博士論文

論文題目　Some topics on analysis of
holomorphic discrete series representations
(正則離散系列表現の解析に関するいくつかの話題)

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Contents

Preface

This thesis is a collection of three individual articles

- Integral formula and upper estimate of I and J-Bessel functions on Jordan algebras,
- Norm computation and analytic continuation of vector valued holomorphic discrete series representations,
- Intertwining operators between holomorphic discrete series representations,

all of which are related to the analysis of holomorphic discrete series representations. This thesis is organized by three chapters, and each chapter corresponds to the aforementioned article.

The holomorphic discrete series representations are introduced by Harish-Chandra in 1950's, and are one of the easiest class of representations to study deeply, among all infinite-dimensional unitary representations of real reductive Lie groups. For example, this class of representations have highest weight vectors, and this allows us to treat this representations parallelly to finite-dimensional representations in some sense. Moreover, these representations have several explicit realizations, with inner products given by explicit converging integrals, and this enables us to compute several quantities such as reproducing kernels explicitly. The holomorphic series representations also connects with various theories, such as analysis on symmetric cones, Hardy spaces, modular forms, and physics.

Now we review some explicit realizations of the holomorphic discrete series representations in the simplest case, namely, in $G = SL(2,\mathbb{R})$ case. The first realization is given by the space of holomorphic functions $\mathcal{O}(\mathbf{D})$ on the unit disk $\mathbf{D} := \{w \in \mathbb{C} : |w| < 1\}$. For any $\lambda \in \mathbb{C}$, the universal covering group $\widetilde{SU}(1,1)$ of $SU(1,1) \simeq SL(2,\mathbb{R})$ acts on $\mathcal{O}(\mathbf{D})$ by the linear fractional transformation

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}^{-1}\right) f(w) := (\bar{b}w + \bar{a})^{-\lambda} f\left(\frac{aw+b}{\bar{b}w+\bar{a}}\right) \tag{0.0.1}$$

(Here the function $(\bar{b}w + \bar{a})^{-\lambda}$ is not well-defined on $SU(1,1) \times \mathbf{D}$, but is well-defined as a function on the universal covering space $\widetilde{SU}(1,1) \times \mathbf{D}$). When $\lambda \in \mathbb{R}$ and $\lambda > 1$, this action preserves the inner product

$$\langle f, g\rangle_{\lambda,\mathbf{D}} := \frac{\lambda-1}{\pi}\int_{\mathbf{D}} f(w)\overline{g(w)}(1-|w|^2)^{\lambda-2}dw \tag{0.0.2}$$

where dw is the Lebesgue measure on $\mathbb{C}$. Thus the corresponding Hilbert subspace in $\mathcal{O}(\mathbf{D})$ gives the first realization of the holomorphic discrete series representation of $SU(1,1) \simeq SL(2,\mathbb{R})$. Since $\mathbf{D}$ is biholomorphically diffeomorphic to the upper half plane $\mathbf{H} := \mathbb{R} + \sqrt{-1}\mathbb{R}_{>0}$ via the Cayley transform, $\mathcal{O}(\mathbf{D})$ is isomorphic to the space of holomorphic functions $\mathcal{O}(\mathbf{H})$ on $\mathbf{H}$, and this gives the second realization of the holomorphic discrete series representation of $SL(2,\mathbb{R})$, with the inner product

$$\langle f,g\rangle_{\lambda,\mathbf{H}} := \frac{\lambda-1}{4\pi}\int_{\mathbf{H}} f(z)\overline{g(z)}(\mathrm{Im}(z))^{\lambda-2}dz.$$

Moreover, via the Laplace transform, the Hilbert subspace in $\mathcal{O}(\mathbf{H})$ is isomorphic to the space of square-integrable functions on the half line $\mathbb{R}_{>0}$, with the inner product

$$\langle f,g\rangle_{\lambda,\mathbb{R}_{>0}} := \frac{1}{\Gamma(\lambda)}\int_0^\infty f(x)\overline{g(x)}x^{\lambda-1}dx.$$

Then the Hilbert space $L^2(\mathbb{R}_{>0},x^{\lambda-1}dx)$ gives the third realization of the holomorphic discrete series representations. We note that $\widetilde{SL}(2,\mathbb{R})$ does not act on the geometry $\mathbb{R}_{>0}$, but it acts on the function space $L^2(\mathbb{R}_{>0},x^{\lambda-1}dx)$, and its infinitesimal action of $\mathfrak{sl}(2,\mathbb{R})$ is given by at most 2nd order differential operators.

In general, let G be a real reductive group of Hermitian type, that is, the Riemannian symmetric space G/K has a natural complex structure, where K is a maximal compact subgroup of G. Then G/K is diffeomorphic to a bounded domain D in a complex vector space $V^{\mathbb{C}}=\mathfrak{p}^+$ ($V^{\mathbb{C}}$ is a notation in Chapter 1, $\mathfrak{p}^+$ is a notation in Chapter 2, 3), which is called the bounded symmetric domain. Therefore, the universal covering group $\tilde{G}$ acts on the space of holomorphic sections of a vector bundle on D. Since the complex domain D is contractible, the vector bundle is isomorphic to the direct product bundle, and thus the space of holomorphic sections is isomorphic to the space of vector-valued holomorphic functions on D. If this action preserves an inner product given by a converging integral on D, then the corresponding Hilbert space gives the first realization of the holomorphic discrete series representations. Moreover, if G is of tube type, that is, the symmetric space G/K is also diffeomorphic to a tube domain $T_\Omega=V+\sqrt{-1}\Omega$ over a symmetric cone Ω, the holomorphic discrete series representation is also realized on the space of holomorphic functions on the tube domain T_Ω (the second realization), and on the space of square-integrable functions on the symmetric cone Ω (the third realization). In the first realization, the K-finite vectors are given by polynomials, and it is easy to treat algebraically. On the other hand, in the third realization, we can construct a rich theory for analysis on symmetric cones, sometimes with the aid of the second realization.

In chapter 1, we deal with the third realization, the symmetric cone picture. There are various special functions on symmetric cones which are the natural generalization of ordinary special functions of one variable. Among these, we deal with the multivariate Bessel function, which was introduced by Dib ([5] of Chapter 1). This Bessel function is used as the kernel function of the Hankel transform, which is a variant of the usual Fourier transform. It is well-known that the usual Fourier transform is the unitary operator on $L^2(\mathbb{R}^n)$, and this appears in the (Segal-Shale-)Weil representation of the metaplectic group $Mp(n,\mathbb{R})$ (the double covering group of the symplectic group $Sp(n,\mathbb{R})$) as the action of the conformal inversion element w_0 (the element interchanging the maximal parabolic subgroup and the opposite parabolic subgroup via the inner automorphism). Likewise, the Hankel transform appears in the holomorphic discrete series representation on $L^2(\Omega)$ (under a suitable measure) as the action of the conformal inversion element. The Fourier transform on $L^2(\mathbb{R}^n)$ also appears as the special value of the Hermite semigroup. The Hermite semigroup is the family of operators $\tilde{\tau}(t)$ on $L^2(\mathbb{R}^n)$, where t runs over the right half plane $\{t\in\mathbb{C}:\mathrm{Re}\,t\geq 0\}$, satisfying $\tilde{\tau}(s)\tilde{\tau}(t)=\tilde{\tau}(s+t)$. When $\mathrm{Re}\,t=0$, $\tilde{\tau}(t)$ is a unitary operator, and it coincides with the restriction of the Weil representation to the center of the maximal compact subgroup $U(n)$ in $Mp(n,\mathbb{R})$. This extends analytically to the right half plane, and when $\mathrm{Re}\,t>0$, $\tilde{\tau}(t)$ gives a Hilbert-Schmidt operator. The special value $\tilde{\tau}(\pi\sqrt{-1}/4)$ coincides with the usual Fourier transform (up to scalar multiple). A similar phenomenon also occurs on $L^2(\Omega)$, that is, the restriction of the holomorphic discrete series representation to the center of the maximal compact subgroup K of G extends to the analytic semigroup on the half plane, and it gives a Hilbert-Schmidt operator when the parameter t satisfies $\operatorname{Re} t > 0$. The multivariate Bessel function appears in the kernel functions of these operators. The program for such problems understanding the highest weight representations of real Lie groups from the viewpoint of representations of complex analytic semigroups was suggested by Gelfand-Gindikin (1977), and the general theory of this program was completed by Stanton (1986) and Ol'shanskiĭ (1981, 91, 95). Moreover, this theory led to the theory of Laguerre semigroups by Kobayashi-Mano (2007), and generated the theories of global analysis on minimal representations and the deformation of Fourier transforms.

The author's result in Chapter 1 is about the upper estimate of the multivariate Bessel functions $\mathcal{I}_\lambda(x)$. In general, for any symmetric cone Ω, there exists a natural Euclidean Jordan algebra which contains Ω as an open subset. Then this is a special function defined on $V^{\mathbb{C}}$. In this chapter the author has proved a new integral expression of $\mathcal{I}_\lambda(x^2)$, and using this, proved the upper estimate of Dib's multivariate Bessel function $\mathcal{I}_\lambda(x^2)$,

$$|\mathcal{I}_\lambda(x^2)| \le C_{\lambda,k}\left(1+|x|_1^{\max\{2n-r\lambda,0\}}\right)e^{2|\operatorname{Re} x|_1},$$

where $|\cdot|_1$ is a suitable norm on $V^{\mathbb{C}}$, and r is the rank of the Jordan algebra V. Especially, it is of polynomial growth on $\sqrt{-1}V \subset V^{\mathbb{C}}$, and from this result we can show that the 1-dimensional analytic semigroup in the previous paragraph maps functions with polynomial growth to functions with exponential decay, and can also reconfirm that it gives the Hilbert-Schmidt operator, without using representation theory.

In Chapters 2 and 3, we deal with the first realization, the bounded symmetric domain picture. In this picture the holomorphic discrete series representation is realized on the space of holomorphic functions on the bounded symmetric domain D, and the corresponding Hilbert space has the reproducing kernel. For example, when $G = SU(1,1)$, the representation (0.0.1) gives the holomorphic discrete series if $\lambda > 1$, and the reproducing kernel with respect to the inner product (0.0.2) is given by

$$K_\lambda(z,w) = (1 - z\bar{w})^{-\lambda}.$$

Now, this reproducing kernel is expanded as

$$K_\lambda(z,w) = \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!}(z\bar{w})^m,$$

where $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$ is the usual shifted factorial. From this expression it follows that the kernel function $K_\lambda(z,w)$ is of positive type if $\lambda \ge 0$, that is, there exists a non-zero Hilbert space with the reproducing kernel $K_\lambda(z,w)$ if $\lambda \ge 0$, on which $\widetilde{SU}(1,1)$ acts unitarily via (0.0.1), even though the integral (0.0.2) converges only when $\lambda > 1$. The corresponding Hilbert spaces for $0 \le \lambda \le 1$ can be regarded as the analytic continuation of the holomorphic discrete series representations for $\lambda > 1$. The similar phenomena also occur for other Lie groups, are studied by e.g. Berezin (1975), Vergne-Rossi (1976) and Wallach (1976), and completely classified by Enright-Howe-Wallach (1983) and Jakobsen (1983). After that, other proofs with analytic methods are given by e.g. Clerc (1995) and Faraut-Korányi (1990) for partial results. Among these studies, Faraut-Korányi ([6] of Chapter 2) computed the expansion of the reproducing kernels explicitly for holomorphic discrete series representations of scalar type of any simple Lie groups of Hermitian type. In Chapter 2 of this thesis the author has generalized the above results of Faraut-Korányi for vector-valued holomorphic discrete series representations such that their K-type decompostion are multiplicity-free. In more detail, in the bounded symmetric domain picture, the space of K-finite vectors is equal to the space of polynomials, and its K-type decomposition is independent of the continuous parameter λ. Thus the reproducing kernel of the Hilbert space is expanded in terms of the reproducing kernel of each K-type, and the author has computed how the coefficients in this expansion depends on the parameter λ. From this result we can determine when the analytic continuation of the holomorphic discrete series representation is unitarizable, and can also determine the underlying $(\mathfrak{g}, K)$-modules of the representation spaces. This argument gives an analytical proof for a part of the results of Enright-Howe-Wallach and Jakobsen.

We can also view the result in Chapter 2 that it determines explicitly how the holomorphic discrete series representation behaves when it is restricted to the maximal compact subgroup K. Then the next natural question is how it behaves when it is restricted to other subgroups. In 1990's, the general theories on discrete decomposability and multiplicity-freeness of restriction of representations were established by Kobayashi, and he suggested the importance of problems of writing down the decomposition explicitly (see [18] of Chapter 3 (2015)), and these problems are studied by e.g. Clerc-Kobayashi-Ørsted-Pevzner (2011), Kobayashi-Ørsted-Somberg-Souček (2015), Kobayashi-Pevzner (2015), Kobayashi-Speh (2015), Möllers-Oshima (2015) and Peng-Zhang (2004). In general, when we consider an irreducible representation $\mathcal{H}$ of a reductive Lie group G, and restrict it to a subgroup $G_1 \subset G$, it may behaves very wildly, for example, the multiplicities in $\mathcal{H}|_{G_1}$ may become infinite, or it may contain continuous spectrums, even if (G, G_1) is a symmetric pair. However, if G is of Hermitian type, $\mathcal{H}$ is a holomorphic discrete series representation, and $G_1 \subset G$ is also of Hermitian type such that the embedding map $G_1/K_1 \hookrightarrow G/K$ of Riemannian symmetric spaces is holomorphic, then $\mathcal{H}|_{G_1}$ decomposes discretely, and moreover all multiplicities are finite and uniformly bounded if (G, G_1) is a symmetric pair (Kobayashi, 2007). In this case we also know what kind of representations of G_1 appears in $\mathcal{H}|_{G_1}$. Thus our next interest is to determine explicitly how each representation of G_1 is embedded in $\mathcal{H}|_{G_1}$, that is, to write down explicitly the G_1-intertwining operators between each representation of G_1 and $\mathcal{H}|_{G_1}$. In Chapter 3, the author has studied this problem, and got the integral expressions of the G_1-intertwining operators for general holomorphic discrete series representations of G_1 and G. From this result the author has also deduced the (infinite-order) differential expressions of the G_1-intertwining embedding maps from the representation of G_1 to that of G in the case both G and G_1 are classical groups and both representations of G and G_1 are of "almost scalar type". In the proof the author has used the series expansion of integrands and the results on the norm computation by Faraut-Korányi.

Finally, the author would like to express his gratitude to his supervisor professor T. Kobayashi for his attentive guidance, and also for professors T. Kubo and B. Ørsted for many helpful advices. He also thank his colleagues, especially for M. Kitagawa and Y. Tanaka for many helpful discussions. He would also like to thank Grant-in-Aid for JSPS Fellows for financial support.

Chapter 1

Integral formula and upper estimate of I and J-Bessel functions on Jordan algebras

In this chapter we give a new integral expression of I and J-Bessel functions on simple Euclidean Jordan algebras, integrating on a bounded symmetric domain. From this we easily get the upper estimate of Bessel functions. As an application we give an upper estimate of the integral kernel function of the holomorphic 1-dimensional semi-group acting on the space of square integrable functions on symmetric cones.

Keywords: Euclidean Jordan algebras; Bessel functions; holomorphic discrete series representations; holomorphic semigroups.
AMS subject classification: 33C10; 33C67; 17C30; 22E45; 47D06.

1.1 Introduction and main results

In this chapter we find in Theorem 1.3.1 a new integral expression of I and J-Bessel functions $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ on a Jordan algebra V. J-Bessel functions are first introduced by Faraut and Travaglini [9] for special cases, associating to self-adjoint representations of Jordan algebras (see also (1.4.2)), and generalized by Dib [5] (for $V = \mathrm{Sym}(r, \mathbb{R})$ case see also [12] and [18]). It is well-known that $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ are the holomorphic functions on $V^{\mathbb{C}}$ for λ in open dense subset of $\mathbb{C}$. On the other hand, for countable singular λ they are still well-defined on certain subvarieties. These are defined by the series expansion (see Section 1.3), and satisfy the following differential equation

$$\mathcal{B}_\lambda \mathcal{I}_\lambda - e\mathcal{I}_\lambda = 0, \qquad \mathcal{B}_\lambda \mathcal{J}_\lambda + e\mathcal{J}_\lambda = 0$$

where $\mathcal{B}_\lambda : C^2(V) \to C(V) \otimes V^{\mathbb{C}}$ is the $V^{\mathbb{C}}$-valued 2nd order differential operator defined in [8, Section XV.2], and e is the unit element on V (see [5, Proposition 1.7] or [8, Theorem XV.2.6]). Also $\mathcal{I}_\lambda$ and $\mathcal{J}_\lambda$ have the following integral expression

$$\mathcal{I}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n} \int_{e+iV} e^{\mathrm{tr}\, w} e^{(w^{-1}|x)} \Delta(w)^{-\lambda} dw, \tag{1.1.1}$$

$$\mathcal{J}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n} \int_{e+iV} e^{\mathrm{tr}\, w} e^{-(w^{-1}|x)} \Delta(w)^{-\lambda} dw \tag{1.1.2}$$

(see [5, Définition 1.2] or [8, Theorem XV.2.2]. For notations tr, $(\cdot|\cdot)$, Δ and $\Gamma_\Omega(\lambda)$ see Section 1.2.1 and (1.2.3)). There are some attempts to generalize these Bessel functions to operator-valued ones (see *e.g.* [6] and references therein), but it is still not very well-understood. In this paper we only treat scalar-valued ones.

Now we briefly state our theorem. Let V be a simple Euclidean Jordan algebra (*i.e.*, V is one of the $\mathrm{Sym}(r,\mathbb{R})$, $\mathrm{Herm}(r,\mathbb{C})$, $\mathrm{Herm}(r,\mathbb{H})$, $\mathbb{R}^{1,n-1}$ or $\mathrm{Herm}(3,\mathbb{O})$). We assume $\dim V = n$, $\operatorname{rank} V = r$. We prove

Theorem 1.1.1. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$ (see (1.2.1) and (1.2.6)), take $k \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$. Then, we have the integral expressions*

$$\mathcal{I}_\lambda\left(x^2\right) = c_{\lambda+k} \int_D {}_1F_1(-k,\lambda;-x,w) e^{2(x|\operatorname{Re} w)} h(w,w)^{\lambda+k-\frac{2n}{r}} dw,$$
$$\mathcal{J}_\lambda\left(x^2\right) = c_{\lambda+k} \int_D {}_1F_1(-k,\lambda;-ix,w) e^{2i(x|\operatorname{Re} w)} h(w,w)^{\lambda+k-\frac{2n}{r}} dw,$$

where c_λ is a constant and ${}_1F_1(-k,\lambda;x,w)$ is a polynomial of degree rk with respect to both x and w.

Here $\mathcal{X}_l$ are the $L = \mathrm{Str}(V^{\mathbb{C}})_0$-orbits. $\overline{\mathcal{X}_l}$ are also characterized as the supports of some distributions on $V^{\mathbb{C}}$ (see [3] and (1.2.2)). $D \subset V^{\mathbb{C}}$ is the *bounded symmetric domain* and $h(w,w)$ is the *generic norm* on $V^{\mathbb{C}}$ (see Section 1.2.1). For the explicit forms of c_λ and ${}_1F_1(-k,\lambda;x,w)$ see Theorem 1.3.1. Especially if $\operatorname{Re}\lambda > \frac{2n}{r} - 1$ we can take $k = 0$ and

$$\mathcal{I}_\lambda\left(x^2\right) = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)} \int_D e^{2(x|\operatorname{Re} w)} h(w,w)^{\lambda-\frac{2n}{r}} dw$$

and $\mathcal{J}_\lambda$ is similar.

Now D is naturally identified with $G/K = \mathrm{Bihol}(D)/\mathrm{Stab}(0) = \mathrm{Co}(V)_0/\mathrm{Aut}_{\mathrm{JTS}}(V)_0$. For $\lambda > \frac{2n}{r} - 1$, the universal covering group $\tilde{G}$ acts unitarily on $\mathcal{O}(D) \cap L^2(D, h(w,w)^{\lambda-\frac{2n}{r}} dw)$ by left translation. This defines the holomorphic discrete series representation of $\tilde{G}$. This is analytically continued with respect to $\lambda \in \mathbb{C}$, and become unitary when $\lambda \in \mathcal{W}$, the *(Berezin–)Wallach set* (see (1.2.7) and [25], [4]). The trivial representation corresponds to $\lambda = 0$.

From now we set $V = \mathbb{R}$. Let $I_\lambda(x)$ be the classical I-Bessel function (see [2, (4.12.2)]), and we set $\tilde{I}_\lambda(x) = \left(\frac{x}{2}\right)^{-\lambda} I_\lambda(x)$. Then $\tilde{I}_\lambda$ and $\mathcal{I}_\lambda$ on $\mathbb{R}$ are related as

$$\tilde{I}_\lambda(x) = \frac{1}{\Gamma(\lambda+1)} \mathcal{I}_{\lambda+1}\left(\frac{x^2}{4}\right).$$

Therefore the above theorem is rewritten as

$$\tilde{I}_\lambda(x) = \frac{\lambda+k}{\pi\Gamma(\lambda+1)} \int_{|w|<1} {}_1F_1(-k,\lambda+1;-xw) e^{x\operatorname{Re} w} \left(1-|w|^2\right)^{\lambda+k-1} dw.$$

where ${}_1F_1(-k,\lambda+1;x)$ is the classical hypergeometric polynomial. This formula seems to be new even for $V = \mathbb{R}$ case. On the other hand, the formula (1.1.1) is rewritten as

$$\tilde{I}_\lambda(x) = \frac{1}{2i\pi\lambda} \int_{1+i\mathbb{R}} e^{w+\frac{x^2}{w}} w^{-\lambda-1} dw.$$

These two integral formulas are mutually independent, and cannot easily deduce one from another.

Again let V be a general Jordan algebra. Since D is bounded, we can prove from this formula the following corollary.

Corollary 1.1.2. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, if $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k} > 0$ such that*

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k}\left(1 + |x|_1^{rk}\right) e^{2|\operatorname{Re} x|_1}, \qquad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k}\left(1 + |x|_1^{rk}\right) e^{2|\operatorname{Im} x|_1}$$

where $|x|_1$ is the norm defined in Definition 1.2.1.

In [17, Lemma 3.1] an upper estimate of $\mathcal{J}_\lambda(x)$ is given by another method, but our estimate is sharper. For detail see Remark 1.3.3. When $V = \mathbb{R}$, this corollary implies that if $\operatorname{Re}\lambda > -k$ for some $k \in \mathbb{Z}_{\geq 0}$,

$$|\tilde{I}_\lambda(x)| = \frac{1}{|\Gamma(\lambda+1)|}\left|\mathcal{I}_{\lambda+1}\left(\frac{x^2}{4}\right)\right| \leq C'_{\lambda,k}\left(1 + |x|^k\right) e^{|\operatorname{Re} x|}.$$

On the other hand, we have the asymptotic expansion

$$\tilde{I}_\lambda(x) \sim \frac{\left(\frac{x}{2}\right)^{-\lambda}}{\sqrt{2\pi x}}\left(e^x \sum_{m=0}^{\infty} \frac{(-1)^m(\lambda,m)}{(2x)^m} + e^{-x+(\lambda+\frac{1}{2})\pi i} \sum_{m=0}^{\infty} \frac{(\lambda,m)}{(2x)^m}\right)$$

where (λ, m) are some numbers (see [2, (4.12.7)]), and this implies that

$$|\tilde{I}_\lambda(x)| \leq C''_\lambda\left(1 + |x|^{\max\left\{-\lambda-\frac{1}{2},0\right\}}\right) e^{|\operatorname{Re} x|}.$$

Therefore our result is not the sharpest when $\operatorname{Re}\lambda \leq 0$, but it still seems to be sufficiently sharp.

This chapter is organized as follows: In Section 1.2, we recall some notations and facts about Euclidean Jordan algebras. In Section 1.3 we prove our main theorem, the integral formula and upper estimates. In Section 1.4, as an application of the inequality (Corollary 1.1.2), we give an upper estimate of the integral kernel function of the 1-dimensional semigroup on the functions on the symmetric cones.

1.2 Preliminaries

1.2.1 Simple Euclidean Jordan algebras

Let V be a simple Euclidean Jordan algebra of dimension n, rank r. We denote the unit element by e. Also let $V^{\mathbb{C}}$ be its complexification. For $x, y, z \in V^{\mathbb{C}}$, we write

$$\begin{aligned}
L(x)y &:= xy, \\
x\Box y &:= L(xy) + [L(x), L(y)], \\
P(x,z) &:= L(x)L(z) + L(z)L(x) - L(xz), \\
P(x) &:= P(x,x) = 2L(x)^2 - L(x^2), \\
B(x,y) &:= I_{V^{\mathbb{C}}} - 2x\Box\bar{y} + P(x)P(\bar{y})
\end{aligned}$$

where $y \mapsto \bar{y}$ is the complex conjugation with respect to the real form V. Also, we write

$$\{x,y,z\} := (x\Box\bar{y})z = P(x,z)\bar{y} = (x\bar{y})z + x(\bar{y}z) - (xz)\bar{y}.$$

Then $V^{\mathbb{C}}$ becomes a positive Hermitian Jordan triple system with this triple product.

We denote the *Jordan trace* and the *Jordan determinant* of the complex Jordan algebra $V^{\mathbb{C}}$ by $\operatorname{tr}(x)$ and $\Delta(x)$ respectively. Also let $h(x,y)$ be the *generic norm* of the Jordan triple system $V^{\mathbb{C}}$. These can be expressed by $L(x)$, $P(x)$, and $B(x,y)$ (see [8, Proposition III.4.2], [7, Part V, Proposition VI.3.6]):

$$\begin{gathered}\operatorname{Tr} L(x)=\frac{n}{r}\operatorname{tr}(x),\\ \operatorname{Det} P(x)=\Delta(x)^{\frac{2n}{r}},\\ \operatorname{Det} B(x,y)=h(x,y)^{\frac{2n}{r}}\end{gathered}$$

where Tr and Det stand for the usual trace and determinant of complex linear operators on $V^{\mathbb{C}}$. Using the Jordan trace we define the inner product on $V^{\mathbb{C}}$:

$$(x|y):=\operatorname{tr}(x\bar{y}),\qquad x,y\in V^{\mathbb{C}}.$$

Then this is positive definite since V is Euclidean. Also we define the *symmetric cone* Ω and the *bounded symmetric domain* D by

$$\begin{gathered}\Omega:=\{x^2 : x\in V,\ \Delta(x)\neq 0\},\\ D:=(\text{connected component of } \{w\in V^{\mathbb{C}} : h(w,w)>0\} \text{ which contains } 0).\end{gathered}$$

Then Ω is self-dual, *i.e.*,

$$\Omega=\{x\in V : (x|y)>0 \text{ for any } y\in\Omega\},$$

and D is biholomorphically equivalent to $V+\sqrt{-1}\Omega\subset V^{\mathbb{C}}$.

Let K_L and K be the identity components of *automorphism groups* of the Jordan algebra V and the Jordan triple system $V^{\mathbb{C}}$. Similarly let L and $L^{\mathbb{C}}$ be the identity components of *structure groups* of V and $V^{\mathbb{C}}$. Also let G be the identity component of *conformal group* of V:

$$\begin{gathered}K_L:=\operatorname{Aut}_{\mathrm{J.Alg}}(V)_0=\{k\in GL(V) : k(xy)=kx\cdot ky,\ \forall x,y\in V\}_0,\\ K:=\operatorname{Aut}_{\mathrm{JTS}}(V^{\mathbb{C}})_0=\{k\in GL(V^{\mathbb{C}}) : k\{x,y,z\}=\{kx,ky,kz\},\ \forall x,y,z\in V^{\mathbb{C}}\}_0,\\ L:=\operatorname{Str}(V)_0=\{l\in GL(V) : l\{x,y,z\}=\{lx,{}^t l^{-1}y,lz\},\ \forall x,y,z\in V\}_0,\\ L^{\mathbb{C}}:=\operatorname{Str}(V^{\mathbb{C}})_0=\{l\in GL(V^{\mathbb{C}}) : l\{x,y,z\}=\{lx,(l^*)^{-1}y,lz\},\ \forall x,y,z\in V^{\mathbb{C}}\}_0,\\ G:=\operatorname{Co}(V)_0=\operatorname{Bihol}(D)_0\simeq\operatorname{Bihol}(V+\sqrt{-1}\Omega)_0\end{gathered}$$

where ${}^t l$ and l^* stand for the transpose with respect to the bilinear form $\operatorname{tr}(xy)$ and the sesquilinear form $\operatorname{tr}(x\bar{y})=(x|y)$. Then Ω and D are naturally identified with L/K_L and G/K respectively. For the classification of these groups see [13, Table 1] or [17, Table 1].

1.2.2 Spectral decomposition and some norms on $V^{\mathbb{C}}$

From now on we fix a *Jordan frame* $\{c_1,\dots,c_r\}\subset V$, *i.e.*,

$$c_jc_k=\delta_{jk}c_j,\qquad \sum_{j=1}^r c_j=e,$$

$$\text{and if } d_{j1},d_{j2}\in V \text{ satisfy } c_j=d_{j1}+d_{j2},\ d_{jk}d_{jl}=\delta_{kl}d_{jk}, \text{ then } d_{j1}=0 \text{ or } d_{j2}=0.$$

Then for any $x\in V^{\mathbb{C}}$ there exist the unique numbers $t_1\geq\cdots t_r\geq 0$ and the element $k\in K$ such that $x=k\sum_{j=1}^r t_jc_j$ ([8, Proposition X.3.2]). Using this, we define the p-norm on $V^{\mathbb{C}}$.

Definition 1.2.1. *For $1 \leq p \leq \infty$ and for $x = k\sum_{j=1}^r t_j c_j \in V^{\mathbb{C}}$, we define*

$$|x|_p := \begin{cases} \left(\displaystyle\sum_{j=1}^r |t_j|^p\right)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \displaystyle\max_{j \in \{1,\dots,r\}} |t_j| & (p = \infty). \end{cases}$$

For example, we have $(x|x) = |x|_2^2$. Also if $x \in \Omega$ then all eigenvalues (in the sense of Jordan algebras. For $V = \mathrm{Sym}(r, \mathbb{R})$ or $\mathrm{Herm}(r, \mathbb{C})$ this coincides with the usual one) are positive and $|x|_1 = \operatorname{tr} x$ holds. In addition, we can define D by $D = \{w \in V^{\mathbb{C}} : |w|_\infty < 1\}$. This norm satisfies the following properties.

Proposition 1.2.2 ([23, Theorem V.4, V.5] for $V = \mathrm{Herm}(r, \mathbb{C})$ case)**.** *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements hold.*

(1) For $x, y \in V^{\mathbb{C}}$, $|(x|y)| \leq |x|_p |y|_q$.

(2) For $x \in V^{\mathbb{C}}$, $|x|_p = \displaystyle\max_{y \in V^{\mathbb{C}} \setminus \{0\}} \frac{|(x|y)|}{|y|_q}$.

(3) $x \mapsto |x|_p$ is a norm on $V^{\mathbb{C}}$.

To prove this, we quote the following lemma (see [7, Part V, Proposition VI.2.1]):

Lemma 1.2.3. *For $x, y \in V^{\mathbb{C}}$, if $x \Box \bar{y} = y \Box \bar{x}$, then there exists an element $k \in K$ such that both x and y belong to $\mathbb{R}$-$\mathrm{span}\{kc_1, \dots, kc_r\}$.*

Proof of Proposition 1.2.2. (1) We note that $|(x|y)| \leq \max_{k \in K} |(kx|y)| = \max_{k \in K} \mathrm{Re}(kx|y)$ since $e^{i\theta} I_{V^{\mathbb{C}}} \in K$ for any $\theta \in \mathbb{R}$. We take $k_0 \in K$ such that $\mathrm{Re}(kx|y)$ $(k \in K)$ attains its maximum at $k = k_0 \in K$. We put $k_0 x =: x_0$. Then for any $D \in \mathfrak{k} = Lie(K)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Re}(e^{tD} x_0 | y) = \mathrm{Re}(D x_0 | y) = 0.$$

In the case when $D = u \Box \bar{v} - v \Box \bar{u}$ with $u, v \in V^{\mathbb{C}}$,

$$\begin{aligned} 0 &= \mathrm{Re}((u \Box \bar{v}) x_0 | y) - \mathrm{Re}((v \Box \bar{u}) x_0 | y) = \mathrm{Re}((x_0 \Box \bar{v}) u | y) - \mathrm{Re}((x_0 \Box \bar{u}) v | y) \\ &= \mathrm{Re}(u | (v \Box \bar{x_0}) y) - \mathrm{Re}(v | (u \Box \bar{x_0}) y) = \mathrm{Re}(u | (y \Box \bar{x_0}) v) - \mathrm{Re}(v | (y \Box \bar{x_0}) u) \\ &= \mathrm{Re}((x_0 \Box \bar{y}) u | v) - \mathrm{Re}(v | (y \Box \bar{x_0}) u) = \mathrm{Re}((x_0 \Box \bar{y} - y \Box \bar{x_0}) u | v). \end{aligned}$$

Since $u, v \in V^{\mathbb{C}}$ are arbitrary and $(\cdot|\cdot)$ is non-degenerate, $x_0 \Box \bar{y} = y \Box \bar{x_0}$. Therefore by Lemma 1.2.3 there exists $k \in K$ such that $x_0, y \in \mathbb{R}$-$\mathrm{span}\{kc_1, \dots, kc_r\}$. Let $x = k' \sum_{j=1}^r t_j c_j$, $y = k \sum_{j=1}^r s_j c_j$. Then

$$\begin{aligned} |(x|y)| &\leq \max_{k \in K} \mathrm{Re}(kx|y) = \mathrm{Re}(x_0|y) = \mathrm{Re}\left(k \sum_{j=1}^r t_j c_j \,\middle|\, k \sum_{j=1}^r s_j c_j \right) \\ &= \sum_{j=1}^r t_j s_j \leq \left(\sum_{j=1}^r |t_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^r |s_j|^q\right)^{\frac{1}{q}} = |x|_p |y|_q. \end{aligned}$$

(2) ($\geq$) Clear from (1).
($\leq$) For $x = k\sum_{j=1}^r t_j c_j \in V^{\mathbb{C}}$ ($t_1 \geq \cdots t_r \geq 0$), we find a $y \in V^{\mathbb{C}}$ which attains the equality. We set

$$y := \begin{cases} k\sum_{j=1}^r t_j^{p-1} c_j & (1 \leq p < \infty), \\ kc_1 & (p = \infty). \end{cases}$$

Then,

$$|y|_q = \begin{cases} \left(\sum_{j=1}^r t_j^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_{j=1}^r t_j^p\right)^{\frac{p-1}{p}} = |x|_p^{p-1} & (1 < p < \infty), \\ 1 & (p = 1, \infty), \end{cases}$$

and

$$(x|y) = \begin{cases} \sum_{j=1}^r t_j^p = |x|_p^p = |x|_p |x|_p^{p-1} = |x|_p |y|_q & (1 \leq p < \infty), \\ t_1 = |x|_\infty = |x|_\infty |y|_1 & (p = \infty). \end{cases}$$

(3) Positivity and homogeneity are clear. For triangle inequality, by (2), for $x, y \in V^{\mathbb{C}}$,

$$|x+y|_p = \max_{|z|_q=1} |(x+y|z)| \leq \max_{|z|_q=1} |(x|z)| + \max_{|z|_q=1} |(y|z)| = |x|_p + |y|_p$$

and this completes the proof. □

We set

$$\mathcal{X}_l := \left\{ k\sum_{j=1}^l t_j c_j : k \in K,\ t_j > 0 \right\} = L^{\mathbb{C}} \cdot \sum_{j=1}^l e_j \subset V^{\mathbb{C}} \quad (l = 0, \ldots, r). \tag{1.2.1}$$

Then $\overline{\mathcal{X}_l} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_l$ holds. $\overline{\mathcal{X}_l}$ are also characterized as the supports of the distributions which are the analytic continuation of $|\Delta(x)|^{2\left(\lambda - \frac{n}{r}\right)} dx$:

$$\operatorname{supp}\left(|\Delta(x)|^{2\left(\lambda-\frac{n}{r}\right)} dx\Big|_{\lambda = l\frac{d}{2}} \right) = \overline{\mathcal{X}_l}, \qquad l = 0, 1, \ldots, r-1 \tag{1.2.2}$$

(see [3, Proposition 5.5]).

1.2.3 Peirce decomposition and generalized power function

As before we fix a Jordan frame $\{c_1, \ldots, c_r\} \subset V$. Then V is decomposed as

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{jk} \quad \text{where} \quad V_{jk} = \left\{ x \in V : L(c_l)x = \frac{\delta_{jl} + \delta_{kl}}{2} x \right\}.$$

Moreover $V_{jj} = \mathbb{R}c_j$ holds, and all V_{jk}'s ($j \neq k$) have the same dimension (see [8, Theorem IV.2.1, Corollary IV.2.6]). We write $\dim V_{jk} = d$. Then $\dim V = n = r + \frac{1}{2}r(r-1)d$ holds.

Let $V_{(l)}^{\mathbb{C}} := \bigoplus_{1 \leq j \leq k \leq l} V_{jk}^{\mathbb{C}}$ ($l = 1, \ldots, r$) and $P_{(l)}$ be the orthogonal projection on $V_{(l)}^{\mathbb{C}}$. We denote by $\det_{(l)}(x)$ the Jordan determinant on the Jordan algebra $V_{(l)}^{\mathbb{C}}$. We set $\Delta_l(x) := \det_{(l)}(P_{(l)}(x))$ for $x \in V^{\mathbb{C}}$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, the *generalized power function* on $V^{\mathbb{C}}$ is defined by

$$\Delta_{\mathbf{s}}(x) := \Delta_1^{s_1 - s_2}(x) \Delta_2^{s_2 - s_3}(x) \cdots \Delta_{r-1}^{s_{r-1} - s_r}(x) \Delta_r^{s_r}(x).$$

Then, the *Gindikin Gamma function* and *Pochhammer symbol* are defined as follows: for $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$,

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\operatorname{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}} dx, \qquad (\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})}. \tag{1.2.3}$$

This integral converges for $\operatorname{Re} s_j > (j-1)\frac{d}{2}$, and both functions are extended meromorphically on $\mathbb{C}^r$ (see [8, Theorem VII.1.1] or [11, Theorem 2.1]). Moreover, we have

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^r \left(s_j - (j-1)\frac{d}{2} \right)_{m_j} \qquad \text{where} \quad (s)_m = s(s+1)\cdots(s+m-1).$$

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we set $\mathbf{s}^* = (s_r, \ldots, s_1)$. Then we can prove easily

$$(\mathbf{s})_{\mathbf{m}+\mathbf{n}} = (\mathbf{s})_{\mathbf{m}} (\mathbf{s}+\mathbf{m})_{\mathbf{n}}, \qquad (-\mathbf{s}^*)_{\mathbf{m}} = (-1)^{|\mathbf{m}|} \left(\mathbf{s} - \mathbf{m}^* + \frac{n}{r} \right)_{\mathbf{m}^*} \tag{1.2.4}$$

where $|\mathbf{m}| = m_1 + \cdots + m_r$. Here we identify $\lambda \in \mathbb{C}$ and $(\lambda, \ldots, \lambda) \in \mathbb{C}^r$.

1.2.4 Polynomials on $V^{\mathbb{C}}$

We set $\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1, \ldots, m_r) \in (\mathbb{Z}_{\geq 0})^r : m_1 \geq m_2 \geq \cdots m_r \geq 0\}$, and denote the space of holomorphic polynomials on $V^{\mathbb{C}}$ by $\mathcal{P}(V^{\mathbb{C}})$. For $\mathbf{m} \in \mathbb{Z}^r_{++}$, we define $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) := \mathbb{C}\text{-}\operatorname{span}\{\Delta_{\mathbf{m}} \circ l : l \in L^{\mathbb{C}}\}$. Then clearly $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ becomes a $L^{\mathbb{C}}$-module. Moreover, we have

Theorem 1.2.4 (Hua–Kostant–Schmid, see [8, Theorem XI.2.4])**.**

$$\mathcal{P}(V^{\mathbb{C}}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}).$$

These $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$'s are mutually inequivalent, and irreducible as $L^{\mathbb{C}}$-modules.

Since Δ_l vanishes on $\overline{\mathcal{X}_{l-1}}$, all polynomials in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ vanish on $\overline{\mathcal{X}_{l-1}}$ if and only if $m_l \neq 0$.

We write $d_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$, and $\Phi_{\mathbf{m}}(x) := \int_{K_L} \Delta_{\mathbf{m}}(kx) dk$. Then the K_L-fixed subspace in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ is spanned by $\Phi_{\mathbf{m}}$ (see [8, Proposition XI.3.1]).

1.2.5 Inner products on $\mathcal{P}(V^{\mathbb{C}})$

For $f, g \in \mathcal{P}(V^{\mathbb{C}})$, we denote the *Fischer inner product* by $\langle f, g \rangle_F$:

$$\langle f, g \rangle_F := \frac{1}{\pi^n} \int_{V^{\mathbb{C}}} f(w) \overline{g(w)} e^{-(w|w)} dw = f\left(\frac{\partial}{\partial w} \right) \bar{g}(w) \bigg|_{w=0}$$

(For the second equality see [8, Proposition XI.1.1]). Then the reproducing kernel of $\overline{\mathcal{P}(V^{\mathbb{C}})}^F$ (Hilbert completion of $\mathcal{P}(V^{\mathbb{C}})$) is given by $e^{(z|w)}$. We denote by $K^{\mathbf{m}}(z, w) = K^{\mathbf{m}}_w(z)$ the reproducing kernel of $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ with respect to $\langle \cdot, \cdot \rangle_F$. Then clearly,

$$e^{(z|w)} = \sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} K^{\mathbf{m}}(z, w),$$

Also, by [8, Proposition XI.3.3, Propsition XI.4.1.(ii)], we have

$$K^{\mathbf{m}}(gz, w) = K^{\mathbf{m}}(z, g^* w) \qquad \text{for any } g \in \mathrm{Str}(V^{\mathbb{C}}),$$
$$K_e^{\mathbf{m}}(z) = \frac{1}{\|\Phi_{\mathbf{m}}\|_F^2}\Phi_{\mathbf{m}}(z) = \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(z)$$

and

$$K^{\mathbf{m}}(x, \bar{x}) = K^{\mathbf{m}}(x^2, e)$$

for $x \in V$, and therefore for any $x \in V^{\mathbb{C}}$ by analytic continuation.

Also, for $\lambda > \frac{2n}{r} - 1$, we denote the *weighted Bergman inner product* on D by $\langle \cdot, \cdot \rangle_\lambda$:

$$\langle f, g \rangle_\lambda := \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)} \int_D f(w)\overline{g(w)} h(w, w)^{\lambda - \frac{2n}{r}} dw.$$

Then, these two inner products are related as follows:

Theorem 1.2.5 (Faraut–Korányi, see [8, Theorem XIII.2.7])**.** *If $f, g \in \mathcal{P}(V^{\mathbb{C}})$ are decomposed as $f = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} f_{\mathbf{m}}$, $g = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} g_{\mathbf{m}}$ ($f_{\mathbf{m}}, g_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$), then*

$$\langle f, g \rangle_\lambda = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}} \rangle_F. \tag{1.2.5}$$

Although the left hand side is only defined for $\lambda > \frac{2n}{r} - 1$, the right hand side extends meromorphically for $\lambda \in \mathbb{C}$. Therefore we can redefine $\langle \cdot, \cdot \rangle_\lambda$ with this formula for any $\lambda \in \mathbb{C}$ by restricting the domain. For $\lambda \in \mathbb{C}$ we set

$$\begin{aligned} \operatorname{rank}\lambda &:= \max\left\{ l \in \{0, 1, \ldots, r\} : (\lambda)_{\mathbf{m}} \neq 0 \text{ for any } \mathbf{m} \in \mathbb{Z}_{++}^r \cap \{m_{l+1} = 0\} \right\} \\ &= \begin{cases} l & \text{if } \lambda \in \left(l\frac{d}{2} + \mathbb{Z}_{\le 0}\right) \setminus \bigcup_{j=0}^{l-1} \left(j\frac{d}{2} + \mathbb{Z}_{\le 0}\right) \quad (l = 0, 1, \ldots, r-1), \\ r & \text{if } \lambda \notin \bigcup_{j=0}^{r-1} \left(j\frac{d}{2} + \mathbb{Z}_{\le 0}\right). \end{cases} \end{aligned} \tag{1.2.6}$$

For example, if $d = 2$, *i.e.*, $V = \mathrm{Herm}(r, \mathbb{C})$, then

$$\operatorname{rank}\lambda = \begin{cases} 0 & (\lambda \in \mathbb{Z}_{\le 0}), \\ l & (\lambda = l,\ l = 1, \ldots, r-1), \\ r & (\lambda \notin r - 1 + \mathbb{Z}_{\le 0}). \end{cases}$$

Then $\langle \cdot, \cdot \rangle_\lambda$ defines a sesquilinear form on $\bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r,\ m_{\operatorname{rank}\lambda + 1} = 0} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$. This form $\langle \cdot, \cdot \rangle_\lambda$ is positive definite if and only if

$$\lambda \in \mathcal{W} := \left\{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\right\} \cup \left((r-1)\frac{d}{2}, \infty\right). \tag{1.2.7}$$

This set $\mathcal{W}$ is called the *(Berezin–)Wallach set* (see [25] or [4]).

1.2.6 Invariant differential operators

For $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{\ge 0}$, we recall the differential operators $D^{(k)}$ from [8, Section XIV.2]:

$$D^{(k)}(\lambda) := \Delta(x)^{\frac{n}{r} - \lambda} \Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda - \frac{n}{r} + k}$$

where $\Delta\left(\frac{\partial}{\partial x}\right)$ is the differential operator characterized by $\Delta\left(\frac{\partial}{\partial x}\right) e^{(x|y)} = \Delta(y) e^{(x|y)}$. Then these operators commute with the $L^{\mathbb{C}}$-action (*i.e.*, $D^{(k)}(\lambda)(f \circ l) = (D^{(k)}(\lambda) f) \circ l$ for $f \in \mathcal{P}(V^{\mathbb{C}})$ and $l \in L^{\mathbb{C}}$). Moreover, we have

Proposition 1.2.6.

$$D^{(k)}(\lambda)e^{(x|y)} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} (-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}}K^{\mathbf{m}}(x,y)e^{(x|y)},$$

and if $(\lambda)_{\mathbf{m}} \neq 0$ *for any* $\mathbf{m}\in\mathbb{Z}^r_{++}$, $|\mathbf{m}|\le rk$*,*

$$D^{(k)}(\lambda)e^{(x|y)} = (\lambda)_{k\,1}F_1(-k,\lambda;-x,y)e^{(x|y)}$$

where

$$ {}_1F_1(-k,\lambda;-x,y) := \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} \frac{(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}K^{\mathbf{m}}(x,y). \tag{1.2.8}$$

Proof. We follow the proof of [8, Proposition XIV.1.5]. For $x\in\Omega$ and $\lambda < -k+1$,

$$\begin{aligned}
&D^{(k)}(\lambda)e^{(x|e)} = \Delta(x)^{\frac{n}{r}-\lambda}\Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda-\frac{n}{r}+k}e^{(x|e)}\\
&= \Delta(x)^{\frac{n}{r}-\lambda}\Delta\left(\frac{\partial}{\partial x}\right)^k \frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\int_\Omega e^{(x|e-y)}\Delta(y)^{-\lambda+\frac{n}{r}-k}\Delta(y)^{-\frac{n}{r}}dy\\
&= \Delta(x)^{\frac{n}{r}-\lambda}\frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\int_\Omega e^{(x|e-y)}\Delta(e-y)^k\Delta(y)^{-\lambda-k}dy\\
&= \Delta(x)^{\frac{n}{r}-\lambda}\frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} d_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\int_\Omega e^{(x|e-y)}\Phi_{\mathbf{m}}(y)\Delta(y)^{-\lambda-k}dy\\
&= \Delta(x)^{\frac{n}{r}-\lambda}\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} d_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{\Gamma_\Omega\left(\mathbf{m}-\lambda+\frac{n}{r}-k\right)}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\Phi_{\mathbf{m}}(x^{-1})\Delta(x)^{\lambda-\frac{n}{r}+k}e^{(x|e)}\\
&= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk}\frac{d_{\mathbf{m}}(-k)_{\mathbf{m}}\left(-\lambda+\frac{n}{r}-k\right)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{k-\mathbf{m}^*}(x)e^{(x|e)}\\
&= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk}\frac{d_{k-\mathbf{m}^*}(-k)_{k-\mathbf{m}^*}\left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}}\Phi_{\mathbf{m}}(x)e^{(x|e)}.
\end{aligned}$$

Here we used [8, Lemma XI.2.3] at the 2nd and 5th equalities, and [8, Corollary XII.1.3] at the 4th equality. At the 6th equality we used $\Phi_{\mathbf{m}}(x^{-1})\Delta(x)^k = \Phi_{k-\mathbf{m}^*}(x)$, which follows from the linear isomorphism $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) \to \mathcal{P}_{k-\mathbf{m}^*}(V^{\mathbb{C}})$, $p\mapsto \Delta(x)^kp(x^{-1})$. Now, $d_{\mathbf{m}} = d_{k-\mathbf{m}^*}$ holds by this isomorphism, and by (1.2.4),

$$\frac{(-k)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|k-\mathbf{m}^*|}\left(\frac{n}{r}+\mathbf{m}\right)_{k-\mathbf{m}}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|k-\mathbf{m}^*|}\left(\frac{n}{r}\right)_k}{\left(\frac{n}{r}\right)_{\mathbf{m}}\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|k-\mathbf{m}^*|}(-k)_{\mathbf{m}}}{(-1)^{|\mathbf{m}|}\left(\frac{n}{r}\right)_{\mathbf{m}}},$$
$$\left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*} = (-1)^{|k-\mathbf{m}^*|}(\lambda+\mathbf{m})_{k-\mathbf{m}}.$$

Therefore,

$$D^{(k)}(\lambda)e^{(x|e)} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk}(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x)e^{(x|e)}.$$

By the $L^{\mathbb{C}}$-invariance of $D^{(k)}(\lambda)$, for $y \in \Omega$,

$$
\begin{aligned}
&D^{(k)}(\lambda)e^{(x|y)} = D^{(k)}(\lambda)e^{(P(y^{\frac{1}{2}})x|e)} \\
&= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} (-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(P(y^{\frac{1}{2}})x)e^{(P(y^{\frac{1}{2}})x|e)} \\
&= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le rk} (-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}} K^{\mathbf{m}}(x,y)e^{(x|y)}.
\end{aligned}
$$

This holds for any $x, y \in V^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ by analytic continuation. The second equality follows from

$$
(\lambda+\mathbf{m})_{k-\mathbf{m}} = \frac{(\lambda)_k}{(\lambda)_{\mathbf{m}}}. \qquad \square
$$

Using these differential operators, we can calculate $\langle f, g\rangle_\lambda$ for $\lambda \in \mathbb{C}$: for $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$ and $f, g \in \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++},\ m_{\operatorname{rank}\lambda+1}=0} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$,

$$
\langle f, g\rangle_\lambda = \begin{cases} \dfrac{c_{\lambda+k}}{(\lambda)_k} \displaystyle\int_D (D^{(k)}(\lambda)f)(w)\overline{g(w)}h(w,w)^{\lambda+k-\frac{2n}{r}}\,dw & (\operatorname{rank}\lambda = r) \\ \displaystyle\lim_{\mu\to\lambda} \frac{c_{\mu+k}}{(\mu)_k} \int_D (D^{(k)}(\mu)f)(w)\overline{g(w)}h(w,w)^{\mu+k-\frac{2n}{r}}\,dw & (\operatorname{rank}\lambda < r) \end{cases} \tag{1.2.9}
$$

where $c_\lambda = \frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)}$ (see [8, Proposition XIV.2.2, Proposition XIV.2.5]). We can prove easily that this equality holds not only for polynomials, but also for holomorphic functions $f, g \in \mathcal{O}(D)$ with $D^{(k)}(\lambda)f$ and g bounded on $\overline{D}$.

1.3 Proof for main theorem

For $\lambda \in \mathbb{C}$ with $\operatorname{rank}\lambda = r$, the I and J-Bessel functions are defined by

$$
\begin{aligned}
\mathcal{I}_\lambda(x) &:= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x), \\
\mathcal{J}_\lambda(x) &:= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) = \mathcal{I}_\lambda(-x).
\end{aligned}
$$

If $\operatorname{rank}\lambda < r$, then $(\lambda)_{\mathbf{m}} = 0$ for some $\mathbf{m}$, so we cannot define these functions on entire $V^{\mathbb{C}}$. However, if $x \in \overline{\mathcal{X}_l}$, $\Phi_{\mathbf{m}}(x) = 0$ for $m_{l+1} \neq 0$, and therefore for any $\lambda \in \mathbb{C}$ we can define I and J-Bessel functions for $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$ (see (1.2.1) and (1.2.6)) by

$$
\begin{aligned}
\mathcal{I}_\lambda(x) &:= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ m_{\operatorname{rank}\lambda+1}=0} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x), \\
\mathcal{J}_\lambda(x) &:= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ m_{\operatorname{rank}\lambda+1}=0} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) = \mathcal{I}_\lambda(-x).
\end{aligned}
$$

Now we are ready to state the main theorem.

Theorem 1.3.1. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, take $k \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$. Then we have the integral expressions*

$$\mathcal{I}_\lambda\left(x^2\right) = c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-x,w)e^{2(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw,$$
$$\mathcal{J}_\lambda\left(x^2\right) = c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-ix,w)e^{2i(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw.$$

where

$$c_\lambda = \frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda-\frac{n}{r}\right)}, \quad {}_1F_1(-k,\lambda;x,w) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\leq rk,\\ m_{\operatorname{rank}\lambda+1}=0}} \frac{(-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}K^{\mathbf{m}}(x,w).$$

When $\operatorname{rank}\lambda = r$, the definition of ${}_1F_1$ clearly coincides with the one in (1.2.8).

Proof. We calculate $\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda$ in two ways. By (1.2.5),

$$\begin{aligned}\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda &= \left\langle \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} K^{\mathbf{m}}_{\bar{x}}, \sum_{\mathbf{n}\in\mathbb{Z}^r_{++}} K^{\mathbf{n}}_{x}\right\rangle_\lambda = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}}\left\langle K^{\mathbf{m}}_{\bar{x}}, K^{\mathbf{m}}_{x}\right\rangle_F \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}}K^{\mathbf{m}}(x,\bar{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}}K^{\mathbf{m}}(x^2,e) \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x^2) = \mathcal{I}(x^2).\end{aligned}$$

On the other hand, by (1.2.9) and Proposition 1.2.6,

$$\begin{aligned}\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda &= \lim_{\mu\to\lambda}\frac{c_{\mu+k}}{(\mu)_k}\int_D \left(D^{(k)}(\mu)e^{(w|\bar{x})}\right)\overline{e^{(w|x)}}h(w,w)^{\mu+k-\frac{2n}{r}}dw \\ &= \lim_{\mu\to\lambda} c_{\mu+k}\int_D {}_1F_1(-k,\mu;-x,w)e^{(w|\bar{x})}\overline{e^{(w|x)}}h(w,w)^{\mu+k-\frac{2n}{r}}dw \\ &= c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-x,w)e^{2(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw.\end{aligned}$$

The formula for $\mathcal{J}_\lambda\left(x^2\right)$ follows by replacing x by ix. □

From this theorem we can easily deduce the following corollary.

Corollary 1.3.2. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, if $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k} > 0$ such that*

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k}\left(1+|x|_1^{rk}\right)e^{2|\operatorname{Re}x|_1}, \qquad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k}\left(1+|x|_1^{rk}\right)e^{2|\operatorname{Im}x|_1}$$

where $|x|_1$ is the norm defined in Definition 1.2.1.

Proof. By Proposition 1.2.2, for $w \in D$, $x \in V^{\mathbb{C}}$,

$$|(\operatorname{Re}x|\operatorname{Re}w)| \leq |\operatorname{Re}x|_1|\operatorname{Re}w|_\infty \leq |\operatorname{Re}x|_1\frac{|w|_\infty + |\bar{w}|_\infty}{2} \leq |\operatorname{Re}x|_1.$$

Also, since ${}_1F_1(-k,\lambda;-x,w)$ is a polynomial of degree rk with respect to both x and w,

$$|{}_1F_1(-k,\lambda;-x,w)| \leq C'_{\lambda,k}\left(1+|x|_1^{rk}\right)\left(1+|w|_\infty^{rk}\right) \leq 2C'_{\lambda,k}\left(1+|x|_1^{rk}\right).$$

Therefore, by Theorem 1.3.1,

$$\begin{aligned}|\mathcal{I}_\lambda(x^2)| &\le |c_{\lambda+k}| \int_D |{}_1F_1(-k,\lambda;-x,w)| e^{2(\operatorname{Re} x|\operatorname{Re} w)} h(w,w)^{\operatorname{Re}\lambda+k-\frac{2n}{r}} dw \\ &\le 2|c_{\lambda+k}| C'_{\lambda,k}\left(1+|x|_1^{rk}\right) e^{2|\operatorname{Re} x|_1} \int_D h(w,w)^{\operatorname{Re}\lambda+k-\frac{2n}{r}} dw \\ &= C_{\lambda,k}\left(1+|x|_1^{rk}\right) e^{2|\operatorname{Re} x|_1}.\end{aligned}$$

The proof for $\mathcal{J}_\lambda\left(x^2\right)$ is similar. □

Remark 1.3.3. *In [17, Lemma 3.1] Möllers gave another estimate of $\mathcal{J}_\lambda(x)$:*

$$\left|\mathcal{J}_\lambda\left(x^2\right)\right| \le C\left(1+|x|_2^2\right)^{\frac{r(2n-1)}{4}} e^{2r|x|_2} \qquad \textit{for any } \lambda\in\mathcal{W},\ x\in\overline{\mathcal{X}_{\operatorname{rank}\lambda}}\subset V^{\mathbb{C}}.$$

However, our estimate is sharper because our leading term is given by $e^{2|\operatorname{Im} x|_1}$. Especially in our estimate $\mathcal{J}_\lambda(x)$ is uniformly bounded on V if $\operatorname{Re}\lambda$ is sufficiently large. This difference comes from that of methods of proofs: in [17] the Taylor expansion was used, while in this paper we use the integral formula. However, in general Taylor series is not strong enough for L^∞ estimates. For example, the bound of cosine function is calculated as follows:

$$|\cos x| = \left|\sum_{m=0}^\infty \frac{(-1)^m}{(2m)!}x^{2m}\right| \le \sum_{m=0}^\infty \frac{1}{(2m)!}|x|^{2m} \le \sum_{m=0}^\infty \frac{1}{m!}|x|^m = e^{|x|}.$$

However, it is well-known that cosine function is bounded unformly on $\mathbb{R}$. So this bound is not sharp.

1.4 Applications

For $\lambda > \frac{n}{r}-1$, $t\in\mathbb{C}\setminus\pi i\mathbb{Z}$, $\operatorname{Re} t\ge 0$, we define a integral operator on Ω: for a measurable function $\varphi:\Omega\to\mathbb{C}$, we define

$$\tau_\lambda(t)\varphi(x) := \frac{1}{\Gamma_\Omega(\lambda)}\int_\Omega \varphi(y)\frac{e^{-\coth t(\operatorname{tr} x+\operatorname{tr} y)}}{\sinh^{r\lambda} t}\mathcal{I}_\lambda\left(\frac{1}{\sinh^2 t}P(x^{\frac12})y\right)\Delta(y)^{\lambda-\frac{n}{r}}dy.$$

Since $\mathcal{I}_\lambda$ is K-invariant, by [8, Lemma XIV.1.2] we can replace $P(x^{\frac12})y$ by $P(y^{\frac12})x$.

Remark 1.4.1. *For $\lambda > \frac{2n}{r}-1$, the Laplace transform*

$$\mathcal{L}_\lambda : L^2(\Omega,\Delta(x)^{\lambda-\frac{n}{r}}dx) \longrightarrow L^2(V+\sqrt{-1}\Omega,\Delta(\operatorname{Im} z)^{\lambda-\frac{2n}{r}}dz)\cap\mathcal{O}(V+\sqrt{-1}\Omega)$$

is defined by

$$\mathcal{L}_\lambda\varphi(z) := \frac{2^n}{\Gamma_\Omega(\lambda)}\int_\Omega e^{i(z|x)}\varphi(x)\Delta(2x)^{\lambda-\frac{n}{r}}dx.$$

Then we can prove by the similar method to [8, Theorem XV.4.1] that

$$\begin{aligned}\mathcal{L}_\lambda\tau_\lambda(t)\mathcal{L}_\lambda^{-1}F(z) &= \Delta(-\sin(it)z+\cos(it)e)^{-\lambda} \\ &\qquad\times F\left((\cos(it)z+\sin(it)e)(-\sin(it)z+\cos(it)e)^{-1}\right).\end{aligned}$$

If t is purely imaginary, then this coincides with the restriction of the holomorphic discrete series representation of the simple Hermitian Lie group $\operatorname{Bihol}(V+\sqrt{-1}\Omega)$, to the center of the maximal compact subgroup $\operatorname{Stab}(ie)$. That is, τ_λ can be regarded as the natural complexification of the action of $Z(\operatorname{Stab}(ie))\subset\operatorname{Bihol}(V+\sqrt{-1}\Omega)$. Especially, $\tau_\lambda(s)\tau_\lambda(t) = \tau_\lambda(s+t)$ holds for $\lambda > \frac{2n}{r}-1$.

Remark 1.4.2. *Let E be an Euclidean vector space of dimension N with inner product $(\cdot|\cdot)_E$. Then the Hermite semigroup on $L^2(E)$ is given by*

$$\tilde{\tau}(t)f(\xi) := \frac{1}{(2\pi \sinh t)^{\frac{N}{2}}} \int_E f(\eta) \exp\left(-\frac{1}{2} \coth t(|\xi|_E^2 + |\eta|_E^2) + \frac{1}{\sinh t}(\xi|\eta)_E \right) d\eta \quad (1.4.1)$$

for $f \in L^2(E)$, $t \in \mathbb{C} \setminus \pi i\mathbb{Z}$, $\mathrm{Re}\, t \geq 0$ (see, e.g., [10, Section 5.2]). From now on we assume there exists an self-adjoint representation $\phi : V \to \mathrm{End}(E)$. We also assume $N > r(r-1)d$. Let $Q : E \to V$ be the quadratic map defined by

$$(\phi(x)\xi|\xi)_E = (x|Q(\xi))_V \qquad \text{for any } x \in V,\ \xi \in E.$$

Let $\Sigma := Q^{-1}(e) \subset E$ be the Stiefel manifold. Then we have

$$\int_\Sigma e^{-i(\xi|\sigma)} d\sigma = \mathcal{J}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2} \right) \right) \quad (1.4.2)$$

(see [8, Proposition XVI.2.3]). We extend Q to $Q : E^{\mathbb{C}} \to V^{\mathbb{C}}$ bilinearly. Then since $\mathcal{J}_\lambda(x) = \mathcal{I}_\lambda(-x)$ we have

$$\int_\Sigma e^{(\xi|\sigma)} d\sigma = \mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2} \right) \right).$$

If $f \in L^2(E)$ is written as $f(\xi) = F\left(\frac{1}{2}Q(\xi)\right)$ with a function F on V, then (1.4.1) can be rewritten as

$$\begin{aligned}
&\tilde{\tau}(t)f(\xi) = \frac{1}{(2\pi \sinh t)^{\frac{N}{2}}} \int_E F\left(\frac{1}{2}Q(\eta) \right) \exp\left(-\frac{1}{2} \coth t(|\xi|_E^2 + |\eta|_E^2) + \frac{1}{\sinh t}(\xi|\eta)_E \right) d\eta \\
&= \frac{1}{(\pi \sinh t)^{\frac{N}{2}}} \int_E F(Q(\eta)) \exp\left(-\coth t \left(\frac{1}{2}|\xi|_E^2 + |\eta|_E^2 \right) + \frac{\sqrt{2}}{\sinh t}(\xi|\eta)_E \right) d\eta \\
&= \frac{1}{\Gamma_\Omega(\frac{N}{2r}) \sinh^{\frac{N}{2}} t} \int_\Omega \int_\Sigma F(Q(\phi(y^{\frac{1}{2}})\sigma)) \exp\left(-\coth t \left(\frac{1}{2}|\xi|_E^2 + |\phi(y^{\frac{1}{2}})\sigma|_E^2 \right) \right) \\
&\qquad \times \exp\left(\frac{\sqrt{2}}{\sinh t}(\xi|\phi(y^{\frac{1}{2}})\sigma)_E \right) \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} d\sigma dy \\
&= \frac{1}{\Gamma_\Omega(\frac{N}{2r})} \int_\Omega \int_\Sigma F(y) \frac{\exp\left(-\coth t \left(\frac{1}{2}|\xi|_E^2 + \mathrm{tr}\, y \right) \right)}{\sinh^{\frac{N}{2}} t} \exp\left(\frac{\sqrt{2}}{\sinh t}(\phi(y^{\frac{1}{2}})\xi|\sigma)_E \right) \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} d\sigma dy \\
&= \frac{1}{\Gamma_\Omega(\frac{N}{2r})} \int_\Omega F(y) \frac{\exp\left(-\coth t \left(\frac{1}{2}|\xi|_E^2 + \mathrm{tr}\, y \right) \right)}{\sinh^{\frac{N}{2}} t} \mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{1}{\sqrt{2}\sinh t}\phi(y^{\frac{1}{2}})\xi \right) \right) \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} dy \\
&= \frac{1}{\Gamma_\Omega(\frac{N}{2r})} \int_\Omega F(y) \frac{\exp\left(-\coth t \left(\frac{1}{2}\mathrm{tr}\, Q(\xi) + \mathrm{tr}\, y \right) \right)}{\sinh^{\frac{N}{2}} t} \mathcal{I}_{\frac{N}{2r}}\left(\frac{1}{2\sinh^2 t} P(y^{\frac{1}{2}})Q(\xi) \right) \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} dy \\
&= \tau_{\frac{N}{2r}}(t) F\left(\frac{1}{2}Q(\xi) \right)
\end{aligned}$$

where we used [8, Proposition XVI.2.1] at the 3rd equality and [8, Lemma XVI.2.2.(ii)] at the 4th, 6th equalities. Therefore $\tau_{\frac{N}{2r}}(t)$ coincides with the action of the Hermite semigroup on radial functions on E.

