博士論文

論文題目

The determinant and the discriminant of a complete intersection of even dimension
(偶数次元完全交叉の行列式と判別式)

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THE DETERMINANT AND THE DISCRIMINANT OF A COMPLETE INTERSECTION OF EVEN DIMENSION

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Abstract. The determinant of the Galois action on the ℓ-adic cohomology of the middle degree of a proper smooth variety of even dimension defines a quadratic character of the absolute Galois group of the base field. In this article, we show that for a complete intersection of even dimension in a projective space, the character is computed via the square root of the discriminant of the defining polynomials of the variety.

Introduction

Let k be a field, $\bar{k}$ an algebraic closure of k and k^s the separable closure of k contained in $\bar{k}$. Let $\Gamma_k = \mathrm{Gal}(k^s/k) = \mathrm{Aut}_k(\bar{k})$.

Let X be a proper smooth variety of even dimension m over k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology $V = H^m(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{m}{2}))$ defines an orthogonal representation of the absolute Galois group Γ_k. The determinant

$$\det V : \Gamma_k \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$$

is independent of the choice of ℓ (Corollary 2.2).

In this introduction we assume that the characteristic of k is not 2. Let $f_1, \ldots, f_r$ be homogeneous polynomials of $n+1$ variables of degrees $d_1, \ldots, d_r$ of coefficients in k. Let X be the intersection of r hypersurfaces defined by these polynomials in a projective space of dimension n. In 2012, O. Benoist[1] studied the *discriminant of a complete intersection* and gave an explicit formula of its degree. The discriminant, here denoted by $\mathrm{disc}(f_1, \ldots, f_r)$, is a polynomial of the coefficients of $f_1, \ldots, f_r$, and is defined in [1] up to sign by the property that X is smooth of dimension $n-r$ if and only if $\mathrm{disc}(f_1, \ldots, f_r) \neq 0$.

Further, we assume that $n-r$ is even. In this case, we determine the sign of the discriminant by the property that the discriminant modulo 4

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is a square (Theorem 2.3.1). Let us denote the discriminant defined in these steps by $\mathrm{disc}_\sigma(f_1, \dots, f_r)$. We shall prove below (Theorem 2.3.2):

Theorem 0.1. *Assume that X is smooth of dimension $m = n - r$. Then the quadratic character $\det V$ is defined by the square root of $\mathrm{disc}_\sigma(f_1, \dots, f_r)$.*

In other words, the kernel of $\det V : \Gamma_k \to \{\pm 1\}$ is the subgroup of Γ_k corresponding to the field extension $k(\sqrt{\mathrm{disc}_\sigma(f_1, \dots, f_r)})/k$.

Let us briefly outline the contents of this paper. In Section 1, we study the discriminant $\mathrm{disc}(f_1, \dots, f_r)$ of a complete intersection. We follow the method of Benoist in [1]. However, we see the variety X_A in [1] as a projective space bundle over the projective space in order to give another calculation of the degree of the discriminant, therefore we recall the detail. We construct the universal family of intersections of hypersurfaces in the projective space, and consider the subset of the parameter space consisting of the points corresponding to singular fibers. We show the subset is identified with the underlying set of the projective dual of a smooth projective variety. This variety is equal to the projective toric variety X_A in [1] (Remark 1.10), though we treat it as a projective space bundle over the projective space. We then verify that the projective dual is an irreducible divisor in the parameter space (Corollary 1.15). We define the discriminant of complete intersections as the defining polynomial of the divisor.

In Subsection 1.22, we calculate the degree of the discriminant in a different way from that in [1]. We give a new explicit presentation of the degree, though we do not know the relation between this and Benoist's formula.

In Section 2, we prove the main theorem. We first recall the quadratic character of the absolute Galois group defined by the determinant of the ℓ-adic representation of the middle degree of a proper smooth variety defined over a field. In [8], T. Saito showed that, for a smooth hypersurface of even dimension, the character is computed via the square root of the discriminant of a defining polynomial of the hypersurface. We adapt his method to extend the result to our case of smooth complete intersections of even dimension. By the same argument on universal families as in the case of hypersurface, the theorem is true up to a sign of the discriminant. Then the sign is determined by properties of the discriminant modulo 4.

Finally, in Section 3, we give an explicit presentation of the discriminant of intersections of two quadrics (Theorem 3.6). Let $F_1 = \sum_{0 \le i \le j \le n} C_{ij}^{(1)} X_i X_j$ and $F_2 = \sum_{0 \le i \le j \le n} C_{ij}^{(2)} X_i X_j$ be universal homogeneous polynomials of degree 2. Let $R = \mathbb{Z}[t_1, t_2]$ be the polynomial ring with variables t_1, t_2. We see $t_1F_1 + t_2F_2$ as a quadratic form with variables $X_0, \dots, X_n$ and denote its discriminant by $\operatorname{disc}(t_1F_2+t_2F_2) \in R[(C_{ij}^{(l)})]$. Further we see $\operatorname{disc}(t_1F_2 + t_2F_2)$ as a binary form with variables t_1, t_2 and denote its discriminant by $\operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2)) \in \mathbb{Z}[(C_{ij}^{(l)})]$.

Theorem 0.2. *1. Let $n \geq 2$ be an even integer. Then the equation*

$$\operatorname{disc}(F_1, F_2) = \operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2))$$

holds up to sign.

2. Let $n \geq 3$ be an odd integer. Then the equation

$$\operatorname{disc}(F_1, F_2) = 2^{-2(n+1)} \operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2))$$

holds up to sign.

The discriminant of a quadratic polynomial is the determinant of the symmetric matrix corresponding to the quadratic form. Further, the discriminant of a binary polynomial is given by the Sylvester's determinant. Thus the above equality give explicit presentation of the discriminant of the complete intersection of two quadrics.

The author was imformed by Takeshi Saito that Jean-Pierre Serre suggested him that the discriminant of a complete intersection of two quadrics should be given by those of a binary polynomial and a quadratic polynomial.

The cohomology of such an intersection is generated by algebraic classes of linear subspaces. The intersection theory of these classes is studied in detail in [6] and [7]. We give an application of the main theorem to this subject.

1. Discriminant

1.1. **Ordinary quadratic singularity.** We recall the definition of an ordinary quadratic singularity [3, Exposé XV, 1.2], [3, Exposé XVII, 1.1], [2, Exposé VI, 6.6]. Let k be an algebraically closed field. A quadratic form with $n+1$ variables over k is called a *ordinary quadratic form* if the hypersurface of $\mathbb{P}^n_k$ defined by the vanishing of the form is smooth over k if $n \geq 1$, and is non-zero if $n = 0$.

Definition 1.2. *Let k be a field. Let X be a k-scheme of finite type, x be a closed point of X, and n be a dimension of X at x.*

1. For k algebraically closed, we call x an ordinary quadratic singularity *of X if the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring of X at x is* *isomorphic to the quotient* $k[[x_0, \ldots, x_n]]/(g)$ *of the ideal generated by a formal series* $g(x_0, \ldots, x_n)$ *such that*

$$g(x_0, \ldots, x_n) = q(x_0, \ldots, x_n) + (\text{terms of degree} > 2) \tag{1}$$

where $q(x_0, \ldots, x_n)$ *is an ordinary quadratic form.*

2. If $\bar{k}$ *is an algebraic closure of* k*, we call a closed point* $x \in X$ *is an* ordinary quadratic singularity *if the points of* $\overline{X} = X \times_k \bar{k}$ *on* x *are ordinary quadratic singularities of* $\overline{X}$*.*

If k is algebraically closed, and if char $k \neq 2$ or n is even, the condition (1) is equivalent to the condition :

$$\widehat{\mathfrak{m}}_{X,x} = \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right). \tag{2}$$

1.3. **Dual variety.** We recall the formalism of the dual variety [3, Exposé XVII, 3.1, 5.1]. Let S be $\operatorname{Spec} \mathbb{Z}$ or $\operatorname{Spec} k$ for a field k. Let $\mathbb{P}^N$ denote the N-dimensional projective space over S, and $\mathbb{P}^\vee$ denote the dual projective space. A point in $\mathbb{P}^\vee$ corresponds to a hyperplane H in $\mathbb{P}^N$.

Let Z be a proper smooth irreducible S-scheme, purely of dimension $n \geq 1$ over S, with closed immersion $Z \hookrightarrow \mathbb{P}^N$. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^N}$ be the ideal sheaf defining Z. Let $\mathcal{N} = (\mathcal{I}/\mathcal{I}^2)^\vee$ denote the normal sheaf and let $\mathbb{P}(\mathcal{N}) = \operatorname{Proj} S^\bullet \mathcal{N}$ denote the associated projective space bundle over Z.

We view the projective bundle $\mathbb{P}(\mathcal{N})$ over Z as a closed subscheme of $\mathbb{P}^N \times \mathbb{P}^\vee$. We have an exact sequence of coherent sheaves on Z

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^N} \otimes \mathcal{O}_Z \to \Omega_Z \to 0. \tag{3}$$

On the other hand, we have an exact sequence of sheaves on $\mathbb{P}^N$

$$0 \to \Omega_{\mathbb{P}^N/S} \to (\mathcal{O}_{\mathbb{P}^N}(-1))^{N+1} \to \mathcal{O}_{\mathbb{P}^N} \to 0$$

and its restriction to Z

$$0 \to \Omega_{\mathbb{P}^N/S} \otimes \mathcal{O}_Z \to (\mathcal{O}_Z(-1))^{N+1} \to \mathcal{O}_Z \to 0. \tag{4}$$

By (3) and (4), we have an injection $\mathcal{I}/\mathcal{I}^2 \to (\mathcal{O}_Z(-1))^{N+1}$ and its dual gives a surjection $\mathcal{O}_Z(1)^{N+1} \to \mathcal{N}$. Hence we have an closed immersion

$$\mathbb{P}(\mathcal{N}) \hookrightarrow \mathbb{P}(\mathcal{O}_Z(1)^{N+1}) \cong Z \times \mathbb{P}^\vee \hookrightarrow \mathbb{P}^N \times \mathbb{P}^\vee.$$

We define $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee$ by the composition $\mathbb{P}(\mathcal{N}) \hookrightarrow \mathbb{P}^n \times \mathbb{P}^\vee \to \mathbb{P}^\vee$.

We denote the reduced induced closed subscheme structure of the image of φ by $Z^\vee$ and call it the *dual variety* of Z (with respect to the immersion $Z \hookrightarrow \mathbb{P}^N$).

Proposition 1.4. (c.f. [3, Exposé XVII, Proposition 3.1.4]) *The dual variety $Z^\vee$ is an irreducible and proper S-scheme of relative dimension $\leq N-1$.*

Proof. Since $\mathbb{P}(\mathcal{N})$ is a projective space bundle over the irreducible scheme Z, it is irreducible and its image $Z^\vee$ is also irreducible.

The S-scheme $\mathbb{P}(\mathcal{N})$ is of relative dimension $N-1$ if $Z \neq \mathbb{P}^N$, and is empty if $Z = \mathbb{P}^N$. □

Let k be an algebraically closed field and let $\operatorname{Spec} k \to S$ be a geometric point. Let the suffix $_k$ denote the base change to $\operatorname{Spec} k$ over S. By [3, Exposé XVII, (3.1.1)], the set of k-valued points $\mathbb{P}(\mathcal{N})_k(k) \subset \mathbb{P}^N_k(k) \times \mathbb{P}^\vee_k(k)$ consists of the pairs $(x, H) \in \mathbb{P}^N_k(k) \times \mathbb{P}^\vee_k(k)$ such that H is hyperplanes tangent to Z_k at x.

The set of k-valued points of the dual variety $Z^\vee_k$ is the set of hyperplanes tangent to Z_k.

Let $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee$ be the canonical morphism. By [3, Exposé XV, 1.3.4], there exists an open subset W of $\mathbb{P}(\mathcal{N})$ consisting of the points w such that w is an ordinary quadratic singularity in the intersection $Z \cap H_{\varphi(w)}$ where $H_{\varphi(w)}$ is the hyperplane in $\mathbb{P}^N$ corresponding to the point $\varphi(w)$.

