mathbb{T}_N}$. We assume the following conditions for jump rates $c_{x,x+1}$:

(2.1) Spatial uniformity : $c_{x,x+1}(\eta) = c_{0,1}(\tau_x \eta)$.

(2.2) Non-degeneracy : $c_{0,1}(\eta) > 0$ for any configuration η.

(2.3) Locality : $c_{0,1}$ depends only on finite coordinates of η.

(2.4) Symmetry : $c_{0,1}(\eta) = c_{0,1}(\eta^{0,1})$.

In the condition (2.1), $\{\tau_x\}_{x \in \mathbb{Z}}$ stands for a translation group acting on the configuration space $\{0,1\}^{\mathbb{T}_N}$:

$$(\tau_x \eta)(y) = \eta(x+y), \ y \in \mathbb{T}_N \ ,$$

where, for each $x \in \mathbb{Z}$ and $y \in \mathbb{T}_N$, $x+y$ is considered modulo N. We use the same notation $\{\tau_x\}_{x \in \mathbb{Z}}$ as a translation group acting on the configuration space $\{0,1\}^{\mathbb{Z}}$.

We now briefly discuss the limiting behavior of the empirical measure, so-called hydrodynamic limit. Define the empirical measure π_t^N by

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t^N(x) \delta_{\frac{x}{N}}(du),$$

where δ_u stands for the Dirac measure which has a point mass at $u \in \mathbb{T}$. We define the diffusion coefficient $D : [0,1] \to \mathbb{R}$ as follows. Let ν_ρ be the Bernoulli product measure with density ρ defined on $\{0,1\}^{\mathbb{Z}}$. From the condition (2.4), these measures are reversible under our process. The expectation with respect to ν_ρ will be denoted by $E^{\nu_\rho}[\cdot]$. For each $x \in \mathbb{Z}$ and each local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, define

$$(\tau_x f)(\eta) = f(\tau_x \eta),$$
$$(\sigma_{0,1} f)(\eta) = f(\eta^{0,1}) - f(\eta).$$

For each density $0 < \rho < 1$ and each local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, define

$$D(\rho; f) = \frac{1}{2\chi(\rho)} E^{\nu_\rho}[\{\eta(0) - \eta(1) - \sum_{y \in \mathbb{Z}} \{\sigma_{0,1}(\tau_y f)\}(\eta)\}^2 c_{0,1}(\eta)],$$
$$\chi(\rho) = \rho - \rho^2,$$

and the diffusion coefficient D by the variational formula

$$D(\rho) = \inf\{D(\rho; f) : f \text{ are local functions on } \{0,1\}^{\mathbb{Z}}\}.$$

REMARK 2.1. *It is known that the diffusion coefficient D is continuously extended to two endpoints 0 and 1, cf. Funaki and Uchiyama* [**27**]. *Furthermore Bernardin* [**2**] *proved under the conditions (2.1)-(2.4) that the diffusion coefficient D is infinitely differentiable on the interval $[0,1]$. Moreover, from the condition (2.2), we can show that D is strictly positive on the interval $[0,1]$. Note that, from the Lipschitz continuity and positivity of the diffusion coefficient D, the H^{-1} method gives the uniqueness of the weak solution of the Cauchy problem (2.5) below for any measurable initial profile $\rho_0 : \mathbb{T} \to [0,1]$, cf. Kipnis and Landim* [**31**].

The next result is known. We refer to [**27, 28, 31**] for its proof.

THEOREM 2.2. *Assume that there exists a measurable function $\rho_0 : \mathbb{T} \to [0,1]$ such that for every smooth function $J : \mathbb{T} \to [0,1]$, it holds that*

$$\lim_{N\to\infty} \int_{\mathbb{T}} J(u)\pi_0^N(du) \; = \int_{\mathbb{T}} J(u)\rho_0(u)du, \text{ in probability.}$$

Then for every $t > 0$ and every smooth function $J : \mathbb{T} \to [0,1]$, we have

$$\lim_{N\to\infty} \int_{\mathbb{T}} J(u)\pi_t^N(du) \; = \int_{\mathbb{T}} J(u)\rho(t,u)du, \text{ in probability,}$$

where $\rho(t,u) : [0,\infty) \times \mathbb{T} \to [0,1]$ is determined as a unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho(t,u) = \partial_u (D(\rho(t,u))\partial_u \rho(t,u)), \\ \rho(0,u) = \rho_0(u). \end{cases} \tag{2.5}$$

Let μ^N be the initial distribution of the exclusion process η_t^N with speed change. As we explained in the introduction, the moving boundary actually consists of two points. We note that if one of the moving boundaries is determined, then the other moving boundary is determined by the conservation law for the total number of red and blue particles (see Remark 3.1). Therefore we concentrate on the analysis of one tagged particle. We now assume that there is a particle sitting at site 0 at time 0, that is, $\mu^N(\eta; \eta(0) = 1) = 1$. Moreover we also assume that the assumption of Theorem 2.1 holds with a positive $C^1(\mathbb{T})$-smooth function ρ_0. Trace the particle sitting at site 0 at time 0 and denote its position at time t by $X_t^N \in \mathbb{T}_N$. Denote by $\mathbb{P}^N$ the probability measure on the Skorokhod space $D([0,\infty), \{0,1\}^{\mathbb{T}_N})$ induced by the process $\{\eta_t^N : t \geq 0\}$ with the initial measure μ^N. The expectation with respect to $\mathbb{P}^N$ is denoted by $\mathbb{E}^N[\cdot]$. We are interested in the behavior of the rescaled position of the tagged particle defined by $u_t^N = \frac{1}{N} X_t^N$. The following theorem is a main result of this chapter. Denote by $\rho(t,u) : [0,\infty) \times \mathbb{T} \to [0,1]$ the unique weak solution of (2.5).

THEOREM 2.3. *The tagged particle X_t^N starts from 0, that is, $X_t^N = 0$. Then, every $t \geq 0$, we have*

$$\lim_{N\to\infty} u_t^N = u_t, \text{ in probability,}$$

where u_t is defined as follows. We first consider the solution, $\tilde{u}_t \in \mathbb{R}$ of the implicit equation

$$\begin{cases} \int_0^{\tilde{u}_t} \rho(t,u)du = -\int_0^t (D(\rho)\partial_u\rho)(s,0)ds, \\ \tilde{u}_0 = 0. \end{cases} \tag{2.6}$$

and u_t is defined as the element in $\mathbb{T}$ which satisfies $u_t \equiv \tilde{u}_t$ mod 1.

REMARK 2.4. *Bertsch et al.* **[3]** *considered the following Stefan free boundary problem on the finite interval $[-L, L]$ under Neumann boundary conditions and initial conditions:*
(2.7)

$$\begin{cases} w_t = (w(\chi(w))_x)_x + w(1-w) & \text{if } -L < x < \zeta(t), t > 0, \\ w_t = d(w(\chi(w))_x)_x + \gamma w(1-w/k) & \text{if } \zeta(t) < x < L, t > 0, \\ \zeta'(t) = -(\chi(w))_x(\zeta(t)^-, t) = -d(\chi(w))_x(\zeta(t)^+, t) & \text{for } t > 0. \end{cases}$$

We now relate our results and the system (2.7). Consider the system (2.7) with the function χ such that $\alpha\chi'(\alpha) = D(\alpha)$, $d = 1$ and without reaction terms, then the equations on w and ζ can be written as

$$\begin{cases} w_t = (w_x D(w))_x, \\ \zeta'(t) = -\dfrac{(w_x D(w))(t,\zeta(t))}{w(t,\zeta(t))}. \end{cases}$$

As we mentioned in the introduction, we treat the state space of the system to have a periodic boundary. Although we have to consider another moving boundary by its topological effect, from Theorem 2.1, 2.2 and Lemma 3.2, we can derive the system (2.7) without reaction terms and $d = 1$ from the microscopic particle system. The similar derivation is given in **[33]** *for the linear case $D \equiv 1$ and general $d > 0$. The derivation of the system (2.7) for the general case $d > 0$ becomes more difficult. Note that Bertsch et al.* **[3]** *studied the only case $d = 1$ in their Theorem 1.2. The analysis of the general case remains an open problem.*

REMARK 2.5. *Let $\rho(t,u) : [0,\infty) \times \mathbb{T} \to [0,1]$ be a solution of (1.1):*

$$\partial_t \rho(t,u) = \partial_u(D(\rho(t,u))\partial_u\rho(t,u)).$$

Let $\tilde{u}_t^1$ and $\tilde{u}_t^2$ be the moving boundaries as considered in the introduction and assume that $\tilde{u}_t^1$ satisfies the equation (1.2):

$$\frac{d}{dt}\tilde{u}_t^1 = -\frac{(D(\rho)\partial_u\rho)(t,\tilde{u}_t^1)}{\rho(t,\tilde{u}_t^1)}.$$

Then the other moving boundary $\tilde{u}_t^2$ also satisfies the same equation (1.2). Indeed, since the total mass between $\tilde{u}_t^1$ and $\tilde{u}_t^2$ is conserved under the dynamics, we have

$$\int_{\tilde{u}_t^1}^{\tilde{u}_t^2} \rho(t,u)du \equiv \textit{constant}. \tag{2.8}$$

Differentiating (2.8) in t, we can conclude that the other moving boundary $\tilde{u}_t^2$ also satisfies the same equation (1.2).

REMARK 2.6. *From the similar computation as we did in Remark 2.3, we can easily drive the ordinary differential equation (1.2) under the 0-Neumann boundary condition. Indeed, let $\rho(t,u) : [0,\infty)\times[0,1]\to[0,1]$ be a solution of (1.1):*

$$\begin{cases} \partial_t \rho(t,u) = \partial_u(D(\rho(t,u))\partial_u\rho(t,u)), & \textit{for } t>0,\ u\in[0,1], \\ \partial_u\rho(t,0)=\partial_u\rho(t,1)=0, & \textit{for } t>0. \end{cases}$$

Assume that u_t satisfies the conservation law

$$\int_0^{u_t} \rho(t,u)du \equiv \textit{constant}. \tag{2.9}$$

Then, differentiating (2.9) in t, we can deduce that u_t actually satisfies the ordinary differential equation (1.2). Note that the same derivation can not apply to the Stefan problem under the periodic boundary condition due to the lack of the conservation law similar to (2.9).

3. Current

In this section, we consider the asymptotic behavior of the current across the bond $(-1,0)$. For a bond $(x,x+1)$, the current $J_{x,x+1}^N(t)$ up to time t over this bond is defined as the total number of jumps of particles from x to $x+1$ in the time interval $[0,t]$ minus the total number of jumps of particles from $x+1$ to x in the time interval $[0,t]$. The main result of this section is the following law of large numbers for the diffusively scaled current $\frac{1}{N}J_{-1,0}^N(t)$. Recall that $\rho(t,u) : [0,\infty)\times\mathbb{T}\to[0,1]$ is determined as the unique weak solution of (2.5).

THEOREM 3.1. *For each time $t\geq 0$, we have*

$$\lim_{N\to\infty}\frac{1}{N}J_{-1,0}^N(t) = \int_0^1 (1-u)(\rho(t,u)-\rho(0,u))du, \ \textit{in probability}.$$

PROOF. For each $x\in\mathbb{T}_N$, since simultaneous jumps of two or more particles do not occur with probability one, the martingales defined by

$$M_{x,x+1}^N(t) := J_{x,x+1}^N(t) - N^2\int_0^t c_{x,x+1}(\eta_s^N)\{\eta_s^N(x)-\eta_s^N(x+1)\}ds \tag{3.1}$$

are orthogonal, that is, their cross variations are equal to 0. Moreover, its quadratic variation $\langle M^N_{x,x+1}\rangle_t$ is given by

$$\langle M^N_{x,x+1}\rangle_t = N^2 \int_0^t c_{x,x+1}(\eta^N_s)\{\eta^N_s(x) - \eta^N_s(x+1)\}^2 ds. \tag{3.2}$$

We define the function $G : \mathbb{T} \to (0, 1]$ by $G(u) = 1 - u$. Since, for each $t \geq 0$ and $x \in \mathbb{T}_N$,

$$\eta^N_t(x) - \eta^N_0(x) = J^N_{x-1,x}(t) - J^N_{x,x+1}(t),$$

we have

$$\begin{aligned}\langle \pi^N_t, G\rangle - \langle \pi^N_0, G\rangle &= \frac{1}{N} \sum_{x\in\mathbb{T}_N} (1 - \frac{x}{N})(\eta^N_t(x) - \eta_0(x)) \\ &= \frac{1}{N} \sum_{x\in\mathbb{T}_N} (1 - \frac{x}{N})(J^N_{x-1,x}(t) - J^N_{x,x+1}(t)).\end{aligned}$$

Therefore the summation by parts formula gives us

$$\frac{1}{N} J^N_{-1,0}(t) = \langle \pi^N_t, G\rangle - \langle \pi^N_0, G\rangle + \frac{1}{N^2} \sum_{x\in\mathbb{T}_N} J^N_{x,x+1}(t). \tag{3.3}$$

From Theorem 2.1, it is easy to see that the difference of the first two terms on the right hand side of (3.3) converges to

$$\int_0^1 (1 - u)(\rho(t, u) - \rho(0, u))du, \text{ in probability.}$$

Hence it suffices to show that the last term on the right hand side of (3.3) vanishes as N tends to ∞.

From the martingale decomposition (3.1), we have

(3.4)

$$\begin{aligned}\frac{1}{N^2} \sum_{x\in\mathbb{T}_N} J^N_{x,x+1}(t) = &\frac{1}{N^2} \sum_{x\in\mathbb{T}_N} M^N_{x,x+1}(t) \\ &+ \sum_{x\in\mathbb{T}_N} \int_0^t c_{x,x+1}(\eta^N_s)\{\eta^N_s(x) - \eta^N_s(x+1)\}ds.\end{aligned}$$

We can easily show that the first sum on the right hand side of (3.4) converges to 0. Indeed, since the jump rates are bounded and from (3.2), we have

$$\mathbb{E}^N[(\frac{1}{N^2} \sum_{x\in\mathbb{T}_N} M^N_{x,x+1}(t))^2] = \frac{1}{N^4}\mathbb{E}^N[\sum_{x\in\mathbb{T}_N} \langle M^N_{x,x+1}\rangle_t] \leq \frac{Ct}{N},$$

for some universal constant $C > 0$.

On the other hand, we need some technical result, so-called gradient replacement, to treat the second sum on the right hand side of (3.4). We refer to [**27, 31**] for details. The gradient replacement asserts that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N[|\int_0^t\sum_{x\in\mathbb{T}_N}\Big(c_{x,x+1}\{\eta_s^N(x)-\eta_s^N(x+1)\}+D(\eta_s^{\varepsilon N}(x))\{\eta_s^{\varepsilon N}(x+1)-\eta_s^{\varepsilon N}(x)\}\Big)ds|]=0,$$

where $\eta^{\varepsilon N}(x)$ is defined by $\eta^{\varepsilon N}(x) := \frac{1}{\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\eta(y)$. Hence, to complete the proof of the theorem, it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}^N[|\int_0^t\sum_{x\in\mathbb{T}_N}D(\eta_s^{\varepsilon N}(x))\{\eta_s^{\varepsilon N}(x+1)-\eta_s^{\varepsilon N}(x)\}ds|]=0. \tag{3.5}$$

Denote by d the integral of D: $d(\rho) = \int_0^\rho D(\alpha)d\alpha$, $\rho \in [0,1]$. Recall Remark 2.1. Since D is continuous on the interval $[0,1]$, we have

$$D(\eta^{\varepsilon N}(x))\{\eta^{\varepsilon N}(x+1)-\eta^{\varepsilon N}(x)\} = d(\eta^{\varepsilon N}(x+1))-d(\eta^{\varepsilon N}(x))+N^{-1}r_N(\eta), \tag{3.6}$$

where $r_N(\eta)$ represents a term that converges uniformly in η to 0 as N tends to ∞. The summation by parts formula with (3.6) gives (3.5), which completes the proof of Theorem 3.1. □

The following lemma easily follows from the definition of the solution of (2.5).

LEMMA 3.2. *(1) Let $\rho(t,u) : [0,\infty)\times\mathbb{T}\to[0,1]$ be the unique solution of (2.5). Then, we have*

$$\int_0^1(1-u)(\rho(t,u)-\rho(0,u))du = -\int_0^t(D(\rho)\partial_u\rho)(s,0)ds.$$

(2) Let $\tilde{u}_t$ be the solution of the implicit equation:

$$\begin{cases}\int_0^{\tilde{u}_t}\rho(t,u)du = -\int_0^t(D(\rho)\partial_u\rho)(s,0)ds,\\ \tilde{u}_0 = 0.\end{cases}$$

Then, it solves an ordinary differential equation:

$$\frac{d}{dt}\tilde{u}_t = -\frac{(D(\rho)\partial_u\rho)(t,\tilde{u}_t)}{\rho(t,\tilde{u}_t)}. \tag{3.7}$$

Notice that the last ordinary differential equation (3.7) is just introduced in (1.2).

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. We present it for the sake of completeness although the strategy of the proof is essentially the same as that of Jara and Landim [**29**].

We first periodically extend the process η_t^N to the $\{0,1\}^{\mathbb{Z}}$-valued process $\tilde{\eta}_t^N$ defined by $\tilde{\eta}_t^N(x+nN) = \eta_t^N(x)$, for $n\in\mathbb{Z}$ and $x\in\{0,\cdots,N-1\}$. Since $\eta_0^N(0) = 1$, we can tag this particle and denote the position of this particle at time t by $\tilde{X}_t^N\in\mathbb{Z}$. To prove Theorem 2.2, it is enough to show that the re-scaled process $\tilde{u}_t^N = \frac{1}{N}\tilde{X}_t^N$ converges to $\tilde{u}_t$ in probability for each time t.

Due to the exclusive constraint, the position of the tagged particle $\tilde{X}_t^N$ is written in terms of the empirical measure π_t^N and the current $J_{-1,0}^N(t)$. More precisely, for each $n \in \mathbb{N}$, we have

$$\{\tilde{X}_t^N \geq n\} = \{J_{-1,0}^N(t) \geq \sum_{x=0}^{n-1} \tilde{\eta}_t^N(x)\}. \tag{4.1}$$

Fix $u > 0$ and take $n = \lceil uN \rceil$, then the relation (4.1) shows

$$\{\tilde{u}_t^N \geq u\} = \{\frac{1}{N} J_{-1,0}^N(t) \geq \frac{1}{N} \sum_{x=0}^{\lceil uN \rceil} \tilde{\eta}_t^N(x) + r_N\},$$

where r_N uniformly converges to 0 as $N \to \infty$. From Theorem 2.1, $\frac{1}{N}\sum_{x=0}^{\lceil uN \rceil} \tilde{\eta}_t^N(x)$ converges to $\int_0^u \rho(t,u)du$ in probability. Therefore, from Theorem 3.1 and Lemma 3.2-(1), we have

$$\lim_{N\to\infty} \mathbb{P}^N(\tilde{u}_t^N \geq u) = \begin{cases} 1 & \text{if } -\int_0^t (D(\rho)\partial_u \rho)(s,0)ds > \int_0^u \rho(t,u)du, \\ 0 & \text{if } -\int_0^t (D(\rho)\partial_u \rho)(s,0)ds < \int_0^u \rho(t,u)du. \end{cases}$$

From the symmetry around the origin, we can show the similar statement for $u < 0$. Therefore, for any $\delta > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}^N(|\tilde{u}_t^N - \tilde{u}_t| \geq \delta) = 0,$$

which completes the proof of Theorem 2.2.

CHAPTER 3

Hydrostatics and dynamical large deviations for a reaction-diffusion model

1. Introduction to Chapter 3

In recent years, the large deviations of interacting particle systems have attracted much attention as an important step in the foundation of a thermodynamic theory of nonequilibrium stationary states [**19, 6, 10, 8**]. Notwithstanding the absence of explicit expressions for the stationary states, large deviations principles for the empirical measure under the stationary state have been derived from a dynamical large deviations principle [**7, 23, 13**], extending to an infinite-dimensional setting [**11, 22**] Freidlin and Wentzell approach [**21**].

We consider in this chapter interacting particle systems in which a symmetric simple exclusion dynamics, speeded-up diffusively, is superposed to a non-conservative Glauber dynamics. De Masi, Ferrari and Lebowitz [**15**] proved that the macroscopic evolution of the empirical measure is described by the solutions of the reaction-diffusion equation

$$\partial_t \rho \;=\; (1/2)\Delta\rho + B(\rho) - D(\rho)\;. \tag{1.1}$$

where Δ is the Laplacian and $F = B - D$ is a reaction term determined by the stochastic dynamics. They also proved that the equilibrium fluctuations evolve as generalized Ornstein-Uhlenbeck processes.

A large deviation principle for the empirical measure has been obtained in [**30**] in the case where the initial distribution is a local equilibrium. The lower bound of the large deviations principle was achieved only for smooth trajectories. More recently, [**13**] extended the large deviations principle to a one-dimensional dynamics in contact with reservoirs and proved the lower bound for general trajectories in the case where the birth and the death rates, $B(\rho)$ and $D(\rho)$, respectively, are monotone, concave functions.

In this chapter, we first present a law of large numbers for the empirical measure under the stationary state [**20, 32**]. More precisely, denote by μ_N the stationary state on a one-dimensional torus with N points of the superposition of a Glauber dynamics with a symmetric simple exclusion dynamics speeded-up by N^2. This probability measure is not known explicitly and it exhibits long range correlations [**4**]. Let V_ϵ denote an ϵ-neighborhood of the set of solutions of the elliptic equation

$$(1/2)\Delta\rho + F(\rho) \;=\; 0\;. \tag{1.2}$$

Theorem 2.1 asserts that for any $\epsilon > 0$, $\mu_N(V_\epsilon^c)$ vanishes as $N \to \infty$. In contrast with previous results, equation (1.2) may not have a unique solution so that equation (1.1) may not have a global attractor, what prevents the use of the techniques developed in [**23, 37**]. This result solves partially a conjecture raised in [**12**].

The main results of this chapter concern the large deviations of the Glauber+Kawasaki dynamics. We first prove a full large deviations principle for the empirical measure under the sole assumption that B and D are concave functions. These assumptions encompass the case in which the potential $F(\rho) = B(\rho) - D(\rho)$ presents two or more wells, and open the way to the investigation of the metastable behavior of this dynamics. Previous results in this directions include [**16, 17, 5**].

Comments on the proof. The proof of the law of large numbers for the empirical measure under the stationary state μ_N borrows ideas from [**23, 37**]. On the one hand, by [**15**], the evolution of the empirical measure is described by the solutions of the reaction-diffusion equation (1.1). On the other hand, by [**14**], for any density profile γ, the solution ρ_t of (1.1) with initial condition γ converges to some solution of the semilinear elliptic equation (1.2). Assembling these two facts, we show in the proof of Theorem 2.1 that the empirical measure eventually reaches a neighborhood of the set of all solutions of the semilinear elliptic equation (1.2).

The proof that the rate function I is lower semicontinuous and has compact level set is divided in two steps. Denote by $Q(\pi)$ the energy of a trajectory π, defined in (2.2). Following [**38**], we first show in Proposition 4.2 that the energy of a trajectory π is bounded by the sum of its rate function with a constant: $Q(\pi) \le C_0(I(\pi) + 1)$. It is not difficult to show that a sequence in the set $\{\pi : Q(\pi) \le a\}$, $a > 0$, which converges weakly also converges in L^1. The lower semicontinuity of the rate function I follows from these two facts. Let π_n be a sequence which converges weakly to π. We may, of course, assume that the sequence $I(\pi_n)$ is bounded. In this case, by the two results presented above, π_n converges to π in L^1. As the rate function $I(\cdot)$, defined in (2.3), is given by $\sup_G J_G(\cdot)$, where the supremum is carried over smooth functions, and since for each such function J_G is continuous for the L^1 topology, $J_G(\pi) = \lim_n J_G(\pi_n) \le \liminf_n I(\pi_n)$. To conclude the proof of the lower semicontinuity of I, it remains to maximize over G. The proof that the level sets are compact is similar.

