博士論文

論文題目

Studies on spaces of initial conditions for nonautonomous mappings of the plane and singularity confinement

(平面上の非自励写像に対する初期値空間と特異点閉じ込めの研究)

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STUDIES ON SPACES OF INITIAL CONDITIONS FOR NONAUTONOMOUS MAPPINGS OF THE PLANE AND SINGULARITY CONFINEMENT

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Abstract. We study nonautonomous mappings of the plane by means of singularity confinement and spaces of initial conditions. First we introduce what we call the full-deautonomisation approach. This is a new method to predict the algebraic entropy of an equation with all singularities confined, only by means of a deautonomisation procedure and singularity confinement. Next we introduce the notion of a space of initial conditions for nonautonomous systems and we study the basic properties of general equations that have spaces of initial conditions. Finally we consider the minimization of spaces of initial conditions for nonautonomous systems and we show that if a nonautonomous mapping of the plane with a space of initial conditions and unbounded degree growth has zero algebraic entropy, then it must be one of the discrete Painlevé equations in the Sakai classification.

Contents

1. Introduction

Mappings of the plane are among the main objects of interest in the field of discrete integrable systems. Such a mapping can be thought of as defining the equation

$$\varphi_n \colon (x_n, y_n) \mapsto (x_{n+1}, y_{n+1}),$$

where x_{n+1} and y_{n+1} are functions of x_n and y_n (and n). A three point mapping, in which x_{n+1} is determined by x_n and x_{n-1}, can be transformed to the above form by introducing $y_n = x_{n+1}$.

In this thesis, we deal with mappings of the plane that can be rationally solved in the opposite direction. Such an equation defines a (family of) birational automorphism(s) (Definition A.2) on $\mathbb{P}^2$ (or on $\mathbb{P}^1 \times \mathbb{P}^1$).

How to detect the integrability of discrete equations has been a major problem in the field of integrable systems for more than a quarter century.

Singularity confinement was first proposed by Grammaticos, Ramani and Papageorgiou [14] as a discrete analogue of the Painlevé property in continuous systems. Where the Painlevé property requires all movable singularities to be at most poles, singularity confinement requires every singularity (i.e. disappearance of information on the initial values) to be confined after finite iterates. That is, an equation is said to enter a singularity when loosing information on the initial values, and is said to exit from a singularity when recovering the lost information. Singularity confinement is so powerful that many discrete Painlevé equations have been discovered by deautonomising QRT mappings with the help of singularity confinement [13].

However, Hietarinta and Viallet presented an equation that passes the singularity confinement test but which exhibits chaotic behavior [19]. Their counterexample is

$$x_{n+1} + x_{n-1} = x_n + \frac{a}{x_n^2}, \tag{1.1}$$

which is called the Hietarinta-Viallet equation. In order to test the integrability more precisely, Bellon and Viallet defined the algebraic entropy and showed that the entropy of the above equation is $\log((3+\sqrt{5})/2)$ [4].

Definition 1.1 (algebraic entropy [4], dynamical degree)**.** The limits

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\deg \varphi^n\right) \text{ and } \lim_{n\to\infty} \left(\deg \varphi^n\right)^{1/n},$$

if they exist, are called the *algebraic entropy* and the *dynamical degree* of the equation, respectively. We denote by φ^n the n-th iterates and by deg the degree as a rational function of the initial values.

It is obvious that the entropy coincides with the logarithm of the dynamical degree.

Even in autonomous cases ("autonomous" means that the φ does not depend on n), it is difficult to calculate the exact value of the entropy for a concrete equation. However, the integrability test based on zero algebraic entropy is empirically accurate. Hereafter, we shall call an equation with zero algebraic entropy integrable.

Remark 1.2. It is known that in the autonomous case, the entropy exists and that is invariant under coordinate changes [4]. However, as in Example 3.1, this does not hold in nonautonomous cases.

In nonautonomous cases, the degree of n-th iterate is

$$\deg \varphi^n = \deg(\varphi_{\ell+n-1} \circ \cdots \circ \varphi_\ell),$$

which in general depends on the starting index ℓ. However, we rarely think of $\deg \varphi^n$ (or the entropy) as a function of ℓ. It is usual to fix the starting index (for example $\ell = 0$, as in Example 3.1) or only consider the cases where $\deg \varphi^n$ do not depend on ℓ for all n. If so, then the algebraic entropy always exists for the same reason in the autonomous case.

It has become quite clear that there are a lot of nonintegrable systems that pass the singularity confinement test [3, 36, 15]. Moreover, most linearizable mappings, which are by definition integrable, do not pass the singularity confinement test [35].

Besides singularity confinement and algebraic entropy, some integrability criteria have been proposed.

Based on Diophantine approximations, Halburd proposed a new integrability criterion, called Diophantine integrability. This approach is useful when we numerically estimate the value of the entropy.

The coprimeness condition was proposed to reinterpret singularity confinement from an algebraic viewpoint [22, 24, 23]. This criterion focuses on the factorization of iterates as rational functions of the initial values and tries to transform the equation to another one with the Laurent property [12]. This method has been recognized as a technique to calculate the exact value of the algebraic entropy [25].

Remark 1.3. There are several degrees of mappings of the plane.

The degree as a birational automorphism on $\mathbb{P}^2$ (Definition A.20) is most standard. We will mainly use this degree in this thesis.

If φ is written as

$$\varphi(x,y) = \left(\frac{\varphi_{11}(x,y)}{\varphi_{21}(x,y)}, \frac{\varphi_{12}(x,y)}{\varphi_{22}(x,y)} \right),$$

where φ_{1i} and φ_{2i} have no common factors for $i = 1, 2$, then the degree of φ as a birational automorphism on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by

$$\deg \varphi = \max(\deg \varphi_{11}, \deg \varphi_{12}, \deg \varphi_{21}, \deg \varphi_{22}).$$

This degree is convenient when we consider three point mappings.

It is known that, while these two degrees are different, their growth as functions of n is the same.

Example 1.4. Consider the equation

$$\varphi(x,y) = \left(\frac{1}{y}, \frac{1}{x} \right).$$

It immediately follows from the above expression that the degree of φ as a birational automorphism on $\mathbb{P}^1 \times \mathbb{P}^1$ is 1.

On the other hand, φ can be written in homogeneous coordinates on $\mathbb{P}^2$ as

$$\varphi(z_1 : z_2 : z_3) = (z_1 z_3 : z_2 z_3 : z_1 z_2).$$

Therefore, the degree of φ as a birational automorphism on $\mathbb{P}^2$ is 2.

Since $\varphi^2 = \mathrm{id}$, the degree growth of φ^n is bounded in both cases.

Since all equations in this thesis are (families of) birational automorphisms, geometric methods are useful to analyze them. The most important and powerful tool is the so-called space of initial conditions, which was first introduced by Okamoto to analyze the continuous Painlevé equations [33].

Sakai focused on a close relation between singularity confinement and a space of initial conditions. Using a special type of algebraic surface, he has classified all discrete Painlevé equations [37].

Takenawa performed the blow-ups for (regularized as an automorphism on a surface) the Hietarinta-Viallet equation to obtain a space of initial conditions [38]. He revealed a correspondence between the singularity pattern and the motion of specific curves, and recalculated the algebraic entropy by computing the maximum eigenvalue of the linear transformation induced on the Picard group. He also considered blow-ups of nonautonomous systems and showed, by using specific bases introduced by Sakai, that the degree growth of every discrete Painlevé equation is at most quadratic [39].

Let us recall the close relationship between singularity confinement and the space of initial conditions.

Example 1.5. Consider the equation

$$x_{n+1}x_{n-1} = \frac{a(x_n+1)}{x_n^2}, \tag{1.2}$$

where a is a nonzero constant. This is one of the so-called QRT mappings [34].

First let us explain singularity confinement on the above equation. Let ε be an infinitesimal quantity and assume that while x_{n-1} is a regular finite value, x_n becomes $-1+\varepsilon$. Then we obtain

$$\begin{aligned} x_{n+1} &= \frac{a}{x_{n-1}}\varepsilon + o(\varepsilon), \\ x_{n+2} &= \frac{-x_{n-1}^2}{a}\varepsilon^{-2} + o(\varepsilon^{-2}), \\ x_{n+3} &= \frac{-a}{x_{n-1}}\varepsilon + o(\varepsilon), \\ x_{n+4} &= -1 + o(1), \\ x_{n+5} &= x_{n-1} + o(1), \end{aligned}$$

where "$o(\varepsilon^k)$" means the Landau symbol, i.e. $\lim_{\varepsilon\to 0} o(\varepsilon^k)/\varepsilon^k = 0$. Since the leading order of x_{n+5} is degree 0 and the leading coefficient depends on the initial value x_{n-1}, we say that this singularity is confined and its pattern is

$$\{-1, \varepsilon, \varepsilon^{-2}, \varepsilon, -1\}. \tag{1.3}$$

This equation has two more patterns, $\{\varepsilon, \varepsilon^{-2}, \varepsilon\}$ and $\{\varepsilon^{-1}, \varepsilon, \varepsilon^{-1}\}$, which are cyclically connected. Thus we have a pattern with period 8:

$$\{\varepsilon, \varepsilon^{-2}, \varepsilon, \mathrm{REG}, \varepsilon^{-1}, \varepsilon, \varepsilon^{-1}, \mathrm{REG}\}, \tag{1.4}$$

where "REG" means a regular value depending on the initial value.

Next, we blow up $\mathbb{P}^1\times\mathbb{P}^1$ to obtain a space of initial conditions. A blow-up procedure for this equation was first given in [21]. Although the figure of the space of initial conditions (Figure 2) in [21] is wrong, the calculation and explanation are detailed, and we therefore omit the calculation of the blow-ups and only give the result.