Now we consider the case that $S = \operatorname{Spec} k$ for an algebraically closed field k. Recall the notion of a *multiple of an immersion* $i : Z \hookrightarrow \mathbb{P}^N$. Let $d \geq 2$ be an integer. Then we can obtain an immersion of Z into a projective space by the composition of $i : Z \hookrightarrow \mathbb{P}^N$ and Segre embedding $S_d : \mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$ where $N' = \begin{pmatrix} N+d \\ d \end{pmatrix} - 1$. We call this composition by *The d-uple embedding of i.*

We will use the following proposition due to N. Katz.

Proposition 1.5. *Let $S = \operatorname{Spec} k$ for an algebraically closed field k. Let $d \geq 2$ be an even integer and $i_d : Z \hookrightarrow \mathbb{P}^N$ be the d-uple embedding of a immersion of Z into a projective space over k. Further let $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee$ be the canonical morphism associated to i_d.*

1. The open set W is dense in $\mathbb{P}(\mathcal{N})$.

2. Assume that $n = \dim_k Z$ is even or $\operatorname{char} k \neq 2$. Then, the canonical morphism $\varphi : \mathbb{P}(\mathcal{N}) \to Z^\vee$ is birational. More precisely, the following two open sets of $Z^\vee$ are equal ;

a) the maximum open subset V of $Z^\vee$ where the morphism $\varphi : \mathbb{P}(\mathcal{N}) \to Z^\vee$ induce an isomorphism $\varphi^{-1}(V) \cong V$,

b) the subset of $Z^\vee$ consisting of the points corresponding to hyperplanes H, such that the intersection $Z \cap H$ has a unique singular point, which is ordinary quadratic.

Proof. 1. By [3, Exposé XVII, (3.7.1)], there exists a hyperplane H such that $Z \cap H$ has an ordinary quadratic singularity and hence the open subscheme W is non-empty. Since $\mathbb{P}(\mathcal{N})$ is irreducible, it is dence.

2. By the assertion 1 and [3, Exposé XVII, Proposition 3.3], The morphism $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee$ is generically unramified. Hence, the assertion follows from [3, Exposé XVII, Proposition 3.5]. $\square$

1.6. **The universal family of intersections of hypersurfaces.** We fix integers $0 \leq r \leq n$. We consider the polynomial ring $\mathbb{Z}[X_0, \ldots, X_n]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^n \mathbb{Z} \cdot X_i$. For an integer $d \geq 1$, we identify the d-th symmetric power $S^d E$ defined over $\mathbb{Z}$ with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree d in $\mathbb{Z}[X_0, \ldots, X_n]$. If $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ is a multi-index, we put $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in \mathbb{Z}[X_0, \ldots, X_n]$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. The monomials X^α of degree $|\alpha| = d$ form a basis of $S^d E$.

We put $\mathbb{P}^n = \mathbb{P}(E) = \operatorname{Proj} \mathbb{Z}[X_0, \ldots, X_n]$ and fix integers $d_1, \ldots, d_r \geq 1$. We assume that $d_l \geq 2$ for an index l $(1 \leq l \leq r)$. Further we put $V = \bigoplus_{1 \leq j \leq r} S^{d_j} E$ and let $\mathbb{P}^\vee = \mathbb{P}(V^\vee) = \operatorname{Proj}(S^\bullet(V^\vee))$ be the projective space defined by the dual $V^\vee = \operatorname{Hom}(V, \mathbb{Z})$. Let $(C_\alpha^{(j)})_{|\alpha| = d_j}$ be the dual basis of $(S^{d_j} E)^\vee$ and define the universal polynomials $F_j = \sum_{|\alpha| = d_j} C_\alpha^{(j)} X^\alpha$. Then we define a closed subscheme $X \subset \mathbb{P}^n \times \mathbb{P}^\vee$ by the equations $F_1 = \cdots = F_r = 0$. This is the universal family of intersections of r hypersurfaces.

Let k be an algebraically closed field and let $s : \operatorname{Spec} k \to \mathbb{P}^\vee$ be a geometric point. Then this s corresponds to a sequence of homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ of coefficients in k. The geometric fiber X_s of π is isomorphic to the intersection of r hypersurfaces in $\mathbb{P}_k^n$ defined by the polynomials $f_1, \ldots, f_r$.

Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^\vee}$ defined by all the $r \times r$ minor determinants of the Jacobian matrix

$$J(F_1, \ldots, F_r) = \left(\left(\frac{\partial F_j}{\partial X_i} \right)_{0 \leq i \leq n, 1 \leq j \leq r} \right)$$

of the universal polynomials $F_1, \ldots, F_r$. We define a closed subscheme $\Delta_X \subset X$ by the ideal sheaf $\mathcal{J} \cdot \mathcal{O}_X$.

Let $\pi : X \subset \mathbb{P}^n \times \mathbb{P}^\vee \to \mathbb{P}^\vee$ be the canonical map. By the Jacobian criterion, the complement $U = X - \Delta_X$ is the maximum open subscheme of X on which the morphism $\pi : X \to \mathbb{P}^\vee$ is smooth of relative dimension $n - r$.

Proposition 1.7. *Let k be an algebraically closed field. Let $(x, s) \in (\Delta_X)_k \in \mathbb{P}_k^n \times \mathbb{P}_k^\vee$ be a closed point and let $(f_1, \ldots, f_r)$ be homogeneous* *polynomials of degrees $d_1, \dots, d_r$ of coefficients in k corresponding to the closed point $s \in \mathbb{P}^\vee_k$. Then, the following two conditions are equivalent.*

1. The point x is a closed point at which the fiber X_s is of dimension $n-r$, and is an ordinary quadratic singularity of X_s.

2. There exists an index l $(1 \leq l \leq r)$ such that x is a closed point at which the intersection $V((f_1, \dots, f_{l-1}, f_{l+1}, \dots, f_r))$ of $r-1$ hypersurfaces in $\mathbb{P}^n_k$ is smooth of dimension $n-r+1$, and x is an ordinary quadratic singularity of X_s.

If char $\neq 2$ or $n-r$ is odd, the conditions 1, 2 are further equivalent to :

3. The morphism $\pi|_{(\Delta_X)_k} : (\Delta_X)_k \to \mathbb{P}^\vee_k$ is unramified at $(x,s) \in (\Delta_X)_k$.

Proof. $1 \Rightarrow 2$. The assumption implies that the rank of the Jacobian matrix $J(f_1, \dots, f_r)$ at x is $r-1$.

$2 \Rightarrow 1$. Obvious.

$1 \Rightarrow 3$. We assume that x is a closed point at which X_s is of dimension $n-r$, and is an ordinary quadratic singularity of X_s. We assume $x \in D_+(X_i)_k$ for a fixed i $(0 \leq i \leq n)$. We write $\phi_j = X_i^{-d_j} f_j$. We identify the fiber $\mathbb{P}^n_k \times s$ with $\mathbb{P}^n_k$. Then by 1.1.(2), there is an isomorphism

$$\widehat{\mathcal{O}}_{\mathbb{P}^n_k, x} \cong k[[x_1, \dots, x_n]]$$

such that

$$\widehat{\mathcal{O}}_{X_s, x} = \widehat{\mathcal{O}}_{\mathbb{P}^n_k, x}/(\phi_1, \dots, \phi_r) \cong k[[x_1, \dots, x_{n-r+1}]]/(g(x_1, \dots, x_{n-r+1}))$$

and $\widehat{\mathfrak{m}}_{X_s, x} \cong \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n-r+1}}\right)$. Hence we have $\mathcal{J}_{\mathbb{P}^n_k, x} \cdot \widehat{\mathcal{O}}_{X_s, x} = \widehat{\mathfrak{m}}_{X_s, x}$. This implies $\widehat{\mathcal{O}}_{(\Delta_X)_k, (x,s)} \otimes_{\widehat{\mathcal{O}}_{\mathbb{P}^\vee_k, s}} \kappa(s) \cong \kappa(s) (= k)$.

$3 \Rightarrow 1$. We assume that $\pi|_{(\Delta_X)_k} : (\Delta_X)_k \to \mathbb{P}^\vee_k$ is unramified at (x,s). Let v denote the rank of the Jacobian matrix $J(\phi_1, \dots, \phi_r)(x)$ at x. Then there are v integers $\{k_1, \dots, k_v\} \subset \{1, \dots, r\}$ such that the intersection $V((\phi_{k_1}, \dots, \phi_{k_v})) \subset D_+(X_i)_k \subset \mathbb{P}^n_k$ is smooth of dimension $n-r+v$ at x and $V((\phi_{k_1}, \dots, \phi_{k_v}, \phi_h))$ is not smooth at x for $h \notin \{k_1, \dots, k_v\}$. So the ideal $\mathcal{J}_{\mathbb{P}^n_k, x} \cdot \widehat{\mathcal{O}}_{\mathbb{P}^n_k, x} \subset \widehat{\mathcal{O}}_{\mathbb{P}^n_k, x}$ generated by the $r \times r$ minor determinants of $J(\phi_1, \dots, \phi_r)$ is contained in $(\widehat{\mathfrak{m}}_{\mathbb{P}^n_k, x})^{r-v}$. By the assumption, we have $\mathcal{J}_{\mathbb{P}^n_k, x} \cdot \widehat{\mathcal{O}}_{\mathbb{P}^n_k, x} + (\phi_1, \dots, \phi_r) = \widehat{\mathfrak{m}}_{\mathbb{P}^n_k, x}$ and hence $r - v = 1$. So we have an isomorphism

$$\widehat{\mathcal{O}}_{\mathbb{P}^n_k, x} \cong k[[x_1, \dots, x_n]]$$

and an integer l such that

$$\widehat{\mathcal{O}}_{\mathbb{P}^n_k, x}/(\phi_1, \dots, \phi_{l-1}, \phi_{l+1}, \dots, \phi_r) \cong k[[x_r, \dots, x_n]]$$

and $\phi_l \mapsto g$ with $\frac{\partial g}{\partial x_i}(0) = 0$ $(1 \leq i \leq n)$, and further $\widehat{\mathfrak{m}}_{\mathbb{P}^n_k,x} \cong (x_1, \ldots, x_{r-1}, g, \frac{\partial g}{\partial x_r}, \ldots, \frac{\partial g}{\partial x_n})$. Hence we have $\widehat{\mathcal{O}}_{X_s,x} \cong k[[x_r, \ldots, x_n]]/(g)$ with $\widehat{\mathfrak{m}}_{X_s,x} \cong \left(\frac{\partial g}{\partial x_r}, \ldots, \frac{\partial g}{\partial x_n}\right)$. By 1.1 (2), the assertion follows. $\square$

1.8. **The dual variety and the discriminant.** We define a closed subscheme $D_X \subset \mathbb{P}^\vee$ as the image $\pi(\Delta_X)$ with the reduced structure. For an algebraically closed field k, the set of k-valued points $D_X(k)$ consists of the sequences of homogeneous polynomial $(f_1, \ldots, f_r)$ of degrees $d_1, \ldots, d_r$ of coefficients in k such that the intersections $V((f_1, \ldots, f_r)) \subset \mathbb{P}^n_k$ are singular. We show that D_X is an irreducible divisor (Corollary 1.15). We can reduce this problem on complete intersections of r hypersurfaces in $\mathbb{P}^n$ to that on hypersurfaces in a $\mathbb{P}^{r-1}$-bundle $T = \mathbb{P}(\mathcal{E}) = \operatorname{Proj} S^\bullet \mathcal{E}$ on $\mathbb{P}^n$ associated to a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{E} = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)$.

We identify

$$\Gamma(T, \mathcal{O}_T(1)) = \Gamma(\mathbb{P}^n, \mathcal{E}) = \Gamma(\mathbb{P}^n, \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)) = V.$$

Let $((S_{\alpha,1}, |\alpha| = d_1), \ldots, (S_{\alpha,r}, |\alpha| = d_r))$ denote the basis $((X^\alpha, |\alpha| = d_1), \ldots, (X^\alpha, |\alpha| = d_r))$ of $V = \Gamma(T, \mathcal{O}_T(1))$. We consider the section

$$s = \sum_{|\alpha|=d_1} C^{(1)}_{|\alpha|} S_{\alpha,1} + \cdots + \sum_{|\alpha|=d_r} C^{(r)}_{|\alpha|} S_{\alpha,r}$$
$$\in V \otimes V^\vee = \Gamma(T \times \mathbb{P}^\vee, \mathcal{O}_T(1) \otimes \mathcal{O}_{\mathbb{P}^\vee}(1)).$$

We define a closed subscheme Y of $T \times \mathbb{P}^\vee$ by the equation $s = 0$. Let $\psi : Y \subset T \times \mathbb{P}^\vee \to \mathbb{P}^\vee$ be the canonical map.