Note that the previous argument does not require a bound of the H_{-1} norm of $\partial_t \pi$ in terms of $I(\pi)$ and $Q(\pi)$. Actually, such a bound does not hold in the present context. For example, let ρ represent the solution of the hydrodynamic equation (1.1) starting from some initial condition γ. Due to the reaction term, the H_{-1} norm of $\partial_t \rho$ might be infinite, while $I(\rho) = 0$ and $Q(\rho) < \infty$. The fact that a bound on the H_{-1} norm of $\partial_t \pi$ is not used, may simplify the earlier proofs of the regularity of the rate function in the case of conservative dynamics [**9, 23**].

The main difficulty in the proof of the lower bound lies in the I-density of smooth trajectories: each trajectory π with finite rate function should be approachable by a sequence of smooth trajectories π_n such that $I(\pi_n)$ converges to $I(\pi)$. We use in this step the hydrodynamic equation and several convolutions with mollifiers to smooth the paths. The concavity of B and D are used in this step and only in this one. It is possible that the theory of Orlicz spaces may allow to weaken these assumptions. Similar difficulties appeared in the investigation of the large deviations of a random walk driven by an exclusion process and of the exclusion process with a slow bond [**1, 25**].

This chapter is organized as follows. In Section 2, we introduce a reaction-diffusion model and state the main results. In Section 3 we prove the law of large numbers for the empirical measure under the stationary state. In Section 4, we present the main properties of the rate function I. In Section 5, we prove that the smooth trajectories are I-dense and we prove Theorem 2.5, the main result of the chapter. In Section 6, we recall some results on the solution of the hydrodynamic equation (1.1).

2. Notation and Results

Throughout this chapter, we use the following notation. $\mathbb{N}_0$ stands for the set $\{0, 1, \cdots\}$. For a function $f : X \to \mathbb{R}$, defined on some space X, let $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We will use $C_0 > 0$ and $C > 0$ as a notation for a generic positive constant which may change from line to line.

2.1. Reaction-diffusion model. We fix some notation and define the model. Let $\mathbb{T}_N$ be the one-dimensional discrete torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \cdots, N-1\}$. The state space of our process is given by $X_N = \{0, 1\}^{\mathbb{T}_N}$. Let η denote a configuration in X_N, x a site in $\mathbb{T}_N$, $\eta(x) = 1$ if there is a particle at site x, otherwise $\eta(x) = 0$.

We consider in the set $\mathbb{T}_N$ the superposition of the symmetric simple exclusion process (Kawasaki) with a spin-flip dynamics (Glauber). This model was introduced by De Masi, Ferrari and Lebowitz in [**15**] to derive a reaction-diffusion equation from a microscopic dynamics. More precisely, the stochastic dynamics is a Markov process on X_N whose generator $\mathcal{L}_N$ acts on functions $f : X_N \to \mathbb{R}$ as

$$\mathcal{L}_N f \;=\; (N^2/2)\mathcal{L}_K f + \mathcal{L}_G f \;,$$

where $\mathcal{L}_K$ is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_K f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} [f(\eta^{x,x+1}) - f(\eta)] \;,$$

and where $\mathcal{L}_G$ is the generator of a spin flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} c(x,\eta)[f(\eta^x) - f(\eta)] \;.$$

In these formulas, $\eta^{x,x+1}$ (resp. η^x) represents the configuration obtained from η by exchanging (resp. flipping) the occupation variables $\eta(x)$, $\eta(x+1)$ (resp. $\eta(x)$):

$$\eta^x(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x\ , \\ 1-\eta(z) & \text{if } z = x\ , \end{cases} \quad \eta^{x,y}(z) \;=\; \begin{cases} \eta(y) & \text{if } z = x\ , \\ \eta(x) & \text{if } z = y\ , \\ \eta(z) & \text{otherwise}\ . \end{cases}$$

Moreover, $c(x,\eta) = c(\eta(x-M),\cdots,\eta(x+M))$, for some $M \geq 1$ and some strictly positive cylinder function $c(\eta)$, that is, a function which depends only on a finite number of variables $\eta(y)$. Note that the exclusion dynamics has been speeded-up by a factor N^2, and that the Markov process generated by $\mathcal{L}_N$ is irreducible because $c(\eta)$ is a strictly positive function.

2.2. Hydrodynamic limit. We briefly discuss in this subsection the limiting behavior of the empirical measure.

Denote by $\mathbb{T}$ the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$. Let $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of nonnegative measures on $\mathbb{T}$, whose total mass bounded by 1, endowed with the weak topology. For a measure π in $\mathcal{M}_+$ and a continuous function $G : \mathbb{T} \to \mathbb{R}$, denote by $\langle \pi, G\rangle$ the integral of G with respect to π:

$$\langle \pi, G\rangle \;=\; \int_{\mathbb{T}} G(u)\pi(du)\ .$$

The space $\mathcal{M}_+$ is metrizable. Indeed, if $f_{2k}(u) = \cos(\pi k u)$ and $f_{2k+1}(u) = \sin(\pi k u)$, $k \in \mathbb{N}_0$, one can define the distance d on $\mathcal{M}_+$ as

$$d(\pi_1,\pi_2) \;:=\; \sum_{k=0}^{\infty} \frac{1}{2^k} |\langle \pi_1, f_k\rangle - \langle \pi_2, f_k\rangle|\ .$$

Denote by $C^m(\mathbb{T})$, m in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions on $\mathbb{T}$ which are m times differentiable and whose m-th derivative is continuous. Given a function G in $C^2(\mathbb{T})$, we shall denote by ∇G and ΔG the first and second derivative of G, respectively.

Let $\{\eta_t^N : N \geq 1\}$ be the continuous-time Markov process on X_N whose generator is given by $\mathcal{L}_N$. Let $\pi^N : X_N \to \mathcal{M}_+$ be the function which associates to a configuration η the positive measure obtained by assigning mass N^{-1} to each particle of η,

$$\pi^N(\eta) \;=\; \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x)\delta_{x/N}\ ,$$

where δ_u stands for the Dirac measure which has a point mass at $u \in \mathbb{T}$. Denote by π_t^N the empirical measure process $\pi^N(\eta_t^N)$.

Fix arbitrarily $T > 0$. For a topological space X and an interval $I = [0,T]$ or $[0,\infty)$, denote by $C(I,X)$ the set of all continuous trajectories from I to X endowed with the uniform topology. Let $D(I,X)$ be the space of all right-continuous trajectories from I to X with left-limits, endowed with the Skorokhod topology. For a probability measure ν in X_N, denote by $\mathbb{P}^N_\nu$ the measure on $D([0,T], X_N)$ induced by the process η^N_t starting from ν.

Let $\nu_\rho = \nu^N_\rho$, $0 \le \rho \le 1$, be the Bernoulli product measure with the density ρ. Define the continuous functions $B, D : [0,1] \to \mathbb{R}$ by

$$B(\rho) \;=\; \int [1-\eta(0)]\, c(\eta)\, d\nu_\rho \;, \quad D(\rho) \;=\; \int \eta(0)\, c(\eta)\, d\nu_\rho \;.$$

Since $B(1) = 0$, $D(0) = 0$ and B, D are polynomials in ρ,

$$B(\rho) \;=\; (1-\rho)\, \tilde{B}(\rho) \;, \quad D(\rho) \;=\; \rho\, \tilde{D}(\rho) \;, \tag{2.1}$$

where $\tilde{B}(\rho)$, $\tilde{D}(\rho)$ are polynomials.

The next result was proved by De Masi, Ferrari and Lebowitz in [**15**] for the first time. We refer to [**15, 30, 31**] for its proof.

THEOREM 2.1. *Fix $T > 0$ and a measurable function $\gamma : \mathbb{T} \to [0,1]$. Let $\nu = \nu_N$ be a sequence of probability measures on X_N associated to γ, in the sense that*

$$\lim_{N\to\infty} \nu_N \Big(|\langle \pi^N, G\rangle - \int_{\mathbb{T}} G(u)\gamma(u)du| > \delta \Big) \;=\; 0 \;,$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T} \to \mathbb{R}$. Then, for every $t \ge 0$, every $\delta > 0$ and every continuous function $G : \mathbb{T} \to \mathbb{R}$, we have

$$\lim_{N\to\infty} \mathbb{P}^N_\nu \Big(|\langle \pi^N_t, G\rangle - \int_{\mathbb{T}} G(u)\rho(t,u)du| > \delta \Big) \;=\; 0 \;,$$

where $\rho : [0,\infty) \times \mathbb{T} \to [0,1]$ is the unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho \;=\; (1/2)\Delta\rho + F(\rho) \;\; \text{on} \;\; \mathbb{T} \;, \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \;, \end{cases} \tag{2.2}$$

where $F(\rho) = B(\rho) - D(\rho)$.

The definition, existence and uniqueness of weak solutions of the Cauchy problem (3.1) are discussed in Section 6.

2.3. Hydrostatic limit. We examine in this subsection the asymptotic behavior of the empirical measure under the stationary state. Fix $N \ge 1$ large enough. Since the Markov Process η^N_t is irreducible and the cardinality of the state space X_N is finite, there exists a unique invariant probability measure for the process η^N_t, denoted by μ_N. Let $\mathcal{P}_N$ be the probability measure on $\mathcal{M}_+$ defined by $\mathcal{P}_N = \mu_N \circ (\pi^N)^{-1}$.

For each $p \ge 1$, let $L^p(\mathbb{T})$ be the space of all real p-th integrable functions $G : \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_{\mathbb{T}} |G(u)|^p du < \infty$. The corresponding norm is denoted by $\|\cdot\|_p$:

$$\|G\|_p^p \;:=\; \int_{\mathbb{T}} |G(u)|^p du \;.$$

In particular, $L^2(\mathbb{T})$ is a Hilbert space equipped with the inner product

$$\langle G, H \rangle \;=\; \int_{\mathbb{T}} G(u)H(u)du \;.$$

For a function G in $L^2(\mathbb{T})$, we also denote by $\langle G \rangle$ the integral of G with respect to the Lebesgue measure: $\langle G \rangle := \int_{\mathbb{T}} G(u)du$.

Let $\mathcal{E}$ be the set of all classical solutions of the semilinear elliptic equation:

$$(1/2)\Delta\rho + F(\rho) \;=\; 0 \;\text{ on }\; \mathbb{T} \;. \tag{2.3}$$

Classical solution means a function $\rho : \mathbb{T} \to [0,1]$ in $C^2(\mathbb{T})$ which satisfies the equation (2.1) for any $u \in \mathbb{T}$. We sometimes identify $\mathcal{E}$ with the set of all absolutely continuous measures whose density are a classical solution of (2.1):

$$\{\pi \in \mathcal{M}_+ : \pi(du) = \rho(u)du,\ \rho \text{ is a classical solution of the equation (2.1)}\}.$$

THEOREM 2.2. *The measure $\mathcal{P}_N$ asymptotically concentrates on the set $\mathcal{E}$. Namely, for any $\delta > 0$, we have*

$$\lim_{N\to\infty} \mathcal{P}_N(\pi \in \mathcal{M}_+ : \inf_{\bar\pi \in \mathcal{E}} d(\pi, \bar\pi) \geq \delta) \;=\; 0 \;.$$

If the set $\mathcal{E}$ is a singleton, it follows from Theorem 2.1 that the sequence $\{\mathcal{P}_N : N \geq 1\}$ converges:

COROLLARY 2.3. *Assume that there exists a unique classical solution $\overline{\rho} : \mathbb{T} \to [0,1]$ of the semilinear elliptic equation* (2.1)*. Then $\mathcal{P}_N$ converges to the Dirac measure concentrated on $\overline{\rho}(u)du$ as $N \to \infty$.*

REMARK 2.4. *In* [**16, 17**]*, De Masi et al. examined the dynamics introduced above in the case of the double well potential $F(\rho) = -V'(\rho) = a(2\rho-1) - b(2\rho-1)^3$, a, $b > 0$, which is symmetric around the density $1/2$. They proved that, starting from a product measure with mean $1/2$, the unstable equilibrium of the ODE $\dot{x}(t) = -V'(x(t))$, the empirical density remains in a neighborhood of $1/2$ in a time scale of order $\log N$. Bodineau and Lagouge in* [**12**] *conjectured that Theorem 2.1 remains true if we replace $\mathcal{E}$ by the set of all stable equilibrium solutions of the equation* (2.1)*. This conjecture is proved in* [**24**] *and follows from the large deviation principle for the sequence $\{\mathcal{P}_N : N \geq 1\}$.*

2.4. Dynamical large deviations. Denote by $\mathcal{M}_{+,1}$ the closed subset of $\mathcal{M}_+$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_{+,1} \;=\; \{\pi \in \mathcal{M}_+(\mathbb{T}) : \pi(du) = \rho(u)du,\ 0 \leq \rho(u) \leq 1 \; a.e.\; u \in \mathbb{T}\} \;.$$

Fix $T > 0$, and denote by $C^{m,n}([0,T] \times \mathbb{T})$, m, n in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions defined on $[0,T] \times \mathbb{T}$ which are m times differentiable in the first variable and n times on the second one, and whose derivatives are continuous. Let $Q_\eta = Q^N_\eta$, $\eta \in X_N$, be the probability measure on $D([0,T], \mathcal{M}_+)$ induced by the measure-valued process π^N_t starting from $\pi^N(\eta)$.

Fix a measurable function $\gamma : \mathbb{T} \to [0,1]$. For each path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$, define the energy $\mathcal{Q} : D([0,T],\mathcal{M}_{+,1}) \to [0,\infty]$ as

(2.4)

$$\mathcal{Q}(\pi) \;=\; \sup_{G\in C^{0,1}([0,T]\times\mathbb{T})} \Big\{ 2\int_0^T dt\, \langle \rho_t, \nabla G_t\rangle - \int_0^T dt\, \int_{\mathbb{T}^d} du\, G^2(t,u) \Big\} \,.$$

It is known that the energy $\mathcal{Q}(\pi)$ is finite if and only if ρ has a generalized derivative and this generalized derivative is square integrable on $[0,T]\times\mathbb{T}$:

$$\int_0^T dt\, \int_{\mathbb{T}} du\, |\nabla\rho(t,u)|^2 < \infty \,.$$

Moreover, it is easy to see that the energy $\mathcal{Q}$ is convex and lower semicontinuous.

For each function G in $C^{1,2}([0,T]\times\mathbb{T})$, define the functional $\bar J_G : D([0,T],\mathcal{M}_{+,1}) \to \mathbb{R}$ by

$$\begin{aligned}\bar J_G(\pi) \;&=\; \langle \pi_T, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T dt\, \langle \pi_t, \partial_t G_t + \frac{1}{2}\Delta G_t\rangle \\ &\quad - \frac{1}{2}\int_0^T dt\, \langle \chi(\rho_t), (\nabla G_t)^2\rangle - \int_0^T dt\, \big\{\langle B(\rho_t), e^{G_t}-1\rangle + \langle D(\rho_t)\,, e^{-G_t}-1\rangle\big\},\end{aligned}$$

where $\chi(r) = r(1-r)$ is the mobility. Let $J_G : D([0,T],\mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$J_G(\pi) \;=\; \begin{cases} \bar J_G(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}_{+,1})\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

We define the large deviation rate function $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}_+) \to [0,\infty]$ as

$$I_T(\pi|\gamma) \;=\; \begin{cases} \sup J_G(\pi) & \text{if } \mathcal{Q}(\pi) < \infty\,, \\ \infty & \text{otherwise}\,, \end{cases} \tag{2.5}$$

where the supremum is taken over all functions G in $C^{1,2}([0,T]\times\mathbb{T})$.

THEOREM 2.5. *Assume that the functions B and D are concave on $[0,1]$. Fix $T>0$ and a measurable function $\gamma : \mathbb{T}\to[0,1]$. Assume that a sequence η^N of initial configurations in X_N is associated to γ, in the sense that*

$$\lim_{N\to\infty} \langle \pi^N(\eta^N), G\rangle \;=\; \int_{\mathbb{T}} G(u)\gamma(u)du$$

for every continuous function $G : \mathbb{T}\to\mathbb{R}$. Then, the measure Q_{η^N} on $D([0,T],\mathcal{M}_+)$ satisfies a large deviation principle with the rate function $I_T(\cdot|\gamma)$. That is, for each closed subset $\mathcal{C}\subset D([0,T],\mathcal{M}_+)$,

$$\varlimsup_{N\to\infty} \frac{1}{N}\log Q_{\eta^N}(\mathcal{C}) \;\le\; -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma)\,,$$

and for each open subset $\mathcal{O} \subset D([0,T], \mathcal{M}_+)$,

$$\varliminf_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathcal{O}) \;\geq\; -\inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma)\ .$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

REMARK 2.6. *Jona-Lasinio, Landim and Vares* [**30**] *proved the dynamical large deviations principle stated above, but the lower bound was obtained only for smooth trajectories. Bodineau and Lagouge* [**13**] *proved the lower bound for one-dimensional reaction-diffusion models in contact with reservoirs in the case where B and D are concave, monotone functions.*

REMARK 2.7. *Proposition 4.2 asserts that there exists a finite constant C_0 such that if π is a trajectory with finite energy, $Q(\pi) < \infty$, then $Q(\pi) \leq C_0(I_T(\pi|\gamma) + 1)$. In the case where B and D are concave functions, we can use Theorem 5.2, which asserts that the smooth trajectories are $I_T(|\gamma)$-dense, to prove the same bound without the assumption that the trajectory π has finite energy. In particular, in this case we can define the rate function $I_T(\ |\gamma)$ simply as*

$$I_T(\pi|\gamma) \;=\; \sup_G J_G(\pi)\ .$$

REMARK 2.8. *In the proof that the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets we do not use a bound on the H_{-1} norm of $\partial_t\rho$ in terms of its rate function $I_T(\pi|\gamma)$. Actually, as mentioned in the introduction, such a bound does not hold for reaction-diffusion models. Therefore, the arguments presented here permit to simplify the proof of the regularity of the rate function in other models, such as the weakly asymmetric simple exclusion process* [**5, 23**].

3. Proof of Theorem 2.1

We prove in this section Theorem 2.1. Our approach is a generalization of the one developed in [**23, 37**], but it does not require the existence of a global attractor for the underlying dynamical system. The method can be applied to any dynamics which fulfills two conditions: the macroscopic evolution of the empirical measure is described by a hydrodynamic equation, and for any initial condition the solution of this equation converges to a stationary profile as time goes to infinity. For instance, the boundary driven reaction-diffusion models examined in [**13**].

Recall from Subsection 2.3 the definition of the measure μ_N on X_N, the map π^N from X_N to $\mathcal{M}_+$ and the measure $\mathcal{P}_N = \mu_N \circ (\pi^N)^{-1}$ on $\mathcal{M}_+$. Denote by $\mathbf{Q}^N$ the probability measure on the Skorokhod space $D([0,\infty), \mathcal{M}_+)$ induced by the measure-valued process π^N_t under the initial distribution $\mathcal{P}_N$. Since the measure μ_N is stationary under the dynamics, $\mathcal{P}_N(\mathcal{B}) = \mathbf{Q}^N(\pi : \pi_T \in \mathcal{B})$, for each $T > 0$ and Borel set $\mathcal{B} \subset \mathcal{M}_+$.

LEMMA 3.1. *The sequence* $\{\mathbf{Q}^N : N \geq 1\}$ *is tight and all its limit points* $\mathbf{Q}^*$ *are concentrated on absolutely continuous paths* $\pi(t, du) = \rho(t, u)du$ *whose density* ρ *is nonnegative and bounded above by* 1 *:*

$$\mathbf{Q}^*\{\pi : \pi(t, du) = \rho(t, u)du \text{ , for } t \in [0, \infty)\} \;=\; 1 \;,$$
$$\mathbf{Q}^*\{\pi : 0 \leq \rho(t, u) \leq 1 \text{ , for } (t, u) \in [0, \infty) \times \mathbb{T}\} \;=\; 1 \;.$$

The proof of this lemma is similar to the one of Proposition 3.3 in [**23**].

Let $\mathcal{A}$ be the set of all trajectories $\pi(t, du) = \rho(t, u)du$ in $D([0, \infty), \mathcal{M}_{+,1})$ whose density ρ is a weak solution to the Cauchy problem (3.1) for some initial profile $\rho_0 : \mathbb{T} \to [0, 1]$.

LEMMA 3.2. *All limit points* $\mathbf{Q}^*$ *of the sequence* $\{\mathbf{Q}^N : N \geq 1\}$ *are concentrated on paths* $\pi(t, du) = \rho(t, u)du$ *in* $\mathcal{A}$ *:*

$$\mathbf{Q}^*(\mathcal{A}) \;=\; 1 \;.$$

The proof of this lemma is similar to the one of Lemma A.1.1 in [**31**].

PROOF OF THEOREM 2.1. Fix a positive $\delta > 0$. Let $\mathcal{E}_\delta$ be the δ-neighborhood of $\mathcal{E}$ in $\mathcal{M}_+$:

$$\mathcal{E}_\delta \;:=\; \{\pi \in \mathcal{M}_+ : \inf_{\bar{\pi} \in \mathcal{E}} d(\pi, \bar{\pi}) < \delta\} \;.$$

Denote by $\mathcal{E}_\delta^c$ the complement of the set $\mathcal{E}_\delta$. The assertion of Theorem 2.1 can be rephrased as

$$\lim_{N \to \infty} \mathcal{P}_N(\mathcal{E}_\delta^c) \;=\; 0 \;.$$

Therefore, to conclude the theorem it is enough to show that any limit point of the sequence $\mathcal{P}_N(\mathcal{E}_\delta^c)$ is equal to zero.

Fix $T > 0$. Since the measure μ_N is invariant under the dynamics,

$$\mathcal{P}_N(\mathcal{E}_\delta^c) \;=\; \mathbf{Q}^N(\pi : \pi_T \in \mathcal{E}_\delta^c) \;. \tag{3.1}$$

Let $\mathbf{Q}^*$ be a limit point of $\{\mathbf{Q}^N : N \geq 1\}$ and take a subsequence N_k so that the sequence $\{\mathbf{Q}^{N_k} : k \geq 1\}$ converges to $\mathbf{Q}^*$ as $k \to \infty$. Note that the set $\{\pi : \pi_T \in \mathcal{E}_\delta^c\}$ is not closed in $D([0, \infty), \mathcal{M}_+)$. However, we claim that

$$\overline{\lim_{k \to \infty}}\, \mathbf{Q}^{N_k}(\pi : \pi_T \in \mathcal{E}_\delta^c) \;\leq\; \mathbf{Q}^*(\{\pi : \pi_T \in \mathcal{E}_\delta^c\} \cap \mathcal{A}) \;, \tag{3.2}$$

where $\mathcal{A}$ is the set introduced just before Lemma 3.2. Indeed, denote by $\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}}$ the closure of the set $\{\pi : \pi_T \in \mathcal{E}_\delta^c\}$ under the Skorokhod topology. By definition of the weak topology and by Lemma 3.2,

$$\overline{\lim_{k \to \infty}}\, \mathbf{Q}^{N_k}(\pi : \pi_T \in \mathcal{E}_\delta^c) \;\leq\; \mathbf{Q}^*(\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}}) \;=\; \mathbf{Q}^*(\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}} \cap \mathcal{A}) \;.$$

It remains to prove that

$$\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}} \cap \mathcal{A} \;=\; \{\pi : \pi_T \in \mathcal{E}_\delta^c\} \cap \mathcal{A} \;.$$

Let π be a path in $\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}} \cap \mathcal{A}$. Then there exists a sequence $\{\pi^n : n \geq 1\}$ such that π^n converges to π in $D([0, \infty), \mathcal{M}_+)$ as $n \to \infty$ and π_T^n belongs to $\mathcal{E}_\delta^c$ for any $n \geq 1$. Since $\mathcal{A}$ is contained in $C([0, \infty), \mathcal{M}_{+,1})$, the sequence $\{\pi^n : n \geq 1\}$ converges to π under the uniform topology. Hence

π_T^n converges to π_T. Since $\mathcal{E}_\delta^c$ is closed in $\mathcal{M}_+$, π_T also belongs to $\mathcal{E}_\delta^c$, which proves (3.4).