Remark 1.4.3. *For $x \in \overline{\mathcal{X}_1}$ (see (1.2.1)), $\mathcal{I}_\lambda(x) = \Gamma(\lambda)\tilde{I}_{\lambda-1}(2\sqrt{|x|_2})$ holds (see [17, Example 3.3]), and by analytic continuation the distribution $\frac{1}{\Gamma_\Omega(\lambda)}\Delta(x)^{\lambda-\frac{n}{r}}\mathbf{1}_\Omega dx$ at $\lambda = \frac{d}{2}$*

gives the semi-invariant measure on $\overline{\mathcal{X}_1} \cap \overline{\Omega}$ (see [8, Proposition VII.2.3]). Therefore for $V = \mathbb{R}^{1,n-1}$ the action τ_λ at $\lambda = \frac{d}{2}$ coincides with the action of the holomorphic semigroup on the minimal representation of $O(p,2)$ (see [14, Theorem B] or [15, Theorem 5.1.1]).

Remark 1.4.4. *We set*

$$H_\lambda \varphi(x) := i^{r\lambda} \tau_\lambda \left(\frac{\pi i}{2}\right) \varphi(x) = \frac{1}{\Gamma_\Omega(\lambda)} \int_\Omega \varphi(y) \mathcal{J}\left(P(x^{\frac{1}{2}})y\right) \Delta(y)^{\lambda - \frac{n}{r}} dy.$$

This is called the generalized Hankel transform ([8, Section XV.4]). Similar to Remark 1.4.2, this is regarded as a variant of the Fourier transform. Therefore it is expected that this Hankel transform has similar properties as the Fourier transform such as a Paley-Wiener type theorem, which determines the image of the compactly supported functions. This is done by, e.g., [1], [16, Remark 5.4] for classical $V = \mathbb{R}$ case, but not for generalized case. In this paper we don't touch this topic in detail.

We set $K_\lambda(x,y;t) := e^{-\coth t(\operatorname{tr} x + \operatorname{tr} y)} \mathcal{I}_\lambda\left(\sinh^{-2} t P(x^{\frac{1}{2}})y\right)$, the kernel function of $\tau_\lambda(t)$. Then we can deduce from Theorem 1.3.2 that

Theorem 1.4.5. *Take $k \in \mathbb{Z}_{\geq 0}$ such that $\lambda + k > \frac{2n}{r} - 1$. Then if $t = u + iv$, $u, v \in \mathbb{R}$, $u \geq 0$,*

$$|K_\lambda(x,y;t)| \leq C_{\lambda,t}\left(1 + (\operatorname{tr} x \operatorname{tr} y)^{\frac{rk}{2}}\right) \exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr} x + \operatorname{tr} y)\right).$$

Especially, if $u = \operatorname{Re} t > 0$ then the integral defining $\tau_\lambda(t)$ converges if φ is of polynomial growth, and the resulting $\tau_\lambda(t)\varphi$ has exponential decay. Even if $u = \operatorname{Re} t = 0$, if $\lambda > \frac{2n}{r} - 1$ and $t \notin \pi i \mathbb{Z}$, the integral converges if $\varphi \in L^1(\Omega, \Delta(x)^{\lambda - \frac{n}{r}} dx)$, and the resulting $\tau_\lambda(t)\varphi$ is bounded. In order to prove this theorem, we prepare the following lemma.

Lemma 1.4.6. *(1) For $x \in \Omega$ the directional derivative of $x \mapsto \sqrt{x}$ is*

$$D_u \sqrt{x} = \frac{1}{2} L\left(\sqrt{x}\right)^{-1} u.$$

(2) For $x, y \in V$ if $[L(x), L(y)] = 0$, then there exists a Jordan frame $\{c_1, \dots, c_r\}$ such that $x, y \in \mathbb{R}\text{-}\operatorname{span}\{c_1, \dots, c_r\}$.

(3) For $x, y \in \Omega$, $\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y} \leq \sqrt{\operatorname{tr} x \operatorname{tr} y} \leq \dfrac{\operatorname{tr} x + \operatorname{tr} y}{2}$.

Proof. (1) $u = D_u x = D_u\left(\sqrt{x}\right)^2 = 2\sqrt{x} D_u \sqrt{x} = 2L\left(\sqrt{x}\right) D_u \sqrt{x}$ and then $D_u \sqrt{x} = \frac{1}{2} L\left(\sqrt{x}\right)^{-1} u$ follows.
(2) See [8, Lemma X.2.2].
(3) The second inequality is clear. For the first inequality, we take $k_0 \in K$ such that $\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})ky}$ ($k \in K_L$) attains its maximum at $k = k_0$. We put $k_0 y =: y_0$. Then for any $D \in \mathfrak{k}_\mathfrak{l} = Lie(K_L)$,

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \operatorname{tr}\sqrt{P(x^{\frac{1}{2}})e^{tD} y_0} = \frac{1}{2} \operatorname{tr}\left(L\left(\sqrt{P(x^{\frac{1}{2}})y_0}\right)^{-1} P(x^{\frac{1}{2}}) D y_0\right) \\ &= \frac{1}{2}\left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \middle| P(x^{\frac{1}{2}}) D y_0\right) = \frac{1}{2}\left(P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \middle| D y_0\right). \end{aligned}$$

We put $P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} =: z$. If $D = [L(u), L(v)]$ $(u, v \in V)$, then

$$\begin{aligned}
0 &= (z|[L(u), L(v)]y_0) = (z|u(vy_0)) - (z|v(uy_0)) = (zu|vy_0) - (zv|uy_0) \\
&= (y_0(zu)|v) - (v|(uy_0)z) = ([L(y_0), L(z)]u|v).
\end{aligned}$$

Since $(\cdot|\cdot)$ is non-degenerate, $[L(y_0), L(z)] = 0$. Also,

$$\begin{aligned}
P(z)y_0 &= P\left(P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\right)y_0 \\
&= P(x^{\frac{1}{2}})P\left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\right)P(x^{\frac{1}{2}})y_0 = P(x^{\frac{1}{2}})e = x.
\end{aligned}$$

So especially $[L(x), L(y_0)] = 0$. Let $x = \sum_{j=1}^r t_j c_j$, $y = \sum_{j=1}^r s_j d_j$ $(t_j, s_j > 0$, and $\{c_j\}_{j=1}^r, \{d_j\}_{j=1}^r$ are Jordan frames). Then,

$$\begin{aligned}
\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y} \leq \operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y_0} &= \operatorname{tr}\sqrt{P\left(\sum_{j=1}^r t_j^{\frac{1}{2}} c_j\right)\sum_{j=1}^r s_j c_j} \\
&= \sum_{j=1}^r \sqrt{t_j s_j} \leq \sqrt{\left(\sum_{j=1}^r t_j\right)\left(\sum_{j=1}^r s_j\right)} = \sqrt{\operatorname{tr}x \operatorname{tr}y}
\end{aligned}$$

and the proof is completed. □

Now we are ready to prove Theorem 1.4.5.

Proof of Theorem 1.4.5. By Corollary 1.3.2,

$$\begin{aligned}
|K_\lambda(x, y; t)| &\leq C'_\lambda e^{-\operatorname{Re}\coth t(\operatorname{tr}x + \operatorname{tr}y)}\left(1 + \left|\frac{1}{\sinh t}\sqrt{P(x^{\frac{1}{2}})y}\right|_1^{rk}\right) e^{2\left|\operatorname{Re}\frac{1}{\sinh t}\sqrt{P(x^{\frac{1}{2}})y}\right|_1} \\
&= C'_\lambda e^{-\operatorname{Re}\coth t(\operatorname{tr}x + \operatorname{tr}y)}\left(1 + \frac{1}{|\sinh t|^{rk}}\operatorname{tr}\left(\sqrt{P(x^{\frac{1}{2}})y}\right)^{rk}\right) e^{2\left|\operatorname{Re}\frac{1}{\sinh t}\right|\operatorname{tr}\left(\sqrt{P(x^{\frac{1}{2}})y}\right)} \\
&\leq C_{\lambda,t}\exp\left(-\frac{\cosh u \sinh u}{\cosh^2 u - \cos^2 v}(\operatorname{tr}x + \operatorname{tr}y)\right)\left(1 + \sqrt{\operatorname{tr}x \operatorname{tr}y}^{rk}\right) \\
&\qquad \times \exp\left(\frac{\sinh u |\cos v|}{\cosh^2 u - \cos^2 v}(\operatorname{tr}x + \operatorname{tr}y)\right) \\
&= C_{\lambda,t}\left(1 + (\operatorname{tr}x \operatorname{tr}y)^{\frac{rk}{2}}\right)\exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr}x + \operatorname{tr}y)\right)
\end{aligned}$$

and this completes the proof. □

Acknowledgements

The author would like to thank T. Kobayashi for many helpful advices on the topic of this chapter. He also thanks his colleagues for many helpful discussions.

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Chapter 2

Norm computation and analytic continuation of vector valued holomorphic discrete series representations

In this chapter we compute explicitly the norm of the vector-valued holomorphic discrete series representations, when its K-type is "almost multiplicity-free". As an application, we discuss the properties of highest weight modules, such as unitarizability, reducibility and composition series.

Keywords: holomorphic discrete series representations; highest weight modules; Jordan triple systems; composition series.
AMS subject classification: 22E45; 43A85; 17C30.

2.1 Introduction

The purpose of this chapter is to compute explicitly the norm of the vector-valued holomorphic discrete series representations, and to study the properties of the highest weight modules, such as unitarizabily, reducibility and composition series.

Let G be a simple Lie group, such that its maximal compact subgroup K has a non-discrete center. Then it is known that there exist a linear subspace $\mathfrak{p}^+ \subset \mathfrak{g}^{\mathbb{C}}$ and a bounded domain $D \subset \mathfrak{p}^+$ such that the symmetric space G/K is diffeomorphic to D. Therefore G/K becomes a complex manifold. Let (τ, V) be a finite-dimensional holomorphic representation of $K^{\mathbb{C}}$, and $\chi^{-\lambda}$ be a suitable character of the universal covering group $\tilde{K}^{\mathbb{C}}$. Then we can consider the representation of the universal covering group $\tilde{G}$ on the space of holomorphic sections of the equivariant vector bundle on G/K with fiber $V \otimes \chi^{-\lambda}$,

$$\tilde{G} \curvearrowright \Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})).$$

Since $D \simeq G/K$ is contractible, this space is isomorphic to the space of V-valued holomorphic functions on D,

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})) \simeq \mathcal{O}(D, V).$$

Then the infinitesimal action of the Lie subalgebra $\mathfrak{p}^+ \subset \mathfrak{g}^{\mathbb{C}}$ on $\mathcal{O}(D, V)$ is given by 1st order differential operators with constant coefficients, and thus it annihilates constant functions in $\mathcal{O}(D,V)$. Such representations are called the highest weight representations. Also, if $\lambda \in \mathbb{R}$ is sufficiently large, then this representation preserves an inner product which is given by an explicit integral on D. Such representations are called the holomorphic discrete series representations.

For example, let $G := Sp(r,\mathbb{R})$, realized explicitly as

$$Sp(r,\mathbb{R}) = \left\{ g \in GL(2r,\mathbb{C}) : g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},\ g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \bar{g} \right\}.$$

Then $G/K = Sp(r,\mathbb{R})/U(r)$ is diffeomorphic to

$$D := \{ w \in \mathrm{Sym}(r,\mathbb{C}) : I_r - ww^* \text{ is positive definite.} \}.$$

Let (τ, V) be a representation of $K^{\mathbb{C}} = GL(r,\mathbb{C})$. Then the universal covering group $\tilde{G} = \widetilde{Sp}(r,\mathbb{R})$ acts on $\mathcal{O}(D,V)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(w) = \det(cw+d)^{-\lambda} \tau\left({}^t(cw+d) \right) f\left((aw+b)(cw+d)^{-1} \right).$$

We note that $\det(cw+d)^{-\lambda}$ is not well-defined as a function on $G \times D$, but is well-defined as a function on the universal covering space $\tilde{G} \times D$. If $\operatorname{Re}\lambda$ is sufficiently large, then this preserves the sesquilinear form

$$\langle f, h \rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^{r(r+1)/2}} \int_D \left(\tau((I - ww^*)^{-1}) f(w), h(w) \right)_\tau \det(I - ww^*)^{\lambda-(r+1)} dw,$$

that is, $\langle \tau_\lambda(g) f, \tau_{\bar{\lambda}}(g) h \rangle_{\lambda,\tau} = \langle f, h \rangle_{\lambda,\tau}$ holds for any $f, h \in \mathcal{O}(D,V)$ with finite norms, and for any $g \in \tilde{G}$. Therefore τ_λ gives a holomorphic discrete series representation of $\tilde{G}$ if $\lambda \in \mathbb{R}$ and the above norm converges for some nonzero function in $\mathcal{O}(D,V)$. In this case the corresponding Hilbert space $\mathcal{H}_\lambda(D,V) \subset \mathcal{O}(D,V)$ has the reproducing kernel

$$K_{\lambda,\tau}(z,w) := \det(I_r - zw^*)^{-\lambda} \tau(I_r - zw^*) \in \mathcal{O}(D \times \overline{D}, \mathrm{End}(V)),$$

if we choose the normalizing constant c_λ suitably. When $r = 1$, then we have $G = SU(1,1)$ and $D = \{ w \in \mathbb{C} : |w| < 1 \}$, and the action τ_λ of $\widetilde{SU}(1,1)$ on $\mathcal{O}(D)$ reduces to the simplest example

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(w) = (cw+d)^{-\lambda} f\left(\frac{aw+b}{cw+d} \right),$$

with the invariant inner product and the reproducing kernel

$$\langle f, h \rangle_\lambda = \frac{\lambda - 1}{\pi} \int_{|w|<1} f(w) \overline{h(w)} (I - |w|^2)^{\lambda-2} dw, \tag{2.1.1}$$

$$K_\lambda(z,w) = (1 - z\bar{w})^{-\lambda} \in \mathcal{O}(D \times \overline{D}). \tag{2.1.2}$$

We return to the general case. The question of when the highest weight representations are unitarizable is studied by e.g. Berezin [2], Clerc [3], Vergne-Rossi [28], and Wallach [29], and completely classified by Enright-Howe-Wallach [4] and Jakobsen [13] by different methods. In [4] and [13] they used purely algebraic methods.

On the other hand, the analytical proof, the proof using explicit norm computation, was only partially successful. When the fiber (τ, V) is trivial, this is studied by e.g. Hua [11], Upmeier [27], and Ørsted [19], and completely done by Faraut-Korányi [6]. However, vector-valued cases are not computed yet except for a few cases, e.g. the case when (τ, V) is a defining representation of $K^{\mathbb{C}} = GL(s, \mathbb{C})$ (Ørsted-Zhang [20], [21]), and the case when G is of real rank 1 (Hwang-Liu-Zhang [12]).

Now we explain how the explicit norm computation gives informations on unitarizability and reducibility in the simplest example. Let $G = SU(1,1)$. Then the $\tilde{G}$-invariant inner product (2.1.1) converges for any polynomial $f, h \in \mathcal{P}(\mathbb{C})$ if $\operatorname{Re}\lambda > 1$, but does not converge for any non-zero polynomial $f, h \in \mathcal{P}(\mathbb{C})$ if $\operatorname{Re}\lambda \le 1$. Suppose f, h has a Taylor expansion $f(w) = \sum_m a_m w^m$, $h(w) = \sum_m b_m w^m$. Then for $\operatorname{Re}\lambda > 1$, we can compute $\langle f, h\rangle_\lambda$ explicitly as

$$\langle f, h\rangle_\lambda = \sum_{m=0}^{\infty} \frac{m!}{(\lambda)_m} a_m \overline{b_m},$$

where $(\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)$. This expression is available even if $\operatorname{Re}\lambda \le 1$, and is also $(\mathfrak{g}, \tilde{K})$-invariant. As a result, the reproducing kernel $K_\lambda(z, w)$ in (2.1.2) is expanded as

$$K_\lambda(z, w) = (1 - z\bar{w})^{-\lambda} = \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} z^m \bar{w}^m.$$

This expression is also available when $\operatorname{Re}\lambda \le 1$. This kernel function is positive definite if $\lambda \ge 0$, and thus $(\tau_\lambda, \mathcal{O}(D))$ is unitarizable if $\lambda \ge 0$. Here, when $\lambda = 0$, the corresponding Hilbert space consists of only 0th order polynomials, and is of 1-dimensional. Also, for $\lambda = -l \in \mathbb{Z}_{\le 0}$, the sesquilinear forms

$$\langle f, h\rangle_{-l} = \sum_{m=0}^{l} \frac{m!}{(-l)_m} a_m \overline{b_m}, \tag{2.1.3}$$

$$\lim_{\lambda \to -l} (\lambda + l)\langle f, h\rangle_\lambda = \frac{1}{(-l)_l} \sum_{m=l+1}^{\infty} \frac{m!}{(1)_{m-l-1}} a_m \overline{b_m} \tag{2.1.4}$$

are well-defined and $(\mathfrak{g}, K)$-invariant on $\mathcal{P}_{\le l}(\mathbb{C})$, the space of polynomials of order at most l, and on $\mathcal{P}(\mathbb{C})/\mathcal{P}_{\le l}(\mathbb{C})$ respectively. Moreover (2.1.4) is definite. Therefore $\mathcal{P}_{\le l}(\mathbb{C})$ gives a $(\mathfrak{g}, K)$-submodule, and $\mathcal{P}(\mathbb{C})/\mathcal{P}_{\le l}(\mathbb{C})$ gives a infinitesimally unitary $(\mathfrak{g}, K)$-module.

To compute the norm for general G, we use the K-type decomposition of $\mathcal{O}(D, V)_K = \mathcal{P}(\mathfrak{p}^+, V)$ instead of the Taylor expansion, fix a K-invariant norm $\|\cdot\|_{F,\tau}$ on $\mathcal{P}(\mathfrak{p}^+, V)$ independent of λ (see (2.3.2)), and compare $\|\cdot\|_{\lambda,\tau}$ and $\|\cdot\|_{F,\tau}$ on each K-type. Let

$$\mathcal{O}(D, V)_K = \mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_i W_i$$

be a K-type decomposition such that each W_i is orthogonal to the others with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$. Then since $\|\cdot\|_{\lambda,\tau}$ and $\|\cdot\|_{F,\tau}$ are both K-invariant, the ratio of two norms are constant on W_i. We denote this ratio by $R_i(\lambda)$. Moreover, if $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$ implies $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{\lambda,\tau}$ (for example, if $\mathcal{P}(\mathfrak{p}^+, V)$ is K-multiplicity free), then we have

$$\|f\|_{\lambda,\tau}^2 = \sum_i R_i(\lambda)\|f_i\|_{F,\tau}^2 \qquad (f \in \mathcal{O}(\mathfrak{p}^+, V))$$

where f_i is the orthogonal projection of f onto W_i, and the reproducing kernel $K_{\lambda,\tau}(z, w)$ is expanded as

$$K_{\lambda,\tau}(z, w) = \sum_i R_i(\lambda)^{-1} K_i(z, w),$$

where $K_i(z,w)$ is the reproducing kernel of W_i with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$. Similarly to the $SU(1,1)$ case, if we compute $R_i(\lambda)$ explicitly, then we can determine completely when the representation is unitarizable, or reducible, and can get some informations on composition series.

Since the above argument is available only if $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$ implies $W_i \perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{\lambda,\tau}$, we specialize our interest to (G,V)'s in the following table.

G	K	V	Where
$Sp(r,\mathbb{R})$	$U(r)$	$\bigwedge^k(\mathbb{C}^r)^\vee \quad (0\le k\le r-1)$	Thm 2.4.2
$SU(q,s)$	$S(U(q)\times U(s))$	$\mathbb{C}\boxtimes V' \quad (V'\text{: any irrep of } U(s))$	Thm 2.4.3 ($q\ge s$) Thm 2.5.1 ($q<s$)
$SO^*(2s)$	$U(s)$	$S^k(\mathbb{C}^s)^\vee$ $S^k(\mathbb{C}^s)\otimes\det^{-k/2} \quad (k\in\mathbb{Z}_{\ge 0})$	Thm 2.4.5 (s even) Thm 2.5.2, 2.5.5 (s odd)
$Spin_0(2,n)$	$(Spin(2)\times$ $Spin(n))/\mathbb{Z}_2$	$\mathbb{C}_{-k}\boxtimes V_{(k,\dots,k,\pm k)} \quad (k\in\frac{1}{2}\mathbb{Z}_{\ge 0},\ n\text{ even})$ $\mathbb{C}_{-k}\boxtimes V_{(k,\dots,k)} \quad (k\in\{0,\frac{1}{2}\},\ n\text{ odd})$	Thm 2.4.7
$E_{6(-14)}$	$SO(2)\times Spin(10)$	$\mathbb{C}_{-k/2}\boxtimes\mathcal{H}^k(\mathbb{R}^{10}) \quad (k\in\mathbb{Z}_{\ge 0})$	Prop 2.5.8, Conj 2.5.11
$E_{7(-25)}$	$SO(2)\times E_6$	$\mathbb{C}$	Already done in [7]

In the above cases, except for $G=SU(q,s)$ case, $\mathcal{P}(\mathfrak{p}^+,V)$ is multiplicity-free under K, which is proved by direct computation of K-type decomposition. We can also prove multiplicity-freeness a priori by using [14, Theorem 2]. In $G=SU(q,s)$ case, $\mathcal{P}(\mathfrak{p}^+,V)$ is not multiplicity-free in general, but each K-isotypic component sits in a single polynomial space, and thus the arguments explained above is still available.

When G is of tube type or $G=SU(q,s)$ with $q\ge s$, which we deal with in Section 2.4, we can compute the norm in a uniform way, by generalizing the technique used by Faraut-Korányi [7]. For these cases, the fibers V in the above table satisfy the condition that they remain irreducible even if restricted to some subgroup K_L of K, and this condition allows us to compute the norm explicitly. The same condition also appears in e.g. [3], [10]. In these papers they got some necessary condition on the unitarizability of highest weight representations, by considering when the reproducing kernel on the tube domain becomes a Laplace transform of some measure. Under the assumption that $V|_{K_L}$ is irreducible, the necessary and sufficient condition is also computable, and therefore this assumption seems to be natural.

However, when G is of non-tube type, there is no such uniform way to compute the norm at this time, and we do this by purely case-by-case analysis. For example, we use an embedding of G into a larger group, or use an embedding of some smaller subgroup into G. We deal with such cases in Section 2.5.

We enumerate the main results of this chapter.

Theorem 2.1.1 (Theorem 2.4.2). *When $G=Sp(r,\mathbb{R})$, and $(\tau,V)=(\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k},V^\vee_{\varepsilon_1+\cdots+\varepsilon_k})$ $(k=0,1,\dots,r-1)$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>r$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by*

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\ \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\,|\mathbf{k}|=k\\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_+}} V^\vee_{2\mathbf{m}+\mathbf{k}},$$

and for $f \in V^\vee_{2\mathbf{m}+\mathbf{k}}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}}{\|f\|^2_{F,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}} &= \frac{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)\right)}{\prod_{j=1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}} \\ &= \frac{1}{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)+1\right)_{m_j+k_j-1} \prod_{j=k+1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}}.\end{aligned}$$

Theorem 2.1.2 (Theorem 2.4.3, 2.5.1)**.** *When $G = SU(q,s)$, and $(\tau, V) = (\mathbf{1}^{(q)} \boxtimes \tau^{(s)}_{\mathbf{k}}, \mathbb{C} \otimes V^{(s)}_{\mathbf{k}})$ ($\mathbf{k} \in \mathbb{Z}^s_{++}$), $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda + k_s > q+s-1$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by*

$$\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V^{(s)}_{\mathbf{k}}\right) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^s_{++}} \bigoplus_{\mathbf{n}\in\mathbf{m}+\mathrm{wt}(\mathbf{k})} c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}} V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}},$$

and for $f \in V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\mathbf{1}^{(q)}\boxtimes\tau^{(s)}_{\mathbf{k}}}}{\|f\|^2_{F,\mathbf{1}^{(q)}\boxtimes\tau^{(s)}_{\mathbf{k}}}} = \frac{\prod_{j=1}^s (\lambda-(j-1))_{k_j}}{\prod_{j=1}^s (\lambda-(j-1))_{n_j}} = \frac{1}{\prod_{j=1}^s (\lambda-(j-1)+k_j)_{n_j-k_j}}.$$

Theorem 2.1.3 (Theorem 2.4.5)**.** *When $G = SO^*(4r)$, and $(\tau, V) = (\tau^\vee_{(k,0,\ldots,0)}, V^\vee_{(k,0,\ldots,0)})$ ($k \in \mathbb{Z}_{\geq 0}$), $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda > 4r-3$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by*

$$\mathcal{P}(\mathfrak{p}^+) \otimes V^\vee_{(k,0,\ldots,0)} = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{k}|=k \\ 0\leq k_j\leq m_{j-1}-m_j}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r)},$$

and for $f \in V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau^\vee_{(k,0,\ldots,0)}}}{\|f\|^2_{F,\tau^\vee_{(k,0,\ldots,0)}}} = \frac{(\lambda)_k}{\prod_{j=1}^r (\lambda-2(j-1))_{m_j+k_j}} = \frac{1}{(\lambda+k)_{m_1+k_1-k} \prod_{j=2}^r (\lambda-2(j-1))_{m_j+k_j}}.$$

When $G = SO^*(4r)$, and $(\tau, V) = (\tau^\vee_{(k/2,\ldots,k/2,-k/2)}, V^\vee_{(k/2,\ldots,k/2,-k/2)})$ ($k \in \mathbb{Z}_{\geq 0}$), $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda > 4r-3$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+) \otimes V^\vee_{\left(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2}\right)} = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{k}|=k \\ 0\leq k_j\leq m_j-m_{j+1}}} V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r)+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)},$$

and for $f \in V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r)+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{(k/2,\ldots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^\vee_{(k/2,\ldots,k/2,-k/2)}}} &= \frac{\prod_{j=1}^{r-1} (\lambda-2(j-1))_k}{\prod_{j=1}^r (\lambda-2(j-1))_{m_j-k_j+k}} \\ &= \frac{1}{\prod_{j=1}^{r-1} (\lambda+k-2(j-1))_{m_j-k_j} (\lambda-2(r-1))_{m_r-k_r+k}}.\end{aligned}$$

Theorem 2.1.4 (Theorem 2.5.2, 2.5.5)**.** *When $G=SO^*(4r+2)$ and $(\tau,V)=(\tau^\vee_{(k,0,\ldots,0)},V^\vee_{(k,0,\ldots,0)})$ $(k\in\mathbb{Z}_{\ge 0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-1$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by*

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(k,0,\ldots,0)}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge 0})^{r+1};|\mathbf{k}|=k\\ 0\le k_j\le m_{j-1}-m_j}}V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r,k_{r+1})},$$

and for $f\in V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r,k_{r+1})}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{(k,0,\ldots,0)}}}{\|f\|^2_{F,\tau^\vee_{(k,0,\ldots,0)}}}&=\frac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}}\\ &=\frac{1}{(\lambda+k)_{m_1+k_1-k}\prod_{j=2}^r(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}}.\end{aligned}$$

When $G=SO^*(4r+2)$ and $(\tau,V)=(\tau^\vee_{(k/2,\ldots,k/2,-k/2)},V^\vee_{(k/2,\ldots,k/2,-k/2)})$ $(k\in\mathbb{Z}_{\ge 0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-1$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{\left(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2}\right)}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge 0})^{r+1};|\mathbf{k}|=k\\ 0\le k_j\le m_j-m_{j+1}\\ 0\le k_r\le m_r}}V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)},$$

and for $f\in V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{(k/2,\ldots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^\vee_{(k/2,\ldots,k/2,-k/2)}}}&=\frac{\prod_{j=1}^r(\lambda-2(j-1))_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j-k_j+k}(\lambda-2r+1)_{k-k_{r+1}}}\\ &=\frac{1}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}}.\end{aligned}$$

Theorem 2.1.5 (Theorem 2.4.7)**.** *When $G=Spin_0(2,n)$ and*

$$(\tau,V)=\begin{cases}(\chi^{-k}\boxtimes\tau_{(k,\ldots,k,\pm k)},\mathbb{C}_{-k}\otimes V_{(k,\ldots,k,\pm k)}) & \left(k\in\frac{1}{2}\mathbb{Z}_{\ge 0}\right)\quad (n:\textit{even}),\\ (\chi^{-k}\boxtimes\tau_{(k,\ldots,k)},\mathbb{C}_{-k}\otimes V_{(k,\ldots,k)}) & \left(k=0,\frac{1}{2}\right)\quad (n:\textit{odd}),\end{cases}$$

$\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>n-1$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V=\begin{cases}\displaystyle\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{-k\le l\le k\\ m_1-m_2+l\ge k}}\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l)} & (n:\textit{even}),\\ \displaystyle\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{-k\le l\le k\\ m_1-m_2+l\ge k}}\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,|l|)} & (n:\textit{odd}),\end{cases}$$

and for $f\in\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l)}$ or $\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,|l|)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}=\frac{(\lambda)_{2k}}{(\lambda)_{m_1+k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}=\frac{1}{(\lambda+2k)_{m_1-k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}.$$

We also state the conjecture on $E_{6(-14)}$ in Section 2.5.5. From these theorems we can get informations on unitarizability, reducibility and composition series.

This chapter is organized as follows. In Section 2.2 we prepare some notations and review some facts on Lie algebras of Hermitian type and Jordan triple systems. In Section 2.3 we state and prove the theorems (Theorem 2.3.1, Corollary 2.3.4) which plays a key role in this chapter. In Section 2.4 and 2.5 we compute the norm explicitly. In Section 2.4 we deal with the cases that the norm is computable directly from the theorem in Section 2.3, and in Section 2.5 we deal with the cases that need more techniques. In Section 2.6 we apply the results on norm computation to the problems on unitarizabily, reducibility and composition series.

2.2 Preliminaries

2.2.1 Root decomposition

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a simple Hermitian Lie algebra, that is, the maximal compact part $\mathfrak{k}$ has a 1-dimensional center. We take an element z from the center of $\mathfrak{k}$ such that the eigenvalues of $\mathrm{ad}(z)$ are $+\sqrt{-1}$, 0, $-\sqrt{-1}$, and let

$$\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}^{+}\oplus\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^{-}$$

be the corresponding eigenspace decomposition. We denote the Cartan involution of $\mathfrak{g}^{\mathbb{C}}$ (the anti-holomorphic extension of the Cartan involution on $\mathfrak{g}$) by ϑ. Then $\mathfrak{p}^{+}$ has a Hermitian Jordan triple system structure with the product

$$(x,y,z)\longmapsto\{x,y,z\}:=-\frac{1}{2}[[x,\vartheta y],z],\qquad x,y,z\in\mathfrak{p}^{+}.$$

We take a maximal abelian subalgebra $\mathfrak{h}\subset\mathfrak{k}$. Then $\mathfrak{h}^{\mathbb{C}}$ becomes simaltaneously a Cartan subalgebra of both $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$. Let $\Delta=\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$ be the root system. We denote by $\Delta_{\mathfrak{p}^{\pm}}$, $\Delta_{\mathfrak{k}^{\mathbb{C}}}$ the all roots α such that the corresponding root space $\mathfrak{g}^{\mathbb{C}}_{\alpha}$ is contained in $\mathfrak{p}^{\pm}$, $\mathfrak{k}^{\mathbb{C}}$ respectively. Also, we take a positive root system $\Delta_{+}=\Delta_{+}(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$ such that $\Delta_{\mathfrak{p}^{+}}\subset\Delta_{+}$, and we denote $\Delta_{\mathfrak{k}^{\mathbb{C}},+}:=\Delta_{\mathfrak{k}^{\mathbb{C}}}\cap\Delta_{+}$. We set $n:=\dim\mathfrak{p}^{+}$, $r:=\mathrm{rank}_{\mathbb{R}}\,\mathfrak{g}$.

We take the set of strongly orthogonal roots $\{\gamma_1,\ldots,\gamma_r\}\subset\Delta_{\mathfrak{p}^{+}}$ such that

(1) γ_1 is the highest root in $\Delta_{\mathfrak{p}^{+}}$,

(2) γ_k is the root in $\Delta_{\mathfrak{p}^{+}}$ which is highest among the roots strongly orthogonal to each γ_j with $1\le j\le k-1$,

and for each j, we take $e_j\in\mathfrak{g}^{\mathbb{C}}_{\gamma_j}$ such that $-[[e_j,\vartheta e_j],e_j]=2e_j$. Then $\mathfrak{a}:=\bigoplus_{j=1}^{r}\mathbb{R}(e_j-\vartheta e_j)\subset\mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{p}$, and $\{e_1,\ldots,e_r\}$ is a Jordan frame on $\mathfrak{p}^{+}$. We set $e:=\sum_{j=1}^{r}e_j\in\mathfrak{p}^{+}$ (a maximal tripotent), and $h:=-[e,\vartheta e]\in\sqrt{-1}\mathfrak{h}$. Then $ad(h)$ has eigenvalues $2,1,0,-1,-2$. We set

$$\begin{aligned}\mathfrak{p}^{\pm}_{\mathrm{T}}&:=\{x\in\mathfrak{p}^{\pm}:[h,x]=\pm 2x\}\subset\mathfrak{p}^{\pm},\\ \mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}&:=[\mathfrak{p}^{+}_{\mathrm{T}},\mathfrak{p}^{-}_{\mathrm{T}}]\subset\mathfrak{k}^{\mathbb{C}},\\ \mathfrak{g}^{\mathbb{C}}_{\mathrm{T}}&:=\mathfrak{p}^{+}_{\mathrm{T}}\oplus\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}\oplus\mathfrak{p}^{-}_{\mathrm{T}},\\ \mathfrak{g}_{\mathrm{T}}&:=\mathfrak{g}^{\mathbb{C}}_{\mathrm{T}}\cap\mathfrak{g}.\end{aligned}$$

Then, $\mathfrak{p}_{\mathrm{T}}^{+}$ becomes a complex simple Jordan algebra with the product

$$x \cdot y := \{x, e, y\} = -\frac{1}{2}[[x, \vartheta e], y], \tag{2.2.1}$$

and $\mathfrak{g}_{\mathrm{T}}$ becomes a Lie algebra of tube type.

We define the Cayley transform $c : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ by $c := Ad(e^{\frac{\pi i}{4}(e-\vartheta e)})$, and set ${}^{c}\mathfrak{g} := c(\mathfrak{g})$, ${}^{c}\mathfrak{g}_{\mathrm{T}} := c(\mathfrak{g}_{\mathrm{T}})$. Then ${}^{c}\mathfrak{g}_{\mathrm{T}} \subset \mathfrak{g}_{\mathrm{T}}^{\mathbb{C}}$ is fixed by the involution $\sigma\vartheta := Ad(e^{\frac{\pi}{2}(e+\vartheta e)}) \circ \vartheta$. By direct computation we have

$$\begin{aligned} \sigma\vartheta|_{\mathfrak{p}_{\mathrm{T}}^{+}} &= \frac{1}{2} ad(e)^2 \circ \vartheta : \mathfrak{p}_{\mathrm{T}}^{+} \longrightarrow \mathfrak{p}_{\mathrm{T}}^{+}, \\ \sigma\vartheta|_{\mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}} &= (\mathrm{id}_{\mathfrak{k}^{\mathbb{C}}} + ad(e)ad(\vartheta e)) \circ \vartheta : \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}} \longrightarrow \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}, \\ \sigma\vartheta|_{\mathfrak{p}_{\mathrm{T}}^{-}} &= \frac{1}{2} ad(\vartheta e)^2 \circ \vartheta : \mathfrak{p}_{\mathrm{T}}^{-} \longrightarrow \mathfrak{p}_{\mathrm{T}}^{-}. \end{aligned}$$

That is, $\sigma\vartheta$ preserves the grading. Therefore we denote

$${}^{c}\mathfrak{g}_{\mathrm{T}} = \mathfrak{n}^{+} \oplus \mathfrak{l} \oplus \mathfrak{n}^{-} \subset \mathfrak{p}_{\mathrm{T}}^{+} \oplus \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}} \oplus \mathfrak{p}_{\mathrm{T}}^{-} = \mathfrak{g}_{\mathrm{T}}^{\mathbb{C}}.$$

Then the real form $\mathfrak{n}^{+}$ of $\mathfrak{p}_{\mathrm{T}}^{+}$ becomes a Euclidean simple Jordan algebra.

We set $\mathfrak{a}_{\mathfrak{l}} := c(\mathfrak{a}) = \sqrt{-1}\mathfrak{h} \cap \mathfrak{l} = \bigoplus_{j=1}^{r} \mathbb{R} h_j$, where $h_j := -[e_j, \vartheta e_j]$. Then the restricted root system $\Sigma = \Sigma({}^{c}\mathfrak{g}, \mathfrak{a}_{\mathfrak{l}})$ is given by

$$\Sigma = \begin{cases} \left\{ \frac{1}{2}(\gamma_j - \gamma_k)\Big|_{\mathfrak{a}_{\mathfrak{l}}} : \begin{matrix} 1 \le j, k \le r, \\ j \ne k \end{matrix} \right\} \cup \left\{ \pm\frac{1}{2}(\gamma_j + \gamma_k)\Big|_{\mathfrak{a}_{\mathfrak{l}}} : 1 \le j \le k \le r \right\} & (\mathfrak{g} = \mathfrak{g}_{\mathrm{T}}), \\ (\text{as above}) \cup \left\{ \pm\frac{1}{2}\gamma_j\Big|_{\mathfrak{a}_{\mathfrak{l}}} : 1 \le j \le r \right\} & (\mathfrak{g} \ne \mathfrak{g}_{\mathrm{T}}). \end{cases}$$

We define the positive restricted roots Σ_{+} by

$$\Sigma_{+} = \begin{cases} \left\{ \frac{1}{2}(\gamma_j - \gamma_k)\Big|_{\mathfrak{a}_{\mathfrak{l}}} : 1 \le j < k \le r \right\} \cup \left\{ \frac{1}{2}(\gamma_j + \gamma_k)\Big|_{\mathfrak{a}_{\mathfrak{l}}} : 1 \le j \le k \le r \right\} & (\mathfrak{g} = \mathfrak{g}_{\mathrm{T}}), \\ (\text{as above}) \cup \left\{ \frac{1}{2}\gamma_j\Big|_{\mathfrak{a}_{\mathfrak{l}}} : 1 \le j \le r \right\} & (\mathfrak{g} \ne \mathfrak{g}_{\mathrm{T}}). \end{cases}$$

Then Σ_{+} and Δ_{+} are compatible, that is, $\alpha \in \Delta_{+}$ implies $\alpha|_{\mathfrak{a}_{\mathfrak{l}}} \in \Sigma_{+} \cup \{0\}$. We set

$$\begin{aligned} \mathfrak{l}_{jk} &:= \left\{ X \in {}^{c}\mathfrak{g}_{\mathrm{T}} : ad(H)X = \frac{1}{2}(\gamma_j - \gamma_k)(H)X \text{ for any } H \in \mathfrak{a}_{\mathfrak{l}} \right\} && (1 \le j, k \le r,\ j \ne k), \\ \mathfrak{m}_{\mathfrak{l}} &:= \left\{ X \in {}^{c}\mathfrak{g}_{\mathrm{T}}^{\vartheta} : ad(H)X = 0 \text{ for any } H \in \mathfrak{a}_{\mathfrak{l}} \right\}, \\ \mathfrak{n}_{jk}^{\pm} &:= \left\{ X \in {}^{c}\mathfrak{g}_{\mathrm{T}} : ad(H)X = \pm\frac{1}{2}(\gamma_j + \gamma_k)(H)X \text{ for any } H \in \mathfrak{a}_{\mathfrak{l}} \right\} && (1 \le j \le k \le r), \\ \mathfrak{p}_{jk}^{\pm} &:= (\mathfrak{n}_{jk}^{\pm})^{\mathbb{C}} && (1 \le j \le k \le r), \\ \mathfrak{p}_{0j}^{\pm} &:= \left\{ X \in \mathfrak{p}^{\pm} : ad(H)X = \pm\frac{1}{2}\gamma_j(H)X \text{ for any } H \in \mathfrak{a}_{\mathfrak{l}} \right\} && (1 \le j \le r), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{k}_{\mathfrak{l}} &:= \mathfrak{l}^{\vartheta} = \{ X \in \mathfrak{l} : \vartheta X = Ad(e^{\frac{\pi}{2}(e+\vartheta e)})X = X \}, \\ \mathfrak{n}_{\mathfrak{l}}^{-} &:= \bigoplus_{1 \le k < j \le r} \mathfrak{l}_{jk}. \end{aligned}$$

Then we have

$$\mathfrak{l} = \mathfrak{a}_{\mathfrak{l}} \oplus \mathfrak{m}_{\mathfrak{l}} \oplus \bigoplus_{j\neq k} \mathfrak{l}_{jk} = \mathfrak{k}_{\mathfrak{l}} \oplus \mathfrak{a}_{\mathfrak{l}} \oplus \mathfrak{n}_{\mathfrak{l}}^-,$$

$$\mathfrak{n}^{\pm} = \bigoplus_{1\le j\le k\le r} \mathfrak{n}_{jk}^{\pm}, \qquad \mathfrak{p}_{\mathrm{T}}^{\pm} = \bigoplus_{1\le j\le k\le r} \mathfrak{p}_{jk}^{\pm}, \qquad \mathfrak{p}^{\pm} = \bigoplus_{\substack{0\le j\le k\le r\\ (j,k)\neq(0,0)}} \mathfrak{p}_{jk}^{\pm}.$$

The decomposition $\mathfrak{n}^+ = \bigoplus_{j\le k} \mathfrak{n}_{jk}^+$, or $\mathfrak{p}^+ = \bigoplus_{j\le k} \mathfrak{p}_{jk}^+$, coincides with the Peirce decomposition of the Jordan algebra $\mathfrak{n}^+$, or the Jordan triple system $\mathfrak{p}^+$, with respect to the Jordan frame $\{e_1, \ldots, e_r\}$. We set $d := \dim_{\mathbb{C}} \mathfrak{p}_{12}^+$, $b := \dim_{\mathbb{C}} \mathfrak{p}_{01}^+$, and $n_{\mathrm{T}} := \dim_{\mathbb{C}} \mathfrak{p}_{\mathrm{T}}^+$. Then $n = r + \frac{1}{2}r(r-1)d + br$ and $n_{\mathrm{T}} = r + \frac{1}{2}r(r-1)d$ holds. Also we set $p := 2 + (r-1)d + b$.

Throughout this chapter, let $G^{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and let $G, {}^cG_{\mathrm{T}}, K, K^{\mathbb{C}}, K_{\mathrm{T}}^{\mathbb{C}}$ be the connected Lie subgroups with Lie algebras $\mathfrak{g}, {}^c\mathfrak{g}_{\mathrm{T}}, \mathfrak{k}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}$ respectively. Also we set $L := K^{\mathbb{C}} \cap {}^cG_{\mathrm{T}}$, $K_L := K \cap L$ (possibly non-connected, with Lie algebras $\mathfrak{l}, \mathfrak{k}_{\mathfrak{l}}$), let A_L, N_L^- be the connected Lie subgroups of L with Lie algebras $\mathfrak{a}_{\mathfrak{l}}, \mathfrak{n}_{\mathfrak{l}}^-$ respectively, and let M_L be the centralizer of $\mathfrak{a}_{\mathfrak{l}}$ in K_L.

We write

$$\begin{aligned}
\bar{x} &:= \sigma\vartheta x = \frac{1}{2}ad(e)^2(\vartheta x) && (x \in \mathfrak{p}_{\mathrm{T}}^+),\\
l^* &:= -\vartheta l && (l \in \mathfrak{k}^{\mathbb{C}}),\\
{}^t l &:= -\sigma l = -(\mathrm{id}_{\mathfrak{k}^{\mathbb{C}}} + ad(e)ad(\vartheta e))(l) && (l \in \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}),\\
\bar{l} &:= \sigma\vartheta l = (\mathrm{id}_{\mathfrak{k}^{\mathbb{C}}} + ad(e)ad(\vartheta e)) && (l \in \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}).
\end{aligned}$$

Then these are (anti-)involutions on $\mathfrak{p}_{\mathrm{T}}^+$, $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}$, which preserves $\mathfrak{n}^+$, $\mathfrak{k}$, $(\mathfrak{k}_{\mathfrak{l}})^{\mathbb{C}}$ and $\mathfrak{l}$ respectively. Also, we denote by the same symbols *, ^{t} and $^-$ the corresponding (anti-)involutions on $K^{\mathbb{C}}$ and $K_{\mathrm{T}}^{\mathbb{C}}$. Also, for $x \in \mathfrak{p}^+$ and $l \in K^{\mathbb{C}}$ or $\mathfrak{k}^{\mathbb{C}}$, we abbreviate $Ad(l)x$ or $ad(l)x$ as lx.

2.2.2 Some operations and polynomials on Jordan algebras

As in the previous subsection, $\mathfrak{p}^+$ has a Jordan triple system structure, and $\mathfrak{p}_{\mathrm{T}}^+, \mathfrak{n}^+$ has a Jordan algebra structure. For $x, y \in \mathfrak{p}^+$, we define $x\Box y$, $B(x,y) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{p}^+)$ by, for $z \in \mathfrak{p}^+$,

$$\begin{aligned}
(x\Box y)z &:= \{x,y,z\} = -\frac{1}{2}ad([x,\vartheta y])z,\\
B(x,y)z &:= x - 2\{x,y,z\} + \{x,\{y,z,y\},x\} = \left(I_{\mathfrak{p}^+} + ad([x,\vartheta y]) + \frac{1}{4}ad(x)^2 ad(\vartheta y)^2\right) z.
\end{aligned}$$

These depends holomorphically on x, and anti-holomorphically on y. Also, for $x \in \mathfrak{p}_{\mathrm{T}}^+$, we define $L(x), P(x) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{p}_{\mathrm{T}}^+)$ by, for $y \in \mathfrak{p}_{\mathrm{T}}^+$,

$$\begin{aligned}
L(x)y &:= xy = -\frac{1}{2}ad([x,\vartheta e])y,\\
P(x)y &:= 2x(xy) - (x^2)y = \frac{1}{4}ad(x)^2 ad(\vartheta e)^2 y.
\end{aligned}$$

Then for $x, y \in \mathfrak{p}^+$ and $l \in K^{\mathbb{C}}$,

$$\begin{aligned}
lx\Box(l^*)^{-1}y &= l(x\Box y)l^{-1},\\
B(lx,(l^*)^{-1}y) &= lB(x,y)l^{-1}
\end{aligned}$$

holds, and for $x \in \mathfrak{p}^+_{\mathrm{T}}$, $l \in K^{\mathbb{C}}_{\mathrm{T}}$,

$$\begin{aligned} P(lx) &= lP(x)^t l, \\ B(x,\overline{x})|_{\mathfrak{p}^+_{\mathrm{T}}} &= P(e-x^2) \end{aligned}$$

holds. We define an inner product $(\cdot|\cdot)$ on $\mathfrak{p}^+$ by

$$(x|y) := \frac{2}{p}\operatorname{Tr}(x\Box y : \mathfrak{p}^+ \to \mathfrak{p}^+).$$

Then for $l \in K^{\mathbb{C}}$, $(lx|y) = (x|l^*y)$ holds. This inner product is proportional to the restriction of the Killing form on $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{p}^+ \times \mathfrak{p}^-$, under the identification of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ through ϑ. Also, let $\operatorname{tr}(x)$, $\det(x)$ be the trace and determinant polynomials of the Jordan algebra $\mathfrak{p}^+_{\mathrm{T}}$, and let $h(x,y)$ be the generic norm of the Jordan triple system $\mathfrak{p}^+$. Then these polynomials are expressed by

$$\begin{aligned} \frac{n_{\mathrm{T}}}{r}\operatorname{tr}(x) &= \operatorname{Tr}(L(x) : \mathfrak{p}^+_{\mathrm{T}} \to \mathfrak{p}^+_{\mathrm{T}}), \\ (\det(x))^{2n_{\mathrm{T}}/r} &= \operatorname{Det}(P(x) : \mathfrak{p}^+_{\mathrm{T}} \to \mathfrak{p}^+_{\mathrm{T}}), \\ (h(x,y))^p &= \operatorname{Det}(B(x,y) : \mathfrak{p}^+ \to \mathfrak{p}^+). \end{aligned}$$

$\operatorname{tr}(x)$ is a linear form satisfying $\operatorname{tr}(x) = (x|e)$, and $\det(x)$, $h(x,y)$ are polynomials of degree r with respect to each variable. These polynomials satisfy

$$\begin{aligned} \det(lx) &= \det(le)\det(x) && (l \in K^{\mathbb{C}}_{\mathrm{T}},\ x \in \mathfrak{p}^+_{\mathrm{T}}), \\ h(lx,(l^*)^{-1}y) &= h(x,y) && (l \in K^{\mathbb{C}},\ x,y \in \mathfrak{p}^+), \\ h(x,\overline{x}) &= \det(e-x^2) && (x \in \mathfrak{p}^+_{\mathrm{T}}). \end{aligned}$$

From now we abbreviate $B(x,x) = B(x)$, $h(x,x) = h(x)$, and $(x|x) = |x|^2$ for $x \in \mathfrak{p}^+$. Then $B(x)$ is self-adjoint on $\mathfrak{p}^+$, and therefore $h(x)$ is real-valued. Also we set

$$\begin{aligned} \Omega &:= \{x^2 \in \mathfrak{n}^+ : x \in \mathfrak{n}^+,\ \det(x) \neq 0\}, \\ D &:= (\text{connected componet of } \{w \in \mathfrak{p}^+ : h(w) > 0\} \text{ which contains } 0). \end{aligned}$$

Then L acts on Ω by linear transformation, and G acts on $D \subset \mathfrak{p}^+$ via Borel embedding, which we will review later. Moreover we have

$$\Omega \simeq L/K_L, \qquad D \simeq G/K.$$

For $x \in \Omega$, $P(x)$ is positive definite on $\mathfrak{n}^+$, and there exists a unique element $l \in \exp(\mathfrak{l}^{-\vartheta}) \subset L$ such that $P(x) = Ad(l)|_{\mathfrak{n}^+}$. We denote such $l \in L$ by the same $P(x)$. Similarly, for $z,w \in D$, $B(z,w)$ is invertible on $\mathfrak{p}^+$, and there exists an element $l \in K^{\mathbb{C}}$ such that $B(z,w) = Ad(l)|_{\mathfrak{p}^+}$. So we define the holomorphic map $B : D \times \overline{D} \to K^{\mathbb{C}}$ (with the same symbol B) such that $Ad(B(z,w))|_{\mathfrak{p}^+} = B(z,w)$ and $B(0,0) = \mathbf{1}$. Clearly $P(x)$ and $B(z,w)$ are also well-defined as elements of the universal covering groups $\tilde{L}$, $\tilde{K}^{\mathbb{C}}$.

Now we recall the Peirce decomposition

$$\mathfrak{p}^+ = \bigoplus_{\substack{0 \le j \le k \le r \\ (j,k) \neq (0,0)}} \mathfrak{p}^+_{jk}.$$

We set

$$\mathfrak{p}^+_{(l)} := \bigoplus_{1 \le j \le k \le l} \mathfrak{p}^+_{jk}$$

for $l = 1, 2, \ldots, r$. Then each $\mathfrak{p}^+_{(l)}$ is again a unital Jordan algebra. For each l, let $\det_{(l)}$ be the determinant polynomial of $\mathfrak{p}^+_{(l)}$, $P_l : \mathfrak{p}^+ \to \mathfrak{p}^+_{(l)}$ be the orthogonal projection, and we set

$$\Delta_l(x) := \det_{(l)}(P_l(x)).$$

For $l = r$ we also write

$$\Delta(x) = \Delta_r(x) = \det(x).$$

Using these, for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we set

$$\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_{r-1}(x)^{s_{r-1} - s_r} \Delta_r(x)^{s_r}.$$

If $\mathbf{m} \in \mathbb{Z}^r$ and $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$, then $\Delta_{\mathbf{m}}$ is a polynomial of degree $m_1 + \cdots + m_r$. We denote this condition by $\mathbb{Z}^r_{++}$:

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r : m_1 \ge \cdots \ge m_r \ge 0\}.$$

For later use, we prepare another set $\mathbb{Z}^r_+$:

$$\mathbb{Z}^r_+ := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r : m_1 \ge \cdots \ge m_r\}.$$

Now for $q \in (M_L A_L N_L^-)^{\mathbb{C}}$, since q preserves each $\mathfrak{p}^+_{(l)}$, we have

$$\Delta_{\mathbf{s}}(qx) = \Delta_{\mathbf{s}}(qe)\Delta_{\mathbf{s}}(x).$$

That is, for any $\mathbf{m}$, $\Delta_{\mathbf{m}}$ is a lowest weight vector with lowest weight $-m_1\gamma_1 - \cdots - m_r\gamma_r$ under the representation

$$L \longrightarrow \mathrm{End}(\mathcal{P}(\mathfrak{p}^+)), \qquad l \longmapsto (f(x) \longmapsto f(l^{-1}x))$$

where $\mathcal{P}(\mathfrak{p}^+)$ denotes the space of all holomorphic polynials on $\mathfrak{p}^+$. In fact, we have

Theorem 2.2.1 (Hua-Kostant-Schmid, [5, Part III, Theorem V.2.1])**.**

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$$

where $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ is the irreducible representation of $K^{\mathbb{C}}$ with lowest weight $-m_1\gamma_1 - \cdots - m_r\gamma_r$.

We quote another theorem here.

Theorem 2.2.2 ([7, Theorem XII.2.2])**.** *The irreducible representation V of L has a K_L-fixed vector if and only if the lowest weight $-\lambda$ is of the form $-\lambda = -m_1\gamma_1 - \cdots - m_r\gamma_r$ with $(m_1, \ldots, m_r) \in \mathbb{Z}^r_+$.*

For $l = 0, 1, \ldots, r$ we set

$$\mathcal{O}_l := Ad(K^{\mathbb{C}})(e_1 + \cdots + e_l) \subset \mathfrak{p}^+. \tag{2.2.2}$$

Then $K^{\mathbb{C}}$ acts on each $\mathcal{O}_l$ transitively, and we have the orbit decomposition

$$\mathfrak{p}^+ = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r.$$

For each orbit $\mathcal{O}_l$, its closure $\overline{\mathcal{O}_l}$ is given by

$$\overline{\mathcal{O}_l} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_l.$$

Also, since the polynomial $\Delta_{l+1}(x)$ vanishes on $\overline{\mathcal{O}_l}$, the polynomial space on $\overline{\mathcal{O}_l}$ decomposes under $K^{\mathbb{C}}$ as

$$\mathcal{P}(\overline{\mathcal{O}_l}) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++} \\ m_{l+1}=m_{l+2}=\cdots=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+). \tag{2.2.3}$$

Each orbit $\mathcal{O}_l$ has the dimension

$$\dim_{\mathbb{C}} \mathcal{O}_l = l + \frac{1}{2} l(2r-l-1)d + lb \tag{2.2.4}$$

since the tangent space of $\mathcal{O}_l$ at $e_1 + \cdots + e_l$ is given by

$$T_{e_1+\cdots+e_l}\mathcal{O}_l = \bigoplus_{\substack{0\le j\le k\le r \\ j\le l,\, (j,k)\neq(0,0)}} \mathfrak{p}^+_{jk}.$$

Now we recall the generalized Gamma function, which was introduced by Gindikin [8]. For $\mathbf{s} \in \mathbb{C}^n$ this is defined as

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\operatorname{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n_{\mathrm{T}}}{r}} dx.$$

This integral converges if $\operatorname{Re} s_j > (j-1)\frac{d}{2}$, and we have the following equality

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n_{\mathrm{T}}-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{d}{2}\right)$$

([7, Corollary VII.1.3]), and this is meromorphically extended on $\mathbb{C}^n$. Also we denote

$$(\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{j=1}^{r} \left(s_j - (j-1)\frac{d}{2}\right)_{m_j}.$$

For $\mathbf{s} = (\lambda, \ldots, \lambda)$, we abbreviate $(\lambda, \ldots, \lambda) =: \lambda$. For example, we denote

$$\Gamma_\Omega((\lambda, \ldots, \lambda)) = \Gamma_\Omega(\lambda), \qquad ((\lambda, \ldots, \lambda))_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda+\mathbf{m})}{\Gamma_\Omega(\lambda)} = (\lambda)_{\mathbf{m}}.$$

2.3 Norm computation: General theory

2.3.1 Holomorphic discrete series representation

In this subsection we recall the explicit realization of the holomorphic series representation of the universal covering group $\tilde{G}$. First we recall the Borel embedding.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, "\wr"'] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

We consider maps $\pi^+ : G \times D \to D \subset \mathfrak{p}^+$, $\kappa : G \times D \to K^{\mathbb{C}}$, $\pi^- : G \times D \to \mathfrak{p}^-$ such that

$$g\exp(w) = \exp(\pi^+(g,w))\kappa(g,w)\exp(\pi^-(g,w)) \qquad (g \in G, w \in D).$$

Then π^+ gives the action of G on D, so we abbreviate $\pi^+(g,w) =: gw$. On $K \subset G$ this coincides with the adjoint action. Also, κ satisfies the cocycle condition

$$\kappa(gh,w) = \kappa(g,hw)\kappa(h,w) \qquad (g,h \in G,\ w \in D),$$

and for $k \in K$, $\kappa(k,w) = k$ holds. $Ad(\kappa(g,w))|_{\mathfrak{p}^+} \in \mathrm{End}(\mathfrak{p}^+)$ coincides with the tangent map of $w \mapsto gw = \pi^+(g,w)$ at $w \in \mathfrak{p}^+$. We naturally lift κ to the universal covering group, and we denote this map by the same symbol $\kappa : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$.

Let (τ, V) be a finite dimensional irreducible complex representation of $K^{\mathbb{C}}$, and we fix a K-invariant inner product $(\cdot,\cdot)_\tau$ on V. Also, let χ^λ be the character of $\tilde{K}^{\mathbb{C}}$ such that $\chi(k)^\lambda = \mathrm{Det}(Ad(k)|_{\mathfrak{p}^+})^{\lambda/p}$. We consider the space of holomorphic sections

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})).$$

Then since $G/K \simeq D$ is contractible, this is isomorphic to $\mathcal{O}(D,V)$, the space of V-valued holomorphic functions. Under this identification, the natural action τ_λ of $\tilde{G}$ on $\mathcal{O}(D,V)$ is written as

$$\tau_\lambda(g)f(w) = \chi(\kappa(g^{-1},w))^\lambda \tau(\kappa(g^{-1},w))^{-1} f(g^{-1}w) \qquad (g \in \tilde{G}, w \in D, f \in \mathcal{O}(D,V)).$$

Its differential representation is given by, for $u + l - \vartheta v \in \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^- = \mathfrak{g}^{\mathbb{C}}$,

$$\begin{aligned} d\tau_\lambda(u + l - \vartheta v)f(w) &= -\lambda d\chi(l + [w,\vartheta v])f(w) + d\tau(l + [w,\vartheta v])f(w) \\ &\qquad + \left.\frac{d}{dt}\right|_{t=0} f\left(w - t\left(u + ad(l)w - \frac{1}{2}ad(w)^2\vartheta v\right)\right). \end{aligned}$$

Then since $\kappa(g,w)B(w)\kappa(g,w)^* = B(gw)$ holds for any $g \in \tilde{G}$, $w \in D$ (see [16, Lemma 2.11]), this action preserves the following *weighted Bergman inner product*

$$\langle f, g\rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^n}\int_D \left(\tau(B(w)^{-1})f(w), g(w)\right)_\tau h(w)^{\lambda-p}dw \qquad (f, g \in \mathcal{O}(D,V)), \qquad (2.3.1)$$

where c_λ is a constant defined such that $\|v\|_{\lambda,\tau} = |v|_\tau$ holds for any constant functions $z \mapsto v \in V$ (i.e. for any element of the minimal K-type). Let $\mathcal{H}_\lambda(D,V) \subset \mathcal{O}(D,V)$ be the unitary subrepresentation of $\tilde{G}$ under τ_λ. Then $\mathcal{H}_\lambda(D,V)$ is non-zero if $\lambda \in \mathbb{R}$ is sufficiently large so that the above inner product converges. On the other hand, we cannot know a priori whether $\mathcal{H}_\lambda(D,V)$ is zero or non-zero if λ is small. In any case, if $\mathcal{H}_\lambda(D,V)$ is non-zero, the reproducing kernel is proportional to $K_{\mathrm{Re}\,\lambda,\tau}(z,w)$, where

$$K_{\lambda,\tau}(z,w) := h(z,w)^{-\lambda}\tau(B(z,w)) \in \mathcal{O}(D \times \overline{D}, \mathrm{End}(V)).$$

This is because the reproducing kernel $K(z,w)$ is characterized by

$$\chi(\kappa(g,z))^\lambda \tau(\kappa(g,z))^{-1}K(gz,gw)\tau(\kappa(g,w))^{*-1}\overline{\chi(\kappa(g,w))^\lambda} = K(z,w),$$

and such $K(z,w)$ is unique up to constant multiple, since $\tilde{G}$ acts transitively on the totally real submanifold $\mathrm{diag}(D) \subset D \times \overline{D}$, which allows the value at origin $K(0,0)$ to determine the whole $K(z,w)$, and $K(0,0) \in \mathrm{End}(V)$ is proportional to identity since this commutes with $\tilde{K}$-action. When $\lambda \in \mathbb{R}$ is sufficiently large, then the reproducing kernel corresponding to the inner product (2.3.1) is precisely $K_{\lambda,\tau}(z,w)$ by the normalization assumption.

2.3.2 Key theorem

The norm $\|\cdot\|_{\lambda,\tau}$ in the previous subsection is $\tilde{G}$-invariant, and therefore $\tilde{K}$-invariant. From now on we observe how the norm varies as the parameter λ varies on each K-type. In order to compare, we consider another K-invariant norm which is independent of λ.

We recall the *Fischer inner product* $\langle\cdot,\cdot\rangle_{F,\tau}$ on $\mathcal{P}(\mathfrak{p}^+,V)$, the space of V-valued holomorphic polynomials on $\mathfrak{p}^+$.

$$\langle f,g\rangle_{F,\tau}:=\frac{1}{\pi^n}\int_{\mathfrak{p}^+}(f(w),g(w))_\tau e^{-|w|^2}dw \qquad (f,g\in\mathcal{P}(\mathfrak{p}^+,V)). \tag{2.3.2}$$

This inner product is invariant under the following representation $(\hat{\tau},\mathcal{P}(\mathfrak{p}^+,V))$:

$$\left(\hat{\tau}(k)f\right)(w):=\tau(k)f(k^{-1}w) \qquad (k\in K^{\mathbb{C}},\ f\in\mathcal{P}(\mathfrak{p}^+,V),\ w\in\mathfrak{p}^+),$$

that is, $\langle\hat{\tau}(k)f,g\rangle_{F,\tau}=\langle f,\hat{\tau}(k^*)g\rangle_{F,\tau}$ holds. Let $W\subset\mathcal{P}(\mathfrak{p}^+,V)=\mathcal{O}(D,V)_K$ be a $K^{\mathbb{C}}$-irreducible subspace. Then since both $\|\cdot\|_{F,\tau}$ and $\|\cdot\|_{\lambda,\tau}$ are K-invariant, the ratio of these two norms are constant on W. Therefore we aim to compute this ratio of two norms.

In order to state the key theorem, we prepare some notations. Let

$$(\tau,V)|_{K_{\mathrm{T}}^{\mathbb{C}}}=\bigoplus_i(\tau_i,V_i)$$

be the decomposition of the $K^{\mathbb{C}}$-module (τ,V) into $K_{\mathrm{T}}^{\mathbb{C}}$-irreducible submodules, and for each i we denote by $(\bar{\tau}_i,\overline{V_i})$ the complex conjugate representation of V_i with respect to the real form $L\subset K_{\mathrm{T}}^{\mathbb{C}}$, that is, there exists a conjugate linear isomorphism $\bar{\cdot}:V_i\to\overline{V_i}$, and $\bar{\tau}_i$ is given by $\bar{\tau}_i(l)\bar{v}=\overline{\tau_i(\bar{l})v}$. Let

$$\mathrm{rest}:\mathcal{P}(\mathfrak{p}^+,V)\to\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,V)=\bigoplus_i\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,V_i)$$

be the restriction map, and for each i we take $K_{\mathrm{T}}^{\mathbb{C}}$-submodules $W_{ij}\subset\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,V_i)$ such that

$$\mathrm{rest}(W)\subset\bigoplus_i\bigoplus_j W_{ij}$$

holds.

Theorem 2.3.1. *Let* $(\tau,V)|_{K_{\mathrm{T}}^{\mathbb{C}}}=\bigoplus_i(\tau_i,V_i)$*, and suppose each* (τ_i,V_i) *has a restricted lowest weight* $-\left(\frac{k_{i,1}}{2}\gamma_1+\cdots+\frac{k_{i,r}}{2}\gamma_r\right)\Big|_{\mathfrak{a}_{\mathfrak{l}}}$*. Let* $W\subset\mathcal{P}(\mathfrak{p}^+,V)$ *be a* $K^{\mathbb{C}}$*-irreducible subspace, with* $\mathrm{rest}(W)\subset\bigoplus_i\bigoplus_j W_{ij}\subset\bigoplus_i\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,V_i)$ *as above. We assume*

(A1) $(\tau_i,V_i)|_{K_L}$ *still remains irreducible for each* i*.*

(A2) For each i,j*, all the* K_L*-spherical irreducible subspaces in* $W_{ij}\otimes\overline{V_i}$ *have the same lowest weight* $-(n_{ij,1}\gamma_1+\cdots+n_{ij,r}\gamma_r)$*.*

Then the integral $\|f\|_{\lambda,\tau}^2$ *converges for any* $f\in W$ *if* $\mathrm{Re}(\lambda)+k_{i,r}>p-1$ *for all* i*. Moreover, there exist non-negative numbers* a_{ij} *such that, for any* $f\in W$*,*

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\frac{c_\lambda}{\sum_{ij}a_{ij}}\sum_{ij}a_{ij}\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{n}_{ij})},$$

where

$$c_\lambda^{-1}=\frac{1}{\dim V}\sum_i(\dim V_i)\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{k}_i)}.$$

In the rest of this section we prove this theorem. We set $\|f\|_{\lambda,\tau}^2/\|f\|_{F,\tau}^2 =: R_W(\lambda)$ for $f \in W$, and compute this ratio $R_W(\lambda)$.