For $V = S^{d_1}E \oplus \cdots \oplus S^{d_r}E$, we put $N = \dim(V) - 1$ and $\mathbb{P}^N = \mathbb{P}(V) = \operatorname{Proj}(S^\bullet V)$. The projective space $\mathbb{P}^\vee = \mathbb{P}(V^\vee)$ is the dual of $\mathbb{P}^N$ parametrizing hyperplanes in $\mathbb{P}^N$. The $\mathbb{Z}$-module V is identified with the space of global sections $\Gamma(T, \mathcal{O}_T(1))$ of the invertible sheaf $\mathcal{O}_T(1)$ on T.

Lemma 1.9. *The invertible sheaf $\mathcal{O}_T(1)$ is very ample relatively to* $\operatorname{Spec} \mathbb{Z}$. *More explicitly, the global sections*

$$(S_{\alpha,1}, |\alpha| = d_1), \ldots, (S_{\alpha,r}, |\alpha| = d_r)$$

define a closed immersion $v : T \hookrightarrow \mathbb{P}^N = \mathbb{P}(V)$.

Proof. For the sections $S_{\alpha,j}$, we define open sets $U_{\alpha,j} \subset T$ by $U_{\alpha,j} = \{x \in T \mid (S_{\alpha,j})_x \notin \mathfrak{m}_x \mathcal{O}_T(1)_x\}$.

Let $p : T \to \mathbb{P}^n$ denote the canonical map and $D_+(X_i) \subset \mathbb{P}^n$ $(0 \leq i \leq n)$ denote the fundamental open sets. Then we have $D_+(X_i) \cong \operatorname{Spec} \mathbb{Z}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ where $x_k = \frac{X_k}{X_i}(0 \leq k \leq n, k \neq i)$. On the open subscheme $D_+(X_i)$, the section $X_i^{d_j} \in \Gamma(D_+(X_i), \mathcal{O}(d_j))$ gives the trivialization $\mathcal{O}(d_j)|_{D_+(X_i)} \cong \mathcal{O}_{D_+(X_i)} (1 \leq j \leq r)$. Let T_j denote this generator $X_i^{d_j}$. Then we have an isomorphism

$$p^{-1}(D_+(X_i)) \cong D_+(X_i) \times \operatorname{Proj} \mathbb{Z}[T_1, \ldots, T_r] = D_+(X_i) \times \mathbb{P}^{r-1}_{\mathbb{Z}}.$$

For $0 \leq i \leq n$ and $1 \leq j \leq r$, we define multi-indices $\alpha^{i,j} = (\alpha_0, \ldots, \alpha_n)$ by $\alpha_k = 0$ $(k \neq i)$ and $\alpha_i = d_j$. Then we have $U_{\alpha^{i,j},j} \cong D_+(X_i) \times D_+(T_j) \subset D_+(X_i) \times \mathbb{P}^{r-1}_{\mathbb{Z}}$ for any fixed i. Thus the open sets $(U_{\alpha^{i,j},j})_{0 \leq i \leq n, 1 \leq j \leq r}$ cover T.

We show that for each $U_{\alpha^{i,j},j}$, the global sections define a closed immersion $U_{\alpha^{i,j},j} \to \mathbb{P}^N$. We have an isomorphism

$$U_{\alpha^{i,j},j} \cong \operatorname{Spec} \mathbb{Z}\left[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \frac{T_0}{T_j}, \ldots, \frac{T_r}{T_j}\right]$$

where $x_k = \frac{X_k}{X_i}$ as above. For each i and j, we define a ring homomorphism

$$\mathbb{Z}\left[(s_{\alpha,j'})_{1 \leq j' \leq r, |\alpha| = d_{j'}, (\alpha, j') \neq (\alpha^{i,j}, j)}\right] \to \Gamma(U_{\alpha^{i,j},j}, \mathcal{O}_T)$$

by the indeterminate $s_{\alpha,j'}$ mapping to the element $S_{\alpha,j'}/S_{\alpha^{i,j},j}$. Then this morphism is surjective. In fact, in the isomorphism

$$\Gamma(U_{\alpha^{i,j},j}, \mathcal{O}_T) \cong \mathbb{Z}\left[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \frac{T_0}{T_j}, \ldots, \frac{T_r}{T_j}\right],$$

the indeterminate $s_{\alpha^{i,j'},j'}$ maps to $\frac{T_{j'}}{T_j}$. Further, for the multi-indices $\alpha^{i,j,l} = (\alpha_0, \ldots, \alpha_n)$ $(0 \leq l \leq n, l \neq i)$ defined by $\alpha_k = 0 (k \neq i, l)$ and $\alpha_i = d_j - 1$ and $\alpha_l = 1$, the indeterminate $s_{\alpha^{i,j,l},j}$ maps to x_l. □

We consider T as a closed subscheme of $\mathbb{P}^N$ by the immersion v.

Remark 1.10. For any algebraically closed field k, the base change $T_k \subset \mathbb{P}^N_k$ is equal to the projective toric variety $X_A \subset \mathbb{P}^N_k$ introduced in [1]. We recall the definition of the variety X_A. We consider the finite set $A = \{Y_j X^\alpha\}_{1 \leq j \leq r, |\alpha| = d_j}$ of monomials in $n + r + 1$ variables $Y_1, \ldots, Y_r, X_0, \ldots, X_n$. Each monomial $Y_j X^\alpha$ in A defines the function

$$(k^\times)^{n+r+1} \to k^\times : (y_1, \ldots, y_r, x_0, \ldots, x_n) \mapsto y_j x^\alpha.$$

The variety $X_A \subset \mathbb{P}^N_k$ is defined by the closure of the set

$$X_A^0 = \{[y_1 x^\alpha : \ldots : y_r x^\alpha] : 1 \leq j \leq r, |\alpha| = d_j, (y_1, \ldots, x_n) \in (k^\times)^{n+r+1}\}.$$

Then the set X_A^0 is included in T_k by the definition of the embedding $v : T_k \hookrightarrow \mathbb{P}^N_k$. Since T_k is irreducible, we have $X_A = T_k$.

As in Subsection 1.3, we consider the projective space bundle $\mathbb{P}(\mathcal{N})$ over T with respect to the immersion v. We define a closed subvariety $\Delta_Y \subset \mathbb{P}^N \times \mathbb{P}^\vee$ by $\Delta_Y = \mathbb{P}(\mathcal{N})$.

We have an explicit presentation of Δ_Y. Let $p : T \to \mathbb{P}^n$ denote the canonical map. Recall that

$$p^{-1}(D_+(X_i)) \cong D_+(X_i) \times \mathbb{P}^{r-1}_{\mathbb{Z}}.$$

Let $\Phi_j = X_i^{-d_j} F_j$. Then $Y \times_{T \times \mathbb{P}^\vee} p^{-1}(D_+(X_i)) \times \mathbb{P}^\vee$ is defined in $p^{-1}(D_+(X_i)) \times \mathbb{P}^\vee$ by the equation $\Phi_1 T_1 + \cdots + \Phi_r T_r = 0$. Hence the closed subscheme $\Delta_Y \times_{T \times \mathbb{P}^\vee} p^{-1}(D_+(X_i)) \times \mathbb{P}^\vee$ is defined in $p^{-1}(D_+(X_i)) \times \mathbb{P}^\vee$ by the equations of the row vectors

$$(\Phi_1, \ldots, \Phi_r) = 0, (T_1, \ldots, T_r) J(\Phi_1, \ldots, \Phi_r) = 0$$

where $J(\Phi_1, \ldots, \Phi_r) = \left(\frac{\partial \Phi_j}{\partial x_l}\right)_{0 \le l \le n, l \ne i, 1 \le j \le r}$.

Lemma 1.11. *The restriction of the morphism $T \times \mathbb{P}^\vee \to \mathbb{P}^n \times \mathbb{P}^\vee$ to Δ_Y induce a surjective morphism $\delta : \Delta_Y \to \Delta_X$. In particular, Δ_X is irreducible.*

Proof. Recall that $\pi : X \to \mathbb{P}^\vee$ is the universal family. Let k be an algebraically closed field. Let $(x, s) \in \mathbb{P}^n_k \times \mathbb{P}^\vee_k$ be a closed point. Then the point $s \in \mathbb{P}^\vee_k$ corresponds to a sequence of homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ of coefficients in k. We assume $x \in D_+(X_i)_k \subset \mathbb{P}^n(k)$ for a fixed i $(0 \le i \le n)$ and write $\phi_j = X_i^{-d_j} f_j$. Then (x, s) is in $(\Delta_X)_k$ if and only if $\phi_1(x) = \cdots = \phi_r(x) = 0$ and the Jacobian matrix $J(\phi_1, \ldots, \phi_r)$ has rank $< r$ at x. This is equivalent to that $\phi_1(x) = \cdots = \phi_r(x) = 0$ and there exists a non-trivial vector value $(t_1, \ldots, t_r) \in k^r - \{0\}$ such that

$$(t_1, \ldots, t_r) J(\phi_1(x), \ldots, \phi_r(x)) = 0.$$

Thus Δ_Y maps onto Δ_X as sets.

Since $\Delta_Y = \mathbb{P}(\mathcal{N})$ is a projective space bundle over T, it is reduced and irreducible. Hence the map $\delta : \Delta_Y \to \Delta_X$ as sets induces a morphism of schemes, and its image is irreducible. □

Recall that the projective dual $T^\vee$ is the image of $\varphi : \Delta_Y (= \mathbb{P}(\mathcal{N})) \to \mathbb{P}^\vee$ with the reduced induced closed subscheme structure.

Corollary 1.12. (c.f. [1, Proposition 3.1]) *The scheme D_X is isomorphic to $T^\vee$. In particular, D_X is irreducible.*

Proof. The latter assertion follows from Proposition 1.4. □

Let $\psi : Y \to \mathbb{P}^\vee$ denote the canonical morphism. By [3, Exposé XV, Corollaire 1.3.4], there exists an open subset W_Y of Δ_Y consisting of points w such that w is an ordinary quadratic singularity in the fiber $Y_{\psi(w)}$.

Further, let $\pi : X \to \mathbb{P}^\vee$ be the canonical morphism. In the same way as above, there exists an open subset W_X of Δ_X consisting of points u such that u is an ordinary quadratic singularity in the fiber $X_{\pi(u)}$.

Lemma 1.13. *For every algebraically closed field k, the geometric fiber $(W_X)_k$ is dense in $(\Delta_X)_k$.*

Proof. Let l $(1 \le l \le r)$ be an integer such that $d_l \ge 2$. Let $f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_r$ be homogeneous polynomials of degrees $d_1, \ldots, d_{l-1}, d_{l+1}, \ldots, d_r$ of coefficients in k such that the intersection

$$Z = V((f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_r)) \subset \mathbb{P}^n_k$$

is smooth of dimension $n-r+1$. Then by Proposition 1.5.1, there exists a homogeneous polynomial f_l of degree d_l such that the intersection $Z \cap (f_l = 0) = V((f_1, \ldots, f_r))$ has an ordinary quadratic singularity at a point $x \in \mathbb{P}^n_k$. If we denote the closed point corresponding to $(f_1, \ldots, f_r)$ by $s \in \mathbb{P}^\vee_k$, the closed point $(x, s) \in \mathbb{P}^n_k \times \mathbb{P}^\vee_k$ belongs to $(W_X)_k$.

Since $(\Delta_X)_k$ is irreducible by Lemma 1.11, $(W_X)_k$ is dense. □

Proposition 1.14. *Let k be an algebraically closed field.*

1. The inverse image $\delta^{-1}((W_X)_k)$ is included in $(W_Y)_k$. In particular, the open set $(W_Y)_k$ is dense in $(\Delta_Y)_k$.

2. Assume that $n-r$ is even or char $k \neq 2$. Let $(W'_Y)_k$ be the subset of $(W_Y)_k \subset (\Delta_Y)_k$ consisting of the images of geometric points w of $(W_Y)_k$ that is a unique singular point in the geometric fiber $Y_{\psi(w)}$.