The equation can be written as

$$\varphi\colon \mathbb{P}^1\times\mathbb{P}^1 \dashrightarrow \mathbb{P}^1\times\mathbb{P}^1, \quad (x,y)\mapsto \left(y, \frac{a(y+1)}{xy^2}\right). \tag{1.5}$$

Introducing the variables $s=1/x$ and $t=1/y$, $\mathbb{P}^1\times\mathbb{P}^1$ is covered with 4 copies of $\mathbb{C}^2$ as follows:

$$\mathbb{P}^1\times\mathbb{P}^1 = (x,y)\cup(s,y)\cup(x,t)\cup(s,t)\,.$$

Let X be the surface obtained by blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at the following 8 points (Figure 1):

(1) $(x,y)=(0,-1)$,
(2) $(x,y)=(-1,0)$,
(3) $(x,t)=(0,0)$,
(4) $\left(x, \frac{t}{x}\right) = (0,0)$,

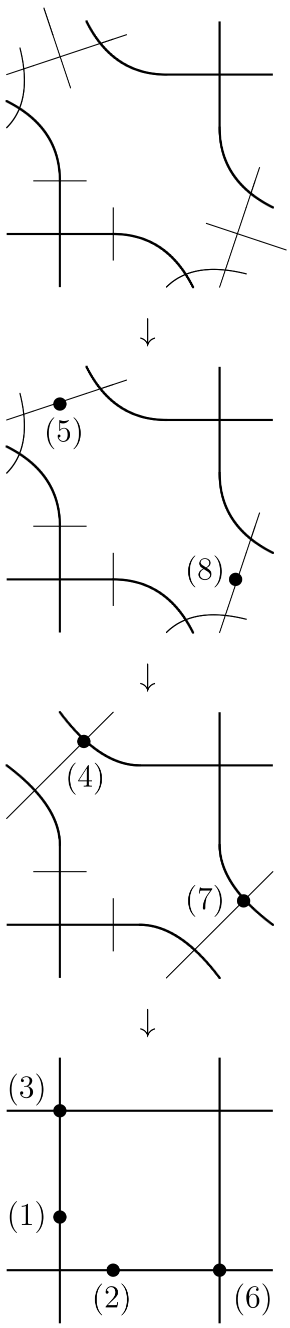


FIGURE 1. Diagram showing the blow-ups needed to obtain a space of initial conditions for the mapping (1.5).

(5) $\left(x, \dfrac{t}{x^2}\right) = \left(0, -\dfrac{1}{a}\right)$,
(6) $(s, y) = (0, 0)$,
(7) $\left(\dfrac{s}{y}, y\right) = (0, 0)$,
(8) $\left(\dfrac{s}{y^2}, y\right) = \left(-\dfrac{1}{a}, 0\right)$.

Then, φ becomes an automorphism on X.

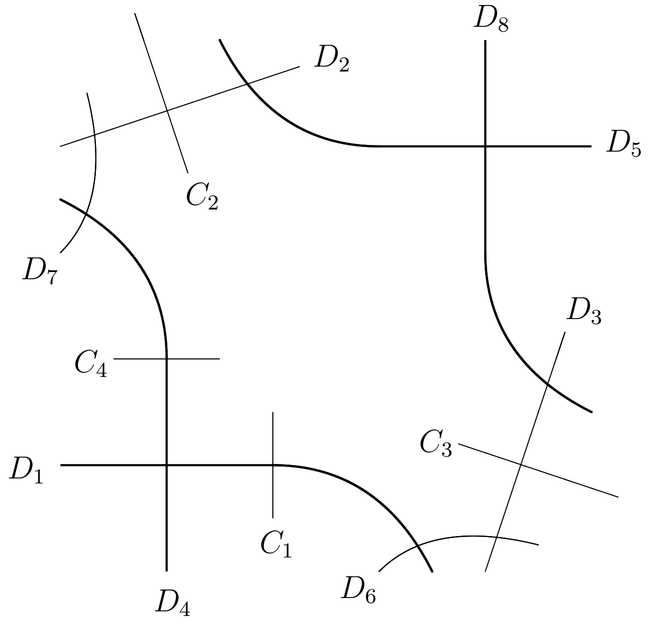


FIGURE 2. Space of initial conditions for the mapping (1.5). The curves $D_1, \ldots, D_8$ are all (-2)-curves and compose the fundamental chain (1.6). The motion of the exceptional curves $C_1, \ldots, C_4$ corresponds to the singularity pattern (1.3).

Let $D_1, \ldots, D_8, C_1, C_2, C_3, C_4$ be the curves in Figure 2 and let $\{x = -1\}, \{y = -1\} \subset X$ be the strict transforms of the corresponding lines in $\mathbb{P}^1 \times \mathbb{P}^1$. These curves move under φ as follows:

$$D_1 \to D_2 \to \cdots \to D_8 \to D_1, \tag{1.6}$$

$$\{y = -1\} \to C_1 \to C_2 \to C_3 \to C_4 \to \{x = -1\}. \tag{1.7}$$

We find an exact correspondence between the cyclic pattern (1.4) and the cyclic motion of curves (1.6). On the other hand, the singularity pattern (1.3) corresponds to the motion of curves (1.7). In $\mathbb{P}^1 \times \mathbb{P}^1$, these curves become points such as (1), (2), (3) and (6). After several steps, however, they again become curves. This phenomenon corresponds to the recovery of the information of the initial value, and this is a geometric interpretation of singularity confinement.

We will see in §4 how to calculate the degree growth of the equation from the linear action on $\operatorname{Pic} X$. According to Takenawa [38], the maximum eigenvalue of the linear action gives the dynamical degree of the equation. Using

$$D_1 + D_3 + D_6 + C_1 + C_3 \sim 2D_2 + D_5 + D_7 + 2C_2,$$

we have

$$C_3 \sim -D_1 + 2D_2 - D_3 + D_5 - D_6 + D_7 - C_1 + 2C_2,$$

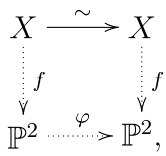
where "$\sim$" means the linear equivalence (Definitions A.7 and A.11). Thus, the matrix of $\varphi_*\colon \operatorname{Pic} X \to \operatorname{Pic} X$ with respect to the basis $D_1, \dots, D_8, C_1, C_2$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}.$$

Since the eigenvalues of this matrix all have modulus 1, the entropy of this equation is 0.

In the above example, we started with $\mathbb{P}^1 \times \mathbb{P}^1$ and only used blow-ups to obtain a space of initial conditions. However, it is possible to start with $\mathbb{P}^2$ (or a Hirzebruch surface $\mathbb{F}_a$) and, in general, blow-downs are also necessary to obtain a space of initial conditions. If we admit the use of blow-downs, we can take an arbitrary rational surface as a starting point. Therefore, the definition of a space of initial conditions is as follows:

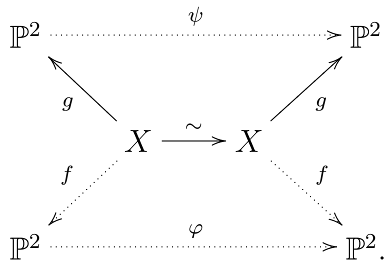
Definition 1.6 (space of initial conditions for autonomous systems)**.** If for an autonomous equation $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, there exist a rational surface X and a birational map $f\colon X \dashrightarrow \mathbb{P}^2$ such that $f^{-1} \circ \varphi \circ f$ is an automorphism on X:



then X is called a *space of initial conditions* for φ. That is, an autonomous equation has a space of initial conditions if it can be regularized as an automorphism on some rational surface.

It is important to note that in general f is a composition of a finite number of blow-ups and blow-downs (Proposition A.5).

Remark 1.7. Consider an autonomous equation $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with a space of initial conditions $f\colon X \dashrightarrow \mathbb{P}^2$ and assume that the degree of φ^n is unbounded. In this case, X has infinitely many exceptional curves of first kind and thus Theorem A.38 implies that there exists a birational morphism $g\colon X \to \mathbb{P}^2$. Let $\psi = g \circ f^{-1} \circ \varphi \circ f \circ g^{-1}$. Then ψ is a birational automorphism on $\mathbb{P}^2$:



If we identify two equations that are transformed to each other by a coordinate change of $\mathbb{P}^2$, then φ and ψ are the same equation. Therefore, by changing coordinates on $\mathbb{P}^2$

appropriately, we can think of f in Definition 1.6 as a composition of blow-ups as long as the degree growth of the equation is unbounded.

All birational automorphisms on surfaces have been classified by Diller and Favre [8]. Extracting the classification of birational automorphisms on rational surfaces from their theorem and interpreting it from the viewpoint of integrable systems, we have the following classification of autonomous equations of the plane:

Theorem 1.8. *Autonomous equations φ of the plane are classified into the following* 5 *classes:*

class 1: *The degree of φ^n is bounded.*
This type of equation has a space of initial conditions.
For example, projective transformations on $\mathbb{P}^2$ and periodic mappings belong to this class.

class 2: *The degree of φ^n grows linearly.*
This type of equation does not have a space of initial conditions.
Most linearizable mappings belong this class.

class 3: *The degree of φ^n grows quadratically.*
This type of equation has a space of initial conditions. It is an elliptic surface and φ preserves the elliptic fibration on the surface.
For example, the QRT mappings belong to this class [34, 41, 11]*.*

class 4: *The degree of φ^n grows exponentially but the equation has a space of initial conditions.*
Its Picard number is greater than 10*.*
For example, the Hietarinta-Viallet equation belongs to this class.

class 5: *The degree of φ^n grows exponentially and the equation does not have a space of initial conditions.*
"Most" equations belong to this class.

Moreover, Diller and Favre showed that the value of the dynamical degree of an equation is quite restricted.

Definition 1.9. A reciprocal quadratic integer is a root of $\lambda^2 - a\lambda + 1 = 0$ for some integer a. A real algebraic integer $\lambda > 1$ is a Pisot number if all its conjugates have modulus less than 1. A real algebraic integer $\lambda > 1$ is a Salem number if $1/\lambda$ is a conjugate and all (but at least one) of the other conjugates lie on the unit circle.

Remark 1.10. It goes without saying that reciprocal quadratic integers greater than 1 and Salem numbers are by definition irrational.

Theorem 1.11 (Diller-Favre [8])**.** *The dynamical degree of an autonomous equation of the plane is* 1*, a Pisot number or a Salem number.*

Theorem 1.12 (Diller-Favre [8])**.** *If an autonomous equation of the plane has a space of initial conditions, then its dynamical degree must be* 1*, a reciprocal quadratic integer greater than* 1 *or a Salem number. If the dynamical degree is* 1*, then the degree growth is bounded or quadratic. In particular, this implies that if the degree grows linearly, then the equation does not have a space of initial conditions.*

Theorem 1.12 says that if a mapping has a space of initial conditions, then the value of its dynamical degree (and algebraic entropy) is strongly restricted. Thus, it is sometimes possible to prove the nonexistence of a space of initial conditions by calculating the algebraic entropy [25].