Let $K_W(z,w) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ be the reproducing kernel of W with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$, that is, for an orthonormal basis $\{f_i\}$ of W with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$,

$$K_W(z,w)v := \sum_i (v, f_i(w))_\tau f_i(z) \qquad (v \in V),$$

which does not depend on the choice of $\{f_i\}$. Then the ratio $R_W(\lambda)$ is computed as

$$\begin{aligned} R_W(\lambda) &= \frac{c_\lambda \displaystyle\sum_i \int_D \left(\tau(B(w)^{-1})f_i, f_i\right)_\tau h(w)^{\lambda-p} dw}{\displaystyle\sum_i \int_{\mathfrak{p}^+} (f_i, f_i)_\tau\, e^{-|w|^2} dw} \\ &= \frac{c_\lambda \displaystyle\int_D \mathrm{Tr}_V\left(\tau(B(w)^{-1})K_W(w,w)\right) h(w)^{\lambda-p} dw}{\displaystyle\int_{\mathfrak{p}^+} \mathrm{Tr}_V(K_W(w,w)) e^{-|w|^2} dw}, \end{aligned}$$

and if the numerator converges, then $\|f_i\|_{\lambda,\tau}^2$ converges for any i, and so does $\|f\|_{\lambda,\tau}^2$ for any $f \in W$. To proceed the computation, we use the following lemma.

Lemma 2.3.2. *For any integrable, or non-negative-valued measurable function f on $\mathfrak{p}^+$, we have*

$$\frac{1}{\pi^n}\int_{\mathfrak{p}^+} f(w)dw = \frac{1}{\Gamma_\Omega\left(\frac{n}{r}\right)}\int_\Omega\int_K f(kx^{\frac{1}{2}})\Delta(x)^b dk dx,$$

where $x^{\frac{1}{2}}$ is the square root with respect to the Jordan algebra structure (2.2.1) on $\Omega \subset \mathfrak{n}^+$.

Proof. For tube type case ($b = 0$) see [7, Proposition X.3.4]. Even for $b \neq 0$ case we can prove this similarly. $\square$

Since the integrand of $R_W(\lambda)$ is non-negative-valued, by this lemma, this is equal to

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_{\Omega\cap(e-\Omega)}\int_K \mathrm{Tr}_V\left(\tau(B(kx^{\frac{1}{2}})^{-1})K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}})\right) h(kx^{\frac{1}{2}})^{\lambda-p}\Delta(x)^b dk dx}{\displaystyle\int_\Omega\int_K \mathrm{Tr}_V\left(K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}})\right) e^{-|kx^{\frac{1}{2}}|^2}\Delta(x)^b dk dx}.$$

Since the reproducing kernel satisfies

$$K_W(kz, k^{*-1}w) = \tau(k)K_W(z,w)\tau(k^{-1}) \qquad (z, w \in \mathfrak{p}^+,\ k \in K^{\mathbb{C}}),$$

we have,

$$\begin{aligned} K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}}) &= \tau(k)K_W(P(x^{-\frac{1}{4}})x, P(x^{\frac{1}{4}})e)\tau(k^{-1}) \\ &= \tau(k)\tau(P(x^{-\frac{1}{4}}))K_W(x,e)\tau(P(x^{\frac{1}{4}}))\tau(k^{-1}) \qquad (x \in \Omega, k \in K). \end{aligned}$$

Therefore we have

$$\mathrm{Tr}_V\left(K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}})\right) = \mathrm{Tr}_V(K_W(x,e)).$$

Also, since $k^{-1}B(kx^{\frac{1}{2}})^{-1}k = B(x^{\frac{1}{2}})^{-1} = P(e-x)^{-1}$ and $P(e-x)^{-1}$ commutes with $P(x^{\frac{1}{4}})$, we have

$$\mathrm{Tr}_V\left(\tau(B(kx^{\frac{1}{2}})^{-1})K_W(kx^{\frac{1}{2}}, kx^{\frac{1}{2}})\right) = \mathrm{Tr}_V\left(\tau(P(e-x)^{-1})K_W(x,e)\right).$$

By these and $h(kx^{\frac{1}{2}}) = \Delta(e-x)$, $|kx^{\frac{1}{2}}|^2 = \mathrm{tr}(x)$, we have

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_V\left(\tau(P(e-x)^{-1})K_W(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx}{\displaystyle\int_\Omega \mathrm{Tr}_V(K_W(x,e))e^{-\mathrm{tr}(x)}\Delta(x)^b dx}.$$

By the assumption, we can rewrite $K_W(z,w)$ by using $K_{W_{ij}}(z,w)$, the reproducing kernels of W_{ij}, when $z, w \in \mathfrak{p}^+_{\mathrm{T}}$:

$$K_W(z,w) = \sum_{ij} \tilde{a}_{ij} K_{W_{ij}}(z,w) \in \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}} \times \overline{\mathfrak{p}^+_{\mathrm{T}}}, \mathrm{End}(V)) \qquad (z, w \in \mathfrak{p}^+_{\mathrm{T}}),$$

using some non-negative numbers $\tilde{a}_{ij}$. Therefore we have

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\sum_{ij}\tilde{a}_{ij}\int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})K_{W_{ij}}(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx}{\displaystyle\sum_{ij}\tilde{a}_{ij}\int_\Omega \mathrm{Tr}_{V_i}(K_{W_{ij}}(x,e))e^{-\mathrm{tr}(x)}\Delta(x)^b dx}.$$

Now we set

$$B_{ij}(\lambda) := \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})K_{W_{ij}}(x,e)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx,$$
$$\Gamma_{ij} := \int_\Omega \mathrm{Tr}_{V_i}(K_{W_{ij}}(x,e))e^{-\mathrm{tr}(x)}\Delta(x)^b dx$$

so that $R_W(\lambda) = c_\lambda\left(\sum_{ij}\tilde{a}_{ij}B_{ij}(\lambda)\right)\Big/\left(\sum_{ij}\tilde{a}_{ij}\Gamma_{ij}\right)$. Now, we regard $K_{W_{ij}}(x,e) \in \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, \mathrm{End}(V_i))$ as a function of x. We define the action $\tilde{\tau}_i$ of $K_T^{\mathbb{C}}$ on $\mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, \mathrm{End}(V_i))$ by

$$(\tilde{\tau}_i(k)F)(x) := \tau_i(k)F(k^{-1}x)\tau_i({}^tk) \qquad (k \in K^{\mathbb{C}}_{\mathrm{T}}, F \in \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, \mathrm{End}(V_i)), x \in \mathfrak{p}^+_{\mathrm{T}}).$$

Then $K_{W_{ij}}(x,e)$ is K_L-invariant under $\tilde{\tau}_i$. Now we identify

$$(\tilde{\tau}_i, \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, \mathrm{End}(V_i))) \simeq (\hat{\tau}|_{K^{\mathbb{C}}_{\mathrm{T}}} \otimes \bar{\tau}_i, \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, V_i) \otimes \overline{V_i}).$$

Then under this identification $K_{W_{ij}}(x,e)$ sits in $W_{ij} \otimes \overline{V_i}$, and therefore by (A2) this sits in the space with lowest weight $-(n_{ij,1}\gamma_1 + \cdots + n_{ij,r}\gamma_r)$. That is, there exists a function $F_{ij} \in \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, \mathrm{End}(V_i))$ such that

$$(\tilde{\tau}_i(q)F_{ij})(x) = \Delta_{\mathbf{n}_{ij}}(q^{-1}e)F_{ij}(x) \qquad (q \in A_L N_L^-, x \in \mathfrak{p}^+_{\mathrm{T}}),$$
$$\int_{K_L}(\tilde{\tau}(k)F_{ij})(x)dk = K_{W_{ij}}(x,e).$$

We note that $\int_{K_L}(\tilde{\tau}(k)F_{ij})(x)dk$ is non-zero for any non-zero N_L^--fixed vector F_{ij}, since we have $(F_{ij}, K_{W_{ij}}(\cdot, e))_\tau \neq 0$, which is proved by using the Iwasawa decomposition $L = K_L A_L N_L^-$.

From now, we compute $B_{ij}(\lambda)$ formally, allowing variable changes. By using F_{ij}, we rewrite $B_{ij}(\lambda)$ and Γ_{ij}.

$$B_{ij}(\lambda) = \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-x)^{-1})F_{ij}(x)\right)\Delta(e-x)^{\lambda-p}\Delta(x)^b dx,$$
$$\Gamma_{ij} := \int_\Omega \mathrm{Tr}_{V_i}(F_{ij}(x))e^{-\mathrm{tr}(x)}\Delta(x)^b dx.$$

For $y \in \Omega$ we set

$$I(y) := \int_{\Omega\cap(y-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(y-x)^{-1})F_{ij}(x)\right)\Delta(y-x)^{\lambda-p}\Delta(x)^b dx \tag{2.3.3}$$

so that $I(e) = B_{ij}(\lambda)$. We take $q \in A_L N_L^-$ such that $y = qe$, and set $x = qz$. Then

$$\begin{aligned}
I(y) &= \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(q.(e-z))^{-1})F_{ij}(qz)\right)\Delta(q.(e-z))^{\lambda-p}\Delta(qz)^b\Delta(qe)^{\frac{n_\mathrm{T}}{r}}dz\\
&= \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i({}^t q^{-1})\tau_i(P(e-z)^{-1})\tau_i(q^{-1})F_{ij}(qz)\right)\Delta(e-z)^{\lambda-p}\Delta(z)^b\Delta(qe)^{\lambda-p+b+\frac{n_\mathrm{T}}{r}}dz\\
&= \int_{\Omega\cap(e-\Omega)} \mathrm{Tr}_{V_i}\left(\tau_i(P(e-z)^{-1})F_{ij}(z)\right)\Delta_{\mathbf{n}_{ij}}(qe)\Delta(e-z)^{\lambda-p}\Delta(z)^b\Delta(qe)^{\lambda-\frac{n_\mathrm{T}}{r}}dz\\
&= I(e)\Delta_{\mathbf{n}_{ij}}(y)\Delta(y)^{\lambda-\frac{n_\mathrm{T}}{r}} = B_{ij}(\lambda)\Delta_{\lambda+\mathbf{n}_{ij}}(y)\Delta(y)^{-\frac{n_\mathrm{T}}{r}}.
\end{aligned}$$

Now we calculate $\int_\Omega I(y)e^{-\mathrm{tr}(y)}dy$ by two ways.

$$\int_\Omega I(y)e^{-\mathrm{tr}(y)}dy = B_{ij}(\lambda)\int_\Omega e^{-\mathrm{tr}(y)}\Delta_{\lambda+\mathbf{n}_{ij}}(y)\Delta(y)^{-\frac{n_\mathrm{T}}{r}}dy = B_{ij}(\lambda)\Gamma_\Omega(\lambda+\mathbf{n}_{ij}),$$

$$\begin{aligned}
\int_\Omega I(y)e^{-\mathrm{tr}(y)}dy &= \iint_{x\in\Omega, y-x\in\Omega} e^{-\mathrm{tr}(y)}\,\mathrm{Tr}_{V_i}\left(\tau_i(P(y-x)^{-1})F_{ij}(x)\right)\Delta(y-x)^{\lambda-p}\Delta(x)^b dxdy\\
&= \iint_{x\in\Omega, z\in\Omega} e^{-\mathrm{tr}(x+z)}\,\mathrm{Tr}_{V_i}\left(\tau_i(P(z)^{-1})F_{ij}(x)\right)\Delta(z)^{\lambda-p}\Delta(x)^b dxdz\\
&= \mathrm{Tr}_{V_i}\left(\int_\Omega e^{-\mathrm{tr}(z)}\tau_i(P(z)^{-1})\Delta(z)^{\lambda-p}dz\int_\Omega e^{-\mathrm{tr}(x)}F_{ij}(x)\Delta(x)^b dx\right).
\end{aligned}$$

Therefore, formally

$$B_{ij}(\lambda)\Gamma_\Omega(\lambda+\mathbf{n}_{ij}) = \mathrm{Tr}_{V_i}\left(\int_\Omega e^{-\mathrm{tr}(z)}\tau_i(P(z)^{-1})\Delta(z)^{\lambda-p}dz\int_\Omega e^{-\mathrm{tr}(x)}F_{ij}(x)\Delta(x)^b dx\right)$$

holds. By Fubini's theorem, variable changes are verified and the above equality exactly holds if

$$\iint_{x\in\Omega, z\in\Omega} e^{-\mathrm{tr}(x+z)}\left|\mathrm{Tr}_{V_i}\left(\tau_i(P(z)^{-1})F_{ij}(x)\right)\right|\Delta(z)^{\mathrm{Re}(\lambda)-p}\Delta(x)^b dxdz < \infty$$

is verified, and since all norms on the finite-dimensional vector space $\mathrm{End}(V_i)$ are equivalent, this holds if

$$\int_\Omega e^{-\mathrm{tr}(z)}\left|\tau_i(P(z)^{-1})\right|_{\tau_i,\mathrm{op}}\Delta(z)^{\mathrm{Re}(\lambda)-p}dz < \infty, \tag{2.3.4}$$
$$\int_\Omega e^{-\mathrm{tr}(x)}\left|F_{ij}(x)\right|_{\tau_i,\mathrm{op}}\Delta(x)^b dx < \infty \tag{2.3.5}$$

hold, where $|\cdot|_{\tau_i,\mathrm{op}}$ denotes the operator norm. Since

$$|F_{ij}(x)|_{\tau_i,\mathrm{op}} = \max_{u,v\in V_i\setminus\{0\}} \frac{|(F_{ij}(x)u,v)_{\tau_i}|}{|u|_{\tau_i}|v|_{\tau_i}}$$

holds and $(F_{ij}(x)u,v)_\tau$ is a polynomial on Ω for any $u,v\in V_i$, (2.3.5) exactly holds. Also, since $\tau_i(P(z)^{-1})$ is self-adjoint and positive definite for $z\in\Omega$, we have

$$\left|\tau_i(P(z)^{-1})\right|_{\tau_i,\mathrm{op}} = \max_{u\in V_i\setminus\{0\}} \frac{\left|(\tau_i(P(z)^{-1})u,u)_{\tau_i}\right|}{|u|_{\tau_i}^2},$$

and elements $v\in V_i$ such that

$$\int_\Omega e^{-\operatorname{tr}(z)}\left|(\tau_i(P(z)^{-1})v,v)_{\tau_i}\right|\Delta(z)^{\mathrm{Re}(\lambda)-p}dz<\infty \tag{2.3.6}$$

forms a K_L-invariant vector subspace, by the triangle inequality and the K_L-invariance of the integral. By assumption (A1), such vector subspace is either V_i or $\{0\}$. Thus (2.3.4) holds if and only if (2.3.6) holds for some non-zero $v\in V_i$. Moreover, again by assumption (A1), the integral

$$\Gamma_i'(\lambda) := \int_\Omega e^{-\operatorname{tr}(z)}\tau_i(P(z)^{-1})\Delta(z)^{\lambda-p}dz \tag{2.3.7}$$

is proportional to the identity operator I_{V_i} if (2.3.6) holds, since this $\Gamma_i'(\lambda)$ commutes with K_L-action. Now we prove (2.3.6) for $v\in V_i$ lowest weight vector, assuming $\mathrm{Re}(\lambda)+k_{i,r}>p-1$. Since the restricted lowest weight of V_i is $-\frac{k_{i,1}}{2}\gamma_1-\cdots-\frac{k_{i,r}}{2}\gamma_r\Big|_{\mathfrak{a}_\mathfrak{l}}$, for $q\in A_LN_L^-$ we have

$$(\tau_i(P(qe)^{-1})v,v)_{\tau_i} = (\tau_i({}^tq^{-1}q^{-1})v,v)_{\tau_i} = |\tau_i(q^{-1})v|_{\tau_i}^2 = \Delta_{-\frac{\mathbf{k}_i}{2}}(q^{-1}e)^2|v|_{\tau_i}^2 = \Delta_{\mathbf{k}_i}(qe)|v|_{\tau_i}^2,$$

and this is positive valued. Therefore we have

$$\begin{aligned}(\Gamma_i'(\lambda)v,v)_{\tau_i} &= \int_\Omega e^{-\operatorname{tr}(z)}(\tau_i(P(z)^{-1})v,v)_{\tau_i}\Delta(z)^{\lambda-p}dz\\ &= \int_\Omega e^{-\operatorname{tr}(z)}\Delta_{\mathbf{k}_i}(z)\Delta(z)^{\lambda-\frac{n}{r}-\frac{n_\mathrm{T}}{r}}dz|v|_{\tau_i}^2\\ &= \Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)|v|_{\tau_i}^2 \end{aligned}\tag{2.3.8}$$

if $\mathrm{Re}(\lambda)+k_{i,r}>p-1$. That is, (2.3.4) is verified, and $\Gamma_i'(\lambda)=\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)I_{V_i}$ holds. Therefore,

$$B_{ij}(\lambda) = \frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{n}_{ij})}\operatorname{Tr}_V\left(\int_\Omega e^{-\operatorname{tr}(x)}\Delta(x)^bF_{ij}(x)dx\right) = \frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{n}_{ij})}\Gamma_i,$$

exactly holds, and

$$R_W(\lambda) = \frac{c_\lambda}{\sum_{ij}\tilde{a}_{ij}\Gamma_{ij}}\sum_{ij}\tilde{a}_{ij}\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{n}_{ij})}\Gamma_{ij}.$$

By putting $\tilde{a}_{ij}\Gamma_{ij}=:a_{ij}$, we get the desired formula.

When $W=V$, clearly we have $\mathrm{rest}(V)=\oplus_i V_i$, and $K_V(z,w)=I_V$, $K_{V_i}(z,w)=I_{V_i}$. Thus, the coefficients

$$\begin{aligned} a_i=\Gamma_i &= \int_\Omega \mathrm{Tr}_{V_i}(K_{V_i}(x,e))e^{-\mathrm{tr}(x)}\Delta(x)^b dx \\ &= \int_\Omega \mathrm{Tr}_{V_i}(I_{V_i})e^{-\mathrm{tr}(x)}\Delta(x)^b dx = (\dim V_i)\Gamma_\Omega\left(\frac{n}{r}\right). \end{aligned}$$

Also, by assumption (A1), K_L-spherical vectors in $(\tilde{\tau},\mathrm{End}(V_i))\simeq(\tau_i\otimes\overline{\tau_i},V_i\otimes\overline{V_i})$ is proportional to I_{V_i}, that is, $\dim\mathrm{End}(V_i)^{K_L}=1$. Therefore, assumption (A2) is automatically satisfied, with $\mathbf{n}_i=\mathbf{k}_i$. Since c_λ is determined such that $R_{V,\lambda}=1$, we have

$$\begin{aligned} c_\lambda^{-1} &= \frac{1}{\sum_i(\dim V_i)\Gamma_\Omega\left(\frac{n}{r}\right)}\sum_i(\dim V_i)\Gamma_\Omega\left(\frac{n}{r}\right)\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{k}_i)} \\ &= \frac{1}{\dim V}\sum_i(\dim V_i)\frac{\Gamma_\Omega\left(\lambda+\mathbf{k}_i-\frac{n}{r}\right)}{\Gamma_\Omega(\lambda+\mathbf{k}_i)}, \end{aligned}$$

and this completes the proof. □

Remark 2.3.3. *The integral $\Gamma'_{i,\lambda}$ in (2.3.7) is essentially the same as the "Gamma function" in [9, Definition 3.1], [10, Section 4] on* $\mathrm{End}(V_i)$*, or the integral with the measure R_μ in [3, Theorem 3.4], and the property of $\Gamma'_{i,\lambda}$ or the finiteness of (2.3.4) have been already proved. However, since the notation is different, the author wrote the proof for completeness.*

If $(\tau,V)|_{\mathfrak{k}_\mathrm{T}^\mathbb{C}}$ is still irreducible and $\mathrm{rest}(W)\subset\mathcal{P}(\mathfrak{p}_\mathrm{T}^+,V)$ consists of one irreducible $K_\mathrm{T}^\mathbb{C}$-module, then Theorem 2.3.1 becomes easier.

Corollary 2.3.4. *Suppose $(\tau,V)|_{K_\mathrm{T}^\mathbb{C}}$ has a restricted lowest weight $-\left(\frac{k_1}{2}\gamma_1+\cdots+\frac{k_r}{2}\gamma_r\right)\Big|_{\mathfrak{a}_\mathfrak{l}}$. Let $W\subset\mathcal{P}(\mathfrak{p}^+,V)$ be a $K^\mathbb{C}$-irreducible subspace. We assume*

(A0) $\mathrm{rest}(W)\subset\mathcal{P}(\mathfrak{p}_\mathrm{T}^+,V)$ *is irreducible as a $K_\mathrm{T}^\mathbb{C}$-module.*

(A1') $(\tau,V)|_{K_L}$ still remains irreducible.

(A2') All the K_L-spherical irreducible subspaces in $\mathrm{rest}(W)\otimes\overline{V}$ *have the same lowest weight $-(n_1\gamma_1+\cdots+n_r\gamma_r)$.*

Then the integral $\|f\|_{\lambda,\tau}^2$ converges for any $f\in W$ if $\mathrm{Re}(\lambda)+k_r>p-1$. *Moreover, we have*

$$c_\lambda=\frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega\left(\lambda+\mathbf{k}-\frac{n}{r}\right)},$$

and for any $f\in W$, we have

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega(\lambda+\mathbf{n})}=\frac{(\lambda)_\mathbf{k}}{(\lambda)_\mathbf{n}}=\frac{1}{(\lambda+\mathbf{k})_{\mathbf{n}-\mathbf{k}}}.$$

The assumption (A0) is automatically satisfied if

- $G=G_\mathrm{T}$ i.e. G is of tube type, or
- $G=SU(q,r)$ $(q\le r)$, and $V=\mathbb{C}\boxtimes V'$ as a $K=S(U(q)\times U(r))$-module.

In Section 2.4, we deal with these cases explicitly, and in Section 2.5, we deal with the cases such that Corollary 2.3.4 is not applicable. To remove the ambiguity of the action of the center, we assume $k_{i,r}\ge 0$ for any i, and $k_{i,r}=0$ for some i.

2.4 Norm computation: Tube type case

2.4.1 Explicit roots

Before starting the computation of norms, we fix the notation about roots of classical Lie algebras of Hermitian type.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a classical simple Lie algebra of Hermitian type, i.e. one of $\mathfrak{sp}(r, \mathbb{R})$, su$(q, s)$, $\mathfrak{so}^*(2s)$, or $\mathfrak{so}(2, n)$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$. Then $\mathfrak{h}$ automatically becomes a Cartan subalgebra of $\mathfrak{g}$. We take a basis

$$\begin{aligned}
\{t_1, t_2, \ldots, t_r\} &\subset \sqrt{-1}\mathfrak{h} && (\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})),\\
\{t_1, t_2, \ldots, t_{q+s}\} &\subset (\sqrt{-1}\mathfrak{h}) \oplus \mathbb{R} && (\mathfrak{g} = \mathfrak{su}(q, s)),\\
\{t_1, t_2, \ldots, t_s\} &\subset \sqrt{-1}\mathfrak{h} && (\mathfrak{g} = \mathfrak{so}^*(2s)),\\
\{t_0, t_1, \ldots, t_{\lfloor n/2 \rfloor}\} &\subset \sqrt{-1}\mathfrak{h} && (\mathfrak{g} = \mathfrak{so}(2, n)),
\end{aligned}$$

with the dual basis $\{\varepsilon_j\}$, such that the simple systems $\Pi_{\mathfrak{g}^\mathbb{C}}$, $\Pi_{\mathfrak{k}^\mathbb{C}}$ of positive roots $\Delta_+(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$, $\Delta_+(\mathfrak{k}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$ are given by

$$\Pi_{\mathfrak{k}^\mathbb{C}} = \begin{cases}
\{\varepsilon_j - \varepsilon_{j+1} : j = 1, \ldots, r-1\} & (\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})),\\
\{\varepsilon_j - \varepsilon_{j+1} : j = 1, \ldots, q-1\} & \\
\qquad \cup \{\varepsilon_{j+1} - \varepsilon_j : j = q+1, \ldots, q+s-1\} & (\mathfrak{g} = \mathfrak{su}(q, s)),\\
\{\varepsilon_j - \varepsilon_{j+1} : j = 1, \ldots, s-1\} & (\mathfrak{g} = \mathfrak{so}^*(2s)),\\
\{\varepsilon_j - \varepsilon_{j+1} : j = 1, \ldots, s-1\} \cup \{\varepsilon_{s-1} + \varepsilon_s\} & (\mathfrak{g} = \mathfrak{so}(2, 2s)),\\
\{\varepsilon_j - \varepsilon_{j+1} : j = 1, \ldots, s-1\} \cup \{\varepsilon_s\} & (\mathfrak{g} = \mathfrak{so}(2, 2s+1)),
\end{cases}$$

$$\Pi_{\mathfrak{g}^\mathbb{C}} = \Pi_{\mathfrak{k}^\mathbb{C}} \cup \begin{cases}
\{2\varepsilon_r\} & (\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})),\\
\{\varepsilon_q - \varepsilon_{q+s}\} & (\mathfrak{g} = \mathfrak{su}(q, s)),\\
\{\varepsilon_{s-1} + \varepsilon_s\} & (\mathfrak{g} = \mathfrak{so}^*(2s)),\\
\{\varepsilon_0 - \varepsilon_1\} & (\mathfrak{g} = \mathfrak{so}(2, n)).
\end{cases}$$

Then the central character $d\chi$ of $\mathfrak{k}^\mathbb{C}$ is given by

$$d\chi = \begin{cases}
\varepsilon_1 + \cdots + \varepsilon_r & (\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})),\\
\varepsilon_1 + \cdots + \varepsilon_q = -(\varepsilon_{q+1} + \cdots + \varepsilon_{q+s}) & (\mathfrak{g} = \mathfrak{su}(q, s)),\\
\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_s) & (\mathfrak{g} = \mathfrak{so}^*(2s)),\\
\varepsilon_0 & (\mathfrak{g} = \mathfrak{so}(2, n)),
\end{cases}$$

and the maximal set of strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_{\operatorname{rank}_\mathbb{R} \mathfrak{g}}\}$ is given by

$$\begin{aligned}
&\gamma_j = 2\varepsilon_j && (j = 1, \ldots, r) && (\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})),\\
&\gamma_j = \varepsilon_j - \varepsilon_{q+j} && (j = 1, \ldots, \min\{q, s\}) && (\mathfrak{g} = \mathfrak{su}(q, s)),\\
&\gamma_j = \gamma_{2j-1} + \gamma_{2j} && (j = 1, \ldots, \lfloor s/2 \rfloor) && (\mathfrak{g} = \mathfrak{so}^*(2s)),\\
&\gamma_1 = \varepsilon_0 + \varepsilon_1, \quad \gamma_2 = \varepsilon_0 - \varepsilon_1 && && (\mathfrak{g} = \mathfrak{so}(2, n)).
\end{aligned}$$

When $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$, $\mathfrak{su}(r, r)$, $\mathfrak{so}^*(4r)$ or $\mathfrak{so}(2, n)$, $\mathfrak{g}$ is of tube type, i.e. $\mathfrak{g} = \mathfrak{g}_\mathrm{T}$ holds. On the other hand, when $\mathfrak{su}(q, s)$ $(q \neq s)$ or $\mathfrak{g} = \mathfrak{so}^*(4r+2)$, $\mathfrak{g}$ is of non-tube type, and we have $\mathfrak{g}_{\mathrm{T}} = \mathfrak{su}(r,r)$ ($r := \min\{q,s\}$), or $\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}^*(4r)$ respectively. Let $\mathfrak{h}_{\mathrm{T}} := \mathfrak{h} \cap \mathfrak{g}_{\mathrm{T}}$. Then we have

$$\sqrt{-1}\mathfrak{h}_{\mathrm{T}} = \operatorname{span}(\{t_j - t_{j+1} : j = 1,\dots,r-1, q+1,\dots,q+r-1\} \cup \{t_r - t_{q+r}\}) \qquad (\mathfrak{g} = \mathfrak{su}(q,s)),$$

$$\sqrt{-1}\mathfrak{h}_{\mathrm{T}} = \operatorname{span}\{t_1,\dots,t_{2r}\} \qquad (\mathfrak{g} = \mathfrak{so}^*(4r+2)).$$

Also, $\mathfrak{a}_{\mathfrak{l}} \subset \sqrt{-1}\mathfrak{h}_{\mathrm{T}}$ is given by

$$\mathfrak{a}_{\mathfrak{l}} = \begin{cases} \sqrt{-1}\mathfrak{h} & (\mathfrak{g}_{\mathrm{T}} = \mathfrak{sp}(r,\mathbb{R})), \\ \operatorname{span}\{t_j - t_{q+j} : j = 1,\dots,r\} & (\mathfrak{g}_{\mathrm{T}} = \mathfrak{su}(r,r)), \\ \operatorname{span}\{t_{2j-1} + t_{2j} : j = 1,\dots,r\} & (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}^*(4r)), \\ \operatorname{span}\{t_0, t_1\} & (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n)). \end{cases}$$

In general, we consider $\mathfrak{gl}(s,\mathbb{C})$ or $\mathfrak{so}(n,\mathbb{C})$, and parametrize their irreducible representations. We fix the positive root system of $\mathfrak{gl}(s,\mathbb{C})$ such that its simple system is given by $\{\varepsilon_j - \varepsilon_{j+1} : j = 1,\dots,s-1\}$, and for $\mathbf{m} \in \mathbb{Z}^s_+$, let $(\tau^{(s)}_{\mathbf{m}}, V^{(s)}_{\mathbf{m}})$, $(\tau^{(s)\vee}_{\mathbf{m}}, V^{(s)\vee}_{\mathbf{m}})$ be the finite-dimensional irreducible representation of $\mathfrak{gl}(s,\mathbb{C})$ with highest weight $m_1\varepsilon_1 + \cdots + m_s\varepsilon_s$, $-m_s\varepsilon_1 - \cdots - m_1\varepsilon_s$ respectively. Similarly, we fix the positive root system of $\mathfrak{so}(n,\mathbb{C})$ such that its simple system is given by

$$\{\varepsilon_j - \varepsilon_{j+1} : j = 1,\dots,s-1\} \cup \{\varepsilon_{s-1} + \varepsilon_s\} \qquad (n = 2s),$$
$$\{\varepsilon_j - \varepsilon_{j+1} : j = 1,\dots,s-1\} \cup \{\varepsilon_s\} \qquad (n = 2s+1),$$

and for $\mathbf{m} \in \mathbb{Z}^s \cup \left(\mathbb{Z} + \frac{1}{2}\right)^s$ with

$$m_1 \ge m_2 \ge \cdots \ge m_{s-1} \ge |m_s| \qquad (n = 2s),$$
$$m_1 \ge m_2 \ge \cdots \ge m_{s-1} \ge m_s \ge 0 \qquad (n = 2s+1),$$

let $(\tau^{[n]}_{\mathbf{m}}, V^{[n]}_{\mathbf{m}})$ be the finite-dimensional irreducible representation of $\mathfrak{so}(n,\mathbb{C})$ with highest weight $m_1\varepsilon_1 + \cdots + m_s\varepsilon_s$. Then $(\tau^{(r)\vee}_{\mathbf{m}}, V^{(r)\vee}_{\mathbf{m}})$, $(\tau^{(q)\vee}_{\mathbf{m}} \boxtimes \tau^{(s)}_{\mathbf{n}}, V^{(q)\vee}_{\mathbf{m}} \otimes V^{(s)}_{\mathbf{n}})$, $(\tau^{(s)\vee}_{\mathbf{m}}, V^{(s)\vee}_{\mathbf{m}})$ and $(\chi^{m_0} \boxtimes \tau^{[n]}_{\mathbf{m}}, \mathbb{C}_{m_0} \otimes V^{[n]}_{\mathbf{m}})$ are naturally identified with the representation of $\mathfrak{k}^{\mathbb{C}}$ for $\mathfrak{g} = \mathfrak{sp}(r,\mathbb{R})$, $\mathfrak{su}(q,s)$, $\mathfrak{so}^*(2s)$ and $\mathfrak{so}(2,n)$ respectively. Their restricted lowest weights are given by

$$-\frac{1}{2}(m_1\gamma_1 + \cdots + m_r\gamma_r)\Big|_{\mathfrak{a}_{\mathfrak{l}}} \qquad (\mathfrak{g} = \mathfrak{sp}(r,\mathbb{R}), \quad V = V^{(r)\vee}_{\mathbf{m}}),$$

$$-\frac{1}{2}((m_1 - n_1)\gamma_1 + \cdots + (m_r - n_r)\gamma_r)\Big|_{\mathfrak{a}_{\mathfrak{l}}} \qquad (\mathfrak{g} = \mathfrak{su}(q,s), \quad V = V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}),$$

$$-\frac{1}{2}((m_1 + m_2)\gamma_1 + \cdots + (m_{2r-1} + m_{2r})\gamma_r)\Big|_{\mathfrak{a}_{\mathfrak{l}}} \qquad (\mathfrak{g} = \mathfrak{so}^*(2s), \quad V = V^{(s)\vee}_{\mathbf{m}}),$$

$$-\frac{1}{2}((m_0 + m_1)\gamma_1 + (m_0 - m_1)\gamma_2)\Big|_{\mathfrak{a}_{\mathfrak{l}}} \qquad (\mathfrak{g} = \mathfrak{so}(2,n), \quad V = \mathbb{C}_{m_0} \boxtimes V^{[n]}_{\mathbf{m}}).$$

We will omit the superscript (s) or $[n]$ if there is no confusion.

Next we determine $(\bar{\tau}, \bar{V})$ for each representation (τ, V) of $\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}$. As in Section 2.2.1, let $\bar{\cdot}$ be the involution of $\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}$ fixing $\mathfrak{l}$. Then $\bar{\cdot}$ acts on $\mathfrak{h}^{\mathbb{C}}_{\mathrm{T}}$ anti-linearly, and fixes $\mathfrak{a}_{\mathfrak{l}} \oplus (\mathfrak{m}_{\mathfrak{l}} \cap \mathfrak{h})$.

Therefore $\bar{\cdot}|_{\mathfrak{h}_{\mathrm{T}}^{\mathbb{C}}}$ is characterized by

$$\begin{aligned}
&\overline{t_j} = t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{sp}(r,\mathbb{R})),\\
&\overline{t_j} = -t_{q+j},\ \overline{t_{q+j}} = -t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{su}(r,r)),\\
&\overline{t_{2j-1}} = t_{2j},\ \overline{t_{2j}} = t_{2j-1} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}^*(4r)),\\
&\overline{t_j} = \begin{cases} t_j & (j=0,1) \\ -t_j & (j=2,\dots,s) \end{cases} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ s=\lfloor n/2\rfloor).
\end{aligned}$$

We take an element $w \in N_K(\mathfrak{h}) \subset K$ (the normalizer of $\mathfrak{h}$ in K, or the "Weyl group" of $\mathfrak{h}$) such that

$$\begin{aligned}
&Ad(w)t_j = t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{sp}(r,\mathbb{R}), \mathfrak{su}(r,r)),\\
&Ad(w)t_{2j-1} = t_{2j},\ Ad(w)t_{2j} = t_{2j-1} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}^*(4r)),\\
&Ad(w)t_j = \begin{cases} t_j & (j=0,1,s) \\ -t_j & (j=2,3,\dots,s-1) \end{cases} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\in 4\mathbb{N},\ s=\lfloor n/2\rfloor),\\
&Ad(w)t_j = \begin{cases} t_j & (j=0,1) \\ -t_j & (j=2,3,\dots,s) \end{cases} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\notin 4\mathbb{N},\ s=\lfloor n/2\rfloor).
\end{aligned}$$

Then we have

$$\begin{aligned}
&Ad(w)\overline{t_j} = t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{sp}(r,\mathbb{R}), \mathfrak{so}^*(4r)),\\
&Ad(w)\overline{t_j} = -t_{q+j},\ Ad(w)\overline{t_{q+j}} = -t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{su}(r,r)),\\
&Ad(w)\overline{t_j} = \begin{cases} t_j & (j=0,1,\dots,s-1) \\ -t_s & (j=s) \end{cases} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\in 4\mathbb{N},\ s=\lfloor n/2\rfloor),\\
&Ad(w)\overline{t_j} = t_j && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\notin 4\mathbb{N},\ s=\lfloor n/2\rfloor),
\end{aligned}$$

and thus $Ad(w)\bar{\cdot}|_{\mathfrak{h}_{\mathrm{T}}^{\mathbb{C}}}$ preserves the positive Weyl chamber. This implies $Ad(w)\bar{\cdot}$ preserves the Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}$. Let (τ, V) be an irreducible $\mathfrak{k}_{\mathrm{T}}$-module with highest weight $\mu \in (\mathfrak{h}_{\mathrm{T}}^{\mathbb{C}})^\vee$ and we extend μ on $\mathfrak{b}$ such that it is trivial on the nilradical. Let $v \in V$ be the highest weight vector. Then for $b \in \mathfrak{b}$ we have

$$d\bar{\tau}(b)(\overline{\tau(w^{-1})v}) = \overline{d\tau(\bar{b})\tau(w^{-1})v} = \overline{\tau(w^{-1})d\tau(Ad(w)\bar{b})v} = \overline{\mu(Ad(w)\bar{b})}\,\overline{\tau(w^{-1})v}.$$

Therefore $(\bar{\tau}, \bar{V})$ has the highest weight vector $\overline{\tau(w^{-1})v}$ with highest weight $t \mapsto \overline{\mu(Ad(w)\bar{t})}$ $(t \in \mathfrak{h}_{\mathrm{T}}^{\mathbb{C}})$. Thus we conclude

$$\begin{aligned}
\overline{V_{\mathbf{m}}^{(r)\vee}} &\simeq V_{\mathbf{m}}^{(r)\vee} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{sp}(r,\mathbb{R})),\\
\overline{V_{\mathbf{m}}^{(r)\vee} \boxtimes V_{\mathbf{n}}^{(r)}} &\simeq V_{\mathbf{n}}^{(r)\vee} \boxtimes V_{\mathbf{m}}^{(r)} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{su}(r,r)),\\
\overline{V_{\mathbf{m}}^{(2r)\vee}} &\simeq V_{\mathbf{m}}^{(2r)\vee} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}^*(4r)),\\
\overline{\mathbb{C}_{m_0} \boxtimes V_{(m_1,\dots,m_{s-1},m_s)}^{[n]}} &\simeq \mathbb{C}_{m_0} \boxtimes V_{(m_1,\dots,m_{s-1},-m_s)}^{[n]} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\in 4\mathbb{N},\ s=\lfloor n/2\rfloor),\\
\overline{\mathbb{C}_{m_0} \boxtimes V_{(m_1,\dots,m_{s-1},m_s)}^{[n]}} &\simeq \mathbb{C}_{m_0} \boxtimes V_{(m_1,\dots,m_{s-1},m_s)}^{[n]} && (\mathfrak{g}_{\mathrm{T}} = \mathfrak{so}(2,n),\ n\notin 4\mathbb{N},\ s=\lfloor n/2\rfloor).
\end{aligned}$$

In the following sections, we compute the ratio of norms by using Corollary 2.3.4.

2.4.2 $Sp(r,\mathbb{R})$

In this subsection we set $G = Sp(r,\mathbb{R})$. This is of tube type, and we have

$$K \simeq U(r), \quad \mathfrak{p}^\pm \simeq \mathrm{Sym}(r,\mathbb{C}), \quad L \simeq GL(r,\mathbb{R}), \quad K_L \simeq O(r),$$
$$r = r, \quad n = \frac{1}{2}r(r+1), \quad d = 1, \quad p = r+1.$$

We want to calculate the norm $\|\cdot\|_{\lambda,\tau}$ of $\mathcal{O}(D,V)$ in the case $V = V^\vee_{\varepsilon_1+\cdots+\varepsilon_k} \simeq \bigwedge^k(\mathbb{C}^r)^\vee$ $(k = 0,1,\ldots,r-1)$. These V have the restricted lowest weight $-\frac{1}{2}(\gamma_1+\cdots+\gamma_s)\big|_{\mathfrak{a}_\mathfrak{l}}$, and remain irreducible even if restricted to $K_L = O(r)$, i.e. satisfy assumption (A1') of corollary 2.3.4. Thus the norm $\|\cdot\|^2_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}$ converges if $\mathrm{Re}\,\lambda > r$, and the normalizing constant c_λ is given by

$$c_\lambda = \frac{\Gamma_\Omega(\lambda+\varepsilon_1+\cdots+\varepsilon_k)}{\Gamma_\Omega\left(\lambda+\varepsilon_1+\cdots+\varepsilon_k-\frac{r+1}{2}\right)} = \frac{\prod_{j=1}^k\Gamma\left(\lambda-\frac{j-1}{2}+1\right)\prod_{j=k+1}^r\Gamma\left(\lambda-\frac{j-1}{2}\right)}{\prod_{j=1}^k\Gamma\left(\lambda-\frac{j+r}{2}+1\right)\prod_{j=k+1}^r\Gamma\left(\lambda-\frac{j+r}{2}\right)}.$$

First we compute the K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$. To do this, we quote the following lemma.

Lemma 2.4.1 ([30, §79, Example 3])**.**

$$V^\vee_{\mathbf{m}}\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k} = \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\,|\mathbf{k}|=k\\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_+}} V^\vee_{\mathbf{m}+\mathbf{k}}.$$

By this lemma and Theorem 2.2.1, we have

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k} &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V^\vee_{2\mathbf{m}}\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}\\ &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\,|\mathbf{k}|=k\\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_+}} V^\vee_{2\mathbf{m}+\mathbf{k}}.\end{aligned}$$

Second, for each K-type $V^\vee_{2\mathbf{m}+\mathbf{k}}$, we compute $V^\vee_{2\mathbf{m}+\mathbf{k}}\otimes\overline{V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}} \simeq V^\vee_{2\mathbf{m}+\mathbf{k}}\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$.

$$V^\vee_{2\mathbf{m}+\mathbf{k}}\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k} = \bigoplus_{\substack{\mathbf{k}'\in\{0,1\}^r,\,|\mathbf{k}'|=k\\ 2\mathbf{m}+\mathbf{k}+\mathbf{k}'\in\mathbb{Z}^r_+}} V^\vee_{2\mathbf{m}+\mathbf{k}+\mathbf{k}'}.$$

By Theorem 2.2.2, $V^\vee_{2\mathbf{m}+\mathbf{k}+\mathbf{k}'}$ is K_L-spherical if and only if each component of $2\mathbf{m}+\mathbf{k}+\mathbf{k}'$ is even, that is, $\mathbf{k} = \mathbf{k}'$. Thus, the only K_L-spherical submodule in $V^\vee_{2\mathbf{m}+\mathbf{k}}\otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$ is $V^\vee_{2\mathbf{m}+2\mathbf{k}}$, and $V^\vee_{2\mathbf{m}+\mathbf{k}}$ satisfies the assumption (A2') of Corollary 2.3.4 with $\mathbf{n} = \mathbf{m}+\mathbf{k}$. Therefore by Corollary 2.3.4, for $f\in V^\vee_{2\mathbf{m}+\mathbf{k}}$ we have

$$\frac{\|f\|^2_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}}{\|f\|^2_{F,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}} = \frac{(\lambda)_{\varepsilon_1+\cdots+\varepsilon_k}}{(\lambda)_{\mathbf{m}+\mathbf{k}}} = \frac{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)\right)}{\prod_{j=1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}}.$$

We summarize this subsection.

Theorem 2.4.2. *When $G = Sp(r,\mathbb{R})$, and $(\tau, V) = (\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}, V^\vee_{\varepsilon_1+\cdots+\varepsilon_k})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda > r$, the normalizing constant c_λ is given by*

$$c_\lambda = \frac{\prod_{j=1}^k \Gamma\left(\lambda - \frac{j-1}{2} + 1\right) \prod_{j=k+1}^r \Gamma\left(\lambda - \frac{j-1}{2}\right)}{\prod_{j=1}^k \Gamma\left(\lambda - \frac{j+r}{2} + 1\right) \prod_{j=k+1}^r \Gamma\left(\lambda - \frac{j+r}{2}\right)},$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+) \otimes V^\vee_{\varepsilon_1+\cdots+\varepsilon_k} = \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\, |\mathbf{k}|=k \\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_{+}}} V^\vee_{2\mathbf{m}+\mathbf{k}},$$

and for $f \in V^\vee_{2\mathbf{m}+\mathbf{k}}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}}{\|f\|^2_{F,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}} &= \frac{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)\right)}{\prod_{j=1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}} \\ &= \frac{1}{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)+1\right)_{m_j+k_j-1} \prod_{j=k+1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}}.\end{aligned}$$

2.4.3 $SU(q,s)$

In this subsection we set $G = SU(q,s)$, with $q \geq s$. Then we have

$$\begin{gathered} K \simeq S(U(q)\times U(s)), \quad \mathfrak{p}^\pm \simeq M(q,s;\mathbb{C}), \quad G_T \simeq SU(s,s), \quad K_T \simeq S(U(s)\times U(s)), \\ L \simeq \{l \in GL(s,\mathbb{C}) : \det l \in \mathbb{R}^\times\}, \quad K_L \simeq \{k \in U(s) : \det k = \pm 1\}, \\ r = s, \quad n = qs, \quad d = 2, \quad p = q+s. \end{gathered}$$

We want to calculate the norm $\|\cdot\|_{\lambda,\tau}$ of $\mathcal{O}(D,V)$ in the case $(\tau, V) = (\tau^{(q)\vee}_{\mathbf{0}} \boxtimes \tau^{(s)}_{\mathbf{k}}, V^{(q)\vee}_{\mathbf{0}} \otimes V^{(s)}_{\mathbf{k}}) = (\mathbf{1}^{(q)} \boxtimes \tau^{(s)}_{\mathbf{k}}, \mathbb{C} \otimes V^{(s)}_{\mathbf{k}})$ ($\mathbf{k} \in \mathbb{Z}^s_{++}$). These V have the restricted lowest weight $-\frac{1}{2}(k_1\gamma_1 + \cdots + k_s\gamma_s)\big|_{\mathfrak{a}_\mathfrak{l}}$, and remain irreducible even if restricted to $K_L = \operatorname{diag}(\{\pm 1\} \times SU(s))$ i.e. satisfy assumption (A1') of corollary 2.3.4. Thus $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda + k_s > q+s-1$, and the normalizing constant c_λ is given by

$$c_\lambda = \frac{\Gamma_\Omega(\lambda+\mathbf{k})}{\Gamma_\Omega(\lambda+\mathbf{k}-q)} = \prod_{j=1}^s (\lambda - (j-1) + k_j - q)_q.$$

First, we compute the K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V^{(s)}_{\mathbf{k}}\right)$. By Theorem 2.2.1 we have

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V^{(s)}_{\mathbf{k}}\right) &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^s_{++}} \left(V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{m}}\right) \otimes \left(\mathbb{C} \boxtimes V^{(s)}_{\mathbf{k}}\right) \\ &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^s_{++}} \bigoplus_{\mathbf{n}\in\mathbf{m}+\mathrm{wt}(\mathbf{k})} c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}} V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}.\end{aligned}$$

where $V^{(q)\vee}_{\mathbf{m}}$ is the abbreviation of $V^{(q)\vee}_{(m_1,\ldots,m_s,0,\ldots,0)}$, $\mathrm{wt}(\mathbf{k})$ is the set of all weights in the $GL(s,\mathbb{C})$-module $V^{(s)}_{\mathbf{k}}$, and $c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}}$ are some non-negative integers. Second, let rest : $\mathcal{P}(\mathfrak{p}^+) \otimes V \to \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}) \otimes V$ be the restriction map, as in Section 2.3.2. Then we have

$$\operatorname{rest}\left(V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}\right) = V^{(s)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}},$$

so each K-type $V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$ satisfies the assumption (A0) in Corollary 2.3.4. Third, we compute the tensor product with $\overline{\mathbb{C} \boxtimes V_{\mathbf{n}}^{(s)}} \simeq V_{\mathbf{n}}^{(s)} \boxtimes \mathbb{C}$.

$$\left(V_{\mathbf{m}}^{(s)\vee} \boxtimes V_{\mathbf{n}}^{(s)}\right) \otimes \left(V_{\mathbf{k}}^{(s)\vee} \boxtimes \mathbb{C}\right) = \bigoplus_{\mathbf{n}' \in \mathbf{m}+\mathrm{wt}(\mathbf{k})} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}'} V_{\mathbf{n}'}^{(s)\vee} \boxtimes V_{\mathbf{n}}^{(s)}.$$

By Theorem 2.2.2, $V_{\mathbf{n}'}^{(s)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$ is K_L-spherical if and only if $\mathbf{n}' = \mathbf{n}$, so all irreducible K_L-spherical submodules in $\left(V_{\mathbf{m}}^{(s)\vee} \boxtimes V_{\mathbf{n}}^{(s)}\right) \otimes \left(V_{\mathbf{k}}^{(s)\vee} \boxtimes \mathbb{C}\right)$ are isomorphic to $V_{\mathbf{n}}^{(s)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$, which has the lowest weight $-(n_1\gamma_1 + \cdots + n_s\gamma_s)$. Therefore each K-type satisfies the assumption (A2'), and by Corollary 2.3.4, for $f \in V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$ we have

$$\frac{\|f\|^2_{\lambda,\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}}}{\|f\|^2_{F,\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}}} = \frac{(\lambda)_{\mathbf{k}}}{(\lambda)_{\mathbf{n}}} = \frac{\prod_{j=1}^s (\lambda - (j-1))_{k_j}}{\prod_{j=1}^s (\lambda - (j-1))_{n_j}}.$$

We summarize this subsection.

Theorem 2.4.3. *When $G = SU(q,s)$ $(q \geq s)$, and $(\tau, V) = (\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}, \mathbb{C} \otimes V_{\mathbf{k}}^{(s)})$ $(\mathbf{k} \in \mathbb{Z}_{++}^s)$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda + k_s > q + s - 1$, the normalizing constant c_λ is given by*

$$c_\lambda = \prod_{j=1}^s (\lambda - (j-1) + k_j - q)_q,$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}\right) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^s} \bigoplus_{\mathbf{n} \in \mathbf{m}+\mathrm{wt}(\mathbf{k})} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}} V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)},$$

and for $f \in V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}}}{\|f\|^2_{F,\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}}} = \frac{\prod_{j=1}^s (\lambda - (j-1))_{k_j}}{\prod_{j=1}^s (\lambda - (j-1))_{n_j}} = \frac{1}{\prod_{j=1}^s (\lambda - (j-1) + k_j)_{n_j - k_j}}.$$

2.4.4 $SO^*(4r)$

In this subsection we set $G = SO^*(4r)$. Then we have

$$K \simeq U(2r), \quad \mathfrak{p}^\pm \simeq \mathrm{Skew}(2r,\mathbb{C}), \quad L \simeq GL(r,\mathbb{H}), \quad K_L \simeq Sp(r),$$
$$r = r, \quad n = r(2r-1), \quad d = 4, \quad p = 2(2r-1).$$

We want to calculate the norm $\|\cdot\|_{\lambda,\tau}$ of $\mathcal{O}(D,V)$ in the case $V = V_{(k,0,\ldots,0)}^\vee \simeq S^k(\mathbb{C}^r)^\vee$, or $V = V_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2})}^\vee \simeq S^k(\mathbb{C}^r) \otimes \det^{-k/2}$ $(k = 0,1,2\ldots)$ (the latter is not defined as the representation of $U(2r)$ if k is odd, so in this case we consider the double covering group $K = \widetilde{U}^2(r) \subset G = \widetilde{SO^*}^2(4r) \subset Spin(4r,\mathbb{C})$). These V have the restricted lowest weight $-\frac{k}{2}\gamma_1\big|_{\mathfrak{a}_\mathfrak{l}}$ and $-\frac{k}{2}(\gamma_1 + \cdots + \gamma_{r-1})\big|_{\mathfrak{a}_\mathfrak{l}}$ respectively. Also, these V remain irreducible even if restricted to $K_L = Sp(r)$, i.e. satisfy assumption (A1') of corollary 2.3.4.

First, we deal with $V = V^\vee_{(k,0,\dots,0)}$ case. Then $\|\cdot\|^2_{\lambda,\tau^\vee_{(k,0,\dots,0)}}$ converges if $\operatorname{Re}\lambda > 4r-3$, and the normalizing constant c_λ is given by

$$c_\lambda = \frac{\Gamma_\Omega(\lambda+(k,0,\dots,0))}{\Gamma_\Omega(\lambda+(k,0,\dots,0)-(2r-1))} = (\lambda+k)_{2r-1}\prod_{j=2}^{r}(\lambda-2(j-1)-(2r-1))_{2r-1}.$$

To begin with, we compute the K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(k,0,\dots,0)}$. To do this, we quote the following lemma.

Lemma 2.4.4 ([30, §79, Example 4])**.**

$$V^\vee_{\mathbf{m}}\otimes V^\vee_{(k,0,\dots,0)} = \bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^{2r},\ |\mathbf{k}|=k\\ 0\leq k_j\leq m_{j-1}-m_j}} V^\vee_{\mathbf{m}+\mathbf{k}}.$$

Using this and Theorem 2.2.1, we get

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(k,0,\dots,0)} &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V^\vee_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r)}\otimes V^\vee_{(k,0,\dots,0)}\\ &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\ \bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^{r},\ |\mathbf{k}|=k\\ 0\leq k_j\leq m_{j-1}-m_j}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r)}.\end{aligned}$$

Next, for each K-type $V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)}$, we compute the tensor product with $\overline{V^\vee_{(k,0,\dots,0)}}\simeq V^\vee_{(k,0,\dots,0)}$.

$$\begin{aligned}&V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r)}\otimes V^\vee_{(k,0,\dots,0)}\\ &= \bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{2r},\ |\mathbf{l}|=k\\ 0\leq l_{2j-1}\leq m_{j-1}-m_j-k_j\\ 0\leq l_{2j}\leq k_j}} V^\vee_{(m_1+k_1+l_1,m_1+l_2,m_2+k_2+l_3,m_2+l_4,\dots,m_r+k_r+l_{2r-1},m_r+l_{2r})}.\end{aligned}$$

By Theorem 2.2.2, $V^\vee_{(m_1+k_1+l_1,m_1+l_2,\dots,m_r+k_r+l_{2r-1},m_r+l_{2r})}$ is K_L-spherical if and only if the $(2j-1)$-th component of its lowest weight is equal to the $2j$-th component for each j, that is, $l_{2j-1}=0$ and $l_{2j}=k_j$. Thus, the only K_L-spherical submodule in $V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)}\otimes V^\vee_{(k,0,\dots,0)}$ is $V^\vee_{(m_1+k_1,m_1+k_1,\dots,m_r+k_r,m_r+k_r)}$, and $V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)}$ satisfies the assumption (A2') of Corollary 2.3.4 with $\mathbf{n}=\mathbf{m}+\mathbf{k}$. Therefore by Corollary 2.3.4, for $f\in V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)}$ we have

$$\frac{\|f\|^2_{\lambda,\tau^\vee_{(k,0,\dots,0)}}}{\|f\|^2_{F,\tau^\vee_{(k,0,\dots,0)}}} = \frac{(\lambda)_{(k,0,\dots,0)}}{(\lambda)_{\mathbf{m}+\mathbf{k}}} = \frac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}}.$$

Second, we deal with $V = V^\vee_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}$ case. Then $\|\cdot\|^2_{\lambda,\tau^\vee_{(k,0,\dots,0)}}$ converges if $\operatorname{Re}\lambda > 4r-3$, and the normalizing constant c_λ is given by

$$\begin{aligned}c_\lambda &= \frac{\Gamma_\Omega(\lambda+(k,\dots,k,0))}{\Gamma_\Omega(\lambda+(k,\dots,k,0)-(2r-1))}\\ &= \prod_{j=1}^{r-1}(\lambda-2(j-1)+k-(2r-1))_{2r-1}(\lambda-2(r-1)-(2r-1))_{2r-1}.\end{aligned}$$

Similar to the previous arguments, K-type decomposition of $\mathcal{O}(D,V)_K=\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}$ is given by

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}V^\vee_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r)}\otimes V^\vee_{(0,\dots,0,-k)}\otimes V^\vee_{(\frac{k}{2},\dots,\frac{k}{2})}\\&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge0})^r,\ |\mathbf{k}|=k\\0\le k_j\le m_j-m_{j+1}}}V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})},\end{aligned}$$

and for each K-type, we can show that the only K_L-spherical submodule in

$$V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})}\otimes\overline{V^\vee_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}}$$

is $V^\vee_{(m_1-k_1,m_1-k_1,\dots,m_r-k_r,m_r-k_r)+(k,\dots,k)}$. Thus $V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})}$ satisfies the assumption (A2') of Corollary 2.3.4 with $\mathbf{n}=\mathbf{m}-\mathbf{k}+(k,\dots,k)$. Therefore by Corollary 2.3.4, for $f\in V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})}$ we have

$$\frac{\|f\|^2_{\lambda,\tau^\vee_{(k/2,\dots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^\vee_{(k/2,\dots,k/2,-k/2)}}}=\frac{(\lambda)_{(k,\dots,k,0)}}{(\lambda)_{\mathbf{m}-\mathbf{k}+k}}=\frac{\prod_{j=1}^{r-1}(\lambda-2(j-1))_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j-k_j+k}}.$$

We summarize this subsection.

Theorem 2.4.5. *When $G=SO^*(4r)$, and $(\tau,V)=(\tau^\vee_{(k,0,\dots,0)},V^\vee_{(k,0,\dots,0)})$ $(k\in\mathbb{Z}_{\ge0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-3$, the normalizing constant c_λ is given by*

$$c_\lambda=(\lambda+k)_{2r-1}\prod_{j=2}^r(\lambda-2(j-1)-(2r-1))_{2r-1},$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(k,0,\dots,0)}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge0})^r,\ |\mathbf{k}|=k\\0\le k_j\le m_{j-1}-m_j}}V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r)},$$

and for $f\in V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau^\vee_{(k,0,\dots,0)}}}{\|f\|^2_{F,\tau^\vee_{(k,0,\dots,0)}}}=\frac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}}=\frac{1}{(\lambda+k)_{m_1+k_1-k}\prod_{j=2}^r(\lambda-2(j-1))_{m_j+k_j}}.$$

When $G=SO^*(4r)$, and $(\tau,V)=(\tau^\vee_{(k/2,\dots,k/2,-k/2)},V^\vee_{(k/2,\dots,k/2,-k/2)})$ $(k\in\mathbb{Z}_{\ge0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-3$, the normalizing constant c_λ is given by

$$c_\lambda=\prod_{j=1}^{r-1}(\lambda-2(j-1)+k-(2r-1))_{2r-1}(\lambda-2(r-1)-(2r-1))_{2r-1},$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^\vee_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge0})^r,\ |\mathbf{k}|=k\\0\le k_j\le m_j-m_{j+1}}}V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})},$$

and for $f \in V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r)+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}$, *the ratio of norms is given by*

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^\vee_{(k/2,\dots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^\vee_{(k/2,\dots,k/2,-k/2)}}} &= \frac{\prod_{j=1}^{r-1}(\lambda-2(j-1))_k}{\prod_{j=1}^{r}(\lambda-2(j-1))_{m_j-k_j+k}} \\ &= \frac{1}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2(r-1))_{m_r-k_r+k}}.\end{aligned}$$

2.4.5 $Spin_0(2,n)$

In this subsection we set $G = Spin_0(2,n)$, the identity component of the indefinite spin group. This is of tube type, and we have

$$\begin{gathered} K \simeq (Spin(2)\times Spin(n))/\{(1,1),(-1,-1)\}, \qquad \mathfrak{p}^\pm \simeq \mathbb{C}^n, \\ r=2, \quad n=n, \quad d=n-2, \quad p=n. \end{gathered}$$

Let $\pi : K^\mathbb{C} = (Spin(2,\mathbb{C})\times Spin(n,\mathbb{C}))/\{(1,1),(-1,-1)\} \to SO(2,\mathbb{C})\times SO(n,\mathbb{C})$ be the covering map. Then we have

$$\begin{gathered} \pi(L) \simeq SO_0(1,1)\times SO_0(1,n-1)\cup SO_-(1,1)\times SO_-(1,n-1), \\ \pi(K_L) \simeq \{+I_2\}\times SO(n-1)\cup\{-I_2\}\times O_-(n-1), \end{gathered}$$

where $SO_-(p,q), O_-(q)$ are the connected component of $SO(p,q), O(q)$ which does not contain the unit element. Each representation of $K^\mathbb{C}$ is of the form $(\chi^{m_0}\boxtimes\tau^{[n]}_{\mathbf{m}}, \mathbb{C}_{m_0}\otimes V^{[n]}_{\mathbf{m}})$, and sometimes we abbreviate this to $(\tau_{(m_0;\mathbf{m})}, V_{(m_0;\mathbf{m})})$.

Now we want to calculate the norm $\|\cdot\|_{\lambda,\tau}$ of $\mathcal{O}(D,V)$ in the case

$$(\tau,V) = \begin{cases} (\chi^{-k}\boxtimes\tau_{(k,\dots,k,\pm k)}, \mathbb{C}_{-k}\otimes V_{(k,\dots,k,\pm k)}) & \left(k\in\frac{1}{2}\mathbb{Z}_{\ge 0}\right) \quad (n:\text{even}), \\ (\chi^{-k}\boxtimes\tau_{(k,\dots,k)}, \mathbb{C}_{-k}\otimes V_{(k,\dots,k)}) & \left(k=0,\frac{1}{2}\right) \quad (n:\text{odd}). \end{cases}$$

These (τ,V) have the restricted lowest weight $-k\gamma_1$, and remain irreducible even if restricted to K_L, i.e. satisfy assumption (A1') of corollary 2.3.4. Thus $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda > n-1$, and the normalizing constant c_λ is given by

$$c_\lambda = \frac{\Gamma_\Omega(\lambda+(k,0))}{\Gamma_\Omega\left(\lambda+(k,0)-\frac{n}{2}\right)} = \frac{\Gamma(\lambda+k)\,\Gamma\left(\lambda-\frac{n-2}{2}\right)}{\Gamma\left(\lambda+k-\frac{n}{2}\right)\Gamma(\lambda-(n-1))}.$$

First we compute the K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+)\otimes V$. To do this, we use the following lemma, which comes from the "multi-minuscule rule" [25, Corollary 2.16].

Lemma 2.4.6. *(1) Let* $m\in\mathbb{Z}_{\ge 0}$ *and* $k\in\frac{1}{2}\mathbb{Z}_{\ge 0}$. *For two representations* $V_{(m,0,\dots,0)}$ *and* $V_{(k,\dots,k,\pm k)}$ *of* $\mathfrak{so}(2s,\mathbb{C})$,

$$V_{(m,0,\dots,0)}\otimes V_{(k,\dots,k,\pm k)} = \bigoplus_{l=\max\{-k,k-m\}}^{k} V_{(m+l,k,\dots,k,\pm l)}$$

(double sign corresponds) holds.

(2) Let $m\in\mathbb{Z}_{>0}$. *For two representations* $V_{(m,0,\dots,0)}$ *and* $V_{\left(\frac{1}{2},\dots,\frac{1}{2}\right)}$ *of* $\mathfrak{so}(2s+1,\mathbb{C})$,

$$V_{(m,0,\dots,0)}\otimes V_{\left(\frac{1}{2},\dots,\frac{1}{2}\right)} = V_{\left(m+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)}\oplus V_{\left(m-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)}$$

holds.

By Theorem 2.2.1,

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}} \mathbb{C}_{-(m_1+m_2)} \boxtimes V_{(m_1-m_2,0,\dots,0)}$$

holds, and combining with the above lemma, we have

$$\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C}_{-k} \boxtimes V_{(k,\dots,k,\pm k)}\right) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}} \bigoplus_{\substack{-k\le l\le k \\ m_1-m_2+l\ge k}} \mathbb{C}_{-(m_1+m_2+k)} \boxtimes V_{(m_1-m_2+l,k,\dots,k,\pm l)}$$

for $n=2s$ even case, $k\in\frac{1}{2}\mathbb{Z}_{\ge 0}$, and

$$\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C}_{-k} \boxtimes V_{(k,\dots,k)}\right) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}} \bigoplus_{\substack{-k\le l\le k \\ m_1-m_2+l\ge k}} \mathbb{C}_{-(m_1+m_2+k)} \boxtimes V_{(m_1-m_2+l,k,\dots,k,|l|)}$$

for $n=2s+1$ odd case, $k=0,\frac{1}{2}$.

Second, we seek K_L-spherical subspace in the tensor product of each K-type and $\bar{V}$. To begin with, we deal with $n=2s$ even, $V=V_{(-k;k,\dots,k,k)}$ case. Suppose

$$V_{(-(n_1+n_2):n_1-n_2,0,\dots,0)} \subset V_{(-(m_1+m_2+k);m_1-m_2+l,k,\dots,k,l)} \otimes \overline{V_{(-k;k,\dots,k)}},$$

where $(n_1,n_2)\in\mathbb{Z}^2_+$. This implies that $(-(n_1+n_2)+(m_1+m_2+k);(n_1-n_2)-(m_1-m_2+l),-k,\dots,-k,-l)$ is a weight of $\overline{V_{(-k;k,\dots,k)}}$. However, the weight of this form is only $(-k;l,-k,\dots,-k,-l)$, since $\overline{V_{(-k;k,\dots,k,k)}}$ has the lowest weight $(-k;-k,\dots,-k,k)$, and root vectors $x_{\varepsilon_1-\varepsilon_s}, x_{\varepsilon_1+\varepsilon_s}\in\mathfrak{so}(2s)$ commute with each other. Therefore we have

$$\begin{cases} (n_1+n_2)-(m_1+m_2+k)=k, \\ (n_1-n_2)-(m_1-m_2+l)=l. \end{cases} \qquad \therefore \begin{cases} n_1=m_1+k+l, \\ n_2=m_2+k-l. \end{cases}$$

Thus all K_L-spherical irreducible submodule in $V_{(-(m_1+m_2+k);m_1-m_2+l,k,\dots,k,l)}\otimes\overline{V_{(-k;k,\dots,k)}}$ have the same lowest weight $-(n_1\gamma_1+n_2\gamma_2)$ with $(n_1,n_2)=(m_1+k+l,m_2+k-l)$, and all K-types satisfy the assumption (A2') of Corollary 2.3.4. The same argument holds for $V=V_{(-k;k,\dots,k,-k)}$ case, and also for n odd case, noting that only $k=0,\frac{1}{2}$ is allowed, and $n_1,n_2\in\mathbb{Z}$. Therefore by Corollary 2.3.4, for $f\in V_{(-(m_1+m_2+k);m_1-m_2+l,k,\dots,k,\pm l)}$ or $V_{(-(m_1+m_2+k);m_1-m_2+l,k,\dots,k,|l|)}$, we have

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} = \frac{(\lambda)_{(2k,0)}}{(\lambda)_{(m_1+k+l,m_2+k-l)}} = \frac{(\lambda)_{2k}}{(\lambda)_{m_1+k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}.$$

We summarize this subsection.