Fix a path $\pi(t, du) = \rho(t, u)du$ in $\mathcal{A}$. By Proposition 4.3, there exists a density profile ρ_∞ in $\mathcal{E}$ such that ρ_t converges to ρ_∞ in $C^2(\mathbb{T})$. Hence,

$$\mathcal{A} \subset \bigcup_{j\geq 1}\bigcap_{k\geq j}\{\pi_k \in \mathcal{E}_\delta\} \; . \tag{3.3}$$

By (3.1) and (3.4),

$$\overline{\lim_{N\to\infty}} \mathcal{P}_N(\mathcal{E}_\delta^c) \;\leq\; \mathbf{Q}^*(\{\pi : \pi_k \in \mathcal{E}_\delta^c\} \cap \mathcal{A}) \quad \text{for all } k \geq 1 \; .$$

Since this bound holds for any $k \geq 1$,

$$\overline{\lim_{N\to\infty}} \mathcal{P}_N(\mathcal{E}_\delta^c) \;\leq\; \overline{\lim_{k\to\infty}} \mathbf{Q}^*(\{\pi_k \in \mathcal{E}_\delta^c\}\cap\mathcal{A}) \;\leq\; \mathbf{Q}^*\Big(\bigcap_{j\geq 1}\bigcup_{k\geq j}\{\pi_k \in \mathcal{E}_\delta^c\}\cap\mathcal{A}\Big) \; .$$

This latter set is empty in view of (3.3), which completes the proof of the theorem. □

4. The rate function $I_T(\cdot|\gamma)$

We prove in this section that the large deviations rate function is lower semicontinuous and has compact level sets. These properties play a fundamental role in the proof of the static large deviation principle, cf. [**11, 22**]. One of the main steps in the proof of these properties is Proposition 4.2. It asserts that there exists a finite constant C_0 such that for all trajectory $\pi(t, dx) = \rho(t, x)$ whose density ρ has finite energy is such that $Q(\pi) \leq C_0(I_T(\pi|\gamma) + 1)$. Such bound was first proved in [**38**].

PROPOSITION 4.1. *Let π be a path in $D([0, T], \mathcal{M}_+)$ such that $I_T(\pi|\gamma)$ is finite. Then $\pi(0, du) = \gamma(u)du$ and π belongs to $C([0, T], \mathcal{M}_{+,1})$.*

PROOF. The proof of this proposition is similar to the one of Lemma 4.1 in [**23**]. Actually, the computation performed in the proof of Lemma 4.1 in [**23**] gives that, for any g in $C^2(\mathbb{T})$ and any $0 \leq s < t \leq T$,

$$|\langle \pi_t, g\rangle - \langle \pi_s, g\rangle| \;\leq\; C\alpha_{s,r}\{I_T(\pi|\gamma) + 1\} \; , \tag{4.1}$$

for some positive constant $C = C(g)$, which depends only on g. In the inequality (4.6), the constant $\alpha_{s,r}$ is given by $(\log{(r - s)^{-1}})^{-1}$. (4.6) implies the desired continuity. □

The next proposition plays an important role in the proof of Theorem 4.11.

PROPOSITION 4.2. *There exists a constant $C_0 > 0$ such that, for any path $\pi(t, du) = \rho(t, u)du$ in $D([0, T], \mathcal{M}_{+,1})$ with finite energy, we have*

$$\int_0^T dt \; \int_{\mathbb{T}} du \, \frac{|\nabla\rho(t, u)|^2}{\chi(\rho(t, u))} \;\leq\; C_0 \left\{I_T(\pi|\gamma) + 1\right\} \; .$$

We fix some notation before proving Proposition 4.2.

Let $H^1(\mathbb{T})$ be the Sobolev space of functions G with generalized derivatives ∇G in $L^2(\mathbb{T})$. $H^1(\mathbb{T})$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$, defined by

$$\langle G, H \rangle_{1,2} \;=\; \langle G, H \rangle + \langle \nabla G, \nabla H \rangle \;,$$

is a Hilbert space. The corresponding norm is denoted by $\| \cdot \|_{1,2}$:

$$\|G\|_{1,2}^2 \;:=\; \int_{\mathbb{T}} |G(u)|^2 du + \int_{\mathbb{T}} |\nabla G(u)|^2 du \;.$$

For a Banach space $(\mathbb{B}, \| \cdot \|_{\mathbb{B}})$ and $T > 0$, we denote by $L^2([0,T], \mathbb{B})$ the Banach space of measurable functions $U : [0,T] \to \mathbb{B}$ for which

$$\|U\|_{L^2([0,T],\mathbb{B})}^2 \;=\; \int_0^T \|U_t\|_{\mathbb{B}}^2 \, dt < \infty$$

holds. For each $p \geq 1$ and $T > 0$, let $L^p([0,T] \times \mathbb{T})$ be the space of all real p-th integrable functions $U : [0,T] \times \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_0^T dt \int_{\mathbb{T}} |U(t,u)|^p du < \infty$.

Fix a path $\pi(t, du) = \rho(t,u) du$ in $D([0,T], \mathcal{M}_{+,1})$ with finite energy. For a smooth function $G : [0,T] \times \mathbb{T} \to \mathbb{R}$ and a for bounded function H in $L^2([0,T], H^1(\mathbb{T}))$, define the functionals

$$\begin{aligned} L_G(\pi) \;&=\; \langle \pi_T, G_T \rangle - \langle \pi_0, G_0 \rangle - \int_0^T dt \, \langle \pi_t, \partial_t G_t \rangle \;, \\ B_H^1(\pi) \;&=\; \frac{1}{2} \int_0^T dt \, \langle \nabla \rho_t, \nabla H_t \rangle \;-\; \frac{1}{2} \int_0^T dt \, \langle \chi(\rho_t), (\nabla H_t)^2 \rangle \;, \\ B_H^2(\pi) \;&=\; \int_0^T dt \, \Big\{ \langle B(\rho_t), e^{H_t} - 1 \rangle + \langle D(\rho_t), e^{-H_t} - 1 \rangle \Big\} \;. \end{aligned}$$

Note that, for paths $\pi(t, du)$ such that $\pi(0, du) = \gamma(u) du$,

$$\sup_{H \in C^{1,2}([0,T] \times \mathbb{T})} \Big\{ L_H(\pi) + B_H^1(\pi) - B_H^2(\pi) \Big\} \;=\; I_T(\pi | \gamma) \;. \tag{4.2}$$

Consider the function $\phi : \mathbb{R} \to [0, \infty)$ defined by

$$\phi(r) \;:=\; \begin{cases} \dfrac{1}{Z} \exp\{-\dfrac{1}{(1-r^2)}\} & \text{if } |r| < 1 \;, \\ 0 & \text{otherwise} \;, \end{cases}$$

where the constant Z is chosen so that $\int_{\mathbb{R}} \phi(r) dr = 1$. For each $\delta > 0$, let

$$\phi^\delta(r) \;:=\; \frac{1}{\delta} \, \phi\Big(\frac{r}{\delta}\Big) \;.$$

Since the support of the function ϕ^δ is contained in $[-\delta, \delta]$, the function ϕ^δ can be regarded as a function on $\mathbb{T}$. To distinguish convolution in time from convolution in space, we denote by $\psi^\delta : \mathbb{T} \to [0, \infty)$ the function ϕ^ε defined on $\mathbb{T}$ with $\varepsilon = \delta$.

Denote by $f * g$ the space or time convolution of two functions f, g:

$$(f * g)(a) \;=\; \int f(a-b)\, g(b)\, db \;,$$

where the integral runs over $\mathbb{R}$ in the case where f, g are functions of time and over $\mathbb{T}$ in the case where f and g are functions of space.

Throughout this section, we adopt the following notation: For a bounded measurable function $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$, define the smooth approximation in space, time and space-time by

$$\begin{aligned}
\rho^\varepsilon(t,u) \;&:=\; [\rho(t,\cdot) * \psi^\varepsilon](u) \;=\; \int_{\mathbb{T}} \rho(t,u+v)\psi^\varepsilon(v)dv \;,\\
\rho^\delta(t,u) \;&:=\; [\rho(\cdot,u) * \phi^\delta](t) \;=\; \int_{-\delta}^{\delta} \rho(t+r,u)\phi^\delta(r)dr \;,\\
\rho^{\varepsilon,\delta}(t,u) \;&:=\; \int_{-\delta}^{\delta} dr \int_{\mathbb{T}} dv\; \rho(t+r,u+v)\psi^\varepsilon(v)\phi^\delta(r) \;.
\end{aligned}$$

In the above formulas, we extend the definition of ρ to $[-1,T+1]$ by setting $\rho_t = \rho_0$ for $-1 \le t \le 0$ and $\rho_t = \rho_T$ for $T \le t \le T+1$. Remark that we use similar notation, ρ^ε and ρ^δ, for different objects. However, ρ^ε and ρ^δ always represent a smooth approximation of ρ in space and time, respectively. For each $\pi(t,du) = \rho(t,u)du$, we also define paths $\pi^\varepsilon(t,du) = \rho^\varepsilon(t,u)du$, $\pi^\delta(t,du) = \rho^\delta(t,u)du$ and $\pi^{\varepsilon,\delta}(t,du) = \rho^{\varepsilon,\delta}(t,u)du$.

We summarize some properties of ρ^ε in the next proposition. The proof is elementary and is thus omitted.

PROPOSITION 4.3. *Let $\rho : [0,T]\times\mathbb{T} \to \mathbb{R}$ be a function in $L^2([0,T], H^1(\mathbb{T}))$. Then, for each $\varepsilon > 0$, ρ^ε and $\nabla\rho^\varepsilon$ converges to ρ and $\nabla\rho$ in $L^2([0,T] \times \mathbb{T})$, respectively. Moreover, if ρ is bounded in $[0,T] \times \mathbb{T}$ and the application $\langle \rho_t, g\rangle$ is continuous on the time interval $[0,T]$ for any function g in $C^\infty(\mathbb{T})$, then, for each $\varepsilon > 0$, ρ^ε is uniformly continuous on $[0,T] \times \mathbb{T}$.*

For each $a > 0$, define the functions $h = h_a$ and χ_a on $[0,1]$ by

$$\begin{aligned}
h(\rho) \;&:=\; \frac{1}{2(1+2a)} \Big\{ (\rho+a)\log(\rho+a) + (1-\rho+a)\log(1-\rho+a) \Big\} \;,\\
\chi_a(\rho) \;&:=\; (\rho+a)(1-\rho+a) \;.
\end{aligned}$$

Note that $h'' = (2\chi_a)^{-1}$.

Until the end of this section, $0 < C_0 < \infty$ represents a constant independent of ε, δ and a and which may change from line to line.

LEMMA 4.4. *Let $R^{\varepsilon,\delta}$ be the difference between $L_H(\pi^{\varepsilon,\delta})$ and $L_{H^{\varepsilon,\delta}}(\pi)$:*

$$R^{\varepsilon,\delta} \;=\; L_H(\pi^{\varepsilon,\delta}) \;-\; L_{H^{\varepsilon,\delta}}(\pi) \;,$$

where $H = h_a'(\rho^{\varepsilon,\delta})$. Then, for any fixed $\varepsilon > 0$, $R^{\varepsilon,\delta}$ converges to 0 as $\delta \downarrow 0$.

PROOF. Keep in mind that $H = h_a'(\rho^{\varepsilon,\delta})$ depends on ε and δ, although this does not appears in the notation, and recall that C_0 represents a constant independent of ε, δ and a which may change from line to line. A change of variables shows that

$$\begin{aligned} L_H(\pi^{\varepsilon,\delta}) &= \langle \rho^\delta_T, H^\varepsilon_T\rangle - \langle \rho^\delta_0, H^\varepsilon_0\rangle - \int_0^T dt\ \langle \rho^\delta_t, \partial_t H^\varepsilon_t\rangle \\ &= \langle \rho_T, H^{\varepsilon,\delta}_T\rangle - \langle \rho_0, H^{\varepsilon,\delta}_0\rangle - \int_0^T dt\ \langle \rho^\delta_t, \partial_t H^\varepsilon_t\rangle + R_1^{\varepsilon,\delta}\ , \end{aligned}$$

where

$$R_1^{\varepsilon,\delta} := R^{\varepsilon,\delta,T} - R_0^{\varepsilon,\delta,0} \quad \text{and} \quad R^{\varepsilon,\delta,t} := \langle \rho^\delta_t - \rho_t, H^\varepsilon_t\rangle + \langle \rho_t, H^\varepsilon_t - H^{\varepsilon,\delta}_t\rangle$$

for $0 \le t \le T$.

From a simple computation it is easy to see that

$$\int_0^T dt\ \langle \rho^\delta_t, \partial_t H^\varepsilon_t\rangle \ = \ \int_0^T dt\ \langle \rho_t, \partial_t H^{\varepsilon,\delta}_t\rangle + R_2^{\varepsilon,\delta}\ ,$$

where $|R_2^{\varepsilon,\delta}| \le C_0\delta\|\partial_t H^\varepsilon\|_\infty$. To conclude the proof, it is enough to show that, for each fixed $\varepsilon > 0$, $R_1^{\varepsilon,\delta}$ and $\delta\|\partial_t H^\varepsilon\|_\infty$ converge to zero as $\delta \downarrow 0$.

Fix $\varepsilon > 0$. We first prove that

$$\lim_{\delta\downarrow 0} R^{\varepsilon,\delta,t} \ = \ 0 \quad \text{for} \quad t = 0 \ \text{ and } \ t = T\ . \tag{4.3}$$

We prove this assertion for $t = T$, the argument being similar for $t = 0$. A change of variables shows that

$$R^{\varepsilon,\delta,T} \ = \ \langle \rho^{\varepsilon,\delta}_T - \rho^\varepsilon_T, H_T\rangle + \langle \rho^\varepsilon_T, H_T - H^\delta_T\rangle\ .$$

By Proposition 4.3, $\rho^\varepsilon(\cdot,u)$ is continuous for any $u \in \mathbb{T}$. Therefore, for any $(t,u) \in [0,T]\times\mathbb{T}$,

$$\begin{aligned} \lim_{\delta\downarrow 0} \rho^{\varepsilon,\delta}(t,u) \ &= \ \rho^\varepsilon(t,u)\ , \\ \lim_{\delta\downarrow 0} H^\delta(T,u) \ = \ h'_a(\rho^\varepsilon(T,u)) \ &= \ \lim_{\delta\downarrow 0} H(T,u)\ . \end{aligned} \tag{4.4}$$

Since h' is bounded and continuous on $[0,1]$, (4.3) is proved by letting $\delta \downarrow 0$ and by the bounded convergence theorem.

It remains to show that $\delta\|\partial_t H^\varepsilon\|_\infty$ converges to 0 as $\delta \downarrow 0$. An elementary computation gives that, for any $(t,u) \in [0,T]\times\mathbb{T}$,

$$\partial_t H^\varepsilon(t,u) \ = \ \int_{\mathbb{T}} dv\, h''(\rho^{\varepsilon,\delta}(t,u{+}v))\psi^\varepsilon(v) \int_{-\delta}^{\delta} dr\ \rho^\varepsilon(t{+}r,u{+}v)(\phi^\delta)'(r)\ .$$

Since ϕ^δ is a symmetric function, a change of variables shows that

$$\int_{-\delta}^{\delta} dr\ \rho^\varepsilon(t{+}r,u{+}v)(\phi^\delta)'(r) \ = \ \int_{-\delta}^{0} dr\ \{\rho^\varepsilon(t{+}r,u{+}v) - \rho^\varepsilon(t{-}r,u{+}v)\}(\phi^\delta)'(r)\ .$$

By Proposition 4.3, ρ^ε is uniformly continuous on $[-1,T+1]\times\mathbb{T}$. On the other hand, $\delta\int_{-\delta}^0 (\phi^\delta)'(r)dr = \phi(0)$. Therefore, the last expression multiplied by δ converges to 0 as $\delta \downarrow 0$ uniformly in $(t,u) \in [0,T]\times\mathbb{T}$. Since h'' and ψ^ε are uniformly bounded, $\delta\|\partial_t H^\varepsilon\|_\infty$ converges to 0 as $\delta \downarrow 0$. □

LEMMA 4.5. *For any path $\pi(t, du) = \rho(t, u)du$ such that $\mathcal{Q}(\pi) < \infty$ and for $i = 1, 2$,*

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} B^i_{H^{\varepsilon,\delta}}(\pi) \;=\; B^i_{h'(\rho)}(\pi)\ .$$

Moreover, there exists a positive constant $C_0 < \infty$, independent of $a > 0$, such that

$$\int_0^T dt\ \int_{\mathbb{T}} du\ \frac{(\nabla\rho(t,u))^2}{\chi_a(\rho(t,u))} \;\le\; C_0\, B^1_{h'(\rho)}(\pi)\ , \quad |B^2_{h'(\rho)}(\pi)| \;\le\; C_0\ . \tag{4.5}$$

PROOF. Throughout this proof, $C(a)$ expresses a constant depending only on $a > 0$ which may change from line to line.

Let $\pi(t, du) = \rho(t, u)du$ be a path in $D([0, T], \mathcal{M}_{+,1})$ such that $Q(\pi) < \infty$. We first show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} B^1_{H^{\varepsilon,\delta}}(\pi) \;=\; B^1_{h'(\rho)}(\pi)\ . \tag{4.6}$$

Since $\nabla\rho^\varepsilon = \rho * \nabla\psi^\varepsilon$, by Proposition 4.3, $\nabla\rho^\varepsilon$ is uniformly continuous in $[0, T]\times\mathbb{T}$. Therefore, for any $(t, u) \in [0, T] \times \mathbb{T}$, we have

$$\lim_{\delta\downarrow 0} \nabla\rho^{\varepsilon,\delta}(t,u) \;=\; \nabla\rho^\varepsilon(t,u)\ ,$$

$$\lim_{\delta\downarrow 0} \nabla H^{\varepsilon,\delta}(t,u) \;=\; \int_{\mathbb{T}} dv\ \psi^\varepsilon(v) h_a''(\rho^\varepsilon(t,u+v))\nabla\rho^\varepsilon(t,u+v)\ .$$

Hence, by the bounded convergence theorem and a change of variables,

$$\lim_{\delta\downarrow 0} B^1_{H^{\varepsilon,\delta}}(\pi) \;=\; \frac{1}{2}\int_0^T dt\ \big\{\langle\nabla\rho^\varepsilon_t, h_a''(\rho^\varepsilon_t)\nabla\rho^\varepsilon_t\rangle - \langle\chi(\rho_t), ([h_a''(\rho^\varepsilon_t)\nabla\rho^\varepsilon_t]^\varepsilon)^2\rangle\big\}\ . \tag{4.7}$$

On the one hand, since for any fixed $a > 0$ h_a'' is bounded, and since by Proposition 4.3, $\nabla\rho^\varepsilon$ converges to $\nabla\rho$ in $L^2([0, T] \times \mathbb{T})$,

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\ \big\langle h_a''(\rho^\varepsilon_t)\,\big[\nabla\rho^\varepsilon_t - \nabla\rho_t\big]^2\big\rangle \;=\; 0\ .$$

As ρ has finite energy and h_a'' is bounded, the family $\{h_a''(\rho^\varepsilon)[\nabla\rho]^2; \varepsilon > 0\}$ is uniformly integrable. Moreover, since h_a'' is Lipschitz continuous, by Proposition 4.3, $h_a''(\rho^\varepsilon)$ converges to $h_a''(\rho)$ as $\varepsilon \downarrow 0$ in measure, that is, for any $b > 0$, the Lebesgue measure of the set $\{(t, u) \in [0, T] \times \mathbb{T}; |h_a''(\rho^\varepsilon(t, u)) - h_a''(\rho(t, u))| \ge b\}$ converges to 0 as $\varepsilon \downarrow 0$. Therefore

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\ \big\langle h_a''(\rho^\varepsilon_t)\,\big[\nabla\rho_t\big]^2\big\rangle \;=\; \int_0^T dt\ \big\langle h_a''(\rho_t)\,\big[\nabla\rho_t\big]^2\big\rangle\ . \tag{4.8}$$

On the other hand, by Schwarz inequality,

$$\limsup_{\varepsilon\downarrow 0}\int_0^T dt\ \big\langle\chi(\rho_t)\,\big\{[h_a''(\rho^\varepsilon_t)\nabla\rho^\varepsilon_t - h_a''(\rho_t)\nabla\rho_t]^\varepsilon\big\}^2\big\rangle$$

$$\le\; \limsup_{\varepsilon\downarrow 0}\int_0^T dt\ \big\langle\chi(\rho_t)\,\big\{h_a''(\rho^\varepsilon_t)\nabla\rho^\varepsilon_t - h_a''(\rho_t)\nabla\rho_t\big\}^2\big\rangle\ .$$

We may now repeat the arguments presented to estimate the first term on the right hand side of (4.7) to show that the last expression vanishes.

Since χ is a bounded function, to complete the proof of (4.6), it remains to show that

$$\limsup_{\varepsilon\downarrow 0} \int_0^T dt \, \big\langle \big\{ [h_a''(\rho_t)\nabla\rho_t]^\varepsilon - h_a''(\rho_t)\nabla\rho_t \big\}^2 \big\rangle \;=\; 0 \; .$$

We estimate the previous integral by the sum of two terms, the first one being

$$\begin{aligned} &\int_0^T dt \, \big\langle \big\{ [h_a''(\rho_t)\nabla\rho_t]^\varepsilon - [h_a''(\rho_t)]^\varepsilon \nabla\rho_t \big\}^2 \big\rangle \\ &\qquad \le\; C(a) \int_0^T dt \int_{\mathbb{T}} dv \, \psi^\varepsilon(v) \, \big\langle \big\{ \nabla\rho_t(u+v) - \nabla\rho_t(u) \big\}^2 \big\rangle \; , \end{aligned}$$

where we used Schwarz inequality and the fact that h_a'' is uniformly bounded. This expression vanishes as $\varepsilon \to 0$ because $\nabla\rho$ belongs to $L^2([0,T]\times\mathbb{T})$. The second term in the decomposition is

$$\int_0^T dt \, \big\langle [\nabla\rho_t]^2 \big\{ [h_a''(\rho_t)]^\varepsilon - h_a''(\rho_t) \big\}^2 \big\rangle \; . \tag{4.9}$$

By the argument leading to (4.8), the expression (4.9) converges to 0 as $\varepsilon \downarrow 0$.