It is well-known that there is a close relation between the degree growth of an equation and the Picard number of a space of initial conditions:

Proposition 1.13. *If the Picard number of a space of initial conditions is less than* 10 *(resp.* 11*), then the degree growth of the equation is bounded (resp. at most quadratic). Moreover, if the degree growth is quadratic and a space of initial conditions is minimal (Definition 5.1), then its Picard number is* 10*.*

All autonomous mappings with quadratic degree growth have been classified in [6].

Moreover, there is a strong result about equations with bounded degree:

Theorem 1.14 (Blanc-Déserti [5])**.** *Let φ be a nonperiodic equation with bounded degree growth and let X be its space of initial conditions. Then, φ can be minimized from X to either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$ with $a \neq 1$. Furthermore, φ is birationally conjugate to a projective transformation on $\mathbb{P}^2$ (Definition A.19).*

Therefore, besides periodic mappings, all autonomous integrable (zero algebraic entropy) equations of the plane are characterized by a minimal space of initial conditions with Picard number less than 11.

Using Theorem 1.8, one approach to test the integrability of an autonomous equation of the plane is as follows:

(1) Using singularity confinement, we verify if the equation has a space of initial conditions.
(2) In the case where the equation does not have a space of initial conditions, we somehow verify if the degree growth of the equation is linear. If so, then the equation belongs to Class 2 in Theorem 1.8. It is integrable and linearizable. Otherwise, the equation belongs to Class 5 and is not integrable.
(3) In the case where the equation has a space of initial conditions, we somehow calculate the algebraic entropy. If it is zero, then the equation belongs to Class 1 or 3 and is integrable. Otherwise, the equation belongs to Class 4 and is nonintegrable.

The most difficult step in this approach is (3), especially to distinguish Classes 3 and 4. In order to test the integrability of a concrete equation, it is of course sufficient to calculate the algebraic entropy at the beginning. However, computing the algebraic entropy for a general equation is extremely difficult and practically impossible. If an equation passes the singularity confinement test, then we can use some techniques such as constructing a space of initial conditions. Therefore, in order to test the integrability of a concrete equation, it would be better to apply singularity confinement first of all.

In this thesis, we will propose a new approach to "calculate" (or at least predict) the algebraic entropy of an equation that passes the singularity confinement test, which we will call "full-deautonomisation." This approach looks strange at first sight because its first step is to add nonautonomous coefficients to an autonomous equation. However, this is the essence of full-deautonomisation since this approach is based on a conjecture about a close relation between the algebraic entropy of a "sufficiently nonautonomous" equation and the condition on the nonautonomous coefficients for the equation to pass the singularity confinement test. We will introduce this approach, which is one of the main results in this thesis, in §2.

Therefore, nonautonomous equations are not only interesting for themselves but also have applications to autonomous systems. Besides full-deautonomisation, for example, the classification of autonomous equations with quadratic degree growth [6] was performed using generalized Halphen surfaces, which are in general spaces of initial conditions for discrete Painlevé equations.

While the most famous class of nonautonomous equations that have a space of initial conditions is that of the discrete Painlevé equations, there are a lot of other examples. For instance, using an algebro-geometric method, Takenawa considered a nonautonomous extension of the Hietarinta-Viallet equation [38, 39, 40]. In addition, one of the most important and powerful methods to find a nonautonomous equation with all singularities confined is late confinement, which was first reported in [20]. This method provides us with a family of nonautonomous equations that pass the singularity confinement test. We will perform a detailed algebro-geometric analysis of late confinements of the q-$\mathrm{P_{II}}$ equation in §2.2.

Unfortunately, there has been almost no general theory of nonautonomous equations with a space of initial conditions. One of the main aims of this thesis is a classification of integrable equations with a space of initial conditions. It is known that all discrete Painlevé equations have a space of initial conditions (by definition) and that they are integrable (as shown by Takenawa). Then, is it conceivable that there exists an integrable equation that is not a discrete Painlevé equation but has a space of initial conditions?

The reason why there has been almost no general theory of a space of initial conditions in nonautonomous cases is the difficulty in setting up a starting point. In autonomous cases, an equation with a space of initial conditions is reduced to one automorphism on one rational surface. However, even if a nonautonomous system such as a discrete Painlevé equation has a space of initial conditions, it is in general not reduced to an automorphism on a surface. Furthermore, in nonautonomous cases, even the centers of the blow-ups and therefore the obtained surface do depend on n. As a result, a space of initial conditions is not a single surface in a strict sense but a family of surfaces. Therefore, choosing appropriate φ_n, we can obtain many artificial examples. It is true that this kind of problem does not matter when we consider a concrete example such as a discrete Painlevé equation or a nonautonomous extension of the Hietarinta-Viallet equation. However, if we are interested in a classification, we cannot avoid setting up an appropriate starting point. In §3, we shall first describe several artificial examples and then define a space of initial conditions for nonautonomous equations. We will also recall the space of initial conditions in Sakai's sense and show that these two definitions are equivalent.

§4 mainly contains preliminaries. We shall see that, under our definition of a space of initial conditions, many analogues of the properties of autonomous equations still hold.

As in the autonomous case, in order to use the Picard number of a space of initial conditions in a classification, we must consider a minimization since increasing the Picard number of a space of initial conditions is possible. A minimization was considered by Carstea and Takenawa [7], but general nonautonomous cases have not been considered. In §5, we shall see that a minimization of a space of initial conditions in nonautonomous cases in fact is similar to that in autonomous cases.

§5.1 is one of the main parts of this thesis. We consider a minimization of an integrable equation with unbounded degree growth and a space of initial conditions to classify all such equations. As a result, we will obtain the main theorem of this thesis (Theorem 5.6), which says that an integrable mapping of the plane with unbounded degree growth and a space of initial conditions must be one of the discrete Painlevé equations. We also show the uniqueness of the minimization (Proposition 5.12).

§5.2 contains some additional results on the minimization of a space of initial conditions in nonintegrable cases. We will not classify such equations but give a procedure to minimize a space of initial conditions and show the uniqueness of the minimization.

In Appendix A, we describe the notations we use throughout the thesis and recall basic results on algebraic surfaces. Appendix B is an elementary but involved proof of a fundamental fact in linear algebra (Lemma 4.7).

2. Full-deautonomisation

In this section, we introduce what we call the full-deautonomisation, which is a new approach to predict (or, in a sense, calculate) the algebraic entropy of an equation by singularity confinement.

In §2.1, we first recall the notion of deautonomisation procedure and introduce so-called late confinement. In §2.2, we perform a detailed algebro-geometric analysis of all late confinements of an equation in order to establish an important correspondence between the condition on the nonautonomous coefficients and the linear action induced on the Picard group of a space of initial conditions. This correspondence enables us to predict the algebraic entropy of an equation by singularity confinement. In §2.3, we introduce the full-deautonomisation approach and apply it to several examples. When performing this approach, we must disregard gauge freedom. In §2.4, we see how to find such gauge freedom. In §2.5, we consider a family of late confinements and show that the dynamical degree in the nonconfining case can be estimated from the roots of the characteristic polynomial for the conditions on the nonautonomous coefficients.

In this section, we use the correspondence between the entropy of an equation and the maximum eigenvalue of the linear action induced on the Picard group of a space of initial conditions, which was first reported by Takenawa (Corollary 4.3).

2.1. **Deautonomisation and late confinement.** First let us review the deautonomisation procedure through the same example as in Example 1.5.

Example 2.1. We change the constant a in (1.2) to a nonvanishing function of n to deautonomise the equation:

$$x_{n+1}x_{n-1} = \frac{a_n(x_n+1)}{x_n^2}. \tag{2.1}$$

As in the autonomous case, this equation has three singularity patterns. A straightforward calculation shows that for all a_n, the singularity patterns that start with $x_n = \varepsilon$ or $x_n = \varepsilon^{-1}$ are the same as those in the autonomous case, respectively. However, the singularity starting with $x_n = -1+\varepsilon$ is not confined for general a_n. Let us see that the pattern of this singularity is the same as (1.3) if and only if a_n satisfies $a_n^2 = a_{n-1}a_{n+1}$.

A calculation similar to that in Example 1.5 leads to

$$\begin{aligned}
x_{n+1} &= \frac{a_n}{x_{n-1}}\varepsilon + o(\varepsilon),\\
x_{n+2} &= \frac{-a_{n+1}x_{n-1}^2}{a_n^2}\varepsilon^{-2} + o(\varepsilon^{-2}),\\
x_{n+3} &= \frac{-a_{n+2}a_n}{a_{n+1}x_{n-1}}\varepsilon + o(\varepsilon),\\
x_{n+4} &= -\frac{a_{n+3}a_{n+1}}{a_{n+2}^2} + o(1),\\
x_{n+5} &= \frac{a_{n+4}a_{n+2}(a_{n+3}a_{n+1}-a_{n+2}^2)}{a_{n+3}^2a_{n+1}a_n}\varepsilon^{-1} + o(\varepsilon^{-1}).
\end{aligned}$$

Thus, the condition

$$a_{n+3}a_{n+1} - a_{n+2}^2 = 0 \tag{2.2}$$

is necessary for x_{n+5} to become regular as in the autonomous case. Conversely, if a_n satisfies this condition for all n, then we have

$$x_{n+5} = \frac{a_n^3(-a_n^2 + a_n a_{n-1} + a_{n-1}x_{n-1})}{a_{n-1}^4} + o(1),$$

which is a regular value that depends explicitly on x_{n-1}. Solving (2.2), we conclude that the deautonomised equation (2.1) has the same pattern as of the original mapping if and only if

$$a_n = \alpha\beta^n,$$

where α and β are nonzero constants.

Thus, we obtained a nonautonomous equation that passes the singularity confinement test by a deautonomisation. The obtained equation

$$x_{n+1}x_{n-1} = \frac{\alpha\beta^n(x_n + 1)}{x_n^2}$$

is called the q-$\mathrm{P_I}$ equation [13].

Remark 2.2. When using the expression "same singularity pattern," we only require the leading order of each iterate with respect to an infinitesimal quantity to coincide with that of the original mapping, i.e. it is *not* necessary for the finite values in a pattern themselves (such as "-1" in the above example) to coincide with those in the autonomous case. If we require that, then it is often impossible to deautonomise an equation since a finite value in a singularity pattern can contain information on the nonautonomous coefficients.

In the above example, we showed that (2.1) has the same singularity patterns if and only if a_n satisfies (2.2). However, this condition is *not* necessary for (2.1) to pass the singularity confinement test. It is possible for the singularity starting with $x_n = -1 + \varepsilon$ to end at a later stage. This kind of phenomenon was first reported by Hietarinta-Viallet [20]. We shall call it a "late confinement." Let us review their example.