Theorem 2.4.7. *When $G=Spin_0(2,n)$ and*

$$(\tau,V) = \begin{cases} (\chi^{-k}\boxtimes\tau_{(k,\dots,k,\pm k)}, \mathbb{C}_{-k}\otimes V_{(k,\dots,k,\pm k)}) & \left(k\in\frac{1}{2}\mathbb{Z}_{\ge0}\right) \quad (n:\textit{even}), \\ (\chi^{-k}\boxtimes\tau_{(k,\dots,k)}, \mathbb{C}_{-k}\otimes V_{(k,\dots,k)}) & \left(k=0,\frac{1}{2}\right) \quad (n:\textit{odd}), \end{cases}$$

$\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>n-1$, the normalizing constant c_λ is given by

$$c_\lambda = \frac{\Gamma(\lambda+k)\,\Gamma\left(\lambda-\frac{n-2}{2}\right)}{\Gamma\left(\lambda+k-\frac{n}{2}\right)\Gamma(\lambda-(n-1))},$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V=\begin{cases}\displaystyle\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{-k\le l\le k\\ m_1-m_2+l\ge k}}\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l)} & (n:\textit{even}),\\ \displaystyle\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{-k\le l\le k\\ m_1-m_2+l\ge k}}\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,|l|)} & (n:\textit{odd}),\end{cases}$$

and for $f\in\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l)}$ or $\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,|l|)}$, the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}=\frac{(\lambda)_{2k}}{(\lambda)_{m_1+k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}=\frac{1}{(\lambda+2k)_{m_1-k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}.$$

2.5 Norm computation: Non-tube type case

When G is of non-tube type, we cannot compute the norm by just using Theorem 2.3.1, because it is difficult to determine the constants a_{ij} in Theorem 2.3.1. Thus we have to use other informations to compute the norm. In this section we compute the norm in the case

- $(G,V)=(SU(q,s),\mathbb{C}\boxtimes V')$ $(q<s)$, by direct computation,
- $(G,V)=(SO^*(4r+2),S^k(\mathbb{C}^{2r+1})^\vee)$, by using the embedding $SO^*(4r+2)\subset SO^*(4r+4)$,
- $(G,V)=(SO^*(4r+2),S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$, by combining Theorem 2.3.1 and the embedding $SU(1,2r)\subset SO^*(4r+2)$.

Also, for $G=E_{6(-14)}$, we try to compute the norm as best we can, by using Theorem 2.3.1.

2.5.1 Explicit realization of G

Before starting the computation, we fix the realization of $G=SU(q,s)$, $SO^*(2s)$. We realize $SU(q,s)$, $SO^*(2s)$ as

$$SU(q,s):=\left\{g\in SL(q+s,\mathbb{C}):g\begin{pmatrix}I_q&0\\0&-I_s\end{pmatrix}g^*=\begin{pmatrix}I_q&0\\0&-I_s\end{pmatrix}\right\},\tag{2.5.1}$$

$$SO^*(2s):=\left\{g\in GL(2s,\mathbb{C}):g\begin{pmatrix}0&I_s\\I_s&0\end{pmatrix}{}^tg=\begin{pmatrix}0&I_s\\I_s&0\end{pmatrix},\ g\begin{pmatrix}0&I_s\\-I_s&0\end{pmatrix}=\begin{pmatrix}0&I_s\\-I_s&0\end{pmatrix}\bar{g}\right\},\tag{2.5.2}$$

and realize $K^{\mathbb{C}}$, $\mathfrak{p}^\pm$ as

$$K^{\mathbb{C}}:=\left\{\begin{pmatrix}a&0\\0&d\end{pmatrix}:\begin{matrix}(a,d)\in S(GL(q,\mathbb{C})\times GL(s,\mathbb{C})) & (G=SU(q,s))\\ a\in GL(s,\mathbb{C}),\ d={}^ta^{-1} & (G=SO^*(2s))\end{matrix}\right\},$$

$$\mathfrak{p}^+:=\left\{\begin{pmatrix}0&b\\0&0\end{pmatrix}:\begin{matrix}b\in M(q,s;\mathbb{C}) & (G=SU(q,s))\\ b\in \mathrm{Skew}(s,\mathbb{C}) & (G=SO^*(2s))\end{matrix}\right\},$$

$$\mathfrak{p}^-:=\left\{\begin{pmatrix}0&0\\c&0\end{pmatrix}:\begin{matrix}c\in M(s,q;\mathbb{C}) & (G=SU(q,s))\\ c\in \mathrm{Skew}(s,\mathbb{C}) & (G=SO^*(2s))\end{matrix}\right\}.$$

Then under the identification $\mathfrak{p}^+ \simeq M(q,s;\mathbb{C})$ or $\mathrm{Skew}(2s,\mathbb{C})$ by $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mapsto b$, we have

$$D = \{w \in M(q,s;\mathbb{C}) : I_q - ww^* \text{ is positive definite.}\} \qquad (G = SU(q,s)), \tag{2.5.3}$$
$$D = \{w \in \mathrm{Skew}(s,\mathbb{C}) : I_s - ww^* \text{ is positive definite.}\} \qquad (G = SO^*(2s)). \tag{2.5.4}$$

For a representation $(\tau_1 \boxtimes \tau_2, V_1 \otimes V_2)$ of $K^{\mathbb{C}} = S(GL(q,\mathbb{C}) \times GL(s,\mathbb{C}))$, the universal covering group $\widetilde{SU}(q,s)$ acts on $\mathcal{O}(D, V_1 \otimes V_2)$ by

$$\begin{aligned} \tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w) = \det(cw+d)^{-\lambda} \left(\tau_1\left(a^* + wb^*\right) \boxtimes \tau_2\left((cw+d)^{-1}\right)\right) & \\ \times f\left((aw+b)(cw+d)^{-1}\right), & \quad (2.5.5) \end{aligned}$$

and for a representation (τ, V) of $K^{\mathbb{C}} = GL(s,\mathbb{C})$, the universal covering group $\widetilde{SO^*}(2s)$ acts on $\mathcal{O}(D,V)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w) = \det(cw+d)^{-\lambda/2} \tau\left({}^t(cw+d)\right) f\left((aw+b)(cw+d)^{-1}\right), \tag{2.5.6}$$

We note that we have the identities, for $w \in M(q,s;\mathbb{C})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(q,s)$,

$$\det(I_q - ww^*) = \det(I_s - w^*w), \qquad \det(a^* + wb^*) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \det(cw+d).$$

Therefore, on $SU(q,s)$, $\det(a^*+wb^*) = \det(cw+d)$ holds. We also note that $\det(cw+d)^{-\lambda}$ is not well-defined on G for general $\lambda \in \mathbb{C}$, but is well-defined on the universal covering group $\tilde{G}$. These representations preserve the inner product

$$\begin{aligned} \langle f, g\rangle_{\lambda,\tau} = \frac{c_\lambda}{\pi^{qs}} \int_D \left(\left(\tau_1\left((I_q - ww^*)^{-1}\right) \boxtimes \tau_2\left(I_s - w^*w\right)\right) f(w), g(w)\right)_{\tau_1\boxtimes\tau_2} & \\ \times \det(I_q - ww^*)^{\lambda-(q+s)} dw, & \quad (2.5.7) \\ \langle f, g\rangle_{\lambda,\tau} = \frac{c_\lambda}{\pi^{s(s-1)/2}} \int_D \left(\tau\left((I_s - ww^*)^{-1}\right) f(w), g(w)\right)_\tau \det(I_s - ww^*)^{\frac{1}{2}(\lambda - 2(s-1))} dw. & \\ & \quad (2.5.8) \end{aligned}$$

respectively. Let $\mathfrak{h} \subset \mathfrak{g}$ be the subspace which consists of all diagonal matrices, and define the linear form ε_i on $\mathfrak{h}^{\mathbb{C}}$ by $\varepsilon_i(E_{jj}) = \delta_{ij}$. We define the positive system $\Delta_+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ as in Section 2.4.1.

2.5.2 $SU(q,s)$

In this subsection we set $G = SU(q,s)$, with $q < s$, which is realized explicitly as (2.5.1). Then we have

$$\begin{gathered} K \simeq S(U(q) \times U(s)), \quad \mathfrak{p}^\pm \simeq M(q,s;\mathbb{C}), \quad G_T \simeq SU(q,q), \quad K_T \simeq S(U(q) \times U(q)), \\ L \simeq \{l \in GL(q,\mathbb{C}) : \det l \in \mathbb{R}^\times\}, \quad K_L \simeq \{k \in U(q) : \det k = \pm 1\}, \\ r = q, \quad n = qs, \quad d = 2, \quad p = q+s. \end{gathered}$$

We set $(\tau, V) = (\tau_{\mathbf{0}}^{(q)\vee} \boxtimes \tau_{\mathbf{k}}^{(s)}, V_{\mathbf{0}}^{(q)\vee} \otimes V_{\mathbf{k}}^{(s)}) = (\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}, \mathbb{C} \otimes V_{\mathbf{k}}^{(s)})$ ($\mathbf{k} \in \mathbb{Z}_{++}^s$). In this case, the inner product is given by

$$\langle f, g\rangle_{\lambda, \mathbf{1}^{(q)}\boxtimes\tau_{\mathbf{k}}^{(s)}} = \frac{c_\lambda}{\pi^{qs}} \int_D \left(\left(\tau_{\mathbf{k}}^{(s)}\left(I_s - w^*w\right)\right) f(w), g(w)\right)_{\tau_{\mathbf{k}}^{(s)}} \det(I_s - w^*w)^{\lambda-(q+s)} dw.$$

The goal of this subsection is to prove the following theorem.

Theorem 2.5.1. *When $G = SU(q, s)$ $(q < s)$ and $(\tau, V) = (\mathbf{1}^{(q)} \boxtimes \tau_{\mathbf{k}}^{(s)}, \mathbb{C} \otimes V_{\mathbf{k}}^{(s)})$ ($\mathbf{k} \in \mathbb{Z}_{++}^s$), $\|\cdot\|_{\lambda,\tau}^2$ converges if $\operatorname{Re}\lambda + k_s > q + s - 1$, the normalizing constant c_λ is given by*

$$c_\lambda = \prod_{j=1}^{s} (\lambda - (j-1) + k_j - q)_q,$$

the K-type decomposition of $\mathcal{O}(D, V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}\right) = \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^q} \bigoplus_{\mathbf{n}\in\mathbf{m}+\mathrm{wt}(\mathbf{k})} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}} V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)},$$

and for $f \in V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$, the ratio of norms is given by

$$\frac{\|f\|_{\lambda,\mathbf{1}^{(q)}\boxtimes\tau_{\mathbf{k}}^{(s)}}^2}{\|f\|_{F,\mathbf{1}^{(q)}\boxtimes\tau_{\mathbf{k}}^{(s)}}^2} = \frac{\prod_{j=1}^{s}(\lambda - (j-1))_{k_j}}{\prod_{j=1}^{s}(\lambda - (j-1))_{n_j}} = \frac{1}{\prod_{j=1}^{s}(\lambda - (j-1) + k_j)_{n_j - k_j}}.$$

Before beginning the proof, we prepare some more notations. For $k \in \mathbb{N}$, $\mathbf{m} \in \mathbb{C}^k$ and for $x \in M(k, \mathbb{C})$, we write

$$\Delta_{\mathbf{m}}(x) := \prod_{l=1}^{k-1} \det\left((x_{ij})_{1\le i,j\le l}\right)^{m_l - m_{l+1}} \det(x)^{m_k}.$$

For $k \in \mathbb{N}$, let $Q_k \subset GL(k, \mathbb{C})$ be the set of upper triangular matrices with positive diagonal entries. Then for $l_1, l_2 \in Q_k$, $\mathbf{m} \in \mathbb{C}^k$, $\Delta_{\mathbf{m}}(l_1)\Delta_{\mathbf{m}}(l_2) = \Delta_{\mathbf{m}}({}^t l_1 l_2)$ holds, and for $l_1 \in Q_k$, $l_2 \in M(k, l; \mathbb{C})$, $l_3 \in Q_l$ and $\mathbf{m} \in \mathbb{C}^k$, $\mathbf{n} \in \mathbb{C}^l$, $\Delta_{\mathbf{m}}(l_1)\Delta_{\mathbf{n}}(l_3) = \Delta_{(\mathbf{m},\mathbf{n})}\begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix}$ holds. Also we set

$$\begin{aligned}
(\mathfrak{p}_{\mathrm{T}}^+)^\perp &:= M(q, s-q; \mathbb{C}), \\
\Omega &:= \{x \in \mathrm{Herm}(q, \mathbb{C}) : x \text{ is positive definite.}\}, \\
\tilde{\Omega} &:= \{x \in \mathrm{Herm}(s, \mathbb{C}) : x \text{ is positive definite.}\}.
\end{aligned}$$

Now we start the proof. To begin with, we compute the K-type decomposition of $\mathcal{O}(D, V)_K = \mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}\right)$.

$$\begin{aligned}
\mathcal{P}(\mathfrak{p}^+) \otimes \left(\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}\right) &= \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^q} \left(V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{m}}^{(s)}\right) \otimes \left(\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}\right) \\
&= \bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^q} \bigoplus_{\mathbf{n}\in\mathbf{m}+\mathrm{wt}(\mathbf{k})} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}} V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}.
\end{aligned}$$

where $V^{(s)}_{\mathbf{m}}$ is the abbreviation of $V^{(s)}_{(m_1,\dots,m_q,0,\dots,0)}$, $\mathrm{wt}(\mathbf{k})$ is the set of all weights in the $GL(s,\mathbb{C})$-module $V^{(s)}_{\mathbf{k}}$, and $c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}}$ are some non-negative integers. We note that, for $\mathbf{n} \in \mathbb{Z}^s_{++}$, there exists $\mathbf{m} \in \mathbb{Z}^q_{++}$ such that $c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}} \neq 0$ if and only if

$$n_j \geq k_j \ (1 \leq j \leq q) \quad \text{and} \quad k_{j-q} \leq n_j \leq k_j \ (j \geq q+1),$$

which can be proved by using Littlewood-Richardson rule.

For each K-type $V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}$, let $K_{\mathbf{m},\mathbf{n}}(z,w) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V^{(s)}_{\mathbf{k}}))$ be the reproducing kernel of the $K^{\mathbb{C}}_{\mathrm{T}}$-submodule $V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(q)}_{\mathbf{n}'} \subset V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}$, where $\mathbf{n}' := (n_1,\dots,n_q) \in \mathbb{Z}^q_{++}$. Then since $V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(q)}_{\mathbf{n}'} \subset V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}$ is the lowest submodule, we have

$$\tau^{(s)}_{\mathbf{k}}\begin{pmatrix} l_2 & l_3 \\ 0 & l_4 \end{pmatrix} K_{\mathbf{m},\mathbf{n}}\left(l_1 z \begin{pmatrix} l_2 & l_3 \\ 0 & l_4 \end{pmatrix}, l_1^{*-1} w \begin{pmatrix} l_2^{*-1} & l_5 \\ 0 & l_6 \end{pmatrix} \right) \tau^{(s)}_{\mathbf{k}}\begin{pmatrix} l_2^{-1} & 0 \\ l_5^* & l_6^* \end{pmatrix} = \Delta_{\mathbf{n}''}(l_6^* l_4) K_{\mathbf{m},\mathbf{n}}(z,w)$$
$$(z,w \in M(q,s;\mathbb{C}),\ l_1,l_2 \in GL(q,\mathbb{C}),\ l_3,l_5 \in M(q,s-q;\mathbb{C}),\ l_4,l_6 \in Q_{s-q}),$$

where $\mathbf{n}'' := (n_{s-q+1},\dots,n_s)$. Using this $K_{\mathbf{m},\mathbf{n}}(z,w)$, we can rewrite the ratio of norms. That is, for $f \in V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{n}}$, the ratio of norms $\|f\|^2_{\lambda,\mathbf{1}^{(q)}\boxtimes\tau^{(s)}_{\mathbf{k}}} / \|f\|^2_{F,\mathbf{1}^{(q)}\boxtimes\tau^{(s)}_{\mathbf{k}}}$ is equal to

$$R_{\mathbf{m},\mathbf{n}}(\lambda) := \frac{c_\lambda \displaystyle\int_D \mathrm{Tr}_{V^{(s)}_{\mathbf{k}}}\left(\tau^{(s)}_{\mathbf{k}}(I_s - w^*w) K_{\mathbf{m},\mathbf{n}}(w,w) \right) \det(I_s - w^*w)^{\lambda-(q+s)} dw}{\displaystyle\int_{\mathfrak{p}^+} \mathrm{Tr}_{V^{(s)}_{\mathbf{k}}}(K_{\mathbf{m},\mathbf{n}}(w,w)) e^{-\mathrm{tr}(w^*w)} dw}.$$

Now similarly to Lemma 2.3.2, for any non-negative measurable function f on $M(q,s;\mathbb{C})$, we have

$$\frac{1}{\pi^{qs}} \int_{\mathfrak{p}^+} f(w) dw = \frac{1}{\Gamma_\Omega(q)} \int_{\substack{x\in\Omega, y\in(\mathfrak{p}^+_{\mathrm{T}})^\perp \\ k_1,k_2\in U(q)}} f((k_1 x^{\frac{1}{2}} k_2, k_1 y)) dk_1 dk_2 dx dy.$$

Using this and the K_{T}-invariance of $K_{\mathbf{m},\mathbf{n}}(z,w)$

$$\begin{aligned} &K_{\mathbf{m},\mathbf{n}}((k_1 x^{\frac{1}{2}} k_2, k_1 y), (k_1 x^{\frac{1}{2}} k_2, k_1 y)) \\ &= \tau^{(s)}_{\mathbf{k}} \begin{pmatrix} k_2^{-1} & 0 \\ 0 & I_{s-q} \end{pmatrix} K_{\mathbf{m},\mathbf{n}}((x^{\frac{1}{2}}, y), (x^{\frac{1}{2}}, y)) \tau^{(s)}_{\mathbf{k}} \begin{pmatrix} k_2 & 0 \\ 0 & I_{s-q} \end{pmatrix} \\ &\qquad\qquad (x \in \Omega,\ y \in (\mathfrak{p}^+_{\mathrm{T}})^\perp,\ k_1,k_2 \in U(q)), \end{aligned}$$

we have

$$R_{\mathbf{m},\mathbf{n}}(\lambda) = \frac{\begin{aligned} c_\lambda \int_{\substack{x\in\Omega, y\in(\mathfrak{p}^+_{\mathrm{T}})^\perp \\ (x^{1/2},y)\in D}} \mathrm{Tr}_{V^{(s)}_{\mathbf{k}}}\left(\tau^{(s)}_{\mathbf{k}}\left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right) K_{\mathbf{m},\mathbf{n}}((x^{\frac{1}{2}}, y), (x^{\frac{1}{2}}, y)) \right) \\ \times \det\left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right)^{\lambda-(q+s)} dx dy \end{aligned}}{\displaystyle\int_{x\in\Omega, y\in(\mathfrak{p}^+_{\mathrm{T}})^\perp} \mathrm{Tr}_{V^{(s)}_{\mathbf{k}}}(K_{\mathbf{m},\mathbf{n}}((x^{\frac{1}{2}}, y), (x^{\frac{1}{2}}, y))) e^{-\mathrm{tr}\begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix}} dx dy}.$$

$K_{\mathbf{m},\mathbf{n}}((x^{\frac{1}{2}}, y), (x^{\frac{1}{2}}, y))$ is transformed as below.

$$\begin{aligned} &K_{\mathbf{m},\mathbf{n}}((x^{\frac{1}{2}}, y), (x^{\frac{1}{2}}, y)) = K_{\mathbf{m},\mathbf{n}}\left(x^{-\frac{1}{2}}(x,0) \begin{pmatrix} I_q & x^{-1/2}y \\ 0 & I_{s-q} \end{pmatrix}, x^{\frac{1}{2}}(I_q,0) \begin{pmatrix} I_q & x^{-1/2}y \\ 0 & I_{s-q} \end{pmatrix} \right) \\ &= \tau^{(s)}_{\mathbf{k}} \begin{pmatrix} I_q & -x^{-1/2}y \\ 0 & I_{s-q} \end{pmatrix} K_{\mathbf{m},\mathbf{n}}((x,0),(I_q,0)) \tau^{(s)}_{\mathbf{k}} \begin{pmatrix} I_q & 0 \\ -y^*x^{-1/2} & I_{s-q} \end{pmatrix}. \end{aligned}$$

Then $K_{\mathbf{m},\mathbf{n}}((\cdot,0),(I_q,0))$ is $K_L=\mathrm{diag}(\{\pm 1\}\times SU(q))$-invariant under the representation $\tilde{\tau}$ of $K_{\mathrm{T}}^{\mathbb{C}}$ on $\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,\mathrm{End}(V_{\mathbf{k}}^{(s)}))=\mathcal{P}(M(q,s),\mathrm{End}(V_{\mathbf{k}}^{(s)}))$, where

$$(\tilde{\tau}(l_1,l_2))F(x):=\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} l_2 & 0\\ 0 & I_{s-q}\end{pmatrix}F(l_1^{-1}xl_2)\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} l_1^{-1} & 0\\ 0 & I_{s-q}\end{pmatrix}.$$

That is, $K_{\mathbf{m},\mathbf{n}}((\cdot,0),(I_q,0))\in\left(\left(V_{\mathbf{m}}^{(q)\vee}\boxtimes V_{\mathbf{n}'}^{(q)}\right)\otimes\left(V_{\mathbf{k}}^{(s)\vee}\Big|_{U(q)}\boxtimes\mathbb{C}\right)\right)^{K_L}=\left(V_{\mathbf{n}'}^{(q)\vee}\boxtimes V_{\mathbf{n}'}^{(q)}\right)^{K_L}$. Therefore there exists an $F_{\mathbf{m},\mathbf{n}}(x)\in\mathcal{P}(\mathfrak{p}_{\mathrm{T}}^+,\mathrm{End}(V_{\mathbf{k}}^{(s)}))$ such that

$$\int_{U(q)}\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} k & 0\\ 0 & I_{s-q}\end{pmatrix}F_{\mathbf{m},\mathbf{n}}(k^{-1}xk)\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} k^{-1} & 0\\ 0 & I_{s-q}\end{pmatrix}dk=K_{\mathbf{m},\mathbf{n}}((x,0),(I_q,0)),$$
$$\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} l_2 & 0\\ 0 & l_4\end{pmatrix}F_{\mathbf{m},\mathbf{n}}({}^tl_1xl_2)\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} {}^tl_1 & 0\\ 0 & {}^tl_3\end{pmatrix}=\Delta_{\mathbf{n}'}({}^tl_1l_2)\Delta_{\mathbf{n}''}({}^tl_3l_4)F_{\mathbf{m},\mathbf{n}}(x)$$
$$(x\in\mathfrak{p}_{\mathrm{T}}^+,\ l_1,l_2\in Q_q,\ l_3,l_4\in Q_{s-q}).$$

We define

$$\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y):=\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} I_q & -x^{-1/2}y\\ 0 & I_{s-q}\end{pmatrix}F_{\mathbf{m},\mathbf{n}}(x)\tau_{\mathbf{k}}^{(s)}\begin{pmatrix} I_q & 0\\ -y^*x^{-1/2} & I_{s-q}\end{pmatrix}.$$

Then we have

$$R_{\mathbf{m},\mathbf{n}}(\lambda)=\frac{\begin{aligned}c_\lambda\int_{\substack{x\in\Omega,y\in(\mathfrak{p}_{\mathrm{T}}^+)^\perp\\(x^{1/2},y)\in D}}&\mathrm{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\tau_{\mathbf{k}}^{(s)}\left(I_s-\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}\right)\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y)\right)\\&\times\det\left(I_s-\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}\right)^{\lambda-(q+s)}dxdy\end{aligned}}{\displaystyle\int_{x\in\Omega,y\in(\mathfrak{p}_{\mathrm{T}}^+)^\perp}\mathrm{Tr}_{V_{\mathbf{k}}^{(s)}}(\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y))e^{-\mathrm{tr}\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}}dxdy}.$$

We set

$$\begin{aligned}B_{\mathbf{m},\mathbf{n}}(\lambda):=\int_{\substack{x\in\Omega,y\in(\mathfrak{p}_{\mathrm{T}}^+)^\perp\\(x^{1/2},y)\in D}}&\mathrm{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\tau_{\mathbf{k}}^{(s)}\left(I_s-\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}\right)\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y)\right)\\&\times\det\left(I_s-\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}\right)^{\lambda-(q+s)}dxdy,\\
\Gamma_{\mathbf{m},\mathbf{n}}:=\int_{x\in\Omega,y\in(\mathfrak{p}_{\mathrm{T}}^+)^\perp}&\mathrm{Tr}_{V_{\mathbf{k}}^{(s)}}(\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y))e^{-\mathrm{tr}\begin{pmatrix} x & x^{1/2}y\\ y^*x^{1/2} & y^*y\end{pmatrix}}dxdy,\end{aligned}$$

so that $R_{\mathbf{m},\mathbf{n}}(\lambda)=c_\lambda B_{\mathbf{m},\mathbf{n}}(\lambda)/\Gamma_{\mathbf{m},\mathbf{n}}$. We want to compute $B_{\mathbf{m},\mathbf{n}}(\lambda)$ explicitly. To do this, similarly to (2.3.3), for $z\in\tilde{\Omega}$ we define

$$\begin{aligned}J(z):=\int_{E(z)}&\mathrm{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\tau_{\mathbf{k}}^{(s)}\left(z-\begin{pmatrix} x' & (x')^{1/2}y'\\ (y')^*(x')^{1/2} & (y')^*y'\end{pmatrix}\right)\tilde{F}_{\mathbf{m},\mathbf{n}}(x',y')\right)\\&\times\det\left(z-\begin{pmatrix} x' & (x')^{1/2}y'\\ (y')^*(x')^{1/2} & (y')^*y'\end{pmatrix}\right)^{\lambda-(q+s)}dx'dy',\end{aligned}$$

where

$$E(z) := \left\{ (x,y) \in \Omega \times (\mathfrak{p}_{\mathrm{T}}^+)^\perp : z - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \text{ is positive definite.} \right\},$$

so that $E(I_s)$ coincides with the domain of integration of $B_{\mathbf{m},\mathbf{n}}(\lambda)$, and $J(I_s) = B_{\mathbf{m},\mathbf{n}}(\lambda)$ holds. To compute $J(z)$, we take $l_1 \in Q_q$, $l_2 \in M(q, s-q; \mathbb{C})$ and $l_3 \in Q_{s-q}$ such that

$$z = \begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix},$$

and we change variables x, y to

$$x' = l_1^* x l_1, \qquad y' = (l_1^* x l_1)^{-1/2} l_1^* x^{1/2}(y l_3 + x^{1/2} l_2),$$

so that

$$\begin{aligned}
\begin{pmatrix} x' & (x')^{1/2}y' \\ (y')^*(x')^{1/2} & (y')^*y' \end{pmatrix} &= \begin{pmatrix} l_1^* x l_1 & l_1^* x^{1/2}(y l_3 + x^{1/2} l_2) \\ (l_3^* y^* + l_2^* x^{1/2}) x^{1/2} l_1 & (l_3^* y^* + l_2^* x^{1/2})(y l_3 + x^{1/2} l_2) \end{pmatrix} \\
&= \begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix}.
\end{aligned}$$

Then under this change of variables, we have

$$\begin{aligned}
&\tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \tilde{F}_{\mathbf{m},\mathbf{n}}(x', y') \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \\
&= \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & -(x')^{-1/2}y' \\ 0 & I_{s-q} \end{pmatrix} F_{\mathbf{m},\mathbf{n}}(x') \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & 0 \\ -(y')^*(x')^{-1/2} & I_{s-q} \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \\
&= \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & -l_1^{-1} x^{-1/2}(y l_3 + x^{1/2} l_2) \\ 0 & I_{s-q} \end{pmatrix} F_{\mathbf{m},\mathbf{n}}(l_1^* x l_1) \\
&\qquad\qquad \times \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & 0 \\ -(l_3^* y^* + l_2^* x^{1/2}) x^{-1/2} l_1^{*-1} & I_{s-q} \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \\
&= \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & -x^{-1/2}y \\ 0 & I_{s-q} \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1 & 0 \\ 0 & l_3 \end{pmatrix} F_{\mathbf{m},\mathbf{n}}(l_1^* x l_1) \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} l_1^* & 0 \\ 0 & l_3^* \end{pmatrix} \tau_{\mathbf{k}}^{(s)} \begin{pmatrix} I_q & 0 \\ -y^* x^{-1/2} & I_{s-q} \end{pmatrix} \\
&= \Delta_{\mathbf{n}} \left(\begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \right) \tilde{F}_{\mathbf{m},\mathbf{n}}(x, y).
\end{aligned}$$

Thus we can compute $J(z)$ as

$$\begin{aligned}
J(z) &= \int_{E(I_s)} \mathrm{Tr}_{V_{\mathbf{k}}^{(s)}} \left(\tau_{\mathbf{k}}^{(s)} \left(\begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right) \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \right) \tilde{F}_{\mathbf{m},\mathbf{n}}(x', y') \right) \\
&\qquad \times \det \left(\begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right) \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \right)^{\lambda-(q+s)} \\
&\qquad \times \det(l_1)^{2q} \det(l_3)^{2q} dx dy \\
&= \int_{E(I_s)} \mathrm{Tr}_{V_{\mathbf{k}}^{(s)}} \left(\tau_{\mathbf{k}}^{(s)} \left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right) \tilde{F}_{\mathbf{m},\mathbf{n}}(x, y) \right) \\
&\qquad \times \det \left(I_s - \begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y \end{pmatrix} \right)^{\lambda-(q+s)} \Delta_{\lambda+\mathbf{n}-s} \left(\begin{pmatrix} l_1^* & 0 \\ l_2^* & l_3^* \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ 0 & l_3 \end{pmatrix} \right) dx dy \\
&= B_{\mathbf{m},\mathbf{n}}(\lambda) \Delta_{\lambda+\mathbf{n}-s}(z).
\end{aligned}$$

Next we compute $\int_{\tilde{\Omega}} J(z)e^{-\operatorname{tr}(z)}dz$ in two ways.

$$\int_{\tilde{\Omega}} J(z)e^{-\operatorname{tr}(z)}dz = B_{\mathbf{m},\mathbf{n}}(\lambda)\int_{\tilde{\Omega}} \Delta_{\lambda+\mathbf{n}-s}(z)e^{-\operatorname{tr}(z)} = B_{\mathbf{m},\mathbf{n}}(\lambda)\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{n}),$$

$$\begin{aligned}
&\int_{\tilde{\Omega}} J(z)e^{-\operatorname{tr}(z)}dz \\
&= \iint_{E(z)} \operatorname{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\tau_{\mathbf{k}}^{(s)}\left(z-\begin{pmatrix} x' & (x')^{1/2}y' \\ (y')^*(x')^{1/2} & (y')^*y'\end{pmatrix}\right)\tilde{F}_{\mathbf{m},\mathbf{n}}(x',y')\right) \\
&\qquad\qquad \times \det\left(z-\begin{pmatrix} x' & (x')^{1/2}y' \\ (y')^*(x')^{1/2} & (y')^*y'\end{pmatrix}\right)^{\lambda-(q+s)} e^{-\operatorname{tr}(z)}dx'dy'dz \\
&= \iint_{\substack{x'\in\Omega, y'\in(\mathfrak{p}_{\mathrm{T}}^{+})^{\perp}, \\ z'\in\tilde{\Omega}}} \operatorname{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\tau_{\mathbf{k}}^{(s)}(z')\tilde{F}_{\mathbf{m},\mathbf{n}}(x',y')\right)\det(z')^{\lambda-(q+s)}e^{-\operatorname{tr}\left(z'+\begin{pmatrix} x' & (x')^{1/2}y' \\ (y')^*(x')^{1/2} & (y')^*y'\end{pmatrix}\right)}dx'dy'dz' \\
&= \operatorname{Tr}_{V_{\mathbf{k}}^{(s)}}\left(\int_{\tilde{\Omega}}\tau_{\mathbf{k}}^{(s)}(z)\det(z)^{\lambda-(q+s)}e^{-\operatorname{tr}(z)}dz\int_{\Omega\times(\mathfrak{p}_{\mathrm{T}}^{+})^{\perp}}\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y)e^{-\operatorname{tr}\begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y\end{pmatrix}}dxdy\right).
\end{aligned}$$

Since $V_{\mathbf{k}}^{(s)}$ is $U(s)$-invariant and $\int_{\tilde{\Omega}}\tau_{\mathbf{k}}^{(s)}(z)\det(z)^{\lambda-(q+s)}e^{-\operatorname{tr}(z)}dz$ commutes with $U(s)$-action, this is proportional to the identity map. Also, similar to (2.3.8), we can show

$$\int_{\tilde{\Omega}}\tau_{\mathbf{k}}^{(s)}(z)\det(z)^{\lambda-(q+s)}e^{-\operatorname{tr}(z)}dz = \Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)I_{V_{\mathbf{k}}^{(s)}}$$

when $\operatorname{Re}\lambda + k_s > q+s-1$. Therefore we have

$$\begin{aligned}
\int_{\tilde{\Omega}} J(z)e^{-\operatorname{tr}(z)}dz &= \Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)\int_{\Omega\times(\mathfrak{p}_{\mathrm{T}}^{+})^{\perp}}\operatorname{Tr}_{V_{\mathbf{k}}^{(s)}}(\tilde{F}_{\mathbf{m},\mathbf{n}}(x,y))e^{-\operatorname{tr}\begin{pmatrix} x & x^{1/2}y \\ y^*x^{1/2} & y^*y\end{pmatrix}}dxdy \\
&= \Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)\Gamma_{\mathbf{m},\mathbf{n}},
\end{aligned}$$

and thus we get

$$\begin{aligned}
B_{\mathbf{m},\mathbf{n}}(\lambda) &= \frac{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)}{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{n})}\Gamma_{\mathbf{m},\mathbf{n}}, \\
R_{\mathbf{m},\mathbf{n}}(\lambda) &= c_\lambda\frac{B_{\mathbf{m},\mathbf{n}}(\lambda)}{\Gamma_{\mathbf{m},\mathbf{n}}} = c_\lambda\frac{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)}{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{n})}.
\end{aligned}$$

Since the norm is normalized so that $R_{\mathbf{0},\mathbf{k}}(\lambda)=1$, we have

$$c_\lambda = \frac{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k})}{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k}-q)} = \prod_{j=1}^{s}(\lambda-(j-1)+k_j-q)_q,$$

and consequently we get

$$R_{\mathbf{m},\mathbf{n}}(\lambda) = \frac{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{k})}{\Gamma_{\tilde{\Omega}}(\lambda+\mathbf{n})} = \frac{\prod_{j=1}^{s}(\lambda-(j-1))_{k_j}}{\prod_{j=1}^{s}(\lambda-(j-1))_{n_j}},$$

and we have completed the proof of Theorem 2.5.1. $\square$

2.5.3 $SO^*(4r+2)$, $V = S^k(\mathbb{C}^{2r+1})^\vee$

In this subsection we set $G = SO^*(4r+2)$, which is realized explicitly as (2.5.2) with $s = 2r+1$. Then we have

$$\begin{gathered} K \simeq U(2r+1), \quad \mathfrak{p}^\pm \simeq \mathrm{Skew}(2r+1, \mathbb{C}), \\ G_{\mathrm{T}} \simeq SO^*(4r), \quad L \simeq GL(r, \mathbb{H}), \quad K_L \simeq Sp(r), \\ r = r, \quad n = r(2r+1), \quad d = 4, \quad p = 4r. \end{gathered}$$

We set $V = V^{(2r+1)\vee}_{(k,0,\dots,0)} \simeq S^k(\mathbb{C}^{2r+1})^\vee$. The goal of this subsection is to prove the following theorem.

Theorem 2.5.2. *When $G = SO^*(4r+2)$ and $(\tau, V) = (\tau^{(2r+1)\vee}_{(k,0,\dots,0)}, V^{(2r+1)\vee}_{(k,0,\dots,0)})$ $(k \in \mathbb{Z}_{\ge 0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if* $\operatorname{Re}\lambda > 4r-1$, *the normalizing constant c_λ is given by*

$$c_\lambda = (\lambda - (2r+1))(\lambda + k - 2r)_{2r} \prod_{j=2}^{r} (\lambda - (2r+1) - 2(j-1))_{2r+1},$$

the K-type decomposition of $\mathcal{O}(D, V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+) \otimes V^{(2r+1)\vee}_{(k,0,\dots,0)} = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r_{++}} \bigoplus_{\substack{\mathbf{k} \in (\mathbb{Z}_{\ge 0})^{r+1}; |\mathbf{k}| = k \\ 0 \le k_j \le m_{j-1} - m_j}} V^{(2r+1)\vee}_{(m_1+k_1, m_1, m_2+k_2, m_2, \dots, m_r+k_r, m_r, k_{r+1})},$$

and for $f \in V^{(2r+1)\vee}_{(m_1+k_1, m_1, m_2+k_2, m_2, \dots, m_r+k_r, m_r, k_{r+1})}$, the ratio of norms is given by

$$\begin{aligned} \frac{\|f\|^2_{\lambda, \tau^{(2r+1)\vee}_{(k,0,\dots,0)}}}{\|f\|^2_{F, \tau^{(2r+1)\vee}_{(k,0,\dots,0)}}} &= \frac{(\lambda)_k}{\prod_{j=1}^r (\lambda - 2(j-1))_{m_j + k_j} (\lambda - 2r)_{k_{r+1}}} \\ &= \frac{1}{(\lambda + k)_{m_1 + k_1 - k} \prod_{j=2}^r (\lambda - 2(j-1))_{m_j + k_j} (\lambda - 2r)_{k_{r+1}}}. \end{aligned}$$

To begin with, we determine the normalizing constant c_λ. Since $V|_{K^{\mathbb{C}}_{\mathrm{T}}}$ is decomposed as

$$V^{(2r+1)\vee}_{(k,0,\dots,0)}\Big|_{K^{\mathbb{C}}_{\mathrm{T}}} = \bigoplus_{l=0}^{k} V^{(2r)\vee}_{(l,0,\dots,0)},$$

and $V^{(2r)\vee}_{(l,0,\dots,0)}$ has the restricted lowest weight $-\frac{l}{2}\gamma_1\big|_{\mathfrak{a}_{\mathfrak{l}}}$, and remains irreducible when restricted to $K_L = Sp(r)$, by Theorem 2.3.1 $\|\cdot\|^2_{\lambda, \tau^{(2r+1)\vee}_{(k,0,\dots,0)}}$ converges if $\operatorname{Re}\lambda > 4r-1$, and we have

$$\begin{aligned} c_\lambda^{-1} &= \frac{1}{\dim V^{(2r+1)\vee}_{(k,0,\dots,0)}} \sum_{l=0}^{k} \left(\dim V^{(2r)\vee}_{(l,0,\dots,0)}\right) \frac{\Gamma_\Omega(\lambda + (l,0,\dots,0) - (2r+1))}{\Gamma_\Omega(\lambda + (l,0,\dots,0))} \\ &= \frac{1}{\binom{2r+k}{k}} \sum_{l=0}^{k} \frac{\binom{2r+l-1}{l}}{(\lambda + l - (2r+1))_{2r+1}} \frac{1}{\prod_{j=2}^r (\lambda - (2r+1) - 2(j-1))_{2r+1}} \\ &= \frac{1}{(\lambda - (2r+1))(\lambda + k - 2r)_{2r} \prod_{j=2}^r (\lambda - (2r+1) - 2(j-1))_{2r+1}}. \end{aligned}$$

To compute the norm on each K-type, we consider $G' := SO^*(4r+4)$, which is realized explicitly as (2.5.2) with $s = 2r+2$, and embed $G \hookrightarrow G'$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c, d \in M(2r+1, \mathbb{C})).$$

We realize $(\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}, V^{(2r+1)\vee}_{(k,0,\ldots,0)})$ as

$$\begin{aligned} V^{(2r+1)\vee}_{(k,0,\ldots,0)} &= \mathcal{P}_k(\mathbb{C}^{2r+1}) = \{\text{Homogeneous holomorphic polynomials on } \mathbb{C}^{2r+1} \text{ of degree } k\}, \\ \tau^{(2r+1)\vee}_{(k,0,\ldots,0)}(l)p(v) &= p(l^{-1}v) \qquad (l \in GL(2r+1, \mathbb{C}),\ v \in \mathbb{C}^{2r+1},\ p \in \mathcal{P}_k(\mathbb{C}^{2r+1})), \end{aligned}$$

with the inner product

$$(p_1, p_2)_{\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}} := \frac{1}{\pi^{2r+1}} \int_{\mathbb{C}^{2r+1}} p_1(v)\overline{p_2(v)} e^{-|v|^2} dv \qquad (p_1, p_2 \in \mathcal{P}_k(\mathbb{C}^{2r+1})).$$

Then $\tilde{G} = \widetilde{SO^*}(4r+2)$ acts on $\mathcal{O}(D, \mathcal{P}_k(\mathbb{C}^{2r+1}))$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w, v) := \det(cw+d)^{-\lambda/2} f\left((aw+b)(cw+d)^{-1}, {}^t(cw+d)^{-1}v\right)$$
$$(w \in D \subset \mathrm{Skew}(2r+1, \mathbb{C}),\ v \in \mathbb{C}^{2r+1}).$$

On the other hand, the scalar type representation of $\tilde{G}' = \widetilde{SO^*}(4r+4)$ on $\mathcal{O}(D')$ (D' is realized as (2.5.4) with $s = 2r+2$) is given by

$$\tau'_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) f(w) := \det(cw+d)^{-\lambda/2} f\left((aw+b)(cw+d)^{-1}\right)$$
$$(w \in D' \subset \mathrm{Skew}(2r+2, \mathbb{C})).$$

If we restrict this representation to $\tilde{G}$, we have

$$\tau'_\lambda\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}\right) f\begin{pmatrix} w & v \\ -{}^tv & 0 \end{pmatrix} = \det(cw+d)^{-\lambda} f\begin{pmatrix} (aw+b)(cw+d)^{-1} & {}^t(cw+d)^{-1}v \\ -{}^tv(cw+d)^{-1} & 0 \end{pmatrix}$$
$$(w \in \mathrm{Skew}(2r+1, \mathbb{C}),\ v \in \mathbb{C}^{2r+1}).$$

Therefore if we define the embedding map $\iota : \mathcal{O}(D, \mathcal{P}_k(\mathbb{C}^{2r+1})) \to \mathcal{O}(D')$ by

$$(\iota(f))\begin{pmatrix} w & v \\ -{}^tv & 0 \end{pmatrix} := f(w, v) \qquad (w \in \mathrm{Skew}(2r+1, \mathbb{C}),\ v \in \mathbb{C}^{2r+1}),$$

then ι intertwines two actions τ_λ and $\tau'_\lambda|_{\tilde{G}}$. Also, since Fischer inner products on $\mathcal{P}(\mathfrak{p}^+, \mathcal{P}_k(\mathbb{C}^{2r+1}))$ and $\mathcal{P}(\mathfrak{p}^{+\prime})$ ($\mathfrak{p}^+ = \mathrm{Skew}(2r+1, \mathbb{C})$, $\mathfrak{p}^{+\prime} = \mathrm{Skew}(2r+2, \mathbb{C})$) are given by

$$\begin{aligned} \langle f, g\rangle_{F, \tau^{(2r+1)\vee}_{(k,0,\ldots,0)}} &= \frac{1}{\pi^{(r+1)(2r+1)}} \int_{\mathrm{Skew}(2r+1, \mathbb{C})} \int_{\mathbb{C}^{2r+1}} f(w, v)\overline{g(w, v)} e^{-\frac{1}{2}\mathrm{tr}(ww^*)} e^{-|v|^2} dv dw, \\ \langle f, g\rangle_{F, \mathbf{1}^{(2r+2)}} &= \frac{1}{\pi^{(r+1)(2r+1)}} \int_{\mathrm{Skew}(2r+2, \mathbb{C})} f(w)\overline{g(w)} e^{-\frac{1}{2}\mathrm{tr}(ww^*)} dw, \end{aligned}$$

ι is an isometry with respect to the Fischer inner product.

Next, we compute the K-type decomposition of $\mathcal{O}(D,\mathcal{P}_k(\mathbb{C}^{2r+1}))_K=\mathcal{P}(\mathfrak{p}^+)\otimes\mathcal{P}_k(\mathbb{C}^{2r+1})$ and $\mathcal{O}(D')_{K'}=\mathcal{P}(\mathfrak{p}^{+\prime})$.

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+)\otimes\mathcal{P}_k(\mathbb{C}^{2r+1})&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\ldots,m_r,m_r,0)}\otimes V^{(2r+1)\vee}_{(k,0,\ldots,0)}\\&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^{r+1},\ |\mathbf{k}|=k\\ 0\leq k_j\leq m_{j-1}-m_j}}V^{(2r+1)\vee}_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r,k_{r+1})},\\\mathcal{P}(\mathfrak{p}^{+\prime})&=\bigoplus_{\mathbf{n}\in\mathbb{Z}^{r+1}_{++}}V^{(2r+2)\vee}_{(n_1,n_1,n_2,n_2,\ldots,n_{r+1},n_{r+1})}.\end{aligned}$$

Each $K'^{\mathbb{C}}=GL(2r+2,\mathbb{C})$-module $V^{(2r+2)\vee}_{(n_1,n_1,n_2,n_2,\ldots,n_{r+1},n_{r+1})}$ is decomposed under $K^{\mathbb{C}}=GL(2r+1,\mathbb{C})$ as

$$V^{(2r+2)\vee}_{(n_1,n_1,n_2,n_2,\ldots,n_{r+1},n_{r+1})}\Big|_{K^{\mathbb{C}}}=\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ n_j\geq m_j\geq n_{j+1}}}V^{(2r+1)\vee}_{(n_1,m_1,n_2,m_2,\ldots,n_r,m_r,n_{r+1})},$$

which follows from the following lemma about the branching law of $GL(s,\mathbb{C})\downarrow GL(s-1,\mathbb{C})$.

Lemma 2.5.3 ([30, §66, Theorem 2])**.** *For* $\mathbf{m}\in\mathbb{Z}^s_+$,

$$V^{(s)\vee}_{\mathbf{m}}\Big|_{GL(s-1,\mathbb{C})}=\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^{s-1}_+\\ m_j\geq n_j\geq m_{j+1}}}V^{(s-1)\vee}_{\mathbf{n}}.$$

Therefore it follows that

$$\iota\left(V^{(2r+1)\vee}_{(m_1+k_1,m_1,\ldots,m_r+k_r,m_r,k_{r+1})}\right)\subset V^{(2r+2)\vee}_{(m_1+k_1,m_1+k_1,\ldots,m_r+k_r,m_r+k_r,k_{r+1},k_{r+1})}. \tag{2.5.9}$$

Therefore, for any $f\in V^{(2r+1)\vee}_{(m_1+k_1,m_1,m_2+k_2,m_2,\ldots,m_r+k_r,m_r,k_{r+1})}$, the ratio of norm is given by

$$\frac{\|\iota(f)\|^2_{\lambda,\mathbf{1}^{(2r+2)}}}{\|\iota(f)\|^2_{F,\mathbf{1}^{(2r+2)}}}=\frac{1}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}}.$$

Since ι intertwines $\tilde{G}$-action, $\|\cdot\|_{\lambda,\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}}$ is proportional to $\|\iota(\cdot)\|_{\lambda,\mathbf{1}^{(2r+2)}}$. Also, since ι preserves the Fischer norm, and $\|\cdot\|_{\lambda,\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}}$ is normalized such that it coincides with the Fischer norm on the minimal K-type, we have

$$\frac{\|f\|^2_{\lambda,\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}}}{\|f\|^2_{F,\tau^{(2r+1)\vee}_{(k,0,\ldots,0)}}}=\frac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}},$$

and we have proved Theorem 2.5.2. □

Remark 2.5.4. *We can also prove the former part of Theorem 2.4.5 ($G=SO^*(4r)$), or Theorem 2.4.3, 2.5.1 ($G=SU(q,s)$) by this method, by embedding*

$$SO^*(4r)\hookrightarrow SO^*(4r+2),\qquad \mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}),\mathcal{P}_k(\mathbb{C}^{2r}))\hookrightarrow\mathcal{P}(\mathrm{Skew}(2r+1,\mathbb{C})),$$
$$U(p)\times U(q,s)\hookrightarrow U(p+q,s),\qquad V^{(p)\vee}_{\mathbf{k}}\boxtimes\mathcal{P}(M(q,s,\mathbb{C}),V^{(s)}_{\mathbf{k}})\hookrightarrow\mathcal{P}(M(p+q,s,\mathbb{C})),$$

but we cannot determine the normalizing constant c_λ in this way.

2.5.4 $SO^*(4r+2)$, $V=S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2}$

In this subsection we continue to set $G=SO^*(4r+2)$, which is realized explicitly as (2.5.2). We set $V=V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}\simeq S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2}$. The goal of this subsection is to prove the following theorem.

Theorem 2.5.5. *When $G=SO^*(4r+2)$ and $(\tau,V)=(\tau^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)},V^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)})$ $(k\in\mathbb{Z}_{\ge0})$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-1$, the normalizing constant c_λ is given by*

$$c_\lambda=\prod_{j=1}^{r-1}(\lambda+k-(2r+1)-2(j-1))_{2r+1}(\lambda-4r+1)_{2r}(\lambda+k-2r+1),$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+)\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\ge0})^{r+1};|\mathbf{k}|=k\\0\le k_j\le m_j-m_{j+1}\\0\le k_r\le m_r}}V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})},$$

and for $f\in V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)}}}&=\frac{\prod_{j=1}^r(\lambda-2(j-1))_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j-k_j+k}(\lambda-2r+1)_{k-k_{r+1}}}\\&=\frac{1}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}}.\end{aligned}$$

To begin with, we determine the normalizing constant c_λ. Since $V|_{K_T^{\mathbb{C}}}$ is decomposed as

$$V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}\Big|_{K_T^{\mathbb{C}}}=\bigoplus_{l=0}^k V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)},$$

and $V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)}$ has the restricted lowest weight $-\left(\frac{k}{2}(\gamma_1+\cdots+\gamma_{r-1})+\frac{k-l}{2}\gamma_r\right)\big|_{\mathfrak{a}_\mathfrak{l}}$ and remains irreducible when restricted to $K_L=Sp(r)$, by Theorem 2.3.1 $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>4r-1$, and we have

$$\begin{aligned}c_\lambda^{-1}&=\frac{1}{\dim V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}}\sum_{l=0}^k\left(\dim V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)}\right)\frac{\Gamma_\Omega(\lambda+(k,\dots,k,k-l)-(2r+1))}{\Gamma_\Omega(\lambda+(k,\dots,k,k-l))}\\&=\frac{1}{\binom{2r+k}{k}}\frac{1}{\prod_{j=1}^{r-1}(\lambda+k-(2r+1)-2(j-1))_{2r+1}}\sum_{l=0}^k\frac{\binom{2r+l-1}{l}}{(\lambda+k-l-(4r-1))_{2r+1}}\\&=\frac{1}{\prod_{j=1}^{r-1}(\lambda+k-(2r+1)-2(j-1))_{2r+1}(\lambda-4r+1)_{2r}(\lambda+k-2r+1)}\\&=\frac{(\lambda-2r+1)_k}{\prod_{j=1}^{r-1}(\lambda+k-(2r+1)-2(j-1))_{2r+1}(\lambda-4r+1)_{2r+1+k}}\\&=\frac{\Gamma_\Omega(\lambda+(k,\dots,k,0)-(2r+1))(\lambda-2r+1)_k}{\Gamma_\Omega(\lambda+(k,\dots,k,k))}.\end{aligned}$$

Next we compute the K-type decomposition of $\mathcal{O}(D,V)_K=\mathcal{P}(\mathfrak{p}^+)\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}$.

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+)\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)}\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}\\&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^{r+1},\ |\mathbf{k}|=k\\0\leq k_j\leq m_j-m_{j+1}\\0\leq k_r\leq m_r}}V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})}.\end{aligned}$$

To apply Theorem 2.3.1 for each K-type, we determine the image of each K-type under $\mathrm{rest}:\mathcal{P}(\mathfrak{p}^+,V)\to\mathcal{P}(\mathfrak{p}^+_{\mathrm{T}},V)$. Since we have

$$\begin{aligned}&\mathrm{rest}\left(V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)}\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}\right)\\&=V^{(2r)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r)}\otimes V^{(2r+1)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2})}\Big|_{K^{\mathbb{C}}_{\mathrm{T}}}=V^{(2r)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r)}\otimes\bigoplus_{l=0}^{k}V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)}\\&=\bigoplus_{l=0}^{k}\bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=l\\0\leq l_j\leq m_j-m_{j+1}}}V^{(2r)\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\dots,m_r,m_r-l_r)+(\frac{k}{2},\dots,\frac{k}{2})},\end{aligned}$$

and the abstract decomposition of $K^{\mathbb{C}}$-modules under $K^{\mathbb{C}}_{\mathrm{T}}$ is given by Lemma 2.5.3, we have

$$\begin{aligned}&\mathrm{rest}\left(V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})}\right)\\&\subset\bigoplus_{l=k-k_{r+1}}^{k}\bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=l\\k_j\leq l_j\leq m_j-m_{j+1}}}V^{(2r)\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\dots,m_r,m_r-l_r)+(\frac{k}{2},\dots,\frac{k}{2})}.\end{aligned}$$

Then, the only $K_L=Sp(r)$-spherical submodule in

$$\begin{aligned}&V^{(2r)\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\dots,m_r,m_r-l_r)+(\frac{k}{2},\dots,\frac{k}{2})}\otimes\overline{V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)}}\\&\simeq V^{(2r)\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\dots,m_r,m_r-l_r)+(\frac{k}{2},\dots,\frac{k}{2})}\otimes V^{(2r)\vee}_{(\frac{k}{2},\dots,\frac{k}{2},\frac{k}{2}-l)}\end{aligned}$$

is $V^{(2r)\vee}_{(m_1-l_1,m_1-l_1,m_2-l_2,m_2-l_2,\dots,m_r-l_r,m_r-l_r)+(k,\dots,k)}$, which has the lowest weight $-((m_1-l_1+k)\gamma_1+\cdots+(m_r-l_r+k)\gamma_r)$. Therefore by Theorem 2.3.1, there exist non-negative numbers $a_{\mathbf{m},\mathbf{k},\mathbf{l}}$ such that for $f\in V_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}&=\frac{c_\lambda}{\sum_{\mathbf{l}}a_{\mathbf{m},\mathbf{k},\mathbf{l}}}\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=l\\k_j\leq l_j\leq m_{j+1}-m_j}}a_{\mathbf{m},\mathbf{k},\mathbf{l}}\frac{\Gamma_\Omega\left(\lambda+(k,\dots,k,k-l)-(2r+1)\right)}{\Gamma_\Omega(\lambda+\mathbf{m}-\mathbf{l}+(k,\dots,k))}\\&=\frac{1}{\sum_{\mathbf{l}}a_{\mathbf{m},\mathbf{k},\mathbf{l}}}\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=l\\k_j\leq l_j\leq m_{j+1}-m_j}}\frac{a_{\mathbf{m},\mathbf{k},\mathbf{l}}(\lambda-4r+1)_{k-l}}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-l_j}(\lambda-2r+1)_k}.\end{aligned}$$

It is difficult to know the exact values of $a_{\mathbf{m},\mathbf{k},\mathbf{l}}$, but at least we have proved

Lemma 2.5.6. *For $f \in V^{(2r+1)\vee}_{(m_1,m_1-k_1,\ldots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\ldots,\frac{k}{2})}$, the ratio of norms is*

$$\frac{\|f\|^2_{\lambda,\tau^{(2r+1)\vee}_{(k/2,\ldots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^{(2r+1)\vee}_{(k/2,\ldots,k/2,-k/2)}}}=\frac{(\textit{monic polynomial of degree } k_{r+1})}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k}.$$

Next we consider $G_{\mathrm{A}} := SU(2r,1)$, which is realized as (2.5.1), and embed $G_{\mathrm{A}} \hookrightarrow G$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & \bar{d} & -\bar{c} & 0 \\ 0 & -\bar{b} & \bar{a} & 0 \\ c & 0 & 0 & d \end{pmatrix} \qquad \begin{pmatrix} a \in M(2r,\mathbb{C}),\ b \in M(2r,1;\mathbb{C}), \\ c \in M(1,2r;\mathbb{C}),\ d \in \mathbb{C} \end{pmatrix}.$$

Then the positive root system $\Delta_+(\mathfrak{g}_{\mathrm{A}}^{\mathbb{C}},(\mathfrak{h}\cap\mathfrak{g}_{\mathrm{A}})^{\mathbb{C}})$ of $\mathfrak{g}_{\mathrm{A}}$, induced from $\Delta_+(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$, has the simple system

$$\{\varepsilon_j-\varepsilon_{j+1} : j=1,2,\ldots,2r-1\}\cup\{\varepsilon_{2r}+\varepsilon_{2r+1}\}.$$

Each representation of $K_{\mathrm{A}}^{\mathbb{C}} = S(GL(2r,\mathbb{C})\times GL(1,\mathbb{C}))$ is of the form $(\tau^{(2r)\vee}_{\mathbf{m}}\boxtimes\tau^{(1)\vee}_{m_0}, V^{(2r)\vee}_{\mathbf{m}}\otimes V^{(1)\vee}_{m_0})$, and we sometimes abbreviate this to $(\tau^{(2r,1)\vee}_{(\mathbf{m};m_0)}, V^{(2r,1)\vee}_{(\mathbf{m};m_0)})$. Clearly $V^{(2r,1)\vee}_{(\mathbf{m}+(c,\ldots,c);m_0-c)} \simeq V^{(2r,1)\vee}_{(\mathbf{m};m_0)}$ holds as $K_{\mathrm{A}}^{\mathbb{C}}$-modules for any c. The representation τ_λ of $\tilde{G}$ on $\mathcal{O}(D,V)$ is given by (2.5.6), and if we restrict this representation to $\tilde{G}_{\mathrm{A}}$, we have

$$\begin{aligned}
&\tau_\lambda\left(\begin{pmatrix} a & 0 & 0 & b \\ 0 & \bar{d} & -\bar{c} & 0 \\ 0 & -\bar{b} & \bar{a} & 0 \\ c & 0 & 0 & d \end{pmatrix}^{-1}\right) f\begin{pmatrix} w & v \\ -{}^tv & 0 \end{pmatrix} \\
&= \det(a^*+vb^*)^{-\lambda/2}\det(cv+d)^{-\lambda/2}\tau^{(2r+1)\vee}_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2})}\begin{pmatrix} a^*+vb^* & -w{}^tc \\ 0 & {}^t(cv+d) \end{pmatrix} \\
&\qquad\qquad \times f\begin{pmatrix} (a^*+vb^*)^{-1}w{}^t(a^*+vb^*)^{-1} & (av+b)(cv+d)^{-1} \\ -{}^t((av+b)(cv+d)^{-1}) & 0 \end{pmatrix} \\
&= \det(cv+d)^{-\lambda}\tau^{(2r+1)\vee}_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2})}\begin{pmatrix} a^*+vb^* & -w{}^tc \\ 0 & {}^t(cv+d) \end{pmatrix} \\
&\qquad\qquad \times f\begin{pmatrix} (a^*+vb^*)^{-1}w{}^t(a^*+vb^*)^{-1} & (av+b)(cv+d)^{-1} \\ -{}^t((av+b)(cv+d)^{-1}) & 0 \end{pmatrix} \\
&\qquad\qquad\qquad\qquad (w \in \mathrm{Skew}(2r,\mathbb{C}),\ v \in \mathbb{C}^{2r}).
\end{aligned}$$

For $N \in \mathbb{N}$, let $\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))$ be the space of polynomials on $\mathrm{Skew}(2r,\mathbb{C})$ whose degree is smaller than or equal to N, and let $D_{\mathrm{A}} \subset \mathbb{C}^{2r}$ be the unit disk. Also, let $\mathrm{incl} : V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} = V^{(2r,1)\vee}_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2};\frac{k}{2})} \hookrightarrow V^{(2r+1)\vee}_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2})}$ be the K_{A}-equivariant inclusion. Then by the above computation, the map

$$\iota : \mathcal{O}(D_{\mathrm{A}}, (\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)}) \to \mathcal{O}(D, V^{(2r+1)\vee}_{(\frac{k}{2},\ldots,\frac{k}{2},-\frac{k}{2})}),$$

$$\iota(f)\begin{pmatrix} w & v \\ -{}^tv & 0 \end{pmatrix} := \mathrm{incl}(f(v,w))$$

intertwines the G_{A} action, and we can also prove that ι preserves the Fischer norm. Thus we study the space

$$\begin{aligned}
&\mathcal{O}(D_{\mathrm{A}}, (\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)})_{K_{\mathrm{A}}} \\
&= \mathcal{P}(\mathbb{C}^{2r})\otimes(\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} \\
&\simeq \bigoplus_{m_0=0}^{\infty} V^{(2r,1)\vee}_{(m_0,0,\ldots,0;m_0)} \otimes \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ |\mathbf{m}|\le N}} V^{(2r,1)\vee}_{(m_1,m_1,m_2,m_2,\ldots,m_r,m_r;0)} \otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)}.
\end{aligned}$$

This space is not irreducible under G_{A}. For $\mathbf{m}\in\mathbb{Z}^r_{++}$ and $\mathbf{l}\in\mathbb{Z}^r_{\ge 0}$ we define

$$\begin{aligned}
F_{\mathbf{m},\mathbf{l}} &:= V^{(2r,1)\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\ldots,m_r,m_r-l_r;0)+(k,\ldots,k;0)} \\
&\subset V^{(2r,1)\vee}_{(m_1,m_1,m_2,m_2,\ldots,m_r,m_r;0)} \otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} \\
&\subset (\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)},
\end{aligned}$$

so that

$$(\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ |\mathbf{m}|\le N}} \bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\ge 0},\ |\mathbf{l}|=k\\ 0\le l_j\le m_j-m_{j+1}}} F_{\mathbf{m},\mathbf{l}},$$

$$\mathcal{O}(D_{\mathrm{A}}, (\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)}) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ |\mathbf{m}|\le N}} \bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\ge 0},\ |\mathbf{l}|=k\\ 0\le l_j\le m_j-m_{j+1}}} \mathcal{O}(D_{\mathrm{A}}, F_{\mathbf{m},\mathbf{l}}).$$

Also, for $\mathbf{m}\in\mathbb{Z}^r_{++}$ and $\mathbf{k}\in\mathbb{Z}^{r+1}_{\ge 0}$ we set

$$\begin{aligned}
W_{\mathbf{m},\mathbf{k}} &:= V^{(2r,1)\vee}_{(m_1-k_1,m_2,m_2-k_2,m_3,\ldots,m_{r-1}-k_{r-1},m_r,m_r-k_r,-k_{r+1};m_1)+(k,\ldots,k;0)} \\
&\subset V^{(2r,1)\vee}_{(m_1,m_2,m_2,m_3,\ldots,m_{r-1},m_r,m_r,0;m_1)} \otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} \\
&\subset V^{(2r,1)\vee}_{(m_1,0,\ldots,0;m_1)} \otimes V^{(2r,1)\vee}_{(m_2,m_2,m_3,m_3,\ldots,m_r,m_r,0,0;0)} \otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)} \\
&\subset \mathcal{P}(\mathbb{C}^{2r})\otimes(\mathcal{P}_{\le N}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{(2r,1)\vee}_{(k,\ldots,k,0;0)}.
\end{aligned}$$

Then we have the following.

Lemma 2.5.7. *(1)* $\iota(W_{\mathbf{m},\mathbf{k}}) \subset V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)}$.

(2) $W_{\mathbf{m},\mathbf{k}} \subset \displaystyle\bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{l}|=k\\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}} \mathcal{O}(D_{\mathrm{A}}, F_{(m_2,\ldots,m_r,0),\mathbf{l}})$.

(3) $\iota(F_{\mathbf{m},\mathbf{l}}) \subset \displaystyle\bigoplus_{\substack{\mathbf{n}\in(\mathbb{Z}_{\ge 0})^{r+1},\ |\mathbf{n}|=k\\ n_j\le l_j,\ n_{r+1}\ge l_r-m_r}} V^{(2r+1)\vee}_{(m_1,m_1-n_1,m_2,m_2-n_2,\ldots,m_r,m_r-n_r,-n_{r+1})+\left(\frac{k}{2},\ldots,\frac{k}{2}\right)}$.

Proof. (1) The polynomial space $\mathcal{P}(\mathbb{C}^{2r})\otimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})$ is decomposed as

$$\begin{aligned}
\mathcal{P}(\mathbb{C}^{2r})\otimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C}) &= \bigoplus_{m_0=0}^{\infty} V^{(2r,1)\vee}_{(m_0,0,\ldots,0;m_0)} \otimes \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} V^{(2r,1)\vee}_{(m_1,m_1,m_2,m_2,\ldots,m_r,m_r;0)} \\
&= \bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}} \bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{l}|=m_0\\ 0\le l_j\le m_{j-1}-m_j}} V^{(2r,1)\vee}_{(m_1+l_1,m_1,m_2+l_2,m_2,\ldots,m_r+l_r,m_r;m_0)},
\end{aligned}$$

and similarly to (2.5.9), we have

$$V^{(2r,1)\vee}_{(m_1+l_1,m_1,m_2+l_2,m_2,\dots,m_r+l_r,m_r;m_0)} \subset V^{(2r+1)\vee}_{(m_1+l_1,m_1+l_1,m_2+l_2,m_2+l_2,\dots,m_r+l_r,m_r+l_r)}.$$

Therefore we have

$$\begin{aligned}
&\iota\left(V^{(2r,1)\vee}_{(m_1+l_1,m_1,m_2+l_2,m_2,\dots,m_r+l_r,m_r;m_0)} \otimes V^{(2r,1)\vee}_{(k,\dots,k,0;0)}\right)\\
&\subset V^{(2r+1)\vee}_{(m_1+l_1,m_1+l_1,m_2+l_2,m_2+l_2,\dots,m_r+l_r,m_r+l_r,0)} \otimes \mathrm{incl}\left(V^{(2r,1)\vee}_{(k,\dots,k,0;0)}\right)\\
&\subset V^{(2r+1)\vee}_{(m_1+l_1,m_1+l_1,m_2+l_2,m_2+l_2,\dots,m_r+l_r,m_r+l_r,0)} \otimes V^{(2r+1)\vee}_{\left(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2}\right)}.
\end{aligned} \tag{2.5.10}$$

Especially, by putting $\mathbf{l}=\mathbf{0}$ we have

$$\begin{aligned}
W_{\mathbf{m},\mathbf{k}} &\subset V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)} \otimes \mathrm{incl}\left(V^{(2r,1)\vee}_{(k,\dots,k,0;0)}\right)\\
&\subset V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)} \otimes V^{(2r+1)\vee}_{\left(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2}\right)}
\end{aligned}$$

Let $v \in W_{\mathbf{m},\mathbf{k}}$ be the highest weight vector. Then

$$\begin{aligned}
\iota(v) = \sum_i v_{1,i} \otimes v_{2,i} &\in V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)} \otimes \mathrm{incl}\left(V^{(2r,1)\vee}_{(k,\dots,k,0;0)}\right)\\
&\subset V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r,0)} \otimes V^{(2r+1)\vee}_{\left(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2}\right)}
\end{aligned}$$

has the weight $-(-k_{r+1}, m_r-k_r, m_r, \dots, m_2-k_2, m_2, m_1-k_1, m_1) - \left(\frac{k}{2},\dots,\frac{k}{2}\right)$, vanishes under root vectors $x \in \mathfrak{k}^{\mathbb{C}}_{\varepsilon_j-\varepsilon_{j+1}}$ $(j=1,\dots,2r-1)$ since v is the highest under $K^{\mathbb{C}}_{\mathrm{A}}$, and also vanishes under root vectors $x \in \mathfrak{k}^{\mathbb{C}}_{\varepsilon_{2r}-\varepsilon_{2r+1}}$ since each $v_{1,i}, v_{2,i}$ has the weight $(*,\dots,*,-m_1)$ and $(*,\dots,*,0)-\left(\frac{k}{2},\dots,\frac{k}{2}\right)$ respectively, where $*$ are some integers. Thus $\iota(v)$ becomes a highest weight vector of $V^{(2r+1)\vee}_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}$.