Then, $(W'_Y)_k$ is the maximum open subscheme of Δ_Y where the restriction of the morphism $\psi|_{(\Delta_Y)_k} : (\Delta_Y)_k \to \mathbb{P}^\vee_k$ is an immersion. Consequently, the canonical morphism $(\Delta_Y)_k \to (D_X)_{k,\mathrm{red}}$ to the maximum reduced subscheme of $(D_X)_k = T^\vee_k$ is birational.

Proof. 1. Let k be an algebraically closed field. By Lemma 1.13, the open set $(W_X)_k$ is not empty. Let $(x, s) \in (W_X)_k$ be a point and let $(f_1, \ldots, f_r)$ be homogeneous polynomials of degrees $d_1, \ldots, d_r$ of coefficients in k corresponding to the s. Then by $1 \Leftrightarrow 2$ in Proposition 1.7, there exists an integer l $(1 \le l \le r)$ such that the intersection $V((f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_r)) \subset \mathbb{P}^n_k$ is smooth of dimension $n - r + 1$ and the intersection $V((f_1, \ldots, f_r)) = X_s$ has an ordinary quadratic singularity at x. We may assume $x \in D_+(X_i)_k$ for some i $(0 \le i \le n)$.

Let us write $\phi_j = X_i^{-d_j} f_j$ for $1 \le j \le r$. Then the Jacobian matrix $J(\phi_1, \dots, \phi_{l-1}, \phi_{l+1}, \dots, \phi_r)$ has the full rank $(= r-1)$ at x. Hence the equation $(T_1, \dots, T_r)J(\phi_1(x), \dots, \phi_r(x)) = 0$ has one dimensional roots space. Thus there exists a unique point $y \in Y_s$ which maps to $x \in X_s$.

Further, if $(t_1, \dots, t_r) \in k^r - \{0\}$ is a non-trivial root of the above equation, we have $t_l \neq 0$.

We show that the point y is an ordinary quadratic singularity of the fiber Y_s. The open subscheme $Y_s \times_{T_k} p^{-1}(D_+(X_i)_k)$ of Y_s is defined in $p^{-1}(D_+(X_i)_k)$ by the equation $\phi_1 T_1 + \ldots + \phi_r T_r = 0$. On the other hand, by the assumptions on $\phi_1, \dots, \phi_r$, there is an isomorphism of complete local rings

$$\widehat{\mathcal{O}}_{\mathbb{P}^n_k, x} \cong k[[x_1, \dots, x_n]]$$

such that $\phi_j \mapsto x_j$ $(1 \le j < l)$, $\phi_j \mapsto x_{j-1}$ $(l < j \le r)$ and $\phi_l \mapsto g(x_r, \dots, x_n)$ where $g = q(x_r, \dots, x_n) +$ (terms of degree > 2) with q being an ordinary quadratic form. Hence we have an isomorphism

$$\widehat{\mathcal{O}}_{Y_s, y} \cong k[[x_1, \dots, x_n, \tau_1, \dots, \tau_{r-1}]]/(x_1\tau_1 + \cdots + x_{r-1}\tau_{r-1} + g(x_r, \dots, x_n)).$$

Since the degree 2 part of the series $x_1\tau_1 + \cdots + x_{r-1}\tau_{r-1} + g(x_r, \dots, x_n)$ is an ordinary quadratic form, the point y is an ordinary quadratic singularity in Y_s. Thus the point $(y, s) \in T_k \times \mathbb{P}^\vee_k$ belongs to $(W_Y)_k$, and hence $\delta^{-1}((W_X)_k) \subset (W_Y)_k$.

The latter assertion follows from that $(\Delta_Y)_k$ is irreducible.

2. By assertion 1 and [3, Exposé XVII, Proposition 3.3], the morphism $\varphi : (\Delta_Y)_k \to \mathbb{P}^\vee_k$ is generically unramified. Hence, the assertion follows from [3, Exposé XVII, Proposition 3.5]. ☐

Corollary 1.15. ([1, Lemme 4.4.(ii)]) *1. Let k be an algebraically closed field. We further assume that char $k \neq 2$ or $n - r$ is odd. Then the subscheme $(D_X)_k$ is irreducible and of codimension one in $\mathbb{P}^\vee_k$.*

2. The subscheme D_X is irreducible and of codimension one in $\mathbb{P}^\vee$.

Proof. 1. The scheme $(\Delta_Y)_k$ is a projective space bundle over T_k, hence irreducible. Hence its image $T^\vee_k = (D_X)_k$ is also irreducible.

Since the dimension of the irreducible scheme $(\Delta_Y)_k$ equals to the dimension of $\mathbb{P}^\vee_k$ minus one, its image $T^\vee_k = (D_X)_k$ is of codimension one by Proposition 1.14.2. Thus the closed subscheme $(D_X)_k \subset \mathbb{P}^\vee_k$ is irreducible and of codimension one.

2. Since the scheme Δ_Y over $\mathbb{Z}$ is a projective space bundle over T, the irreducibility of D_X follows from the same way as in 1. The latter assertion follows from applying 1 to $k = \overline{\mathbb{Q}}$. ☐

Definition 1.16. *By Corollary 1.15.2 , there exists a geometrically irreducible polynomial in $(C_I^{(j)})_{1 \le j \le r, |I| = d_j}$ with coefficients in $\mathbb{Z}$ uniquely* *defined up to* ± 1*, such that it defines the closed subscheme* $D_X \subset \mathbb{P}^\vee$*. We call this homogeneous polynomial defined up to sign the* discriminant of complete intersetions *and we denote it by* $\mathrm{disc}(F_1, \ldots, F_r)$*.*

By specialization, the definition of the discriminant gives a meaning to $\mathrm{disc}(f_1, \ldots, f_r)$ for every homogeneous polynomials $f_1, \ldots, f_r$ in $n+1$ variables over a commutative ring R. Then the discriminant satisfies the following smoothness criterion.

Proposition 1.17. *Let* $f_1, \ldots f_r$ *be homogeneous polynomials of degrees* $d_1, \ldots, d_r$ *in* $n+1$ *variables of coefficients in a commutative ring* R*. Then, the discriminant* $\mathrm{disc}(f_1, \ldots, f_r)$*, defined up to a sign, is invertible in* R *if and only if the corresponding intersection* $V((f_1, \ldots, f_r))$ *is a smooth complete intersection in the projective space* $\mathbb{P}^n_R$ *over* R*.*

We deduce the irreducibility of the reduction of the discriminant modulo p.

Proposition 1.18. (c.f. [1, Théorème 1.7]) *Let* p *be a prime. Except for* $p = 2$ *and* $n - r$ *being even, the polynomial* $\mathrm{disc}(F_1, \ldots, F_r) \mod p$ *is geometrically irreducible in the* C_I*. Consequently, the canonical morphism* $(\Delta_Y)_{\mathbb{F}_p} \to (D_X)_{\mathbb{F}_p}$ *is birational.*

Proof. Since $(\Delta_Y)_{\overline{\mathbb{F}}_p}$ is a projective space bundle over $T_{\overline{\mathbb{F}}_p}$, it is reduced and irreducible. Hence by the same proof as of [8, Proposition 2.11], the fiber $(D_X)_{\mathbb{F}_p} = T^\vee_{\mathbb{F}_p}$ is a divisor of $\mathbb{P}^\vee_{\mathbb{F}_p}$ defined by a geometrically irreducible polynomial.

The latter assertion follows from Proposition 1.14.2 and that $(D_X)_{\mathbb{F}_p}$ is reduced. □

1.19. **Properties of Δ_X.** Let k be an algebraically closed field.

Proposition 1.20. *Assume that char* $k \neq 2$ *or* $n - r$ *is odd. Then the morphism* $\delta|_{\delta^{-1}((W_X)_k)} : \delta^{-1}((W_X)_k) \to (W_X)_k$ *is an isomorphism. In particular, the scheme* $(W_X)_k$ *is smooth.*

Proof. First we show that $(W_X)_k$ is reduced. Let $u \in (W_X)_k$ be a closed point. Let A be the henselization of $\mathcal{O}_{\mathbb{P}^\vee_k, \pi(u)}$. Since X is regular, the base change X_A to the henselization is also regular. By [3, Exposé XV, 1.3.2.(i)], there exists an ordinary quadratic form q of $n-r+1$ variables $(x_0, \ldots, x_{n-r})$ with coefficients in A, and an element b in the maximal ideal $\mathfrak{m}_A$ of A, such that the henselization of X at u is isomorphic to a henselization at the origin of the subvariety of $\mathbb{A}^{n-r+1}_A$ defined by the equation $q - b = 0$. Since X_A is regular, we have $b \notin \mathfrak{m}_A^2$. Since q is an ordinary quadratic form, we have the isomorphism $\mathcal{J}_{\mathcal{O}_A} \cong$

$(x_0, \dots, x_{n-r})$. Hence $\mathcal{O}_{(\Delta_X)_A,u} \cong A/(b)$, and it is reduced. Hence $(\Delta_X)_k$ is reduced at u.

Let $(x, s) \in (W_X)_k$ be a closed point. By Proposition 1.14.1, there exists a unique point $y \in Y_s$ which maps to x, which is an ordinary quadratic singularity of the fiber Y_s. Then by [3, Exposé XVII, Proposition 3.3], the morphism $\varphi : (\Delta_Y)_k \to (D_X)_k = T_k^\vee$ is unramified at (y, s). This implies that the morphism $\delta : (\Delta_Y)_k \to (\Delta_X)_k$ is unramified at (y, s). Thus the morphism $\delta|_{\delta^{-1}((W_X)_k)} : \delta^{-1}((W_X)_k) \to (W_X)_k$ is unramified. On the other hand, since the scheme $(\Delta_X)_k$ is irreducible, the reduced open set $(W_X)_k$ is integral. Since the morphism $\delta|_{\delta^{-1}((W_X)_k)}$ is of relative dimension 0 and unramified, thus it is étale. Further, since $\delta|_{\delta^{-1}((W_X)_k)}$ is radiciel, it is an open immersion, and hence an isomorphism. Since $(\Delta_Y)_k$ is projective space over T_k, it is smooth and the open subset $\delta^{-1}((W_X)_k)$ is also smooth. 2. By Proposition 1.7, the morphism $\pi|_{(W'_X)_k} : (W'_X)_k \to (D_X)_k$ is unramified. Further, since this morphism is relative dimension 0 and $(D_X)_k$ is integral by Proposition 1.18, it is étale. On the other hand, it is radiciel by the definition of $(W_X)_k$. Hence it is open immersion. □

Corollary 1.21. *Assume that char $k \neq 2$ or $n - r$ is odd. Then the schemes $(\Delta_X)_k$, $(\Delta_Y)_k$ and $(D_X)_k$ are birational to each other.*

Proof. The assertion follows from Proposition 1.14.2 and Proposition 1.20.1 or 2. □

1.22. **The degree of the discriminant of complete intersections.** To compute the degree of D_X, we define a homogeneous polynomial $P(H, K) \in \mathbb{Z}[H, K]$ by

$$P(H, K) = (d_1 H - K) \cdots (d_r H - K).$$

We put $\bar{d} = d_1 \cdots d_r$, $\check{d}_i = d_1 \cdots d_{i-1} \cdot d_{i+1} \cdots d_r$ for $i = 1, \dots, r$ and $\check{d} = \check{d}_1 + \cdots + \check{d}_r$.

Lemma 1.23. *The degree of the discriminant is the coefficient of $h^n k^{r-1}$ of the element*

(5)
$$\bar{d} \cdot h^r \sum_{i=0}^{n-r} \binom{n+1}{i} (n-i) k^{n-1-i} (-h)^i + \check{d} \cdot h^{r-1} \sum_{i=0}^{n-r+1} \binom{n+1}{i} k^{n-i} (-h)^i$$

in the ring $\mathbb{Z}[h, k]/(h^{n+1}, P(h, k))$ with respect to the basis $(h^i k^j; i = 0, \dots, n, j = 0, \dots, r-1)$.