Example 2.3 (Hietarinta-Viallet [20])**.** Consider the equation

$$x_{n+1} + x_{n-1} = x_n + \frac{a_n}{x_n} + b, \tag{2.3}$$

where a_n and b never become 0. This mapping enters a singularity when x_n becomes 0. In the autonomous case, i.e. $a_n = a$ for all n, then the singularity pattern is

$$\{\varepsilon, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon\},$$

where ε is an infinitesimal quantity. Requiring (2.3) to have the same singularity pattern as above, we obtain the confinement condition

$$a_{n+2} - a_{n+1} - a_n + a_{n-1} = 0,$$

under which (2.3) is integrable and called the d-$\mathrm{P_I}$ equation [13]. This condition is, however, not the only one under which the singularity is confined. According to [20], the singularity of (2.3) is confined if a_n satisfies

$$a_{n-1} - (a_n + a_{n+1} - a_{n+2}) - \cdots - (a_{n+3\ell} + a_{n+3\ell+1} - a_{n+3\ell+2}) = 0 \tag{2.4}$$

for some $\ell \geq 0$. In [20], the equation was conjectured to be nonintegrable for all $\ell \geq 1$.

In this example, the length of the shortest singularity pattern is the same as that in the autonomous case. However, some deautonomised equations have shorter singularity patterns than the original equations. We shall call this kind of phenomenon an "early confinement."

Definition 2.4 (late confinement, early confinement, standard confinement)**.** Consider a deautonomised equation. If the length of its singularities is longer (resp. shorter) than that of the original mapping, then we call this situation a *late (resp. early) confinement*. If the pattern is the same as that of the original mapping, then we call this situation the *standard confinement*.

It was proved in [30] that, under the condition (2.4), the dynamical degree of (2.3) is given by the greatest root of the polynomial

$$\lambda^{3\ell+3} - \lambda^{3\ell+2} - \lambda^{3\ell+1} + \cdots + \lambda^3 - \lambda^2 - \lambda + 1. \tag{2.5}$$

The proof was done by constructing a space of initial conditions.

One can easily notice that (2.5) coincides with the characteristic polynomial of the condition (2.4). This fact is not a coincidence and can be seen through an algebro-geometric analysis. However, we do not give such an analysis in this thesis. Instead, we will calculate a more complicated example in the following subsection. A detailed calculation of late confinements of (2.3) is given in [30].

2.2. **An algebro-geometric analysis of late confinements.** Let us consider the equation [28]

$$x_{n+1}x_{n-1} = a_n \frac{x_n - b_n}{x_n - 1}, \tag{2.6}$$

where $a_n \neq 0, b_n \neq 0, 1$. This mapping has two singularity patterns for general values of the parameters. Although this equation can become periodic for specific values of a_n and b_n, we shall discard such cases in the standard deautonomisation approach since we are interested in mappings with unbounded degree growth.

The first singularity appears when x_{n-1} becomes $1 + \varepsilon$, where ε is an infinitesimal quantity. In this case, the singularity pattern is

$$\left\{1, \varepsilon^{-1}, a_{n-1}, \varepsilon, \frac{a_{n+1}b_{n+1}}{a_{n-1}}\right\},$$

which ends at this step with the confinement constraint

$$a_{n+1}b_{n+1} = a_{n-1}b_{n+2}.$$

The second singularity appears with $x_{n-1} = b_{n-1} + \varepsilon$ and its pattern is

$$\left\{b_{n-1}, \varepsilon, \frac{a_n b_n}{b_{n-1}}, \varepsilon^{-1}, \frac{a_{n+2}b_{n-1}}{a_n b_n}\right\},$$

which ends at this step with the constraint

$$a_n b_n = a_{n+2} b_{n-1}.$$

Note that these two conditions are satisfied in the autonomous case. We can solve these two relations for a_n and b_n:

$$\log a_n = \alpha n + \beta + \gamma(-1)^n + \delta j^n + \zeta j^{2n},$$
$$\log b_n = 2\alpha n + \eta - \delta j^n - \zeta j^{2n},$$

where j is a primitive third root of unity. Thus, there are 6 degrees of freedom.

However, there is another possibility. For example, we can choose to confine earlier in one of the two patterns (as a consequence, the other pattern will be longer). This can be done either by assuming that $a_{n-1} = 1$ for all n in the first pattern, or by assuming that

$$\frac{a_n b_n}{b_{n-1}} = b_{n+1}$$

in the second pattern. Note however that this second choice is just the dual of the first one. Indeed, introducing

$$z_n = \frac{b_n}{x_n},$$

we have

$$z_{n+1} z_{n-1} = \frac{b_{n-1} b_{n+1}}{a_n b_n} \times \frac{z_n - b_n}{z_n - 1},$$

which is (2.6) with a_n replaced by

$$\frac{b_{n+1} b_{n-1}}{a_n b_n}.$$

In the $a_n = 1$ case, the second pattern is

$$\left\{ b_{n-1}, \varepsilon, \frac{b_n}{b_{n-1}}, \varepsilon^{-1}, \frac{b_{n-1}}{b_n}, \varepsilon, \frac{b_{n+4} b_n}{b_{n-1}} \right\}$$

and its confinement condition is

$$b_{m+5} b_{m-1} = b_{m+4} b_m,$$

which holds in the autonomous case. The solution of this relation is

$$\log b_n = \alpha n + \beta + \sum_{m=1}^{4} \gamma_m k^{mn},$$

where k is a primitive fifth root of unity. Thus, again we have 6 degrees of freedom. In both cases, the total length of two singularity patterns is 10 (either $5+5$ or $3+7$).

Let us rewrite (2.6) as follows:

$$\text{(2.7)} \qquad \phi_n\colon \quad \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad (x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) = \left(y_n, \frac{a_n (y_n - b_n)}{x_n (y_n - 1)} \right),$$

where $a_n \neq 0$ and $b_n \neq 0, 1$. Let $s_n = 1/x_n$ and $t_n = 1/y_n$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is covered with four copies of $\mathbb{C}^2$:

$$\mathbb{P}^1 \times \mathbb{P}^1 = (x_n, y_n) \cup (x_n, t_n) \cup (s_n, y_n) \cup (s_n, t_n).$$

It is easily seen that ϕ_n becomes indeterminate at $(x_n, y_n) = (0, b_n)$ and $(s_n, y_n) = (0, 1)$, and its inverse ϕ_n^{-1} at the points $(x_{n+1}, y_{n+1}) = (b_n, 0)$ and $(x_{n+1}, t_{n+1}) = (1, 0)$. For convenience, we introduce the following notation:

$$P_n\colon (x_n, t_n) = (1, 0), \qquad Q_n\colon (x_n, y_n) = (b_{n-1}, 0),$$
$$R_n\colon (s_{n+1}, y_{n+1}) = (0, 1), \qquad S_n\colon (x_{n+1}, y_{n+1}) = (0, b_{n+1}).$$

Note that R_n and S_n are the indeterminate points of ϕ_{n+1} and P_n and Q_n are those of ϕ_{n-1}^{-1}.

We first blow up the n-th $\mathbb{P}^1 \times \mathbb{P}^1$ at R_{n-1} and S_{n-1} and the $(n+1)$-st $\mathbb{P}^1 \times \mathbb{P}^1$ at P_{n+1} and Q_{n+1}. Since ϕ_{n-1}^{-1} is indeterminate at P_n and Q_n (and ϕ_{n+1} at R_n and S_n), we must perform blow-ups at these points as well. The resulting surfaces and exceptional lines are depicted in Figure 3.

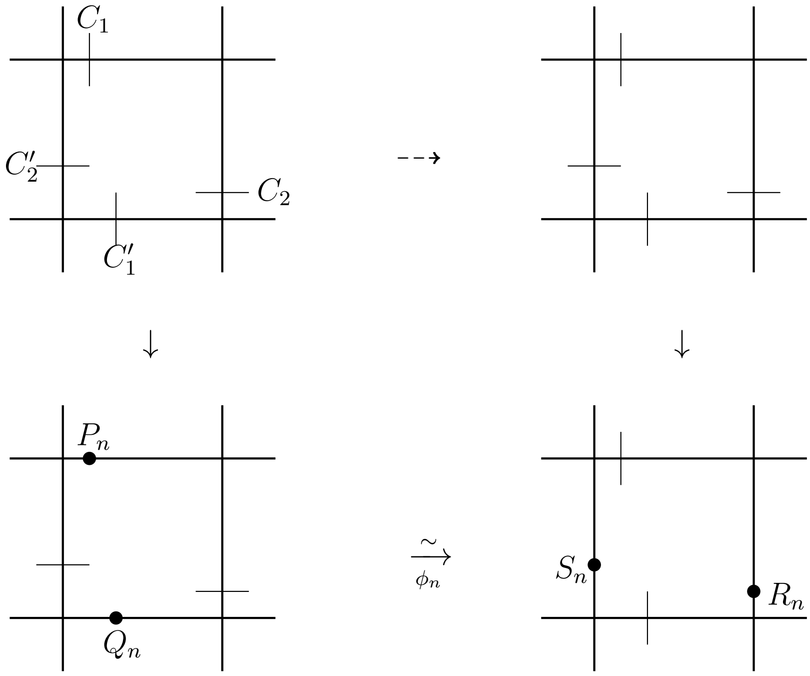


FIGURE 3. The left column shows the n-th surface and the right shows the $(n+1)$-st. The exceptional lines C_1, C_2, C_1', C_2' come from the blow-ups at $P_n, R_{n-1}, Q_n, S_{n-1}$, respectively.

Next, we calculate the images of P_n and Q_n under the mapping ϕ_n. In general, the points

$$\phi_n(P_n)\colon (s_{n+1}, y_{n+1}) = (0, a_n), \quad \phi_n(Q_n)\colon (x_{n+1}, y_{n+1}) = \left(0, \frac{a_n b_n}{b_{n-1}}\right)$$

are indeterminate points for the mapping $\phi_{n-1}^{-1}\phi_n^{-1}$. This is where a first opportunity to regularize the mapping arises. However, since the obtained mapping will become periodic, we discard this case in the standard deautonomisation approach. Indeed, if we require

$$\phi_n(P_n) = R_n \quad \text{and} \quad \phi_n(Q_n) = S_n,$$

then the mapping ϕ_n needs no further blow-ups. In this case, the conditions on the parameters are

$$a_n = 1 \quad \text{and} \quad b_{n+1}b_{n-1} = b_n,$$

which means that b_n is periodic with period 6. Moreover, the mapping ϕ_n itself becomes periodic with period 12 for these a_n and b_n and, in particular, its degree growth is bounded.