(2) We have

$$\begin{aligned}
W_{\mathbf{m},\mathbf{l}} &\subset V^{\vee}_{(m_1,\dots,0;m_1)} \otimes V^{(2r,1)\vee}_{(m_2,m_2,m_3,m_3,\dots,m_r,m_r,0,0;0)} \otimes V^{(2r,1)\vee}_{(k,\dots,k,0;0)}\\
&= \bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\geq 0},\ |\mathbf{l}|=k\\ 0\leq l_j\leq m_{j+1}-m_{j+2}}} V^{\vee}_{(m_1,\dots,0;m_1)} \otimes V^{(2r,1)\vee}_{(m_2,m_2-l_1,m_3,m_3-l_2,\dots,m_r,m_r-l_{r-1},0,-l_r;0)+(k,\dots,k;0)}\\
&= \bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\geq 0},\ |\mathbf{l}|=k\\ 0\leq l_j\leq m_{j+1}-m_{j+2}}} V^{\vee}_{(m_1,\dots,0;m_1)} \otimes F_{(m_2,\dots,m_r,0),\mathbf{l}},
\end{aligned}$$

and abstractly

$$\begin{aligned}
W_{\mathbf{m},\mathbf{l}} &\simeq V^{(2r,1)\vee}_{(m_1-k_1,m_2,m_2-k_2,m_3,\dots,m_{r-1}-k_{r-1},m_r,m_r-k_r,-k_{r+1};m_1)+(k,\dots,k;0)}\\
&\subset V^{(2r,1)\vee}_{(m_1,0,\dots,0;m_1)} \otimes V^{(2r,1)\vee}_{(m_2,m_2-l_1,m_3,m_3-l_2,\dots,m_r,m_r-l_{r-1},0,-l_r;0)+(k,\dots,k;0)}
\end{aligned}$$

holds only if $l_j \leq k_{j+1}$, $l_r \geq k_{r+1}$ holds.

(3) By (2.5.10) with $\mathbf{l}=\mathbf{0}$ we have

$$\begin{aligned}
\iota(F_{\mathbf{m},\mathbf{l}}) &\subset V^{(2r+1)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r)} \otimes V^{(2r+1)\vee}_{\left(\frac{k}{2},\dots,\frac{k}{2},-\frac{k}{2}\right)}\\
&= \bigoplus_{\substack{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^{r+1},\ |\mathbf{n}|=k\\ n_j\leq m_j-m_{j+1}}} V^{(2r+1)\vee}_{(m_1,m_1-n_1,m_2,m_2-n_2,\dots,m_r,m_r-n_r,-n_{r+1})+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}.
\end{aligned}$$

Combining with the abstract branching rule under $K^{\mathbb{C}} \supset K_{\mathrm{A}}^{\mathbb{C}}$ (Lemma 2.5.3), we get the desired formula. $\square$

Now we want to show that, on $V^{(2r+1)\vee}_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}$ the ratio is given by

$$\frac{\|f\|^2_{\lambda,\tau^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)}}}{\|f\|^2_{F,\tau^{(2r+1)\vee}_{(k/2,\dots,k/2,-k/2)}}} = \frac{1}{\prod_{j=1}^{r}\left(\lambda+k-2(j-1)\right)_{m_j-k_j}\left(\lambda-2r+1\right)_{k-k_{r+1}}} \tag{2.5.11}$$

by induction on $\min\{j : m_j = 0\}$.

First, when $\mathbf{m} = \mathbf{0}$ i.e. on $V^{\vee}_{(0,\dots,0,-k)+\frac{k}{2}}$, (2.5.11) clearly holds by the normalization assumption. Second, we assume (2.5.11) holds when $m_j = 0$, and prove this also holds on $V^{(2r+1)\vee}_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,-k_{r+1})+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}$ when $m_{j+1} = 0$.

By Lemma 2.5.7 (1), it suffices to compute $\|\iota(f)\|^2_{\lambda,\tau}/\|\iota(f)\|^2_{F,\tau}$ for $f \in W_{\mathbf{m},\mathbf{k}}$. For any $\mathbf{l}$, let $f_{\mathbf{l}}$ be the orthogonal of f onto $\mathcal{O}(D_{\mathrm{A}}, F_{\mathbf{m}',\mathbf{l}})$, where $\mathbf{m}' := (m_2,\dots,m_r,0)$. Then by Lemma 2.5.7 (2), we have

$$f = \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=k \\ l_j\leq k_{j+1},\ l_r\geq k_{r+1}}} f_{\mathbf{l}},$$

and there exist $b_{\mathbf{l}} \geq 0$ such that $\|\iota(f_{\mathbf{l}})\|^2_F = b_{\mathbf{l}}\|\iota(f)\|^2_F$ holds. Next, by Theorem 2.5.1, we have

$$\begin{aligned}
&\frac{\|\iota(f_{\mathbf{l}})\|_{\lambda,\tau}}{\|\iota(f_{\mathbf{l}})\|_{F,\tau}} \times \frac{\|\iota(v_{\mathbf{l}})\|_{F,\tau}}{\|\iota(v_{\mathbf{l}})\|_{\lambda,\tau}} \\
&= \frac{\prod_{j=1}^{r-1}((\lambda-(2j-2))_{m_{j+1}+k}(\lambda-(2j-1))_{m_{j+1}-l_j+k})(\lambda-(2r-1))_{-l_r+k}}{\begin{array}{r}\prod_{j=1}^{r-1}((\lambda-(2j-2))_{m_j-k_j+k}(\lambda-(2j-1))_{m_{j+1}+k}) \\ \times(\lambda-(2r-2))_{m_r-k_r+k}(\lambda-(2r-1))_{-k_{r+1}+k}\end{array}} \\
&= \frac{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}\prod_{j=2}^{r}(\lambda+k-(2j-3))_{m_j-l_{j-1}}(\lambda-2r+1)_{k-l_r}}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^{r}(\lambda+k-(2j-3))_{m_j}(\lambda-2r+1)_{k-k_{r+1}}},
\end{aligned}$$

where $v_{\mathbf{l}}$ is any non-zero element in the minimal K_{A}-type $F_{\mathbf{m}',\mathbf{l}}$. Next, let $v_{\mathbf{l},\mathbf{n}}$ be the orthogonal projection of $\iota(v_{\mathbf{l}})$ onto $V^{(2r+1)\vee}_{(m_2,m_2-n_1,m_3,m_3-n_2,\dots,m_r,m_r-n_{r-1},0,0,-n_r)+\left(\frac{k}{2},\dots,\frac{k}{2}\right)}$, so that

$$\iota(v_{\mathbf{l}}) = \sum_{\substack{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{n}|=k \\ n_j\leq l_j,\ n_r\geq l_r}} v_{\mathbf{l},\mathbf{n}}$$

by Lemma 2.5.7 (3). Then there exist $c_{\mathbf{l},\mathbf{n}} \geq 0$ such that $\|v_{\mathbf{l},\mathbf{n}}\|^2_{F,\tau} = c_{\mathbf{l},\mathbf{n}}\|\iota(v_{\mathbf{l}})\|^2_{F,\tau}$ holds. Next, by the induction hypothesis (2.5.11), for each $\mathbf{n}$ we have

$$\frac{\|v_{\mathbf{l},\mathbf{n}}\|^2_{\lambda,\tau}}{\|v_{\mathbf{l},\mathbf{n}}\|^2_{F,\tau}} = \frac{1}{\prod_{j=1}^{r-1}\left(\lambda+k-2(j-1)\right)_{m_{j+1}-n_j}\left(\lambda-2r+1\right)_{k-n_r}}.$$

Thus for each $\mathbf{l}$ we get

$$\begin{aligned}\frac{\|\iota(v_{\mathbf{l}})\|_{\lambda,\tau}^2}{\|\iota(v_{\mathbf{l}})\|_{F,\tau}^2} &= \sum_{\substack{\mathbf{n}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{n}|=k\\ n_j\le l_j,\ n_r\ge l_r}} c_{\mathbf{l},\mathbf{n}} \frac{\|v_{\mathbf{l},\mathbf{n}}\|_{\lambda,\tau}^2}{\|v_{\mathbf{l},\mathbf{n}}\|_{F,\tau}^2}\\ &= \sum_{\substack{\mathbf{n}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{n}|=k\\ n_j\le l_j,\ n_r\ge l_r}} \frac{c_{\mathbf{l},\mathbf{n}}}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}-n_j}(\lambda-2r+1)_{k-n_r}}\\ &= \frac{(\text{monic polynomial of degree } k-l_r)}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}(\lambda-2r+1)_{k-l_r}},\end{aligned}$$

and therefore we get

$$\begin{aligned}&\frac{\|\iota(f)\|_{\lambda,\tau}^2}{\|\iota(f)\|_{F,\tau}^2} = \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{l}|=k\\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}} b_{\mathbf{l}} \frac{\|f_{\mathbf{l}}\|_{\lambda,\tau}^2}{\|f_{\mathbf{l}}\|_{F,\tau}^2}\\ &= \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\ge 0})^r,\ |\mathbf{l}|=k\\ l_j\le k_{j+1},\ l_r\ge k_{r+1}}} b_{\mathbf{l}} \Biggl(\frac{(\text{monic polynomial of degree } k-l_r)}{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}(\lambda-2r+1)_{k-l_r}}\\ &\qquad\times \frac{\prod_{j=1}^{r-1}(\lambda+k-2(j-1))_{m_{j+1}}\prod_{j=2}^{r}(\lambda+k-2(j-1)+1)_{m_j-l_{j-1}}(\lambda-2r+1)_{k-l_r}}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^{r}(\lambda+k-(2j-3))_{m_j}(\lambda-2r+1)_{k-k_{r+1}}}\Biggr)\\ &= \frac{(\text{monic polynomial of degree } k_2+\cdots+k_r)}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^{r}(\lambda+k+m_j-k_j-(2j-3))_{k_j}(\lambda-2r+1)_{k-k_{r+1}}}.\end{aligned}$$

On the other hand, by Lemma 2.5.6 we have

$$\frac{\|\iota(f)\|_{\lambda,\tau}^2}{\|\iota(f)\|_{F,\tau}^2} = \frac{(\text{monic polynomial of degree } k_{r+1})}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k},$$

so combining these two formulas, we get

$$\frac{\|\iota(f)\|_{\lambda,\tau}^2}{\|\iota(f)\|_{F,\tau}^2} = \frac{1}{\prod_{j=1}^{r}(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}},$$

and the induction continues. Thus we have proved (2.5.11) for any $\mathbf{m}$, and proved Theorem 2.5.5. $\square$

2.5.5 Conjecture on $E_{6(-14)}$

In this subsection we set $G = E_{6(-14)}$. Then we have

$$\begin{gathered}\mathfrak{k}\simeq\mathfrak{so}(2)\oplus\mathfrak{so}(10),\quad \mathfrak{p}^{\pm}\simeq M(2,1;\mathbb{O}_{\mathbb{C}}),\quad \mathfrak{g}_{\mathrm{T}}\simeq\mathfrak{so}(2,8),\quad \mathfrak{l}\simeq\mathbb{R}\oplus\mathfrak{so}(1,7),\quad \mathfrak{k}_{\mathfrak{l}}\simeq\mathfrak{so}(7),\\ r=2,\quad n=16,\quad d=6,\quad p=12.\end{gathered}$$

We take a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{k}$. Then we can take a basis $\{t_0,t_1,\ldots,t_5\}\subset\sqrt{-1}\mathfrak{h}$ and $\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_5\}\subset(\sqrt{-1}\mathfrak{h})^\vee$, such that

$$\varepsilon_0(t_j)=\frac{4}{3}\delta_{0,j},\quad \varepsilon_i(t_j)=\delta_{i,j}\quad (i=1,\ldots,5,\ j=0,1,\ldots,5),$$

and the simple system of positive roots $\Delta_+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ is given by

$$\left\{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_4 + \varepsilon_5,\ \frac{3}{4}\varepsilon_0 + \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5)\right\},$$

where $\frac{3}{4}\varepsilon_0 + \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5)$ is the unique non-compact simple root, and the central character of $\mathfrak{k}^{\mathbb{C}}$ is given by $d\chi = \varepsilon_0$. The set of strongly orthogonal roots $\{\gamma_1, \gamma_2\} \subset \Delta_{\mathfrak{p}^+}$ is given by

$$\gamma_1 = \frac{3}{4}\varepsilon_0 + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5), \quad \gamma_2 = \frac{3}{4}\varepsilon_0 + \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5),$$

and $\mathfrak{h}_{\mathrm{T}} := \mathfrak{h} \cap \mathfrak{g}_{\mathrm{T}}$, $\mathfrak{a}_{\mathfrak{l}}$ is given by

$$\sqrt{-1}\mathfrak{h}_{\mathrm{T}} = \operatorname{span}\left\{\frac{3}{4}t_0 + \frac{1}{2}t_1,\ t_2,\ t_3,\ t_4,\ t_5\right\}, \ \mathfrak{a}_{\mathfrak{l}} = \operatorname{span}\left\{\frac{3}{4}t_0 + \frac{1}{2}t_1,\ \frac{1}{2}(t_2 + t_3 + t_4 + t_5)\right\}.$$

We denote the restriction of ε_j to $\sqrt{-1}\mathfrak{h}_{\mathrm{T}}$ by the same symbol ε_j ($j = 2, 3, 4, 5$), and define $\varepsilon_1' \in (\sqrt{-1}\mathfrak{h}_{\mathrm{T}})^\vee$ by

$$\varepsilon_1'\left(\frac{3}{4}t_0 + \frac{1}{2}t_1\right) = 1, \quad \varepsilon_1'(t_j) = 0 \quad (j = 2, 3, 4, 5),$$

so that $(m_0\varepsilon_0 + m_1\varepsilon_1)|_{\sqrt{-1}\mathfrak{h}_{\mathrm{T}}} = \left(m_0 + \frac{1}{2}m_1\right)\varepsilon_1'$ holds. Also, we define $\varepsilon_2^\omega, \varepsilon_3^\omega, \varepsilon_4^\omega, \varepsilon_5^\omega \in (\sqrt{-1}\mathfrak{h}_{\mathrm{T}})^\vee$ such that they satisfy the relations

$$\begin{aligned} \varepsilon_2^\omega &= \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5), & \frac{1}{2}(\varepsilon_2^\omega + \varepsilon_3^\omega + \varepsilon_4^\omega + \varepsilon_5^\omega) &= \varepsilon_2, \\ \varepsilon_2^\omega + \varepsilon_3^\omega &= \varepsilon_2 + \varepsilon_3, & \frac{1}{2}(\varepsilon_2^\omega + \varepsilon_3^\omega + \varepsilon_4^\omega - \varepsilon_5^\omega) &= \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5), \end{aligned}$$

so that $\gamma_1|_{\sqrt{-1}\mathfrak{h}_{\mathrm{T}}} = \varepsilon_1' + \varepsilon_2^\omega$, $\gamma_2|_{\sqrt{-1}\mathfrak{h}_{\mathrm{T}}} = \varepsilon_1' - \varepsilon_2^\omega$ holds.

For $(m_0; \mathbf{m}) \in \mathbb{C} \times \left(\mathbb{Z}^5 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^5\right)$ with $m_1 \geq \cdots \geq m_4 \geq |m_5|$, let $(\tau^{[2,10]}_{(m_0;\mathbf{m})}, V^{[2,10]}_{(m_0;\mathbf{m})}) = (\chi^{m_0} \boxtimes \tau^{[10]}_{\mathbf{m}}, \mathbb{C}_{m_0} \otimes V^{[10]}_{\mathbf{m}})$ be the irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight $m_0\varepsilon_0 + m_1\varepsilon_1 + \cdots + m_5\varepsilon_5$. Also, for $(m_0; m_1; m_2, \ldots, m_5) \in \mathbb{C} \times \mathbb{C} \times \left(\mathbb{Z}^4 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^4\right)$ with $m_2 \geq m_3 \geq m_4 \geq |m_5|$, let $(\tau^{[2,2,8]}_{(m_0;m_1;m_2,\ldots,m_5)}, V^{[2,2,8]}_{(m_0;m_1;m_2,\ldots,m_5)})$, $(\tau^{[2,8]}_{(m_1;m_2,\ldots,m_5)}, V^{[2,8]}_{(m_1;m_2,\ldots,m_5)})$ and $(\tau^{[2,8]\omega}_{(m_1;m_2,\ldots,m_5)}, V^{[2,8]\omega}_{(m_1;m_2,\ldots,m_5)})$ be the irreducible $\mathfrak{k}_{\mathrm{T}}^{\mathbb{C}}$-module with highest weight $m_0\varepsilon_0 + m_1\varepsilon_1 + m_2\varepsilon_2 + \cdots + m_5\varepsilon_5$, $m_1\varepsilon_1' + m_2\varepsilon_2 + \cdots + m_5\varepsilon_5$, and $m_1\varepsilon_1' + m_2\varepsilon_2^\omega + \cdots + m_5\varepsilon_5^\omega$ respectively. Then as in Section 2.4.1, we can show

$$(\overline{\tau^{[2,8]\omega}_{(m_1;m_2,m_3,m_4,m_5)}}, \overline{V^{[2,8]\omega}_{(m_1;m_2,m_3,m_4,m_5)}}) \simeq (\tau^{[2,8]\omega}_{(m_1;m_2,m_3,m_4,-m_5)}, V^{[2,8]\omega}_{(m_1;m_2,m_3,m_4,-m_5)}).$$

We set $V = V^{[2,10]}_{(-\frac{k}{2};k,0,0,0,0)}$. The goal of this subsection is to prove the following proposition.

Proposition 2.5.8. *When* $G = E_{6(-14)}$ *and* $(\tau, V) = (\chi_{-k/2} \boxtimes \tau^{[10]}_{(k,0,0,0,0)}, \mathbb{C}_{-k/2} \otimes V^{[10]}_{(k,0,0,0,0)})$ $(k \in \mathbb{Z}_{\geq 0})$, $\|\cdot\|^2_{\lambda,\tau}$ *converges if* $\operatorname{Re}\lambda > 11$, *the normalizing constant* c_λ *is given by*

$$c_\lambda = (\lambda - 7 + k)_7(\lambda - 8)(\lambda - 11)_7(\lambda - 4 + k),$$

the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\begin{aligned}
&\mathcal{P}(\mathfrak{p}^+)\otimes\left(\mathbb{C}_{-k/2}\boxtimes V^{[10]}_{(k,0,0,0,0)}\right)\\
&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^4,\ |\mathbf{k}|=k\\ k_2+k_4\leq m_2\\ k_3\leq m_1-m_2}}\mathbb{C}_{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}}\boxtimes V^{[10]}_{\left(\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)},
\end{aligned}$$

and for $f\in\mathbb{C}_{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}}\boxtimes V^{[10]}_{\left(\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}$, the ratio of norms is of the form

$$\begin{aligned}
\frac{\|f\|^2_{\lambda,\chi_{-k/2}\boxtimes\tau^{[10]}_{(k,0,0,0,0)}}}{\|f\|^2_{F,\chi_{-k/2}\boxtimes\tau^{[10]}_{(k,0,0,0,0)}}}&=\frac{(\lambda)_k(\lambda-3)_k(\text{monic polynomial of degree }2k_1+k_2+k_3)}{(\lambda)_{m_1+k_1+k_2}(\lambda-3)_{m_2+k_1+k_3}(\lambda-4)_k(\lambda-7)_k}\\
&=\frac{(\text{monic polynomial of degree }2k_1+k_2+k_3)}{(\lambda+k)_{m_1+k_1+k_2-k}(\lambda+k-3)_{m_2+k_1+k_3-k}(\lambda-4)_k(\lambda-7)_k}.
\end{aligned}$$

Before starting the proof, we quote the following lemma about the restriction of the representation $V^{[2s+2]}$ of $\mathfrak{so}(2s+2)$ to $\mathfrak{so}(2)\oplus\mathfrak{so}(2s)$.

Lemma 2.5.9 ([26, Theorem 1.1])**.**

$$V^{[2s+2]}_{(m_0,m_1,\ldots,m_s)}\Big|_{\mathfrak{so}(2)\oplus\mathfrak{so}(2s)}\simeq\bigoplus_{\substack{m_{i-1}\geq n_i\geq|m_{i+1}|\\ m_{s-1}\geq|n_s|}}\bigoplus_{n_0}c^{(m_0,m_1,\ldots,m_s)}_{(n_1,\ldots,n_s)}(n_0)V^{[2,2s]}_{(n_0;n_1,\ldots,n_s)},$$

where $c^{(m_0,m_1,\ldots,m_s)}_{(n_1,\ldots,n_s)}(n_0)\in\mathbb{Z}_{\geq 0}$ is the coefficient of X^{n_0} of the polynomial

$$X^{a_s}\prod_{j=0}^{s-1}\frac{X^{a_j+1}-X^{-a_j-1}}{X-X^{-1}},$$

where

$$\begin{aligned}
a_0&=m_0-\max\{m_1,n_1\},\\
a_j&=\min\{m_j,n_j\}-\max\{|m_{j+1}|,|n_{j+1}|\}\qquad\qquad (j=1,\ldots,s-1),\\
a_s&=(\operatorname{sgn}m_s)(\operatorname{sgn}n_s)\min\{|m_s|,|n_s|\}.
\end{aligned}$$

From this lemma we can easily deduce the following.

Lemma 2.5.10.

$$V^{[2s+2]}_{(k,0,\ldots,0)}\Big|_{\mathfrak{so}(2)\oplus\mathfrak{so}(2s)}=\bigoplus_{l_1=0}^{k}\bigoplus_{\substack{l_0\in\mathbb{Z},\ |l_0|\leq k-l_1\\ k-l_0-l_1\in 2\mathbb{Z}}}V^{[2,2s]}_{(l_0;l_1,0,\ldots,0)}.$$

Now we start the proof. To begin with, we determine the normalizing constant c_λ. Since $V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}$ is decomposed under $\mathfrak{k}_{\mathrm{T}}$ as

$$\begin{aligned}
V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}\Big|_{\mathfrak{k}_{\mathrm{T}}}&=\bigoplus_{l_1=0}^{k}\bigoplus_{\substack{l_0\in\mathbb{Z},\ |l_0|\leq k-l_1\\ k-l_0-l_1\in 2\mathbb{Z}}}V^{[2,2,8]}_{\left(-\frac{k}{2};l_0;l_1,0,0,0\right)}=\bigoplus_{l_1=0}^{k}\bigoplus_{\substack{l_0\in\mathbb{Z},\ |l_0|\leq k-l_1\\ k-l_0-l_1\in 2\mathbb{Z}}}V^{[2,8]}_{\left(\frac{-k+l_0}{2};l_2,0,0,0\right)}\\
&=\bigoplus_{\substack{k_1,k_2\in\mathbb{Z}_{\geq 0}\\ k\geq k_1\geq k_2\geq 0}}V^{[2,8]}_{\left(-\frac{k_1+k_2}{2};k_1-k_2,0,0,0\right)}=\bigoplus_{\substack{k_1,k_2\in\mathbb{Z}_{\geq 0}\\ k\geq k_1\geq k_2\geq 0}}V^{[2,8]\omega}_{\left(-\frac{k_1+k_2}{2};\frac{k_1-k_2}{2},\frac{k_1-k_2}{2},\frac{k_1-k_2}{2},\frac{k_1-k_2}{2}\right)},
\end{aligned}$$

each $V^{[2,8]\omega}_{\left(-\frac{k_1+k_2}{2};\frac{k_1-k_2}{2},\frac{k_1-k_2}{2},\frac{k_1-k_2}{2},\frac{k_1-k_2}{2}\right)}$ remains irreducible under $\mathfrak{k}_\mathfrak{l}=\mathfrak{so}(7)$, and has the restricted lowest weight $-\frac{1}{2}(k_1\gamma_1+k_2\gamma_2)\big|_{\mathfrak{a}_\mathfrak{l}}$, by Theorem 2.3.1, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda>11$, and c_λ is given by

$$\begin{aligned}c_\lambda^{-1}&=\frac{1}{\dim V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}}\sum_{\substack{k_1,k_2\in\mathbb{Z}_{\geq 0}\\ k\geq k_1\geq k_2\geq 0}}\left(\dim V^{[2,8]\omega}_{\left(-\frac{k_1+k_2}{2};\frac{k_1-k_2}{2},\ldots,\frac{k_1-k_2}{2}\right)}\right)\frac{\Gamma_\Omega(\lambda+(k_1,k_2)-8)}{\Gamma_\Omega(\lambda+(k_1,k_2))}\\&=\frac{1}{\binom{k+9}{9}-\binom{k+7}{9}}\sum_{\substack{k_1,k_2\in\mathbb{Z}_{\geq 0}\\ k\geq k_1\geq k_2\geq 0}}\frac{\binom{k_1-k_2+7}{7}-\binom{k_1-k_2+5}{7}}{(\lambda+k_1-8)_8(\lambda+k_2-11)_8}.\end{aligned}$$

For $l\in\mathbb{Z}_{\geq 0}$, we define

$$F(\lambda,l):=\sum_{\substack{k_1,k_2\in\mathbb{Z}_{\geq 0}\\ l\geq k_1\geq k_2\geq 0}}\frac{\binom{k_1-k_2+7}{7}-\binom{k_1-k_2+5}{7}}{(\lambda+k_1-8)_8(\lambda+k_2-11)_8}.$$

Then it satisfies

$$\begin{aligned}&F(\lambda,l+1)\\&=\left(\sum_{l\geq k_1\geq k_2\geq 0}+\sum_{l+1\geq k_1\geq k_2\geq 1}-\sum_{l\geq k_1\geq k_2\geq 1}+\sum_{(k_1,k_2)=(l+1,0)}\right)\frac{\binom{k_1-k_2+7}{7}-\binom{k_1-k_2+5}{7}}{(\lambda+k_1-8)_8(\lambda+k_2-11)_8}\\&=F(\lambda,l)+F(\lambda+1,l)-F(\lambda+1,l-1)+\frac{\binom{l+8}{7}-\binom{l+6}{7}}{(\lambda+l-7)_8(\lambda-11)_8}.\end{aligned}$$

Solving this recurrence relation, we get

$$F(\lambda,l)=\frac{\binom{l+9}{9}-\binom{l+7}{9}}{(\lambda-7+l)_7(\lambda-8)(\lambda-11)_7(\lambda-4+l)},$$

and thus we have

$$\begin{aligned}c_\lambda&=(\lambda-7+k)_7(\lambda-8)(\lambda-11)_7(\lambda-4+k)=\frac{(\lambda-8)_{k+8}(\lambda-11)_{k+8}}{(\lambda-7)_k(\lambda-4)_k}\\&=\frac{\Gamma_\Omega(\lambda+k)}{\Gamma_\Omega(\lambda-8)(\lambda-4)_k(\lambda-7)_k}.\end{aligned}$$

Next we compute the K-type decomposition of $\mathcal{O}(D,V)_K=\mathcal{P}(\mathfrak{p}^+)\otimes V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}$. By Theorem 2.2.1 and the "multi-minuscule rule" [25, Corollary 2.16], we have

$$\begin{aligned}&\mathcal{P}(\mathfrak{p}^+)\otimes V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}\\&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2);\frac{m_1+m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}\right)}\otimes V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}\\&=\bigoplus_{\mathbf{m}\in\mathbb{Z}^2_{++}}\bigoplus_{\substack{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^4,\ |\mathbf{k}|=k\\ k_2+k_4\leq m_2\\ k_3\leq m_1-m_2}}V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2)-\frac{k}{2};\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}.\end{aligned}$$

In order to apply Theorem 2.3.1, we observe the image of each K-type under rest : $\mathcal{P}(\mathfrak{p}^+, V) \to \mathcal{P}(\mathfrak{p}^+_{\mathrm{T}}, V)$. For each $\mathbf{m} \in \mathbb{Z}^2_{++}$, we have

$$\begin{aligned}
&\operatorname{rest}\left(V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2);\frac{m_1+m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}\right)} \otimes V^{[2,10]}_{\left(-\frac{k}{2};k,0,0,0,0\right)}\right) \\
&= V^{[2,8]}_{\left(-(m_1+m_2);\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2}\right)} \otimes \bigoplus_{\substack{k_1',k_2'\in\mathbb{Z}_{\geq 0}\\ k\geq k_1'\geq k_2'\geq 0}} V^{[2,8]}_{\left(-\frac{k_1'+k_2'}{2};k_1'-k_2',0,0,0\right)} \\
&= \bigoplus_{\substack{k_1',k_2'\in\mathbb{Z}_{\geq 0}\\ k\geq k_1'\geq k_2'\geq 0}} \bigoplus_{\substack{l_1,l_2\in\mathbb{Z}_{\geq 0}\\ l_2\leq m_1-m_2\\ l_1+l_2=k_1'-k_2'}} V^{[2,8]}_{\left(-\left(m_1+m_2+\frac{k_1'+k_2'}{2}\right);\frac{m_1-m_2}{2}+l_1,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2}-l_2\right)}.
\end{aligned}$$

We write $k_1' + k_2' =: l_0$, so that $k_1' = \frac{1}{2}(l_0 + l_1 + l_2)$, $k_2' = \frac{1}{2}(l_0 - l_1 - l_2)$. By Lemma 2.5.9,

$$\begin{aligned}
\operatorname{rest}&\left(V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2)-\frac{k}{2};\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}\right) \\
&\cap V^{[2,8]}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);\frac{m_1-m_2}{2}+l_1,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2}-l_2\right)} \neq \{0\}
\end{aligned}$$

implies

$$0 \leq l_1 \leq m_2 + k_1 - k_4, \qquad 0 \leq l_2 \leq m_1 - m_2,$$

and the coefficient of $X^{2\left(-\left(m_1+m_2+\frac{l_0}{2}\right)+\left(\frac{3}{4}(m_1+m_2)+\frac{k}{2}\right)\right)} = X^{-\frac{m_1+m_2}{2}-l_0+k}$ of the polynomial

$$X^{a_4}\frac{X^{a_0+1}-X^{-a_0-1}}{X-X^{-1}}\frac{X^{a_1+1}-X^{-a_1-1}}{X-X^{-1}}\frac{X^{a_3+1}-X^{-a_3-1}}{X-X^{-1}},$$

does not vanish, where

$$\begin{aligned}
a_0 &= \frac{m_1+m_2}{2} + k_1 - k_4 - \max\left\{\frac{m_1-m_2}{2}+k_2, \frac{m_1-m_2}{2}+l_1\right\} \\
&= m_2 + k_1 - k_4 - \max\{k_2, l_1\}, \\
a_1 &= \min\left\{\frac{m_1-m_2}{2}+k_2, \frac{m_1-m_2}{2}+l_1\right\} - \frac{m_1-m_2}{2} \\
&= \min\{k_2, l_1\}, \\
a_3 &= \frac{m_1-m_2}{2} - \max\left\{\left|\frac{m_1-m_2}{2}-k_3\right|, \left|\frac{m_1-m_2}{2}-l_2\right|\right\}, \\
a_4 &= \operatorname{sgn}\left(-\frac{m_1-m_2}{2}+k_3\right)\operatorname{sgn}\left(\frac{m_1-m_2}{2}-l_2\right)\min\left\{\left|\frac{m_1-m_2}{2}-k_3\right|, \left|\frac{m_1-m_2}{2}-l_2\right|\right\}.
\end{aligned}$$

This condition is satisfied only if

$$\begin{aligned}
-\frac{m_1+m_2}{2} - l_0 + k &\geq -a_0 - a_1 - a_3 + a_4 \\
&= -\frac{m_1+m_2}{2} - k_1 + k_4 + |k_2 - l_1| + |k_3 - l_2| \\
\therefore l_0 &\leq k + k_1 - k_4 - |k_2 - l_1| - |k_3 - l_2| \\
&= 2k_1 + k_2 + k_3 - |k_2 - l_1| - |k_3 - l_2|.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\operatorname{rest}\left(V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2)-\frac{k}{2};\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}\right)\\
&\subset \bigoplus_{\substack{l_0,l_1,l_2\in\mathbb{Z}_{\ge 0},\ l_0-l_1-l_2\in 2\mathbb{Z}_{\ge 0}\\ l_1\le m_2+k_1-k_4,\ l_2\le m_1-m_2\\ l_0\le 2k_1+k_2+k_3-|k_2-l_1|-|k_3-l_2|}} V^{[2,8]}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);\frac{m_1-m_2}{2}+l_1,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2}-l_2\right)}.
\end{aligned}$$

For each m_1,m_2,l_0,l_1,l_2, we have

$$V^{[2,8]}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);\frac{m_1-m_2}{2}+l_1,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},\frac{m_1-m_2}{2}-l_2\right)}=V^{[2,8]\omega}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);m_1-m_2+\frac{l_1-l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1-l_2}{2}\right)},$$

and as in Section 2.4.5, $\mathfrak{k}_\mathfrak{l}=\mathfrak{so}(7)$-spherical irreducible submodules in

$$\begin{aligned}
&V^{[2,8]\omega}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);m_1-m_2+\frac{l_1-l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1-l_2}{2}\right)}\otimes\overline{V^{[2,8]\omega}_{\left(-\frac{l_0}{2};\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2}\right)}}\\
&\simeq V^{[2,8]\omega}_{\left(-\left(m_1+m_2+\frac{l_0}{2}\right);m_1-m_2+\frac{l_1-l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1-l_2}{2}\right)}\otimes V^{[2,8]\omega}_{\left(-\frac{l_0}{2};\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},\frac{l_1+l_2}{2},-\frac{l_1+l_2}{2}\right)}
\end{aligned}$$

are isomorphic to $V^{[2,8]\omega}_{(-(m_1+m_2+l_0);m_1-m_2+l_1-l_2,0,0,0)}$, which has the lowest weight

$$-\left(m_1+\frac{l_0+l_1-l_2}{2}\right)\gamma_1-\left(m_2+\frac{l_0-l_1+l_2}{2}\right)\gamma_2.$$

Therefore for $f\in V^{[2,10]}_{\left(-\frac{3}{4}(m_1+m_2)-\frac{k}{2};\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}$, by Theorem 2.3.1, the ratio of norms is given by

$$\begin{aligned}
\frac{\|f\|_{\lambda,\tau}}{\|f\|_{F,\tau}}&=\frac{c_\lambda}{\sum_{\mathbf{l}}a_{\mathbf{m},\mathbf{k},\mathbf{l}}}\sum_{\substack{l_0,l_1,l_2\in\mathbb{Z}_{\ge 0},\ l_0-l_1-l_2\in 2\mathbb{Z}_{\ge 0}\\ l_1\le m_2+k_1-k_4,\ l_2\le m_1-m_2\\ l_0\le 2k_1+k_2+k_3-|k_2-l_1|-|k_3-l_2|}}\frac{a_{\mathbf{m},\mathbf{k},\mathbf{l}}\Gamma_\Omega\left(\lambda+\left(\frac{l_0+l_1+l_2}{2},\frac{l_0-l_1-l_2}{2}\right)-8\right)}{\Gamma_\Omega\left(\lambda+\left(m_1+\frac{l_0+l_1-l_2}{2},m_2+\frac{l_0-l_1+l_2}{2}\right)\right)}\\
&=\frac{1}{\sum_{\mathbf{l}}a_{\mathbf{m},\mathbf{k},\mathbf{l}}}\sum_{\substack{l_0,l_1,l_2\in\mathbb{Z}_{\ge 0},\ l_0-l_1-l_2\in 2\mathbb{Z}_{\ge 0}\\ l_1\le m_2+k_1-k_4,\ l_2\le m_1-m_2\\ l_0\le 2k_1+k_2+k_3-|k_2-l_1|-|k_3-l_2|}}\frac{a_{\mathbf{m},\mathbf{k},\mathbf{l}}(\lambda)_k(\lambda-3)_k(\lambda-8)_{\frac{l_0+l_1+l_2}{2}}(\lambda-11)_{\frac{l_0-l_1-l_2}{2}}}{(\lambda)_{m_1+\frac{l_0+l_1-l_2}{2}}(\lambda-3)_{m_2+\frac{l_0-l_1+l_2}{2}}(\lambda-4)_k(\lambda-7)_k},
\end{aligned}$$

using some non-negative numbers $a_{\mathbf{m},\mathbf{k},\mathbf{l}}$. Now, since

$$\begin{aligned}
l_0+l_1-l_2&\le 2k_1+k_2+k_3-|k_2-l_1|-|k_3-l_2|+l_1-l_2\\
&\le 2k_1+2k_2-(k_2-l_1)-|k_2-l_1|+(k_3-l_2)-|k_3-l_2|\le 2(k_1+k_2),\\
l_0-l_1+l_2&\le 2k_1+k_2+k_3-|k_2-l_1|-|k_3-l_2|-l_1+l_2\\
&\le 2k_1+2k_3+(k_2-l_1)-|k_2-l_1|-(k_3-l_2)-|k_3-l_2|\le 2(k_1+k_3),
\end{aligned}$$

we have

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}=\frac{(\lambda)_k(\lambda-3)_k(\text{monic polynomial of degree }2k_1+k_2+k_3)}{(\lambda)_{m_1+k_1+k_2}(\lambda-3)_{m_2+k_1+k_3}(\lambda-4)_k(\lambda-7)_k},$$

and we have proved Proposition 2.5.8. □

By $k_2+k_4\le m_2$ and $k_3\le m_1-m_2$, we have the inequality

$$m_1+k_1+k_2\ge m_2+k_1+k_3\ge k_2+k_3+k_4\ge k_4.$$

Thus the author conjectures the following.

Conjecture 2.5.11. *For* $f \in \mathbb{C}_{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}} \boxtimes V^{[10]}_{\left(\frac{m_1+m_2}{2}+k_1-k_4, \frac{m_1-m_2}{2}+k_2, \frac{m_1-m_2}{2}, \frac{m_1-m_2}{2}, -\frac{m_1-m_2}{2}+k_3\right)}$, *the ratio of norms is given by*

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\chi_{-k/2}\boxtimes\tau^{[10]}_{(k,0,0,0,0)}}}{\|f\|^2_{F,\chi_{-k/2}\boxtimes\tau^{[10]}_{(k,0,0,0,0)}}} &= \frac{(\lambda)_k(\lambda-3)_k}{(\lambda)_{m_1+k_1+k_2}(\lambda-3)_{m_2+k_1+k_3}(\lambda-4)_{k_2+k_3+k_4}(\lambda-7)_{k_4}} \\ &= \frac{1}{(\lambda+k)_{m_1+k_1+k_2-k}(\lambda+k-3)_{m_2+k_1+k_3-k}(\lambda-4)_{k_2+k_3+k_4}(\lambda-7)_{k_4}}.\end{aligned}$$

2.6 Analytic continuation of holomorphic discrete series

In the previous sections, we calculated the norms of the holomorphic discrete series representations. Using this, we see how the highest weight modules behave as the parameter λ goes small, following the arguments in [6] and [19].

For example, when $G = Sp(r,\mathbb{R})$ and $V = V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$ with $k = 0, 1, \ldots, r-1$, by Theorem 2.4.2, the norm $\|\cdot\|_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}$ is written as

$$\|f\|^2_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \sum_{\substack{\mathbf{k}\in\{0,1\}^r,\, |\mathbf{k}|=k \\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_+}} \frac{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)\right)}{\prod_{j=1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}} \|f_{\mathbf{m},\mathbf{k}}\|^2_{F,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}$$

for $\lambda > r$, where $f_{\mathbf{m},\mathbf{k}}$ is the orthogonal projection of f onto $V^\vee_{2\mathbf{m}+\mathbf{k}}$. Then as in [7, Theorem XIII.2.4], the reproducing kernel $K_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}$ is written by the converging sum

$$K_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}(z,w) = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \sum_{\substack{\mathbf{k}\in\{0,1\}^r,\, |\mathbf{k}|=k \\ \mathbf{m}+\mathbf{k}\in\mathbb{Z}^r_+}} \frac{\prod_{j=1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}}{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)\right)} K_{\mathbf{m},\mathbf{k}}(z,w)$$

where $K_{\mathbf{m},\mathbf{k}}(z,w)$ is the reproducing kernel of $V^\vee_{2\mathbf{m}+\mathbf{k}}$ with respect to the Fischer norm $\|\cdot\|^2_{F,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}$. This is continued analytically for smaller λ, and by [7, Lemma XIII.2.6], this is positive definite if and only if each coefficient is positive, that is,

$$\lambda \in \left\{\frac{k}{2}, \frac{k+1}{2}, \ldots, \frac{r-1}{2}\right\} \cup \left(\frac{r-1}{2}, \infty\right).$$

The positive definite function automatically becomes a reproducing kernel of some Hilbert space $\mathcal{H}_\lambda(D,V)$, and this $\mathcal{H}_\lambda(D,V)$ gives the unitary representation of $\tilde{G}$. Conversely, if there exists a unitary subrepresentation $\mathcal{H}_\lambda(D,V) \subset \mathcal{O}(D,V)$ for some $\lambda \in \mathbb{R}$, then its reproducing kernel is automatically proportional to $K_{\lambda,\tau^\vee_{\varepsilon_1+\cdots+\varepsilon_k}}(z,w)$ by the arguments in Section 2.3.1, and thus the above condition on λ is precisely the necessary and sufficient condition for unitarizability. Using this idea, we get the following result.

Theorem 2.6.1. *(1) When $G = Sp(r,\mathbb{R})$ and $V = V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$ with $k = 0, 1, \ldots, r-1$, $(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable when $\lambda > r$, contains a non-zero unitary submodule $\mathcal{H}_\lambda(D,V)$ if and only if*

$$\lambda \in \left\{\frac{k}{2}, \frac{k+1}{2}, \ldots, \frac{r-1}{2}\right\} \cup \left(\frac{r-1}{2}, \infty\right).$$

(2) When $G = SU(q,s)$ and $V = \mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}$ with $\mathbf{k} \in \mathbb{Z}_{++}^s$ ($k_l \neq 0,\ k_{l+1} = 0,\ l = 0, \dots, s-1$), $(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable when $\lambda > q+s-1$, contains a non-zero unitary submodule $\mathcal{H}_\lambda(D,V)$ if and only if

$$\lambda \in \big\{l, l+1, \dots, \min\{q+l, s\} - 1\big\} \cup \big(\min\{q+l, s\} - 1, \infty\big).$$

(3) When $G = SO^*(2s)$ and $V = V_{(k,0,\dots,0)}^\vee$ with $k \in \mathbb{Z}_{\geq 0}$, $(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable when $\lambda > 2s-3$, contains a non-zero unitary submodule $\mathcal{H}_\lambda(D,V)$ if and only if

$$\lambda \in \begin{cases} \{0, 2, 4, \dots, 2\left(\lfloor \frac{s}{2} \rfloor - 1\right)\} \cup \left(2\left(\lfloor \frac{s}{2} \rfloor - 1\right), \infty\right) & (k = 0), \\ \{2, 4, \dots, 2\left(\lceil \frac{s}{2} \rceil - 1\right)\} \cup \left(2\left(\lceil \frac{s}{2} \rceil - 1\right), \infty\right) & (k \geq 1). \end{cases}$$

(4) When $G = SO^*(2s)$ and $V = V_{(k/2,\dots,k/2,-k/2)}^\vee$ with $k \in \mathbb{Z}_{>0}$, $(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable when $\lambda > 2s-3$, contains a non-zero unitary submodule $\mathcal{H}_\lambda(D,V)$ if and only if

$$\lambda \in \{s-2\} \cup (s-2, \infty).$$

(5) When $G = Spin_0(2,n)$ and

$$V = \begin{cases} \mathbb{C}_k \boxtimes V_{(k,\dots,k,\pm k)} & (k \in \frac{1}{2}\mathbb{Z}_{\geq 0}) \quad (n : \textit{even}), \\ \mathbb{C}_k \boxtimes V_{(k,\dots,k,k)} & (k = 0, \frac{1}{2}) \quad (n : \textit{odd}), \end{cases}$$

$(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable when $\lambda > n-1$, contains a non-zero unitary submodule $\mathcal{H}_\lambda(D,V)$ if and only if

$$\lambda \in \begin{cases} \left\{0, \frac{n-2}{2}\right\} \cup \left(\frac{n-2}{2}, \infty\right) & (k = 0), \\ \left\{\frac{n-2}{2}\right\} \cup \left(\frac{n-2}{2}, \infty\right) & (k \geq \frac{1}{2}). \end{cases}$$

From the explicit norm computation, we can also determine completely when the representation is reducible, and get some informations on the composition series, as in [6], [19]. We denote the K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+, V)$ by

$$\mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_m W_m,$$

and for $f \in W_m$ we denote the ratio of norms by $\|f\|_{\lambda,\tau}^2 / \|f\|_{F,\tau}^2 =: R_m(\lambda)$, so that

$$\langle f, g \rangle_{\lambda,\tau} = \sum_m R_m(\lambda) \langle f_m, g_m \rangle_{F,\tau}.$$

If λ is not a pole for all $R_m(\lambda)$, then the above sesquilinear form is well-defined, and non-degenerate for our cases because the numerator of each $R_m(\lambda)$ is one. From this we can show $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ is irreducible, because if $\mathcal{P}(\mathfrak{p}^+, V)$ has a proper submodule M, then its orthogonal complement $M^\perp$ also becomes a submodule, and both M and $M^\perp$ contain a $\mathfrak{p}^+$-invariant vector i.e. contain the minimal K-type V, which is a contradiction. We note that in our cases the sesquilinear form is always definite on each K-isotypic component, and thus $M^\perp$ is precisely a complement vector space.

On the other hand, if λ is a pole for some $R_m(\lambda)$, then $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ is reducible. In fact, for $j \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ we define $\tilde{M}_j(\lambda)$ as the direct sum of W_m's such that $R_m(\lambda)$ has a pole of order at most j at λ. Then the sesquilinear form

$$\lim_{\lambda' \to \lambda} (\lambda' - \lambda)^j \langle f, g \rangle_{\lambda',\tau} \tag{2.6.1}$$

is $(\mathfrak{g},K)$-invariant under the representation $d\tau_\lambda$ on $\tilde{M}_j(\lambda)$, which vanishes on $\tilde{M}_{j-1}(\lambda)$. Thus $\tilde{M}_j(\lambda)$ is a $(\mathfrak{g},K)$-submodule of $\mathcal{P}(\mathfrak{p}^+,V)$. Clearly $\tilde{M}_j(\lambda)/\tilde{M}_{j-1}(\lambda)$ is infinitesimally unitary if the sesquilinear form (2.6.1) is definite. This gives the following theorem.

Theorem 2.6.2. *(1) When $G=Sp(r,\mathbb{R})$ and $V=V^\vee_{\varepsilon_1+\cdots+\varepsilon_k}$ with $k=0,1,\dots,r-1$, for $\lambda\in\mathbb{R}$ and $j=1,2,\dots,r$, we define*

$$M_j(\lambda):=\bigoplus_{m_j+k_j<\frac{j}{2}-\lambda+\frac{1}{2}} V^\vee_{2\mathbf{m}+\mathbf{k}}\subset\mathcal{P}(\mathfrak{p}^+,V).$$

Then $(d\tau_\lambda,\mathcal{P}(\mathfrak{p}^+,V))$ is reducible if and only if $\lambda\le\frac{r-1}{2}$ and $\lambda\in\frac{1}{2}\mathbb{Z}$. In this case we have the sequence of submodules

$$\{0\}\subset M_a(\lambda)\subset M_{a+2}(\lambda)\subset\cdots\subset M_b(\lambda)\subset\mathcal{P}(\mathfrak{p}^+,V),$$

where

$$a=\begin{cases}2\lambda+1 & (\frac{k}{2}\le\lambda\le\frac{r-1}{2}),\\ 2\lambda+3 & (0\le\lambda\le\frac{k-1}{2}),\\ 1 & (\lambda\le-\frac{1}{2},\ \lambda\in\mathbb{Z}),\\ 2 & (\lambda\le-\frac{1}{2},\ \lambda\in\mathbb{Z}+\frac{1}{2}),\end{cases}\qquad b=\begin{cases}r-1 & (2\lambda\equiv r \mod 2),\\ r & (2\lambda\not\equiv r \mod 2).\end{cases}$$

$M_{2\lambda+1}(\lambda)$ $(\lambda=\frac{k}{2},\frac{k+1}{2},\dots,\frac{r-1}{2})$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_r(\lambda)$ $(\lambda\le\frac{r-1}{2},\ 2\lambda\not\equiv r \mod 2)$ are infinitesimally unitary.

(2) When $G=SU(q,s)$ and $V=\mathbb{C}\boxtimes V^{(s)}_{\mathbf{k}}$ with $\mathbf{k}\in\mathbb{Z}^s_{++}$ $(k_l\ne0,\ k_{l+1}=0,\ l=0,\dots,s-1)$, for $\lambda\in\mathbb{R}$ and $j=1,2,\dots,s$, we define

$$M_j(\lambda):=\bigoplus_{n_j<j-\lambda} c^{\mathbf{n}}_{\mathbf{k},\mathbf{m}}V^{(q)\vee}_{\mathbf{m}}\boxtimes V^{(s)}_{\mathbf{n}}\subset\mathcal{P}(\mathfrak{p}^+,V).$$

Then $(d\tau_\lambda,\mathcal{P}(\mathfrak{p}^+,V))$ is reducible if and only if $\lambda\le\min\{q+l,s\}-1$, $\lambda\in\mathbb{Z}$ and there is no $j=q+1,\dots,s$ such that $\lambda=j-k_j=j-k_{j-q+1}$ holds. In this case we have the sequence of submodules

$$\{0\}\subset M_a(\lambda)\subset M_{a+1}(\lambda)\subset\cdots\subset M_b(\lambda)\subset\mathcal{P}(\mathfrak{p}^+,V),$$

where

$$a=\begin{cases}j+1 & (j-k_j\le\lambda\le j-k_{j+1})\quad(1\le j\le\min\{q+l,s\}-1),\\ 1 & (\lambda\le-k_1),\end{cases}$$

and $b=s$ if $q\ge s$,

$$b=\begin{cases}\min\{q+l,s\} & (\min\{q+l,s\}-k_{\min\{l,s-q\}}\le\lambda\le\min\{q+l,s\}-1),\\ j & (j-k_{j-q}\le\lambda\le j-k_{j-q+1})\quad(q+1\le j\le\min\{q+l,s\}-1),\\ q & (\lambda\le q-k_1)\end{cases}$$

if $q<s$.

If $q\ge s$ or $\mathbf{k}=\mathbf{0}$, then $M_{\lambda+1}(\lambda)$ $(\lambda=l,l+1,\dots,\min\{q,s\}-1)$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_{\min\{q,s\}}(\lambda)$ $(\lambda\le\min\{q,s\}-1,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

If $q<s$ and $\mathbf{k}\ne\mathbf{0}$, then $M_{\lambda+1}(\lambda)$ $(\lambda=l,l+1,\dots,\min\{q+l,s\}-1)$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_{\min\{q+l,s\}}(\lambda)$ $(\min\{q+l,s\}-k_{\min\{l,s-q\}}\le\lambda\le\min\{q+l,s\}-1,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

(3) *When* $G = SO^*(4r)$ *and* $V = V^{\vee}_{(k,0,\dots,0)}$ *with* $k \in \mathbb{Z}_{\geq 0}$*, for* $\lambda \in \mathbb{R}$ *and* $j = 1, 2, \dots, r$*, we define*

$$M_j(\lambda) := \bigoplus_{m_j+k_j<2j-\lambda-1} V^{\vee}_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)} \subset \mathcal{P}(\mathfrak{p}^+, V).$$

Then $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ *is reducible if and only if* $\lambda \leq 2r-2$ *and* $\lambda \in \mathbb{Z}$*. In this case we have the sequence of submodules*

$$\{0\} \subset M_a(\lambda) \subset M_{a+1}(\lambda) \subset \cdots \subset M_r(\lambda) \subset \mathcal{P}(\mathfrak{p}^+, V),$$

where

$$a = \begin{cases} \left\lceil \frac{\lambda}{2} \right\rceil + 1 & (3 \leq \lambda \leq 2r-2), \\ 2 & (-k+1 \leq \lambda \leq 2), \\ 1 & (\lambda \leq -k). \end{cases}$$

$M_{\frac{\lambda}{2}+1}(\lambda)$ $(\lambda = 2, 4, \dots, 2r-2$ *if* $k \geq 1$*,* $\lambda = 0, 2, \dots, 2r-2$ *if* $k = 0)$ *and* $\mathcal{P}(\mathfrak{p}^+, V)/M_r(\lambda)$ $(\lambda \leq 2r-2,\ \lambda \in \mathbb{Z})$ *are infinitesimally unitary.*

(4) *When* $G = SO^*(4r)$ *and* $V = V^{\vee}_{(k/2,\dots,k/2,-k/2)}$ *with* $k \in \mathbb{Z}_{>0}$*, for* $\lambda \in \mathbb{R}$ *and* $j = 1, 2, \dots, r$*, we define*

$$M_j(\lambda) := \bigoplus_{m_j-k_j+k<2j-\lambda-1} V^{\vee}_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(k/2,\dots,k/2)} \subset \mathcal{P}(\mathfrak{p}^+, V).$$

Then $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ *is reducible if and only if* $\lambda \leq 2r-2$ *and* $\lambda \in \mathbb{Z}$*. In this case we have the sequence of submodules*

$$\{0\} \subset M_a(\lambda) \subset M_{a+1}(\lambda) \subset \cdots \subset M_r(\lambda) \subset \mathcal{P}(\mathfrak{p}^+, V),$$

where

$$a = \begin{cases} r & (2r-3-k \leq \lambda \leq 2r-2), \\ \left\lceil \frac{\lambda+k}{2} \right\rceil + 1 & (-k+1 \leq \lambda \leq 2r-4-k), \\ 1 & (\lambda \leq -k). \end{cases}$$

$M_r(2r-2)$ *and* $\mathcal{P}(\mathfrak{p}^+, V)/M_r(\lambda)$ $(\lambda \leq 2r-2,\ \lambda \in \mathbb{Z})$ *are infinitesimally unitary.*

(5) *When* $G = SO^*(4r+2)$ *and* $V = V^{\vee}_{(k,0,\dots,0)}$ *with* $k \in \mathbb{Z}_{\geq 0}$*, for* $\lambda \in \mathbb{R}$ *and* $j = 1, 2, \dots, r+1$*, we define*

$$M_j(\lambda) := \bigoplus_{m_j+k_j<2j-\lambda-1} V^{\vee}_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)} \subset \mathcal{P}(\mathfrak{p}^+, V) \quad (j = 1, \dots, r),$$

$$M_{r+1}(\lambda) := \bigoplus_{k_{r+1}<2r-\lambda+1} V^{\vee}_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r)} \subset \mathcal{P}(\mathfrak{p}^+, V).$$

Then $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ *is reducible if and only if* $\lambda \leq \begin{cases} 2r & (k \geq 1) \\ 2r-2 & (k = 0) \end{cases}$*,* $\lambda \in \mathbb{Z}$ *and* $(r, \lambda) \neq (1, -k+1)$*. In this case we have the sequence of submodules*

$$\{0\} \subset M_a(\lambda) \subset M_{a+1}(\lambda) \subset \cdots \subset M_b(\lambda) \subset \mathcal{P}(\mathfrak{p}^+, V),$$

where

$$a=\begin{cases}\lceil\frac{\lambda}{2}\rceil+1 & (3\le\lambda\le 2r),\\ 2 & (-k+1\le\lambda\le 2),\\ 1 & (\lambda\le -k),\end{cases}\qquad b=\begin{cases} r+1 & (2r+1-k\le\lambda\le 2r),\\ r & (\lambda\le 2r-k).\end{cases}$$

If $k=0$, then $M_{\frac{\lambda}{2}+1}(\lambda)$ $(\lambda=0,2,\dots,2r-2)$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_r(\lambda)$ $(\lambda\le 2r-2,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

If $k\ge 1$, then $M_{\frac{\lambda}{2}+1}(\lambda)$ $(\lambda=2,4,\dots,2r)$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_{r+1}(\lambda)$ $(2r+1-k\le\lambda\le 2r,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

(6) *When $G=SO^*(4r+2)$ and $V=V^\vee_{(k/2,\dots,k/2,-k/2)}$ with $k\in\mathbb{Z}_{>0}$, for $\lambda\in\mathbb{R}$ and $j=1,2,\dots,r+1$, we define*

$$M_j(\lambda):=\bigoplus_{m_j-k_j+k<2j-\lambda-1} V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(k/2,\dots,k/2)}\subset\mathcal{P}(\mathfrak{p}^+,V)\ \ (j=1,\dots,r),$$
$$M_{r+1}(\lambda):=\bigoplus_{k-k_{r+1}<2r-\lambda} V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r)+(k/2,\dots,k/2)}\subset\mathcal{P}(\mathfrak{p}^+,V).$$

Then $(d\tau_\lambda,\mathcal{P}(\mathfrak{p}^+,V))$ is reducible if and only if $\lambda\le 2r-1$, $\lambda\in\mathbb{Z}$ and $\lambda\ne 2r-k-1$. In this case we have the sequence of submodules

$$\{0\}\subset M_a(\lambda)\subset M_{a+1}(\lambda)\subset\cdots\subset M_b(\lambda)\subset\mathcal{P}(\mathfrak{p}^+,V),$$

where

$$(a,b)=\begin{cases}(r+1,r+1) & (2r-k\le\lambda\le 2r-1),\\ (\lceil\frac{\lambda+k}{2}\rceil+1,r) & (-k+1\le\lambda\le 2r-2-k),\\ (1,r) & (\lambda\le -k).\end{cases}$$

$M_{r+1}(2r-1)$ and $\mathcal{P}(\mathfrak{p}^+,V)/M_{r+1}(\lambda)$ $(2r-k\le\lambda\le 2r-1,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

(7) *When $G=Spin_0(2,2s)$ and $V=\mathbb{C}_k\boxtimes V_{(k,\dots,k,\pm k)}$ with $k\in\frac{1}{2}\mathbb{Z}_{\ge 0}$, for $\lambda\in\mathbb{R}$ and $j=1,2$, we define*

$$M_1(\lambda):=\bigoplus_{m_1+k+l<1-\lambda}\mathbb{C}_{m_1+m_2+k}\boxtimes V_{(m_1-m_2+l,k,\dots,k,\pm l)}\subset\mathcal{P}(\mathfrak{p}^+,V),$$
$$M_2(\lambda):=\bigoplus_{m_2+k-l<\frac{n}{2}-\lambda}\mathbb{C}_{m_1+m_2+k}\boxtimes V_{(m_1-m_2+l,k,\dots,k,\pm l)}\subset\mathcal{P}(\mathfrak{p}^+,V).$$

Then $(d\tau_\lambda,\mathcal{P}(\mathfrak{p}^+,V))$ is reducible if and only if $\lambda\le s-1$ and $\lambda\in\mathbb{Z}$. In this case we have the sequence of submodules

$$\begin{aligned}\{0\}\subset M_2(\lambda)\subset\mathcal{P}(\mathfrak{p}^+,V)&\qquad(1-2k\le\lambda\le s-1),\\ \{0\}\subset M_1(\lambda)\subset M_2(\lambda)\subset\mathcal{P}(\mathfrak{p}^+,V)&\qquad(\lambda\le -2k).\end{aligned}$$

$M_2(s-1)$, $M_1(0)$ (only when $k=0$), and $\mathcal{P}(\mathfrak{p}^+,V)/M_2(\lambda)$ $(\lambda\le s-1,\ \lambda\in\mathbb{Z})$ are infinitesimally unitary.

(8) *When* $G = Spin_0(2,2s+1)$ *and* $V = \mathbb{C}_k \boxtimes V_{(k,\dots,k)}$ *with* $k = 0, \frac{1}{2}$, *for* $\lambda \in \mathbb{R}$ *and* $j = 1, 2$, *we define*

$$M_1(\lambda) := \bigoplus_{m_1+k+l<1-\lambda} \mathbb{C}_{m_1+m_2+k} \boxtimes V_{(m_1-m_2+l,k,\dots,k,|l|)} \subset \mathcal{P}(\mathfrak{p}^+, V),$$
$$M_2(\lambda) := \bigoplus_{m_2+k-l<\frac{n}{2}-\lambda} \mathbb{C}_{m_1+m_2+k} \boxtimes V_{(m_1-m_2+l,k,\dots,k,|l|)} \subset \mathcal{P}(\mathfrak{p}^+, V).$$

Then $(d\tau_\lambda, \mathcal{P}(\mathfrak{p}^+, V))$ *is reducible if and only if* $\lambda \leq s - \frac{1}{2}$ *and* $\lambda \in \mathbb{Z} + \frac{1}{2}$, *or* $\lambda \leq -2k$ *and* $\lambda \in \mathbb{Z}$. *In this case we have the sequence of submodules*

$$\begin{aligned} &\{0\} \subset M_2(\lambda) \subset \mathcal{P}(\mathfrak{p}^+, V) && (\lambda \leq s - \frac{1}{2},\ \lambda \in \mathbb{Z} + \frac{1}{2}), \\ &\{0\} \subset M_1(\lambda) \subset \mathcal{P}(\mathfrak{p}^+, V) && (\lambda \leq -2k,\ \lambda \in \mathbb{Z}). \end{aligned}$$

$M_2(s - \frac{1}{2})$, $M_1(0)$ *(only when* $k = 0$*), and* $\mathcal{P}(\mathfrak{p}^+, V)/M_2(\lambda)$ $(\lambda \leq s - \frac{1}{2},\ \lambda \in \mathbb{Z} + \frac{1}{2})$ *are infinitesimally unitary.*

By [15, Lemma 4.8], we can determine the associated variety of each subquotient module by comparing the asymptotic K-support of each subquotient module and (2.2.3). In fact, we have

$$\begin{aligned} \mathcal{V}_\mathfrak{g}(M_{l+1}(\lambda)/M_{l\ (\mathrm{or}\ l-1)}(\lambda)) &= \begin{cases} \overline{\mathcal{O}_l} & (l = 0, 1, \dots, r-1), \\ \overline{\mathcal{O}_r} = \mathfrak{p}^+ & (l \geq r), \end{cases} \\ \mathcal{V}_\mathfrak{g}(\mathcal{P}(\mathfrak{p}^+, V)/M_{b\ (\mathrm{or}\ r)}(\lambda)) &= \overline{\mathcal{O}_r} = \mathfrak{p}^+, \end{aligned}$$

where we set $M_0(\lambda) = M_{-1}(\lambda) = \{0\}$, $\mathcal{O}_l$ are defined in (2.2.2), and $r = \operatorname{rank}_\mathbb{R} G$. These and (2.2.4) give the Gelfand-Kirillov dimension of each subquotient module.

$$\begin{aligned} \mathrm{DIM}(M_{l+1}(\lambda)/M_{l\ (\mathrm{or}\ l-1)}(\lambda)) &= \begin{cases} l + \frac{1}{2}l(2r-l-1)d + lb & (l = 0, 1, \dots, r-1), \\ r + \frac{1}{2}r(r-1)d + rb = n & (l \geq r), \end{cases} \\ \mathrm{DIM}(\mathcal{P}(\mathfrak{p}^+, V)/M_{b\ (\mathrm{or}\ r)}(\lambda)) &= r + \frac{1}{2}r(r-1)d + rb = n. \end{aligned}$$

Also, we can show that the smallest submodule $M_a(\lambda)$ is irreducible in any case, by the same argument for the irreducibility of $\mathcal{P}(\mathfrak{p}^+, V)$ for λ generic case. However, we cannot determine whether the other subquotient modules are irreducible or not, by the norm computation, and we need some other techniques to determine the full composition series, such as the techniques used in e.g. [17], [22], [23], or [1].

Acknowledgments

The author would like to thank his supervisor T. Kobayashi, and professor B. Ørsted for a lot of helpful advice on this chapter. He also thanks his colleagues, especially M. Kitagawa for a lot of helpful discussion.

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Chapter 3

Intertwining operators between holomorphic discrete series representations

In this chapter we explicitly construct the G_1-intertwining operator between a holomorphic discrete series representation of some Lie group G and that of some subgroup $G_1 \subset G$. More precisely, we construct a G_1-intertwining projection operator from $\mathcal{H}$ of G onto $\mathcal{H}_1$ of G_1 as a differential operator, in the case $(G, G_1) = (G_0 \times G_0, \Delta G_0)$ and both $\mathcal{H}$, $\mathcal{H}_1$ are of "almost scalar type", and also construct a G_1-intertwining embedding operator from $\mathcal{H}_1$ of G_1 into $\mathcal{H}$ of G as an "infinite-order differential operator", in the case both G, G_1 are classical groups and both $\mathcal{H}$, $\mathcal{H}_1$ are of "almost scalar type". In the actual computation we make use of a series expansion of integral kernels and the result of Faraut-Korányi [5] on norm computation.

Keywords: branching laws; intertwining operators; symmetry breaking operators; symmetric pair; holomorphic discrete series representations; highest weight modules.
AMS subject classification: 22E45; 43A85; 17C30.

3.1 Introduction

The purpose of this chapter is to study the intertwining operator between a holomorphic discrete series representation of some Lie group G and that of some subgroup $G_1 \subset G$, and write down such an operator explicitly.

Let G be a Lie group, G_1 be a subgroup of G, and consider a representation $(\hat{\tau}, \mathcal{H})$ of G. Then it is a fundamental problem to understand how the representation $(\hat{\tau}, \mathcal{H})$ of G behaves when it is restricted to the subgroup G_1. Recently Kobayashi [18] proposed a program for such problems in the following three stages.