We turn to the proof that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{\delta\downarrow 0}} \, |B^2_{H^{\varepsilon,\delta}}(\pi) - B^2_{h'(\rho)}(\pi)| \;=\; 0 \; . \tag{4.10}$$

Since B, D and h' are bounded functions, the difference appearing in the previous formula is less than or equal to

$$\begin{aligned} &C(a) \Big\{ \int_0^T \| e^{H_t^{\varepsilon,\delta}} - e^{h'(\rho_t)} \|_1 \, dt \;+\; \int_0^T \| e^{-H_t^{\varepsilon,\delta}} - e^{-h'(\rho_t)} \|_1 \, dt \Big\} \\ &\quad \le\; C(a) \int_0^T \| H_t^{\varepsilon,\delta} - h'(\rho_t) \|_1 \, dt \; . \end{aligned}$$

By Proposition 4.3, ρ^ε is uniformly continuous in $[0,T]\times\mathbb{T}$. Therefore letting $\delta \to 0$, the previous expression converges to

$$\begin{aligned} &C(a) \int_0^T dt \, \| \, [h'(\rho_t^\varepsilon)]^\varepsilon - h'(\rho_t) \|_1 \, dt \\ &\quad \le\; C(a) \Big\{ \int_0^T \| [h'(\rho_t^\varepsilon)]^\varepsilon - h'(\rho_t^\varepsilon) \|_1 \, dt + \int_0^T \| h'(\rho_t^\varepsilon) - h'(\rho_t) \|_1 \, dt \Big\} \; . \end{aligned}$$

Since h' is Lipschitz continuous and ρ^ε converges to ρ in $L^2([0,T]\times\mathbb{T})$, the second integral vanishes in the limit as $\varepsilon \downarrow 0$. On the other hand, the first integral is bounded above by

$$
\begin{aligned}
C(a) \int_0^T dt \ \int_{\mathbb{T}} dv \ \psi^\varepsilon(v) \int_{\mathbb{T}} du \ |\rho_t^\varepsilon(u+v) - \rho_t^\varepsilon(u)| \\
\le \ C(a) \int_0^T dt \ \int_{\mathbb{T}} dv \ \psi^\varepsilon(v) \int_{\mathbb{T}} du \ |\rho_t(u+v) - \rho_t(u)| \ .
\end{aligned}
$$

This last integral vanishes in the limit as $\varepsilon \downarrow 0$ because ρ belongs to $L^2([0,T]\times \mathbb{T})$.

To proof of the first bound in (4.5) is elementary and left to the reader. To prove the second one, recall from (2.1) that there exist polynomials $\tilde{B}, \tilde{D}$ such that $B(\rho) = (1-\rho)\tilde{B}(\rho)$ and $D(\rho) = \rho\tilde{D}(\rho)$. From this fact, it is easy to see that the second bound in (4.5) holds for some finite constant C_0, independent of $a > 0$. □

PROOF OF PROPOSITION 4.2. We may assume, without loss of generality, that $I_T(\pi|\gamma)$ is finite. From the variational formula (4.2) and Lemma 4.4,

$$
L_H(\pi^{\varepsilon,\delta}) + B^1_{H^{\varepsilon,\delta}}(\pi) - B^2_{H^{\varepsilon,\delta}}(\pi) - R^{\varepsilon,\delta} \ \le \ I_T(\pi|\gamma) \ , \tag{4.11}
$$

where H stands for the function $h'(\rho^{\varepsilon,\delta})$.

Since $\rho^{\varepsilon,\delta}$ is smooth, an integration by parts yields the identity

$$
L_H(\pi^{\varepsilon,\delta}) \ = \ \langle h(\rho_T^{\varepsilon,\delta})\rangle - \langle h(\rho_0^{\varepsilon,\delta})\rangle \ .
$$

There exists, therefore, a constant C_0, independent of ε, δ and a, such that

$$
|L_H(\pi^{\varepsilon,\delta})| \ \le \ C_0 \ .
$$

In (4.11), let $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$. It follows from the previous bound, and from Lemmas 4.4 and 4.5 that

$$
\int_0^T dt \ \int_{\mathbb{T}} du \ \frac{|\nabla \rho(t,u)|^2}{\chi_a(\rho(t,u))} \ \le \ C_0\{I_T(\pi|\gamma) + 1\} \ .
$$

It remains to let $a \downarrow 0$ and to use Fatou's lemma. □

COROLLARY 4.6. *The density ρ of a path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem* (3.1) *with initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ is equal to* 0. *Moreover, in that case*

$$
\int_0^T dt \ \int_{\mathbb{T}} du \ \frac{|\nabla \rho(t,u)|^2}{\chi(\rho(t,u))} < \infty. \tag{4.12}
$$

PROOF. If the density ρ of a path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem (3.1), then for any G in $C^{1,2}([0,T]\times$

$\mathbb{T})$ we have

$$J_G(\pi) \;=\; -\frac{1}{2}\int_0^T dt\, \langle \chi(\rho_t), (\nabla G_t)^2\rangle$$
$$- \int_0^T dt\, \{\langle B(\rho_t), e^{G_t} - G_t - 1\rangle + \langle D(\rho_t), e^{-G_t} + G_t - 1\rangle\}\;.$$

Since $e^x - x - 1 \geq 0$ for any x in $\mathbb{R}$, $I_T(\pi|\gamma) = 0$. In addition, the bound (4.12) follows from Proposition 4.2.

On the other hand, if $I_T(\pi|\gamma)$ is equal to 0, then, for any G in $C^{1,2}([0,T]\times\mathbb{T})$ and ε in $\mathbb{R}$, we have $J_{\varepsilon G}(\pi) \leq 0$. Note that $J_0(\pi)$ is equal to 0. Hence the derivative of $J_{\varepsilon G}(\pi)$ in ε at $\varepsilon = 0$ is equal to 0. This implies that the density ρ is a weak solution of the Cauchy problem (3.1). □

THEOREM 4.7. *The function* $I_T(\cdot|\gamma) : D([0,T], \mathcal{M}_+) \to [0,\infty]$ *is lower semicontinuous and has compact level sets.*

PROOF. For each $q \geq 0$, let E_q be the level set of the rate function $I_T(\cdot|\gamma)$:

$$E_q \;:=\; \{\pi \in D([0,T], \mathcal{M}_+) | I_T(\pi|\gamma) \leq q\}\;.$$

Let $\{\pi^n : n \geq 1\}$ be a sequence in $D([0,T], \mathcal{M}_+)$ such that π^n converges to some element π in $D([0,T], \mathcal{M}_+)$. We show that $I_T(\pi|\gamma) \leq \liminf_{n\to\infty} I_T(\pi^n|\gamma)$. If $\liminf I_T(\pi^n|\gamma)$ is equal to ∞, the conclusion is clear. Therefore, we may assume that the set $\{I_T(\pi^n|\gamma) : n \geq 1\}$ is contained in E_q for some $q > 0$. From the lower semicontinuity of the energy $\mathcal{Q}$ and Proposition 4.2, we have

$$\mathcal{Q}(\pi) \;\leq\; \varliminf_{n\to\infty} \mathcal{Q}(\pi^n) \;\leq\; C(q+1) < \infty\;.$$

Since π^n belongs to $D([0,T], \mathcal{M}_{+,1})$, so does π.

Let ρ and ρ^n be the density of π and π^n respectively. We now claim that the sequence $\{\rho^n : n \geq 1\}$ converges to ρ in $L^1([0,T]\times\mathbb{T})$. Indeed, by the triangle inequality,

(4.13)

$$\int_0^T \|\rho_t - \rho_t^n\|_1\, dt$$
$$\leq \int_0^T \|\rho_t - \rho_t^\varepsilon\|_1\, dt + \int_0^T \|\rho_t^\varepsilon - \rho_t^{n,\varepsilon}\|_1\, dt + \int_0^T \|\rho_t^{n,\varepsilon} - \rho_t^n\|_1\, dt\;,$$

where $\rho_t^{n,\varepsilon} = \rho_t^n * \psi^\varepsilon$. The first term on the right hand side in (4.13) can be computed as

$$\int_0^T \|\rho_t - \rho_t^\varepsilon\|_1\, dt \;\leq\; \int_0^T dt \int_{\mathbb{T}} du \int_{\mathbb{T}} dv\, \psi^\varepsilon(v) |\rho(t,u+v) - \rho(t,u)|$$
$$\leq\; \int_0^T dt \int_{\mathbb{T}} du \int_{\mathbb{T}} dv\, \psi^\varepsilon(v) \int_u^{u+v} dw\, |\nabla\rho(t,w)|\;.$$

Note that supp $\psi_\varepsilon \subset [-\varepsilon, \varepsilon]$. From the fundamental inequality $2ab \le A^{-1}a^2 + Ab^2$, for any $A > 0$, the above expression can be bounded above by

$$\frac{\mathcal{Q}(\pi)}{2A} + \frac{AT\varepsilon}{2} .$$

Similarly, the last term on the right hand side in (4.13) can be bounded above by

$$\int_0^T \|\rho_t^{\varepsilon,n} - \rho_t^n\|_1 \, dt \;\le\; \frac{\mathcal{Q}(\pi^n)}{2A} + \frac{AT\varepsilon}{2} .$$

Since, for fixed $\varepsilon > 0$, $\rho_t^{\varepsilon,n}$ converges to ρ_t^ε weakly as $n \to \infty$ for $a.e.\ t \in [0, T]$, letting $n \to \infty$ in (4.13) gives that

$$\overline{\lim_{n\to\infty}} \int_0^T \|\rho_t - \rho_t^n\|_1 \, dt \;\le\; C(q, T)\{\frac{1}{A} + A\varepsilon\} ,$$

for some constant $C(q, T) > 0$ which depends on q and T. Optimizing in A and letting $\varepsilon \downarrow 0$, we complete the proof of the claim made above (4.13).

It follows from this claim that for any function G in $C^{1,2}([0, T] \times \mathbb{T})$,

$$\lim_{n\to\infty} J_G(\pi^n) \;=\; J_G(\pi) .$$

This limit implies that $I_T(\pi|\gamma) \le \liminf_{n\to\infty} I_T(\pi^n|\gamma)$, proving that $I_T(\,\cdot\,|\gamma)$ is lower-semicontinuous.

The same argument shows that E_q is closed in $D([0, T], \mathcal{M}_+)$. Since it is shown in [**30**] that E_q is relatively compact in $D([0, T], \mathcal{M}_+)$, E_q is compact in $D([0, T], \mathcal{M}_+)$, and the proof is completed. □

5. $I_T(\cdot|\gamma)$-Density

The lower bound of the large deviations principle stated in Theorem 2.5 has been established in [**30**] for smooth trajectories. To remove this restriction, we have to show that any trajectory π_t, $0 \le t \le T$, with finite rate function can be approximated by a sequence of smooth trajectories $\{\pi^n : n \ge 1\}$ such that

$$\pi^n \longrightarrow \pi \quad \text{and} \quad I_T(\pi^n|\gamma) \longrightarrow I_T(\pi|\gamma) .$$

This is the content of this section. We first introduce some terminology.

DEFINITION 5.1. *Let A be a subset of $D([0, T], \mathcal{M}_+)$. A is said to be $I_T(\cdot|\gamma)$-dense if for any π in $D([0, T], \mathcal{M}_+)$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in A such that π^n converges to π in $D([0, T], \mathcal{M}_+)$ and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.*

Let Π be the set of all trajectories $\pi(t, du) = \rho(t, u)du$ in $D([0, T], \mathcal{M}_{+,1})$ whose density ρ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho \;=\; \dfrac{1}{2}\Delta\rho - \nabla(\chi(\rho)\nabla H) + B(\rho)e^H - D(\rho)e^{-H} \;\text{ on }\; \mathbb{T} , \\ \rho(0, \cdot) \;=\; \gamma(\cdot) , \end{cases} \tag{5.1}$$

for some function H in $C^{1,2}([0,T]\times\mathbb{T})$.

THEOREM 5.2. *Assume that the functions B and D are concave. Then, the set Π is $I_T(\cdot|\gamma)$-dense.*

The proof of Theorem 5.2 is divided into several steps. Throughout this section, denote by $\lambda : [0,T]\times\mathbb{T}\to[0,1]$ the unique weak solution of the Cauchy problem (3.1) with initial profile γ, and assume that the functions B and D are concave.

Let Π_1 be the set of all paths $\pi(t,du)=\rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ whose density ρ is a weak solution of the Cauchy problem (3.1) in some time interval $[0,\delta]$, $\delta>0$.

LEMMA 5.3. *The set Π_1 is $I_T(\cdot|\gamma)$-dense.*

PROOF. Fix $\pi(t,du)=\rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ such that $I_T(\pi|\gamma)<\infty$. For each $\delta>0$, set the path $\pi^\delta(t,du)=\rho^\delta(t,u)du$ where

$$\rho^\delta(t,u)\;=\;\begin{cases}\lambda(t,u) & \text{if } t\in[0,\delta]\;,\\ \lambda(2\delta-t,u) & \text{if } t\in[\delta,2\delta]\;,\\ \rho(t-2\delta,u) & \text{if } t\in[2\delta,T]\;.\end{cases}$$

It is clear that π^δ converges to π in $D([0,T],\mathcal{M}_+)$ as $\delta\downarrow 0$ and that π^δ belongs to Π_1. To conclude the proof it is enough to show that $I_T(\pi^\delta|\gamma)$ converges to $I_T(\pi|\gamma)$ as $\delta\downarrow 0$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma)\le\liminf_{\delta\to 0}I_T(\pi^\delta|\gamma)$. Note that $\mathcal{Q}(\pi^\delta)\le 2\mathcal{Q}(\lambda)+\mathcal{Q}(\pi)$. From Corollary 4.2, we have $\mathcal{Q}(\pi^\delta)<\infty$. To prove the upper bound $\limsup_{\delta\to 0}I_T(\pi^\delta|\gamma)\le I_T(\pi|\gamma)$, we now decompose the rate function $I_T(\pi^\delta|\gamma)$ into the sum of the contributions on each time interval $[0,\delta]$, $[\delta,2\delta]$ and $[2\delta,T]$. The first contribution is equal to 0 since the density ρ^δ is a weak solution of the equation (3.1) on this interval. The third contribution is bounded above by $I_T(\pi|\gamma)$ since π^δ on this interval is a time translation of the path π.

On the time interval $[\delta,2\delta]$, the density ρ^δ solves the backward reaction-diffusion equation: $\partial_t\rho^\delta=-(1/2)\Delta\rho^\delta-F(\rho^\delta)$. Therefore, the second contribution can be written as

$$\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})}\Big\{\int_0^\delta dt\,\big\{\langle\nabla\lambda_t,\nabla G_t\rangle-\frac{1}{2}\langle\chi(\lambda_t),(\nabla G_t)^2\rangle\big\}$$
$$+\int_0^\delta dt\,\big\{\langle B(\lambda_t),1-e^{G_t}-G_t\rangle+\langle D(\lambda_t),1-e^{-G_t}+G_t\rangle\big\}\Big\}\;.$$

By Schwarz inequality, the first integral inside the supremum is bounded above by

$$\frac{1}{2}\int_0^\delta dt\int_{\mathbb{T}}du\,\frac{|\nabla\lambda(t,u)|^2}{\chi(\lambda(t,u))}\;. \tag{5.2}$$

On the other hand, taking advantage of the relation (2.1) and of the fact that B and D are bounded functions, a simple computation shows that the second integral inside the supremum in the penultimate displayed equation is bounded above by

$$C \int_0^\delta dt \int_{\mathbb{T}} du \, \log \frac{1}{\chi(\lambda(t,u))} + C\delta \, ,$$

for some finite constant C independent of δ. By Corollary 4.2, the expression (5.2) converges to 0 as $\delta \downarrow 0$. Hence, to conclude the proof it suffices to show that

$$\lim_{\delta \downarrow 0} \int_0^\delta dt \int_{\mathbb{T}} du \, \log \chi(\lambda(t,u)) \;=\; 0 \, . \tag{5.3}$$

Let $\lambda^j(t,u) \equiv \lambda^j_t$, $j = 0, 1$, be the weak solution of the equation (3.1) with initial profile $\lambda^j_0(u) \equiv j$. By Proposition 6.5,

$$\lambda^0_t \;\le\; \lambda(t,u) \;\text{ and }\; 1 - \lambda^1_t \;\le\; 1 - \lambda(t,u) \, , \tag{5.4}$$

for any $(t,u) \in [0,\delta] \times \mathbb{T}$. Since λ^j, $j = 1, 2$, solves the ordinary differential equation

$$\frac{d}{dt} \lambda^j_t \;=\; F(\lambda^j_t) \, ,$$

and since $F(1) < 0 < F(0)$, an elementary computation shows that

$$\lim_{\delta \downarrow 0} \int_0^\delta dt \, \log \lambda^0_t \;=\; 0 \quad \text{and} \quad \lim_{\delta \downarrow 0} \int_0^\delta dt \, \log (1 - \lambda^1_t) \;=\; 0 \, . \tag{5.5}$$

By definition of χ and by (5.4),

$$\log \chi(\lambda(t,u)) \;=\; \log \lambda(t,u) + \log (1 - \lambda(t,u)) \ge \log \lambda^0_t + \log (1 - \lambda^1_t) \, .$$

To conclude the proof of (5.3), it remains to recall (5.5). □

Let Π_2 be the set of all paths $\pi(t,du) = \rho(t,u)du$ in Π_1 with the property that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\varepsilon \le \rho(t,u) \le 1 - \varepsilon$ for all $(t,u) \in [\delta, T] \times \mathbb{T}$.

LEMMA 5.4. *The set* Π_2 *is* $I_T(\cdot|\gamma)$*-dense.*

PROOF. Fix $\pi(t,du) = \rho(t,u)du$ in Π_1 such that $I_T(\pi|\gamma) < \infty$. For each $\varepsilon > 0$, set the path $\pi^\varepsilon(t,du) = \rho^\varepsilon(t,u)du$ with $\rho^\varepsilon = (1-\varepsilon)\rho + \varepsilon\lambda$. It is clear that π^ε converges to π in $D([0,T], \mathcal{M}_+)$ as $\varepsilon \downarrow 0$. Let $\lambda^j(t,u) \equiv \lambda^j_t$, $j = 0, 1$, be the weak solution of the equation (3.1) with initial profile $\lambda^j_0(u) \equiv j$. By Proposition 6.5, $\varepsilon\lambda^0 \le \rho^\varepsilon \le (1-\varepsilon) + \varepsilon\lambda^1$. Therefore π^ε belongs to Π_2. To conclude the proof it is enough to show that $I_T(\pi^\varepsilon|\gamma)$ converges to $I_T(\pi|\gamma)$ as $\varepsilon \downarrow 0$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma) \le \liminf_{\varepsilon\downarrow 0} I_T(\pi^\varepsilon|\gamma)$. By the convexity of the energy, $\mathcal{Q}(\pi^\varepsilon) \le \varepsilon\mathcal{Q}(\lambda) + (1-\varepsilon)\mathcal{Q}(\pi)$, hence $\mathcal{Q}(\pi^\varepsilon) < \infty$. Let G be a function in $C^{1,2}([0,T] \times \mathbb{T})$. Since B, D and χ are concave and Lipschitz continuous,

$$J_G(\pi^\varepsilon) \;\le\; (1-\varepsilon) J_G(\pi) + \varepsilon J_G(\lambda) + C_0 \int_0^T \; \|\rho^\varepsilon_t - \rho_t\|_1 \, dt$$

for some finite constant C_0. Therefore,

$$I_T(\pi^\varepsilon|\gamma) \ \le \ (1-\varepsilon)I_T(\pi|\gamma) + \varepsilon I_T(\lambda|\gamma) + C_0 T\varepsilon \ .$$

Letting $\varepsilon \downarrow 0$ gives $\limsup_{\varepsilon\downarrow 0} I_T(\pi^\varepsilon|\gamma) \le I_T(\pi|\gamma)$, which completes the proof. □

Let Π_3 be the set of all paths $\pi(t,du) = \rho(t,u)du$ in Π_2 whose density $\rho(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for any $t \in (0,T]$.

LEMMA 5.5. *The set Π_3 is $I_T(\cdot|\gamma)$-dense.*

PROOF. Fix $\pi(t,du) = \rho(t,u)du$ in Π_2 such that $I_T(\pi|\gamma) < \infty$. Since π belongs to the set Π_1, we may assume that the density solves the equation (3.1) in some time interval $[0,2\delta]$, $\delta > 0$. Take a smooth nondecreasing function $\alpha : [0,T] \to [0,1]$ with the following properties:

$$\begin{cases} \alpha(t) \ = \ 0 & \text{if } \ t \in [0,\delta] \ , \\ 0 \ < \alpha(t) \ < 1 & \text{if } \ t \in (\delta, 2\delta) \ , \\ \alpha(t) \ = \ 1 & \text{if } \ t \in [2\delta, T] \ . \end{cases}$$

Let $\psi(t,u) : (0,\infty)\times\mathbb{T} \to (0,\infty)$ be the transition probability density of the Brownian motion on $\mathbb{T}$ at time t starting from 0. For each $n \in \mathbb{N}$, denote by ψ^n the function

$$\psi^n(t,u) \ := \ \psi(\frac{1}{n}\alpha(t), u)$$

and define the path $\pi^n(t,du) = \rho^n(t,u)du$ where

$$\rho^n(t,u) \ = \ \begin{cases} \rho(t,u) & \text{if } \ t \in [0,\delta] \ , \\ (\rho_t * \psi^n_t)(u) \ = \ \int_{\mathbb{T}} dv\, \rho(t,v)\psi^n(t,u-v) & \text{if } \ t \in (\delta, T] \ . \end{cases}$$

It is clear that π^n converges to π in $D([0,T],\mathcal{M}_+)$ as $n \to \infty$. Since the density ρ^n is a weak solution to the Cauchy problem (3.1) in time interval $[0,\delta]$, by Proposition 3.4, $\rho^n(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for $t \in (0,\delta]$. On the other hand, by the definition of ρ^n, it is clear that $\rho^n(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for $t \in (\delta,T]$. Therefore π^n belongs to Π_3. To conclude the proof it is enough to show that $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$ as $n \to \infty$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma) \le \liminf_{n\to\infty} I_T(\pi^\varepsilon|\gamma)$. Note that the generalized derivative of ρ^n is given by

$$\nabla\rho^n(t,u) \ = \ \begin{cases} \nabla\rho(t,u) & \text{if } \ t \in [0,\delta] \ , \\ (\nabla\rho_t * \psi^n_t)(u) & \text{if } \ t \in (\delta,T] \ . \end{cases}$$

Therefore, by Schwarz inequality, $\mathcal{Q}(\pi^n) \le \mathcal{Q}(\pi) < \infty$.

The strategy of the proof of the upper bound is similar to the one of Lemma 5.3. We decompose the rate function $I_T(\pi^n|\gamma)$ into the sum of the contributions on each time interval $[0,\delta]$, $[\delta,2\delta]$ and $[2\delta,T]$. The first contribution is equal to 0 since the density ρ^n is a weak solution of the Cauchy problem (3.1) on this interval. Since π^n is defined as a spatial average of

π, and since the functions B and D are concave, similar arguments to the ones presented in the proof of Lemma 5.4 yield that the third contribution is bounded above by $I_T(\pi|\gamma) + o_n(1)$. Hence it suffices to show that the second contribution converges to 0 as $n \to \infty$.