2.2.1. *First pattern.* One way to regularize the mapping ϕ_n is to require that $a_n = 1$ for all n (i.e. $\phi_n(P_n) = R_n$) and that

$$\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_n(Q_n) = S_m$$

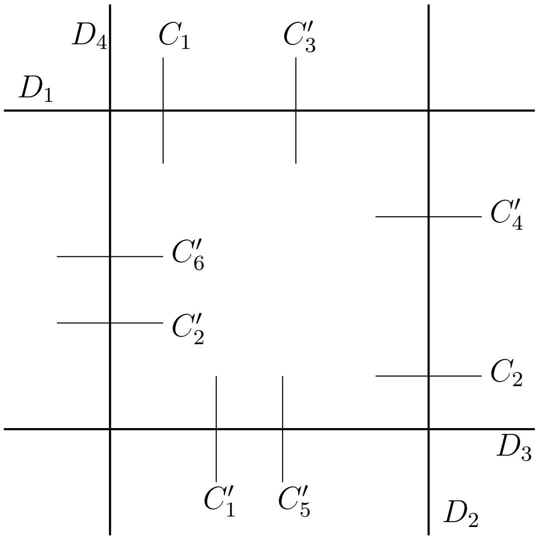


FIGURE 4. Family of surfaces on which the mapping (2.7) acts as an isomorphism under the conditions $a_n = 1$ and $b_{n+5}b_{n-1} = b_n b_{n+4}$. The curves D_1, D_2, D_3, D_4 are all (-2)-curves.

for some $m > n$. Since

$$Q_n \xrightarrow{\phi_n} (x_{n+1}, t_{n+1})\colon \left(0, \frac{b_{n-1}}{b_n}\right) \xrightarrow{\phi_{n+1}} (s_{n+2}, t_{n+2})\colon \left(\frac{b_{n-1}}{b_n}, 0\right)$$
$$\xrightarrow{\phi_{n+2}} (s_{n+3}, y_{n+3})\colon \left(0, \frac{b_{n-1}}{b_n}\right) \xrightarrow{\phi_{n+3}} (x_{n+4}, y_{n+4})\colon \left(\frac{b_{n-1}}{b_n}, 0\right)$$
$$\xrightarrow{\phi_{n+4}} (x_{n+5}, t_{n+5})\colon \left(0, \frac{b_{n-1}}{b_n b_{n+4}}\right),$$

a first opportunity arises by requiring that

$$\phi_{n+4} \circ \phi_{n+3} \circ \phi_{n+2} \circ \phi_{n+1} \circ \phi_n(Q_n) = S_{n+4},$$

which is equivalent to the following condition on b_n:

$$b_{n+5}b_{n-1} = b_n b_{n+4}.$$

Let us perform blow-ups at the points $\phi_n(Q_n), \ldots, \phi_{n+3} \circ \phi_{n+2} \circ \phi_{n+1} \circ \phi_n(Q_n)$ to obtain a space of initial conditions and call the corresponding exceptional curves $C_2', \ldots, C_5'$, respectively, as in Figure 4. We denote by C_6' the exceptional curve of the blow-up at S_n.

The curves D_1, D_2, D_3, D_4 move under the mapping as

$$D_1 \to D_2 \to D_3 \to D_4 \to D_1, \tag{2.8}$$

which in fact always remains true for any value of the parameters. The intersection pattern of these curves is of type $A_3^{(1)}$. Thus, according to Sakai's classification [37], the space depicted in Figure 4 corresponds to a discrete Painlevé equation with symmetry $D_5^{(1)}$. On the other hand, the eight exceptional curves $C_1, C_2, C_1', \ldots, C_6'$ form two separate chains

$$\{y = 1\} \to C_1 \to C_2 \to \{x = 1\} \quad \text{and} \quad \{y = b\} \to C_1' \to \cdots \to C_6' \to \{x = b\},$$

where we denote by $\{y=1\},\{x=1\},\{y=b\},\{x=b\}$ the strict transforms of the corresponding lines in $\mathbb{P}^1\times\mathbb{P}^1$, respectively. These chains correspond to the singularity patterns

$$\{1,\varepsilon,1\} \quad \text{and} \quad \left\{b_{n-1},\varepsilon,\frac{b_n}{b_{n-1}},\varepsilon^{-1},\frac{b_{n-1}}{b_n},\varepsilon,\frac{b_nb_{n+4}}{b_{n-1}},\varepsilon^{-1},\frac{b_{n-1}}{(b_nb_{n+4})}\right\}.$$

Of course it is possible to regularize the mapping ϕ_n by requiring

$$\phi_m\circ\phi_{m-1}\circ\cdots\circ\phi_n(P_n)=R_m$$

for $m>n$. We define the points

$$\begin{aligned} &T_n^{(1)}(\alpha)\colon (s_n,t_n)=(\alpha,0), && T_n^{(2)}(\beta)\colon (s_n,y_n)=(0,\beta),\\ &T_n^{(3)}(\gamma)\colon (x_n,y_n)=(\gamma,0), && T_n^{(4)}(\delta)\colon (x_n,t_n)=(0,\delta), \end{aligned} \tag{2.9}$$

for general $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and note that, in this notation, P_n,Q_n,R_n,S_n can be written as

$$\begin{aligned} P_n&=T_n^{(1)}(1), & Q_n&=T_n^{(3)}(b_{n-1}),\\ R_n&=T_{n+1}^{(2)}(1), & S_n&=T_{n+1}^{(4)}\left(\frac{1}{b_{n+1}}\right). \end{aligned}$$

A direct calculation shows that these points are mapped by ϕ_n as follows:

$$\begin{aligned} &\phi_n\left(T_n^{(1)}(\alpha)\right)=T_{n+1}^{(2)}(a_n\alpha), && \phi_n\left(T_n^{(2)}(\beta)\right)=T_{n+1}^{(3)}(\beta),\\ &\phi_n\left(T_n^{(3)}(\gamma)\right)=T_{n+1}^{(4)}\left(\frac{\gamma}{a_nb_n}\right), && \phi_n\left(T_n^{(4)}(\delta)\right)=T_{n+1}^{(1)}(\delta). \end{aligned} \tag{2.10}$$

The chain starting with P_n is

$$\begin{aligned} T_n^{(1)}(1)&\to T_{n+1}^{(2)}(a_n)\to T_{n+2}^{(3)}(a_n)\to T_{n+3}^{(4)}\left(\frac{a_n}{a_{n+2}b_{n+2}}\right)\to T_{n+4}^{(1)}\left(\frac{a_n}{a_{n+2}b_{n+2}}\right)\\ &\to T_{n+5}^{(2)}\left(\frac{a_na_{n+4}}{a_{n+2}b_{n+2}}\right)\to\cdots\to T_{n+4\ell+1}^{(2)}\left(a_n\prod_{k=1}^{\ell}\frac{a_{n+4k}}{a_{n+4k-2}b_{n+4k-2}}\right), \end{aligned}$$

and that with Q_n is

$$\begin{aligned} T_n^{(3)}(b_{n-1})&\to T_{n+1}^{(4)}\left(\frac{b_{n-1}}{a_nb_n}\right)\to T_{n+2}^{(1)}\left(\frac{b_{n-1}}{a_nb_n}\right)\to T_{n+3}^{(2)}\left(\frac{b_{n-1}a_{n+2}}{a_nb_n}\right)\to T_{n+4}^{(3)}\left(\frac{b_{n-1}a_{n+2}}{a_nb_n}\right)\\ &\to T_{n+5}^{(4)}\left(\frac{b_{n-1}a_{n+2}}{a_nb_na_{n+4}b_{n+4}}\right)\to\cdots\to T_{n+4\ell'+1}^{(4)}\left(\frac{b_{n-1}}{a_nb_n}\prod_{k=1}^{\ell'}\frac{a_{n+4k-2}}{a_{n+4k}b_{n+4k}}\right). \end{aligned}$$

Therefore, for arbitrary nonnegative integers ℓ and ℓ', there is an opportunity to regularize ϕ_n after $4+4\ell+4\ell'$ blow-ups. The required conditions on the parameters are

$$a_n\prod_{k=1}^{\ell}\frac{a_{n+4k}}{a_{n+4k-2}b_{n+4k-2}}=1,\qquad \frac{b_{n-1}}{a_nb_n}\prod_{k=1}^{\ell'}\frac{a_{n+4k-2}}{a_{n+4k}b_{n+4k}}=\frac{1}{b_{n+4\ell'+1}}. \tag{2.11}$$

The family of surfaces X_n obtained after $4+4(\ell+\ell')$ blow-ups is depicted in Figure 5. Note that Figure 4 corresponds to the special case $\ell=0,\ell'=1$.

While the fundamental chain (2.8) is the same as in the case $\ell=0,\ell'=1$, the curves D_1,D_2,D_3,D_4 are all $-(\ell+\ell'+1)$-curves. On the other hand, the exceptional curves

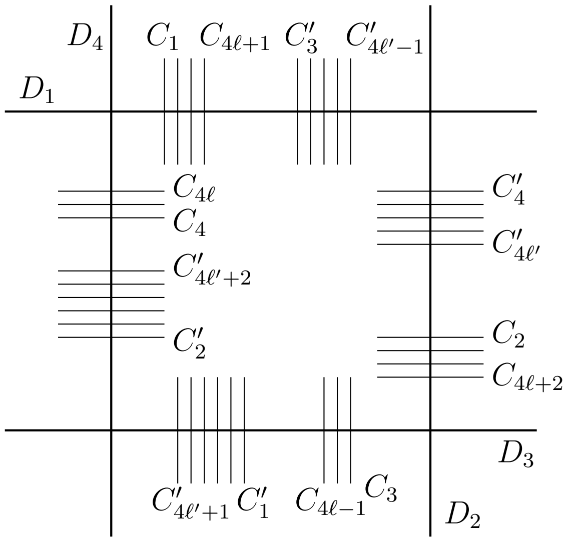


FIGURE 5. Family of surfaces X_n obtained after $4+4\ell+4\ell'$ blow-ups. The mapping (2.7) acts on this family as an isomorphism under the condition (2.11).