Proof. The cycle class of $X \subset \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^\vee$ is given by $[X] = c_r(\mathcal{E}(1_{\mathbb{P}^\vee})) \in CH^r(\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^\vee)$ and that of $\Delta_Y \subset (T \times \mathbb{P}^\vee)_X$ is given by

$$[\Delta_Y] = c_n(\Omega^1_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}(1_T, 1_{\mathbb{P}^\vee})) \in CH^n((T \times \mathbb{P}^\vee)_X).$$

Hence, we have $[\Delta_Y] = c_r(\mathcal{E}(1_{\mathbb{P}^\vee})) \cap c_n(\Omega^1_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}(1_T, 1_{\mathbb{P}^\vee})) \in CH^{r+n}(T \times \mathbb{P}^\vee)$. Since the morphism $\Delta_Y \to D_X$ is birational, the class $[D_X] \in CH^1(\mathbb{P}^\vee)$ is the push-forward of $[\Delta_Y]$. Hence the degree of D_X is equal to the degree of the dimension 0-part

$$\{c(\mathcal{E}) \cap c(\Omega^1_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}(1_T))\}_{\dim 0} \in CH_0(T).$$

Let $h = [c_1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1))]$ and $k = [c_1(\mathcal{O}_T(1))]$ denote the classes of hyperplanes. Then, the Chow ring $CH^\bullet(T)$ is $\mathbb{Z}[h,k]/(h^{n+1}, P(h,k))$. For $i = 1, \ldots, r$, we define a homogeneous polynomial $P_i(H,K)$ of degree $i-1$ by requiring that $P(H,K) - (-K)^{r-i-1} P_i(H,K)$ is of degree $\leq r-i$ in K. Since

$$\begin{aligned} c(\mathcal{E}) \cdot c(\Omega_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}(1_T)) &= (1 + d_1 h) \cdots (1 + d_r h) \cdot (1 - h + k)^{n+1} (1 + k)^{-1} \\ &= \sum_{i=1}^{r} P_i(h,k) \cdot (1 - h + k)^{n+1}, \end{aligned}$$

we obtain

$$\{c(\mathcal{E}) \cap c(\Omega_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}(1_T))\}_{\dim 0} = (n+1) P_r(h,k)(k-h)^n + P_{r-1}(h,k) \cdot (k-h)^{n+1}.$$

Since

$$\begin{aligned} K \cdot P_r(H,K) &= \bar{d} \cdot H^r - P(H,K) \\ K^2 \cdot P_{r-1}(H,K) &= \check{d} \cdot H^{r-1} K - \bar{k} \cdot H^r + P(H,K), \end{aligned}$$

the right hand side is equal to

$$\begin{aligned} &(n+1) \left(\bar{d} \cdot h^r \frac{(k-h)^n - (-h)^n}{k} + P_r(h,k)(-h)^n \right) \\ &+ (\check{d} \cdot h^{r-1} k - \bar{d} \cdot h^r) \cdot \frac{(k-h)^{n+1} - ((n+1) - ((n+1)k(-h)^n + (-h)^{n+1})}{k^2} \\ &\qquad + P_{r-1}(h,k) \cdot ((n+1)k(-h)^n + (-h)^{n+1}) \\ &= \bar{d} \cdot h^r \left((n+1) \frac{(k-h)^n - (-h)^n}{k} - \frac{(k-h)^{n+1} - ((n+2)k(-h)^n + (-h)^{n+1})}{k^2} \right) \\ &\qquad + \bar{d} \cdot h^{r-1} \frac{(k-h)^{n+1} - ((n+1)k(-h)^n + (-h)^{n+1})}{k} \\ &\qquad + (n+1)(P_r(h,k) + P_{r-1}(h,k)k)(-h)^n + P_{r-1}(h,k)(-h)^{n+1}. \end{aligned}$$

On the right hand side, the content of the big parantheses in the first line is

$$(n+1)\sum_{i=0}^{n-1}\binom{n}{i}k^{n-1-i}(-h)^i-\sum_{i=0}^{n-1}\binom{n+1}{i}k^{n-1-i}(-h)^i$$

$$=\sum_{i=0}^{n-1}\binom{n+1}{i}(n-i)k^{n-1-i}(-h)^i.$$

Since $P_r(h,k)+P_{r-1}(h,k)k=\check{d}\cdot h^{r-1}$ and $h^{n+1}=0$, the sum of the remaining two lines is

$$\check{d}\cdot h^{r-1}\cdot\sum_{i=0}^{n}\binom{n+1}{i}k^{n-i}(-h)^i.$$

Since the dimension 0-part is the component generated by $h^n\cdot k^{r-1}$ of degree 1 with respect to the decomposition by the basis $(h^ik^j; i=0,\ldots,n, j=1,\ldots,r)$, the assertion follows. □

Corollary 1.24. *If $d_1=\cdots=d_r=d$, the degree of D_X is*

$$(n-r+2)\binom{n+1}{r-1}d^{r-1}(d-1)^{n-r+1}. \tag{6}$$

If $r\geq 2$ and if $d_1-c=d_2=\cdots=d_r=d$, it is the sum of (6) and

$$d^{r-2}\sum_{j=r}^{n+1}\binom{n+1}{j}c^{j-r+1}(d-1)^{n+1-j} \tag{7}$$

$$+(n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j}c^{j-r+1}(d-1)^{n-j}$$

$$+(r-2)d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}(-1)^p(d-1)^{n-p-j}.$$

If $d_1=\cdots=d_r=d>1$, the degree (6) of D_X is strictly positive.

Proof. We put $d_1=d+c$. Then, we have an isomorphism

$$\mathbb{Z}[h,k]/(h^{n+1},P(h,k))\to\mathbb{Z}[h,l]/(h^{n+1},l^{r-1}(l-ch))$$

sending k to $l+dh$. Hence, the degree of D_X is the coefficient of h^nl^{r-1} of the polynomial obtained by substituting $k=l+dh$ in (5). Since

$\bar{d} = d^{r-1}(d+c)$ and $\check{d} = (r-1)d^{r-1} + (d+c)d^{r-2}$, after substituting $k = l + dh$ and $l^r = cl^{r-1}h$, we see that the coefficient of $h^n l^{r-1}$ is :

(8)

$$d^{r-1}(d+c)\sum_{i=0}^{n-r}\binom{n+1}{i}(n-i)\sum_{j=r-1}^{n-i-1}\binom{n-i-1}{j}c^{j-r-1}d^{n-i-j-1}(-1)^i$$

$$+((r-1)d^{r-1}+(d+c)d^{r-2})\sum_{i=0}^{n-r+1}\binom{n+1}{i}\sum_{j=r-1}^{n-i}\binom{n-i}{j}c^{j-r+1}d^{n-i-j}(-1)^i.$$

Since

$$(d+c)\sum_{j=r-1}^{n-i-1}\binom{n-i-1}{j}c^{j-r+1}d^{n-i-j-1}$$

$$=\binom{n-i-1}{r-1}d^{n+1-i-r}+\sum_{j=r}^{n-i}\binom{n-i}{j}c^{j-r+1}d^{n-i-j}$$

and similarly for $(d+c)\sum_{j=r-1}^{n-i}\binom{n-i}{j}c^{j-r+1}d^{n-i-j}$, (8) is equal to

$$(9)\qquad d^r\cdot\sum_{i=0}^{n-r}\binom{n+1}{i}(n-i)\binom{n-i-1}{r-1}d^{n-r-i}(-1)^i$$

$$+rd^{r-1}\cdot\sum_{i=0}^{n-r+1}\binom{n+1}{i}\binom{n-i}{r-1}d^{n+1-m-i}(-1)^i$$

$$+d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{i=0}^{n-j}\binom{n+1}{i}(n+r+1-i)\binom{n-i}{j}d^{n-i-j}(-1)^i$$

$$+d^{r-2}\sum_{j=r}^{n+1}c^{j-r+1}\sum_{i=0}^{n+1-j}\binom{n+1}{i}\binom{n+1-i}{j}d^{n+1-i-j}(-1)^i.$$

The sum of the first two lines in (9) is

$$rd^{r-1}\sum_{i=0}^{n-r-1}\binom{n+1}{i}\binom{n+1-i}{r}d^{n-r+1-i}(-1)^i$$

$$=rd^{r-1}\binom{n+1}{r}\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}d^{n-r+1-i}(-1)^i$$

$$=(n-r+2)\binom{n+1}{r-1}d^{r-1}(d-1)^{n-r+1}.$$

Similarly, the last line in (9) is equal to

$$d^{r-2}\sum_{j=r}^{n+1}\binom{n+1}{j}c^{j-r+1}(d-1)^{n+1-j}.$$

Since

$$\begin{aligned}&\binom{n+1}{i}(n+r+1-i)\binom{n-i}{j}\\ &=(j+1)\binom{n+1}{i}\binom{n+1-i}{j+1}+(r-2)\binom{n+1}{i}\binom{n-i}{j}\\ &=(n+1)\binom{n}{j}\binom{n-j}{i}+(r-2)\sum_{p=0}^{i}\binom{n-p}{i-p}\binom{n-j}{j},\end{aligned}$$

similarly the third line in (9) is equal to

$$\begin{aligned}&(n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j}c^{j-r+1}\sum_{i=0}^{n-j}\binom{n-j}{i}d^{n-i-j}(-1)^i\\ &+(r-2)d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}\sum_{i-p=0}^{n-j-p}\binom{n-j-p}{i-p}d^{n-i-j}(-1)^i\\ &=(n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j}c^{j-r+1}(d-1)^{n-j}\\ &+(r-2)d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}(-1)^p(d-1)^{n-j-p}.\end{aligned}$$

□

Corollary 1.25. *If $n-r$ is even, the degree of D_X is even.*

Proof. If there exists at least 2 indices such that d_i is even, the integers $\bar{d}=d_1\cdots d_r$ and $\check{d}=\check{d}_1+\cdots+\check{d}_r$ are even.

We consider the case where there exists at most 1 index such that d_i is even. By the same argument as in the proof of Corollary 1.24, the congruences on d_i implies a congruence for the degree of D_X. Hence, if every d_i is congruent to 1, then the degree is even if $n-r$ is even by (6).

Assume there exists exactly 1 index such that d_i is even. We may assume that $i=1$, $d\equiv c\equiv 1 \pmod 2$ in (1.24). Then, (7) is congruent to $1+(n+1)+(r-2)(n-r+1) \pmod 2$ and is even if $n-r$ is even. □

1.26. **The discriminant of hypersurfaces.** Let $r = 1$ and fix a positive integers n and $d = d_1$. Let $F = F_1$ denote the universal polynomial of degree d. We consider the resultant

$$\operatorname{res}\left(\frac{\partial F_1}{\partial X_0}, \dots, \frac{\partial F_1}{\partial X_n}\right)$$

of partial derivatives of F. It is a homogeneous polynomial of degree $m = (n+1)(d-1)^n$ in $(C_I)_{|I|=d_1}$ with integral coefficients. If we put

$$a(n,d) = \frac{(d-1)^{n+1} - (-1)^{n+1}}{d},$$

the greatest common divisor of the coefficients is $d^{a(n,d)}$ by [5, Chap. 13.1.D Proposition 1.7].

Definition 1.27. *We call*

$$\operatorname{disc}_d(F) = \frac{1}{d^{a(n,d)}} \operatorname{res}\left(\frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_n}\right)$$

the divided discriminant *of F.*

The relation between the discriminant of a complete intersection and the divided discriminant of a hypersurface is as follows.

Proposition 1.28. *If $r = 1$ and $d_1 = d$, then the discriminant* $\operatorname{disc}(F)$ *defined in Definition 1.16 equals to* $\operatorname{disc}_d(F)$ *up to sign.*

Proof. The assertion follows from Proposition 1.17 and the smoothness criterion [8, Proposition 2.3] of the divided discriminant of hypersurface. □

2. Determinant

Let S be a normal integral scheme over $\mathbb{Z}$ and $f : X \to S$ be a proper smooth morphism of relative even dimension n. For a prime number ℓ invertible in the function field of S, the cup-product defines a non-degenerate symmetric bilinear form on the smooth $\mathbb{Q}_\ell$-sheaf $R^n f_* \mathbb{Q}_\ell(\frac{n}{2})$ on $S[\frac{1}{\ell}]$. Hence the determinant defines a character $\pi_1(S[\frac{1}{\ell}])^{ab} \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$ of the fundamental group, which we denote by $[\det H^n_\ell(X)]$.