Since $\partial_t \psi = (1/2)\Delta\psi$, an integration by parts yields that in the time interval $(\delta, 2\delta)$,

$$\partial_t \rho^n \;=\; \partial_t \rho * \psi^n + \frac{\alpha'(t)}{2n} \Delta\rho * \psi^n \; .$$

Thus, since in the time interval $[\delta, 2\delta]$ ρ is a weak solution of the hydrodynamic equation (3.1), for any function G in $C^{1,2}([0,T]\times\mathbb{T})$,

$$\begin{aligned}\langle \rho^n_{2\delta}, G_{2\delta}\rangle - \langle \rho^n_\delta, G_\delta\rangle - \int_\delta^{2\delta} dt\, \langle \rho^n_t, \partial_t G_t\rangle \\ = \int_\delta^{2\delta} dt\, \Big\{ \langle \rho^n_t, \frac{1}{2}\Delta G_t\rangle - \frac{\alpha'(t)}{2n}\langle \nabla\rho^n_t, \nabla G_t\rangle + \langle F^n_t, G_t\rangle \Big\}\;,\end{aligned}$$

where $F^n_t = F(\rho_t) * \psi^n_t$. Therefore, the contribution to $I_T(\pi|\gamma)$ of the piece of the trajectory in the time interval $[\delta, 2\delta]$ can be written as

(5.6)

$$\begin{aligned}\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})} \Big\{ \int_\delta^{2\delta} dt\, \Big(-\frac{\alpha'(t)}{2n}\langle\nabla\rho^n_t, \nabla G_t\rangle - \frac{1}{2}\langle \chi(\rho^n_t), (\nabla G_t)^2\rangle\Big) \\ + \int_\delta^{2\delta} dt\, \langle F^n_t G_t - B(\rho^n_t)(e^{G_t}-1) - D(\rho^n_t)(e^{-G_t}-1)\rangle \Big\}\;.\end{aligned}$$

By Schwarz inequality, the first integral inside the supremum is bounded above by

$$\frac{\|\alpha'\|^2_\infty}{8n^2}\int_\delta^{2\delta} dt \int_{\mathbb{T}} du\, \frac{|\nabla\rho^n(t,u)|^2}{\chi(\rho^n(t,u))}\;.$$

Since π belongs to Π_1, there exists a positive constant $C(\delta)$, depending only on δ, such that $C(\delta) \le \rho^n \le 1 - C(\delta)$ on time interval $[\delta, 2\delta]$. This bounds together with the fact that $\mathcal{Q}(\pi^n) \le \mathcal{Q}(\pi)$ permit to prove that the previous expression converges to 0 as $n\to\infty$. On the other hand, the second integral inside the supremum (5.6) is bounded above by

$$\int_\delta^{2\delta} dt\, \langle F^n_t m^n_t - B(\rho^n_t)(e^{m^n_t}-1) - D(\rho^n_t)(e^{-m^n_t}-1)\rangle\;, \tag{5.7}$$

where

$$m^n_t \;=\; \log\frac{F^n_t + \sqrt{(F^n_t)^2 + 4B(\rho^n_t)D(\rho^n_t)}}{2B(\rho^n_t)}\;.$$

Note that m^n_t is well-defined and that the integrand in (5.7) is uniformly bounded in n because in the time interval $[\delta, 2\delta]$ ρ_t is bounded below by a strictly positive constant and bounded above by a constant strictly smaller than 1. Since $m^n(t,u)$ converges to 0 as $n\to\infty$ for any $(t,u)\in[\delta,2\delta]\times\mathbb{T}$, the expression in (5.7) converges to 0 as $n\to\infty$. □

Let Π_4 be the set of all paths $\pi(t, du) = \rho(t, u)du$ in Π_3 whose density ρ belongs to $C^{\infty,\infty}((0, T] \times \mathbb{T})$.

LEMMA 5.6. *The set* Π_4 *is* $I_T(\cdot|\gamma)$*-dense.*

PROOF. Fix $\pi(t, du) = \rho(t, u)du$ in Π_3 such that $I_T(\pi|\gamma) < \infty$. Since π belongs to the set Π_1, we may assume that the density ρ solves the equation (3.1) in the time interval $[0, 3\delta]$ for some $\delta > 0$. Take a smooth non-negative function $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\text{supp}\, \phi \ \subset \ [0, 1] \quad \text{and} \quad \int_0^1 \phi(s)ds \ = \ 1 \ .$$

Let α be the function introduced in the previous lemma. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, let

$$\Phi(\varepsilon, s) \ := \ \frac{1}{\varepsilon}\phi(\frac{s}{\varepsilon}) \ , \quad \alpha_n(t) \ := \ \frac{1}{n}\alpha(t) \ ,$$

and let $\pi^n(t, du) = \rho^n(t, u)du$ where

$$\rho^n(t, u) \ = \ \int_0^1 \rho(t + \alpha_n(t)s, u)\phi(s)ds \ = \ \int_{\mathbb{R}} \rho(t + s, u)\Phi(\alpha_n(t), s)ds \ .$$

In the above formula, we extend the definition of ρ to $[0, T + 1]$ by setting $\rho_t = \tilde{\lambda}_{t-T}$ for $T \le t \le T + 1$, where $\tilde{\lambda} : [0, 1] \times \mathbb{T} \to [0, 1]$ stands for the unique weak solution of the equation (3.1) with initial profile ρ_T.

It is clear that π^n converges to π in $D([0, T], \mathcal{M}_+)$. Since on the time interval $(0, 3\delta)$, the function ρ is smooth in time, for n large enough the function ρ^n is smooth in time on $(0, T] \times \mathbb{T}$. Hence, π^n belongs to Π_4 and $\mathcal{Q}(\pi^n)$ is finite.

The remaining part of the proof is similar to the one of the previous lemma. We only present the arguments leading to the bound $\limsup_{n\to\infty} I_T(\pi^n|\gamma) \le I_T(\pi|\gamma)$. The rate function can be decomposed in three pieces, two of which can be estimated as in Lemma 5.5. We consider the contribution to $I_T(\pi^n|\gamma)$ of the piece of the trajectory corresponding to the time interval $[\delta, 2\delta]$.

The derivative of ρ^n in time on $(\delta, 2\delta)$ is computed as

$$\partial_t\rho^n(t, u) \ = \ \int_{\mathbb{R}} \partial_t\rho(t+s, u)\Phi(\alpha_n(t), s)ds + \int_{\mathbb{R}} \rho(t+s, u)\partial_t[\Phi(\alpha_n(t), s)]ds \ .$$

It follows from this equation and from the fact that the density ρ solves the hydrodynamic equation (3.1) on the time interval $[\delta, 3\delta]$, that for any function G in $C^{1,2}([0, T] \times \mathbb{T})$,

$$\langle\rho^n_{2\delta}, G_{2\delta}\rangle - \langle\rho^n_\delta, G_\delta\rangle - \int_\delta^{2\delta} dt\, \langle\rho^n_t, \partial_t G_t\rangle \ = \ \int_\delta^{2\delta} dt \, \Big\{\langle\rho^n_t, \frac{1}{2}\Delta G_t\rangle + \langle F^n_t + r^n_t, G_t\rangle\Big\} \ ,$$

where

$$F^n(t,u) \;:=\; \int_{\mathbb{R}} F(\rho(t+s,u))\,\Phi(\alpha_n(t),s)\,ds\;,$$
$$r^n(t,u) \;:=\; \int_{\mathbb{R}} \rho(t+s,u)\,\partial_t[\Phi(\alpha_n(t),s)]\,ds\;.$$

Therefore, the second contribution can be bounded above by

(5.8)
$$\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})}\Big\{\int_\delta^{2\delta} dt\;\langle (F_t^n+r_t^n)G_t - B(\rho_t^n)(e^{G_t}-1) - D(\rho_t^n)(e^{-G_t}-1)\rangle\Big\}\;.$$

We now show that $r^n(t,u)$ converges to 0 as $n\to\infty$ uniformly in $(t,u)\in(\delta,2\delta)\times\mathbb{T}$. Let (t,u) in $(\delta,2\delta)\times\mathbb{T}$. Since $\int_{\mathbb{R}}\partial_t[\Phi(\alpha_n(t),s)]ds = \partial_t[\int_{\mathbb{R}}\Phi(\alpha_n(t),s)ds]=0$, $r^n(t,u)$ can be written as

$$\int_{\mathbb{R}}\{\rho(t+s,u)-\rho(t,u)\}\,\partial_t[\Phi(\alpha_n(t),s)]\,ds\;.$$

Since ρ is Lipschitz continuous on $[\delta,3\delta]\times\mathbb{T}$, there exists a positive constant $C(\delta)>0$, depending only on δ, such that

$$|\rho(t+s,u)-\rho(t,u)| \;\le\; C(\delta)s\;,$$

for any $(t,u)\in[\delta,2\delta]\times\mathbb{T}$ and $s\in[0,\delta]$. Therefore $r^n(t,u)$ is bounded above by

$$C(\delta)\int_{\mathbb{R}} s\,\big|\partial_t[\Phi(\alpha_n(t),s)]\big|\,ds\;.$$

It follows from a simple computation and from the change of variables $\alpha_n(t)s=\bar{s}$ that

$$\int_{\mathbb{R}} s\,\big|\partial_t[\Phi(\alpha_n(t),s)]\big|\,ds \;\le\; \frac{\|\alpha'(t)\|_\infty}{n}\int_0^1\big\{s\phi(s)+s^2|\phi'(s)|\big\}\,ds\;.$$

Therefore $r^n(t,u)$ converges to 0 as $n\to\infty$ uniformly in $(t,u)\in(\delta,2\delta)\times\mathbb{T}$.

To complete the proof, it remains to take a supremum in $G\in C^{1,2}([0,T]\times\mathbb{T})$ in formula (5.8) and to let $n\to\infty$. □

PROOF OF THEOREM 5.2. From the previous lemma, all we need is to prove that Π_4 is contained in Π. Let $\pi(t,du)=\rho(t,u)du$ be a path in Π_4. There exists some $\delta>0$ such that the density ρ solves the equation (3.1) on time interval $[0,2\delta]$. In particular, the density ρ also solves the equation (5.1) with $H=0$ on time interval $[0,2\delta]$. On the one hand, since the density ρ is smooth on $[\delta,T]$ and there exists $\varepsilon>0$ such that $\varepsilon\le\rho(t,u)\le 1-\varepsilon$ for any $(t,u)\in[\delta,T]\times\mathbb{T}$, from Lemma 2.1 in [**30**], there exits a unique function H in $C^{1,2}([\delta,T]\times\mathbb{T})$ satisfying the equation (5.1) with ρ on $[\delta,T]$, and it is proved that π belongs to Π. □

PROOF OF THEOREM 2.5. We have already proved in Section 4 that the rate function is lower semicontinuous and that it has compact level sets.

Recall from the beginning of this section the definition of the set Π. It has been proven in [**30**] that for each closed subset $\mathcal{C}$ of $D([0,T],\mathcal{M}_+)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{C}) \ \le\ -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma) \ ,$$

and that for each open subset $\mathcal{O}$ of $D([0,T],\mathcal{M}_+)$,

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathcal{O}) \ \ge\ -\inf_{\pi\in\mathcal{O}\cap\Pi} I_T(\pi|\gamma) \ .$$

Since $\mathcal{O}$ is open in $D([0,T],\mathcal{M}_+)$, by Theorem 5.2,

$$\inf_{\pi\in\mathcal{O}\cap\Pi} I_T(\pi|\gamma) \ =\ \inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma) \ ,$$

which completes the proof. □

6. Appendix

In sake of completeness, we present in this section results on the Cauchy problem (3.1).

DEFINITION 6.1. *A measurable function $\rho:[0,T]\times\mathbb{T}\to[0,1]$ is said to be a weak solution of the Cauchy problem* (3.1) *in the layer $[0,T]\times\mathbb{T}$ if, for every function G in $C^{1,2}([0,T]\times\mathbb{T})$,*

$$\begin{aligned}\langle\rho_T,G_T\rangle - \langle\gamma,G_0\rangle - \int_0^T dt\langle\rho_t,\partial_t G_t\rangle \\ = \frac{1}{2}\int_0^T dt\langle\rho_t,\Delta G_t\rangle + \int_0^T dt\langle F(\rho_t),G_t\rangle \ . \qquad (6.1)\end{aligned}$$

For each $t\ge 0$, let P_t be the semigroup on $L^2(\mathbb{T})$ generated by $(1/2)\Delta$.

DEFINITION 6.2. *A measurable function $\rho:[0,T]\times\mathbb{T}\to[0,1]$ is said to be a mild solution of the Cauchy problem* (3.1) *in the layer $[0,T]\times\mathbb{T}$ if, for any t in $[0,T]$, it holds that*

$$\rho_t \ =\ P_t\gamma + \int_0^t P_{t-s}F(\rho_s)ds \ . \qquad (6.2)$$

The first proposition asserts existence and uniqueness of weak and mild solutions, a well known result in the theory of partial differential equations. We give a brief proof because uniqueness of the Cauchy problem (3.1) plays an important role in the proof of Theorem 2.1.

PROPOSITION 6.3. *Definitions 3.1 and 3.2 are equivalent. Moreover, there exists a unique weak solution of the Cauchy problem* (3.1).

PROOF. Since F is Lipschitz continuous, by the method of successive approximation, there exists a unique mild solution of the Cauchy problem (3.1). Therefore to conclude the proposition it is enough to show that the above two notions of solutions are equivalent.

Assume that $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is a weak solution of the Cauchy problem (3.1). Fix a function g in $C^2(\mathbb{T})$ and $0 \le t \le T$. For each $\delta > 0$, define the function G^δ as

$$G^\delta(s,u) \;=\; \begin{cases} (P_{t-s}g)(u) & \text{if } 0 \le s \le t \;, \\ \delta^{-1}(t+\delta-s)g(u) & \text{if } t \le s \le t+\delta \;, \\ 0 & \text{if } t+\delta \le s \le T \;. \end{cases}$$

One can approximate G^δ by functions in $C^{1,2}([0,T]\times\mathbb{T})$. Therefore, by letting $\delta \downarrow 0$ in (3.2) with G replaced by G^δ and by a summation by parts,

$$\langle \rho_t, g\rangle \;=\; \langle P_t\gamma, g\rangle + \int_0^t \langle P_{t-s}F(\rho_s), g\rangle ds \;. \tag{6.3}$$

Since (6.3) holds for any function g in $C^2(\mathbb{T})$, ρ is a mild solution of the Cauchy problem (3.1).

Conversely, assume that $\rho : [0,T]\times\mathbb{T} \to [0,1]$ is a weak solution of the Cauchy problem (3.1). In this case, (6.3) is true for any function g in $C^2(\mathbb{T})$ and any $0 \le t \le T$. Differentiating (6.3) in t gives that

$$\frac{d}{dt}\langle \rho_t, g\rangle \;=\; \frac{1}{2}\langle \rho_t, \Delta g\rangle + \langle F(\rho_t), g\rangle \;.$$

Therefore (3.2) holds for any function $G(t,u) = g(u)$ in $C^2(\mathbb{T})$. It is not difficult to extend this to any function G in $C^{1,2}([0,T]\times\mathbb{T})$. Hence ρ is a weak solution of the Cauchy problem (3.1). □

The following two propositions assert the smoothness and the monotonicity of weak solutions of the Cauchy problem (3.1).

PROPOSITION 6.4. *Let ρ be the unique weak solution of the Cauchy problem* (3.1)*. Then ρ is infinitely differentiable over* $(0,\infty)\times\mathbb{T}$.

PROPOSITION 6.5. *Let ρ_0^1 and ρ_0^2 be two initial profiles. Let ρ^j, $j = 1, 2$, be the weak solutions of the Cauchy problem* (3.1) *with initial condition ρ_0^j. Assume that*

$$m\{u \in \mathbb{T} : \rho_0^1(u) \le \rho_0^2(u)\} \;=\; 1 \;,$$

where m is the Lebesgue measure on $\mathbb{T}$. Then, for any $t \ge 0$, it holds that

$$m\{u \in \mathbb{T} \;:\; \rho^1(t,u) \le \rho^2(t,u)\} \;=\; 1 \;.$$

The proofs of Propositions 3.4 and 6.5 can be found in the ones of Proposition 2.1 of [**18**].

The last proposition asserts that, for any initial density profile γ, the weak solution ρ_t of the Cauchy problem (3.1) converges to some solution of the semilinear elliptic equation (2.1). Recall, from Subsection 2.3, the definition of the set $\mathcal{E}$.

PROPOSITION 6.6. *Let $\rho : [0,\infty)\times\mathbb{T} \to [0,1]$ be the unique weak solution of the Cauchy problem* (3.1)*. Then there exists a density profile ρ_∞ in $\mathcal{E}$ such that ρ_t converges to ρ_∞ as $t \to \infty$ in $C^2(\mathbb{T})$.*

The proof of this proposition can be found in the one of Proposition 2.1 of [**14**].

CHAPTER 4

Static large deviation principle for a reaction-diffusion model

1. Introduction to Chapter 4

The aim of this chapter is to obtain the static large deviation principle for a reaction-diffusion model, introduced in [**16**]. Our result can be regarded as generalization of one which is shown in [**19, 7**]

We again consider in this chapter the superposition of the symmetric simple exclusion process with a spin-flip dynamics. Landim and Tsunoda [**36**] proved the concentration of the sequence of the stationary measures to the set of all classical solutions to the semilinear elliptic equation

$$(1.1) \qquad (1/2)\Delta\rho + B(\rho) - D(\rho) \;=\; 0 \,,$$

where Δ is the Laplacian and $F = B - D$ is a reaction term determined by the spin-flip dynamics.

We study the large deviations for the sequence of the stationary measures based on results presented in [**36**]. The static large deviation principle for the boundary driven exclusion processes was proved in [**7, 11, 22**]. In contrast with the conservative dynamics, the stationary equation, given by the equation (1.1), may not have a unique solution in general. Therefore the corresponding dynamical system may not have a global attractor. Due to this fact, we need a detailed analysis for the microscopic system to prove the static large deviation principle.

The main idea to prove the static large deviation principle is to reformulate Freidlin and Wentzell approach [**21**] in our infinite-dimensional setting. The basic strategy is the following. We first consider a chain induced from the original one on the union of neighborhoods of all solutions of the equation (1.1). Then we give large deviation type estimates on one-step transition probability for the induced chain. Such estimates give similar bounds for the stationary measure for the induced chain. The next step is to consider the minimal cost which creates a measure from each equilibrium states. This step is somewhat similar to one presented in [**11, 22**]. These two steps give the large deviations bound for the stationary measures of the reaction-diffusion model.

The main difficulty in the proof of the static large deviation is due to the weak topology of the state space. This topology prevents us from connecting two different points by the simple line segment. To avoid this problem, we use several properties of solutions of the hydrodynamic equation.

We mention the necessity of the technical assumption that B and D are concave. The static large deviation principle remains true if the dynamical large deviation principle holds with a lower semicontinuous rate function since we do not additionally use this assumption in the proof of the static large deviation principle. However the dynamical large deviation principle is proved only under this assumption. See [**36**] for more details. We also referred to some references [**4, 13, 8**] for physical aspects of the reaction-diffusion model.

This chapter is organized as follows. In Section 2, we introduce a reaction-diffusion model and state the main result. In Section 3 we study some properties of weak solutions of the Cauchy problem (3.1). In Section 4, we present the main properties of the dynamical and static rate functions. In Section 5, we prove the static large large deviation principle, which is the main result of this chapter.

2. Notation and Results

Throughout this chapter, we use the following notation. $\mathbb{N}_0$ stands for the set $\{0, 1, \cdots\}$. For a function $f : X \to \mathbb{R}$, defined on some space X, let $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We sometimes denote the interval $[0, \infty)$ by $\mathbb{R}_+$.

2.1. Reaction-diffusion model. For each integer $N \geq 1$, Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ be the one-dimensional discrete torus. Denote by $X_N = \{0, 1\}^{\mathbb{T}_N}$ the state space of our process and by η the configurations of X_N. For each $x \in \mathbb{T}_N$, $\eta(x)$ stands for the number of particles sitting at site x for the configuration η. For each $x, y \in \mathbb{T}_N$ with $x \neq y$, we also denote by $\eta^{x,y}$, resp. by η^x, the configuration obtained from η by displacing a particle from x to y, resp. by flipping the occupation variable at site x:

$$\eta^{x,y}(z) \;=\; \begin{cases} \eta(y) & \text{if } z = x\;, \\ \eta(x) & \text{if } z = y\;, \\ \eta(z) & \text{otherwise}\;, \end{cases} \qquad \eta^{x}(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x\;, \\ 1-\eta(z) & \text{if } z = x\;. \end{cases}$$

We consider the superposition of the symmetric simple exclusion process with a spin-flip dynamics. More precisely, the stochastic dynamics is described by the continuous-time Markov process on X_N whose generator acts on functions $f : X_N \to \mathbb{R}$ as

$$\mathcal{L}_N f \;=\; (N^2/2)\mathcal{L}_K f + \mathcal{L}_G f\;,$$

where $\mathcal{L}_K$ is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_K f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} [f(\eta^{x,x+1}) - f(\eta)]\;,$$

and $\mathcal{L}_G$ is the generator of a spin flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} c(x,\eta)[f(\eta^{x}) - f(\eta)]\;.$$

In the last formula, we choose the function $c(x,\eta) = c(\eta(x-M),\cdots,\eta(x+M))$, for some $M \geq 1$, to be strictly positive and local, that is, a function which depends only on a finite number of coordinates $\eta(y)$, $|y| \leq M$. Note that we have already put the time-change factor N^2 in $\mathcal{L}_N$, which corresponds to the diffusive scaling. If the jump rate of the Glauber dynamics is identically equal to 0, then the corresponding Markov process is given by the symmetric simple exclusion process speeded up by N^2. In this case, the static large deviation principle for the boundary driven exclusion process was studied in [**7, 11, 22**].

Fix arbitrarily $T > 0$. For a topological space X and an interval $I = [0,T]$ or $[0,\infty)$, let $D(I,X)$ be the space of all right-continuous trajectories from I to X with left-limits, endowed with the Skorokhod topology. Let $\{\eta_t^N : N \geq 1\}$ be the continuous-time Markov process on X_N whose generator is given by $\mathcal{L}_N$ and $\mathbb{P}_\eta$, $\eta \in X_N$, be the probability measure on $D(\mathbb{R}_+, X_N)$ induced by the process η_t^N starting from η. Denote by $\mathbb{E}_\eta[\cdot]$ the expectation with respect to $\mathbb{P}_\eta$.

2.2. Hydrostatics. We review in this subsection the asymptotic behavior of the empirical measure under the stationary state.

Let $\mathbb{T}$ be the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$ and $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of all nonnegative measures on $\mathbb{T}$, whose total mass bounded by 1, endowed with the weak topology. For a measure ϑ in $\mathcal{M}_+$ and a continuous function $G : \mathbb{T} \to \mathbb{R}$, denote by $\langle \vartheta, G\rangle$ the integral of G with respect to ϑ:

$$\langle \vartheta, G\rangle \;=\; \int_{\mathbb{T}} G(u)\vartheta(du)\ .$$

The space $\mathcal{M}_+$ is metrizable. Indeed, if $e_0(u) = 1$, $e_k(u) = \sqrt{2}\cos(2\pi k u)$ and $e_{-k}(u) = \sqrt{2}\sin(2\pi k u)$, $k \in \mathbb{N}$, then one can define the distance d on $\mathcal{M}_+$ as

$$d(\vartheta_1,\vartheta_2) \;:=\; \sum_{k\in\mathbb{Z}}^{\infty} \frac{1}{2^k}|\langle\vartheta_1, e_k\rangle - \langle\vartheta_2, e_k\rangle|\ .$$

Note that $\mathcal{M}_+$ is compact under the weak topology.

Denote by $C^m(\mathbb{T})$, m in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions on $\mathbb{T}$ which are m times differentiable and whose m-th derivative is continuous. Given a function G in $C^2(\mathbb{T})$, we shall denote by ∇G and ΔG the first and second derivative of G, respectively.