$C_1, \ldots, C_{4\ell+2}, C_1', \ldots, C_{4\ell'+2}'$ form two chains

$$\{y=1\} \to C_1 \to \cdots \to C_{4\ell+2} \to \{x=1\},$$
$$\{y=b\} \to C_1' \to \cdots \to C_{4\ell'+2}' \to \{x=b\}.$$

These correspond, respectively, to the singularity patterns

$$\left\{1, \varepsilon^{-1}, a_n, \varepsilon, \frac{a_{n+2}b_{n+2}}{a_n}, \cdots, \varepsilon^{-1}, a_n \prod_{k=1}^{\ell} \frac{a_{n+4k}}{a_{n+4k-2}b_{n+4k-2}} = 1\right\} \quad \text{and}$$
$$\left\{b_{n-1}, \varepsilon, \frac{a_n b_n}{b_{n-1}}, \varepsilon^{-1}, \frac{b_{n-1}a_{n+2}}{a_n b_n}, \cdots, \varepsilon, \frac{a_n b_n}{b_{n-1}} \prod_{k=1}^{\ell'} \frac{a_{n+4k}b_{n+4k}}{a_{n+4k-2}} = b_{n+4\ell'+1}\right\}.$$

Let us calculate the algebraic entropy of the mapping (2.7) by the linear action induced on the Picard groups (Corollary 4.3). The Picard group $\operatorname{Pic} X_n$ for this surface has rank $6+4\ell+4\ell'$, and we choose

$$D_1, D_2, D_3, D_4, C_2, \ldots, C_{4\ell+1}, C_1', \ldots, C_{4\ell'+2}'$$

as a basis. Since the exceptional curves of first kind $C_{4\ell+2}$ and $\{x=b\}$ are, respectively, linearly isomorphic to the divisors

$$-D_2 + D_4 + C_2' + \sum_{k=1}^{\ell'}(C_{4k+2}' - C_{4k}') + \sum_{k=1}^{\ell}(C_{4k} - C_{4k-2})$$
$$\text{and} \quad D_4 - C_1' + \sum_{k=0}^{\ell'} C_{4k+2}' + \sum_{k=1}^{\ell} C_{4k},$$

the matrix of the linear action ϕ_* with respect to the above basis has the form

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & \\ \hline & & 0 & & N \end{array}\right), \tag{2.12}$$

where N is a square matrix of size $4\ell+4\ell'+2$ defined as follows: the 4ℓ-th and $(4\ell+4\ell'+2)$-nd columns of N are

$$^t\left(-1\ 0\ 1\ 0\ \cdots\ -1\ 0\ 1\ 0\ \middle|\ 0\ 1\ 0\ -1\ \cdots\ 0\ 1\ 0\ -1\ 0\ 1\right),$$
$$^t\left(0\ 0\ 1\ 0\ \cdots\ 0\ 0\ 1\ 0\ \middle|\ -1\ 1\ 0\ 0\ 0\ 1\ \cdots\ 0\ 0\ 0\ 1\right),$$

respectively, where the separator "$|$" lies between the 4ℓ-th and $(4\ell+1)$-st columns, and the m-th column $(m \neq 4\ell, 4\ell+4\ell'+2)$ is

$$^t\left(0\ \cdots\ 0\ 1\ 0\ \cdots\ 0\right),$$

where "1" lies on the $(m+1)$-st row. We have omitted to write the upper right part of (2.12) since it is irrelevant for our purposes.

The upper left submatrix of (2.12) is a permutation matrix and, in particular, its eigenvalues have all modulus 1. Thus, if the matrix (2.12) has an eigenvalue greater than 1, then it must come from the submatrix N. A direct calculation shows that the characteristic polynomial of N is

$$\begin{aligned} f_N(\lambda) =& \lambda^{4(\ell+\ell')+2} + \sum_{j=1}^{\min(\ell',\ell)} \left(-\lambda^{4\ell+4\ell'-4j+5} - \lambda^{4\ell+4\ell'-4j+4} + \lambda^{4\ell+4\ell'-4j+2}\right) \\ & - \sum_{j=0}^{|\ell'-\ell|} \lambda^{4(\min(\ell',\ell)+j)+1} + \lambda^{4\min(\ell',\ell)} + \sum_{j=1}^{\min(\ell',\ell)} \left(-\lambda^{4j-2} - \lambda^{4j-3} + \lambda^{4j-4}\right). \end{aligned} \tag{2.13}$$

In the case $\ell + \ell' \geq 2$, the matrix N has a real eigenvalue greater than 1 since

$$f_N(1) = 1 - (\ell' + \ell) < 0,$$

and thus the mapping is nonintegrable.

On the other hand, in the case $\ell = 1, \ell' = 0$ or $\ell = 0, \ell' = 1$, an easy calculation shows that

$$f_N(\lambda) = (\lambda - 1)^2(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1).$$

While all eigenvalues of N have modulus 1, the matrices N and (2.12) are not diagonalizable, i.e. the degree growth of the mapping is not bounded. As explained above, the cases $\ell = 0, \ell' = 1$ and $\ell = 1, \ell' = 0$ are dual to each other.

In the case $\ell = \ell' = 0$, the matrix (2.12) is in fact a 12-th root of the identity matrix and, as denoted above, the mapping itself is also periodic with period 12.

Combining these results, we obtain the following proposition:

Proposition 2.5. *Let ℓ, ℓ' be nonnegative integers and let a_n, b_n satisfy the system (2.11). Then, the mapping ϕ_n can be regularized after $4\ell + 4\ell' + 4$ blow-ups. If $\ell = \ell' = 0$, then the mapping (2.7) itself is periodic with period* 12*. If $\ell = 1, \ell' = 0$ or $\ell = 0, \ell' = 1$, then the mapping (2.7) has unbounded degree growth but is integrable. If $\ell + \ell' \geq 2$, then the mapping (2.7) is nonintegrable and its dynamical degree is given by the greatest root of* *the polynomial (2.13). In particular, in these three cases, if the number of blow-ups is greater than* 8*, then the mapping (2.7) is nonintegrable.*

2.2.2. *Second pattern.* We have obtained a regularization of (2.7) by requiring that

$$\phi_{n+4\ell} \circ \phi_{n+4\ell-1} \circ \cdots \circ \phi_n(P_n) = R_{n+4\ell},$$
$$\phi_{n+4\ell'} \circ \phi_{n+4\ell'-1} \circ \cdots \circ \phi_n(Q_n) = S_{n+4\ell'}.$$

However, it is also possible to regularize ϕ_n by requiring that

$$\phi_{n+4\ell'+2} \circ \phi_{n+4\ell'+1} \circ \cdots \circ \phi_n(P_n) = S_{n+4\ell'+2},$$
$$\phi_{n+4\ell+2} \circ \phi_{n+4\ell+1} \circ \cdots \circ \phi_n(Q_n) = R_{n+4\ell+2}.$$

Using (2.9) and (2.10), the chains starting with P_n and Q_n can be written as

$$P_n = T_n^{(1)}(1) \to T_{n+1}^{(2)}(a_n) \to T_{n+2}^{(3)}(a_n) \to T_{n+3}^{(4)}\left(\frac{a_n}{a_{n+2}b_{n+2}}\right) \to$$
$$\cdots \to T_{n+4\ell'+3}^{(4)}\left(\prod_{k=0}^{\ell'} \frac{a_{n+4k}}{a_{n+4k+2}b_{n+4k+2}}\right) = T_{n+4\ell'+3}^{(4)}\left(\frac{1}{b_{n+4\ell'+3}}\right) = S_{n+4\ell'+2}$$

and

$$Q_n = T_n^{(3)}(b_{n-1}) \to T_{n+1}^{(4)}\left(\frac{b_{n-1}}{a_n b_n}\right) \to T_{n+2}^{(1)}\left(\frac{b_{n-1}}{a_n b_n}\right) \to T_{n+3}^{(2)}\left(\frac{b_{n-1}a_{n+2}}{a_n b_n}\right) \to$$
$$\cdots \to T_{n+4\ell+3}^{(2)}\left(b_{n-1}\prod_{k=0}^{\ell} \frac{a_{n+4k+2}}{a_{n+4k}b_{n+4k}}\right) = T_{n+4\ell+3}^{(2)}(1) = R_{n+4\ell+2},$$

respectively. Therefore, if the parameters a_n and b_n satisfy

$$\prod_{k=0}^{\ell'} \frac{a_{n+4k}}{a_{n+4k+2}b_{n+4k+2}} = \frac{1}{b_{n+4\ell'+3}}, \quad b_{n-1}\prod_{k=0}^{\ell} \frac{a_{n+4k+2}}{a_{n+4k}b_{n+4k}} = 1, \tag{2.14}$$

then the mapping ϕ_n can be regularized after $8 + 4\ell + 4\ell'$ blow-ups. The obtained family of surfaces $\widetilde{X}_n$ is depicted in Figure 6.

The behavior of the fundamental chain (2.8) is the same as in the case of the first pattern. On the other hand, the exceptional curves $C_1, \ldots, C_{4\ell'+4}, C'_1, \ldots, C_{4\ell+4}$ form the two chains

$$\{y = 1\} \to C'_1 \to \cdots \to C'_{4\ell'+4} \to \{x = b\},$$
$$\{y = b\} \to C_1 \to \cdots \to C_{4\ell+4} \to \{x = 1\},$$

which correspond to the singularity patterns

$$\left\{1, \varepsilon^{-1}, a_n, \varepsilon, \frac{a_{n+2}b_{n+2}}{a_n}, \ldots, \varepsilon^{-1}, a_{n+4\ell'}\prod_{k=0}^{\ell'-1} \frac{a_{n+4k}}{a_{n+4k+2}b_{n+4k+2}}, \varepsilon, b_{n+4\ell'+3}\right\}$$

and $$\left\{b_{n-1}, \varepsilon, \frac{a_n b_n}{b_{n-1}}, \varepsilon^{-1}, \frac{b_{n-1}a_{n+2}}{a_n b_n}, \ldots, \varepsilon, \frac{a_{n+4\ell}b_{n+4\ell}}{b_{n-1}}\prod_{k=0}^{\ell-1} \frac{a_{n+4k}b_{n+4k}}{a_{n+4k+2}}, \varepsilon^{-1}, 1\right\},$$

respectively.