Lemma 2.1 ([8, Lemma 3.2])**.** *There exists a unique character*

$$[\det H^n(X)] : \pi_1(S)^{ab} \to \{\pm 1\}$$

such that, for every prime number ℓ invertible in the function field of S, the composition with the map $\pi_1(S[\frac{1}{\ell}])^{ab} \to \pi_1(S)^{ab}$ induced by the open immersion $S[\frac{1}{\ell}] \to S$ gives $[\det H^n_\ell(X)]$.

Corollary 2.2 ([8, Lemma 3.3])**.** *Let X be a proper smooth scheme of even dimension n over a field k. Then, for a prime number ℓ invertible in k, the character $\det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ of Γ_k is independent of ℓ.*

By applying Lemma 2.1 to the universal family of intersections of r hypersurfaces $\pi_U : X_U \to U$, we define $[\det H^{n-r}(X)] \in H^1(U, \mathbb{Z}/2\mathbb{Z})$. Let now k be a field and let $f_j \in S^{d_j}E \otimes k$ $(1 \leq j \leq r)$ be homogeneous polynomials of degrees $d_1, \ldots, d_r$ which define a smooth complete intersection Y in $\mathbb{P}^n_k$. Then, the pull-back in $H^1(k, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\Gamma_k^{ab}, \mathbb{Z}/2\mathbb{Z})$ of $[\det H^{n-r}(X)]$ by the k-valued point of U corresponding to $(f_1, \ldots, f_r)$ is given by the determinant of the orthogonal representation $H^{n-r}(Y_{\bar{k}}, \mathbb{Q}_\ell(\frac{n-r}{2}))$ for a prime number ℓ invertible in k.

The Kummer sequence gives an exact sequence

(10)
$$0 \to \Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times/(\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times)^2 \xrightarrow{\partial} H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Pic}(U_{\frac{1}{2}})[2] \to 0,$$

where we have written $U_{\frac{1}{2}}$ instead of $U_{\mathbb{Z}[\frac{1}{2}]}$ for typographical reasons, and $\mathrm{Pic}(U_{\frac{1}{2}})[2]$ denotes the subgroup of $\mathrm{Pic}(U_{\frac{1}{2}})$ killed by 2.

Theorem 2.3. *Let $n \geq 1$ and $d_1, \ldots, d_r \geq 1$ be integers. Assume that $n - r$ is even and that $d_j \geq 2$ for an index j $(1 \leq j \leq r)$.*

1. Let $m = \deg(\mathrm{disc}(F_1, \ldots, F_r))$. Then there exists unique choice of sign of the polynomial $\mathrm{disc}(F_1, \ldots, F_r)$ such that, there exist homogeneous polynomials $A \in S^{\frac{m}{2}}(V^\vee)$ and $B \in S^m(V^\vee)$ such that $\mathrm{disc}(F_1, \ldots, F_r) = A^2 + 4B$.

We denote this polynomial by $\mathrm{disc}_\sigma(F_1, \ldots, F_r)$.

2. The square roots of $\mathrm{disc}_\sigma(F_1, \ldots, F_r)$ define a $\mathbb{Z}/2\mathbb{Z}$-torsor on $U_{\frac{1}{2}}$. We denote by $[\mathrm{disc}_\sigma(F_1, \ldots, F_r)]$ the class of this torsor in $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})$. Then
$$[\det H^{n-r}(X)] = [\mathrm{disc}_\sigma(F_1, \ldots, F_r)]$$
in $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})$.

Thus by a standard specialization argument, Theorem 2.3 implies Theorem 0.1.

Proof. We have an exact sequence

$$(11) \quad 0 \to \Gamma(\mathbb{P}^\vee_{\frac{1}{2}}, \mathcal{O})^\times \to \Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times \to \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}}) \to \mathrm{Pic}(U_{\frac{1}{2}}) \to 0.$$

The Picard group $\mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}})$ is canonically identified with $\mathbb{Z}$ by the generator $[\mathcal{O}(1)]$. Then, the map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}})$ is identified with the multiplication $m = \deg(\mathrm{disc}(F_1, \ldots, F_r)) \neq 0$ since it sends 1 to the class

$[\mathcal{O}(m)]$. Thus, we have

$$\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times = \Gamma(\mathbb{P}^\vee_{\frac{1}{2}}, \mathcal{O})^\times = \mathbb{Z}\left[\frac{1}{2}\right]^\times = \langle -1, 2\rangle.$$

It also follows from (11) that $\mathrm{Pic}(U_{\frac{1}{2}}) \cong \mathbb{Z}/m\mathbb{Z}$. Since m is even by Corollary 1.25, this shows $\mathrm{Pic}(U_{\frac{1}{2}})[2] \cong \mathbb{Z}/2\mathbb{Z}$. Thus, by (10), we have $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \cong \langle -1, 2\rangle/\langle -1, 2\rangle^2 \oplus \mathrm{Pic}(U_{\frac{1}{2}})[2] \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Recall that D_X is an irreducible divisor in $\mathbb{P}^\vee$. Let $\bar{\xi}$ be a geometric generic point of D_X and let $I_{\bar{\xi}}$ denote the absolute Galois group of the fraction field of the strict henselization $\mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}$. Since the profinite group $I_{\bar{\xi}}$ is isomorphic to $\widehat{\mathbb{Z}}$, we have $\mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

We recall that $\mathrm{disc}(F_1, \ldots, F_r)$ is defined up to sign as the defining polynomial of D_X. Then the square roots of $\mathrm{disc}(F_1, \ldots, F_r)$ defines a class of $\mathbb{Z}/2\mathbb{Z}$-torsor on $U_{\frac{1}{2}}$ up to sign. We denote by $[\pm \mathrm{disc}]$ this class of torsor in $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})/\langle -1\rangle$.

Since we have $\overline{\mathbb{Q}} \subset \mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}$, the restriction map $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$ induces a map $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})/\langle -1\rangle \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$. We show that the images of $[\det H^{n-r}(X)]$ and $[\pm \mathrm{disc}]$ under this map are both the unique non-trivial element. For the latter $[\pm \mathrm{disc}]$, this follows from that $\mathrm{disc}(F_1, \ldots, F_r)$ is the defining polynomial of the divisor D_X.

For the former $[\det H^{n-r}(X)]$, this follows from the same argument as that in the proof of [8, Theorem 3.5]. Let $\bar{\eta}$ denote the geometric generic point of $\mathrm{Spec}\, \mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}$. We show that the character $\det H^{n-r}(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ of $I_{\bar{\xi}}$ is the unique non-trivial character of order 2. By Proposition 1.20.2, the geometric fiber $X_{\bar{\xi}}$ has a unique singular point which is an ordinary quadratic singularity in $X_{\bar{\xi}}$. Hence, by the Picard-Lefschetz formula [3, Exposé XV, Théorème 3.4 (ii)], we have an exact sequence

$$(12) \qquad 0 \to H^{n-r}(X_{\bar{\xi}}, \mathbb{Q}_\ell) \to H^{n-r}(X_{\bar{\eta}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(\frac{n-r}{2})$$
$$\to H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell) \to H^{n-r+1}(X_{\bar{\eta}}, \mathbb{Q}_\ell) \to 0$$

of ℓ-adic representation of the inertia group $I_{\bar{\xi}}$. Further, since X is regular, the base change $X_{\mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}}$ to the strict henselization is also regular. Hence by [3, Exposé XV, Théorème 3.4 (iii)], the inertia group $I_{\bar{\xi}}$ acts on $\mathbb{Q}_\ell\left(\frac{n-r}{2}\right)$ via the unique non-trivial character $I_{\bar{\xi}} \to \{\pm 1\}$. Since $I_{\bar{\xi}}$ acts trivially on $H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$ and on $H^{n-r}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$, the map $\mathbb{Q}_\ell(\frac{n-r}{2}) \to H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$ in (12) is the zero-map and the character $\det H^{n-r}(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ of $I_{\bar{\xi}}$ is non-trivial.

The composition map

$$\Gamma(U_{\frac{1}{2}}, \mathcal{O})^{\times}/(\Gamma(U_{\frac{1}{2}}, \mathcal{O}^{\times})^2 \to H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$$

is 0 since the strict henselization $\mathcal{O}_{\mathbb{P}^\vee,\bar{\xi}}$ contains $\overline{\mathbb{Q}}$ as a subfield. By (11) we thus have a map $\mathrm{Pic}(U_{\frac{1}{2}})[2] \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$.

Since the images of $[\det H^{n-r}(X)]$ and $[\pm\,\mathrm{disc}]$ in $\mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$ are non-trivial, the map $\mathrm{Pic}(U_{\frac{1}{2}})[2] \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism of groups of order 2. Further by (11), the difference $[\pm\,\mathrm{disc}] - [\det H^{n-r}(X)]$ is in the image of the map

$$\Gamma(U_{\frac{1}{2}}, \mathcal{O})^{\times} = \mathbb{Z}\left[\frac{1}{2}\right]^{\times} \to H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}).$$

Therefore, $[\det H^{n-r}(X)]$ equals either $[\pm\,\mathrm{disc}]$ or $[\pm 2\,\mathrm{disc}]$. We show that the latter case is not possible.

Let K be the local field of $\mathbb{P}^\vee$ at the generic point of the fiber $\mathbb{P}^\vee_{\mathbb{F}_2}$. Then, the character $[\det H^{n-r}(X)]$ induces an unramified character of the absolute Galois group G_K. On the other hand, the class $[\pm 2\,\mathrm{disc}]$ corresponds to a totally ramified quadratic extension of K. Hence we obtain $[\det H^{n-r}(X)] = [\pm\,\mathrm{disc}]$.

Hence there exists a unique homogeneous geometrically irreducible polynomial $\mathrm{disc}_\sigma(F_1, \dots, F_r)$ of degree m such that $\mathrm{disc}_\sigma(F_1, \dots, F_r)$ equals $\mathrm{disc}(F_1, \dots, F_r)$ up to sign and the $\mathbb{Z}/2\mathbb{Z}$-torsor defined by the square roots of disc_σ on $U_{\frac{1}{2}}$ is isomorphic to $[\det H^{n-r}(X)]$.

It remains to show that there exists a homogeneous polynomial A of degree $\frac{m}{2}$ such that $\mathrm{disc}_\sigma(F_1, \dots, F_r) \cong A^2 \pmod 4$.

We use the following fact.

Lemma 2.4. [8, Lemma 4.1] *Let K be a complete discrete valuation field such that 2 is a uniformizer. Let $u \in \mathcal{O}_K^\times$ be a unit which is not a square and let L denote the quadratic extension $K(\sqrt{u})$.*

1. The extension L is unramified over K if and only if there exists a unit $v \in \mathcal{O}_K^\times$ such that $u \cong v^2 \pmod 4$.

2. Assume that the extension L is unramified over K. Then, for every unit v satisfying $u \cong v^2 \pmod 2$, we have $u \cong v^2 \pmod 4$. Further, the corresponding residue field extension is given by the Artin-Schreier equation $t^2 + t = w$, where w is the image of $\frac{1}{4}(uv^{-2} - 1)$ in the residue field.

Let K be the local field of $\mathbb{P}^\vee$ at the generic point ν of the fiber $\mathbb{P}^\vee_{\mathbb{F}_2}$. Namely, K is the fraction field of the completion of the local ring $\mathcal{O}_{\mathbb{P}^\vee,\nu}$. The residue field $F = \kappa(\nu)$ is the function field of $\mathbb{P}^\vee_{\mathbb{F}_2}$. Take a global section $A_1 \in \Gamma(\mathbb{P}^\vee, \mathcal{O}(\frac{m}{2}))$ not divisible by 2. Then the germ of A_1 generates the stalk $(\mathcal{O}(\frac{m}{2}))_\nu$. On the other hand, since the polynomial $\mathrm{disc}_\sigma(F_1,\dots,F_r)$ is not divisible by 2, its germ generates the stalk $(O(m))_\nu$. Hence the ratio $\mathrm{disc}_\sigma(F_1,\dots,F_r)/A_1^2$ is a unit in $\mathbb{O}_{\mathbb{P}^\vee,\nu}$.

