博士論文（要約)

論文題目

The determinant and the discriminant of a complete intersection of even dimension

(偶数次元完全交叉の行列式と判別式)

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THE DETERMINANT AND THE DISCRIMINANT OF A COMPLETE INTERSECTION OF EVEN DIMENSION (SUMMARY)

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Let k be a field, $\bar{k}$ an algebraic closure of k and k^s the separable closure of k contained in $\bar{k}$. Let $\Gamma_k = \mathrm{Gal}(k^s/k) = \mathrm{Aut}_k(\bar{k})$.

Let X be a proper smooth variety of even dimension m over k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology $V = H^m(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{m}{2}))$ defines an orthogonal representation of the absolute Galois group Γ_k. The determinant

$$\det V : \Gamma_k \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$$

is independent of the choice of ℓ.

In this introduction we assume that the characteristic of k is not 2. Let $f_1, \dots, f_r$ be homogeneous polynomials of $n+1$ variables of degrees $d_1, \dots, d_r$ of coefficients in k. Let X be the intersection of r hypersurfaces defined by these polynomials in a projective space of dimension n. In 2012, O. Benoist[1] studied the *discriminant of a complete intersection* and gave an explicit formula of its degree. The discriminant, here denoted by $\mathrm{disc}(f_1, \dots, f_r)$, is a polynomial of the coefficients of $f_1, \dots, f_r$, and is defined in [1] up to sign by the property that X is smooth of dimension $n-r$ if and only if $\mathrm{disc}(f_1, \dots, f_r) \neq 0$.

Further, we assume that $n-r$ is even. In this case, we determine the sign of the discriminant by the property that the discriminant modulo 4 is a square. Let us denote the discriminant defined in these steps by $\mathrm{disc}_\sigma(f_1, \dots, f_r)$. We shall prove below:

Theorem 0.1. *Assume that X is smooth of dimension $m = n - r$. Then the quadratic character $\det V$ is defined by the square root of $\mathrm{disc}_\sigma(f_1, \dots, f_r)$.*

In other words, the kernel of $\det V : \Gamma_k \to \{\pm 1\}$ is the subgroup of Γ_k corresponding to the field extension $k(\sqrt{\mathrm{disc}_\sigma(f_1, \dots, f_r)})/k$.

Let us briefly outline the contents of this paper. In Section 1, we study the discriminant $\mathrm{disc}(f_1, \dots, f_r)$ of a complete intersection. We follow the method of Benoist in [1]. However, we see the variety X_A in [1] as a projective space bundle over the projective space in order to give another calculation of the degree of the discriminant, therefore we recall the detail. We construct the universal family of intersections of hypersurfaces in the projective space, and consider the subset of the parameter space consisting of the points corresponding to singular fibers. We show the subset is identified with the underlying set of the projective dual of a smooth projective variety. This variety is equal to the projective toric variety X_A in [1], though we treat it as a projective space bundle over the projective space. We then verify that the projective dual is an irreducible divisor in the parameter space. We define the discriminant of complete intersections as the defining polynomial of the divisor.

Next we calculate the degree of the discriminant in a different way from that in [1]. We give a new explicit presentation of the degree, though we do not know the relation between this and Benoist's formula.

In Section 2, we prove the main theorem. We first recall the quadratic character of the absolute Galois group defined by the determinant of the ℓ-adic representation of the middle degree of a proper smooth variety defined over a field. In [4], T. Saito showed that, for a smooth hypersurface of even dimension, the character is computed via the square root of the discriminant of a defining polynomial of the hypersurface. We adapt his method to extend the result to our case of smooth complete intersections of even dimension. By the same argument on universal families as in the case of hypersurface, the theorem is true up to a sign of the discriminant. Then the sign is determined by properties of the discriminant modulo 4.

Finally, in Section 3, we give an explicit presentation of the discriminant of intersections of two quadrics. Let $F_1 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(1)} X_i X_j$ and $F_2 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(2)} X_i X_j$ be universal homogeneous polynomials of degree 2. Let $R = \mathbb{Z}[t_1, t_2]$ be the polynomial ring with variables t_1, t_2. We see $t_1F_1 + t_2F_2$ as a quadratic form with variables $X_0, \ldots, X_n$ and denote its discriminant by $\operatorname{disc}(t_1F_2+t_2F_2) \in R[(C_{ij}^{(l)})]$. Further we see $\operatorname{disc}(t_1F_2 + t_2F_2)$ as a binary form with variables t_1, t_2 and denote its discriminant by $\operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2)) \in \mathbb{Z}[(C_{ij}^{(l)})]$.

Theorem 0.2. *1. Let $n \geq 2$ be an even integer. Then the equation*

$$\operatorname{disc}(F_1, F_2) = \operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2))$$

holds up to sign.

2. Let $n \geq 3$ be an odd integer. Then the equation

$$\operatorname{disc}(F_1, F_2) = 2^{-2(n+1)} \operatorname{disc}(\operatorname{disc}(t_1F_1 + t_2F_2))$$

holds up to sign.

The discriminant of a quadratic polynomial is the determinant of the symmetric matrix corresponding to the quadratic form. Further, the discriminant of a binary polynomial is given by the Sylvester's determinant. Thus the above equality give explicit presentation of the discriminant of the complete intersection of two quadrics.

The author was imformed by Takeshi Saito that Jean-Pierre Serre suggested him that the discriminant of a complete intersection of two quadrics should be given by those of a binary polynomial and a quadratic polynomial.

The cohomology of such an intersection is generated by algebraic classes of linear subspaces. The intersection theory of these classes is studied in detail in [2] and [3]. We give an application of the main theorem to this subject.

References

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