Now $\mathrm{Pic}\,\widetilde{X}_n$ has rank $10 + 4\ell + 4\ell'$ and we choose

$$D_1, D_2, D_3, D_4, C_1, \ldots, C_{4\ell+3}, C'_2, \ldots, C'_{4\ell'+4}$$

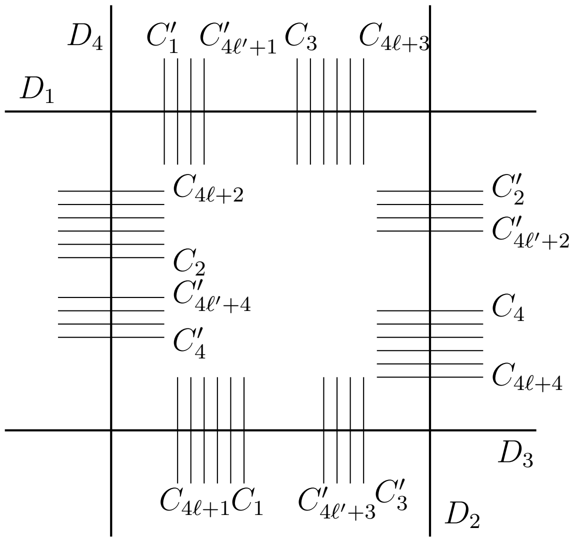


FIGURE 6. Family of surfaces $\widetilde{X}_n$ obtained after $8+4\ell+4\ell'$ blow-ups. The mapping (2.7) acts on this family as an isomorphism under the condition (2.14). Note that on this surface the curves $D_1, \ldots, D_4$ are all $-(\ell+\ell'+2)$-curves.

as a basis. Since the exceptional curves of the first kind $C_{4\ell+4}$ and $\{x = b\}$ are, respectively, linearly isomorphic to the divisors

$$-D_2 + D_4 + \sum_{k=1}^{\ell}(C_{4k-2} - C_{4k}) + C_{4\ell+2} + \sum_{k=0}^{\ell'}(C'_{4k+4} - C'_{4k+2})$$

$$\text{and} \quad D_4 - C_1 + \sum_{k=0}^{\ell} C_{4k+2} + \sum_{k=0}^{\ell'} C'_{4k+4},$$

the matrix of the linear action induced on the Picard group with respect to this basis is

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & \\ \hline & & & & \\ & & 0 & & \widetilde{N} \\ & & & & \end{array}\right), \tag{2.15}$$

where $\widetilde{N}$ is a square matrix of size $4\ell + 4\ell' + 6$ defined as follows: the $(4\ell + 3)$-rd and $(4\ell + 4\ell' + 6)$-th columns of $\widetilde{N}$ are

$${}^t\big(0\ 1\ 0\ {-1}\ \cdots\ 0\ 1\ 0\ {-1}\ 0\ 1\ 0\ \big|\ {-1}\ 0\ 1\ 0\ \cdots\ {-1}\ 0\ 1\ 0\ {-1}\ 0\ 1\big),$$

$${}^t\big({-1}\ 1\ 0\ 0\ 0\ \cdots\ 1\ 0\ 0\ 0\ 1\ 0\big|\ 0\ 0\ 1\ 0\ \cdots\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ \big),$$

respectively, where the separator "$|$" lies between the $(4\ell+3)$-rd and $(4\ell+4)$-th columns, and the m-th column $(m \neq 4\ell + 3, 4\ell + 4\ell' + 6)$ is

$${}^t\big(0\ \cdots\ 0\ 1\ 0\ \cdots\ 0\big),$$

where “1” lies on the $(m+1)$-st row. Again, we have omitted the detail in the upper right part of the matrix (2.15). The characteristic polynomial $f_{\widetilde{N}}(\lambda)$ for $\widetilde{N}$ is

(2.16)

$$f_{\widetilde{N}}(\lambda) = \sum_{j=0}^{\min(\ell,\ell')} \left(\lambda^{4\ell+4\ell'-4j+6} - \lambda^{4\ell+4\ell'-4j+5} - \lambda^{4\ell+4\ell'-4j+4}\right)$$
$$+ \sum_{j=0}^{|\ell'-\ell|-1} \left(\lambda^{4(\min(\ell,\ell'))+6} - \lambda^{4(\min(\ell,\ell')+j)+4}\right) + \sum_{j=0}^{\min(\ell,\ell')} \left(\lambda^{4j+2} + \lambda^{4j+1} - \lambda^{4j}\right).$$

Although this polynomial always vanishes at $\lambda = 1$, its derivative $f'_{\widetilde{N}}$ satisfies

$$f'_{\widetilde{N}}(1) = -2(\ell + \ell' + 2\ell\ell'),$$

which is negative whenever ℓ or ℓ' differs from 0. In such cases, $f_{\widetilde{N}}(\lambda)$ has a real root greater than 1 and thus the algebraic entropy of the mapping is positive.

In the case $\ell = \ell' = 0$, we have

$$f_{\widetilde{N}}(\lambda) = (\lambda+1)(\lambda-1)^3(\lambda^2+\lambda+1),$$

and all eigenvalues of $\widetilde{N}$ have modulus 1 and thus the mapping ϕ_n is integrable. The conditions on the parameters in this case are

$$a_n b_{n+3} = a_{n+2} b_{n+2}, \quad b_{n-1} a_{n+2} = a_n b_n,$$

which we discussed at the beginning.

Combining these results, we obtain the following proposition:

Proposition 2.6. *Let ℓ, ℓ' be nonnegative integers and let a_n, b_n satisfy the system (2.14). Then, the mapping ϕ_n can be regularized after $4\ell + 4\ell' + 8$ blow-ups. If $\ell = \ell' = 0$, then the mapping (2.7) has unbounded degree growth but is integrable. If $\ell + \ell' \geq 1$, then the mapping (2.7) is nonintegrable and its dynamical degree is given by the greatest root of the polynomial (2.16). As in the case of the first pattern, if the number of blow-ups is greater than 8, then the mapping (2.7) is nonintegrable.*

2.2.3. *Conditions on the parameters.* Let us consider the relation between the condition on the parameters and the linear action induced on the Picard group. It is clear that the effect of the duality

$$(x_n, y_n) \to \left(\frac{b_{n-1}}{x_n}, \frac{b_n}{y_n}\right), \quad (a_n, b_n) \to \left(\frac{b_{n+1} b_{n-1}}{a_n b_n}, b_n\right)$$

on the mapping (2.7) is to interchange the roles of ℓ and ℓ'. Therefore, in the following, we may without loss of generality assume that $\ell' \geq \ell \geq 0$. Let $A_n = \log a_n$ and $B_n = \log b_n$.

Let us consider the first pattern. Using A_n and B_n, the condition on the parameter (2.11) can be written as

$$A_{n+1} = \sum_{k=1}^{\ell} (-A_{n-4k+1} + A_{n-4k+3} + B_{n-4k+3}), \tag{2.17}$$

$$B_{n+1} = A_{n-4\ell'} - B_{n-4\ell'-1} + B_{n-4\ell'} + \sum_{k=1}^{\ell'} (-A_{n-4k+2} + A_{n-4k+4} + B_{n-4k+4}). \tag{2.18}$$

As usual, whenever there is a mismatch between the limits in a summation we shall take that sum to be zero.

In the case $\ell = 0$, i.e. $A_n = 0$ for all n, the first condition (2.17) is trivially satisfied and the second is

$$B_{n+1} = -B_{n-4\ell'-1} + \sum_{k=0}^{\ell'} B_{n-4k}.$$

That is, the behavior of B_n can be written by using the matrix N in (2.12) as follows:

$$\begin{pmatrix} B_{n-4\ell'} & B_{n-4\ell'+1} & \cdots & B_{n+1} \end{pmatrix} = \begin{pmatrix} B_{n-4\ell'-1} & B_{n-4\ell'} & \cdots & B_n \end{pmatrix} \times N.$$

Note that in this case the matrix N is in Frobenius normal form since only the last column is special.

Obvious difficulties arise however in the case $\ell' \geq \ell > 0$. While there are $8\ell' + 3$ parameters on the right hand side in the system (2.17, 2.18), the size of the matrix N is equal to $4\ell' + 4\ell + 2$. Using the relation (2.17), however, we can systematically reduce the number of variables in (2.18). As a result we have

$$\begin{aligned} B_{n+1} = &- B_{n-4\ell'-1} + B_{n-4\ell'} + \sum_{k=1}^{\ell'} B_{n-4k+4} \\ &+ \sum_{j=1}^{q} (-1)^{j-1} \sum_{k=1}^{\ell} B_{n+4(j\ell-\ell'-k)+2j} + \sum_{j=0}^{r-1} (-1)^j A_{n-2j}, \end{aligned} \tag{2.19}$$

where the nonnegative integers q and r are the quotient and remainder when dividing $2\ell' + 1$ by $2\ell + 1$:

$$2\ell' + 1 = q(2\ell + 1) + r, \quad r < 2\ell + 1.$$

Now the system (2.17, 2.19) can be expressed as

$$\begin{pmatrix} A_{n-4\ell+2} \cdots A_{n+1} \; B_{n-4\ell'} \cdots B_{n+1} \end{pmatrix} = \begin{pmatrix} A_{n-4\ell+1} \cdots A_n \; B_{n-4\ell'-1} \cdots B_n \end{pmatrix} \times M,$$

where M is a square matrix of size $4\ell' + 4\ell + 2$, which is now the same as that of N. While the matrices M and N have the same size, they are not identical. However, brute force calculations of the first 400 cases (up to $\ell = 20, \ell' = 20$) show that in any case, the matrix M has the same Frobenius normal form as N. This analysis leads us to the following conjecture:

Conjecture 2.7. For all $\ell, \ell' \geq 0$, the matrices M and N are similar to each other. In particular, the behavior of the parameters A_n and B_n can be written by (a part of) the linear action induced on the Picard group.

The case of the second blow-up pattern is more interesting. Rewriting (2.14) in terms of the logarithmic variables, we obtain

$$A_{n+1} = \sum_{k=1}^{\ell} (A_{n-4k-1} - A_{n-4k+1} + B_{n-4k-1}) - B_{n-4\ell-2} + A_{n-1} + B_{n-1}, \tag{2.20}$$

$$B_{n+1} = \sum_{k=0}^{\ell'} (-A_{n-4k-2} + A_{n-4k} + B_{n-4k}). \tag{2.21}$$

When $\ell = \ell'$, the size of the matrix describing the above system coincides with that of the matrix $\widetilde{N}$ but these two matrices are not identical. As we have already seen, the only integrable case of the second pattern is $\ell = \ell' = 0$. Let however

$$B'_n = B_n - A_{n-1}.$$

Then, the system (2.20, 2.21) is equivalent to

$$\begin{aligned} A_{n+1} &= A_{n-1} + B'_n - B'_{n-2}, \\ B'_{n+1} &= A_{n-1} - A_{n-2} + B'_n, \end{aligned}$$

which can be written by the matrix $\widetilde{N}$:

$$\left(A_{n-1}\ A_n\ A_{n+1}\ B'_{n-1}\ B'_n\ B'_{n+1}\right) = \left(A_n\ A_{n+1}\ A_{n+2}\ B'_n\ B'_{n+1}\ B'_{n+2}\right) \times \widetilde{N}.$$

The second relation expresses a conservation law for the quantity $B'_{n+1} - A_{n-1}$ since

$$B'_{n+1} - A_{n-1} = B'_n - A_{n-2}.$$

Using this law, we obtain the equation

$$A_{n+1} = A_{n-1} + A_{n-2} - A_{n-4},$$

which depends only on A_n.