Let $\nu_\rho = \nu_\rho^N$, $0 \leq \rho \leq 1$, be the Bernoulli product measure with the density ρ. Define the continuous functions $B, D : [0,1] \to \mathbb{R}$ by

$$B(\rho) \;=\; \int [1-\eta(0)]\, c(\eta)\, d\nu_\rho\ , \quad D(\rho) \;=\; \int \eta(0)\, c(\eta)\, d\nu_\rho\ .$$

Let S be the set of all classical solutions of the semilinear elliptic equation:

$$(1/2)\Delta\rho + F(\rho) \;=\; 0 \ \text{ on } \ \mathbb{T}\ , \tag{2.1}$$

where $F(\rho) = B(\rho) - D(\rho)$. Classical solution means a function $\rho : \mathbb{T} \to [0,1]$ in $C^2(\mathbb{T})$ which satisfies the equation (2.1) for any $u \in \mathbb{T}$. We also define by $\mathcal{M}_{sol}$ the set of all absolutely continuous measures whose density is a classical solution of (2.1):

$$\mathcal{M}_{sol} \ := \ \{\bar{\vartheta} \in \mathcal{M}_+ : \bar{\vartheta}(du) = \bar{\rho}(u)du, \ \bar{\rho} \in S\} \ .$$

Let $\pi^N : X_N \to \mathcal{M}_+$ be the function which associates to a configuration η the positive measure obtained by assigning mass N^{-1} to each particle of η,

$$\pi^N(\eta) \ = \ \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \delta_{x/N} \ ,$$

where δ_u stands for the Dirac measure which has a point mass at $u \in \mathbb{T}$.

Since the jump rate $c(\eta)$ is strictly positive, the Markov process η_t^N, $t \geq 0$, is irreducible. Therefore there exists a unique stationary probability measure under the dynamics. We denote it by μ_N. We also introduce the probability measure on $\mathcal{M}_+$ defined by $\mathcal{P}_N := \mu_N \circ (\pi^N)^{-1}$.

The following theorem has been established in [**36**].

THEOREM 2.1. *The sequence of measures $\{\mathcal{P}^N : N \geq 1\}$ is asymptotically concentrated on the set $\mathcal{M}_{sol}$. Namely, for any $\delta > 0$, we have*

$$\lim_{N \to \infty} \mathcal{P}^N(\vartheta \in \mathcal{M}_+ : \inf_{\bar{\vartheta} \in \mathcal{M}_{sol}} d(\vartheta, \bar{\vartheta}) \geq \delta) \ = \ 0 \ .$$

2.3. Dynamical and static large deviations. We state in this subsection the dynamical and static large deviation principles. Theorem 2.2 is the main result of this chapter.

Let $\mathcal{M}_{+,1}$ be the closed subset of $\mathcal{M}_+$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_{+,1} \ = \ \{\pi \in \mathcal{M}_+(\mathbb{T}) : \pi(du) = \rho(u)du, \ 0 \leq \rho(u) \leq 1 \ a.e. \ u \in \mathbb{T}\} \ .$$

Fix $T > 0$, and denote by $C^{m,n}([0,T] \times \mathbb{T})$, m, n in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions defined on $[0,T] \times \mathbb{T}$ which are m times differentiable in the first variable and n times on the second one, and whose derivatives are continuous. Let $Q_\eta = Q_\eta^N$, $\eta \in X_N$, be the probability measure on $D([0,T], \mathcal{M}_+)$ induced by the measure-valued process π_t^N starting from $\pi^N(\eta)$.

For each $p \geq 1$, let $L^p(\mathbb{T})$ be the space of all real p-th integrable functions $G : \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_{\mathbb{T}} |G(u)|^p du < \infty$. The corresponding norm is denoted by $\| \cdot \|_p$:

$$\|G\|_p^p \ := \ \int_{\mathbb{T}} |G(u)|^p du \ .$$

In particular, $L^2(\mathbb{T})$ is a Hilbert space equipped with the inner product

$$\langle G, H \rangle \ = \ \int_{\mathbb{T}} G(u)H(u)du \ .$$

For a function G in $L^2(\mathbb{T})$, we also denote by $\langle G\rangle$ the integral of G with respect to the Lebesgue measure: $\langle G\rangle := \int_{\mathbb{T}} G(u)du$.

For each path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$, define the energy $\mathcal{Q}_T$ as
(2.2)

$$\mathcal{Q}_T(\pi) \;=\; \sup_{G\in C^{0,1}([0,T]\times\mathbb{T})} \Big\{ 2\int_0^T dt\, \langle \rho_t, \nabla G_t\rangle - \int_0^T dt \int_{\mathbb{T}^d} du\, G^2(t,u)\Big\}\ .$$

It is known that the energy $\mathcal{Q}_T(\pi)$ is finite if and only if ρ has a generalized derivative and this generalized derivative is square integrable on $[0,T]\times\mathbb{T}$:

$$\int_0^T dt \int_{\mathbb{T}} du\, |\nabla\rho(t,u)|^2 < \infty\ .$$

Moreover, it is easy to see that the energy $\mathcal{Q}_T$ is convex and lower semicontinuous.

For each function G in $C^{1,2}([0,T]\times\mathbb{T})$, define the functional $\bar{J}_G : D([0,T],\mathcal{M}_{+,1}) \to \mathbb{R}$ by

$$\begin{aligned}\bar{J}_G(\pi) \;&=\; \langle \pi_T, G_T\rangle - \langle \pi_0, G_0\rangle - \int_0^T dt\, \langle \pi_t, \partial_t G_t + \frac{1}{2}\Delta G_t\rangle \\ &\quad - \frac{1}{2}\int_0^T dt\, \langle \chi(\rho_t), (\nabla G_t)^2\rangle - \int_0^T dt\, \big\{ \langle B(\rho_t), e^{G_t}-1\rangle + \langle D(\rho_t)\ , e^{-G_t}-1\rangle\big\},\end{aligned}$$

where $\chi(r) = r(1-r)$ is the mobility. Let $J_G : D([0,T],\mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$J_G(\pi) \;=\; \begin{cases} \bar{J}_G(\pi) & \text{if } \pi\in D([0,T],\mathcal{M}_{+,1})\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

We define the functional $I_T : D([0,T],\mathcal{M}_+) \to [0,\infty]$ as

$$I_T(\pi) \;=\; \begin{cases} \sup J_G(\pi) & \text{if } \mathcal{Q}_T(\pi) < \infty\ , \\ \infty & \text{otherwise}\ , \end{cases} \tag{2.3}$$

where the supremum is taken over all functions G in $C^{1,2}([0,T]\times\mathbb{T})$.

For a measurable function $\gamma : \mathbb{T} \to [0,1]$, we define the dynamical large deviation function $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}_+) \to [0,\infty]$ as

$$I_T(\pi|\gamma) \;=\; \begin{cases} I_T(\pi) & \text{if } \pi(0,du) = \gamma(u)du\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

In [**36**], it has been established that the measure-valued process $\pi^N_\cdot$ satisfies a dynamical large deviation principle with the rate function $I_T(\cdot|\gamma)$ under the assumption that the functions B and D are concave on $[0,1]$. The precise statement will be introduced in Subsection 4.1.

We now define the static large deviation rate functional which is defined similar to [**21**][Chapter 6] in spirit.

Assume that there exist density profiles $\bar{\rho}_1, \cdots, \bar{\rho}_l$, $l > 1$, such that any classical solution $\bar{\rho} : \mathbb{T} \to [0,1]$ of the equation (2.1) can be given by $\bar{\rho}(u) = \bar{\rho}_i(u - u_0)$ for some $1 \le i \le l$ and $u_0 \in \mathbb{T}$. In other words, it is equivalent to the following:

$$\mathcal{M}_{sol} \ = \ \{\bar{\rho}_i(\cdot - v)du : 1 \le i \le l, v \in \mathbb{T}\} \ .$$

For each $1 \le i \le l$, let $\mathcal{M}_i$ be the subset of $\mathcal{M}_{sol}$ given by $\mathcal{M}_i = \{\bar{\rho}_i(\cdot - v)du : v \in \mathbb{T}\}$.

For each $1 \le i \le l$, we define the functional $V_i : \mathcal{M}_+ \to [0,\infty]$ by

$$V_i(\vartheta) = \inf\{I_T(\pi|\bar{\rho}) : T > 0, \bar{\rho}(u)du \in \mathcal{M}_i, \pi \in D([0,T], \mathcal{M}_+) \text{ and } \pi_T = \vartheta\} \ ,$$

which is the minimal cost that creates the measure ϑ from the set $\mathcal{M}_i$. For each $1 \le i, j \le l$ with $i \ne j$, let $\bar{\vartheta}_i(du) = \bar{\rho}_i(u)du$ and $v_{ij} = V_i(\bar{\vartheta}_j)$.

To define the static large deviation rate function, we need to recall some notation introduced in [**21**][Chapter 6]. Let L be a finite set and let W be a subset of L. A graph consisting of arrows $m \to n$ ($m \in L\backslash W$, $n \in L$, $n \ne m$) is called a W-graph if it satisfies the following conditions:

i) Every point $m \in L\backslash W$ is the initial point of exactly one arrow,
ii) There are not closed cycles in the graph.

We denote by $G(W)$ the set of all W-graphs. If a graph W is given by the singleton-set $\{i\}$, then we simply denote $G(\{i\})$ by $G(i)$. We regard $L := \{1, \cdots, l\}$ as a graph with weights v_{ij} and, for each $1 \le i \le l$, consider the number

$$w_i = \min_{g \in G(i)} \sum_{(m \to n) \in g} v_{mn} \ ,$$

where the product is taken over all arrows in $G(i)$.

Let $w = \min_{1 \le i \le l} \{w_i\}$. For each $1 \le i \le l$, we define the functions $W_i, W : \mathcal{M}_+ \to [0,\infty]$ by

$$\begin{aligned} W_i(\vartheta) \ &= \ w_i - w + V_i(\vartheta) \ , \\ W(\vartheta) \ &= \ \min_{1 \le k \le l} W_i(\vartheta) \ . \end{aligned}$$

The following theorem is main result of this chapter.

THEOREM 2.2. *Assume that the functions B and D are concave on $[0,1]$. The sequence of measures $\{\mathcal{P}_N; N \ge 1\}$ satisfies a large deviation principle on $\mathcal{M}_+$ with speed N and the rate function W. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}_+$ and each open set $\mathcal{O} \subset \mathcal{M}_+$,*

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log \mathcal{P}_N(\mathcal{C}) \ \le \ -\inf_{\vartheta \in \mathcal{C}} W(\vartheta) \ ,$$

$$\underline{\lim_{N \to \infty}} \frac{1}{N} \log \mathcal{P}_N(\mathcal{O}) \ \ge \ -\inf_{\vartheta \in \mathcal{O}} W(\vartheta) \ .$$

Moreover, the rate functional W is bounded on $\mathcal{M}_{+,1}$, lower semicontinuous and has compact level sets.

3. The hydrodynamic equation

Fix an initial profile $\gamma : \mathbb{T} \to [0,1]$. We discuss in this section several properties of the weak solutions of the Cauchy problem

$$\begin{cases} \partial_t \rho \;=\; (1/2)\Delta\rho + F(\rho) \;\text{ on }\; \mathbb{T}\,, \\ \rho(0,\cdot) \;=\; \gamma(\cdot)\,, \end{cases} \tag{3.1}$$

where $F(\rho) = B(\rho) - D(\rho)$.

3.1. The hydrodynamic limit. We review in this subsection several results on the Cauchy problem (3.1). We first define two concepts of solutions of the Cauchy problem (3.1).

DEFINITION 3.1. *A measurable function* $\rho : [0,T] \times \mathbb{T} \to [0,1]$ *is said to be a weak solution of the Cauchy problem* (3.1) *in the layer* $[0,T] \times \mathbb{T}$ *if, for every function* G *in* $C^{1,2}([0,T] \times \mathbb{T})$*,*

$$\begin{aligned} &\langle \rho_T, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T dt \langle \rho_t, \partial_t G_t\rangle \\ &\qquad = \frac{1}{2}\int_0^T dt\langle \rho_t, \Delta G_t\rangle + \int_0^T dt \langle F(\rho_t), G_t\rangle\,. \end{aligned} \tag{3.2}$$

For each $t \geq 0$, let P_t be the semigroup on $L^2(\mathbb{T})$ generated by $(1/2)\Delta$.

DEFINITION 3.2. *A measurable function* $\rho : [0,T] \times \mathbb{T} \to [0,1]$ *is said to be a mild solution of the Cauchy problem* (3.1) *in the layer* $[0,T] \times \mathbb{T}$ *if, for any t in* $[0,T]$*, it holds that*

$$\rho_t \;=\; P_t\gamma + \int_0^t P_{t-s}F(\rho_s)ds\,. \tag{3.3}$$

The following proposition asserts that two notion of solutions are equivalent.

PROPOSITION 3.3. *Definitions 3.1 and 3.2 are equivalent. Moreover, there exists a unique weak solution of the Cauchy problem* (3.1).

The following three propositions claim fundamental properties of solutions of the Cauchy problem (3.1).

PROPOSITION 3.4. *Let* ρ *be the unique weak solution of the Cauchy problem* (3.1)*. Then* ρ *is infinitely differentiable over* $(0,\infty) \times \mathbb{T}$.

PROPOSITION 3.5. *Let* $\rho : [0,\infty) \times \mathbb{T} \to [0,1]$ *be the unique weak solution of the Cauchy problem* (3.1)*. Then there exists a density profile* ρ_∞ *in* S *such that* ρ_t *converges to* ρ_∞ *as* $t \to \infty$ *in* $C^2(\mathbb{T})$.

See [**36**] and its references for the proof of Propositions 3.3, 3.4 and 3.5.

3.2. Properties of weak solutions of (3.1)**.** We study in this subsection some properties of weak solutions of the Cauchy problem (3.1). Lemmas 3.6 and 3.7 are quite simple but these lemmas play important roles in the proof of the static large deviation principle.

LEMMA 3.6. *There exists a constant $C > 0$ such that for any weak solutions ρ^j, $j = 1, 2$, of the Cauchy problem* (3.1) *with initial profile ρ_0^j and any $t > 0$, we have*

$$\|\rho_t^1 - \rho_t^2\|_2 \;\le\; e^{Ct}\|\rho_0^1 - \rho_0^2\|_2 \, .$$

PROOF. From (3.3), for any $t \ge 0$ and $j = 1, 2$, we have

$$\rho_t^j \;=\; P_t\rho_0^j + \int_0^t P_{t-s}F(\rho_s^j)ds \, .$$

Therefore

$$\begin{aligned}\|\rho_t^1 - \rho_t^2\|_2 \;&\le\; \|P_t(\rho_0^1 - \rho_0^2)\|_2 + \int_0^t \|P_{t-s}(F(\rho_s^1) - F(\rho_s^2))\|_2 ds \\ &\le\; \|\rho_0^1 - \rho_0^2\|_2 + \|F'\|_\infty \int_0^t \|\rho_s^1 - \rho_s^2\|_2 ds \, .\end{aligned}$$

In the last inequality, we use the fact that the operator norm of P_t is equal to 1. Hence, the Gronwall's inequality concludes the lemma. □

LEMMA 3.7. *Let $\rho_0 : \mathbb{T} \to [0, 1]$ be an initial profile and ρ_t be the unique weak solution of the Cauchy problem (3.1) with initial condition ρ_0 and $\bar\rho : \mathbb{T} \to [0, 1]$ be a classical solution to the equation* (2.1). *Set $\bar\vartheta(du) = \bar\rho(u)du$. Then, for any $\delta_1 > 0$ small enough, there exist $\delta > 0$ and $T = T_{\delta_1} > 0$ such that for any $\rho_0(u)du$ in $\mathcal{B}_\delta(\bar\vartheta)$, it holds that $\pi_t \in \mathcal{B}_{\delta_1/2}(\bar\vartheta)$ for any $0 \le t \le 1$ and $\|\rho_T - \bar\rho\|_2 \le \delta_1$.*

PROOF. The first claim of the lemma easily follows from the definition of weak solutions of the Cauchy problem (3.2). We now prove the second claim of the lemma.

Fix $\delta_1 > 0$ and ρ_0 in $\mathcal{B}_\delta(\bar\vartheta)$. δ will be chosen later. From the equation (3.2), we have

$$\begin{aligned}\|\rho_t - \bar\rho\|_2 \;&\le\; \|P_t(\rho_0 - \bar\rho)\|_2 + \int_0^T \|P_{t-s}(F(\rho_s) - F(\bar\rho))\|_2 ds \\ &\le\; \|P_t(\rho_0 - \bar\rho)\|_2 + t\|F'\|_\infty \, , \qquad (3.4)\end{aligned}$$

since the operator norm of P_t is equal to 1. Let $\tilde\rho = \rho_0 - \bar\rho$ and, for each $t > 0$, $\tilde\rho_t = P_t\tilde\rho$. It is easy to see that, for any $k \in \mathbb{Z}$,

$$\langle\tilde\rho_t, e_k\rangle \;=\; \langle\tilde\rho, P_t e_k\rangle \;=\; e^{-2\pi^2k^2t}\langle\tilde\rho, e_k\rangle \, .$$

Therefore, from Parseval's relation, we have

$$\|\tilde\rho_t\|_2^2 \;=\; \sum_{k\in\mathbb{Z}} e^{-4\pi^4k^4t^2}\langle\tilde\rho, e_k\rangle^2 \, . \qquad (3.5)$$

We now set $T = T_{\delta_1} := (\delta_1/2\|F'\|_\infty)$ and choose large $k_{\delta_1} > 0$ so that

$$\sum_{|k|>|k_{\delta_1}|} e^{-4\pi^4 k^4 T^2} \;\le\; \delta_1^2/8 \; .$$

Moreover we define $\delta = (8\sum_{|k|\le k_{\delta_1}} 4^k)^{-1/2}\delta_1$. The conclusion of the lemma follows from these choices, (3.4) and (3.5). □

REMARK 3.8. *It is clear that* $T = T_{\delta_1}$ *appeared in the proof of the previous lemma can be taken to be less than or equal to* 1.

4. The rate functions

We study in this section the dynamical and static rate function.

4.1. The functional I_T. We study several properties of the dynamical rate function. For some details see [**36**]. We start from introducing the dynamical large deviation principle which has been established in [**36**].

THEOREM 4.1. *Assume that the functions B and D are concave on* $[0,1]$. *Fix $T > 0$ and a measurable function $\gamma : \mathbb{T} \to [0,1]$. Assume that a sequence η^N of initial configurations in X_N is associated to γ, in the sense that*

$$\lim_{N\to\infty} \langle \pi^N(\eta^N), G\rangle \;=\; \int_{\mathbb{T}} G(u)\gamma(u)du$$

for every continuous function $G : \mathbb{T} \to \mathbb{R}$. Then, the measure Q_{η^N} on $D([0,T],\mathcal{M}_+)$ satisfies a large deviation principle with the rate function $I_T(\cdot|\gamma)$. That is, for each closed subset $\mathcal{C} \subset D([0,T],\mathcal{M}_+)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{C}) \;\le\; -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma) \; ,$$

and for each open subset $\mathcal{O} \subset D([0,T],\mathcal{M}_+)$,

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{O}) \;\ge\; -\inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma) \; .$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

It is easy to see that the similar computation performed in [**36**], Section 4, gives the similar results for the functional I_T. For the sake of completeness, we review some of them with the functional I_T in the place of $I_T(\cdot|\gamma)$.

The following lemma is proved in [**36**].

LEMMA 4.2. *The density ρ of a path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem* (3.1) *with initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ is equal to* 0. *Moreover, in that case*

$$\int_0^T dt \int_{\mathbb{T}} du \, \frac{|\nabla\rho(t,u)|^2}{\chi(\rho(t,u))} \;<\; \infty.$$

The computation performed in the proof of Theorem 4.7 in [**36**] gives the following lemma.

LEMMA 4.3. *Let* $\{\pi^n(t,du) = \rho^n(t,u)du : n \ge 1\}$ *be a sequence of trajectories in* $D([0,T],\mathcal{M}_{+,1})$ *such that, for some positive constant* C,

$$\sup_{n\ge 1}\{I_T(\pi^n)\} \;\le\; C.$$

If ρ^n *converges to* ρ *weakly in* $L^2(\mathbb{T}\times[0,T])$, *then* ρ^n *converges to* ρ *strongly in* $L^2(\mathbb{T}\times[0,T])$.

For each $\delta >$ and a function ρ in $L^2(\mathbb{T})$, let $\mathbb{B}_\delta(\rho)$ be the δ-open neighborhood of ρ in $L^2(\mathbb{T})$. For each $\delta >$ and a measure ϑ in $\mathcal{M}_+$, let $\mathcal{B}_\delta(\vartheta)$ be the δ-open neighborhood of ϑ in $\mathcal{M}_+$.

For each $\delta > 0$ and each $T > 0$ denote by $\mathbb{D}_{T,\delta}$ the set of trajectories $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_+)$ such that $\rho_t \notin \mathbb{B}_\delta(\bar\rho)$ for all $0 \le t \le T$ and $\bar\rho \in S$. For each $\delta > 0$ and each $T > 0$ denote by $\mathcal{D}_{T,\delta}$ the set of trajectories $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$ such that $\pi_t \notin \mathcal{B}_\delta(\bar\vartheta)$ for all $0 \le t \le T$ and $\bar\vartheta \in \mathcal{M}_{sol}$.

LEMMA 4.4. *For every* $\delta > 0$ *there exists* $T = T(\delta) > 0$ *such that*

$$\inf_{\pi\in\mathbb{D}_{T,\delta}} I_T(\pi) \;>\; 0\,.$$

PROOF. Assume that the conclusion of the lemma is not true. Then there exists some $\delta > 0$ such that, for any $n \in \mathbb{N}$,

$$\inf_{\pi\in\mathbb{D}_{n,\delta}} I_n(\pi) \;=\; 0\,.$$

In this case there exists a sequence of trajectories $\{\pi^n(t,du) = \rho^n(t,u)du : n \ge 1\}$ such that $I_n(\pi^n) \le 1/n$. Since $I_T(\pi)$ has compact level sets, by using a Cantor's diagonal argument and passing to a subsequence if necessary, there exists a path $\pi(t,du) = \rho(t,u)du$ in $D(\mathbb{R}_+,\mathcal{M}_{+,1})$ such that π^n converges to π in $D([0,T],\mathcal{M}_+)$ for any $T > 0$. Moreover, by Lemma 4.3, ρ^n converges to ρ strongly in $L^2([0,T]\times\mathbb{T})$ for any $T > 0$.

From Proposition 3.5, ρ_t converges to some density profile ρ_∞ in S. Therefore there exists some $T > 0$ such that

$$\|\rho_t - \rho_\infty\|_2 \;\le\; \delta/2\,,$$

for any $t \ge T$. Hence

$$\begin{aligned}\int_0^{T+1} \|\rho_t^n - \rho_t\|_2 dt \;&\ge\; \int_T^{T+1} \|\rho_t^n - \rho_t\|_2 dt \\ &\ge\; \int_T^{T+1} \|\rho_t^n - \rho_\infty\|_2 - \|\rho_t - \rho_\infty\|_2 dt \\ &\ge\; \delta - \delta/2 = \delta/2\,,\end{aligned}$$

which contradicts the strong convergence of ρ^n to ρ in $L^2([0,T+1]\times\mathbb{T})$ and we are done. $\square$

From Lemma 4.4, we can obtain similar result for the set $\mathcal{D}_{T,\delta}$.