By Theorem 2.3.2 we have

$$[\det H^{n-r}(X)] = [\mathrm{disc}_\sigma(F_1,\dots,F_r)]$$

in $H^1(U_{\mathbb{Z}[\frac{1}{2}]},\mathbb{Z}/2\mathbb{Z})$. Since the class $[\det H^{n-r}(X)]$ is the restriction of a class of $H^1(U,\mathbb{Z}/2\mathbb{Z})$, the extension of K generated by the square root of $\mathrm{disc}_\sigma(F_1,\dots,F_r)/A_1^2 \in K^\times$ is an unramified extension. Hence by Lemma 2.4.1, there exists a unit $v \in \mathcal{O}_K^\times$ such that $\mathrm{disc}_\sigma(F_1,\dots,F_r) \equiv v^2 \cdot A_1^2 \pmod 4$.

We consider the germ $\bar{A} = v \cdot A_1 \pmod 2$ of the stalk of $\mathcal{O}_{\mathbb{P}^\vee_{\mathbb{F}_2}}(\frac{m}{2})$ at the generic point. Since its square is a germ of polynomial, the germ $\bar{A}$ has the same property and it defines a global section $\Gamma(\mathbb{P}^\vee_{\mathbb{F}_2},\mathcal{O}(\frac{m}{2}))$. Let us choose a lifting $A \in \Gamma(\mathbb{P}^\vee,\mathcal{O}(\frac{m}{2}))$ of this section. Since $\mathrm{disc}_\sigma(F_1,\dots,F_r)/A^2 \equiv 1 \pmod 2$, we have $\mathrm{disc}_\sigma(F_1,\dots,F_r)/A^2 \equiv 1 \pmod 4$ by Lemma 2.4.2. Namely, the difference $\mathrm{disc}_\sigma(F_1,\dots,F_r) - A^2$ is divisible by 4 at ξ and hence divisible on $\mathbb{P}^\vee$. □

2.5. **The determinant in characteristic** 2. We denote $[B \cdot A^{-2}]$ by the class in $H^1(U_{\mathbb{F}_2},\mathbb{Z}/2\mathbb{Z})$ defined by $t^2+t = B\cdot A^{-2}$, and we denote again $[\det H^{n-r}(X)] \in H^1(U_{\mathbb{F}_2},\mathbb{Z}/2\mathbb{Z})$ by the class of the pull-back of $[\det H^{n-r}(X)] \in H^1(U,\mathbb{Z}/2\mathbb{Z})$.

Theorem 2.6. *Let $n, d \geq 2$ be even numbers. Then we have*

$$[B \cdot A^{-2}] = [\det H^{n-r}(X)]$$

in $H^1(U_{\mathbb{F}_2},\mathbb{Z}/2\mathbb{Z})$.

Proof. By Theorem 2.3, the pull-back of $[\det H^{n-r}(X)]$ in $H^1(U_{\mathbb{Z}[\frac{1}{2}]},\mathbb{Z}/2\mathbb{Z})$ is defined by the square roots of

$$\mathrm{disc}_\sigma(F_1,\dots,F_r) \in \Gamma(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})^\times/(\Gamma(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})^\times)^2.$$

Since the polynomial $\mathrm{disc}_\sigma(F_1,\dots,F_r)$ is not divisible by 2, the polynomial A is also not divisible by 2.

Let F be the function field of $\mathbb{P}^\vee_{\mathbb{F}_2}$ as in the proof of Theorem 2.3. Then the restriction map $H^1(U_{\mathbb{F}_2},\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\Gamma_F^{ab},\mathbb{Z}/2\mathbb{Z})$ is injective. By Theorem 2.3 and Lemma 2.4, the classes $[B\cdot A^{-2}]$ and $[\det H^{n-r}(X)]$ maps to the same element in $\mathrm{Hom}(\Gamma_F^{ab},\mathbb{Z}/2\mathbb{Z})$, and hence we have $[B \cdot A^{-2}] = [\det H^{n-r}(X)]$ in $H^1(U_{\mathbb{F}_2},\mathbb{Z}/2\mathbb{Z})$. □

3. The discriminant of the complete intersection of two quadrics

In this section, we give an explicit presentation of the discriminant of the complete intersection of two quadrics, by using the discriminant of a quadric and that of a binary form.

Let F denote the universal homogeneous polynomial of degree $d \geq 2$. Recall that the divided discriminant $\mathrm{disc}_d(F)$ of a hypersurface is defined in 1.26.

Proposition 3.1. *Let $n \geq 1$ and $d \geq 2$ be integers. We assume that n is odd and define the sign $\epsilon(n,d) = \pm 1$ by*

$$\epsilon(n,d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } d \text{ is odd} \\ (-1)^{\frac{d}{2}\frac{n+1}{2}} & \text{if } d \text{ is even.} \end{cases}$$

Then, we have

$$\mathrm{disc}_\sigma(F) = \epsilon(n,d) \cdot \mathrm{disc}_d(F).$$

Proof. By Proposition 1.28, the equality is true up to a sign. For the sign, the assertion follows from Theorem 2.3 and [8, Theorem 4.2]. $\square$

3.2. **Quadrics.** Let $r = 1$ and $d = 2$. Let $F = \sum_{0 \leq i \leq j \leq n} C_{ij} X_i X_j$ and $X = (X_0, \ldots, X_n)$. Let $A \in M_{n+1}(S^\bullet((S^2 E)^\vee))$ be the symmetric matrix such that $XA^tX = 2F$. Then the resultant of the partial derivatives is

$$\mathrm{res}\left(\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n}\right) = \det A.$$

We have $a(n,2) = (1 - (-1)^{n+2})/2$. Thus we have

$$\mathrm{disc}_d(F) = \begin{cases} 2^{-1} \det A & \text{if } n-1 \text{ is odd} \\ \det A & \text{if } n-1 \text{ is even,} \end{cases}$$

$$\deg(\mathrm{disc}_d(F)) = n + 1. \tag{13}$$

3.3. **Binary forms.** Let $n = 1$ and $r = 1$. Let $F = C_0 X_0^d + C_1 X_0^{d-1} X_1 + \cdots + C_d X_1^d$ be the universal binary polynomial of degree $d \geq 2$. The divided discriminant $\mathrm{disc}_d(F)$ is a homogeneous polynomial in (C_i) of degree $m = 2d - 2$ and the sign $\epsilon(1,d)$ is $(-1)^{d(d-1)/2}$. It is well known that the discriminant $\mathrm{disc}_d(F)$ is explicitly presented by the Sylvester's determinant.

If the binary form F is decomposed as $F = \prod_{i=1}^{d}(u_i T_0 - v_i T_1)$, by [8, (5.1.1)], we have

$$\operatorname{disc}_d(F) = \prod_{i \neq j}(u_i v_j - u_j v_i). \tag{14}$$

Further we have $a(1,d) = d-2$ and $\operatorname{disc}_d(F) = d^{-(d-2)} \operatorname{res}\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}\right)$.

We will use following properties of the resultant of two binary forms to calculate the discriminant of a binary form. Let l, $m \geq 1$ be integers and

$$\begin{aligned} G(t_0,t_1) &= a_0 t_0^l + a_1 t_0^{l-1} t_1 + \cdots + a_l t_1^l, \\ H(t_0,t_1) &= b_0 t_0^m + b_1 t_0^{m-1} t_1 + \cdots + b_m t_1^m \end{aligned} \tag{15}$$

be binary forms of degrees l and m over an algebraically closed field k. Further, let

$$g(t) = a_0 + a_1 t + \cdots + a_l t^l, h(t) = b_0 + b_1 t + \cdots + b_m t^m \tag{16}$$

be polynomials in one variable corresponding to (15). They are of degrees at most l and m. Then the resultant $\operatorname{res}(G,H)$ of binary forms equals to the resultant $\operatorname{res}_{l,m}(g,h)$ of polynomials in one variable.

Let $x_1, \ldots, x_l$ be the roots of g and $y_1, \ldots, y_l$ be the roots of h. By [5, Ch12.(1.3)], if $a_l \neq 0$ and $b_m \neq 0$, we have the product formula

$$\operatorname{res}_{l,m}(g,h) = a_l^m b_m^l \prod_{i,j}(x_i - y_j). \tag{17}$$

Further, by [5, Ch12. p400], if $l' \geq l$ we have

$$\operatorname{res}_{l',m}(g,h) = b_m^{l'-l} \operatorname{res}_{l,m}(g,h). \tag{18}$$

3.4. **Intersection of two quadrics.** In this subsection, we consider the case $r = 2$ and $d_1 = d_2 = 2$. Then $V = \Gamma(\mathbb{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(2))$ and we identify the dual $V^\vee$ with the module of pairs of quadratic forms over $\mathbb{Z}$.

Let k be an algebraically closed field. Let $(f_1, f_2) \in V_k^\vee = V^\vee \otimes k$ be pair of quadratic forms of coefficients in k and let $X_{(f_1,f_2)} = V((f_1, f_2)) \subset \mathbb{P}_k^n$ be the intersection of the two quadrics defined by f_1, f_2.

The following proposition is due to M. Reid.

Proposition 3.5. [7, Proposition 2.1] *Let k be an algebraically closed field of characteristic $\neq 2$. Let $(f_1, f_2) \in V_k^\vee$ be non-zero homogeneous polynomials of degree 2 with coefficients in k. Let $M_1, M_2 \in M_{n+1}(k)$ be symmetric matrices such that $XM_1{}^tX = 2f_1$ and $XM_2{}^tX = 2f_2$ where $X = (X_0, \ldots, X_n)$. Then the following two conditions are equivalent.*

1. The intersection $X_{(f_1,f_2)} = V((f_1, f_2))$ *is smooth of dimension* $n-2$.

2. The binary form $\det(t_1M_1+t_2M_2)$ *is not identically zero, and has at most simple roots. In other words, if this binary form is decomposed as* $\det(t_1M_1+t_2M_2) = \prod_{i=1}^{n+1}(u_it_i - v_it_2)$*, we have* $u_iv_j \neq u_jv_i$ *for* $0 \leq i$, $j \leq n$, $i \neq j$.

Let $F_1 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(1)} X_iX_j$ and $F_2 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(2)} X_iX_j$ be universal homogeneous polynomials of degree 2. Let $R = \mathbb{Z}[t_1, t_2]$ be the polynomial ring with variables t_1, t_2. We see $t_1F_1+t_2F_2$ as a quadratic form with variables $X_0, \dots, X_n$ and denote its divided discriminant by $\mathrm{disc}_d(t_1F_2 + t_2F_2) \in R[(C_{ij}^{(l)})]$. Further we see $\mathrm{disc}_d(t_1F_2 + t_2F_2)$ as a binary form with variables t_1, t_2 and denote its divided discriminant by $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)) \in \mathbb{Z}[(C_{ij}^{(l)})]$.

Theorem 3.6. *1. Let* $n \geq 2$ *be an even integer. Then*

$$\mathrm{disc}_\sigma(F_1, F_2) = (-1)^{\frac{n}{2}} \mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)).$$

2. Let $n \geq 3$ *be an odd integer. Then the equation*

$$\mathrm{disc}(F_1, F_2) = 2^{-2(n+1)} \mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2))$$

holds up to sign.

Proof. Let k be an algebraically closed field of char $k \neq 2$. Let $(f_1, f_2) \in V_k^\vee$ be a pair of non-zero homogeneous polynomials of degree 2 and let $M_1, M_2 \in M_{n+1}(k)$ be corresponding symmetric matrices. By Proposition 3.5 and 3.2, the closed subvariety $X_{(f_1,f_2)}$ in $\mathbb{P}_k^n$ defined by the zeros of the two polynomials f_1, f_2 is smooth of dimension $n-2$ if and only if the discriminant $\mathrm{disc}_d(t_1F_1 + t_2F_2)$ is not identically zero and has only simple roots. Further by 3.3. (14), this condition is equivalent to $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)) \neq 0$. Hence we have the equality $V(\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)))_{\mathbb{Z}[\frac{1}{2}]} = (D_X)_{\mathbb{Z}[\frac{1}{2}]}$ as subsets of $\mathbb{P}^\vee_{\mathbb{Z}[\frac{1}{2}]}$. By 3.3, the degrees of the two polynomials $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2))$ and $\mathrm{disc}(F_1, F_2)$ are both $2n(n+1)$. The discriminant $\mathrm{disc}_d(F_1, F_2)$ is geometrically irreducible in characteristic 0 and the greatest common divisor of its coefficients is 1, and hence $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1+t_2F_2))$ is the multiple by a non zero integer of $\mathrm{disc}(F_1, F_2)$. By the above equality as sets, for any prime $p \neq 2$ the polynomial $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2))$ mod p is not identically zero. Thus there exists an integer $s \geq 0$ such that

$$2^s \mathrm{disc}(F_1, F_2) = \mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)). \tag{19}$$

1. First we show $s = 0$. Let $P' = \mathbb{P}((S^2E)^\vee)$ denote the space of quadrics in $\mathbb{P}^n$ and let let $D' \subset P'$ be the divisor defined by the discriminant of quadrics.