(Stage A) Abstract features of the restriction $\hat{\tau}|_{G_1}$.

(Stage B) Branching laws.

(Stage C) Construction of symmetry breaking operators.

In general, the restriction $\hat{\tau}|_{G_1}$ may behave wildly, for example, the multiplicity becomes infinite, or it contains continuous spectrum, even if (G, G_1) is a symmetric pair, and $\hat{\tau}$ is a unitary representation of G. However Kobayashi and his collaborators found conditions for $(G, G_1, \hat{\tau})$ that the restriction $\hat{\tau}|_{G_1}$ behaves nicely, that is, it is discretely decomposable ([9, 11, 12, 14, 22, 23]), its multiplicity becomes finite or uniformly bounded ([17, 19, 21]), or decomposes multiplicity-freely ([13, 15]) (Stage A). Especially, if G is a reductive Lie algebra of *Hermitian type* (i.e. the Riemannian symmetric space G/K has a natural complex structure), (G, G_1) is a symmetric pair of *holomorphic type* (i.e. a symmetric pair such that the embedding map $G_1/K_1 \hookrightarrow G/K$ is holomorphic), and $\hat{\tau}$ is in the nice class of representations, called the *holomorphic discrete series representations* of G, then the restriction $\hat{\tau}|_{G_1}$ decomposes discretely. Moreover, if the holomorphic discrete series representation τ is of *scalar type*, then it decomposes multiplicity-freely. In this case, its branching law

$$\hat{\tau}|_{G_1} \simeq \sum_{\hat{\tau}_1 \in \hat{G}_1}^{\oplus} m(\hat{\tau}, \hat{\tau}_1)\hat{\tau}_1$$

(where $\hat{G}_1$ is the unitary dual of G_1 i.e. the equivalence class of unitary representations of G_1, and $m(\hat{\tau}, \hat{\tau}_1) \in \mathbb{Z}_{\geq 0}$) is also known ([8, 10, 13, 29]) (Stage B). Thus our next interest is to understand the above decomposition explicitly, for example, to construct the G_1-intertwining operator between $\hat{\tau}|_{G_1}$ and $\hat{\tau}_1$ explicitly (Stage C). Such problems have been considered by e.g. Clerc-Kobayashi-Ørsted-Pevzner [1], Kobayashi-Ørsted-Somberg-Souček [20], Kobayashi-Speh [27], Möllers-Ørsted-Oshima [30] and Möllers-Oshima [31] when $\hat{\tau}$ are principal series or complementary series representations, and by e.g. Ibukiyama-Kuzumaki-Ochai [7], Kobayashi-Pevzner [24, 25] and Peng-Zhang [34] when $\hat{\tau}$ are holomorphic discrete series representations. The approach used in [20, 24, 25] is called the "F-method", in which the explicit intertwining operators are determined by solving certain differential equations. This idea first appear in [16]. In this chapter, we also attack this problem when $\hat{\tau}$ are holomorphic discrete series representations, but take an approach different from the F-method, namely, by computing some integrals using series expansion.

Now we review the holomorphic discrete series representations. Let G be a reductive Lie group of Hermitian type, and $K \subset G$ be a maximal compact subgroup. Then there exists a complex subspace $\mathfrak{p}^+ \subset \mathfrak{g}^{\mathbb{C}}$ in the complexified Lie algebra of G and a bounded domain $D \subset \mathfrak{p}^+$ such that the Riemannian symmetric space G/K is diffeomorphic to D, and G/K admits a natural complex structure via this diffeomorphism. Next, let (τ, V) be a finite-dimensional representation of $\tilde{K}^{\mathbb{C}}$, the universal covering group of $K^{\mathbb{C}}$, and consider the space of holomorphic sections of the homogeneous vector bundle $\tilde{G} \times_{\tilde{K}} V$ on G/K. Then since the complex domain $D \simeq G/K$ is contractible, it is isomorphic to the space of V-valued holomorphic functions on D.

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} V) \simeq \mathcal{O}(D, V).$$

Clearly this admits an action of $\tilde{G}$. If (τ, V) is sufficiently "regular", then $\mathcal{O}(D, V)$ admits a $\tilde{G}$-invariant inner product which is given by a converging integral on D. In this case the corresponding Hilbert subspace $\mathcal{H}_\tau(D, V) \subset \mathcal{O}(D, V)$ admits a unitary representation, which is called the *holomorphic discrete series representation.*

We take a subgroup $G_1 \subset G$ which is stable under the Cartan involution of G. We assume that the embedding map $G_1/K_1 \hookrightarrow G/K$ of Riemannian symmetric spaces is holomorphic. Let $\mathfrak{p}_1^+ := \mathfrak{p}^+ \cap \mathfrak{g}_1^{\mathbb{C}}$ be the intersection of $\mathfrak{p}^+$ and the complexfied Lie algebra of G_1, and $\mathfrak{p}_2^+ := (\mathfrak{p}_1^+)^\perp \subset \mathfrak{p}^+$ be the orthogonal complement under a suitable inner product on $\mathfrak{p}^+$. We take a finite dimensional representation (τ_1, V_1) of $\tilde{K}_1^{\mathbb{C}}$, and consider the corresponding holomorphic discrete series representation $\mathcal{H}_{\tau_1}(D_1, V_1)$ of $\tilde{G}_1$. Then $\mathcal{H}_{\tau_1}(D_1, V_1)$ appears in the direct summand of $\mathcal{H}_\tau(D, V)|_{\tilde{G}_1}$ if and only if (τ_1, V_1) appears in the irreducible decomposition of $V \otimes \mathcal{P}(\mathfrak{p}_2^+)$ under K_1, where $\mathcal{P}(\mathfrak{p}_2^+)$ is the space of holomorphic polynomials on $\mathfrak{p}_2^+$. Our aim is to write down the $\tilde{G}_1$ (or $(\mathfrak{g}_1, \tilde{K}_1)$)-intertwining operator between $\mathcal{H}_\tau(D, V)$ and each $\mathcal{H}_{\tau_1}(D_1, V_1)$ explicitly. To do this, we gather such $\mathcal{H}_{\tau_1}(D_1, V_1)$'s, and consider a Hilbert space

$$\mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V) \subset \mathcal{O}(D_1 \times \mathfrak{p}_2^+, V) \approx \mathcal{O}(D_1, \mathcal{P}(\mathfrak{p}_2^+, V))$$

such that each embedding $\mathcal{H}_{\tau_1}(D_1, V_1) \hookrightarrow \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)$ is written easily, and construct the $(\mathfrak{g}_1, \tilde{K}_1)$-intertwining operator between $\mathcal{H}_\tau(D, V)$ and $\mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)$ explicitly.

We calculate the intertwining operator in the following way. First, we find a kernel function $\hat{K}(x; y)$ which is $\tilde{G}_1$-invariant in a suitable sense (Proposition 3.3.1). Then the intertwining operator is given by

$$\begin{aligned} \mathcal{H}_\tau(D, V) \to \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V), \qquad & f \mapsto \langle f, K(\cdot; y)\rangle_{\mathcal{H}_\tau(D,V)}, \\ \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V) \to \mathcal{H}_\tau(D, V), \qquad & g \mapsto \langle g, K(x; \cdot)^*\rangle_{\mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)} \end{aligned}$$

(Corollary 3.3.3). This gives the integral expression of the intertwining operator, and this step is similar to the method used in [30, 26, 27]. However, this expression is a bit complicated. Also, in [24] it is proved that the intertwining operator from $\mathcal{H}_\tau(D, V)$ to $\mathcal{H}_{\tau_1}(D_1, V_1)$ is always given by a differential operator, but we cannot see this fact from the integral expression. Thus we try to rewrite the integral expression to a differential expression by substituting $f(x)$ with $e^{(x|z)}$, $g(y)$ with $e^{(y|w)}$, where $(\cdot|\cdot)$ is a suitable inner product on $\mathfrak{p}^+$. Then we can show that there exists a polynomial $F^*(z_1, z_2; y_2) \in \mathcal{P}(\overline{\mathfrak{p}_1^+ \times \mathfrak{p}_2^+} \times \mathfrak{p}_2^+, \mathrm{End}(V))$ and a function $F(x_2; w_1, w_2) \in \mathcal{O}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_1^+ \times \mathfrak{p}_2^+}, \mathrm{End}(V))$ such that the intertwining operator is given by

$$\begin{aligned} \mathcal{H}_\tau(D, V)_{\tilde{K}} \to \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)_{\tilde{K}_1}, \quad & f(x) \mapsto F^*\left(\overline{\frac{\partial}{\partial x_1}}, \overline{\frac{\partial}{\partial x_2}}; y_2\right)\bigg|_{x_1 = y_1, x_2 = 0} f(x), \\ \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)_{\tilde{K}_1} \to \mathcal{H}_\tau(D, V)_{\tilde{K}}, \quad & g(y) \mapsto F\left(x_2; \overline{\frac{\partial}{\partial y_1}}, \overline{\frac{\partial}{\partial y_2}}\right)\bigg|_{y_1 = x_1, y_2 = 0} g(y) \end{aligned}$$

(Theorem 3.3.5). The latter operator is of infinite order in general, but when g is $\tilde{K}_1$-finite i.e. is a polynomial, then it becomes a finite sum. The functions F and F^* are given by an explicit integral, and actual computation of F and F^* is performed in Section 3.5 case by case, by using the series expansion of integrands and the result of Faraut-Korányi [5] on norm computation. In this way, the author has got the explicit intertwining operators $\mathcal{H}_\tau(D, V) \rightleftarrows \mathcal{H}_{\tau_1}(D_1, V_1)$ in the case

$$\begin{aligned} (G, G_1) = (U(q, s), U(q, s') \times U(s'')),\ & (SO^*(2s), SO^*(2(s-1)) \times SO(2)), \\ & (SO(2, 2s), U(1, s)), \end{aligned}$$

which are given by normal derivatives, the operators $\mathcal{H}_\tau(D, V) \to \mathcal{H}_{\tau_1}(D_1, V_1)$ in the case

$$(G, G_1) = (G_0 \times G_0, \Delta G_0)$$

where G_0 is a simple Lie group of Hermitian type, when (τ, V) is scalar and (τ_1, V_1) is "almost scalar", which gives essentially the same result with [34], and the operators $\mathcal{H}_{\tau_1}(D_1, V_1) \to \mathcal{H}_\tau(D, V)$ in the case

$$\begin{aligned} (G, G_1) = (Sp(s, \mathbb{R}), Sp(s', \mathbb{R}) \times Sp(s'', \mathbb{R})), \quad & (U(q, s), U(q', s') \times U(q'', s'')), \\ (SO^*(2s), SO^*(2s') \times SO^*(2s'')), \quad & (Sp(s, \mathbb{R}), U(s', s'')), \\ (SO^*(2s), U(s', s'')), \quad & (SU(s, s), Sp(s, \mathbb{R})), \\ (SU(s, s), SO^*(2s)), \quad & (SO(2, n), SO(2, n') \times SO(n - n')), \end{aligned}$$

when (τ, V) is scalar and (τ_1, V_1) is "almost scalar".

This chapter is organized as follows. In Section 3.2 we prepare some notations and review some facts on Lie algebras of Hermitian type, Jordan triple systems, and holomorphic discrete series representations. In Section 3.3 we construct a general theory on the intertwining operators between holomorphic discrete series representations. In Section 3.4, as a preparation for case by case analysis, we fix the explicit realization of classical Lie groups, and observe series expansions of some functions. In Section 3.5 we compute the explicit intertwining operators by using the result of Section 3.3 and 3.4.

3.2 Preliminaries for general theory

3.2.1 Root systems

Let $\mathfrak{g}$ be a reductive Lie algebra with Cartan involution ϑ. We decompose $\mathfrak{g}$ into a sum of simple and abelian subalgebras as

$$\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \cdots \oplus \mathfrak{g}_{(m)} \oplus \mathfrak{z}(\mathfrak{g}).$$

We assume that each simple subalgebra $\mathfrak{g}_{(i)}$ is of Hermitian type, that is, its maximal compact subalgebra $\mathfrak{k}_{(i)} := \mathfrak{g}_{(i)}^{\vartheta}$ has a 1-dimensionla center $\mathfrak{z}(\mathfrak{k}_{(i)})$, and also that the abelian part $\mathfrak{z}(\mathfrak{g})$ is fixed by ϑ. For each i, we fix an element $z_{(i)} \in \mathfrak{z}(\mathfrak{k}_{(i)})$ such that $ad(z_{(i)})$ has eigenvalues $+\sqrt{-1}$, 0, $-\sqrt{-1}$, and decompose the complexified Lie algebra $\mathfrak{g}_{(i)}^{\mathbb{C}}$ into eigenspaces under $ad(z_{(i)})^{\mathbb{C}}$ as

$$\mathfrak{g}_{(i)}^{\mathbb{C}} = \mathfrak{p}_{(i)}^{+} \oplus \mathfrak{k}_{(i)}^{\mathbb{C}} \oplus \mathfrak{p}_{(i)}^{-}.$$

We denote

$$\begin{aligned} \mathfrak{p}^{+} &:= \mathfrak{p}_{(1)}^{+} \oplus \cdots \oplus \mathfrak{p}_{(m)}^{+}, & \mathfrak{k}^{\mathbb{C}} &:= \mathfrak{k}_{(1)}^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{k}_{(m)}^{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{g})^{\mathbb{C}}, \\ \mathfrak{p}^{-} &:= \mathfrak{p}_{(1)}^{-} \oplus \cdots \oplus \mathfrak{p}_{(m)}^{-}, & \mathfrak{k} &:= \mathfrak{k}_{(1)} \oplus \cdots \oplus \mathfrak{k}_{(m)} \oplus \mathfrak{z}(\mathfrak{g}) = \mathfrak{g}^{\vartheta}, \end{aligned}$$

so that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{-}.$$

We denote the anti-holomorphic extension of the Cartan involution ϑ on $\mathfrak{g}^{\mathbb{C}}$ by the same symbol ϑ. Also, let $\hat{\vartheta} := \vartheta \circ Ad(e^{\pi z})$ $(z := \sum_i z_{(i)})$ be the anti-holomorphic involution on $\mathfrak{g}^{\mathbb{C}}$ fixing $\mathfrak{g}$.

Next, we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$. Then $\mathfrak{h}^{\mathbb{C}}$ automatically becomes a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We set $\mathfrak{h}_{(i)} := \mathfrak{h} \cap \mathfrak{g}_{(i)}$. Let $\Delta_{\mathfrak{g}_{(i)}^{\mathbb{C}}} = \Delta(\mathfrak{g}_{(i)}^{\mathbb{C}}, \mathfrak{h}_{(i)}^{\mathbb{C}})$ be the root system of $\mathfrak{g}_{(i)}^{\mathbb{C}}$, and let $\Delta_{\mathfrak{p}_{(i)}^{\pm}}$, $\Delta_{\mathfrak{k}_{(i)}^{\mathbb{C}}}$ be the set of roots such that the corresponding root space is contained in $\mathfrak{p}_{(i)}^{\pm}$, $\mathfrak{k}_{(i)}^{\mathbb{C}}$ respectively. We fix a positive system $\Delta_{\mathfrak{g}_{(i)}^{\mathbb{C}},+} \subset \Delta_{\mathfrak{g}_{(i)}^{\mathbb{C}}}$ such that $\Delta_{\mathfrak{p}_{(i)}^{+}} \subset \Delta_{\mathfrak{g}_{(i)}^{\mathbb{C}},+}$, and denote $\Delta_{\mathfrak{k}_{(i)}^{\mathbb{C}},+} := \Delta_{\mathfrak{k}_{(i)}^{\mathbb{C}}} \cap \Delta_{\mathfrak{g}_{(i)}^{\mathbb{C}},+}$. Then we can take a system of strongly orthogonal roots $\{\gamma_{1,(i)}, \ldots, \gamma_{r_{(i)},(i)}\} \subset \Delta_{\mathfrak{p}_{(i)}^{+}}$, where $r_{(i)} = \operatorname{rank}_{\mathbb{R}} \mathfrak{g}_{(i)}$, such that

(1) $\gamma_{1,(i)}$ is the highest root in $\Delta_{\mathfrak{p}_{(i)}^{+}}$,

(2) $\gamma_{k,(i)}$ is the root in $\Delta_{\mathfrak{p}_{(i)}^{+}}$ which is highest among the roots strongly orthogonal to each $\gamma_{j,(i)}$ with $1 \leq j \leq k-1$.

For each j, let $\mathfrak{p}^+_{jj,(i)}$ be the root space corresponding to $\gamma_{j,(i)}$. We take an element $e_{j,(i)} \in \mathfrak{p}^+_{jj,(i)}$ such that

$$-[[e_{j,(i)}, \vartheta e_{j,(i)}], e_{j,(i)}] = 2e_{j,(i)},$$

and set

$$h_{j,(i)} := -[e_{j,(i)}, \vartheta e_{j,(i)}] \in \sqrt{-1}\mathfrak{h}_{(i)}, \qquad e_{(i)} := \sum_{j=1}^{r_{(i)}} e_{j,(i)} \in \mathfrak{p}^+_{(i)}, \qquad e := \sum_{i=1}^{m} e_{(i)} \in \mathfrak{p}^+,$$

$$\mathfrak{a}_{\mathfrak{l},(i)} := \bigoplus_{j=1}^{r_{(i)}} \mathbb{R}h_{j,(i)} \subset \sqrt{-1}\mathfrak{h}_{(i)}, \qquad \mathfrak{a}^+_{(i)} := \bigoplus_{j=1}^{r_{(i)}} \mathbb{R}e_{j,(i)} \subset \mathfrak{p}^+_{(i)}.$$

Then the restricted root system $\Sigma = \Sigma(\mathfrak{g}^{\mathbb{C}}_{(i)}, \mathfrak{a}^{\mathbb{C}}_{\mathfrak{l},(i)})$ is one of

$$\Sigma = \left\{ \frac{1}{2}(\gamma_{j,(i)} - \gamma_{k,(i)})\Big|_{\mathfrak{a}_{\mathfrak{l},(i)}} : \begin{array}{c} 1 \le j,k \le r_{(i)}, \\ j \ne k \end{array} \right\} \cup \left\{ \pm\frac{1}{2}(\gamma_{j,(i)} + \gamma_{k,(i)})\Big|_{\mathfrak{a}_{\mathfrak{l},(i)}} : 1 \le j \le k \le r_{(i)} \right\}$$

(type $C_{r_{(i)}}$), or

$$\Sigma = (\text{as above}) \cup \left\{ \pm\frac{1}{2}\gamma_{j,(i)}\Big|_{\mathfrak{a}_{\mathfrak{l},(i)}} : 1 \le j \le r_{(i)} \right\}$$

(type $BC_{r_{(i)}}$). For $1 \le j \le k \le r_{(i)}$ we set

$$\mathfrak{p}^+_{jk,(i)} := \left\{ x \in \mathfrak{p}^+_{(i)} : ad(l)x = \frac{1}{2}(\gamma_{j,(i)} + \gamma_{k,(i)})(l)x \text{ for all } l \in \mathfrak{a}_{\mathfrak{l},(i)} \right\},$$
$$\mathfrak{p}^+_{0j,(i)} := \left\{ x \in \mathfrak{p}^+_{(i)} : ad(l)x = \frac{1}{2}\gamma_{j,(i)}(l)x \text{ for all } l \in \mathfrak{a}_{\mathfrak{l},(i)} \right\}.$$

Then we have

$$\mathfrak{p}^+_{(i)} = \bigoplus_{\substack{0 \le j \le k \le r_{(i)} \\ (j,k) \ne (0,0)}} \mathfrak{p}^+_{jk,(i)}.$$

We set

$$\mathfrak{p}^+_{\mathrm{T},(i)} := \bigoplus_{1 \le j \le k \le r_{(i)}} \mathfrak{p}^+_{jk,(i)}, \qquad \mathfrak{p}^-_{\mathrm{T},(i)} := \vartheta\mathfrak{p}^+_{\mathrm{T},(i)}, \qquad \mathfrak{p}^+_{\mathrm{T}} := \bigoplus_{i=1}^{m} \mathfrak{p}^+_{\mathrm{T},(i)},$$
$$\mathfrak{k}^{\mathbb{C}}_{\mathrm{T},(i)} := [\mathfrak{p}^+_{\mathrm{T},(i)}, \mathfrak{p}^-_{\mathrm{T},(i)}], \qquad \mathfrak{k}_{\mathrm{T},(i)} := \mathfrak{k}^{\mathbb{C}}_{\mathrm{T},(i)} \cap \mathfrak{k}_{(i)},$$
$$\mathfrak{g}^{\mathbb{C}}_{\mathrm{T},(i)} := \mathfrak{p}^+_{\mathrm{T},(i)} \oplus \mathfrak{k}^{\mathbb{C}}_{\mathrm{T},(i)} \oplus \mathfrak{p}^-_{\mathrm{T},(i)}, \qquad \mathfrak{g}_{\mathrm{T},(i)} := \mathfrak{g}^{\mathbb{C}}_{\mathrm{T},(i)} \cap \mathfrak{g}_{(i)},$$

and we define the integers

$$d_{(i)} := \dim \mathfrak{p}^+_{12,(i)}, \qquad b_{(i)} := \dim \mathfrak{p}^+_{01,(i)},$$
$$n_{(i)} := \dim \mathfrak{p}^+_{(i)} = r_{(i)} + \frac{1}{2}r_{(i)}(r_{(i)} - 1)d_{(i)} + b_{(i)}r_{(i)},$$
$$n := \dim \mathfrak{p}^+ = \sum_{i=1}^{m} n_{(i)},$$
$$n_{\mathrm{T},(i)} := \dim \mathfrak{p}^+_{\mathrm{T},(i)} = r_{(i)} + \frac{1}{2}r_{(i)}(r_{(i)} - 1)d_{(i)},$$
$$p_{(i)} := 2 + (r_{(i)} - 1)d_{(i)} + b_{(i)}.$$

Throughout the chapter, let $G^{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and let G, $K^{\mathbb{C}}$, K, $G^{\mathbb{C}}_{(i)}$, $G_{(i)}$, $K^{\mathbb{C}}_{(i)}$, $K_{(i)}$, $G^{\mathbb{C}}_{\mathrm{T},(i)}$, $G_{\mathrm{T},(i)}$, $K^{\mathbb{C}}_{\mathrm{T},(i)}$, $K_{\mathrm{T},(i)}$ be the connected Lie subgroup with Lie algebras $\mathfrak{g}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{k}$, $\mathfrak{g}^{\mathbb{C}}_{(i)}$, $\mathfrak{g}_{(i)}$, $\mathfrak{k}^{\mathbb{C}}_{(i)}$, $\mathfrak{k}_{(i)}$, $\mathfrak{g}^{\mathbb{C}}_{\mathrm{T},(i)}$, $\mathfrak{g}_{\mathrm{T},(i)}$, $\mathfrak{k}^{\mathbb{C}}_{\mathrm{T},(i)}$, $\mathfrak{k}_{\mathrm{T},(i)}$ respectively. Also, let

$$K_{L,(i)} := \{k \in K_{\mathrm{T},(i)} : Ad(k)e_{(i)} = e_{(i)}\},$$

which is possibly non-connected, and we denote its Lie algebra by $\mathfrak{k}_{\mathfrak{l},(i)}$.

For $k \in K^{\mathbb{C}}$, we write $k^* := (\vartheta k)^{-1}$. Then for each i, there exists a unique Hermitian inner product $(\cdot|\cdot)_{\mathfrak{p}^+_{(i)}}$, holomorphic in the first variable and anti-holomorphic in the second variable, such that

$$\begin{aligned} &(Ad(k)x|y)_{\mathfrak{p}^+_{(i)}} = (x|Ad(k^*)y)_{\mathfrak{p}^+_{(i)}} \qquad (x, y \in \mathfrak{p}^+_{(i)},\ k \in K^{\mathbb{C}}_{(i)}), \\ &(e_{1,(i)}|e_{1,(i)})_{\mathfrak{p}^+_{(i)}} = 1. \end{aligned}$$

This is proportional to the restriction of the Killing form of $\mathfrak{g}^{\mathbb{C}}_{(i)}$ on $\mathfrak{p}^+_{(i)} \times \mathfrak{p}^-_{(i)}$, if we identify $\mathfrak{p}^+_{(i)}$ and $\mathfrak{p}^-_{(i)}$ through ϑ. By summing these inner products, we define

$$(x|y) = (x|y)_{\mathfrak{p}^+} := \sum_{i=1}^{m} (x_i|y_i)_{\mathfrak{p}^+_{(i)}} \qquad \left(x = \sum_{i=1}^{m} x_i, y = \sum_{i=1}^{m} y_i \in \mathfrak{p}^+ = \bigoplus_{i=1}^{m} \mathfrak{p}^+_{(i)} \right). \quad (3.2.1)$$

From now on we omit Ad or ad if there is no confusion, so that $(kx|y)_{\mathfrak{p}^+} = (x|k^*y)_{\mathfrak{p}^+}$.

3.2.2 Operations on Jordan triple systems

$\mathfrak{p}^+$ has a Hermitian positive Jordan triple system structure with the product

$$(x, y, z) \mapsto -\frac{1}{2}[[x, \vartheta y], z].$$

We recall that, for $x, y \in \mathfrak{p}^+$, the *Bergman operator* $B(x, y) \in \mathrm{End}(\mathfrak{p}^+)$ is defined as

$$B(x, y) := I + ad([x, \vartheta y]) + \frac{1}{4} ad(x)^2 ad(\vartheta y)^2 \Big|_{\mathfrak{p}^+} \in \mathrm{End}(\mathfrak{p}^+).$$

We say $(x, y) \in \mathfrak{p}^+ \times \mathfrak{p}^+$ is *quasi-invertible* if $B(x, y)$ (or equivalently $B(y, x)$) is invertible, and in this case the *quasi-inverse* x^y is defined as

$$x^y := B(x, y)^{-1} \left(x + \frac{1}{2} ad(x)^2 \vartheta y \right) \in \mathfrak{p}^+.$$

Then if $B(x, y)$ is invertible, then there exists an element $k \in K^{\mathbb{C}}$ such that $B(x, y)z = Ad(k)z$ holds for any $z \in \mathfrak{p}^+$. Also, $B(x, y)$ and x^y satisfy the following properties. For $x, y, z \in \mathfrak{p}^+$ and $k \in K^{\mathbb{C}}$, if (x, y) is quasi-invertible, then

$$B(kx, k^{*-1}y) = kB(x, y)k^{-1}, \quad (3.2.2)$$

$$B(x, y)B(x^y, z) = B(x, y + z) \quad \text{[4, Part V, Proposition III.3.1, (J6.4)]}, \quad (3.2.3)$$

$$B(z, x^y)B(y, x) = B(y + z, x) \quad \text{[4, Part V, Proposition III.3.1, (J6.4')]}, \quad (3.2.4)$$

$$(kx)^{k^{*-1}y} = k(x^y), \quad (3.2.5)$$

$$x^{y+z} = (x^y)^z \quad \text{[4, Part V, Theorem III.5.1(i)]}, \quad (3.2.6)$$

$$(x + z)^y = x^y + B(x, y)^{-1} z^{(y^x)} \quad \text{[4, Part V, Theorem III.5.1(ii)]} \quad (3.2.7)$$

holds. Here, the equality (3.2.6) holds when one of $(x, y+z)$ or (x^y, z) is quasi-invertible, and the other also becomes quasi-invertible. Similarly, the equality (3.2.7) holds when one of $(x+z, y)$ or (z, y^x) is quasi-invertible, and then the other also is. Next, for each i, let $h_{(i)}(x,y) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+})$ be the generic norm on $\mathfrak{p}^+_{(i)}$. This is the polynomial, holomorphic in x and anti-holomorphic in y, satisfying

$$\mathrm{Det}_{\mathfrak{p}^+_{(i)}}(B(x_i, y_i)) = h_{(i)}(x_i, y_i)^{p_{(i)}} \qquad (x_i, y_i \in \mathfrak{p}^+_{(i)}).$$

If $x_i = \sum_{j=1}^{r_{(i)}} a_j e_{j,(i)}$, $y_i = \sum_{j=1}^{r_{(i)}} b_j e_{j,(i)} \in \mathfrak{a}^+_{(i)} \subset \mathfrak{p}^+_{(i)}$, then $h_{(i)}(x_i, y_i)$ is given by

$$h_{(i)}(x_i, y_i) = \prod_{j=1}^{r_{(i)}} (1 - a_j \overline{b_j}).$$

For later use we abbreviate

$$\mathrm{Det}_{\mathfrak{p}^+}(B(x,y))^{-1} = \prod_{i=1}^{m} h_{(i)}(x_i, y_i)^{-p_{(i)}} =: h(x,y)^{-p}.$$

Also, we abbreviate $B(x,x) =: B(x)$, $h_{(i)}(x_i, x_i) = h_{(i)}(x_i)$. Let

$$D := (\text{connected component of } \{x \in \mathfrak{p}^+ : B(x) \text{ is positive definite.}\} \text{ which contains } 0) \tag{3.2.8}$$

be the bounded symmetric domain, which is diffeomorphic to G/K via the Borel embedding which we will review later. Then if $x, y \in D$, $B(x,y)$ is invertible, and thus it is in the image of $K^{\mathbb{C}}$. Moreover, since D is simply connected, there exists a holomorphic map $\tilde{B} : D \times \overline{D} \to K^{\mathbb{C}}$ (or $\tilde{B} : D \times \overline{D} \to \tilde{K}^{\mathbb{C}}$, where $\tilde{K}^{\mathbb{C}}$ is the universal covering group of $K^{\mathbb{C}}$) such that

$$Ad(\tilde{B}(x,y)) = B(x,y) \in \mathrm{End}(\mathfrak{p}^+), \qquad \tilde{B}(0,0) = \mathbf{1}_{K^{\mathbb{C}}} \in K^{\mathbb{C}} \text{ (resp. } \in \tilde{K}^{\mathbb{C}})$$

holds. From now on we omit the tilde, and use the same symbol B instead of $\tilde{B}$.

Next we consider $\mathfrak{p}^+_{\mathrm{T}}$. This has a complex Jordan algebra structure with the product

$$(x,y) \mapsto x \cdot y := -\frac{1}{2}[[x, \vartheta e], y].$$

We recall the quadratic map $P : \mathfrak{p}^+_{\mathrm{T}} \to \mathrm{End}(\mathfrak{p}^+_{\mathrm{T}})$ by

$$P(x)y := 2x \cdot (y \cdot x) - y \cdot (x \cdot x) = \frac{1}{4} ad(x)^2 ad(\vartheta e) y \qquad (x, y \in \mathfrak{p}^+_{\mathrm{T}}).$$

If y is in the real form $\left\{y \in \mathfrak{p}^+_{\mathrm{T}} : \frac{1}{2} ad(e)^2 \vartheta y = y\right\}$ of $\mathfrak{p}^+_{\mathrm{T}}$, then $P(x)y = -\frac{1}{2}[[x, \vartheta y], x]$ holds. Next we review the determinant polynomials on Jordan algebras. On each simple component $\mathfrak{p}^+_{\mathrm{T},(i)}$ there exists a determinant polynomial $\Delta_{(i)}$, which is the homogeneous polynomial of degree $r_{(i)}$ satisfying

$$\Delta_{(i)}(kx) = \Delta_{(i)}(k e_{(i)}) \Delta_{(i)}(x) \quad \text{for all } k \in K^{\mathbb{C}}_{\mathrm{T},(i)},\ x \in \mathfrak{p}^+_{\mathrm{T},(i)},$$
$$\Delta_{(i)}(e_{(i)}) = 1.$$

The quadratic map P and the determinant polynomials are related as

$$\mathrm{Det}_{\mathfrak{p}^+_{\mathrm{T},(i)}}(P(x_i)) = \Delta(x_i)^{2n_{\mathrm{T},(i)}/r_{(i)}} \qquad (x_i \in \mathfrak{p}^+_{\mathrm{T},(i)}).$$

We extend $\Delta_{(i)}$ on $\mathfrak{p}^+_{(i)}$ such that it does not depend on $(\mathfrak{p}^+_{\mathrm{T},(i)})^\perp = \bigoplus_{j=1}^{r_{(i)}} \mathfrak{p}^+_{0j,(i)}$, and denote by the same symbol $\Delta_{(i)}$. Then the determinant polynomial $\Delta_{(i)}$ and the generic norm $h_{(i)}$ are related as

$$\Delta_{(i)}(e_{(i)} - x) = h_{(i)}(x, e_{(i)}) \qquad (x \in \mathfrak{p}^+_{(i)}).$$

3.2.3 Polynomials on Jordan triple systems

Let $\mathcal{P}(\mathfrak{p}^+)$ be the space of all holomorphic polynomials on $\mathfrak{p}^+$. Then $K^{\mathbb{C}}$ acts on $\mathcal{P}(\mathfrak{p}^+)$ by

$$(Ad|_{\mathfrak{p}^+})^*(k)f(x) := f(k^{-1}x) \qquad (k \in K^{\mathbb{C}},\ f \in \mathcal{P}(\mathfrak{p}^+)).$$

Then clearly we have $\mathcal{P}(\mathfrak{p}^+) \simeq \mathcal{P}(\mathfrak{p}^+_{(1)}) \otimes \cdots \otimes \mathcal{P}(\mathfrak{p}^+_{(m)})$, according to the simple decomposition of the Jordan triple system $\mathfrak{p}^+ = \mathfrak{p}^+_{(1)} \oplus \cdots \oplus \mathfrak{p}^+_{(m)}$. In the rest of this subsection, we assume $\mathfrak{g}$ is simple, and we drop the subscript (i). We set

$$\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r : m_1 \geq \cdots \geq m_r \geq 0\}.$$

Then $\mathcal{P}(\mathfrak{p}^+)$ is decomposed as follows.

Theorem 3.2.1 (Hua-Kostant-Schmid, [4, Part III, Theorem V.2.1])**.** *Under $K^{\mathbb{C}}$-action, $\mathcal{P}(\mathfrak{p}^+)$ is decomposed as*

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$$

where $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ is the irreducible representation of $K^{\mathbb{C}}$ with lowest weight $-m_1\gamma_1 - \cdots - m_r\gamma_r$. Moreover, each $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ has a nonzero K_L-invariant polynomial, which is unique up to scalar multiple.

Let $d^{(d,r,b)}_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, and let $\Phi^{(d,r)}_{\mathbf{m}}$ be the K_L-invariant polynomial in $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ such that $\Phi^{(d,r)}_{\mathbf{m}}(e) = 1$. Especially, when $\mathbf{m} = (m, \ldots, m)$, then $\Phi^{(d,r)}_{(m,\ldots,m)}(x) = \Delta(x)^m$ holds.

Next we recall the *Fischer inner product*. For two holomorphic polynomials $f, g \in \mathcal{P}(\mathfrak{p}^+)$, it is defined as

$$\langle f, g \rangle_F := \frac{1}{\pi^n} \int_{\mathfrak{p}^+} f(x)\overline{g(x)} e^{-|x|^2_{\mathfrak{p}^+}} dx.$$

This integral converges for any polynomial f, g, and the reproducing kernel is given by $e^{(x|y)_{\mathfrak{p}^+}}$. Let $K_{\mathbf{m}}(x, y) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+})$ be the reproducing kernel of $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ with respect to $\langle \cdot, \cdot \rangle_F$, so that $\sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} K_{\mathbf{m}}(x, y) = e^{(x|y)_{\mathfrak{p}^+}}$. Then the following holds.

Proposition 3.2.2 ([4, Part III, Lemma V.3.1(a), Theorem V.3.4])**.**

$$K_{\mathbf{m}}(x, e) = \frac{d^{(d,r,b)}_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m},d}} \Phi^{(d,r)}_{\mathbf{m}}(x).$$

Here, $(\lambda)_{\mathbf{m},d}$ is defined as

$$(\lambda)_{\mathbf{m},d} := \prod_{j=1}^{r} \left(\lambda - \frac{d}{2}(j-1)\right)_{m_j}, \qquad (\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1). \tag{3.2.9}$$

According to [32], we renormalize $\Phi^{(d,r)}_{\mathbf{m}}$ as

$$\tilde{\Phi}^{(d)}_{\mathbf{m}}(x) := |\mathbf{m}|! \frac{d^{(d,r,b)}_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m},d}} \Phi^{(d,r)}_{\mathbf{m}}(x),$$

so that

$$e^{(x|e)_{\mathfrak{p}^+}} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} K_{\mathbf{m}}(x,e) = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(x).$$

Then $\tilde{\Phi}^{(d)}_{\mathbf{m}}(x)$ does not depend on r in the following sense. Since $\tilde{\Phi}^{(d)}_{\mathbf{m}}$ is K_L-invariant, it is determined by the value on $\mathfrak{a}^+ \subset \mathfrak{p}^+$. Thus for $x = a_1e_1 + \cdots + a_re_r \in \mathfrak{a}^+$, we write

$$\tilde{\Phi}^{(d)}_{\mathbf{m}}(x) =: \tilde{\Phi}^{(d)}_{\mathbf{m}}(a_1,\dots,a_r).$$

Then this does not depend on r, that is,

$$\tilde{\Phi}^{(d,r)}_{\mathbf{m}}(a_1,\dots,a_{r-1},0) = \tilde{\Phi}^{(d,r-1)}_{\mathbf{m}}(a_1,\dots,a_{r-1})$$

holds.

Next we recall the Laplace-Beltrami operator from [6, Proposition VI.4.1]. This is a differential operator on the real form $\left\{x\in\mathfrak{p}^+_{\mathrm{T}} : \frac{1}{2}ad(e)^2\vartheta x = x\right\}$ of $\mathfrak{p}^+_{\mathrm{T}}$. We extend this operator to a $K^{\mathbb{C}}$-invariant differential operator on $\mathfrak{p}^+$, so that

$$L := \frac{1}{2}\sum_{\alpha\beta}([[x,-\vartheta e_\alpha],x]|e_\beta)\frac{\partial^2}{\partial x_\alpha\partial x_\beta} + \frac{n_{\mathrm{T}}}{r}\sum_\alpha (x|e_\alpha)\frac{\partial}{\partial x_\alpha}, \tag{3.2.10}$$

where $\{e_\alpha\}\subset\mathfrak{p}^+$ is a basis of $\mathfrak{p}^+$, with the dual basis $\{e^\vee_\alpha\}\subset\mathfrak{p}^+$, and $\frac{\partial}{\partial x_\alpha}$ is the directional derivative along $e^\vee_\alpha$. Then this has the following properties.

Proposition 3.2.3. *(1) ([6, Proposition VI.4.2]) If f is a K_L-invariant function, then using the coordinate $x = a_1e_1+\cdots+a_re_r\in\mathfrak{a}^+$, we have*

$$Lf = \sum_{j=1}^r a_j^2\frac{\partial^2 f}{\partial a_j^2} + d\sum_{j<k}\frac{a_ja_k}{a_j-a_k}\left(\frac{\partial f}{\partial a_j}-\frac{\partial f}{\partial a_k}\right) + \frac{n_{\mathrm{T}}}{r}\sum_{j=1}^r a_j\frac{\partial f}{\partial a_j}.$$

(2) (Corollary of [6, Proposition VI.4.4]) If $f\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, then f is an eigenfunction of L with eigenvalue $\sum_{j=1}^r\left(m_j^2 - \frac{d}{2}(2j-r-1)m_j\right)$.

3.2.4 Holomorphic discrete series representations

In this subsection we recall the explicit realization of the holomorphic discrete series representation of the universal covering group $\tilde{G}$. First we recall the Borel embedding,

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, dashed, "\wr"] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

where $P^\pm := \exp(\mathfrak{p}^\pm)$. When $g\in G^{\mathbb{C}}$ and $x\in\mathfrak{p}^+$ satisfy $g\exp(x)\in P^+K^{\mathbb{C}}P^-$, we write

$$g\exp(x) = \exp(\pi^+(g,x))\kappa(g,x)\exp(\pi^-(g,x)),$$

where $\pi^+(g,x)\in\mathfrak{p}^+$, $\kappa(g,x)\in K^{\mathbb{C}}$, and $\pi^-(g,x)\in\mathfrak{p}^-$. If $g=k\in K^{\mathbb{C}}$, $g=\exp(y)\in P^+$ or $g=\exp(\vartheta y)\in P^-$ with $y\in\mathfrak{p}^+$, we have

$$\begin{aligned} \pi^+(k,x) &= kx, & \kappa(k,x) &= k,\\ \pi^+(\exp(y),x) &= x+y, & \kappa(\exp(y),x) &= \mathbf{1}_{K^{\mathbb{C}}},\\ \pi^+(\exp(\vartheta y),x) &= x^y, & Ad(\kappa(\exp(\vartheta y),x))|_{\mathfrak{p}^+} &= B(x,y)^{-1}. \end{aligned}$$

π^+ gives the birational action of $G^{\mathbb{C}}$ on $\mathfrak{p}^+$, and from now on we abbreviate $\pi^+(g,x) =: gx$. Especially, if $x \in D$ and $g \in G$, then automatically $gx \in D$ and $\kappa(g,x)$ is well-defined, and the action of G on D is transitive. Since D is simply connected, the map $\kappa : G \times D \to K^{\mathbb{C}}$ lifts to the universal covering space, that is, $\kappa : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$ is well-defined. We denote this extended map by the same symbol κ. Then for $x, y \in \mathfrak{p}^+$ and $g \in G^{\mathbb{C}}$,

$$B(gx, (\hat{\vartheta}g)y) = \kappa(g,x)B(x,y)\kappa(\hat{\vartheta}g, y)^* \tag{3.2.11}$$

holds in $\mathrm{End}(\mathfrak{p}^+)$, where $\hat{\vartheta}$ is the anti-holomorphic involution of $G^{\mathbb{C}}$ fixing G, and Ad is omitted. If $g \in G$ (i.e. $g = \hat{\vartheta}g$) and $x, y \in D$, this also holds in $K^{\mathbb{C}}$, regarding $B(x,y)$ as the element of $K^{\mathbb{C}}$. This formula is also verified in $\tilde{K}^{\mathbb{C}}$ if $g \in \tilde{G}$.

Now let (τ, V) be an irreducible holomorphic representation of $\tilde{K}^{\mathbb{C}}$ with $\tilde{K}$-invariant inner product $(\cdot,\cdot)_\tau$. We consider the space of holomorphic sections of the vector bundle on G/K with fiber V. Then since $D \simeq G/K$ is contractible, it is isomorphic to the space of V-valued holomorphic functions on D.

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} V) \simeq \mathcal{O}(D, V).$$

Via this identification, $\tilde{G}$ acts on $\mathcal{O}(D,V)$ by

$$\hat{\tau}(g)f(x) = \tau(\kappa(g^{-1}, x))^{-1} f(g^{-1}x) \qquad (g \in \tilde{G}, x \in D, f \in \mathcal{O}(D,V)).$$

Then since the G-invariant measure on D is given by $h(x)^{-p}dx := \prod_{i=1}^m h_{(i)}(x_i)^{-p_{(i)}}dx = \mathrm{Det}(B(x))^{-1}dx$, $\tilde{G}$ preserves the *weighted Bergman inner product*

$$\langle f, g\rangle_{\hat{\tau}} := \int_D \left(\tau(B(x)^{-1})f(x), g(x)\right)_\tau h(x)^{-p}dx.$$

Let $\mathcal{H}_\tau(D,V)$ be the space of all functions $f \in \mathcal{O}(D,V)$ such that $\|f\|_{\hat{\tau}} < \infty$. If $\mathcal{H}_\tau(D,V)$ is non-trivial, then we call the unitary representation $(\hat{\tau}, \mathcal{H}_\tau(D,V))$ of $\tilde{G}$ the holomorphic discrete series representation. In this case, the space of $\tilde{K}$-finite vectors is equal to the space of polynomials,

$$\mathcal{H}_\tau(D,V)_{\tilde{K}} = \mathcal{O}(D,V)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+, V),$$

and the reproducing kernel of $(\hat{\tau}, \mathcal{H}_\tau(D,V))$ is proportional to $\tau(B(x,y))$.

Now we assume G is simple. Let χ be the character of $\tilde{K}^{\mathbb{C}}$ such that $\chi(k)^p = \mathrm{Det}(Ad(k)|_{\mathfrak{p}^+})$, or $\chi(B(x,y)) = h(x,y)$. Let (τ_0, V) be a fixed irreducleble representation of $K^{\mathbb{C}}$. Then for $\lambda \in \mathbb{R}$, $(\tau, V) = (\tau_0 \otimes \chi^{-\lambda}, V)$ is again a representation of $\tilde{K}^{\mathbb{C}}$. In this case we denote $\mathcal{H}_\tau(D,V) =: \mathcal{H}_\lambda(D,V)$. Then $\mathcal{H}_\lambda(D,V)$ is non-zero if λ is sufficiently large, and the reproducing kernel of this Hilbert space is proportional to $\tau_0 \otimes \chi^{-\lambda}(B(x,y))$. On the other hand, even if λ is smaller so that the integral defining the inner product does not converge, it may happen that the kernel function $\tau_0 \otimes \chi^{-\lambda}(B(x,y))$ is positive definite. In this case we denote the corresponding Hilbert space by the same notation $\mathcal{H}_\lambda(D,V)$. Then this again gives an irreducible unitary representation of $\tilde{G}$, but the underlying $(\mathfrak{g}, \tilde{K})$-module $\mathcal{H}_\lambda(D,V)_{\tilde{K}}$ may be smaller than $\mathcal{P}(\mathfrak{p}^+, V)$.

Now we additionally assume (τ_0, V) is trivial, and review the result of Faraut-Korányi [5] on $\mathcal{H}_\lambda(D,V) =: \mathcal{H}_\lambda(D)$. In this case, the $\tilde{G}$-invariant inner product $\langle f, g\rangle_\lambda$ is given by

$$\langle f, g\rangle_\lambda = \int_D f(x)\overline{g(x)}h(x)^{\lambda-p}dx,$$

and this integral converges for any polynomial f and g if $\lambda > p-1$. Moreover, the following holds.

Theorem 3.2.4 ([5], [4, Part III, Corollary V.3.9, Theorem V.3.10]). *(1) If $f, g \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ ($\mathbf{m} \in \mathbb{Z}^r_{++}$, see Theorem 3.2.1), we have*

$$\langle f, g\rangle_\lambda = \frac{C_{\lambda,d,r,b}}{(\lambda)_{\mathbf{m},d}} \langle f, g\rangle_F,$$

where $(\lambda)_{\mathbf{m},d}$ is as (3.2.9), and

$$C_{\lambda,d,r,b} := \pi^n \frac{\Gamma_{(d,r)}\left(\lambda - \frac{n}{r}\right)}{\Gamma_{(d,r)}(\lambda)}, \qquad \Gamma_{(d,r)}(\lambda) = \pi^{r(r-1)d/4} \prod_{j=1}^{r} \Gamma\left(\lambda - \frac{d}{2}(j-1)\right).$$

(2) The reproducing kernel (under a suitable normalization) is expanded as

$$h(x, y)^{-\lambda} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} (\lambda)_{\mathbf{m},d} K^{(d)}_{\mathbf{m}}(x, y), \tag{3.2.12}$$

where $K^{(d)}_{\mathbf{m}}(x, y) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+})$ is the reproducing kernel of $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ with respect to $\langle\cdot,\cdot\rangle_F$

Then for $f \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ we have

$$\int_D K^{(d)}_{\mathbf{m}}(x, y) f(y) h(y)^{\lambda-p} dy = \frac{C_{\lambda,d,r,b}}{(\lambda)_{\mathbf{m},d}} f(x),$$

and since $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ and $\mathcal{P}_{\mathbf{n}}(\mathfrak{p}^+)$ are perpendicular to each other with respect to both $\langle\cdot,\cdot\rangle_\lambda$ and $\langle\cdot,\cdot\rangle_F$ if $\mathbf{m} \neq \mathbf{n}$, we have

$$\int_D f(y) e^{(x|y)} h(y)^{\lambda-p} dy = \frac{C_{\lambda,d,r,b}}{(\lambda)_{\mathbf{m},d}} f(x). \tag{3.2.13}$$

3.3 Intertwining operators between holomorphic discrete series representations

Let G be a real reductive Lie group such that each simple component is of Hermitian type, as in Section 3.2.1. Let $G_1 \subset G$ be a reductive subgroup which is stable under the Cartan involution ϑ of G. We denote the Lie algebra of G_1 and its Cartan decomposition under ϑ by $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$. We assume

$$\mathfrak{p}_1^{\mathbb{C}} = (\mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^+) \oplus (\mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^-). \tag{3.3.1}$$

We set $\mathfrak{p}_1^+ := \mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^+$, $\mathfrak{p}_1^- := \mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^-$, so that

$$\mathfrak{g}_1^{\mathbb{C}} = \mathfrak{p}_1^+ \oplus \mathfrak{k}_1^{\mathbb{C}} \oplus \mathfrak{p}_1^-.$$

Also, let $\mathfrak{p}_2^+ \subset \mathfrak{p}^+$ be the orthogonal complement of $\mathfrak{p}_1^+$ with respect to the inner product $(\cdot|\cdot)_{\mathfrak{p}^+}$ defined in (3.2.1). We define another inner product $(\cdot|\cdot)_{\mathfrak{p}_1^+}$ on $\mathfrak{p}_1^+$ as in (3.2.1), changing $\mathfrak{g}$ to $\mathfrak{g}_1$, and let $D_1 \subset \mathfrak{p}_1^+$ is the bounded symmetric domain, defined as in (3.2.8).

Let (τ, V) be a representation of $\tilde{K}^{\mathbb{C}}$, and consider the representation $(\hat{\tau}, \mathcal{H}_\tau(D, V))$ of $\tilde{G}$, as in Section 3.2.4. We assume that $\mathcal{H}_\tau(D, V)$ is non-trivial. We want to argue the restriction $\mathcal{H}_\tau(D, V)|_{\tilde{G}_1}$. Then since it is discretely decomposable, the space of $\tilde{K}_1$-finite vectors coincides with the space of $\tilde{K}$-finite vectors (see [18, Theorem 4.5]), which is equal to the space of V-valued polynomials on $\mathfrak{p}^+$.

$$\mathcal{H}_\tau(D,V)_{\tilde{K}_1} = \mathcal{H}_\tau(D,V)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+,V).$$

Since $\mathfrak{p}^+$ acts on $\mathcal{H}_\tau(D,V)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+,V)$ by 1st order differential operators with constant coefficients, every $(\mathfrak{g}_1,\tilde{K}_1)$-submodule in $\mathcal{H}_\tau(D,V)_{\tilde{K}_1} = \mathcal{P}(\mathfrak{p}^+,V)$ has $\mathfrak{p}_1^+$-invariant vectors, and the space of $\mathfrak{p}_1^+$-invariant vectors is equal to

$$\mathcal{H}_\tau(D,V)_{\tilde{K}_1}^{\mathfrak{p}_1^+} = \mathcal{P}(\mathfrak{p}_2^+)\otimes V.$$

Thus if we write the decomposition of the above space under $\tilde{K}_1^{\mathbb{C}}$ as

$$\mathcal{P}(\mathfrak{p}_2^+)\otimes V \simeq \bigoplus_i m(\tau_i')(\tau_i',V_i'),$$

then $\mathcal{H}_\tau(D,V)$ is decomposed under $\tilde{G}_1$ abstractly as

$$\mathcal{H}_\tau(D,V)_{\tilde{K}}|_{(\mathfrak{g}_1,\tilde{K}_1)} \simeq \bigoplus_i m(\tau_i')\mathcal{H}_{\tau_i'}(D_1,V_i')_{\tilde{K}_1},$$
$$\mathcal{H}_\tau(D,V)|_{\tilde{G}_1} \simeq \sum_i{}^{\oplus} m(\tau_i')\mathcal{H}_{\tau_i'}(D_1,V_i')$$

(see [8], [13, Section 8], [29]). Thus we formally gather the space in the right hand side, and consider the space $\mathcal{O}(D_1,\mathcal{P}(\mathfrak{p}_2^+,V))$, with the $\tilde{G}_1$-action

$$\hat{\tau}'(g)f(y_1,y_2) = \tau(\kappa(g^{-1},y_1))^{-1}f(g^{-1}y_1,\kappa(g^{-1},y_1)y_2)$$
$$(g\in\tilde{G},y_1\in D_1,y_2\in\mathfrak{p}_2^+,f\in\mathcal{O}(D_1,\mathcal{P}(\mathfrak{p}_2^+,V))).$$

Then this action preserves the inner product

$$\langle f,g\rangle_{\hat{\tau}'} := \frac{1}{\pi^{n_2}}\iint_{D_1\times\mathfrak{p}_2^+}\left(\tau(B(y_1)^{-1})f(y_1,B(y_1)y_2),g(y_1,y_2)\right)_\tau h_1(y_1)^{-p_1}e^{-|y_2|_{\mathfrak{p}^+}^2}dy_1dy_2,$$

where $n_2 := \dim\mathfrak{p}_2^+$, $h_1(y_1)^{-p_1} := \mathrm{Det}(B(y_1)|_{\mathfrak{p}_1^+})^{-1}$, and dy_1, dy_2 are the Lebesgue measures on $\mathfrak{p}_1^+$, $\mathfrak{p}_2^+$ determined from the inner products $(\cdot|\cdot)_{\mathfrak{p}_1^+}$, $(\cdot|\cdot)_{\mathfrak{p}^+}$ respectively. Let $\mathcal{H}_\tau'(D_1\times\mathfrak{p}_2^+,V)$ be the completion of the pre-Hilbert subspace of functions f such that $\|f\|_{\hat{\tau}'}<\infty$. Our aim is to construct $\tilde{G}_1$-intertwining operators between $\mathcal{H}_\tau(D,V)|_{\tilde{G}_1}$ and $\mathcal{H}_\tau'(D_1\times\mathfrak{p}_2^+,V)$ explicitly.

Let $\mathcal{F}:\mathcal{H}_\tau(D,V)\to\mathcal{H}_\tau'(D_1\times\mathfrak{p}_2^+,V)$ be such an operator. Then for any $y\in D_1\times\mathfrak{p}_2^+$, the linear map $\mathcal{H}_\tau(D,V)\to V$, $f\mapsto(\mathcal{F}f)(y)$ is continuous, and by the Riesz representation theorem, there exists $\hat{K}_y\in\mathcal{H}_\tau(D,V)\otimes\bar{V}$ such that

$$\langle f,\hat{K}_y\rangle_{\hat{\tau}} = (\mathcal{F}f)(y) \qquad (f\in\mathcal{H}_\tau(D,V),y\in D_1\times\mathfrak{p}_2^+).$$

We write $\hat{K}(x;y)=\hat{K}(x;y_1,y_2):=\hat{K}_y(x)$ for $x\in D$, $y=(y_1,y_2)\in D_1\times\mathfrak{p}_2^+$. We identify $V\otimes\bar{V}$ and $\mathrm{End}(V)$ via the inner product of V. Then by the intertwining property, $\hat{K}(x;y)$ must satisfy

$$\hat{K}(gx;gy_1,\kappa(g,y_1)y_2) = \tau(\kappa(g,x))\hat{K}(x;y_1,y_2)\tau(\kappa(g,y_1))^* \tag{3.3.2}$$

for any $g \in \tilde{G}_1$. Thus we seek the kernel function satisfying (3.3.2).

Let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be an operator-valued polynomial satisfying

$$K(kx_2, k^{*-1}y_2) = \tau(k)K(x_2, y_2)\tau(k)^{-1} \qquad (x_2, y_2 \in \mathfrak{p}_2^+,\ k \in \tilde{K}_1^{\mathbb{C}}). \tag{3.3.3}$$

Let $\mathrm{Proj}_2 : \mathfrak{p}^+ \to \mathfrak{p}_2^+$ be the orthogonal projection, and we define an operator-valued function $\hat{K} \in \mathcal{O}(D \times \overline{D_1 \times \mathfrak{p}_2^+}, \mathrm{End}(V))$ by

$$\hat{K}(x; y) = \hat{K}(x_1, x_2; y_1, y_2) := \tau(B(x, y_1))K(\mathrm{Proj}_2(x^{y_1}), y_2)$$
$$(x = (x_1, x_2) \in D \subset \mathfrak{p}^+,\ y_1 \in D_1 \subset \mathfrak{p}_1^+,\ y_2 \in \mathfrak{p}_2^+).$$

Then the following holds.

Proposition 3.3.1. *For any $x \in D$, $y_1 \in D_1$, $y_2 \in \mathfrak{p}_2^+$ and $g \in \tilde{G}_1$, $\hat{K}(x; y)$ satisfies the identity (3.3.2).*

Proof. By (3.2.11), we have

$$\tau(B(gx, gy_1)) = \tau(\kappa(g, x))\tau(B(x, y_1))\tau(\kappa(g, y_1))^*.$$

Thus it suffices to show

$$K(\mathrm{Proj}_2((gx)^{gy_1}), \kappa(g, y_1)y_2) = \tau(\kappa(g, y_1))^{*-1}K(\mathrm{Proj}_2(x^{y_1}), y_2)\tau(\kappa(g, y_1))^*.$$

By $\tilde{K}_1^{\mathbb{C}}$-invariance of $K(\cdot, \cdot)$, this is equivalent to

$$\mathrm{Proj}_2((gx)^{gy_1}) = \kappa(g, y_1)^{*-1}\,\mathrm{Proj}_2(x^{y_1}) \qquad (x \in D,\ y_1 \in D_1,\ g \in G_1).$$

First we show

$$\mathrm{Proj}_2((gx)^{(\hat{\vartheta}g)y_1}) = \kappa(\hat{\vartheta}g, y_1)^{*-1}\,\mathrm{Proj}_2(x^{y_1}) \qquad (x \in \mathfrak{p}^+,\ y_1 \in \mathfrak{p}_1^+) \tag{3.3.4}$$

for $g = k \in K_1^{\mathbb{C}}$ or $g = \exp(-z_1)$, $g = \exp(\vartheta w_1) \in G_1^{\mathbb{C}}$ with $z_1, w_1 \in \mathfrak{p}_1^+$, when one side is well-defined, that is, we show

$$\begin{aligned}
\mathrm{Proj}_2((kx)^{k^{*-1}y_1}) &= k\,\mathrm{Proj}_2(x^{y_1}),\\
\mathrm{Proj}_2((x - z_1)^{(y_1^{z_1})}) &= B(z_1, y_1)\,\mathrm{Proj}_2(x^{y_1}),\\
\mathrm{Proj}_2((x^{w_1})^{y_1 - w_1}) &= \mathrm{Proj}_2(x^{y_1}).
\end{aligned}$$

In fact, these are true by (3.2.6), (3.2.7), and the fact that Proj_2 commutes with $K_1^{\mathbb{C}}$-action and $(x - z_1)^{(y_1^{z_1})} - B(z_1, y_1)x^{y_1} = B(z_1, y_1)z_1^{y_1} \in \mathfrak{p}_1^+$ is annihilated by Proj_2. Since any $g \in G_1$ is written as the form $g = \exp(\vartheta w_1)k\exp(-z_1)$ with $z_1, w_1 \in D_1$ and $k \in K_1^{\mathbb{C}}$ (which is proved by using the KAK-decomposition and [4, Part III, Lemma III.2.4]), the proposition follows from the cocycle condition of κ. □

Also, the function satisfying (3.3.2) is unique for every irreducible submodule of $\mathcal{P}(\mathfrak{p}_2^+) \otimes V$.

Lemma 3.3.2. *We take an irreducible submodule $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+) \otimes V$. Then the function $\hat{K} \in \mathcal{O}(D \times \overline{D_1 \times \mathfrak{p}_2^+}, \mathrm{End}(V))$ satisfying (3.3.2) and*

$$\hat{K}(x; y_1, \cdot) \in V \otimes \overline{W_1} \subset \mathcal{O}(\overline{\mathfrak{p}_2^+}, \mathrm{End}(V)) \qquad (\text{for any } x \in D, y_1 \in D_1)$$

is unique up to scalar multiple.

Proof. By the invariance (3.3.2), if we substitute $x_1 = y_1 = 0$, then the function $K(x_2, \overline{y_2}) := \hat{K}(0, x_2; 0, y_2)$ satisfies (3.3.3), and by the irreducibility of W_1, such function on $\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}$ is unique up to scalar multiple. Then again by (3.3.2), the values of $\hat{K}$ is uniquely determined on

$$S := \left\{(g.(0, x_2); g.0, \kappa(g, 0)y_2) \in D \times \overline{D_1 \times \mathfrak{p}_2^+} : g \in G_1, x_2 \in D_2, y_2 \in \mathfrak{p}_2^+\right\} \subset D \times \overline{D_1 \times \mathfrak{p}_2^+}.$$

Thus it suffices to show S contains a totally real submanifold of full dimension of $D \times \overline{D_1 \times \mathfrak{p}_2^+}$. Let $\mathrm{pr}_1 : D \times \overline{D_1 \times \mathfrak{p}_2^+} \to D$, $\mathrm{pr}_2 : D \times \overline{D_1 \times \mathfrak{p}_2^+} \to \overline{D_1 \times \mathfrak{p}_2^+}$ be the projections. Then since for every $x_2 \in D_2$, $\{\exp(z).(0, x_2) : z \in \mathfrak{p}_1\} \subset D$ intersects transversally with $D \subset \mathfrak{p}_2^+$ at x_2, the differential of $\mathrm{pr}_1|_S$ at $(0, x_2; 0, y_2)$ is surjective. Similarly, since G_1 acts transitively on D_1, the differential of $\mathrm{pr}_2|_S$ at $(0, x_2; 0, \overline{y_2})$ is also surjective. Therefore, $\mathrm{pr}_1|_S$ and $\mathrm{pr}_2|_S$ are both submersive near $\{0\} \times D_2 \times \overline{\{0\} \times \mathfrak{p}_2^+} \subset S$, and $T_{(x;y)}S + JT_{(x;y)}S = T_{(x;y)}(D \times \overline{D_1 \times \mathfrak{p}_2^+})$ holds on this neighborhood, where J is the complex structure of $D \times \overline{D_1 \times \mathfrak{p}_2^+}$. Hence S contains a totally real submanifold of full dimension of $D \times \overline{D_1 \times \mathfrak{p}_2^+}$, and this completes the proof. $\square$

Let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+})$ be a polynomial satisfying (3.3.3), and let $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+) \otimes V$ be a subrepresentation of $\tilde{K}_1^{\mathbb{C}}$ such that $K(\cdot, y_2) \in W_1$ for any $y_2 \in \mathfrak{p}_2^+$. Then by the uniqueness, the function $\hat{K}(x; y)$ becomes the kernel function of the intertwining operator from $\mathcal{H}_\tau(D, V)$ to $\mathcal{H}_{\tau \otimes (Ad|_{\mathfrak{p}_2^+})^*}(D_1, W_1) \subset \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)$. Especially, $\hat{K}(\cdot; y) \in \mathcal{H}_\tau(D, V) \otimes \bar{V}$ holds for any $y \in D_1 \times \mathfrak{p}_2^+$. Similarly, $\hat{K}(x; \cdot)^* \in \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)$ holds for any $x \in D$, and it becomes the kernel function of the intertwining operator of opposite direction. That is, the following holds.

Corollary 3.3.3. *We assume $\mathcal{H}_\tau(D, V)$ is non-trivial.*

(1) The linear map $\mathcal{F}^*_{W_1} : \mathcal{H}_\tau(D, V) \to \mathcal{H}_{\tau \otimes (Ad|_{\mathfrak{p}_2^+})^*}(D_1, W_1) \subset \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)$,

$$(\mathcal{F}^*_{W_1} f)(y_1, y_2) := \int_D \hat{K}(x; y_1, y_2)^* \tau(B(x)^{-1}) f(x) h(x)^{-p} dx$$

intertwines the $\tilde{G}_1$-action.

(2) The linear map $\mathcal{F}_{W_1} : \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V) \supset \mathcal{H}_{\tau \otimes (Ad|_{\mathfrak{p}_2^+})^*}(D_1, W_1) \to \mathcal{H}_\tau(D, V)$,

$$\begin{aligned}&(\mathcal{F}_{W_1} f)(x) \\ &:= \frac{1}{\pi^{n_2}} \iint_{D_1 \times \mathfrak{p}_2^+} \hat{K}(x; y_1, B(y_1) y_2) \tau(B(y_1)^{-1}) f(y_1, y_2) e^{-|y_2|^2_{\mathfrak{p}^+}} h_1(y_1)^{-p_1} dy_1 dy_2\end{aligned}$$

intertwines the $\tilde{G}_1$-action.