LEMMA 4.5. *For every $\delta > 0$ there exists $T > 0$ such that*

$$\inf_{\pi \in \mathcal{D}_{T,\delta}} I_T(\pi) > 0.$$

PROOF. The conclusion follows from Lemma 4.4 and the fact that

$$\{\vartheta(du) = \rho(u)du : \rho \in \mathbb{B}_\delta(\bar\rho)\} \subset \mathcal{B}_\delta(\bar\rho),$$

for every $\bar\rho \in S$ and every $\delta > 0$. □

LEMMA 4.6. *There exists a constant $C > 0$ such that for any $T > 0$, any weak solution ρ of* (3.1) *and any classical solution $\bar\rho$ to the equation* (2.1), *we have*

$$I_T(\pi) \ \le\ C\{T + \|\rho_T - \bar\rho\|_1 + \|\rho_0 - \bar\rho\|_1\}\ ,$$

where π is the trajectory defined by $\pi(t, du) = \rho(T-t, u)du$.

PROOF. For any test function $G \in C^{1,2}([0,T]\times\mathbb{T})$, $J_G(\pi)$ can be rewritten as

$$\begin{aligned} J_G(\pi) \ =\ & \int_0^T dt \int_{\mathbb{T}} du\, \nabla\rho_t \cdot \nabla\widehat{G}_t - \frac{1}{2}\int_0^T dt \int_{\mathbb{T}} du\, \chi(\rho_t)(\nabla\widehat{G}_t)^2 \\ & - \int_0^T dt \int_{\mathbb{T}} du\, B(\rho_t)[\mathrm{e}^{\widehat{G}_t} + \widehat{G}_t - 1] - \int_0^T dt \int_{\mathbb{T}} du\, D(\rho_t)[\mathrm{e}^{-\widehat{G}_t} - \widehat{G}_t - 1]\ . \end{aligned}$$

where $\widehat{G}(t,u) = G(T-t,u)$. The first line is bounded above by

$$\frac{1}{2}\mathcal{E}_T(\rho) \ =\ \int_0^T dt \int_{\mathbb{T}} du\, \frac{1}{2}\frac{|\nabla\rho(t,u)|^2}{\chi(\rho(t,u))}\ .$$

Since for any $0 < r < 1$ and any $s \in \mathbb{R}$

$$\begin{aligned} -B(r)\mathrm{e}^s + D(r)s + D(r) \ &\le\ D(r)\log(D(r)/B(r))\ , \\ -D(r)\mathrm{e}^{-s} - B(r)s + B(r) \ &\le\ B(r)\log(B(r)/D(r))\ , \end{aligned}$$

therefore the second line is bounded above by

$$\begin{aligned} & \int_0^T dt \int_{\mathbb{T}} du\, D(\rho_t)\log(D(\rho_t)/B(\rho_t)) + B(\rho_t)\log(B(\rho_t)/D(\rho_t)) \\ & \qquad \le\ CT - \int_0^T dt \int_{\mathbb{T}} du\, [D(\rho_t)\log(1-\rho_t) + B(\rho_t)\log(\rho_t)] \\ & \qquad \le\ -\frac{1}{2}\mathcal{E}_T(\rho) + C\{T + \|\rho_T - \bar\rho_i\|_1 + \|\rho_0 - \bar\rho_i\|_1\}\ . \end{aligned}$$

The last inequality comes from the equation

$$\begin{aligned} \partial_t[\rho_t\log\rho_t + (1-\rho_t)\log(1-\rho_t)] \ &=\ [\log(\rho_t) - \log(1-\rho_t)]\partial_t\rho_t \\ &=\ [\log(\rho_t) - \log(1-\rho_t)][\frac{1}{2}\Delta\rho_t + F(\rho_t)]\ . \end{aligned}$$

This equation is justified since, from Proposition 3.4, any weak solution of the equation (3.1) is smooth. Therefore

$$
\begin{aligned}
-\int_0^T dt \int_{\mathbb{T}} du\, [D(\rho_t)\log(1-\rho_t) + B(\rho_t)\log(\rho_t)] \\
= -\frac{1}{2}\mathcal{E}_T(\rho) - \int_0^T dt \int_{\mathbb{T}} du\, [D(\rho_t)\log(\rho_t) + B(\rho_t)\log(1-\rho_t)] \\
- \rho_T \log \rho_T - (1-\rho_T)\log(1-\rho_T) + \rho_0 \log \rho_0 + (1-\rho_0)\log(1-\rho_0) \ .
\end{aligned}
$$

It is easy to see that the last four terms can be bounded above by

$$
C\{\|\rho_T - \bar{\rho}\|_1 + \|\rho_0 - \bar{\rho}\|_1\} \ ,
$$

for some $C > 0$, which finishes the proof of the lemma. □

4.2. The static rate functional W. In this subsection we study some properties of the quasi potential W. Throughout the remaining chapter, let $\bar{\vartheta}_i(du) := \bar{\rho}_i(u)du$ for each $1 \le i \le l$. The first main result states that W is continuous at $\bar{\vartheta}_i$ in the $L^2(\mathbb{T})$ topology. The second one states that W is lower semicontinuous.

We start with an estimate on V_i which is the main ingredient in the proof of the former. Let $\mathbb{D}$ be the space of measurable functions on $\mathbb{T}$ bounded below by 0 and bounded above by 1 endowed with the $L^2(\mathbb{T})$ topology:

$$
\mathbb{D} \ = \ \{\rho : \mathbb{T} \to [0,1] \ : \ 0 \le \rho(u) \le 1 \ a.e.\} \ .
$$

For each $1 \le i \le l$. let $\mathbb{V}_i : \mathbb{D} \to [0, +\infty]$ be the functional given by $\mathbb{V}_i(\rho) = V_i(\rho(u)du)$. For each $h > 0$ and each $\delta > 0$, let $\mathbb{D}_\delta^h$ be the subset of $\mathbb{D}$ consisting of those profiles ρ satisfying the following conditions:

i) $\rho \in H^1(\mathbb{T})$.
ii) $\int_{\mathbb{T}} (\nabla \rho(u))^2 du \le h$.
iii) $\delta \le \rho(u) \le 1 - \delta$ a.e. in $\mathbb{T}$.

LEMMA 4.7. *For each $1 \le i \le l$, any $h > 0$, any $\delta > 0$ and any increasing C^1-diffeomorphism $\alpha : [0,1] \to [0,1]$, there exist constants $C_1 = C_1(\delta, h) > 0$ and $C_2 = C_2(\delta, h, \alpha) > 0$ such that*

$$
\mathbb{V}_i(\rho) \ \le \ C_1 \int_0^1 \alpha^2(t) dt + C_2 \|\rho - \bar{\rho}_i\|_1 \ ,
$$

for any ρ in $\mathbb{D}_\delta^h$.

PROOF. Fix $h > 0$ and $\delta > 0$. Let $\rho \in \mathbb{D}_\delta^h$ and let $\alpha : [0,1] \to [0,1]$ be an increasing $\mathcal{C}^1$-diffeomorphism. Consider the path $\pi_t^\alpha(du) = \rho^\alpha(t,u)du$ in $C([0,1], \mathcal{M}_+)$ with density given by $\rho_t^\alpha = (1-\alpha(t))\bar{\rho} + \alpha(t)\rho$. It is clear that π^α belongs to $D([0,1], \mathcal{M}_{+,1})$ and, from the condition i), $\mathcal{Q}_1(\pi^\alpha)$ is finite. From the definition of ρ^α it is easy to see that $\nabla \rho_t^\alpha = \alpha(t)(\nabla\rho - \nabla\bar{\rho}_i) + \nabla\bar{\rho}_i$ and $\partial_t \rho_t^\alpha = \alpha'(t)(\rho - \bar{\rho}_i)$. Since $\bar{\rho}_i$ solves the equation (2.1),

$J_G(\pi^\alpha)$ can be rewritten as

$$
\begin{aligned}
J_G(\pi) &= \frac{1}{2}\int_0^T dt\, \{\langle \alpha(t)(\nabla\rho - \nabla\bar{\rho}_i), \nabla G_t\rangle - \langle \chi(\rho_t^\alpha), (\nabla G_t)^2\rangle\} \\
&\quad + \int_0^T dt\, \langle \{\alpha'(t)(\rho - \bar{\rho}_i) + F(\bar{\rho}_i)\}G_t - B(\rho_t^\alpha)(e^{G_t} - 1) - D(\rho_t^\alpha)(e^{-G_t} - 1)\rangle\ . \qquad (4.1)
\end{aligned}
$$

From the similar argument performed in Lemma 4.7 of [**22**], the first term on the right hand side of (4.1) is bounded by

$$C_1 \int_0^1 \alpha^2(t)\, dt\ ,$$

for some constant $C_1 = C_1(\delta, h)$. To conclude the proof it is enough to prove that the second term on the right hand side of (4.1) is bounded by

$$C_2\|\rho - \bar{\rho}_i\|_1$$

for some constant $C_2 = C_2(\delta, h, \alpha)$.

Consider the function $\Phi : \mathbb{R} \times (0,1) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(H, \rho, G) = HG - B(\rho)(e^G - 1) - D(\rho)(e^{-G} - 1)\ .$$

If we set $H_t = \alpha'(t)(\rho - \bar{\rho}_i) + F(\bar{\rho}_i)$, it is clear that the second term on the right hand side of (4.1) can be expressed as

$$\int_0^T \langle \Phi(H_t, \rho_t^\alpha, G_t)\rangle\, dt\ .$$

From the straightforward computation, for any fixed H in $\mathbb{R}$ and ρ in $(0,1)$, the function $\Phi(H, \rho, \cdot)$ reaches a maximum at

$$G(H, \rho) = \log\left(\frac{H + \sqrt{H^2 + 4B(\rho)D(\rho)}}{2B(\rho)}\right)\ .$$

From the condition $iii)$, there exits a constant $c_\delta > 0$ such that $c_\delta \leq \rho^\alpha \leq 1 - c_\delta$. Note that $\Phi(F(\rho), \rho, G(F(\rho), \rho)) = 0$ for any ρ and $\Phi(H, \rho, G(H, \rho))$ is Lipschitz on $[-\int_0^1 \alpha'(t)dt - \|F\|_\infty, \int_0^1 \alpha'(t)dt + \|F\|_\infty] \times [c_\delta, 1 - c_\delta]$. Therefore

$$
\begin{aligned}
\Phi(H_t, \rho_t^\alpha, G_t) &\leq \Phi(H_t, \rho_t^\alpha, G(H_t, \rho_t^\alpha)) \\
&= \Phi(H_t, \rho_t^\alpha, G(H_t, \rho_t^\alpha)) - \Phi(F(\rho_t^\alpha), \rho_t^\alpha, G(F(\rho_t^\alpha), \rho_t^\alpha)) \\
&\leq C_2|H_t - F(\rho_t^\alpha)| \\
&\leq C_2\alpha'(t)|\rho - \bar{\rho}_i| + C_2\|F'\|_\infty \alpha(t)|\rho - \bar{\rho}_i|\ ,
\end{aligned}
$$

for some constant $C_2 = C_2(\delta, h, \alpha)$. These bounds give the desired conclusion. □

The ideas of Lemma 3.7 and 4.7 give the following corollary.

COROLLARY 4.8. *Fix $1 \leq i \leq l$. For any $\varepsilon > 0$, there exists $\delta_1 > 0$ small enough such that for any $\vartheta(du) = \gamma(u)du$ in $\mathcal{B}_{\delta_1}(\bar{\vartheta}_i)$ there exists a path $\pi(t, du) = \rho(t, u)du$ in $D([0, 1], \mathcal{M}_+)$ such that $\pi_0 = \vartheta$, $\pi_1 = \bar{\vartheta}_i$ and $I_1(\pi) \leq \varepsilon$.*

The following results are proved by using Lemmas 3.6, 4.6, 4.7 and by the arguments performed in [**22**]. Therefore the proofs are omitted.

THEOREM 4.9. *For each $1 \leq i \leq l$, the function $\mathbb{V}_i$ is continuous at $\bar{\rho}_i$ in the $L^2(\mathbb{T})$ norm.*

PROPOSITION 4.10. *The function W is finite if and only if ϑ belongs to $\mathcal{M}_{+,1}$. Moreover,*

$$\sup_{\vartheta \in \mathcal{M}_{+,1}} W(\vartheta) \ < \ \infty \ .$$

THEOREM 4.11. *The rate function W is lower semicontinuous.*

5. Large deviations

We show in this section that the sequence of the stationary measures satisfies the large deviation principle with the rate function W. Recall from Subsection 2.3 that $\bar{\rho}_i$, $1 \leq i \leq l$, is a weak solution of the equation (2.1) and $\bar{\vartheta}_i$, the measure in $\mathcal{M}_{+,1}$ with density $\bar{\rho}_i$, i.e., $\bar{\vartheta}_i(du) = \bar{\rho}_i du$.

5.1. Preliminaries. We study the asymptotic behavior of the stationary measure of some process induced from the original Markov process η^N. The main result of this subsection, Lemma 5.5, plays an important role of the proof of Theorem 2.2.

We start from introducing some notation. For any $\delta_1 > \delta > 0$ small enough, consider the sets defined as follows:

$$B \ = \ \bigcup_{i=1}^{l} B_i \ , \qquad \text{with} \qquad B_i \ = \ \{\vartheta \in \mathcal{M}_+ : \inf_{\bar{\vartheta} \in \mathcal{M}_i} d(\vartheta, \bar{\vartheta}) \leq \delta\} \ .$$

$$\Gamma \ = \ \bigcup_{i=1}^{l} \Gamma_i \ , \qquad \text{with} \qquad \Gamma_i \ = \ \{\vartheta \in \mathcal{M}_+ : \delta_1 \leq \inf_{\bar{\vartheta} \in \mathcal{M}_i} d(\vartheta, \bar{\vartheta}) \leq 2\delta_1\} \ .$$

For each integer $N > 0$ and each subset A of $\mathcal{M}_+$, let $A^N = (\pi^N)^{-1}(A)$ and let $H_A^N : D(\mathbb{R}_+, X_N) \to [0, +\infty]$ be the entry time in A^N:

$$H_A^N \ = \ \inf \left\{ t \geq 0 : \eta_t \in A^N \right\} \ .$$

LEMMA 5.1. *For every $\delta > 0$, there exist $T_0, C_0, N_0 > 0$, which depends on $\delta > 0$, such that*

$$\sup_{\eta \in X_N} \left\{ \mathbb{P}_\eta \left[H_B^N \geq kT_0 \right] \right\} \ \leq \ \exp \left\{ -kC_0 N^d \right\} \ ,$$

for any integers $N > N_0$ and $k > 0$.

PROOF. Fix $\delta > 0$. By Lemma 4.5, there exists $T_0 > 0$ and $C_0 > 0$, which depends on $\delta > 0$, such that

$$\inf_{\pi \in \mathcal{D}} I_{T_0}(\pi) \ > \ C_0 \ ,$$

where $\mathcal{D} = D([0,T_0], \mathcal{M}_+ \backslash B)$. For each integer $N > 0$, consider a configuration η^N in X_N such that

$$\mathbb{P}_{\eta^N}\left[H_B^N \geq T_0\right] \ = \ \sup_{\eta \in X_N} \left\{\mathbb{P}_{\eta}\left[H_B^N \geq T_0\right]\right\} \ .$$

By the compactness of $\mathcal{M}_+$, every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging to some ϑ in $\mathcal{M}_+$. Moreover, since each configuration in X_N has at most one particle per site, ϑ belongs to $\mathcal{M}_{+,1}$. From this and since $\mathcal{D}$ is a closed subset of $D([0,T_0], \mathcal{M}_+)$, by the dynamical large deviations lower bound, there exists a measure $\vartheta(du) = \gamma(u)du$ in $\mathcal{M}_{+,1}$ such that

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}\left[H_B^N \geq T_0\right] \ &\leq \ \limsup_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathcal{D}) \\
&\leq \ - \inf_{\pi \in \mathcal{D}} I_{T_0}(\pi|\gamma) \\
&< \ -C_0 \ .
\end{aligned}$$

In particular, there exists $N_0 > 0$ such that for every integer $N > N_0$,

$$\mathbb{P}_{\eta^N}\left[H_B^N \geq T_0\right] \ \leq \ \exp\{-C_0 N\} \ .$$

To complete the proof, we proceed by induction. Suppose that the statement of the lemma is true until an integer $k-1 > 0$. Let $N > N_0$ and let $\hat{\eta}$ be a configuration in X_N. By the strong Markov property,

$$\begin{aligned}
\mathbb{P}_{\hat{\eta}}\left[H_B^N \geq kT_0\right] \ &= \mathbb{E}_{\hat{\eta}}\left[\mathbf{1}_{\left\{H_B^N \geq T_0\right\}} \mathbb{P}_{\eta_{T_0}}\left[H_B^N \geq (k-1)T_0\right]\right] \\
&\leq \mathbb{P}_{\hat{\eta}}\left[H_B^N \geq T_0\right] \sup_{\eta \in X_N} \left\{\mathbb{P}_{\eta}\left[H_B^N \geq (k-1)T_0\right]\right\} \\
&\leq \ \exp\left\{-kC_0 N\right\} \ ,
\end{aligned}$$

which concludes the proof.

□

Let $\partial B^N = \partial B_\delta^N$ be the set of configurations η in X_N for which there exists a finite sequence of configurations $\{\eta^i : 0 \leq i \leq k\}$ in X_N with η^0 in Γ^N, $\eta^k = \eta$ and such that

i) For every $1 \leq i \leq k$, the configuration η^i can be obtained from η^{i-1} by a jump of the dynamics.

ii) The unique configuration of the sequence that can enter into B^N after a jump is η^k.

We similarly define the set ∂B_i^N for each $1 \le i \le l$. Then it is clear that for N large enough and δ_1 small enough,

$$\partial B^N \;=\; \bigcup_{i=1}^{l} \partial B_i^N \;.$$

Let $\tau = \tau^N : D(\mathbb{R}_+, X_N) \to [0,\infty]$ be the stopping time given by

$$\tau = \inf\left\{t > 0 : \text{ there exists } s < t \text{ such that } \eta_s \in \Gamma^N \text{ and } \eta_t \in \partial B^N\right\} \;.$$

The sequence of stopping times obtained by iterating τ is denoted by τ_k. This sequence generates a irreducible Markov chain X_k on ∂B^N by setting $X_k = \eta_{\tau_k}$. Since the Markov chain X_k on ∂B^N is irreducible, there exists a unique invariant measure under the dynamics. We denote it by ν^N. For more details see [**22**].

Define $\tilde{v}_{ij}$ by

$$\begin{aligned}\tilde{v}_{ij} \;=\; &\inf\{I_T(\pi|\bar\rho) : T > 0, \bar\rho(u)du \in \mathcal{M}_i, \pi \in D([0,T],\mathcal{M}_+), \pi_T = \vartheta_j \\ &\qquad\qquad \text{and } \pi_t \notin \mathcal{M}_{sol} \text{ for any } 0 < t < T\} \;.\end{aligned}$$

LEMMA 5.2. *For every $\varepsilon > 0$, there exist δ and δ_1 with $\delta_1 > \delta > 0$ such that, for any $1 \le i,j \le l$ with $i \neq j$,*

$$\varliminf_{N\to\infty} \frac{1}{N} \log \inf_{\eta\in\partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \;\ge\; -\tilde{v}_{ij} - \varepsilon \;.$$

PROOF. For each integer $N > 0$, let η^N be a configuration in ∂B_i^N such that

$$\mathbb{P}_{\eta^N}(\eta_\tau \in \partial B_j^N) \;=\; \inf_{\eta\in\partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \;.$$

Recall that every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging in $\mathcal{M}_+$ to some ϑ that belongs to $\mathcal{M}_{+,1}$. Therefore we may assume that $\pi^N(\eta^N)$ converges to $\vartheta_\delta(du) = \gamma_\delta(u)du$ in B_i. We may also assume that ϑ_δ belongs to $\mathcal{B}_\delta(\bar\vartheta_i)$ without loss of generality.

Let π be a path in $D([0,T],\mathcal{M}_{+,1})$ such that $\pi_0 = \bar\vartheta_i$, $\pi_T = \bar\vartheta_j$ and $\pi_t \notin \mathcal{M}_{sol}$ for any $0 < t < T$. Let also $\Lambda^{\delta,\delta_1}(\pi)$ be the collection of all trajectories $\bar\pi$ in $D([0,T+2],\mathcal{M}_+)$ such that $I_2(\bar\pi|\gamma_\delta) \le \varepsilon$, $\bar\pi_t = \pi_{t-2}$ for any $2 \le t \le T+2$ and $\pi_t \in \mathcal{B}_{\delta_1/2}(\bar\vartheta_i)$ for any $0 \le t \le 2$. From Corollary 4.8, $\Lambda^{\delta,\delta_1}(\pi)$ is not empty if $\delta_1 > \delta > 0$ are enough small. Denote by $\mathcal{B}^{\delta,\delta_1}(\pi)$ the $\delta_1/2$-open neighborhood of $\Lambda^{\delta,\delta_1}(\pi)$ in $D([0,T+2],\mathcal{M}_+)$. Since $\pi_t \notin \mathcal{M}_{sol}$ for any $0 < t < T$, if $\delta_1 > \delta > 0$ are enough small, then

$$\{\eta_0 \in \partial B_i^N, \pi_\cdot^N \in \mathcal{B}^{\delta,\delta_1}(\pi)\} \;\subset\; \{\eta_0 \in \partial B_i^N, \eta_\tau \in \partial B_j^N\} \;.$$

Therefore by the dynamical large deviations lower bound,

$$\begin{aligned}\varliminf_{N\to\infty} \frac{1}{N} \log \inf_{\eta\in\partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \;&\ge\; -\inf_{\bar\pi\in\mathcal{B}^{\delta,\delta_1}(\pi)} I_{T+2}(\bar\pi|\gamma_\delta) \\ &\ge\; -I_T(\pi) - \varepsilon.\end{aligned}$$

It remains to take a supremum in π and $T > 0$. □

LEMMA 5.3. *For every $\varepsilon > 0$, there exist δ and δ_1 with $\delta_1 > \delta > 0$ such that, for any $1 \le i, j \le l$ with $i \neq j$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \sup_{\eta \in \partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \;\le\; -v_{ij} + \varepsilon \; .$$

PROOF. By the strong Markov property,

$$\sup_{\eta \in \partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \;\le\; \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta(\eta_{H_{\partial B^N}} \in \partial B_j^N) \; .$$

For each integer $N > 0$, fix a configuration η^N in X_N such that

$$\mathbb{P}_{\eta^N}(\eta_{H_{\partial B^N}} \in \partial B_j^N) \;=\; \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta(\eta_{H_{\partial B^N}} \in \partial B_j^N) \; .$$

By Lemma 5.1 and since $v_{ij} < \infty$, for every $\delta > 0$, there exists $T_\delta > 0$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \sup_{\eta \in X_N} \left\{ \mathbb{P}_\eta \left[T_\delta \le H_B^N \right] \right\} \;\le\; -v_{ij} \; .$$

In that case,

$$\begin{aligned}
&\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\eta_{H_{\partial B^N}} \in \partial B_j^N) \\
&\quad \le \max\{ \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\eta_{H_{\partial B^N}} \in \partial B_j^N \wedge H_B^N \le T_\delta), \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_{\eta^N}(H_B^N > T_\delta)\} \\
&\quad \le \max\{ \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_{\eta^N}(H_{\Gamma_j}^N \le T_\delta), -v_{ij}\} \; .
\end{aligned}$$

Let $\mathcal{C}_j = \mathcal{C}_j^{\delta,\delta_1}$ be the subset of $D([0,T], \mathcal{M}_+)$ consisting of all those paths π for which there exists some time $t \in [0, T_\delta]$ such that $\pi(t)$ belongs to Γ_j or $\pi(t-)$ belongs to Γ_j. By the dynamical large deviations lower bound and the compactness of $\mathcal{M}_+$, there exists some $\gamma(u)du = \gamma^{\delta^1}(u)du$ in Γ_i such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{C}_j) \;\le\; -\inf_{\pi \in \mathcal{C}_j} I_T(\pi|\gamma) \; .$$

Thus, in order to conclude the proof, it is enough to check that there exist $\delta_1 > \delta > 0$ such that

$$\inf_{\pi \in \mathcal{C}_j} I_T(\pi|\gamma) \;\ge\; v_{ij} - \varepsilon \; .$$

Assume that this is not true. In that case, for every integer $n > 0$ large enough, there exists a path π^n in $\mathcal{C}_j^{\delta_n, 1/n} \cap C([0, T_{\delta_n}], \mathcal{M}_{+,1})$, with $\delta_n < 1/n$, such that

$$I_{T_{1/n}}(\pi^n|\gamma_{1/n}) \;<\; v_{ij} - \varepsilon \; . \tag{5.1}$$

Moreover, since π^n belongs to $\mathcal{C}_j^{\delta_n, 1/n} \cap C([0, T_{\delta_n}], \mathcal{M}_+)$, there exists $0 < \widetilde{T}_n \le T_{\delta_n}$ such that $\pi^n_{\widetilde{T}_n}$ belongs to the set $\{\vartheta \in \mathcal{M}_+ : \, n^{-1} \le \inf_{\bar\vartheta \in \mathcal{M}_j} d(\vartheta, \bar\vartheta) \le 2n^{-1}\}$.