Let $k = \overline{\mathbb{F}}_2$. The pair (f_1, f_2) defines the line $l_{(f_1,f_2)} = \{t_1f_1 + t_2f_2\} \cong \mathbb{P}^1_k$ in the space P'_k. The intersection $l_{(f_1,f_2)} \cap D'_k$ is isomorphic to the the hypersurface in the line $l_{(f_1,f_2)}$ defined by the binary form $\mathrm{disc}_d(t_1f_1 + t_2f_2)$. Hence by the smooth criterion of the discriminant, the value $\mathrm{disc}_d(\mathrm{disc}_d(t_1f_1 + t_2f_2))$ in k is not equals to zero if and only if $l_{(f_1,f_2)} \cap D'_k$ is smooth. Further, this is equivalent to that the line $l_{(f_1,f_2)}$ intersects with D'_k transversally.

By [3, Exposé XVIII, Théorème 2.5], there exists a *Lefschetz pencil* $l \subset P'_k$. Further by [3, Exposé XVIII, Proposition 3.2.10], the line l intersects with D'_k transversally. We take quadratic forms f_1, f_2 corresponding to two different points on $l \subset P'_k$. (In the above notation, we have $l = l_{(f_1,f_2)}$.) Then by (19), we have

$$2^s \,\mathrm{disc}(f_1, f_2) = \mathrm{disc}_d(\mathrm{disc}_d(t_1f_2 + t_2f_2)) \neq 0 \in k = \overline{\mathbb{F}}_2$$

and hence $s = 0$.

Next we calculate the sign. The degree of the polynomial $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1+ t_2F_2))$ is $n+1$, and the sign is $\epsilon(1, n+1) = (-1)^{n/2}$. Thus the assertion follows from Proposition 3.1.

2. We assume that the dimension $n - 2 \geq 1$ of the complete intersection of two quadrics is odd. We define a pair of quadratic forms $(f_1, f_2) \in V^\vee$ over $\mathbb{Z}$ by

$$\begin{cases} f_1 = \displaystyle\sum_{i=1}^{\frac{n-1}{2}} X_{2i-1}X_{2i} + X_n^2 \\ f_2 = X_0^2 + \displaystyle\sum_{i=0}^{\frac{n-1}{2}} X_{2i}X_{2i+1}. \end{cases} \tag{20}$$

Let $X_{(f_1,f_2)}$ be the intersection defined over $\mathbb{Z}$ of the two quadrics f_1 and f_2. Then we show that its base extension $(X_{(f_1,f_2)})_k$ to $k = \overline{\mathbb{F}}_2$ is smooth of dimension $n - 2$. First we show that the dimension of this scheme is $n - 2$. Since f_1 and f_2 are not constant multiple by any element in k each other, it is sufficient to show that the forms f_1 and f_2 are irreducible over k. Let denote $f_1 = \sum_{0 \leq i,j \leq n} c_{ij}X_iX_j$. Assume that f_1 is decomposed as

$$\left(\sum_{k=0}^{n} a_kX_k\right)\left(\sum_{l=0}^{n} b_lX_l\right) = \sum_{0 \leq i,j \leq n} c_{ij}X_iX_j.$$

Then first we have $a_n \neq 0$. Further we have $b_1 \neq 0$. Since $a_1 b_n + a_n b_1 = c_{n,1} + c_{1,n} = 0$, we have $a_1 \neq 0$ and $a_1 b_1 \neq 0$. This implies $a_1 b_1 = c_{11} = 0$ and it is a contradiction. Thus f_1 is irreducible over $\bar{\mathbb{F}}_2$. The irreducibility of f_2 is showed by the same way. Hence the dimension of $X_{(f_1,f_2)}$ is $n-2$.

Next we show the smoothness. The Jacobian of (f_1, f_2) over k is

$$J(f_1, f_2) =$$

$$\begin{pmatrix} 0 & X_2 & X_1 & \cdots & X_{2i-1} & X_{2i+2} & X_{2i+1} & X_{2i+4} & \cdots & X_{n-3} & 0 \\ X_1 & X_0 & X_3 & \cdots & X_{2i+1} & X_{2i} & X_{2i+3} & X_{2i+2} & \cdots & X_{n-1} & X_{n-2} \end{pmatrix}.$$

There exist following (2×2) minor matrices

$$\begin{pmatrix} 0 & X_1 \\ X_1 & X_3 \end{pmatrix}, \begin{pmatrix} X_{2i-1} & X_{2i+1} \\ X_{2i+1} & X_{2i+3} \end{pmatrix} (1 \leq i \leq \frac{n-2}{2}), \tag{21}$$

$$\begin{pmatrix} X_{n-2} & 0 \\ X_{n-4} & X_{n-2} \end{pmatrix}, \begin{pmatrix} X_{2i} & X_{2i+2} \\ X_{2i-2} & X_{2i} \end{pmatrix} (1 \leq i \leq \frac{n-3}{2}). \tag{22}$$

Their determinants are

$$\begin{cases} X_1^2 \\ X_1 X_5 + X_3^2 \\ \vdots \\ X_{2i-1} X_{2i+3} + X_{2i+1}^2 \\ \vdots \\ X_{n-4} X_n + X_{n-2}^2 \end{cases} \tag{23}$$

$$\begin{cases} X_{n-2}^2 \\ X_{n-1} X_{n-5} + X_{n-3}^2 \\ \vdots \\ X_{2i+2} X_{2i-2} + X_{2i+2}^2 \\ \vdots \\ X_4 X_0 + X_2^2. \end{cases} \tag{24}$$

These polynomials do not have any non-trivial common root in k^{n+1}. Hence by the Jacobian criterion, the variety $(X_{(f_1,f_2)})_k$ is smooth. Thus by the smoothness criterion of the discriminant, we have $\mathrm{disc}_d(f_1, f_2) \equiv 1 \mod 2$.

The $(n+1)\times(n+1)$ symmetric matrices corresponding to the quadratic forms f_1, f_2 are

$$M_1 = \begin{pmatrix} 0 & & & & & & & & \\ & 0 & 1 & & & & & \Large O & \\ & 1 & 0 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & 1 & \\ & \Large O & & & & & 1 & 0 & \\ & & & & & & & & 2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2 & 1 & & & & & & & \\ 1 & 0 & & & & & & \Large O & \\ & & 0 & 1 & & & & & \\ & & 1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & 1 & & \\ & & & & & 1 & 0 & & \\ & \Large O & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix}.$$

Since the dimension n is odd, we have

$$
\begin{aligned}
&\mathrm{disc}_d(t_1 f_1 + t_2 f_2) \\
&= \det(t_1 M_1 + t_2 M_2) \\
&= \det \begin{pmatrix}
2t_2 & t_2 & & & & & & & \\
t_2 & 0 & t_1 & & & & & O & \\
 & t_1 & 0 & t_2 & & & & & \\
 & & t_2 & 0 & t_1 & & & & \\
 & & & t_1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 0 & t_2 & & \\
 & & & & & t_2 & 0 & t_1 & \\
 & O & & & & & t_1 & 0 & t_2 \\
 & & & & & & & t_2 & 2t_1
\end{pmatrix} \\
&= (-1)^{\frac{n-1}{2}} \cdot 4 \cdot t_1^n t_2 + (-1)^{\frac{n+1}{2}} t_2^{n+1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial}{\partial t_1} \mathrm{disc}_d(t_1 f_1 + t_2 f_2) &= (-1)^{\frac{n-1}{2}} 4n t_1^{n-1} t_2, \\
\frac{\partial}{\partial t_2} \mathrm{disc}_d(t_1 f_2 + t_2 f_2) &= (-1)^{\frac{n-1}{2}} 4 t_1^n + (-1)^{\frac{n+1}{2}} (n+1) t_2^n.
\end{aligned}
$$

Let

$$g_1 = (-1)^{\frac{n-1}{2}} 4n t^{n-1}, g_2 = (-1)^{\frac{n+1}{2}} (n+1) + (-1)^{\frac{n-1}{2}} 4 t^n.$$

Then we have

$$\mathrm{res}\left(\frac{\partial}{\partial t_1} \mathrm{disc}_d(t_1 f_2 + t_2 f_2), \frac{\partial}{\partial t_2} \mathrm{disc}_d(t_1 f_1 + t_2 f_2) \right) = \mathrm{res}_{n,n}(g_1, g_2).$$

By 3.3 (18), we have

$$\mathrm{res}_{n,n}(g_1, g_2) = ((-1)^{\frac{n-1}{2}} 4) \,\mathrm{res}_{n-1,n}(g_1, g_2).$$

Let $y_1, \ldots, y_n$ be the roots of the polynomial g_2 in $k = \overline{\mathbb{F}}_2$. Since the polynomial g_1 has 0 as $n-1$-multiple root, by (17) we have

$$\mathrm{res}_{n-1,n}(g_1, g_2) = \{(-1)^{\frac{n-1}{2}} 4n\}^n \{(-1)^{\frac{n-1}{2}} 4\}^{n-1} \left(\prod_{j=0}^{n} (0 - y_j) \right)^{n-1}.$$

Now we have

$$\prod_{j=0}^{n}(-y_j) = \frac{(-1)^{\frac{n+1}{2}}(n+1)}{(-1)^{\frac{n-1}{2}}\cdot 4} = -\frac{n+1}{4}.$$

Thus,

$$\operatorname{res}\left(\frac{\partial}{\partial t_1}\operatorname{disc}_d(t_1f_2+t_2f_2), \frac{\partial}{\partial t_2}\operatorname{disc}_d(t_1f_1+t_2f_2)\right)$$
$$= -2^{2(n+1)}\cdot n^n(n+1)^{n-1}.$$

Further, we have $a(0,n+1) = n-1$ and hence

$$\begin{aligned}&\operatorname{disc}_d(\operatorname{disc}_d(t_1f_1+t_2f_2))\\ &= \frac{1}{(n+1)^{n-1}}(-2^{2(n+1)}\cdot n^n(n+1)^{n-1})\\ &= -2^{2(n+1)}\cdot n^n.\end{aligned}$$

Thus $1 \equiv \operatorname{disc}(f_1,f_2) = 2^{-s}\operatorname{disc}_d(\operatorname{disc}_d(t_1f_1+t_2f_2)) = 2^{-s}(-2^{2(n+1)}\cdot n^n) \pmod 2$. Since the integer n is odd, we have that $s = 2(n+1)$. □

Let $n \geq 2$ be an even integer. Let k be a field. Let $X \subset \mathbb{P}_k^{\vee}$ be an $n-2$-dimensional smooth complete intersection of two quadrics defined by a pair of quadratic forms $(f_1,f_2) \in S^2E_k \oplus S^2E_k$. Then $H^{n-r}(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n-r}{2}))$ is spanned by the classes of $\frac{n-r}{2}$-dimensional linear subspaces of $\mathbb{P}^n_{\bar{k}}$ contained in $X_{\bar{k}}$ [7], [3, Exposé XIX]. The group of $\mathbb{Z}$-lattice spanned by the classes of these linear subspaces permutationg them and preserving the intersection form is isomorphic to the Weyl group $W(D_{n+1})$.

The action of G_k on the linear subspaces defines a homomorphism

$$G_k \to W(D_{n+1}),$$

unique up to conjugation.

Corollary 3.7. *Assume that char $k \neq 2$. Then the composition $G_k \to W(D_{n+1}) \to \{\pm 1\}$ is given by the square root of $(-1)^{\frac{n}{2}}\operatorname{disc}_d(\operatorname{disc}_d(f_1,f_2))$.*

Proof. The assertion follows from Theorem 2.3, Theorem 3.6.1 and specialization. □

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