Next we rewrite these operators. Since the reproducing kernel of $\mathcal{P}(\mathfrak{p}^+, V)$ with respect to the Fischer norm is given by $e^{(x|z)_{\mathfrak{p}^+}}$, we have

$$\begin{aligned}(\mathcal{F}^*_{W_1} f)(y) &= \frac{1}{\pi^n} \int_D \hat{K}(x; y)^* \tau(B(x)^{-1}) \int_{\mathfrak{p}^+} f(z) e^{(x|z)_{\mathfrak{p}^+}} e^{-|z|^2_{\mathfrak{p}^+}} dz \; h(x)^{-p} dx \\ &= \frac{1}{\pi^n} \int_{\mathfrak{p}^+} \int_D \hat{K}(x; y)^* \tau(B(x)^{-1}) e^{(x|z)_{\mathfrak{p}^+}} h(x)^{-p} dx \; f(z) e^{-|z|^2_{\mathfrak{p}^+}} dz.\end{aligned}$$

Now we have

Lemma 3.3.4.

$$\begin{aligned}&\int_D \hat{K}(x;y_1,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx\\&=\int_D \hat{K}(x;0,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx\ e^{(y_1|z)}.\end{aligned}$$

Proof. Since $\mathcal{F}^*_{W_1}$ intertwines the $\tilde{G}_1$-action, it also intertwines the $\mathfrak{g}_1^{\mathbb{C}}$-action. Especially, since $\mathfrak{p}_1^+\subset\mathfrak{g}_1^{\mathbb{C}}$ acts as a 1st-order differential operator with constant coefficients, we have

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0}\int_D \hat{K}(x;y_1+tw_1,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx\\&=\left.\frac{d}{dt}\right|_{t=0}\int_D \hat{K}(x;y_1,y_2)^*\tau(B(x)^{-1})e^{(x+tw_1|z)}h(x)^{-p}dx\\&=\int_D \hat{K}(x;y_1,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx\cdot(w_1|z).\end{aligned}$$

Therefore, as functions of y_1, both

$$\int_D \hat{K}(x;y_1,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx$$

and

$$\int_D \hat{K}(x;0,y_2)^*\tau(B(x)^{-1})e^{(x|z)}h(x)^{-p}dx\ e^{(y_1|z)}$$

satisfy the same differential equation with the same initial condition, and thus they coincide. $\square$

Thus we set

$$\begin{aligned}F^*_{W_1}(z;y_2)=F^*_{W_1}(z_1,z_2;y_2)&:=\int_D \hat{K}(x;0,y_2)^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx\\&=\int_D K(x_2,y_2)^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx.\end{aligned}$$

This is a polynomial anti-holomorphic in z and holomorphic in y_2. Then we have

$$\begin{aligned}(\mathcal{F}^*_{W_1}f)(y)&=\frac{1}{\pi^n}\int_{\mathfrak{p}^+}F^*_{W_1}(z_1,z_2;y_2)e^{(y_1|z)_{\mathfrak{p}^+}}f(z)e^{-|z|^2_{\mathfrak{p}^+}}dz\\&=\left.\frac{1}{\pi^n}\int_{\mathfrak{p}^+}F^*_{W_1}(z_1,z_2;y_2)e^{(x|z)_{\mathfrak{p}^+}}f(z)e^{-|z|^2_{\mathfrak{p}^+}}dz\right|_{x_1=y_1,x_2=0}\\&=F^*_{W_1}\left(\overline{\left.\frac{\partial}{\partial x_1}\right|_{x_1=y_1}},\overline{\left.\frac{\partial}{\partial x_2}\right|_{x_2=0}};y_2\right)\frac{1}{\pi^n}\int_{\mathfrak{p}^+}e^{(x|z)_{\mathfrak{p}^+}}f(z)e^{-|z|^2_{\mathfrak{p}^+}}dz\\&=F^*_{W_1}\left.\left(\overline{\frac{\partial}{\partial x_1}},\overline{\frac{\partial}{\partial x_2}};y_2\right)\right|_{x_1=y_1,x_2=0}f(x).\end{aligned}$$

Here, for anti-holomorphic polynomial $f\in\mathcal{P}(\overline{\mathfrak{p}^+})$, we write

$$f\left(\overline{\frac{\partial}{\partial x}}\right):=\sum_\alpha f(e_\alpha)\frac{\partial}{\partial x_\alpha},$$

where $\{e_\alpha\} \subset \mathfrak{p}^+$ is a basis, with the dual basis $\{e_\alpha^\vee\} \subset \mathfrak{p}^+$ with respect to the inner product $(\cdot|\cdot)_{\mathfrak{p}^+}$, and $\frac{\partial}{\partial x_\alpha}$ is the directional derivative along the direction of $e_\alpha^\vee$. Similarly, we set

$$\begin{aligned}&F_{W_1}(x_2;w) = F_{W_1}(x_2;w_1,w_2)\\ &:= \frac{1}{\pi^{n_2}}\iint_{D_1\times\mathfrak{p}_2^+}\hat{K}(0,x_2;y_1,B(y_1)y_2)\tau(B(y_1)^{-1})e^{((y_1,y_2)|(w_1,w_2))_{\mathfrak{p}^+}}h_1(y_1)^{-p}e^{-|y_2|^2_{\mathfrak{p}^+}}dy_1dy_2\\ &= \int_{D_1}K(0,x_2;y_1,B(y_1)w_2)\tau(B(y_1)^{-1})e^{(y_1|w_1)_{\mathfrak{p}^+}}h_1(y_1)^{-p}dy_1\\ &= \int_{D_1}\tau(B(x_2,y_1))K(\mathrm{Proj}_2((x_2)^{y_1}),B(y_1)w_2)\tau(B(y_1)^{-1})e^{(y_1|w_1)_{\mathfrak{p}^+}}h_1(y_1)^{-p}dy_1.\end{aligned}$$

This is holomorphic in x_2, anti-holomorphic in w, but in general this is not a polynomial. As in $\mathcal{F}^*_{W_1}$ case, we have

$$(\mathcal{F}_{W_1}f)(x) = \frac{1}{\pi^n}\int_{\mathfrak{p}^+}F_K(x_2;w_1,w_2)e^{(x_1|w)_{\mathfrak{p}^+}}f(w)e^{-|w|^2_{\mathfrak{p}^+}}dw.$$

We summarize the above results.

Theorem 3.3.5. *We assume $\mathcal{H}_\tau(D,V)$ is non-trivial. Let $K(x_2,y_2)\in\mathcal{P}(\mathfrak{p}_2^+\times\overline{\mathfrak{p}_2^+},\mathrm{End}(V))$ be an operator-valued polynomial satisfying*

$$K(kx_2,k^{*-1}y_2) = \tau(k)K(x_2,y_2)\tau(k)^{-1} \qquad (x_2,y_2\in\mathfrak{p}_2^+,\ k\in\tilde{K}_1^{\mathbb{C}}). \qquad \text{(3.3.3 reshown)}$$

Let $W_1\subset\mathcal{P}(\mathfrak{p}_2^+)\otimes V$ be a subrepresentation of $\tilde{K}_1^{\mathbb{C}}$ such that $K(\cdot,y_2)\in W$ for any $y_2\in\mathfrak{p}_2^+$.

(1) We set

$$F^*_{W_1}(z;y_2) = F^*_{W_1}(z_1,z_2;y_2) := \int_D K(x_2,y_2)^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx.$$

Then the linear map

$$\begin{aligned}&\mathcal{F}^*_{W_1}:\mathcal{H}_\tau(D,V)_{\tilde{K}}\to\mathcal{H}_{\tau\otimes(Ad|_{\mathfrak{p}_2^+})^*}(D_1,W_1)_{\tilde{K}_1}\subset\mathcal{H}'_\tau(D_1\times\mathfrak{p}_2^+,V)_{\tilde{K}_1},\\ &(\mathcal{F}^*_{W_1}f)(y) = \frac{1}{\pi^n}\int_{\mathfrak{p}^+}F^*_{W_1}(z_1,z_2;y_2)e^{(y_1|z)_{\mathfrak{p}^+}}f(z)e^{-|z|^2_{\mathfrak{p}^+}}dz\\ &\qquad = F^*_{W_1}\left(\overline{\frac{\partial}{\partial x_1}},\overline{\frac{\partial}{\partial x_2}};y_2\right)\bigg|_{x_1=y_1,x_2=0}f(x)\end{aligned}$$

intertwines the $(\mathfrak{g}_1,\tilde{K}_1)$-action.

(2) We set

$$\begin{aligned}&F_{W_1}(x_2;w) = F_{W_1}(x_2;w_1,w_2)\\ &:= \int_{D_1}\tau(B(x_2,y_1))K(\mathrm{Proj}_2((x_2)^{y_1}),B(y_1)w_2)\tau(B(y_1)^{-1})e^{(y_1|w_1)_{\mathfrak{p}^+}}h_1(y_1)^{-p_1}dy_1.\end{aligned}$$

Then the linear map

$$\begin{aligned}&\mathcal{F}_{W_1}:\mathcal{H}'_\tau(D_1\times\mathfrak{p}_2^+,V)_{\tilde{K}_1}\supset\mathcal{H}_{\tau\otimes(Ad|_{\mathfrak{p}_2^+})^*}(D_1,W_1)_{\tilde{K}_1}\to\mathcal{H}_\tau(D,V)_{\tilde{K}},\\ &(\mathcal{F}_{W_1}f)(x) = \frac{1}{\pi^n}\int_{\mathfrak{p}^+}F_{W_1}(x_2;w_1,w_2)e^{(x_1|w)_{\mathfrak{p}^+}}f(w)e^{-|w|^2_{\mathfrak{p}^+}}dw\end{aligned}$$

intertwines the $(\mathfrak{g}_1,\tilde{K}_1)$-action.

The operator $\mathcal{F}_{W_1}$ is not a differential operator of finite order in general, but if $F_{W_1}(x_2;w)$ is expanded as

$$F_{W_1}(x_2;w)=\sum_{k=0}^{\infty}F_k(x_2;w)=\sum_{k=0}^{\infty}F_k(x_2;w_1,w_2),$$

where $F_k(x_2;w)$ is a homogeneous polynomial of degree k in w, then we can write

$$(\mathcal{F}_{W_1}f)(x)=\sum_{k=0}^{\infty}F_k\left(x_2;\overline{\frac{\partial}{\partial y_1}},\overline{\frac{\partial}{\partial y_2}}\right)\bigg|_{y_1=x_1,y_2=0}f(y)$$

for polynomials $f\in\mathcal{H}'_\tau(D_1\times\mathfrak{p}_2^+,V)_{\tilde{K}}=\mathcal{P}(\mathfrak{p}^+,V)$.

Remark 3.3.6. *For $w\in\mathfrak{p}^+$, we define the* $\mathrm{End}(V)$*-valued differential operator $\mathcal{B}_\tau(w)$ on $\overline{\mathfrak{p}^+}$ by*

$$\mathcal{B}_\tau(w)f(z):=\sum_{\alpha\beta}\frac{1}{2}(ad(e_\alpha)ad(e_\beta)\vartheta w|z)_{\mathfrak{p}^+}\frac{\partial^2 f}{\partial\bar{z}_\alpha\partial\bar{z}_\beta}(z)+\sum_\alpha d\tau([e_\alpha,\vartheta w])\frac{\partial f}{\partial\bar{z}_\alpha}(z),$$

where $\{e_\alpha\}$ is a basis of $\mathfrak{p}^+$, with the dual basis $\{e_\alpha^\vee\}$, and $\frac{\partial}{\partial\bar{z}_\alpha}$ is the anti-holomorphic directional derivative along $e_\alpha^\vee$. Then this is a generalization of the Bessel operator $\mathcal{B}_\nu$ in [3] or [6, Section XV.2]. Then for $w_1\in\mathfrak{p}_1^+$, $\mathcal{B}_\tau(w_1)$ annihilates $F_{W_1}^*(z;y_2)$, because

$$\begin{aligned}
&(\mathcal{B}_\tau(w_1))_zF_{W_1}^*(z;y_2)=(\mathcal{B}_\tau(w_1))_z\int_D K(x_2,y_2)^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx\\
&=\int_D K(x_2,y_2)^*\tau(B(x)^{-1})\left(\frac{1}{2}(ad(x)^2\vartheta w_1|z)_{\mathfrak{p}^+}+d\tau([x,\vartheta w_1])\right)e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx\\
&=\int_D K(x_2,y_2)^*\tau(B(x)^{-1})\left(d\hat{\tau}(-\vartheta w_1)_xe^{(x|z)_{\mathfrak{p}^+}}\right)h(x)^{-p}dx\\
&=\int_D (d\hat{\tau}(w_1)_xK(x_2,y_2))^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx\\
&=\int_D \frac{d}{dt}\bigg|_{t=0}K(\mathrm{Proj}_2(x-tw_1),y_2)^*\tau(B(x)^{-1})e^{(x|z)_{\mathfrak{p}^+}}h(x)^{-p}dx=0.
\end{aligned}$$

This differential equation coincides with $\widehat{d\pi_\mu}$ on $\mathfrak{n}_+$ appeared in Proposition 3.10 or Section 4.4, Step 1 of [24], and thus the operator $\mathcal{F}_{W_1}^*$ coincides with the one given by the F-method.

3.4 Preliminaries for examples

3.4.1 Parametrization of representations of $K^{\mathbb{C}}$

In this subsection we fix the realization of root systems and parametrization of irreducible finite-dimensional representations of $K^{\mathbb{C}}$. First we set $K^{\mathbb{C}}:=GL(r,\mathbb{C})$ or $SO(n,\mathbb{C})$. We take a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}\subset\mathfrak{k}^{\mathbb{C}}$, and take a basis $\{t_1,\ldots,t_r\}\subset\mathfrak{h}^{\mathbb{C}}$, with the dual basis $\{\varepsilon_1,\ldots,\varepsilon_r\}\subset(\mathfrak{h}^{\mathbb{C}})^\vee$, where $r=\lfloor\frac{n}{2}\rfloor$ when $K^{\mathbb{C}}=SO(n,\mathbb{C})$, such that the positive root system $\Delta_+(\mathfrak{k}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$ is given by

$$\Delta_+(\mathfrak{k}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})=\begin{cases}\{\varepsilon_j-\varepsilon_k:1\le j<k\le r\} & (K^{\mathbb{C}}=GL(r,\mathbb{C})),\\ \{\varepsilon_j\pm\varepsilon_k:1\le j<k\le r\} & (K^{\mathbb{C}}=SO(2r,\mathbb{C})),\\ \{\varepsilon_j\pm\varepsilon_k:1\le j<k\le r\}\cup\{\varepsilon_j:1\le j\le r\} & (K^{\mathbb{C}}=SO(2r+1,\mathbb{C})).\end{cases}$$

For $\mathbf{m} \in \mathbb{Z}^r$ with $m_1 \geq \cdots \geq m_r$, we denote the irreducible representation of $GL(r,\mathbb{C})$ with highest weight $m_1\varepsilon_1 + \cdots + m_r\varepsilon_r$ by $(\tau^{(r)}_{\mathbf{m}}, V^{(r)}_{\mathbf{m}})$, the irreducible representation of $GL(r,\mathbb{C})$ with highest weight $-m_r\varepsilon_1 - \cdots - m_1\varepsilon_r$ by $(\tau^{(r)\vee}_{\mathbf{m}}, V^{(r)\vee}_{\mathbf{m}})$, and for $\mathbf{m} \in \mathbb{Z}^r$ with $m_1 \geq \cdots \geq m_{r-1} \geq |m_r|$ (when $n = 2r$) or with $m_1 \geq \cdots \geq m_r \geq 0$ (when $n = 2r+1$), we denote the irreducible representation of $SO(n,\mathbb{C})$ with highest weight $m_1\varepsilon_1 + \cdots + m_r\varepsilon_r$ by $(\tau^{[n]}_{\mathbf{m}}, V^{[n]}_{\mathbf{m}})$. We omit the superscript (r) and $[n]$ if there is no confusion.

Next we set $G := Sp(r,\mathbb{C})$, $U(q,s)$, $SO^*(2s)$, or $SO_0(2,n)$, and let $K^{\mathbb{C}}$ be the complexification of their maximal compact subgroups, that is, $K^{\mathbb{C}} = GL(r,\mathbb{C})$, $GL(q,\mathbb{C}) \times GL(s,\mathbb{C})$, $GL(s,\mathbb{C})$ or $SO(2,\mathbb{C}) \times SO(n,\mathbb{C})$ respectively. Then irreducible finite-dimensional representations of $K^{\mathbb{C}}$ are of the form $V^{(r)}_{\mathbf{m}}$, $V^{(q)}_{\mathbf{m}} \boxtimes V^{(s)\vee}_{\mathbf{n}}$, $V^{(s)}_{\mathbf{m}}$, or $\mathbb{C}_{m_0} \boxtimes V^{[n]}_{\mathbf{m}}$ respectively, where we normalize the representation $(\chi^{m_0}, \mathbb{C}_{m_0})$ of $SO(2,\mathbb{C})$ later as in (3.4.2). Also, under the suitable ordering of $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$ in Theorem 3.2.1 is given by

$$\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \begin{cases} V^{(r)\vee}_{(2m_1,2m_2,\dots,2m_r)} & (G = Sp(r,\mathbb{C}),\ \mathbf{m} \in \mathbb{Z}^r_{++}), \\ V^{(q)\vee}_{\mathbf{m}} \boxtimes V^{(s)}_{\mathbf{m}} & (G = U(q,s),\ \mathbf{m} \in \mathbb{Z}^{\min\{q,s\}}_{++}), \\ V^{(s)\vee}_{(m_1,m_1,m_2,m_2,\dots,m_{\lfloor s/2\rfloor},m_{\lfloor s/2\rfloor}(,0))} & (G = SO^*(2s),\ \mathbf{m} \in \mathbb{Z}^{\lfloor s/2\rfloor}_{++}), \\ \mathbb{C}_{m_1+m_2} \boxtimes V^{[n]}_{(m_1-m_2,0,0,\dots,0)} & (G = SO_0(2,n),\ \mathbf{m} \in \mathbb{Z}^2_{++}), \end{cases}$$

where, when $s < q$ and $\mathbf{m} \in \mathbb{Z}^s_{++}$, we denote $V^{(q)}_{(m_1,\dots,m_s,0,\dots,0)} =: V^{(q)}_{\mathbf{m}}$ etc.

3.4.2 Explicit realization of groups and bounded symmetric domains

In this subsection, we review and fix the explicit realization of groups

$$G = Sp(r,\mathbb{R}),\ U(q,s),\ SO^*(2s),\ SO_0(2,n).$$

First we deal with $G = Sp(r,\mathbb{R})$, $U(q,s)$, and $SO^*(2s)$. For these groups we have

$$(r,n,d,p) = \begin{cases} \left(r, \frac{1}{2}r(r+1), 1, r+1\right) & (G = Sp(r,\mathbb{R}), \\ (\min\{q,s\}, qs, 2, q+s) & (G = U(q,s)), \\ \left(\lfloor \frac{s}{2} \rfloor, \frac{1}{2}s(s-1), 4, 2(s-1)\right) & (G = SO^*(2s)) \end{cases}$$

We realize these groups as

$$\begin{aligned} Sp(r,\mathbb{R}) &:= \left\{ g \in GL(2r,\mathbb{C}) : g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},\ g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \bar{g} \right\}, \\ U(q,s) &:= \left\{ g \in GL(q+s,\mathbb{C}) : g \begin{pmatrix} I_q & 0 \\ 0 & -I_s \end{pmatrix} g^* = \begin{pmatrix} I_q & 0 \\ 0 & -I_s \end{pmatrix} \right\}, \\ SO^*(2s) &:= \left\{ g \in GL(2s,\mathbb{C}) : g \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix},\ g \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \bar{g} \right\}. \end{aligned}$$

Then K is isomorphic to $U(r)$, $U(q) \times U(s)$, and $U(s)$ respectively. We embed K into G as

$$\begin{aligned} k &\mapsto \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} && (G = Sp(r,\mathbb{R}),\ SO^*(2s)), \\ (k_1, k_2) &\mapsto \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} && (G = U(q,s)). \end{aligned}$$

Clearly these extends to the embeddings of complexified Lie groups $K^{\mathbb{C}} \to G^{\mathbb{C}}$. When $G = Sp(r, \mathbb{R})$ or $SO^*(2s)$, we sometimes write the elements of K or $K^{\mathbb{C}}$ as $(k, {}^t k^{-1})$, and deal with these inclusions uniformly. Similarly, $\mathfrak{p}^+$ is isomorphic to $\mathrm{Sym}(r, \mathbb{C})$, $M(q, s; \mathbb{C})$ and $\mathrm{Skew}(s, \mathbb{C})$ respectively. We embed $\mathfrak{p}^+$ into $\mathfrak{g}^{\mathbb{C}}$ as $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Then the rational action of G on $\mathfrak{p}^+$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = (ax + b)(cx + d)^{-1} \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \ x \in \mathfrak{p}^+ \right).$$

The Bergman operator $B : D \times \overline{D} \to K$ is given by

$$B(x, y) = \left(I - xy^*, (I - y^* x)^{-1} \right) \qquad (x, y \in \mathfrak{p}^+),$$

the quasi-inverse is given by

$$x^y = x(I - y^* x)^{-1} = (I - xy^*)^{-1} x \qquad (x, y \in \mathfrak{p}^+),$$

and the bounded symmetric domain D is given by

$$D = \{ x \in \mathfrak{p}^+ : I - xx^* \text{ is positive definite.} \}$$

Let (τ, V) be an irreducible representation of $\tilde{K}^{\mathbb{C}}$ with $\tilde{K}$-invariant inner product $(\cdot, \cdot)_\tau$. Then $\tilde{G}$ acts on $\mathcal{O}(D, V)$ as

$$\hat{\tau}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(w) = \tau\left(a^* + xb^*, (cx + d)^{-1} \right) f\left((ax + b)(cx + d)^{-1} \right),$$

where we regard $\left(a^* + xb^*, (cx + d)^{-1} \right)$ as the lift on $\tilde{K}^{\mathbb{C}}$, and this action preserves the inner product

$$\langle f, g \rangle_{\hat{\tau}} = \int_D \left(\tau\left((I - xx^*)^{-1}, I - x^* x \right) f(x), g(x) \right)_\tau \det(I - xx^*)^{-\varepsilon p} dx,$$

where

$$\varepsilon = \begin{cases} 1 & (G = Sp(r, \mathbb{R}),\ U(q, s)), \\ \frac{1}{2} & (G = SO^*(2s)), \end{cases} \qquad p = \begin{cases} r + 1 & (G = Sp(r, \mathbb{R})), \\ q + s & (G = U(q, s)), \\ 2(s - 1) & (G = SO^*(2s)). \end{cases}$$

Especially, for $G = Sp(r, \mathbb{R})$ or $SO^*(2s)$, let $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ be a 1-dimensional representation of $\tilde{K}^{\mathbb{C}}$, normalized as in the latter half of Section 3.2.4, that is,

$$\chi(k) := \det(k)^\varepsilon.$$

Then the $\tilde{G}$-invariant inner product on $\mathcal{H}_\tau(D, \mathbb{C}) = \mathcal{H}_\lambda(D)$ is given by

$$\langle f, g \rangle_\lambda = \int_D f(x) \overline{g(x)} \det(I - xx^*)^{\varepsilon(\lambda - p)} dx, \tag{3.4.1}$$

which converges for any polynomial f, g if $\lambda > p - 1$. When $G = U(q, s)$, we define $(\chi^{-\lambda_1 - \lambda_2}, \mathbb{C})$ as

$$\chi(k_1, k_2) := \det(k_1)^{-\lambda_1} \det(k_2)^{\lambda_2},$$

and write the corresponding representation of $\tilde{G}$ as $\mathcal{H}_{\lambda_1+\lambda_2}(D)$. Then again the $\tilde{G}$-invariant inner product is given by (3.4.1) with $\lambda = \lambda_1 + \lambda_2$.

Next we deal with $G = SO_0(2,n)$ case with $n \geq 3$. In this case, we have

$$(r, n, d, p) = (2, n, n-2, n).$$

We realize this group as

$$SO_0(2,n) := \left\{ g \in SL(2+n, \mathbb{R}) : g \begin{pmatrix} I_2 & 0 \\ 0 & -I_n \end{pmatrix} {}^t g = \begin{pmatrix} I_2 & 0 \\ 0 & -I_n \end{pmatrix} \right\}_0$$

as usual, where the subscript 0 means the identity component. We have $K \simeq SO(2) \times SO(n)$, embedded into G as $(k_1, k_2) \mapsto \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, and $\mathfrak{p}^+ \simeq \mathbb{C}^n$, embedded into $\mathfrak{g}^{\mathbb{C}}$ as

$$x \mapsto \begin{pmatrix} 0 & 0 & {}^t x \\ 0 & 0 & \sqrt{-1}\,{}^t x \\ x & \sqrt{-1}x & 0 \end{pmatrix},$$

where we regard x as a column vector. For $x = {}^t(x_1, \ldots, x_n), y = {}^t(y_1, \ldots, y_n) \in \mathfrak{p}^+$, we write

$$q(x) := x_1^2 + \cdots + x_n^2, \qquad q(x,y) := x_1 y_1 + \cdots + x_n y_n.$$

Then the generic norm is given by

$$h(x,y) = 1 - 2q(x, \bar{y}) + q(x)\overline{q(y)},$$

the quasi-inverse is given by

$$x^y = (1 - 2q(x,\bar{y}) + q(x)\overline{q(y)})^{-1}(x - q(x)\bar{y}),$$

and the bounded symmetric domain D is the connected component of $\{h(x,x) > 0\}$ which contains the origin.

Let $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ be a 1-dimensional representation of $\tilde{K}^{\mathbb{C}}$, where χ is normalized as in the latter half of Section 3.2.4, that is,

$$\chi\left(\exp\left(a \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right), k_2 \right) = e^a \qquad (a \in \mathbb{C},\ k_2 \in SO(n, \mathbb{C})). \tag{3.4.2}$$

Then the $\tilde{G}$-action on $\mathcal{O}(D)$ preserves the inner product

$$\langle f, g \rangle_\lambda = \int_D f(x)\overline{g(x)}(1 - 2q(x,\bar{x}) + |q(x)|^2)^{\lambda - n} dx. \tag{3.4.3}$$

When $n = 1, 2$, we have $\mathfrak{so}(2,1) \simeq \mathfrak{sl}(2,\mathbb{R})$, which is of real rank 1, or $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which is not simple, and thus their properties are a bit different from those of $n \geq 3$ cases. However, for convenience, we use the same inner product as (3.4.3), so that

$$\mathcal{H}_\lambda(D_{SO_0(2,1)}) \simeq \mathcal{H}_{2\lambda}(D_{SL(2,\mathbb{R})}), \qquad \mathcal{H}_\lambda(D_{SO_0(2,2)}) \simeq \mathcal{H}_\lambda(D_{SL(2,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{SL(2,\mathbb{R})}).$$

3.4.3 Polynomials on Jordan triple systems revisited

In this subsection we reconsider the polynomials on $\mathfrak{p}^+ = \mathrm{Sym}(r,\mathbb{C})$, $M(q,s;\mathbb{C})$ and $\mathrm{Skew}(s,\mathbb{C})$. As in (3.2.12) we have

$$h(x,e)^{-\lambda} = \det(I - xe^*)^{-\varepsilon\lambda} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{(\lambda)_{\mathbf{m},d}}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(x),$$

where

$$(d,r) = \begin{cases} (1,r) & (G = Sp(r,\mathbb{R})), \\ (2,\min\{q,s\}) & (G = U(q,s)), \\ (4,\lfloor \frac{s}{2} \rfloor) & (G = SO^*(2s)), \end{cases} \qquad \varepsilon = \begin{cases} 1 & (G = Sp(r,\mathbb{R}),\ U(q,s)), \\ \frac{1}{2} & (G = SO^*(2s)), \end{cases}$$

and e is a maximal tripotent in $\mathfrak{p}^+$, for example,

$$e = I_r \qquad (G = Sp(r,\mathbb{R})), \qquad e = (I_q, 0) \quad (G = U(q,s),\ q \le s),$$

$$e = J_s := \sum_{j=1}^{\lfloor s/2 \rfloor}(E_{2j,2j-1} - E_{2j-1,2j}) \quad (G = SO^*(2s)), \qquad e = \begin{pmatrix} I_s \\ 0 \end{pmatrix} \quad (G = U(q,s),\ q \ge s).$$

Let $x, y \in \mathfrak{p}^+$, and take an element $(k_1,k_2) \in K^{\mathbb{C}}$ such that $y = k_1 e k_2^{-1}$ (such (k_1,k_2) exists if y is in some open dense subset of $\mathfrak{p}^+$). Then we have

$$K^{(d)}_{\mathbf{m}}(x,y) = K^{(d)}_{\mathbf{m}}(k_1^* x k_2^{*-1}, e) = \frac{1}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(k_1^* x k_2^{*-1}).$$

Since $K^{(d)}_{\mathbf{m}}$ is determined by the values on $\mathfrak{a}^+ \subset \mathfrak{p}^+$ (i.e. by the eigenvalues of xe^*), and $k_1^* x k_2^{*-1} e^*$, xy^* and y^*x have the same eigenvalues, we write

$$K^{(d)}_{\mathbf{m}}(x,y) =: \frac{1}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(xy^*) = \frac{1}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(y^*x),$$

following [32], so that

$$h(x,y)^{-\lambda} = \det(I - xy^*)^{-\varepsilon\lambda} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{(\lambda)_{\mathbf{m},d}}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(xy^*).$$

Next we take positive integers q', q'', s', s'', and we consider the sets

$$\mathfrak{p}^+(11,1) := \mathrm{Sym}(s',\mathbb{C}), \qquad \mathfrak{p}^+(22,1) := \mathrm{Sym}(s'',\mathbb{C}), \tag{3.4.4a}$$
$$\mathfrak{p}^+(11,2) := M(q',s';\mathbb{C}), \qquad \mathfrak{p}^+(22,2) := M(q'',s'';\mathbb{C}), \tag{3.4.4b}$$
$$\mathfrak{p}^+(11,4) := \mathrm{Skew}(s',\mathbb{C}), \qquad \mathfrak{p}^+(22,4) := \mathrm{Skew}(s'',\mathbb{C}), \tag{3.4.4c}$$

$$\mathfrak{p}^+(12,1) := \{(x_{12},x_{21}) : x_{12} = {}^t x_{21} \in M(s',s'';\mathbb{C})\}, \tag{3.4.4d}$$
$$\mathfrak{p}^+(12,2) := M(q',s'';\mathbb{C}) \times M(q'',s';\mathbb{C}), \tag{3.4.4e}$$
$$\mathfrak{p}^+(12,4) := \{(x_{12},x_{21}) : x_{12} = -{}^t x_{21} \in M(s',s'';\mathbb{C})\}, \tag{3.4.4f}$$

so that if $(x_{11}, x_{12}, x_{21}, x_{22}) \in \mathfrak{p}^+(11,d) \oplus \mathfrak{p}^+(12,d) \oplus \mathfrak{p}^+(22,d)$, then

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \begin{cases} \mathrm{Sym}(s'+s'',\mathbb{C}) & (d=1), \\ M(q'+q'', s'+s'';\mathbb{C}) & (d=2), \\ \mathrm{Skew}(s'+s'',\mathbb{C}) & (d=4) \end{cases}$$

holds. Now we observe the expansion of $\det(I - x_{11}x_{12}x_{22}x_{21})^{-\varepsilon\lambda}$. This is expanded as

$$\det(I - x_{11}x_{12}x_{22}x_{21})^{-\varepsilon\lambda} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{(\lambda)_{\mathbf{m},d}}{|\mathbf{m}|!}\tilde{\Phi}^{(d)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21})$$

where

$$r := \begin{cases} \min\{s', s''\} & (d = 1), \\ \min\{q', q'', s', s''\} & (d = 2), \\ \min\left\{\left\lfloor \frac{s'}{2}\right\rfloor, \left\lfloor \frac{s''}{2}\right\rfloor\right\} & (d = 4), \end{cases}$$

and each summand is in a single irreducible module, that is,

Lemma 3.4.1. *(1) As a polynomial in x_{11}, $\tilde{\Phi}^{(d)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+(11, d))$.*

(2) As a polynomial in x_{22}, $\tilde{\Phi}^{(d)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21}) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+(22, d))$.

(3) Let $d = 1$. As a polynomial in x_{12}, $\tilde{\Phi}^{(1)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21}) \in \mathcal{P}_{2\mathbf{m}}(\mathfrak{p}^+(12, 1))$, where $2\mathbf{m} = (2m_1, 2m_2, \ldots, 2m_r) \in \mathbb{Z}^{\min\{s',s''\}}_{++}$.

(4) Let $d = 4$. As a polynomial in x_{12}, $\tilde{\Phi}^{(4)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21}) \in \mathcal{P}_{\mathbf{m}^2}(\mathfrak{p}^+(12, 4))$, where $\mathbf{m}^2 = (m_1, m_1, m_2, m_2, \ldots, m_r, m_r(, 0)) \in \mathbb{Z}^{\min\{s',s''\}}_{++}$.

Proof. (1) Clear.

(2) Since $x_{11}x_{12}x_{22}x_{21}$ and $x_{22}x_{21}x_{11}x_{12}$ have the same eigenvalues, we have

$$\tilde{\Phi}^{(d)}_{\mathbf{m}}(x_{11}x_{12}x_{22}x_{21}) = \tilde{\Phi}^{(d)}_{\mathbf{m}}(x_{22}x_{21}x_{11}x_{12}),$$

and the claim follows.

(3) Since $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+(12, 1))$ is $GL(s', \mathbb{C}) \times GL(s'', \mathbb{C})$-invariant, we may assume $x_{11} = I_{s'}$, $x_{22} = I_{s''}$, and consider $\tilde{\Phi}^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12})$. For $x \in \mathrm{Sym}(s', \mathbb{C})$, we set

$$\Delta^{(1)}_{\mathbf{m}}(x) := \prod_{j=1}^{s'-1} \det((x_{kl})_{1\le k,l\le j})^{m_j - m_{j+1}} \det(x)^{m_{s'}}.$$

Then we have

$$\Phi^{(1)}_{\mathbf{m}}(x) = \int_{O(s')} \Delta^{(1)}_{\mathbf{m}}(kx{}^t k)dk,$$

and thus

$$\Phi^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12}) = \int_{O(s')} \Delta^{(1)}_{\mathbf{m}}(kx_{12}{}^t x_{12}{}^t k)dk.$$

Also since $\tilde{\Phi}^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12})$ is proportional to $\Phi^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12})$, $\tilde{\Phi}^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12})$ sits in a $GL(s', \mathbb{C}) \times GL(s'', \mathbb{C})$-module generated by $\Delta^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12})$. Next, for lower triangular matrices $l = (l_{kl})_{1\le l\le k\le s'} \in GL(s', \mathbb{C})$, we have

$$\Delta^{(1)}_{\mathbf{m}}(l^{-1}x_{12}{}^t x_{12}{}^t l^{-1}) = l_{11}^{-2m_1} l_{22}^{-2m_2} \cdots l_{s's'}^{-2m_{s'}} \Delta^{(1)}_{\mathbf{m}}(x_{12}{}^t x_{12}),$$

that is, this is the lowest weight vector with lowest weight $-2m_1\varepsilon_1 - \cdots - 2m_{s'}\varepsilon_{s'}$ under $GL(s', \mathbb{C})$. Since $\mathcal{P}(M(s', s''; \mathbb{C}))$ is decomposed as

$$\mathcal{P}(M(s', s''; \mathbb{C})) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^{\min\{s',s''\}}_{++}} \mathcal{P}_{\mathbf{m}}(M(s', s''; \mathbb{C})) = \bigoplus_{\mathbf{m}\in\mathbb{Z}^{\min\{s',s''\}}_{++}} V^{(s')\vee}_{\mathbf{m}} \boxtimes V^{(s'')}_{\mathbf{m}},$$

the highest weight under $GL(s'',\mathbb{C})$ of submodules in $\mathcal{P}(M(s',s'';\mathbb{C}))$ is uniquely determined by the lowest weight under $GL(s',\mathbb{C})$, and therefore $\Delta^{(1)}_{\mathbf{m}}(x_{12}{}^tx_{12})\in\mathcal{P}_{\mathbf{m}}(M(s',s'';\mathbb{C}))$ holds, and thus $\tilde{\Phi}^{(1)}_{\mathbf{m}}(x_{12}{}^tx_{12})\in\mathcal{P}_{\mathbf{m}}(M(s',s'';\mathbb{C}))$ also holds.

(4) Similarly to (3), we may assume $x_{11}=J_{s'}$, $x_{22}=J_{s''}$. Then by replacing $O(s')$ with $Sp\left(\left\lfloor\frac{s'}{2}\right\rfloor\right)$, and $\Delta^{(1)}_{\mathbf{m}}(x)$ on $\mathrm{Sym}(s',\mathbb{C})$ with

$$\Delta^{(4)}_{\mathbf{m}}(x):=\prod_{j=1}^{\lfloor s'/2\rfloor-1}\mathrm{Pf}((x_{kl})_{1\le k,l\le 2j})^{m_j-m_{j+1}}\,\mathrm{Pf}((x_{kl})_{1\le k,l\le 2\lfloor s_1/2\rfloor})^{m_{\lfloor s'/2\rfloor}}$$

on $\mathrm{Skew}(s',\mathbb{C})$, we can prove parallelly to (3). $\square$

Next, for $x_{\mathrm{s}}\in\mathrm{Sym}(s,\mathbb{C})$ and $x_{\mathrm{a}}\in\mathrm{Skew}(s,\mathbb{C})$, we want to consider the expansion of $\det(I-x_{\mathrm{s}}x_{\mathrm{a}})^{-\lambda}$. Since

$$\det(I-x_{\mathrm{s}}x_{\mathrm{a}})=\det(I-x_{\mathrm{a}}x_{\mathrm{s}})=\det({}^t(I-x_{\mathrm{a}}x_{\mathrm{s}}))=\det(I+x_{\mathrm{s}}x_{\mathrm{a}}),$$

we can rewrite

$$\det(I-x_{\mathrm{s}}x_{\mathrm{a}})^{-\lambda}=\det(I-x_{\mathrm{s}}x_{\mathrm{a}})^{-\lambda/2}\det(I+x_{\mathrm{s}}x_{\mathrm{a}})^{-\lambda/2}=\det(I-(x_{\mathrm{s}}x_{\mathrm{a}})^2)^{-\lambda/2}.$$

If $x_{\mathrm{s}}=I_s$ or $x_{\mathrm{a}}=J_s$, then $\det(I-x_{\mathrm{a}}^2)^{-\lambda/2}$, $\det(I-(x_{\mathrm{s}}J_s)^2)^{-\lambda/2}$ are $O(s)$, $Sp\left(\left\lfloor\frac{s}{2}\right\rfloor\right)$-invariant respectively. We set

$$t_j^{\mathrm{a}}:=\sqrt{-1}(E_{2j,2j-1}-E_{2j-1,2j})\in\mathrm{Skew}(s,\mathbb{C}),\qquad \mathfrak{a}^{\mathrm{a}}:=\bigoplus_{j=1}^{\lfloor s/2\rfloor}\mathbb{R}t_j^{\mathrm{a}}\subset\mathrm{Skew}(s,\mathbb{C}),$$

$$t_j^{\mathrm{s}}:=E_{2j,2j-1}+E_{2j-1,2j}\in\mathrm{Sym}(s,\mathbb{C}),\qquad \mathfrak{a}^{\mathrm{s}}:=\bigoplus_{j=1}^{\lfloor s/2\rfloor}\mathbb{R}t_j^{\mathrm{s}}\subset\mathrm{Sym}(s,\mathbb{C}).$$

Then $O(s)$-invariant functions on $\mathrm{Skew}(s,\mathbb{C})$ and $Sp\left(\left\lfloor\frac{s}{2}\right\rfloor\right)$-invariant functions on $\mathrm{Sym}\left(2\left\lfloor\frac{s}{2}\right\rfloor,\mathbb{C}\right)$ are determined by the values on $\mathfrak{a}^{\mathrm{a}}$ and $\mathfrak{a}^{\mathrm{s}}$ respectively. We note that even when s is odd, we do not have to consider the $\mathrm{Sym}\left(2\left\lfloor\frac{s}{2}\right\rfloor,\mathbb{C}\right)^\perp=\mathrm{Sym}(s-1,\mathbb{C})^\perp:=\bigoplus_{j=1}^{s}\mathbb{C}(E_{s,j}+E_{j,s})$-dependence in this case, because $\det(I-(x_{\mathrm{s}}J_s)^2)^{-\lambda/2}$ does not depend on $\mathrm{Sym}(s-1,\mathbb{C})^\perp$. When $x_{\mathrm{a}}=\sum a_jt_j^{\mathrm{a}}\in\mathfrak{a}^{\mathrm{a}}$ or $x_{\mathrm{s}}=\sum a_jt_j^{\mathrm{s}}\in\mathfrak{a}^{\mathrm{s}}$, then we have

$$\begin{aligned}\det(I-x_{\mathrm{a}}^2)^{-\lambda/2}=\det(I-(x_{\mathrm{s}}J_s)^2)^{-\lambda/2}&=\prod_{j=1}^{\lfloor s/2\rfloor}(1-a_j^2)^{-\lambda}\\&=\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}}\frac{(\lambda)_{\mathbf{m},2}}{|\mathbf{m}|!}\tilde{\Phi}^{(2)}_{\mathbf{m}}(a_1^2,\dots,a_{\lfloor s/2\rfloor}^2).\end{aligned}$$

For $x_{\mathrm{s}}\in\mathrm{Sym}(s,\mathbb{C})$ and $x_{\mathrm{a}}\in\mathrm{Skew}(s,\mathbb{C})$, we take $l_{\mathrm{s}},l_{\mathrm{a}}\in GL(s,\mathbb{C})$ such that $x_{\mathrm{s}}=l_{\mathrm{s}}{}^tl_{\mathrm{s}}$, $x_{\mathrm{a}}=l_{\mathrm{a}}J_s{}^tl_{\mathrm{a}}$. Then we have

$$\det(I-(x_{\mathrm{s}}x_{\mathrm{a}}))^2=\det(I-({}^tl_{\mathrm{s}}x_{\mathrm{a}}l_{\mathrm{s}})^2)=\det(I-({}^tl_{\mathrm{a}}x_{\mathrm{s}}l_{\mathrm{a}}J_s)^2),$$

and a_j's for ${}^tl_{\mathrm{s}}x_{\mathrm{a}}l_{\mathrm{s}}$ and ${}^tl_{\mathrm{a}}x_{\mathrm{s}}l_{\mathrm{a}}$ coincide. Thus using these a_j, we define

$$\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}}x_{\mathrm{a}})^2)=\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{a}}x_{\mathrm{s}})^2):=\tilde{\Phi}^{(2)}_{\mathbf{m}}(a_1^2,\dots,a_{\lfloor s/2\rfloor}^2),$$

so that

$$\det(I-x_{\mathrm{s}}x_{\mathrm{a}})^{-\lambda}=\det(I-(x_{\mathrm{s}}x_{\mathrm{a}})^2)^{-\lambda/2}=\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}}\frac{(\lambda)_{\mathbf{m},2}}{|\mathbf{m}|!}\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_{\mathrm{s}}x_{\mathrm{a}})^2).$$

Then each summand is in a single irreducible module, that is,

Lemma 3.4.2. *(1) As a polynomial in x_{a}, $\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_{\mathrm{s}}x_{\mathrm{a}})^2)\in\mathcal{P}_{2\mathbf{m}}(\mathrm{Skew}(s,\mathbb{C}))$, where $2\mathbf{m}=(2m_1,2m_2,\dots,2m_{\lfloor s/2\rfloor})\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}$.*

(2) As a polynomial in x_{s}, $\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_{\mathrm{s}}x_{\mathrm{a}})^2)\in\mathcal{P}_{\mathbf{m}^2}(\mathrm{Sym}(s,\mathbb{C}))$, where $\mathbf{m}^2=(m_1,m_1,m_2,m_2,\dots,m_{\lfloor s/2\rfloor},m_{\lfloor s/2\rfloor}(,0))\in\mathbb{Z}_{++}^s$.

To prove this, we need the following lemma on Laplace-Beltrami operators (3.2.10),

$$Lf=\sum_{\alpha\beta}\varepsilon\,\mathrm{tr}(xe_\alpha^*xe_\beta^*)\frac{\partial^2}{\partial x_\alpha\partial x_\beta}+\frac{n_{\mathrm{T}}}{r}\sum_\alpha\varepsilon\,\mathrm{tr}(xe^*)\frac{\partial}{\partial x_\alpha},$$

where $\left(\varepsilon,\frac{n_{\mathrm{T}}}{r}\right)=\left(\frac{1}{2},2\left\lfloor\frac{s}{2}\right\rfloor-1\right)$ on $\mathrm{Skew}(s,\mathbb{C})$, $\left(\varepsilon,\frac{n_{\mathrm{T}}}{r}\right)=\left(1,\frac{s+1}{2}\right)$ on $\mathrm{Sym}(s,\mathbb{C})$, $\{e_\alpha\}$ is a basis, with the dual basis $\{e_\alpha^\vee\}$ with respect to the inner product $\varepsilon\,\mathrm{tr}(xy^*)$, and $\frac{\partial}{\partial x_\alpha}$ is the directional derivative along the direction of $e_\alpha^\vee$.

Lemma 3.4.3. *(1) For $O(s)$-invariant functions on $\mathrm{Skew}(s,\mathbb{C})$, using the coordinate $x_{\mathrm{a}}=\sum a_jt_j^{\mathrm{a}}\in\mathfrak{a}^{\mathrm{a}}$, we have*

$$Lf=\sum_{j=1}^{\lfloor s/2\rfloor}a_j^2\frac{\partial^2 f}{\partial a_j^2}+4\sum_{j<k}\frac{a_j^2a_k^2}{a_j^2-a_k^2}\left(\frac{1}{a_j}\frac{\partial f}{\partial a_j}-\frac{1}{a_k}\frac{\partial f}{\partial a_k}\right)+\left(2\left\lfloor\frac{s}{2}\right\rfloor-1\right)\sum_{j=1}^{\lfloor s/2\rfloor}a_j\frac{\partial f}{\partial a_j}.$$

(2) For $Sp\left(\left\lfloor\frac{s}{2}\right\rfloor\right)$-invariant functions on $\mathrm{Sym}(s,\mathbb{C})$, using the coordinate $x_{\mathrm{s}}=\sum a_jt_j^{\mathrm{s}}\in\mathfrak{a}^{\mathrm{s}}$, we have

$$Lf=\frac{1}{2}\sum_{j=1}^{\lfloor s/2\rfloor}a_j^2\frac{\partial^2 f}{\partial a_j^2}+2\sum_{j<k}\frac{a_j^2a_k^2}{a_j^2-a_k^2}\left(\frac{1}{a_j}\frac{\partial f}{\partial a_j}-\frac{1}{a_k}\frac{\partial f}{\partial a_k}\right)+\frac{s-1}{2}\sum_{j=1}^{\lfloor s/2\rfloor}a_j\frac{\partial f}{\partial a_j}.$$

These are proved similarly to [6, Proposition VI.4.2].

Proof of Lemma 3.4.2. (1) We may assume $x_{\mathrm{s}}=I_s$. Then $\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}(x_{\mathrm{a}}^2)$ is $O(s)$-invariant. By the change of variables $a_j^2=b_j$, L on $\mathrm{Skew}(s,\mathbb{C})$ is rewritten as

$$Lf=4\left(\sum_{j=1}^{\lfloor s/2\rfloor}b_j^2\frac{\partial^2 f}{\partial b_j^2}+2\sum_{j<k}\frac{b_jb_k}{b_j-b_k}\left(\frac{\partial f}{\partial b_j}-\frac{\partial f}{\partial b_k}\right)+\left\lfloor\frac{s}{2}\right\rfloor\sum_{j=1}^{\lfloor s/2\rfloor}b_j\frac{\partial f}{\partial b_j}\right).$$

Then since $\tilde{\Phi}_{\mathbf{m}}^{(2)}$ is an eigenfunction of the Laplace-Beltrami operator on $M\left(\left\lfloor\frac{s}{2}\right\rfloor,\mathbb{C}\right)$ with the eigenvalue $\sum_{j=1}^{\lfloor s/2\rfloor}m_j\left(m_j-\left(2j-\left\lfloor\frac{s}{2}\right\rfloor-1\right)\right)$ by Proposition 3.2.3, we have

$$\begin{aligned}L\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}(x_{\mathrm{a}}^2)&=4\sum_{j=1}^{\lfloor s/2\rfloor}m_j\left(m_j-\left(2j-\left\lfloor\frac{s}{2}\right\rfloor-1\right)\right)\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}(x_{\mathrm{a}}^2)\\&=\sum_{j=1}^{\lfloor s/2\rfloor}2m_j\left(2m_j-2\left(2j-\left\lfloor\frac{s}{2}\right\rfloor-1\right)\right)\tilde{\Phi}_{\mathbf{m}}^{(2)\prime}(x_{\mathrm{a}}^2).\end{aligned}$$

Since the highest weight of a finite-dimensional representation is uniquely determined by the action of the Casimir element, we conclude that $\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}(x_{\mathrm{a}}^2) \in \mathcal{P}_{2\mathbf{m}}(\mathrm{Skew}(s, \mathbb{C}))$.

(2) Similarly, we may assume $x_{\mathrm{a}} = J_s$. Then $\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}} J_s)^2)$ is $Sp\left(\left\lfloor \frac{s}{2} \right\rfloor\right)$-invariant. By the change of variables $a_j^2 = b_j$, L on $\mathrm{Sym}(s, \mathbb{C})$ is rewritten as

$$Lf = 2\left(\sum_{j=1}^{\lfloor s/2\rfloor} b_j^2 \frac{\partial^2 f}{\partial b_j^2} + 2\sum_{j<k} \frac{b_j b_k}{b_j - b_k}\left(\frac{\partial f}{\partial b_j} - \frac{\partial f}{\partial b_k}\right) + \left\lfloor \frac{s}{2} \right\rfloor \sum_{j=1}^{\lfloor s/2\rfloor} b_j \frac{\partial f}{\partial b_j}\right) + \left(s - 2\left\lfloor \frac{s}{2} \right\rfloor\right) \sum_{j=1}^{\lfloor s/2\rfloor} b_j \frac{\partial f}{\partial b_j},$$

and therefore

$$\begin{aligned} L\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}} J_s)^2) &= \left(2\sum_{j=1}^{\lfloor s/2\rfloor} m_j\left(m_j - \left(2j - \left\lfloor \frac{s}{2} \right\rfloor - 1\right)\right) + \left(s - 2\left\lfloor \frac{s}{2} \right\rfloor\right)|\mathbf{m}|\right) \tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}} J_s)^2) \\ &= \sum_{j=1}^{\lfloor s/2\rfloor}\left(m_j\left(m_j - \frac{1}{2}(2(2j-1) - s - 1)\right) + m_j\left(m_j - \frac{1}{2}(2(2j) - s - 1)\right)\right) \tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}} J_s)^2). \end{aligned}$$

Thus we conclude that $\tilde{\Phi}^{(2)\prime}_{\mathbf{m}}((x_{\mathrm{s}} J_s)^2) \in \mathcal{P}_{\mathbf{m}^2}(\mathrm{Sym}(s, \mathbb{C}))$. □

3.5 Examples of intertwining operators

3.5.1 Normal derivative case

In this subsection, we seek a sufficient condition for $\mathcal{F}^*_{W_1}$, $\mathcal{F}_{W_1}$ to become a normal derivative, that is, a differential operator for the direction of $\mathfrak{p}_2^+$. Let $G \supset G_1$ be two real reductive groups of Hermitian type satisfying the assumption (3.3.1), (τ, V) be an irreducible finite-dimensional representation of $\tilde{K}^{\mathbb{C}}$ such that $\mathcal{H}_\tau(D, V)$ is non-trivial, and let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be a $\tilde{K}_1^{\mathbb{C}}$-invariant polynomial in the sense of (3.3.3). Let $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+, V)$ be a subrepresentation of $\tilde{K}_1^{\mathbb{C}}$ such that $K(\cdot, y_2) \in W_1$ for any $y_2 \in \mathfrak{p}_2^+$. By taking the projection of K into irreducible subspaces, we may assume W_1 is irreducible. Then the following holds.

Theorem 3.5.1. *(1) Assume that there exists an irreducible subrepresentation $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ of $\tilde{K}$ such that $W_1 \subset W$. Then the linear map*

$$\mathcal{F}^*_{W_1} : \mathcal{H}_\tau(D, V)_{\tilde{K}} \to \mathcal{H}_{\tau \otimes (Ad|_{\mathfrak{p}_2^+})^*}(D_1, W_1)_{\tilde{K}_1} \subset \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)_{\tilde{K}_1},$$

$$(\mathcal{F}^*_{W_1} f)(y_1, y_2) = K\left(\overline{\frac{\partial}{\partial x_2}}, y_2\right)^*\bigg|_{x_2=0} f(y_1, x_2)$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action.

(2) We take a subrepresentation $V_1 \subset V$ such that $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+, V_1)$. Assume that $\mathrm{Proj}_2((x_2)^{y_1}) = x_2$, and $\tau(B(x_2, y_1))|_{V_1} = I_{V_1}$ for any $x_2 \in \mathfrak{p}_2^+$, $y_1 \in \mathfrak{p}_1^+$. Then the linear map

$$\mathcal{F}_{W_1} : \mathcal{H}'_\tau(D_1 \times \mathfrak{p}_2^+, V)_{\tilde{K}} \supset \mathcal{H}_{\tau \otimes (Ad|_{\mathfrak{p}_2^+})^*}(D_1, W_1)_{\tilde{K}_1} \to \mathcal{H}_\tau(D, V)_{\tilde{K}_1},$$

$$(\mathcal{F}_{W_1} f)(x_1, x_2) = K\left(x_2, \overline{\frac{\partial}{\partial y_2}}\right)\bigg|_{y_2=0} f(x_1, y_2)$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$*-action.*

Proof. (1) Since $e^{(x|z)_{\mathfrak{p}^+}} I_V$ is the reproducing kernel of $\mathcal{P}(\mathfrak{p}^+, V)$ with respect to the inner product $\langle \cdot, \cdot \rangle_F$, the projection of $e^{(x|z)_{\mathfrak{p}^+}} I_V$ onto any subrepresentation of $\mathcal{P}(\mathfrak{p}^+, V)$ is nonzero. Let $K_W(x,z) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ be the orthogonal projection of $e^{(x|z)_{\mathfrak{p}^+}} I_V$ onto W with respect to the inner product $\langle \cdot, \cdot \rangle_{\hat{\tau}}$. Then we have

$$\begin{aligned} F_{W_1}^*(z_1, z_2; y_2)^* &= \int_D e^{(z|x)_{\mathfrak{p}^+}} \tau(B(x)^{-1}) K(x_2, y_2) h(x)^{-p} dx \\ &= \int_D K_W(z,x) \tau(B(x)^{-1}) K(x_2, y_2) h(x)^{-p} dx. \end{aligned}$$

Then since the map $f \mapsto \int_D K_W(z,x)\tau(B(x)^{-1}) f(x) h(x)^{-p} dx$ in $\mathrm{End}(W)$ intertwines the $\tilde{K}$-action, by Schur's lemma, there exists a constant C such that

$$F_{W_1}^*(z_1, z_2; y_2)^* = CK(z_2, y_2) \qquad \therefore F_{W_1}^*(z_1, z_2; y_2) = \bar{C} K(z_2, y_2)^*.$$

Since the intertwining property does not change by scalar multiplication, we may omit $\bar{C}$. Then the corresponding $\mathcal{F}_K^*$ intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action, and the claim follows.

(2) By the assumption, we have

$$\begin{aligned} &F_{W_1}(x_2; w_1, w_2) \\ &= \int_{D_1} \tau(B(x_2, y_1)) K(\mathrm{Proj}_2((x_2)^{y_1}), B(y_1) w_2) \tau(B(y_1)^{-1}) e^{(y_1|w_1)_{\mathfrak{p}^+}} h_1(y_1)^{-p_1} dy_1 \\ &= \int_{D_1} K(x_2, B(y_1) w_2) \tau(B(y_1)^{-1}) e^{(y_1|w_1)_{\mathfrak{p}^+}} h_1(y_1)^{-p_1} dy_1 \\ &= \int_{D_1} K(B(y_1) x_2, w_2) \tau(B(y_1)^{-1}) e^{(y_1|w_1)_{\mathfrak{p}^+}} h_1(y_1)^{-p_1} dy_1. \end{aligned}$$

Then $x_1 \mapsto K(B(y_1)x_2, w_2)$ is regarded as a W-valued constant function on $\mathfrak{p}_1^+$, and such functions forms the irreducible subrepresentation of $\tilde{K}_1^{\mathbb{C}}$ in $\mathcal{P}(\mathfrak{p}_1^+, W)$. Thus by the argument similar to (1), we can show that $F_{W_1}(x_2; w_1, w_2)$ is proportional to $K(x_2, w_2)$, and the claim follows. $\square$

The condition in Theorem 3.5.1 (1) is the same as [25, Lemma 5.5 (3)] when (G, G_1) is of split rank 1 (i.e. $(G, G_1) = (U(q,s), U(q, s-1) \times U(1))$, $(SO^*(2s), SO^*(2(s-1)) \times SO(2))$, or $(SO(2,2s), U(1,s))$), and (τ, V) is 1-dimensional. That is also satisfied when $(G, G_1) = (U(q,s), U(q,s') \times U(s''))$ with $s' + s'' = s$, and (τ, V) is 1-dimensional. That is,

Corollary 3.5.2. *Let* $(G, G_1) = (U(q,s), U(q,s') \times U(s''))$*,* $(SO^*(2s), SO^*(2(s-1)) \times SO(2))$*, or* $(SO(2,2s), U(1,s))$*, and* $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ *be 1-dimensional. Then for any subrepresentation* $\mathcal{H}_\lambda(D_1, W_1) \subset \mathcal{H}_\lambda(D)$ *of* $\tilde{G}_1$*, the intertwining operator* $\mathcal{F}_{W_1}^* : \mathcal{H}_\lambda(D) \to \mathcal{H}_\lambda(D_1, W_1)$ *is given by normal derivative.*

Proof. Since it is already proved for $(G, G_1) = (U(q,s), U(1) \times U(q-1, s))$, $(SO^*(2s), SO^*(2(s-1)) \times SO(2))$, or $(SO(2,2s), U(1,s))$ in [25], we only deal with $(G, G_1) = (U(q,s), U(q,s') \times U(s''))$. In this case we have $\mathfrak{p}^+ = M(q,s;\mathbb{C})$, $\mathfrak{p}_1^+ = M(q,s';\mathbb{C})$, $\mathfrak{p}_2^+ = M(q,s'';\mathbb{C})$, and

$$\begin{aligned} \mathcal{P}(\mathfrak{p}^+) &= \bigoplus_{\mathbb{Z}_{++}^{\min\{q,s\}}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) = \bigoplus_{\mathbb{Z}_{++}^{\min\{q,s\}}} V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{m}}^{(s)}, \\ \mathcal{P}(\mathfrak{p}_2^+) &= \bigoplus_{\mathbb{Z}_{++}^{\min\{q,s''\}}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}_2^+) = \bigoplus_{\mathbb{Z}_{++}^{\min\{q,s''\}}} V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{m}}^{(s'')}. \end{aligned}$$

Then by comparing the weights for $GL(q,\mathbb{C})$, we get $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}_2^+) \subset \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, and clearly we also get $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}_2^+) \otimes \chi^{-\lambda} \subset \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \otimes \chi^{-\lambda}$, and therefore the condition in Theorem 3.5.1 (1) is satisfied. $\square$

Next we consider $\mathcal{F}_{W_1}$. We again consider

$$(G, G_1) = \begin{cases} (U(q,s), U(q,s') \times U(s'')) & \text{(Case 1)}, \\ (SO^*(2s), SO^*(2(s-1)) \times SO(2)) & \text{(Case 2)}, \\ (SO(2,2s), U(1,s)) & \text{(Case 3)}. \end{cases}$$

Then $\mathfrak{p}^+ = M(q,s;\mathbb{C})$, $\mathrm{Skew}(s,\mathbb{C})$ and $\mathbb{C}^{2s}$ respectively. We realize $G_1 \subset G$ such that

$$\mathfrak{p}_1^+ = \mathfrak{g}_1 \cap \mathfrak{p}^+ = \begin{cases} \left\{ y_1 = \begin{pmatrix} y & 0 \end{pmatrix} : y \in M(q,s';\mathbb{C}) \right\} & \text{(Case 1)}, \\ \left\{ y_1 = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} : y \in \mathrm{Skew}(s-1,\mathbb{C}) \right\} & \text{(Case 2)}, \\ \left\{ y_1 = \left(\frac{1}{2}y, \frac{\sqrt{-1}}{2}y \right) : y \in \mathbb{C}^s \right\} & \text{(Case 3)}, \end{cases}$$

$$\mathfrak{p}_2^+ = (\mathfrak{p}_1^+)^\perp = \begin{cases} \left\{ x_2 = \begin{pmatrix} 0 & x \end{pmatrix} : x \in M(q,s'';\mathbb{C}) \right\} & \text{(Case 1)}, \\ \left\{ x_2 = \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} : x \in M(s-1,1;\mathbb{C}) \right\} & \text{(Case 2)}, \\ \left\{ x_2 = \left(\frac{1}{2}x, -\frac{\sqrt{-1}}{2}x \right) : x \in \mathbb{C}^s \right\} & \text{(Case 3)}. \end{cases}$$

Then for $(y_1, x_2) \in \mathfrak{p}_1^+ \times \mathfrak{p}_2^+$, we have

$$\begin{aligned} B(x_2,y_1) &= \left(I_q - \begin{pmatrix} 0 & x \end{pmatrix} \begin{pmatrix} y^* \\ 0 \end{pmatrix}, \left(I_s - \begin{pmatrix} y^* \\ 0 \end{pmatrix} \begin{pmatrix} 0 & x \end{pmatrix} \right)^{-1} \right) \\ &= \left(I_q, \begin{pmatrix} I_{s'} & -y^*x \\ 0 & I_{s''} \end{pmatrix}^{-1} \right) && \text{(Case 1)}, \\ B(x_2,y_1) &= I_s - \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} \begin{pmatrix} y^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{s-1} & 0 \\ -{}^t x y^* & 1 \end{pmatrix} && \text{(Case 2)}, \\ h(x_2,y_1) &= 1 - 2q(x_2, \overline{y_1}) + q(x_2)\overline{q(y_1)} = 1 && \text{(Case 3)}, \end{aligned}$$

and

$$\begin{aligned} (x_2)^{y_1} &= \begin{pmatrix} 0 & x \end{pmatrix} \left(I_s - \begin{pmatrix} y^* \\ 0 \end{pmatrix} \begin{pmatrix} 0 & x \end{pmatrix} \right)^{-1} = \begin{pmatrix} 0 & x \end{pmatrix} = x_2 && \text{(Case 1)}, \\ (x_2)^{y_1} &= \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} \left(I_s - \begin{pmatrix} y^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} = x_2 && \text{(Case 2)}, \\ (x_2)^{y_1} &= \left(1 - 2q(x_2,\overline{y_1}) + q(x_2)\overline{q(y_1)} \right)^{-1} (x_2 - q(x_2)\overline{y_1}) = x_2 && \text{(Case 3)}. \end{aligned}$$

Thus $(x_2)^{y_1} = \mathrm{Proj}_2((x_2)^{y_1}) = x_2$ holds, and for the representation

$$V = \chi^{-\lambda} \otimes \begin{cases} V_{\mathbf{k}}^{(q)\vee} \boxtimes V_{\mathbf{m}}^{(s)} & \text{(Case 1)} \\ V_{\mathbf{m}}^{(s)\vee} & \text{(Case 2)} \\ \mathbf{1} & \text{(Case 3)} \end{cases}$$

of $\tilde{K}^{\mathbb{C}}$, if we take the subrepresentation

$$V_1 = \chi^{-\lambda} \otimes \begin{cases} V_{\mathbf{k}}^{(q)\vee} \boxtimes V_{(m_1,\ldots,m'_s)}^{(s')} \boxtimes V_{(m_{s'+1},\ldots,m_s)}^{(s'')} & \text{(Case 1)} \\ V_{(m_1,\ldots,m_{s-1})}^{(s-1)\vee} \boxtimes \mathbb{C}_{-m_s} & \text{(Case 2)} \\ \mathbf{1} & \text{(Case 3)} \end{cases}$$

of $\tilde{K}_1^{\mathbb{C}}$, then $\tau(B(x_2, y_1))|_{V_1} = I_{V_1}$ holds. Thus we proved the following.

Corollary 3.5.3. *(1) Let $(G, G_1) = (U(q, s), U(q, s') \times U(s''))$, and $(\tau, V) = (\chi^{-\lambda_1-\lambda_2} \otimes (\tau_{\mathbf{k}}^{(q)\vee} \boxtimes \tau_{\mathbf{m}}^{(s)}), V_{\mathbf{k}}^{(q)\vee} \otimes V_{\mathbf{m}}^{(s)})$. Then for any subrepresentation $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+, V_{\mathbf{k}}^{(q)\vee} \boxtimes V_{(m_1,\ldots,m'_s)}^{(s')} \boxtimes V_{(m_{s'+1},\ldots,m_s)}^{(s'')})$ of $\tilde{K}_1^{\mathbb{C}}$, the intertwining operator $\mathcal{F}_{W_1} : \mathcal{H}_{\lambda_1+\lambda_2}(D_1, W_1) \to \mathcal{H}_{\lambda_1+\lambda_2}(D, V)$ is given by normal derivative.*

(2) Let $(G, G_1) = (SO^*(2s), SO^*(2(s-1)) \times SO(2))$, and $(\tau, V) = (\chi^{-\lambda} \otimes \tau_{\mathbf{m}}^{(s)\vee}, V_{\mathbf{m}}^{(s)\vee})$. Then for any subrepresentation $W_1 \subset \mathcal{P}(\mathfrak{p}_2^+, V_{(m_1,\ldots,m_{s-1})}^{(s-1)\vee} \boxtimes \mathbb{C}_{-m_s})$ of $\tilde{K}_1^{\mathbb{C}}$, the intertwining operator $\mathcal{F}_{W_1} : \mathcal{H}_\lambda(D_1, W_1) \to \mathcal{H}_\lambda(D, V)$ is given by normal derivative.

(3) Let $(G, G_1) = (SO(2, 2s), U(1, s))$, and $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ be 1-dimensional. Then for any subrepresentation $\mathcal{H}_\lambda(D_1, W_1) \subset \mathcal{H}_\lambda(D)$ of $\tilde{G}_1$, the intertwining operator $\mathcal{F}_{W_1} : \mathcal{H}_\lambda(D_1, W_1) \to \mathcal{H}_\lambda(D)$ is given by normal derivative.