Let us first assume that the sequence of times $\{\widetilde{T}_n : n \geq 1\}$ is bounded above by some $T > 0$. For each integer $n > 0$, let $\tilde{\pi}^n$ be the path in $C([0,T], \mathcal{M}_{+,1})$ given by

$$\tilde{\pi}^n_t \;=\; \begin{cases} \pi^n_t & \text{if } 0 \leq t \leq \widetilde{T}_n \;, \\ \pi^n_{\widetilde{T}_n} & \text{if } \widetilde{T}_n \leq t \leq T \;. \end{cases}$$

Since I_T has compact level sets and since $\pi^n_0(du) = \gamma_{1/n}(u)du$ belongs to $\Gamma_{i,1/n} \cap \mathcal{M}_{+,1}$ for every integer $n > 0$, we may obtain a subsequence of $\tilde{\pi}^n$ converging to some π in $C([0,T], \mathcal{M}_{+,1})$ such that $\pi_0(du) = \bar{\rho}_i(u - u_i)$, $\pi_T = \bar{\rho}_j(u - u_j)$, for some $u_i, u_j \in \mathbb{T}$, and $I_T(\pi) \leq v_{ij} - \varepsilon$, which contradicts the definition of v_{ij} and we are done.

If the sequence of times $\{\widetilde{T}_n : n \geq 1\}$ is not bounded, in this case, by using Lemma 4.4, for large n, we can replace the path π^n by some new path $\bar{\pi}^n$ which satisfies the inequality (5.1) and whose entry time to the set $\{\vartheta \in \mathcal{M}_+ : n^{-1} \leq \inf_{\bar{\vartheta} \in \mathcal{M}_j} d(\vartheta, \bar{\vartheta}) \leq 2n^{-1}\}$ is bounded in n. Therefore performing the argument of the previous paragraph gives the contradiction. This finishes the proof of the lemma. □

The next result is similarly proved by the proof of Lemma 3.1 of chapter 6 in [**21**].

LEMMA 5.4. *Let us be given a Markov chain on a phase space X divided into disjoint sets X_i, where i runs over a finite set L. Suppose that there exist nonnegative numbers $p_{ij}, \tilde{p}_{ij}$ ($j \neq i$, $i, j \in L$) and a number $a > 1$ such that*

$$a^{-1} p_{ij} \leq P(x, X_j) \leq a \tilde{p}_{ij} \;, \qquad \textit{for any } x \in X_i, i \neq j \;,$$

for the transition probabilities of our chain. Furthermore, suppose that very set X_j can be reached from any state x sooner or later. Then

$$a^{2-2l} (\sum_{i \in L} \tilde{Q}_i)^{-1} Q_i \leq \nu(X_i) \leq a^{2l-2} (\sum_{i \in L} Q_i)^{-1} \tilde{Q}_i \;,$$

for any invariant probability measure of our chain, where l is the number of elements in L and Q_i and $\tilde{Q}_i$ are given by

$$Q_i \;=\; \sum_{g \in G(i)} \prod_{(m \to n) \in g} p_{mn} \quad \textit{and} \quad \tilde{Q}_i \;=\; \sum_{g \in G(i)} \prod_{(m \to n) \in g} \tilde{p}_{mn} \;.$$

We now introduce the main result of this subsection.

LEMMA 5.5. *For every $\varepsilon > 0$, there exist δ and δ_1 with $\delta_1 > \delta > 0$ such that, for any $1 \leq i, j \leq l$ with $i \neq j$,*

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log \nu^N(\partial B^N_i) \;\leq\; -w_i + w + \varepsilon \;. \tag{5.2}$$

$$\underline{\lim_{N \to \infty}} \frac{1}{N} \log \nu^N(\partial B^N_i) \;\geq\; -w_i + w - \varepsilon \;. \tag{5.3}$$

PROOF. Let

$$\tilde{w}_i = \min_{g \in G(i)} \sum_{(m \to n) \in g} \tilde{v}_{mn} \ .$$

By the argument presented in the proof of Lemma 4.1 in [**21**], we have $w_i = \tilde{w}_i$ for any $1 \le i \le l$. Therefore the conclusion of the lemma is a straightforward consequence of Lemmas 5.2, 5.3 and 5.4. $\square$

5.2. Lower bound. We show in this subsection the large deviations lower bound, that is, for any closed subset $\mathcal{O}$ of $\mathcal{M}_+$,

$$\varliminf_{N \to \infty} \frac{1}{N} \log \mathcal{P}_N(\mathcal{O}) \ \geq \ -\inf_{\vartheta \in \mathcal{C}} W(\mathcal{O}) \ .$$

Following [**11, 21, 22**], we represent the stationary measure μ_N of a subset A of X_N as

$$\mu_N(A) = \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta \left(\int_0^\tau \mathbf{1}_{\{\eta_s \in A\}} ds \right) d\nu_N(\eta) \ , \tag{5.4}$$

where

$$C_N \ = \ \int_{\partial B^N} \mathbb{E}_\eta(\tau) d\nu_N(\eta) \ .$$

We start from the estimate on the normalizing constant C_N.

LEMMA 5.6. *For any $\varepsilon > 0$, there exist δ and δ_1 with $\delta_1 > \delta > 0$ such that*

$$\varlimsup_{N \to \infty} \frac{1}{N} \log C_N \ \le \ \varepsilon \ .$$

PROOF. By the Strong Markov property,

$$\begin{aligned} C_N \ &= \ \sum_{i=1}^{l} \int_{\partial B_i^N} \mathbb{E}_\eta(\tau) d\nu_N(\eta) \\ &= \ \sum_{i=1}^{l} \int_{\partial B_i^N} \mathbb{E}_\eta(\tau \cdot \mathbf{1}_{\{H_{\Gamma_i}^N < \tau\}}) d\nu_N(\eta) \\ &\le \ \sum_{i=1}^{l} \int_{\partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) d\nu_N(\eta) + \sup_{\eta \in X_N} (H_B^N) \ . \end{aligned}$$

From Lemma 5.7 below, to conclude the lemma it is enough to show that for any $\varepsilon > 0$ there exist $\delta_1 > \delta > 0$ such that, for any $1 \le i \le l$,

$$\varlimsup_{N \to \infty} \frac{1}{N} \log \left(\sup_{\eta \in \partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) \right) \ \le \ \varepsilon \ . \tag{5.5}$$

For each integer $N > 0$, consider a configuration η^N in $\mathcal{B}_{2\delta_1}(\mathcal{M}_i)$ such that

$$\mathbb{P}_{\eta^N}(H_{\Gamma_i}^N < 3) \ = \ \inf_{\eta \in \mathcal{B}_{2\delta_1}^N(\mathcal{M}_i)} \mathbb{P}_\eta(H_{\Gamma_i^N} < 3) \ .$$

Recall that every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging in $\mathcal{M}_+$ to some ϑ that belongs to $\mathcal{M}_{+,1}$. Therefore we may assume that $\pi^N(\eta^N)$ converges to $\vartheta_{\delta_1}(du) = \gamma_{\delta_1}(u)du$ in B_i. We may also assume that ϑ_{δ_1} belongs to $\mathcal{B}_{2\delta_1}(\bar{\vartheta}_i)$ without loss of generality.

Fix $\varepsilon > 0$. Let $\bar{\pi}$ be a path in $D([0,1], \mathcal{M}_{+,1})$ which satisfies the statements of Corollary 4.8. Let also $\tilde{\pi}$ be a path in $D([0,1], \mathcal{M}_{+,1})$ such that $\tilde{\pi}_0 = \bar{\vartheta}_i$, $\tilde{\pi}_1 \in \mathcal{M}_+ \setminus \mathcal{B}_{2\delta_1}(\mathcal{M}_i)$ and $I_1(\tilde{\pi}) \le \varepsilon$. This choice is possible if δ_1 is small enough. Define the path π in $D([0,2], \mathcal{M}_{+,1})$ defined as $\pi_t = \bar{\pi}_t$ if $0 \le t \le 1$ and $\pi_t = \tilde{\pi}_{t-1}$ if $1 \le t \le 2$. For $\epsilon > 0$ small enough, denote by $\Lambda_\epsilon(\pi)$ the ϵ-open neighborhood of π in $D([0,2], \mathcal{M}_{+,1})$.

Then by the dynamical large deviations lower bound, for N large enough and any $\eta \in \mathcal{B}^N_{2\delta_1}(\mathcal{M}_i)$,

$$\begin{aligned} \mathbb{P}_\eta(H^N_{\Gamma_i} < 3) \;&\ge\; \exp\{-N(\inf_{\pi' \in \Lambda_\epsilon(\pi)} I_2(\pi'|\gamma) + \varepsilon)\} \\ &\ge\; \exp\{-3N\varepsilon\}\ . \end{aligned}$$

This bound together with the arguments performed in Lemmas 5.1 and 5.7 gives the bound (5.5) , which finishes the proof. □

In order to prove the lower bound, we first claim that for any open set $\mathcal{O}$ of $\mathcal{M}_+$ containing $\bar{\vartheta}_i$, for some $1 \le i \le l$,

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N(\mathcal{O}) \;\ge\; -w_i + w\ . \tag{5.6}$$

Indeed, fix $\varepsilon > 0$ and let $\delta_1 > \delta > 0$ satisfying (5.3), Lemma 3.7 , Lemma 5.6 and such that $\mathcal{B}_{2\delta_1}(\bar{\vartheta}_i) \subset \mathcal{O}$. Then

$$\begin{aligned} \mathcal{P}_N(\mathcal{O}) \;&=\; \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta\left(\int_0^\tau \mathbf{1}_{\{\eta_s \in \mathcal{O}^N\}} ds\right) d\nu_N(\eta) \\ &\ge\; \frac{1}{C_N} \int_{\partial B^N_i} \mathbb{E}_\eta(H^N_{\Gamma_i}) \nu^N(d\eta) \\ &\ge\; \frac{1}{C_N} \nu^N(\partial B^N_i) \inf_{\eta \in \partial B^N_i} \mathbb{P}_\eta(H^N_{\Gamma_i} \ge 1)\ . \end{aligned}$$

Hence to conclude the claim it suffices to show that

$$\varliminf_{N\to\infty} \frac{1}{N} \log \inf_{\eta \in \partial B^N_i} \mathbb{P}_\eta(H^N_{\Gamma_i} \ge 1) \;\ge\; 0\ .$$

For each integer $N > 0$, let η^N be a configuration in ∂B^N_i such that

$$\mathbb{P}_{\eta^N}(H^N_{\Gamma_i} \ge 1) \;=\; \inf_{\eta \in \partial B^N_i} \mathbb{P}_\eta(H^N_{\Gamma_i} \ge 1)\ .$$

Let also Λ be the collection of trajectories $\pi(t, du) = \rho(t,u)du$ in $C([0,1], \mathcal{M}_+)$ whose densities ρ are weak solutions of the Cauchy problem (3.1) starting at some profile ρ_0 in $\mathcal{B}_\delta(\mathcal{M}_i)$. Consider the open set

$$\mathcal{U} \;=\; \bigcup_{\pi \in \Lambda} \mathcal{B}^{\delta_1/2}_{[0,1]}(\pi)\ ,$$

where $\mathcal{B}^{\delta_1/2}_{[0,1]}(\pi)$ is the open $\delta_1/2$-neighborhood of π in $D([0,1],\mathcal{M}_+)$. Then from Lemma 3.7, for some $\gamma(u)du$ in $\mathcal{B}_\delta(\mathcal{M}_i)$, we have

$$\begin{aligned}\varliminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}(H^N_{\Gamma_i}\geq 1) &\geq \varliminf_{N\to\infty}\frac{1}{N}\log Q_{\eta^N}(\mathcal{U})\\ &\geq -\inf_{\bar\pi\in\mathcal{U}} I_T(\bar\pi|\gamma) \;=\; 0\;,\end{aligned}$$

which implies the claim.

From (5.6), we may deduce that there exists a sequence $\varepsilon_N\to 0$ such that

$$\varliminf_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(\mathcal{B}_{\varepsilon_N}(\mathcal{M}_i)) \;\geq\; -w_i+w\;.$$

Fix now an open set $\mathcal{O}$ of $\mathcal{M}_+$. In order to prove the lower bound, it is enough to prove that for any measure ϑ in $\mathcal{O}\cap\mathcal{M}_+$ and any trajectory $\tilde\pi$ in $D([0,T],\mathcal{M}_+)$ with $\tilde\pi_T=\vartheta$,

$$\varliminf_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(\mathcal{O}) \;\geq\; -w_i+w-I_T(\tilde\pi|\bar\rho_i)\;.$$

Indeed, for each N, let $\mathcal{B}_N=\left(\pi^N\right)^{-1}(\mathcal{B}_{\varepsilon_N}(\mathcal{M}_i))$ and let η^N be a configuration in $\mathcal{B}_N$ such that

$$\mathbb{P}_{\eta^N}\left[\pi^N_T\in\mathcal{O}\right] \;=\; \inf_{\eta\in\mathcal{B}_N}\left\{\mathbb{P}_\eta\left[\pi^N_T\in\mathcal{O}\right]\right\}\;.$$

Let $\mathcal{O}_T=\pi_T^{-1}\mathcal{O}$, then

$$\begin{aligned}\varliminf_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(\mathcal{O}) &= \varliminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\mu_N}\left[\mathbb{P}_{\eta_0}\left(\pi^N_T\in\mathcal{O}\right)\right]\\ &\geq \varliminf_{N\to\infty}\frac{1}{N}\log\left\{\mathcal{P}_N(\mathcal{B}_{\varepsilon_N}(\mathcal{M}_i))\inf_{\eta\in\mathcal{B}_N}\{\mathbb{P}_\eta[\pi^N_T\in\mathcal{O}]\}\right\}\\ &\geq -w_i+w+\varliminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}\left[\pi^N_T\in\mathcal{O}\right]\\ &= -w_i+w+\varliminf_{N\to\infty}\frac{1}{N}\log Q_{\eta^N}(\mathcal{O}_T)\\ &\geq -w_i+w-\inf_{\pi\in\mathcal{O}_T} I_T(\pi|\bar\rho_i)\\ &\geq -w_i+w-I_T(\tilde\pi|\bar\rho_i)\,.\end{aligned}$$

It remains to take a supremum in $\tilde\pi$ and $T>0$, which finishes the proof of the lower bound.

5.3. Upper bound. We show in this subsection the large deviations upper bound, that is, for any closed subset $\mathcal{C}$ of $\mathcal{M}_+$,

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathcal{P}_N(\mathcal{C}) \;\leq\; -\inf_{\vartheta\in\mathcal{C}} W(\mathcal{C})\;.$$

Let us first assume that $\mathcal{C}$ is a closed set of $\mathcal{M}_+$ such that, for each $1\leq i\leq l$, $\bar\vartheta_i\notin\mathcal{C}$. In that case we may assume that $\bigcup_{i=1}^l\mathcal{B}_{2\delta_1}(\mathcal{M}_i)\cap\mathcal{C}=\emptyset$ without loss of generality.

$$\mathcal{P}_N(\mathcal{C}) \;=\; \mu_N(\mathcal{C}^N) \;=\; \frac{1}{C_N}\int_{\partial B^N} \mathbb{E}_\eta\left(\int_0^\tau \mathbf{1}_{\{\eta_s\in\mathcal{C}^N\}}ds\right) d\nu_N(\eta)$$
$$\le\; \frac{1}{C_N}\sum_{i=1}^{l}\nu_N(\partial B_i^N)\sup_{\eta\in\partial B_i^N}\mathbb{E}_\eta\left(\int_0^\tau \mathbf{1}_{\{\eta_s\in\mathcal{C}^N\}}ds\right)\;.$$

Recall that a configuration in X_N can jump by the dynamics to less than other $2N$ configurations and that the jump rates are at most of order N^2. Hence, since any trajectory in $D(\mathbb{R}_+, X_N)$ has to perform at least a jump before the stopping time τ, $C_N \ge 1/CN^3$ for some constant $C > 0$. Hence, by (5.2), in order to prove the upper bound it is enough to show that, for each $1 \le i \le l$,

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\sup_{\eta\in\partial B_i^N}\mathbb{E}_\eta\left(\int_0^\tau \mathbf{1}_{\{\eta_s\in\mathcal{C}^N\}}ds\right) \;\le\; -V_i(\mathcal{C})+\varepsilon\;, \tag{5.7}$$

where $V_i(\mathcal{C}) = \inf_{\vartheta\in\mathcal{C}} V_i(\vartheta)$.

For each configuration η in ∂B_i^N, by the strong Markov property,

$$\mathbb{E}_\eta\left(\int_0^\tau \mathbf{1}_{\{\eta_s\in\mathcal{C}^N\}}ds\right) \;\le\; \mathbb{P}_\eta\left[H_{\mathcal{C}}^N < \tau\right]\sup_{\eta\in\mathcal{C}^N}\left\{\mathbb{E}_\eta\left(\tau\right)\right\}\;.$$

Notice that the jumps of the process $d(\pi^N(\eta_t), \mathcal{M}_i)$ are of order N^{-1}. Thus, for N large enough, any trajectory in $D(\mathbb{R}_+, X_N)$ starting at some configuration in ∂B_i^N, resp. $\mathcal{C}^N$, satisfies $H_{\Gamma_i}^N \le H_{\mathcal{C}}^N$, resp. $\tau \le H_B^N$. Hence, by the strong Markov property, the expression in the left side of (5.7) is bounded above by

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\sup_{\eta\in\Gamma_i^N}\mathbb{P}_\eta\left[H_{\mathcal{C}}^N < H_B^N\right]\sup_{\eta\in\mathcal{C}^N}\left\{\mathbb{E}_\eta\left(H_B^N\right)\right\}\;.$$

Therefore, in order to prove (5.7), it is enough to show the next lemma.

LEMMA 5.7. *For every $\delta > 0$ enough small, we have*

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\sup_{\eta\in X_N}\left\{\mathbb{E}_\eta\left(H_B^N\right)\right\} \;\le\; 0\;. \tag{5.8}$$

For every $\varepsilon > 0$, there exist δ and δ_1 with $\delta_1 > \delta > 0$ such that, for any $1 \le i \le l$, we have

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\sup_{\eta\in\Gamma_i^N}\left\{\mathbb{P}_\eta\left[H_{\mathcal{C}}^N < H_B^N\right]\right\} \;\le\; -V_i(\mathcal{C})+\varepsilon\;. \tag{5.9}$$

PROOF. Let $\delta > 0$ and consider $T_0, C_0, N_0 > 0$ satisfying the statement of Lemma 5.1. For every integer $N > N_0$ and every configuration η in X_N,

$$\mathbb{E}_\eta\left(H_B^N\right) \;\le\; T_0\sum_{k=0}^{\infty}\mathbb{P}_\eta\left(H_B^N \ge kT_0\right) \;\le\; T_0\sum_{k=0}^{\infty}\exp\left\{-kC_0N^d\right\} \;\le\; \frac{T_0}{1-e^{-C_0}}\;,$$

which proves (5.8).

We turn now to the proof of (5.9). Fix $\varepsilon > 0$. By Lemma 5.1 and since $V_i(\mathcal{C}) < \infty$, for every $\delta > 0$, there exists $T_\delta > 0$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \sup_{\eta\in X_N} \left\{\mathbb{P}_\eta \left[T_\delta \;\le\; H_B^N\right]\right\} \;\le\; -V_i(\mathcal{C}) \;.$$

For each integer $N > 0$, consider a configuration η^N in Γ_i^N such that

$$\mathbb{P}_{\eta^N} \left[H_{\mathcal{C}}^N \le T_\delta\right] \;=\; \sup_{\eta\in\Gamma_i^N} \left\{\mathbb{P}_\eta \left[H_{\mathcal{C}}^N \le T_\delta\right]\right\} \;.$$

Let $\mathcal{C}_\delta$ be the subset of $D([0,T_\delta],\mathcal{M}_+)$ consisting of all those paths π for which there exists t in $[0,T_\delta]$ such that $\pi(t)$ or $\pi(t-)$ belongs to $\mathcal{C}$. Notice that $\mathcal{C}_\delta$ is the closure of $\pi^N(\{H_{\mathcal{C}}^N \le T_\delta\})$ in $D([0,T_\delta],\mathcal{M}_+)$.

Recall that every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging in $\mathcal{M}_+$ to some ϑ that belongs to $\mathcal{M}_{+,1}$. Hence, by the dynamical large deviations upper bound, there exists a measure $\vartheta_\delta(du) = \gamma_\delta(u)du$ in $\Gamma_i \cap \mathcal{M}_{+,1}$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_{\eta^N}\big(H_{\mathcal{C}}^N \le T_\delta\big) \;\le\; \overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{C}_\delta) \;\le\; -\inf_{\pi\in\mathcal{C}_\delta} I_{T_\delta}(\pi|\gamma_\delta) \;.$$

Therefore, since

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log\{a_N + b_N\} \;\le\; \max\left\{\overline{\lim_{N\to\infty}} \frac{1}{N} \log a_N,\, \overline{\lim_{N\to\infty}} \frac{1}{N} \log b_N\right\} \;,$$

the left hand side in (5.9) is bounded above by

$$\max\left\{-V_i(\mathcal{C}), -\inf_{\pi\in\mathcal{C}_\delta} I_{T_\delta}(\pi|\gamma_\delta)\right\} \;,$$

for every $\delta > 0$. Thus, in order to conclude the proof, it is enough to check that there exists $\delta > 0$ such that

$$\inf_{\pi\in\mathcal{C}_\delta} I_{T_\delta}(\pi|\gamma_\delta) \;\ge\; V_i(\mathcal{C}) - \varepsilon \;.$$

This bound is proved by the argument as we did in the proof of Lemma 5.3. □

To finish the proof of the large deviations upper bound, it is enough to show that, for each $1 \le i \le l$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathcal{P}_N(\mathcal{B}_\delta(\bar{\vartheta}_i)) \;\le\; -w_i + w \;.$$

However one can prove this bound by Lemma 5.6 and the argument presented in the first part of this subsection.

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