3.5.2 $\mathcal{F}_{W_1}^*$ for $(G, G_1) = (G_0 \times G_0, \Delta G_0)$

In this subsection we seek the operator $\mathcal{F}_{W_1}^*$ for $(G, G_1) = (G_0 \times G_0, \Delta G_0)$, where G_0 is a simple Lie group of Hermitian type, although it is already done by Peng-Zhang [34]. We denote the complexified Lie algebra of G_0 by $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{p}_0^+ \oplus \mathfrak{k}_0^{\mathbb{C}} \oplus \mathfrak{p}_0^-$. Similarly, we denote the objects such as $D \subset \mathfrak{p}^+$, $h(x, y) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+})$, $p \in \mathbb{Z}$ for G_0 by writing the subscript 0. Then we have

$$\mathfrak{p}_1^+ = \{(x_0, x_0) : x_0 \in \mathfrak{p}_0^+\}, \qquad \mathfrak{p}_2^+ = \{(x_0, -x_0) : x_0 \in \mathfrak{p}_0^+\} \subset \mathfrak{p}^+ = \mathfrak{p}_0^+ \oplus \mathfrak{p}_0^+.$$

We identify $\mathfrak{p}_0^+$ and $\mathfrak{p}_1^+$, $\mathfrak{p}_2^+$ via $x_0 \mapsto (x_0, x_0)$ and $x_0 \mapsto (x_0, -x_0)$ respectively. Then for $x = (x_L, x_R) \in \mathfrak{p}$, the projection onto $\mathfrak{p}_2^+$ is given by

$$x_2 = \mathrm{Proj}_2((x_L, x_R)) = \frac{1}{2}(x_L - x_R).$$

Let $(\tau, V) = (\tau_L \boxtimes \tau_R, V_L \otimes V_R)$ be a finite dimensional irreducible representation of $\tilde{K} = \tilde{K}_0 \times \tilde{K}_0$. Let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be a $\tilde{K}^{\mathbb{C}}$-invariant polynomial in the sense of (3.3.3). Then the function $F_{W_1}^*(z_L, z_R; y_2) \in \mathcal{P}(\overline{\mathfrak{p}^+} \times \mathfrak{p}_2^+, \mathrm{End}(V))$ in Theorem 3.3.5 (1) is given by

$$F_{W_1}^*(z_L, z_R; y_2) = \iint_{D_0 \times D_0} K\left(\frac{1}{2}(x_L - x_R), y_2\right)^* (\tau_L(B(x_L)^{-1}) \otimes \tau_R(B(x_R)^{-1})) \\ \times e^{(x_L|z_L)_{\mathfrak{p}_0^+} + (x_R|z_R)_{\mathfrak{p}_0^+}} h_0(x_L)^{-p_0} h_0(x_R)^{-p_0} dx_L dx_R.$$

Especially, when $(\tau, V) = (\chi_0^{-\lambda} \boxtimes \chi_0^{-\mu}, \mathbb{C})$ is 1-dimensional, with $\lambda, \mu > p_0 - 1$, rewriting $K\left(\frac{x_2}{2}, y_2\right)$ as $K(x_2, y_2)$, we get

$$F_{W_1}(z_L, z_R; y_2) = \iint_{D_0 \times D_0} \overline{K(x_L - x_R, y_2)} e^{(x_L|z_L)_{\mathfrak{p}_0^+} + (x_R|z_R)_{\mathfrak{p}_0^+}} h_0(x_L)^{\lambda - p_0} h_0(x_R)^{\mu - p_0} dx_L dx_R.$$

Now we additionally assume that $K(x_2,y_2)$ is proportional to the reproducing kernel of $\mathcal{P}_{(k,\ldots,k)}(\mathfrak{p}_0^+)$ with $k\in\mathbb{Z}_{\ge 0}$. We normalize $K(x_2,y_2)$ such that $K(x_2,y_2)=\Delta(x_2)^k\overline{\Delta(y_2)^k}$ if $x_2,y_2\in\mathfrak{p}_{\mathrm{T},0}^+$. Then for $x_L,x_R,y_2\in\mathfrak{p}_{\mathrm{T},0}^+$, we have

$$\begin{aligned}K(x_L-x_R,y_2)&=\Delta(x_L-x_R)^k\overline{\Delta(y_2)^k}=\Delta(x_L)^k\overline{\Delta(y_2)^k}\Delta\left(e_0-P(x_L^{-1/2})x_R\right)^k\\&=\Delta(x_L)^k\overline{\Delta(y_2)^k}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r_0}}(-k)_{\mathbf{m},d_0}\frac{d_{\mathbf{m}}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}}\Phi_{\mathbf{m}}^{(d_0,r_0)}\left(P(x_L^{-1/2})x_R\right).\end{aligned}$$

By [6, Lemma XIV.1.2], we have $\Delta(x_L)^k\Phi_{\mathbf{m}}^{(d_0,r_0)}\left(P(x_L^{-1/2})x_R\right)=\Delta(x_L)^k\Phi_{\mathbf{m}}^{(d_0,r_0)}\left(P(x_R^{1/2})x_L^{-1}\right)$. This lies in $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}_{\mathrm{T},0}^+)$ as a polynomial in x_R, and lies in $\mathcal{P}_{k-\mathbf{m}^*}(\mathfrak{p}_{\mathrm{T},0}^+)$ as a polynomial in x_L, where $k-\mathbf{m}^*:=(k-m_{r_0},k-m_{r_0-1},\ldots,k-m_1)$. Now let $\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(x_L,x_R;y_2)\in\mathcal{P}(\mathfrak{p}_0^+\times\mathfrak{p}_0^+\times\overline{\mathfrak{p}_0^+})$ be the polynomial satisfying

$$\begin{aligned}\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(lx_L,lx_R;y_2)&=\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(x_L,x_R;l^*y_2) && (x_L,x_R,y_2\in\mathfrak{p}_0^+,\ l\in K_0^{\mathbb{C}}),\\ \Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(x_L,x_R;y_2)&=\Delta(x_L)^k\overline{\Delta(y_2)^k}\Phi_{\mathbf{m}}^{(d_0,r_0)}\left(P(x_L^{-1/2})x_R\right) && (x_L,x_R,y_2\in\mathfrak{p}_{\mathrm{T},0}^+),\end{aligned}$$

and write

$$\overline{\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(x_L,x_R;y_2)}=:\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(y_2;x_L,x_R),$$

so that

$$\overline{K(x_L-x_R,y_2)}=\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r_0}}(-k)_{\mathbf{m},d_0}\frac{d_{\mathbf{m}}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}}\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(y_2;x_L,x_R).$$

Using this expansion, we get

$$\begin{aligned}&F_{W_1}^*(z_L,z_R;y_2)\\&=\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r_0}}(-k)_{\mathbf{m},d_0}\frac{d_{\mathbf{m}}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}}\iint_{D_0\times D_0}\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(y_2;x_L,x_R)e^{(x_L|z_L)_{\mathfrak{p}_0^+}+(x_R|z_R)_{\mathfrak{p}_0^+}}\\&\qquad\qquad\times h_0(x_L)^{\lambda-p_0}h_0(x_R)^{\mu-p_0}dx_Ldx_R\\&=C\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r_0}}\frac{(-k)_{\mathbf{m},d_0}}{(\lambda)_{k-\mathbf{m}^*,d_0}(\mu)_{\mathbf{m},d_0}}\frac{d_{\mathbf{m}}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}}\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(y_2;z_L,z_R),\end{aligned}$$

with some C. Here we used (3.2.13). We note that the sum is finite because $(-k)_{\mathbf{m},d_0}=0$ if $m_1>k$, and the above formula is symmetric under the exchange of (z_L,λ) and (z_R,μ) up to signature, because

$$\begin{aligned}\Psi_{k-\mathbf{m}^*,\mathbf{m}}^{(d_0,r_0)}(y_2;z_L,z_R)&=\Psi_{\mathbf{m},k-\mathbf{m}^*}^{(d_0,r_0)}(y_2;z_R,z_L),\\(-k)_{\mathbf{m},d_0}\frac{d_{\mathbf{m}}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}}&=(-k)_{\mathbf{m},d_0}\frac{d_{\mathbf{m}}^{(d_0,r_0,0)}}{\left(\frac{n_{0,\mathrm{T}}}{r_0}\right)_{\mathbf{m},d_0}}\\&=(-1)^{kr}(-k)_{k-\mathbf{m}^*,d_0}\frac{d_{k-\mathbf{m}^*}^{(d_0,r_0,0)}}{\left(\frac{n_{0,\mathrm{T}}}{r_0}\right)_{k-\mathbf{m}^*,d_0}}=(-1)^{kr}(-k)_{k-\mathbf{m}^*,d_0}\frac{d_{k-\mathbf{m}^*}^{(d_0,r_0,b_0)}}{\left(\frac{n_0}{r_0}\right)_{k-\mathbf{m}^*,d_0}},\end{aligned}$$

the latter of which follows from the proof of [33, Proposition 2.6]. Since the intertwining property does not change under scalar multiplication, we may omit the constant C, and write $\mathcal{F}^*_{\lambda,\mu,k} := C^{-1}\mathcal{F}^*_{W_1}$. Then we have proved the following.

Theorem 3.5.4. *Let $\lambda, \mu > p_0 - 1$, and $k \in \mathbb{Z}_{\geq 0}$. Then the linear map*

$$\mathcal{F}^*_{\lambda,\mu,k} : \mathcal{H}_\lambda(D_0) \boxtimes \mathcal{H}_\mu(D_0)_{\tilde{K}_0\times\tilde{K}_0} \to \mathcal{H}_{\lambda+\mu}(D_0, \mathcal{P}_{(k,\dots,k)}(\mathfrak{p}_0^+))_{\tilde{K}_0},$$

$$\mathcal{F}^*_{\lambda,\mu,k} f(y_1, y_2) := \sum_{\mathbf{m}\in\mathbb{Z}^{r_0}_{++}} \frac{(-k)_{\mathbf{m},d_0}}{(\lambda)_{k-\mathbf{m}^*,d_0}(\mu)_{\mathbf{m},d_0}} \frac{d^{(d_0,r_0,b_0)}_{\mathbf{m}}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}} \times \Psi^{(d_0,r_0)}_{k-\mathbf{m}^*,\mathbf{m}}\left(y_2; \overline{\frac{\partial}{\partial x_L}}, \overline{\frac{\partial}{\partial x_R}}\right)\bigg|_{x_L=x_R=y_1} f(x_L, x_R)$$

intertwines the $(\Delta\mathfrak{g}_0, \Delta\tilde{K}_0)$-action.

This gives essentially the same result with [34]. If G_0 is of tube type, i.e. $G_0 = G_{0,\mathrm{T}}$, then $\mathcal{P}_{(k,\dots,k)}(\mathfrak{p}_0^+)$ is 1-dimensional, and we have $\mathcal{H}_{\lambda+\mu}(D_0, \mathcal{P}_{(k,\dots,k)}(\mathfrak{p}_0^+)) \simeq \mathcal{H}_{\lambda+\mu+2k}(D_0)$ via $f\Delta(y)^k \mapsto f$, and thus it gives the intertwining operator $\mathcal{F}'^*_{\lambda,\mu,k} : \mathcal{H}_\lambda(D_0)\boxtimes\mathcal{H}_\mu(D_0)_{\tilde{K}_0\times\tilde{K}_0} \to \mathcal{H}_{\lambda+\mu+2k}(D_0)_{\tilde{K}_0}$,

$$\mathcal{F}'^*_{\lambda,\mu,k} f(y) := \sum_{\mathbf{m}\in\mathbb{Z}^{r_0}_{++}} \frac{(-k)_{\mathbf{m},d_0}}{(\lambda)_{k-\mathbf{m}^*,d_0}(\mu)_{\mathbf{m},d_0}} \frac{d^{(d_0,r_0,b_0)}_{\mathbf{m}}}{\left(\frac{n_0}{r_0}\right)_{\mathbf{m},d_0}} \Phi^{(d_0,r_0)}_{k-\mathbf{m}^*,\mathbf{m}}\left(\frac{\partial}{\partial x_L}, \frac{\partial}{\partial x_R}\right)\bigg|_{x_L=x_R=y} f(x_L, x_R),$$

where we write

$$\Phi^{(d_0,r_0)}_{k-\mathbf{m}^*,\mathbf{m}}(x_L, x_R) := \overline{\Delta(y_2)^{-k}}\Psi^{(d_0,r_0)}_{k-\mathbf{m}^*,\mathbf{m}}(x_L, x_R; y_2) = \Delta(x_L)^k \Phi^{(d_0,r_0)}_{\mathbf{m}}\left(P(x_L^{-1/2})x_R\right).$$

Also, if $G_0 = U(s,1)$, then $\Psi^{(2,1)}_{k-m,m}(y_2; x_L, x_R) = ({}^t y_2 \overline{x_L})^{k-m}({}^t y_2 \overline{x_R})^m$ holds, and thus $\mathcal{F}^*_{\lambda,\mu,k} : \mathcal{H}_\lambda(D_0) \boxtimes \mathcal{H}_\mu(D_0)_{\tilde{K}_0\times\tilde{K}_0} \to \mathcal{H}_{\lambda+\mu}(D_0, \mathcal{P}_k(\mathbb{C}^s))_{\tilde{K}_0}$ becomes

$$\mathcal{F}^*_{\lambda,\mu,k} f(y_1, y_2) := \sum_{m=0}^{\infty} \frac{(-k)_m}{(\lambda)_{k-m}(\mu)_m} \frac{1}{m!} \left({}^t y_2 \frac{\partial}{\partial x_L}\right)^{k-m} \left({}^t y_2 \frac{\partial}{\partial x_R}\right)^m \bigg|_{x_L=x_R=y_1} f(x_L, x_R).$$

This coincides with the Rankin-Cohen bidifferential operator (see [2, Theorem 7.1], [25, Theorem 8.1 (2)]).

3.5.3 $\mathcal{F}_{W_1}$ for $(G, G_1) = (Sp(s,\mathbb{R}), Sp(s',\mathbb{R}) \times Sp(s'',\mathbb{R}))$, $(U(q,s), U(q',s') \times U(q'',s''))$, $(SO^*(2s), SO^*(2s') \times SO^*(2s''))$

In this subsection we set

$$(G, G_1) = \begin{cases} (Sp(s,\mathbb{R}), Sp(s',\mathbb{R}) \times Sp(s'',\mathbb{R})) & (s = s' + s'') & (\text{Case } d = 1), \\ (U(q,s), U(q',s') \times U(q'',s'')) & (q = q' + q'', s = s' + s'') & (\text{Case } d = 2), \\ (SO^*(2s), SO^*(2s') \times SO^*(2s'')) & (s = s' + s'') & (\text{Case } d = 4). \end{cases}$$

We realize $\mathfrak{g}_1 \subset \mathfrak{g}$ so that

$$\mathfrak{p}_1^+ = \mathfrak{g}_1 \cap \mathfrak{p}^+ = \mathfrak{p}^+(11, d) \oplus \mathfrak{p}^+(22, d), \qquad \mathfrak{p}_2^+ = (\mathfrak{p}_1^+)^\perp = \mathfrak{p}^+(12, d),$$

where $\mathfrak{p}^+(ij,d)$ are as in (3.4.4). In this case, for $y_1=\begin{pmatrix}y_{11}&0\\0&y_{22}\end{pmatrix}\in\mathfrak{p}_1^+$ and $x_2=\begin{pmatrix}0&x_{12}\\x_{21}&0\end{pmatrix}\in\mathfrak{p}_2^+$, we have

$$\begin{aligned}
B(x_2,y_1)&=\left(I-\begin{pmatrix}0&x_{12}\\x_{21}&0\end{pmatrix}\begin{pmatrix}y_{11}^*&0\\0&y_{22}^*\end{pmatrix},\left(I-\begin{pmatrix}y_{11}^*&0\\0&y_{22}^*\end{pmatrix}\begin{pmatrix}0&x_{12}\\x_{21}&0\end{pmatrix}\right)^{-1}\right)\\
&=\left(\begin{pmatrix}I&-x_{12}y_{22}^*\\-x_{21}y_{11}^*&I\end{pmatrix},\begin{pmatrix}I&-y_{11}^*x_{12}\\-y_{22}^*x_{21}&I\end{pmatrix}^{-1}\right)\\
h(x_2,y_1)&=\det\begin{pmatrix}I&-x_{12}y_{22}^*\\-x_{21}y_{11}^*&I\end{pmatrix}^{\varepsilon}=\det(I-x_{12}y_{22}^*x_{21}y_{11}^*)^{\varepsilon},\\
B(y_1)&=\left(I-\begin{pmatrix}y_{11}&0\\0&y_{22}\end{pmatrix}\begin{pmatrix}y_{11}^*&0\\0&y_{22}^*\end{pmatrix},\left(I-\begin{pmatrix}y_{11}^*&0\\0&y_{22}^*\end{pmatrix}\begin{pmatrix}y_{11}&0\\0&y_{22}\end{pmatrix}\right)^{-1}\right)\\
&=\left(\begin{pmatrix}I-y_{11}y_{11}^*&0\\0&I-y_{22}y_{22}^*\end{pmatrix},\begin{pmatrix}I-y_{11}^*y_{11}&0\\0&I-y_{22}^*y_{22}\end{pmatrix}^{-1}\right)\\
h_1(y_1)^{-p_1}&=\det(I-y_{11}y_{11}^*)^{-\varepsilon p'}\det(I-y_{22}y_{22}^*)^{-\varepsilon p''},\\
x_2^{y_1}&=\begin{pmatrix}0&x_{12}\\x_{21}&0\end{pmatrix}\left(I-\begin{pmatrix}y_{11}^*&0\\0&y_{22}^*\end{pmatrix}\begin{pmatrix}0&x_{12}\\x_{21}&0\end{pmatrix}\right)^{-1}\\
&=\begin{pmatrix}x_{12}y_{22}^*x_{21}(I-y_{11}^*x_{12}y_{22}^*x_{21})^{-1}&x_{12}(I-y_{22}^*x_{21}y_{11}^*x_{12})^{-1}\\x_{21}(I-y_{11}^*x_{12}y_{22}^*x_{21})^{-1}&x_{21}y_{11}^*x_{12}(I-y_{22}^*x_{21}y_{11}^*x_{12})^{-1}\end{pmatrix},\\
\mathrm{Proj}_2(x_2^{y_1})&=\begin{pmatrix}0&x_{12}(I-y_{22}^*x_{21}y_{11}^*x_{12})^{-1}\\x_{21}(I-y_{11}^*x_{12}y_{22}^*x_{21})^{-1}&0\end{pmatrix},
\end{aligned}$$

where

$$\varepsilon=\begin{cases}1&(d=1,2),\\\frac{1}{2}&(d=4),\end{cases}\qquad(p',p'')=\begin{cases}(s'+1,s''+1)&(d=1),\\(q'+s',q''+s'')&(d=2),\\(2(s'-1),2(s''-1))&(d=4).\end{cases}$$

Let (τ,V) be a finite-dimensional irreducible representation of $\tilde{K}^{\mathbb{C}}$, and let $K(x_2,y_2)\in\mathcal{P}(\mathfrak{p}^+(12,d)\times\overline{\mathfrak{p}^+(12,d)},\mathrm{End}(V))$ be a $\tilde{K}^{\mathbb{C}}$-invariant polynomial in the sense of (3.3.3). Then the function $F_{W_1}(x_2;w_1,w_2)=F_{W_1}(x_{12},x_{21};w_{11},w_{12},w_{21},w_{22})\in\mathcal{O}(\mathfrak{p}_2^+\times\overline{\mathfrak{p}^+},\mathrm{End}(V))$ in Theorem 3.3.5 (2) is given by

$$\begin{aligned}
&F_{W_1}(x_2;w_1,w_2)\\
&=\iint_{D'\times D''}\tau\left(\begin{pmatrix}I&-x_{12}y_{22}^*\\-x_{21}y_{11}^*&I\end{pmatrix},\begin{pmatrix}I&-y_{11}^*x_{12}\\-y_{22}^*x_{21}&I\end{pmatrix}^{-1}\right)\\
&\quad\times K\left(\begin{pmatrix}0&x_{12}(I-y_{22}^*x_{21}y_{11}^*x_{12})^{-1}\\x_{21}(I-y_{11}^*x_{12}y_{22}^*x_{21})^{-1}&0\end{pmatrix},\right.\\
&\qquad\left.\begin{pmatrix}0&(I-y_{11}y_{11}^*)w_{12}(I-y_{22}^*y_{22})\\(I-y_{22}y_{22}^*)w_{21}(I-y_{11}^*y_{11})&0\end{pmatrix}\right)\\
&\quad\times\tau\left(\begin{pmatrix}I-y_{11}y_{11}^*&0\\0&I-y_{22}y_{22}^*\end{pmatrix}^{-1},\begin{pmatrix}I-y_{11}^*y_{11}&0\\0&I-y_{22}^*y_{22}\end{pmatrix}\right)\\
&\quad\times e^{\varepsilon(\mathrm{tr}(y_{11}w_{11}^*)+\mathrm{tr}(y_{22}w_{22}^*))}\det(I-y_{11}y_{11}^*)^{-\varepsilon p'}\det(I-y_{22}y_{22}^*)^{-\varepsilon p''}dy_{11}dy_{22}
\end{aligned}$$

Now we assume $(\tau,V)=(\chi^{-\lambda},\mathbb{C})=(\chi^{-\lambda_1-\lambda_2},\mathbb{C})$ is 1-dimensional, where $\chi^{-\lambda_1-\lambda_2}(k_1,k_2)=\det(k_1)^{-\varepsilon\lambda_1}\det(k_2)^{\varepsilon\lambda_2}$, with $\lambda>p-1$, and assume $K(\cdot,y_2)\in\mathcal{P}(\mathfrak{p}^+(12,d))$ lies in only one irreducible submodule of $\mathcal{P}(\mathfrak{p}^+(12,d))$. Then we have

$$
\begin{aligned}
&F_{W_1}(x_2;w_1,w_2)\\
&=\iint_{D'\times D''}K\left(\begin{pmatrix}0 & x_{12}(I-y_{22}^*x_{21}y_{11}^*x_{12})^{-1}(I-y_{22}^*y_{22})\\ (I-y_{22}y_{22}^*)(I-x_{21}y_{11}^*x_{12}y_{22}^*)^{-1}x_{21} & 0\end{pmatrix},\right.\\
&\qquad\qquad\left.\begin{pmatrix}0 & (I-y_{11}y_{11}^*)w_{12}\\ w_{21}(I-y_{11}^*y_{11}) & 0\end{pmatrix}\right)\\
&\qquad\qquad\times\det(I-x_{21}y_{11}^*x_{12}y_{22}^*)^{-\varepsilon\lambda}e^{\varepsilon(\mathrm{tr}(y_{11}w_{11}^*)+\mathrm{tr}(y_{22}w_{22}^*))}\\
&\qquad\qquad\times\det(I-y_{11}y_{11}^*)^{\varepsilon(\lambda-p')}\det(I-y_{22}y_{22}^*)^{\varepsilon(\lambda-p'')}dy_{11}dy_{22}\\
&=C\int_{D'}K\left(\begin{pmatrix}0 & x_{12}\\ x_{21} & 0\end{pmatrix},\begin{pmatrix}0 & (I-y_{11}y_{11}^*)w_{12}\\ w_{21}(I-y_{11}^*y_{11}) & 0\end{pmatrix}\right)\\
&\qquad\qquad\times e^{\varepsilon(\mathrm{tr}(y_{11}w_{11}^*)+\mathrm{tr}(x_{21}y_{11}^*x_{12}w_{22}^*))}\det(I-y_{11}y_{11}^*)^{\varepsilon(\lambda-p')}dy_{11},
\end{aligned}
$$

with some $C>0$. Here we have used the reproducing property on $\mathcal{O}(D'',\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+(12,d)))$,

$$
\begin{aligned}
&\int_{D''}f\begin{pmatrix}* & x_{12}(I-y_{22}^*z_{22})^{-1}(I-y_{22}^*y_{22})\\ (I-y_{22}y_{22}^*)(I-z_{22}y_{22}^*)^{-1}x_{21} & y_{22}\end{pmatrix}\\
&\qquad\times\det(I-z_{22}y_{22}^*)^{-\varepsilon\lambda}\det(I-y_{22}y_{22}^*)^{\varepsilon(\lambda-p'')}dy_{22}=Cf\begin{pmatrix}* & x_{12}\\ x_{21} & z_{22}\end{pmatrix},
\end{aligned}
$$

with

$$
\begin{gathered}
f\begin{pmatrix}* & x_{12}\\ x_{21} & x_{22}\end{pmatrix}=K\left(\begin{pmatrix}0 & x_{12}\\ x_{21} & 0\end{pmatrix},\begin{pmatrix}0 & (I-y_{11}y_{11}^*)w_{12}\\ w_{21}(I-y_{11}^*y_{11}) & 0\end{pmatrix}\right)e^{\varepsilon\,\mathrm{tr}(y_{22}w_{22}^*)},\\
z_{22}=x_{21}y_{11}^*x_{12}.
\end{gathered}
$$

Now we assume $s'\le s''$ when $d=1,4$, $q'\le s''$ when $d=2$, and set

$$
K\left(\begin{pmatrix}0 & x_{12}\\ x_{21} & 0\end{pmatrix},\begin{pmatrix}0 & w_{12}\\ w_{21} & 0\end{pmatrix}\right)=\det(x_{12}w_{12}^*)^{k_1}\det(w_{21}^*x_{21})^{k_2},
$$

where $k_1\in\mathbb{Z}_{\ge 0}$, and $k_2=0$ if $d=1,4$ or $d=2$ with $s'\ge q''$, $k_2\in\mathbb{Z}_{\ge0}$ if $d=2$ with $s'\le q''$. Then $F_{W_1}(x_2;w_1,w_2)$ becomes

$$
\begin{aligned}
&F_{W_1}(x_2;w_1,w_2)\\
&=C\int_{D'}\det(x_{12}w_{12}^*)^{k_1}\det(w_{21}^*x_{21})^{k_2}e^{\varepsilon(\mathrm{tr}(y_{11}w_{11}^*)+\mathrm{tr}(x_{21}y_{11}^*x_{12}w_{22}^*))}\\
&\qquad\qquad\qquad\qquad\times\det(I-y_{11}y_{11}^*)^{\varepsilon(\lambda+\varepsilon^{-1}(k_1+k_2)-p')}dy_{11}\\
&=C\det(x_{12}w_{12}^*)^{k_1}\det(w_{21}^*x_{21})^{k_2}\\
&\qquad\times\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}}\frac{1}{|\mathbf{m}|!}\int_{D'}\tilde{\Phi}_{\mathbf{m}}^{(d)}(y_{11}w_{11}^*)e^{\mathrm{tr}(x_{12}w_{22}^*x_{21}y_{11}^*))}\det(I-y_{11}y_{11}^*)^{\varepsilon(\lambda+\varepsilon^{-1}(k_1+k_2)-p')}dy_{11}\\
&=C'\det(x_{12}w_{12}^*)^{k_1}\det({}^tx_{21}\overline{w_{21}})^{k_2}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}}\frac{1}{(\lambda+\varepsilon^{-1}(k_1+k_2))_{\mathbf{m},d}}\frac{1}{|\mathbf{m}|!}\tilde{\Phi}_{\mathbf{m}}^{(d)}(x_{12}w_{22}^*x_{21}w_{11}^*).
\end{aligned}
$$

Here $r' = s'$ when $d = 1$, $r' = \min\{q', s'\}$ when $d = 2$ and $r' = \lfloor \frac{s'}{2} \rfloor$ when $d = 4$, and we have used the equality (3.2.13). Now we have $K(\cdot, y_2) \in W_1$ where

$$\begin{aligned}
&W_1 = \mathcal{P}_{(k_1,\dots,k_1)}(M(s', s''; \mathbb{C})) \simeq \mathbb{C}^{(s')}_{-k_1} \boxtimes V^{(s'')\vee}_{k_1^{s'}} && (d = 1),\\
&W_1 = \mathcal{P}_{(k_1,\dots,k_1)}(M(q', s''; \mathbb{C})) \boxtimes \mathcal{P}_{(k_2,\dots,k_2)}(M(q'', s'; \mathbb{C}))\\
&\quad \simeq \mathbb{C}^{(q')}_{-k_1} \boxtimes \mathbb{C}^{(s')}_{k_2} \boxtimes V^{(q'')\vee}_{k_2^{s'}} \boxtimes V^{(s'')}_{k_1^{q'}} && (d = 2),\\
&W_1 = \mathcal{P}_{(k_1,\dots,k_1)}(M(s', s''; \mathbb{C})) \simeq \mathbb{C}^{(s')}_{-k_1} \boxtimes V^{(s'')\vee}_{k_1^{s'}} && (d = 4),
\end{aligned}$$

where $V^{(s'')\vee}_{k^{s'}} := V^{(s'')\vee}_{(\underbrace{k,\dots,k}_{s'},\underbrace{0,\dots,0}_{s''-s'})}$ etc. Let ι_{k_1} $(d = 1, 4)$ or $\iota_{(k_1,k_2)}$ $(d = 2)$ be this isomorphism from the right hand side to the left hand side. Then we have

$$\begin{aligned}
&\mathcal{H}_{\chi^{-\lambda}\otimes(Ad|_{\mathfrak{p}_2^+})^*}(D' \times D'', W_1) \simeq \mathcal{H}_{\lambda+k_1}(D') \hat{\boxtimes} \mathcal{H}_{\lambda}(D'', V^{(s'')\vee}_{k_1^{s'}}) && (d = 1),\\
&\mathcal{H}_{\chi^{-\lambda_1-\lambda_2}\otimes(Ad|_{\mathfrak{p}_2^+})^*}(D' \times D'', W_1) \simeq \mathcal{H}_{(\lambda_1+k_1)+(\lambda_2+k_2)}(D') \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D'', V^{(q'')\vee}_{k_2^{s'}} \boxtimes V^{(s'')}_{k_1^{q'}})\\
& && (d = 2),\\
&\mathcal{H}_{\chi^{-\lambda}\otimes(Ad|_{\mathfrak{p}_2^+})^*}(D' \times D'', W_1) \simeq \mathcal{H}_{\lambda+\frac{k_1}{2}}(D') \hat{\boxtimes} \mathcal{H}_{\lambda}(D'', V^{(s'')\vee}_{k_1^{s'}}) && (d = 4),
\end{aligned}$$

via $\mathrm{id}_{\mathcal{O}(D'\times D'')} \otimes \iota^{-1}_{(k_1,k_2)}$. Thus we have proved the following.

Theorem 3.5.5. *(1) Let $(G, G_1) = (Sp(s, \mathbb{R}), Sp(s', \mathbb{R}) \times Sp(s'', \mathbb{R}))$ with $s = s' + s''$, $s' \le s''$. Let $\lambda > s$, $k \in \mathbb{Z}_{\ge 0}$. Then the linear map*

$$\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D') \hat{\boxtimes} \mathcal{H}_{\lambda}(D'', V^{(s'')\vee}_{k^{s'}})_{\tilde{K}_1} \to \mathcal{H}_{\lambda}(D)_{\tilde{K}},$$

$$\begin{aligned}
(\mathcal{F}_{\lambda,k} f) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} &= \det\left(x_{12} {}^t\!\left(\frac{\partial}{\partial y_{12}} \right) \right)^k \sum_{\mathbf{m}\in\mathbb{Z}^{s'}_{++}} \frac{1}{(\lambda + k)_{\mathbf{m},1}} \frac{1}{|\mathbf{m}|!}\\
&\times \tilde{\Phi}^{(1)}_{\mathbf{m}} \left(x_{12} \frac{\partial}{\partial y_{22}} {}^t x_{12} \frac{\partial}{\partial y_{11}} \right) \Bigg|_{\substack{y_{11}=x_{11},\\ y_{22}=x_{22},\\ y_{12}=0}} ((\mathrm{id} \otimes \iota_k) f) \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}
\end{aligned}$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action.

(2) Let $(G, G_1) = (U(q, s), U(q', s') \times U(q'', s''))$ with $q = q' + q''$, $s = s' + s''$, $q' \le s''$. Let $\lambda_1 + \lambda_2 > q + s - 1$, $k_1 \in \mathbb{Z}_{\ge 0}$, and $k_2 \in \mathbb{Z}_{\ge 0}$ if $s' \le q''$, $k_2 = 0$ if $s' > q''$. Then the linear map

$$\mathcal{F}_{\lambda,k_1,k_2} : \mathcal{H}_{(\lambda_1+k_1)+(\lambda_2+k_2)}(D') \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D'', V^{(q'')\vee}_{k_2^{s'}} \boxtimes V^{(s'')}_{k_1^{q'}})_{\tilde{K}_1} \to \mathcal{H}_{\lambda_1+\lambda_2}(D)_{\tilde{K}},$$

$$\begin{aligned}
&(\mathcal{F}_{\lambda,k_1,k_2} f) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\\
&= \det\left(x_{12} {}^t\!\left(\frac{\partial}{\partial y_{12}} \right) \right)^{k_1} \det\left({}^t x_{21} \frac{\partial}{\partial y_{21}} \right)^{k_2} \sum_{\mathbf{m}\in\mathbb{Z}^{\min\{q',s'\}}_{++}} \frac{1}{(\lambda + k_1 + k_2)_{\mathbf{m},2}} \frac{1}{|\mathbf{m}|!}\\
&\qquad \times \tilde{\Phi}^{(2)}_{\mathbf{m}} \left(x_{12} \frac{\partial}{\partial y_{22}} x_{21} \frac{\partial}{\partial y_{11}} \right) \Bigg|_{\substack{y_{11}=x_{11},\\ y_{22}=x_{22},\\ y_{12}=y_{21}=0}} ((\mathrm{id} \otimes \iota_{(k_1,k_2)}) f) \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}
\end{aligned}$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action.

(3) *Let* $(G,G_1)=(SO^*(2s),SO^*(2s')\times SO^*(2s''))$ *with* $s=s'+s''$, $s'\le s''$. *Let* $\lambda>2s-3$, $k\in\mathbb{Z}_{\ge 0}$. *Then the linear map*

$$\mathcal{F}_{\lambda,k}:\mathcal{H}_{\lambda+2k}(D')\hat{\boxtimes}\mathcal{H}_\lambda(D'',V_{k^{s'}}^{(s'')\vee})_{\tilde{K}_1}\to\mathcal{H}_\lambda(D)_{\tilde{K}},$$

$$(\mathcal{F}_{\lambda,k}f)\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\det\left(x_{12}{}^t\!\left(\frac{\partial}{\partial y_{12}}\right)\right)^k\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s'/2\rfloor}}\frac{1}{(\lambda+2k)_{\mathbf{m},4}}\frac{1}{|\mathbf{m}|!}$$
$$\times\tilde{\Phi}_{\mathbf{m}}^{(4)}\left(-x_{12}\frac{\partial}{\partial y_{22}}{}^t x_{12}\frac{\partial}{\partial y_{11}}\right)\Bigg|_{\substack{y_{11}=x_{11},\\ y_{22}=x_{22},\\ y_{12}=0}}((\mathrm{id}\otimes\iota_k)f)\begin{pmatrix}y_{11}&y_{12}\\y_{21}&y_{22}\end{pmatrix}$$

intertwines the $(\mathfrak{g}_1,\tilde{K}_1)$*-action.*

If $s'=s''$ $(d=1,4)$ or $q'=s''$, $s'=q''$ $(d=2)$, we have

$$\begin{aligned}
W_1&\simeq\mathbb{C}_{-k_1}^{(s')}\boxtimes\mathbb{C}_{-k_1}^{(s'')} && (d=1),\\
W_1&\simeq\mathbb{C}_{-k_1}^{(q')}\boxtimes\mathbb{C}_{k_2}^{(s')}\boxtimes\mathbb{C}_{-k_2}^{(q'')}\boxtimes\mathbb{C}_{k_1}^{(s'')} && (d=2),\\
W_1&\simeq\mathbb{C}_{-k_1}^{(s')}\boxtimes\mathbb{C}_{-k_1}^{(s'')} && (d=4),
\end{aligned}$$

via $\iota_{(k_1,k_2)}^{-1}:f\mapsto\det\left(\frac{\partial}{\partial y_{12}}\right)^{k_1}\det\left(\frac{\partial}{\partial y_{21}}\right)^{k_2}f$. Thus it gives the intertwining operator

$$\mathcal{F}_{\lambda,k_1,k_2}:\mathcal{H}_{(\lambda_1+\varepsilon^{-1}k_1)+(\lambda_2+\varepsilon^{-1}k_2)}(D')\hat{\boxtimes}\mathcal{H}_{(\lambda_1+\varepsilon^{-1}k_2)+(\lambda_2+\varepsilon^{-1}k_1)}(D'')_{\tilde{K}_1}\to\mathcal{H}_{\lambda_1+\lambda_2}(D)_{\tilde{K}},$$

$$(\mathcal{F}_{\lambda,k_1,k_2}f)\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\det(x_{12})^{k_1}\det(x_{21})^{k_2}\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}}\frac{1}{(\lambda+\varepsilon^{-1}(k_1+k_2))_{\mathbf{m},d}}\frac{1}{|\mathbf{m}|!}$$
$$\times\tilde{\Phi}_{\mathbf{m}}^{(d)}\left(x_{12}\frac{\partial}{\partial x_{22}}x_{21}\frac{\partial}{\partial x_{11}}\right)f(x_{11},x_{22}).$$

3.5.4 $\mathcal{F}_{W_1}$ for $(G,G_1)=(Sp(s,\mathbb{R}),U(s',s''))$, $(SO^*(2s),U(s',s''))$

In this subsection we set

$$(G,G_1)=\begin{cases}(Sp(s,\mathbb{R}),U(s',s'')) & (s=s'+s'')\quad(\text{Case }d=1),\\ (SO^*(2s),U(s',s'')) & (s=s'+s'')\quad(\text{Case }d=4).\end{cases}$$

We realize $\mathfrak{g}_1\subset\mathfrak{g}$ so that

$$\mathfrak{p}_1^+=\mathfrak{g}_1\cap\mathfrak{p}^+=\mathfrak{p}^+(12,d),\qquad\mathfrak{p}_2^+=(\mathfrak{p}_1^+)^\perp=\mathfrak{p}^+(11,d)\oplus\mathfrak{p}^+(22,d),$$

where $\mathfrak{p}^+(ij,d)$ are as in (3.4.4). In this case, for $y_1 = \begin{pmatrix} 0 & y_{12} \\ y_{21} & 0 \end{pmatrix} \in \mathfrak{p}_1^+$ and $x_2 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \in \mathfrak{p}_2^+$, we have

$$\begin{aligned}
B(x_2,y_1) &= I - \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}\begin{pmatrix} 0 & y_{21}^* \\ y_{12}^* & 0 \end{pmatrix} = \begin{pmatrix} I & -x_{11}y_{21}^* \\ -x_{22}y_{12}^* & I \end{pmatrix}, \\
h(x_2,y_1) &= \det\begin{pmatrix} I & -x_{11}y_{21}^* \\ -x_{22}y_{12}^* & I \end{pmatrix}^{\varepsilon} = \det(I - x_{11}y_{21}^*x_{22}y_{12}^*)^{\varepsilon}, \\
B(y_1) &= I - \begin{pmatrix} 0 & y_{12} \\ y_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & y_{21}^* \\ y_{12}^* & 0 \end{pmatrix} = \begin{pmatrix} I - y_{12}y_{12}^* & 0 \\ 0 & I - y_{21}y_{21}^* \end{pmatrix}, \\
h_1(y_1) &= \det(I - y_{12}y_{12}^*), \\
x_2^{y_1} &= \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}\left(I - \begin{pmatrix} 0 & y_{21}^* \\ y_{12}^* & 0 \end{pmatrix}\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}\right)^{-1} \\
&= \begin{pmatrix} x_{11}(I - y_{21}^*x_{22}y_{12}^*x_{11})^{-1} & x_{11}y_{21}^*x_{22}(I - y_{12}^*x_{11}y_{21}^*x_{22})^{-1} \\ x_{22}y_{12}^*x_{11}(I - y_{21}^*x_{22}y_{12}^*x_{11})^{-1} & x_{22}(I - y_{12}^*x_{11}y_{21}^*x_{22})^{-1} \end{pmatrix}, \\
\mathrm{Proj}_2(x_2^{y_1}) &= \begin{pmatrix} x_{11}(I - y_{21}^*x_{22}y_{12}^*x_{11})^{-1} & 0 \\ 0 & x_{22}(I - y_{12}^*x_{11}y_{21}^*x_{22})^{-1} \end{pmatrix}.
\end{aligned}$$

Let (τ,V) be a finite-dimensional irreducible representation of $\tilde{K}^{\mathbb{C}}$, and let $K(x_2,y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be a $\tilde{K}^{\mathbb{C}}$-invariant polynomial in the sense of (3.3.3). Then the function $F_{W_1}(x_2;w_1,w_2) = F_{W_1}(x_{11},x_{22};w_{11},w_{12},w_{22}) \in \mathcal{O}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ in Theorem 3.3.5 (2) is given by

$$\begin{aligned}
&F_{W_1}(x_2;w_1,w_2) \\
&= \int_{D_1} \tau\begin{pmatrix} I & -x_{11}y_{21}^* \\ -x_{22}y_{12}^* & I \end{pmatrix} \\
&\quad \times K\left(\begin{pmatrix} x_{11}(I - y_{21}^*x_{22}y_{12}^*x_{11})^{-1} & 0 \\ 0 & x_{22}(I - y_{12}^*x_{11}y_{21}^*x_{22})^{-1} \end{pmatrix},\right. \\
&\qquad\qquad \left.\begin{pmatrix} (I - y_{12}y_{12}^*)w_{11}(I - y_{21}^*y_{21}) & 0 \\ 0 & (I - y_{21}y_{21}^*)w_{22}(I - y_{12}^*y_{12}) \end{pmatrix}\right) \\
&\quad \times \tau\left(\begin{pmatrix} I - y_{12}y_{12}^* & 0 \\ 0 & I - y_{21}y_{21}^* \end{pmatrix}^{-1}\right) e^{2\varepsilon\,\mathrm{tr}(y_{12}w_{12}^*)} \det(I - y_{12}y_{12}^*)^{-s} dy_{12},
\end{aligned}$$

where $\varepsilon = 1$ when $d = 1$, $\varepsilon = \frac{1}{2}$ when $d = 4$. Now we assume $(\tau,V) = (\chi^{-\lambda},\mathbb{C})$ is 1-dimensional, where $\chi(k) = \det(k)^{\varepsilon}$, and $\lambda > s$ if $d = 1$, $\lambda > 2s - 3$ if $d = 4$. Then we have

$$\begin{aligned}
&F_{W_1}(x_2;w_1,w_2) \\
&= \int_{D_1} K\left(\begin{pmatrix} x_{11}(I - y_{21}^*x_{22}y_{12}^*x_{11})^{-1} & 0 \\ 0 & x_{22}(I - y_{12}^*x_{11}y_{21}^*x_{22})^{-1} \end{pmatrix},\right. \\
&\qquad\qquad \left.\begin{pmatrix} (I - y_{12}y_{12}^*)w_{11}(I - y_{21}^*y_{21}) & 0 \\ 0 & (I - y_{21}y_{21}^*)w_{22}(I - y_{12}^*y_{12}) \end{pmatrix}\right) \\
&\quad \times \det(I - x_{11}y_{21}^*x_{22}y_{12}^*)^{-\varepsilon\lambda} e^{2\varepsilon\,\mathrm{tr}(y_{12}w_{12}^*)} \det(I - y_{12}y_{12}^*)^{2\varepsilon\lambda - s} dy_{12}.
\end{aligned}$$

Now additionally assume that

$$K\left(\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}\right) = \det(x_{11}w_{11}^*)^{\varepsilon k_1} \det(x_{22}w_{22}^*)^{\varepsilon k_2},$$

where $k_i \in \mathbb{Z}_{\geq 0}$ when $d=1$ case or $d=4$ case with $s^{i\prime}$ even, $k_i = 0$ when $d=4$ case with $s^{i\prime}$ odd. Then we have

$$\begin{aligned}
&F_{W_1}(x_2; w_1, w_2) \\
&= \int_{D_1} \det(x_{11}w_{11}^*)^{\varepsilon k_1} \det(x_{22}w_{22}^*)^{\varepsilon k_2} \\
&\qquad \times \det(I - x_{11}y_{21}^* x_{22} y_{12}^*)^{-\varepsilon(\lambda+k_1+k_2)} e^{2\varepsilon \operatorname{tr}(y_{12}w_{12}^*)} \det(I - y_{12}y_{12}^*)^{2\varepsilon(\lambda+k_1+k_2)-s} dy_{12} \\
&= \det(x_{11}w_{11}^*)^{\varepsilon k_1} \det(x_{22}w_{22}^*)^{\varepsilon k_2} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}} \int_{D_1} \frac{(\lambda+k_1+k_2)_{\mathbf{m},d}}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(d)}(x_{11}y_{21}^* x_{22} y_{12}^*) \\
&\qquad\qquad \times e^{2\varepsilon \operatorname{tr}(y_{12}w_{12}^*)} \det(I - y_{12}y_{12}^*)^{2\varepsilon(\lambda+k_1+k_2)-s} dy_{12} \\
&= C \det(x_{11}w_{11}^*)^{\varepsilon k_1} \det(x_{22}w_{22}^*)^{\varepsilon k_2} \\
&\quad \times \begin{cases} \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}} \frac{(\lambda+k_1+k_2)_{\mathbf{m},1}}{(2(\lambda+k_1+k_2))_{2\mathbf{m},2}} \frac{2^{2|\mathbf{m}|}}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(1)}(x_{11}w_{21}^* x_{22} w_{12}^*) & (d=1) \\ \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}} \frac{(\lambda+k_1+k_2)_{\mathbf{m},4}}{(\lambda+k_1+k_2)_{\mathbf{m}^2,2}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(4)}(x_{11}w_{21}^* x_{22} w_{12}^*) & (d=4) \end{cases} \\
&= C \det(x_{11}w_{11}^*)^{\varepsilon k_1} \det(x_{22}w_{22}^*)^{\varepsilon k_2} \\
&\quad \times \begin{cases} \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}} \frac{1}{\left(\lambda+k_1+k_2+\frac{1}{2}\right)_{\mathbf{m},1}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(1)}(x_{11}w_{21}^* x_{22} w_{12}^*) & (d=1) \\ \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{r'}} \frac{1}{(\lambda+k_1+k_2-1)_{\mathbf{m},4}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(4)}(x_{11}w_{21}^* x_{22} w_{12}^*) & (d=4), \end{cases}
\end{aligned}$$

where $r' = \min\{s', s''\}$ when $d=1$, $r' = \min\left\{\left\lfloor \frac{s'}{2} \right\rfloor, \left\lfloor \frac{s''}{2} \right\rfloor\right\}$ when $d=4$. Here we have used (3.2.13) and Lemma 3.4.1. Since

$$K(\cdot, y_2) \in W_1 := \mathcal{P}_{(k_1,\ldots,k_1)}(\mathfrak{p}^+(11,d)) \boxtimes \mathcal{P}_{(k_2,\ldots,k_2)}(\mathfrak{p}^+(22,d)) \simeq \mathbb{C}_{-2\varepsilon k_1} \boxtimes \mathbb{C}_{-2\varepsilon k_2},$$

and

$$\mathcal{H}_{\varepsilon\lambda+\varepsilon\lambda}(D_1, W_1) \simeq \mathcal{H}_{\varepsilon(\lambda+2k_1)+\varepsilon(\lambda+2k_2)}(D_1)$$

via

$$f\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto \det\left(\frac{\partial}{\partial y_{11}}\right)^{\varepsilon k_1} \det\left(\frac{\partial}{\partial y_{22}}\right)^{\varepsilon k_2} \Bigg|_{y_{11}=y_{22}=0} f\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

we have the following.

Theorem 3.5.6. *(1) Let $(G, G_1) = (Sp(s,\mathbb{R}), U(s', s''))$ with $s = s' + s''$. Let $\lambda > s$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. Then the linear map $\mathcal{F}_{\lambda,k_1,k_2} : \mathcal{H}_{(\lambda+2k_1)+(\lambda+2k_2)}(D_1)_{\tilde{K}_1} \to \mathcal{H}_\lambda(D)_{\tilde{K}}$,*

$$\begin{aligned}
(\mathcal{F}_{\lambda,k} f)\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} &= \det(x_{11})^{k_1} \det(x_{22})^{k_2} \\
&\quad \times \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\min\{s',s''\}}} \frac{1}{\left(\lambda+k_1+k_2+\frac{1}{2}\right)_{\mathbf{m},1}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(1)}\left(x_{11} \frac{\partial}{\partial x_{12}} x_{22} {}^t\!\left(\frac{\partial}{\partial x_{12}}\right)\right) f(x_{12})
\end{aligned}$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$*-action.*

(2) *Let* $(G, G_1) = (SO^*(2s), U(s', s''))$ *with* $s = s' + s''$*. Let* $\lambda > 2s - 3$*, and* $k_i \in \mathbb{Z}_{\geq 0}$ *if* $s^{i'}$ *is even,* $k_i = 0$ *if* $s^{i'}$ *is odd. Then the linear map*

$$\mathcal{F}_{\lambda,k_1,k_2} : \mathcal{H}_{\left(\frac{\lambda}{2}+k_1\right)+\left(\frac{\lambda}{2}+k_2\right)}(D_1)_{\tilde{K}_1} \to \mathcal{H}_\lambda(D)_{\tilde{K}},$$

$$(\mathcal{F}_{\lambda,k} f)\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \mathrm{Pf}(x_{11})^{k_1}\,\mathrm{Pf}(x_{22})^{k_2}$$

$$\times \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\min\{\lfloor s'/2\rfloor, \lfloor s''/2\rfloor\}}} \frac{1}{(\lambda + k_1 + k_2 - 1)_{\mathbf{m},4}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(4)}\left(-x_{11}\frac{\partial}{\partial x_{12}} x_{22}\,{}^t\!\left(\frac{\partial}{\partial x_{12}}\right)\right) f(x_{12})$$

intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$*-action.*

3.5.5 $\mathcal{F}_{W_1}$ for $(G, G_1) = (SU(s,s), Sp(s,\mathbb{R}))$, $(SU(s,s), SO^*(2s))$

In this subsection we set

$$(G, G_1) = \begin{cases} (SU(s,s), Sp(s,\mathbb{R})) & (\text{Case } d = 1), \\ (SU(s,s), SO^*(2s)) & (\text{Case } d = 4). \end{cases}$$

We realize $\mathfrak{g}_1 \subset \mathfrak{g}$ so that

$$(\mathfrak{p}_1^+, \mathfrak{p}_2^+) := (\mathfrak{g}_1 \cap \mathfrak{p}^+, (\mathfrak{p}_1^+)^\perp) = \begin{cases} (\mathrm{Sym}(s,\mathbb{C}), \mathrm{Skew}(s,\mathbb{C})) & (\text{Case } d = 1), \\ (\mathrm{Skew}(s,\mathbb{C}), \mathrm{Sym}(s,\mathbb{C})) & (\text{Case } d = 4). \end{cases}$$

Then for $(y_1, x_2) \in \mathfrak{p}_1^+ \times \mathfrak{p}_2^+$, we have

$$B(x_2, y_1) = (I - x_2 y_1^*, (I - y_1^* x_2)^{-1}), \qquad h(x_2, y_1) = \det(I - x_2 y_1^*),$$
$$B(y_1) = (I - y_1 y_1^*, (I - y_1^* y_1)^{-1}), \qquad h_1(y_1) = \det(I - y_1 y_1^*)^\varepsilon,$$

where $\varepsilon = 1$ when $d = 1$, $\varepsilon = \frac{1}{2}$ when $d = 4$, and

$$x_2^{y_1} = x_2(I - y_1^* x_2)^{-1} = (I - x_2 y_1^*)^{-1} x_2,$$
$$\mathrm{Proj}_2(x_2^{y_1}) = \frac{1}{2}\left(x_2(I - y_1^* x_2)^{-1} + (I + x_2 y_1^*)^{-1} x_2\right) = (I + x_2 y_1^*)^{-1} x_2 (I - y_1^* x_2)^{-1}.$$

Let (τ, V) be a finite-dimensional irreducible representation of $\tilde{K}^\mathbb{C}$, and let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be a $\tilde{K}^\mathbb{C}$-invariant polynomial in the sense of (3.3.3). Then the function $F_{W_1}(x_2; w_1, w_2) \in \mathcal{O}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ in Theorem 3.3.5 (2) is given by

$$\begin{aligned} &F_{W_1}(x_2; w_1, w_2) \\ &= \int_{D_1} \tau\left(I - x_2 y_1^*, (I - y_1^* x_2)^{-1}\right) K\left((I + x_2 y_1^*)^{-1} x_2 (I - y_1^* x_2)^{-1}, (I - y_1 y_1^*) w_2 (I - y_1^* y_1)\right) \\ &\qquad \times \tau\left((I - y_1 y_1^*)^{-1}, I - y_1^* y_1\right) e^{\mathrm{tr}(y_1 w_1^*)} \det(I - y_1 y_1^*)^{-\varepsilon p_1}\, dy_1 \end{aligned}$$

where $(\varepsilon, p_1) = (1, r+1)$ when $d = 1$, $(\varepsilon, p_1) = \left(\frac{1}{2}, 2(s-1)\right)$ when $d = 4$. Now we assume $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ is 1-dimensional, where $\chi(k_1, k_2) = \det(k_2)$. Then we have

$$\begin{aligned} F_{W_1}(x_2; w_1, w_2) = \int_{D_1} &K\left((I + x_2 y_1^*)^{-1} x_2 (I - y_1^* x_2)^{-1}, (I - y_1 y_1^*) w_2 (I - y_1^* y_1)\right) \\ &\times \det(I - x_2 y_1^*)^{-\lambda} e^{\mathrm{tr}(y_1 w_1^*)} \det(I - y_1 y_1^*)^{\lambda - \varepsilon p_1}\, dy_1. \end{aligned}$$

Now we additionally assume that

$$K(x_2, w_2) = \det(x_2 w_2^*)^{(2\varepsilon)^{-1}k},$$

where $k \in \mathbb{Z}_{\geq 0}$ when $d = 1$ or $d = 4$ with s even, $k = 0$ when $d = 4$ with s odd. Then we have

$$\begin{aligned}
&F_{W_1}(x_2; w_1, w_2)\\
&= \int_{D_1} \det(x_2 w_2^*)^{(2\varepsilon)^{-1}k} \det(I - x_2 y_1^*)^{-\lambda-\varepsilon^{-1}k} e^{\mathrm{tr}(y_1 w_1^*)} \det(I - y_1 y_1^*)^{\lambda+\varepsilon^{-1}k-\varepsilon p_1} dy_1\\
&= \det(x_2 w_2^*)^{(2\varepsilon)^{-1}k} \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \int_{D_1} \frac{(\lambda+\varepsilon^{-1}k)_{\mathbf{m},2}}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_2 y_1^*)^2) e^{\mathrm{tr}(y_1 w_1^*)}\\
&\qquad\qquad \times \det(I - y_1 y_1^*)^{\varepsilon(\varepsilon^{-1}\lambda+\varepsilon^{-2}k-p_1)} dy_1\\
&= \det(x_2 w_2^*)^{(2\varepsilon)^{-1}k} \times \begin{cases} \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{(\lambda+k)_{\mathbf{m},2}}{(\lambda+k)_{\mathbf{m}^2,1}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_2 w_1^*)^2) & (d=1)\\ \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{(\lambda+2k)_{\mathbf{m},2}}{(2\lambda+4k)_{2\mathbf{m},4}} \frac{2^{2|\mathbf{m}|}}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_2 w_1^*)^2) & (d=4) \end{cases}\\
&= \det(x_2 w_2^*)^{(2\varepsilon)^{-1}k} \times \begin{cases} \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{1}{\left(\lambda+k-\frac{1}{2}\right)_{\mathbf{m},2}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_2 w_1^*)^2) & (d=1)\\ \displaystyle\sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{1}{\left(\lambda+2k+\frac{1}{2}\right)_{\mathbf{m},2}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}((x_2 w_1^*)^2) & (d=4). \end{cases}
\end{aligned}$$

Here we have used (3.2.13) and Lemma 3.4.2. Since $K(\cdot, y_2) \in W_1 := \mathcal{P}_{(k,\ldots,k)}(\mathfrak{p}_2^+) \simeq \mathbb{C}_{-\varepsilon^{-1}k}$, and

$$\mathcal{H}_{\varepsilon^{-1}\lambda}(D_1, W_1) \simeq \mathcal{H}_{\varepsilon^{-1}\lambda+\varepsilon^{-2}k}(D_1)$$

via

$$f(y_1 + y_2) \mapsto \det\left(\frac{\partial}{\partial y_2}\right)^{(2\varepsilon)^{-1}k}\Bigg|_{y_2=0} f(y_1 + y_2),$$

we have the following.

Theorem 3.5.7. *(1) Let $(G, G_1) = (SU(s,s), Sp(s,\mathbb{R}))$. Let $\lambda > 2s-1$, $k \in \mathbb{Z}_{\geq 0}$. Then the linear map $\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D_1)_{\tilde{K}_1} \to \mathcal{H}_\lambda(D)_{\tilde{K}}$,*

$$(\mathcal{F}_{\lambda,k} f)(x_1 + x_2) = \mathrm{Pf}(x_2)^k \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{1}{\left(\lambda+k-\frac{1}{2}\right)_{\mathbf{m},2}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}\left(\left(x_2 \frac{\partial}{\partial x_1}\right)^2\right) f(x_1)$$

$(x_1 \in \mathrm{Sym}(s,\mathbb{C}), x_2 \in \mathrm{Skew}(s,\mathbb{C}))$ intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action.

(2) Let $(G, G_1) = (SU(s,s), SO^*(2s))$. Let $\lambda > 2s-1$, and $k \in \mathbb{Z}_{\geq 0}$ if s is even, $k = 0$ if s is odd. Then the linear map $\mathcal{F}_{\lambda,k} : \mathcal{H}_{2\lambda+4k}(D_1)_{\tilde{K}_1} \to \mathcal{H}_\lambda(D)_{\tilde{K}}$,

$$(\mathcal{F}_{\lambda,k} f)(x_1 + x_2) = \det(x_2)^k \sum_{\mathbf{m}\in\mathbb{Z}_{++}^{\lfloor s/2\rfloor}} \frac{1}{\left(\lambda+2k+\frac{1}{2}\right)_{\mathbf{m},2}} \frac{1}{|\mathbf{m}|!} \tilde{\Phi}_{\mathbf{m}}^{(2)\prime}\left(\left(x_2 \frac{\partial}{\partial x_1}\right)^2\right) f(x_1)$$

$(x_1 \in \mathrm{Skew}(s,\mathbb{C}), x_2 \in \mathrm{Sym}(s,\mathbb{C}))$ intertwines the $(\mathfrak{g}_1, \tilde{K}_1)$-action.

3.5.6 $\mathcal{F}_{W_1}$ for $(G, G_1) = (SO(2,n), SO(2,n') \times SO(n-n'))$

In this subsection we set

$$(G, G_1) = (SO(2,n), SO(2,n') \times SO(n-n')),$$

with $n \geq 3$. Then we have $\mathfrak{p}^+ \simeq \mathbb{C}^n$, $\mathfrak{p}_1^+ \simeq \mathbb{C}^{n'}$, and $\mathfrak{p}_2^+ = (\mathfrak{p}_1^+)^\perp \simeq \mathbb{C}^{n-n'}$. For $y_1 \in \mathfrak{p}_1^+$ and $x_2 \in \mathfrak{p}_2^+$, we have

$$h(x_2, y_1) = 1 + q(x_2)\overline{q(y_1)}, \qquad h_1(y_1) = 1 - 2q(y_1, \bar{y}_1) + |q(y_1)|^2,$$
$$x_2^{y_1} = \left(1 + q(x_2)\overline{q(y_1)}\right)^{-1} (x_2 - q(x_2)\bar{y}_1), \qquad \mathrm{Proj}_2(x_2^{y_1}) = \left(1 + q(x_2)\overline{q(y_1)}\right)^{-1} x_2.$$

Let $(\tau, V) = (\chi^{-\lambda}, \mathbb{C})$ be the 1-dimensional representation of $\tilde{K}^{\mathbb{C}}$, and let $K(x_2, y_2) \in \mathcal{P}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}_2^+}, \mathrm{End}(V))$ be a $\tilde{K}^{\mathbb{C}}$-invariant polynomial in the sense of (3.3.3). Then the function $F_K(x_2; w_1, w_2) \in \mathcal{O}(\mathfrak{p}_2^+ \times \overline{\mathfrak{p}^+})$ in Theorem 3.3.5 (2) is given by

$$F_{W_1}(x_2; w_1, w_2) = \int_{D_1} K\left(\left(1 + q(x_2)\overline{q(y_1)}\right)^{-1} x_2, B(y_1)w_2\right) \left(1 + q(x_2)\overline{q(y_1)}\right)^{-\lambda}$$
$$\times e^{2q(y_1, \bar{w}_1)} \left(1 - 2q(y_1, \bar{y}_1) + |q(y_1)|^2\right)^{\lambda - n'} dy_1$$

Now we additionally assume that $n - n' = 1$ or $n - n' \geq 3$, and

$$K(x_2, w_2) = q(x_2)^k \overline{q(w_2)}^k$$

where $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ when $n - n' = 1$, $k \in \mathbb{Z}_{\geq 0}$ when $n - n' \geq 3$. Then we have

$$\begin{aligned}
&F_{W_1}(x_2; w_1, w_2) \\
&= \int_{D_1} q(x_2)^k \overline{q(w_2)}^k \left(1 + q(x_2)\overline{q(y_1)}\right)^{-\lambda-2k} e^{2q(y_1, \bar{w}_1)} \left(1 - 2q(y_1, \bar{y}_1) + |q(y_1)|^2\right)^{\lambda+2k-n'} dy_1 \\
&= q(x_2)^k \overline{q(w_2)}^k \sum_{m=0}^{\infty} \int_{D_1} \frac{(-1)^m (\lambda+2k)_m}{m!} q(x_2)^m \overline{q(y_1)}^m e^{2q(y_1, \bar{w}_1)} \\
&\qquad\qquad \times \left(1 - 2q(y_1, \bar{y}_1) + |q(y_1)|^2\right)^{\lambda+2k-n'} dy_1 \\
&= C q(x_2)^k \overline{q(w_2)}^k \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda+2k)_m}{(\lambda+2k)_{(m,m),n'-2}} \frac{1}{m!} q(x_2)^m \overline{q(w_1)}^m \\
&= C q(x_2)^k \overline{q(w_2)}^k \sum_{m=0}^{\infty} \frac{(-1)^m}{\left(\lambda + 2k - \frac{n'-2}{2}\right)_m} \frac{1}{m!} q(x_2)^m \overline{q(w_1)}^m.
\end{aligned}$$

Here we have used (3.2.13) and the fact that $q(y_1)^m \in \mathcal{P}_{(m,m)}(\mathbb{C}^{n'})$. Similarly, if we assume $n - n' = 2$ and

$$K(x_2, w_2) = (x_{21} + \sqrt{-1}x_{22})^{k_1} \overline{(w_{21} + \sqrt{-1}w_{22})}^{k_1} (x_{21} - \sqrt{-1}x_{22})^{k_2} \overline{(w_{21} - \sqrt{-1}w_{22})}^{k_2},$$

where $x_2=(x_{21},x_{22}), w_2=(w_{21},w_{22})\in\mathfrak{p}_2^+=\mathbb{C}^2$, and $k_1,k_2\in\mathbb{Z}_{\ge0}$, then we have

$$\begin{aligned}
&F_{W_1}(x_2;w_1,w_2)\\
&=\int_{D_1}(x_{21}+\sqrt{-1}x_{22})^{k_1}\overline{(w_{21}+\sqrt{-1}w_{22})}^{k_1}(x_{21}-\sqrt{-1}x_{22})^{k_2}\overline{(w_{21}-\sqrt{-1}w_{22})}^{k_2}\\
&\qquad\times\left(1+q(x_2)\overline{q(y_1)}\right)^{-\lambda-k_1-k_2}e^{2q(y_1,\bar{w}_1)}\left(1-2q(y_1,\bar{y}_1)+|q(y_1)|^2\right)^{\lambda+k_1+k_2-n'}dy_1\\
&=C(x_{21}+\sqrt{-1}x_{22})^{k_1}\overline{(w_{21}+\sqrt{-1}w_{22})}^{k_1}(x_{21}-\sqrt{-1}x_{22})^{k_2}\overline{(w_{21}-\sqrt{-1}w_{22})}^{k_2}\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(\lambda+k_1+k_2)_m}\frac{1}{m!}q(x_2)^m\overline{q(w_1)}^m.
\end{aligned}$$

Now since $K(\cdot,y_2)\in W_1:=\mathcal{P}_{(k,k)}(\mathfrak{p}_2^+)\simeq\mathbb{C}_{-2k,SO(n')}\boxtimes\mathbf{1}_{SO(n-n')}$ and

$$\mathcal{H}_\lambda(D_1,W_1)\simeq\mathcal{H}_{\lambda+2k}(D_{SO_0(2,n')})\boxtimes\mathbf{1}_{SO(n-n')}$$

via

$$f(y_1,y_2)\mapsto q\left(\frac{\partial}{\partial y_2}\right)^k\bigg|_{y_2=0}f(y_1,y_2)$$

when $n-n'\neq2$, or $K(\cdot,y_2)\in W_1\simeq\mathbb{C}_{-k_1-k_2,SO(n-2)}\boxtimes\mathbb{C}_{k_1-k_2,SO(2)}$ and

$$\mathcal{H}_\lambda(D_1,W_1)\simeq\mathcal{H}_{\lambda+k_1+k_2}(D_{SO_0(2,n')})\boxtimes\mathbb{C}_{k_1-k_2,SO(2)}$$

via

$$f(y_1,y_2)\mapsto\left(\frac{\partial}{\partial y_{21}}-\sqrt{-1}\frac{\partial}{\partial y_{22}}\right)^{k_1}\left(\frac{\partial}{\partial y_{21}}+\sqrt{-1}\frac{\partial}{\partial y_{22}}\right)^{k_2}q\left(\frac{\partial}{\partial y_2}\right)^k\bigg|_{y_2=0}f(y_1,y_2)$$

when $n-n'=2$, we have the following.

Theorem 3.5.8. *Let $(G,G_1)=(SO(2,n),SO(2,n')\times SO(n-n'))$ with $n\ge3$, and let $\lambda>n-1$.*

(1) Let $n-n'=1$, $k\in\frac12\mathbb{Z}_{\ge0}$, or $n-n'\ge3$, $k\in\mathbb{Z}_{\ge0}$. Then the linear map

$$\mathcal{F}_{\lambda,k}:(\mathcal{H}_{\lambda+2k}(D_{SO_0(2,n')})\boxtimes\mathbf{1}_{SO(n-n')})_{\tilde{K}_1}\to\mathcal{H}_\lambda(D_{SO_0(2,n)})_{\tilde{K}},$$

$$(\mathcal{F}_{\lambda,k}f)(x_1,x_2)=q(x_2)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{\left(\lambda+2k-\frac{n'-2}{2}\right)_m}\frac{1}{m!}q(x_2)^mq\left(\frac{\partial}{\partial x_1}\right)^mf(x_1)$$

($x_1\in\mathbb{C}^{n'}, x_2\in\mathbb{C}^{n-n'}$) intertwines the $(\mathfrak{g}_1,\tilde{K}_1)$-action.

(2) Let $n-n'=2$, $k_1,k_2\in\mathbb{Z}_{\ge0}$. Then the linear map

$$\mathcal{F}_{\lambda,k_1,k_2}:(\mathcal{H}_{\lambda+k_1+k_2}(D_{SO_0(2,n-2)})\boxtimes\mathbb{C}_{k_1-k_2,SO(2)})_{\tilde{K}_1}\to\mathcal{H}_\lambda(D_{SO_0(2,n)})_{\tilde{K}},$$

$$\begin{aligned}
(\mathcal{F}_{\lambda,k}f)(x_1,x_2)&=(x_{21}+\sqrt{-1}x_{22})^{k_1}(x_{21}-\sqrt{-1}x_{22})^{k_2}\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(\lambda+k_1+k_2)_m}\frac{1}{m!}q(x_2)^mq\left(\frac{\partial}{\partial x_1}\right)^mf(x_1)
\end{aligned}$$

($x_1\in\mathbb{C}^{n-2}, x_2=(x_{21},x_{22})\in\mathbb{C}^2$) intertwines the $(\mathfrak{g}_1,\tilde{K}_1)$-action.

Acknowledgments

The author would like to thank his supervisor T. Kobayashi for a lot of helpful advice on this chapter. He also thanks his colleagues, especially M. Kitagawa for a lot of helpful discussion.

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