In general cases, the matrix expressing the system (2.20, 2.21) does not have the same size as $\widetilde{N}$. Thus, as in the previous case the number of variables needs to be reduced. Using (2.20) and $\ell' \geq \ell \geq 0$, we have

$$\begin{aligned} B_{n+1} = &\sum_{k=0}^{\ell'} B_{n-4k} + \sum_{i=0}^{q-1} \sum_{k=0}^{\ell} B_{n-4(k+i(\ell+1)+r)-2} \\ &- \sum_{j=1}^{q} B_{n-4j(\ell+1)-4r+1} + \sum_{k=0}^{r-1} \left(A_{n-4k} - A_{n-4k-2}\right), \end{aligned} \tag{2.22}$$

where the nonnegative integers q and r are now the quotient and remainder when dividing $\ell' + 1$ by $\ell + 1$:

$$\ell' + 1 = q(\ell + 1) + r, \quad r < \ell + 1.$$

Now the system (2.20, 2.22) can be expressed as

$$\left(A_{n-4\ell}\ \cdots\ A_{n+1}\ B_{n-4\ell'-2}\ \cdots\ B_{n+1}\right) = \left(A_{n-4\ell-1}\ \cdots\ A_n\ B_{n-4\ell'-3}\ \cdots\ B_n\right) \times \widetilde{M},$$

where $\widetilde{M}$ is a square matrix of size $4\ell' + 4\ell + 6$, which is the same as that of $\widetilde{N}$. Again, brute force calculations show that this matrix is similar to $\widetilde{N}$, at least for all cases up to $\ell = \ell' = 20$. As for the first pattern, this analysis leads us to the following conjecture:

Conjecture 2.8. For all $\ell, \ell' \geq 0$, the matrices $\widetilde{M}$ and $\widetilde{N}$ are similar to each other. In particular, the behavior of the parameters A_n and B_n can be written by (a part of) the linear action induced on the Picard group.

Remark 2.9. In a strict sense, the surfaces X_n, their Picard groups and the curves D_i, C_i, C'_i vary depending on n, and we have naturally identified $\operatorname{Pic} X_n, D_i, C_i, C'_i$ for all n, respectively. When considering blow-ups of nonautonomous systems, it is usual to unconsciously assume that the "basic structure" of the blow-ups does not depend on n. We will consider this kind of problem in §3.

2.3. **Full-deautonomisation.** The algebro-geometric analysis in the above subsection (and that of many other equations) leads us to the following conjecture:

Conjecture 2.10. Consider an equation of the plane with all singularities confined, and its deautonomisation with the same singularity patterns as in the case of the original mapping. Assume that there are sufficiently many nonautonomous coefficients and that we disregard gauge freedom in the coefficients. Then, there is at least one coefficient a_n such that

$$\lim_{n\to+\infty} |a_n|^{1/n} = \lambda \quad \text{or} \quad \lim_{n\to+\infty} |\log a_n|^{1/n} = \lambda,$$

where λ is the dynamical degree of the equation.

Definition 2.11 (full-deautonomisation)**.** Based on the above conjecture, we can "calculate" (or at least predict) the value of the algebraic entropy of an equation. We shall call this procedure *full-deautonomisation.*

When using the full-deautonomisation approach, we must disregard gauge freedom. In §2.4, we will see how to find such gauge freedom.

Let us first apply the full-deautonomisation approach to some examples.

Example 2.12. Consider the mapping

$$x_{n+1}x_{n-1} = \frac{x_n^4 - 1}{x_n^4 + 1},$$

which was introduced in [31]. This equation has 8 singularity patterns:

$$\{\pm 1, \varepsilon, \mp 1\}, \qquad \{\pm j^2, \varepsilon, \pm j^2\},$$
$$\{\pm j, \varepsilon^{-1}, \mp j^3\}, \qquad \{\pm j^3, \varepsilon^{-1}, \mp j\}, \tag{2.23}$$

where ε is an infinitesimal quantity and j is a primitive eighth root of unity. While these singularities are all confined, this equation is nonintegrable since the algebraic entropy is $\log(2+\sqrt{3}) > 0$ [31].

Let us perform the full-deautonomisation method. One deautonomisation is

$$x_{n+1}x_{n-1} = \frac{x_n^4 - q_n^4}{x_n^4 + 1}$$

and, requiring this equation to have the same singularity patterns as (2.23), we obtain the condition on q_n:

$$q_{n+1}q_{n-1} = q_n^4.$$

The characteristic polynomial of $\log q_n$ is

$$\lambda^2 - 4\lambda + 1,$$

whose greater root coincides with $2+\sqrt{3}$, the dynamical degree of the equation.

Example 2.13. Let us consider the Hietarinta-Viallet equation (1.1):

$$x_{n+1} + x_{n-1} = x_n + \frac{a}{x_n^2}.$$

This mapping has a singularity pattern

$$\{\varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon\},$$

where ε is an infinitesimal quantity.

One of the simplest deautonomisations of this equation is

$$x_{n+1} + x_{n-1} = x_n + \frac{a_n}{x_n^2},$$

but the condition for this deautonomised equation to have the same pattern as above is just $a_n = a_{n-3}$. However, this is not the only deautonomisation that has the above pattern. It is possible to deautonomise the Hietarinta-Viallet mapping to [38, 39, 40]

$$x_{n+1} + x_{n-1} = x_n + \frac{b_n}{x_n} + \frac{a}{x_n^2},$$

from which the original Hietarinta-Viallet equation is recovered by taking $b_n = 0$ for all n. Requiring this equation to have the above singularity pattern, we obtain the confinement constraint

$$b_{n+3} - 2b_{n+2} - 2b_{n+1} - b_n = 0.$$

Its characteristic polynomial is

$$\lambda^3 - 2\lambda^2 - 2\lambda - 1$$

and its greatest root is $(3 + \sqrt{5})/2$, which coincides with the dynamical degree of the Hietarinta-Viallet equation.

Example 2.14. An extension of the Hietarinta-Viallet equation was considered in [25]:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^k},$$

where $k \geq 2$. The behavior of this mapping varies depending on the parity of k. Since this equation passes the singularity confinement test only for even k, we assume that k is even. In this case, the equation has a singularity pattern

$$\{\varepsilon, \varepsilon^{-k}, \varepsilon^{-k}, \varepsilon\},$$

where ε is an infinitesimal quantity.

As in the original Hietarinta-Viallet equation, we deautonomise the above equation as follows:

$$x_{n+1} + x_{n-1} = x_n + \frac{b_n}{x_n} + \frac{1}{x_n^k}.$$

Requiring this equation to have the same singularity pattern as above, we obtain the following condition on b_n:

$$b_{n+3} - kb_{n+2} - kb_{n+1} - b_n = 0.$$

Its characteristic polynomial is

$$\lambda^3 - k\lambda^2 - k\lambda + 1$$

and its greatest root is

$$\frac{k + 1 + \sqrt{(k-1)(k+3)}}{2},$$

which coincides with the exact value of the dynamical degree calculated in [25].

2.4. **Gauge freedom.** There are many possible choices when introducing nonautonomous coefficients to an equation. If a deautonomised equation has a nonautonomous coefficient that contains gauge freedom, the full-deautonomisation approach may fail in predicting the exact value of the algebraic entropy. In this subsection, we see through an example how to find such gauge freedom.

Example 2.15. Let us consider the equation

$$x_{n+1}x_{n-1} = a(1 - x_n), \tag{2.24}$$

where a is a nonzero constant. This equation is known to be integrable. Since in the case $a = 1$ the equation has period 5, we mainly consider the case $a \neq 1$. Note that the constant "1" on the right hand side can be changed to any nonzero value by a gauge transformation.

This equation has two singularity patterns, the beginnings of which are $x_n = 1 + \varepsilon$ and $x_n = \varepsilon^{-1}$, respectively, where ε is an infinitesimal quantity. The more essential pattern is

$$\{1, \varepsilon, a, \varepsilon^{-1}, \varepsilon^{-1}, a, \varepsilon, 1\},$$

which we shall focus on.

The simplest deautonomisation of (2.24) is

$$x_{n+1}x_{n-1} = a_n(1 - x_n)$$

and, starting with a regular x_n and $x_{n+1} = 1 + \varepsilon$, we obtain

$$\left\{1, \varepsilon, a_{n+2}, \varepsilon^{-1}, \varepsilon^{-1}, \frac{a_{n+5}a_{n+4}}{a_{n+2}}, \varepsilon, \frac{a_{n+7}a_{n+2}}{a_{n+5}a_{n+4}}\right\}.$$

A straightforward calculation shows that this pattern ends at this step if and only if a_n satisfies

$$a_{n+7}a_{n+2} = a_{n+5}a_{n+4}. \tag{2.25}$$

The characteristic polynomial of $\log a_n$ is

$$\lambda^5 - \lambda^3 - \lambda^2 + 1 = (\lambda - 1)^2(\lambda + 1)(\lambda^2 + \lambda + 1),$$

all roots of which have modulus 1.

However, this deautonomisation is not the only one. Since the constant "1" on the right hand side in (2.24) is not essential, it is possible to consider the following deautonomisation:

$$x_{n+1}x_{n-1} = a_n - a_n^k x_n, \tag{2.26}$$

where k is an integer greater than 1. Requiring this equation to have the same pattern as above, we obtain the condition on a_n:

$$a_{n+4}a_{n+5} = a_{n+2}a_{n+7}(a_{n+1}a_{n+8})^{k-1}. \tag{2.27}$$

In this case, the characteristic polynomial of $\log a_n$ is

$$g_1(\lambda) = (\lambda - 1)^2(\lambda + 1)(\lambda^2 + \lambda + 1)\left((k - 1)(\lambda^2 + 1) + \lambda\right), \tag{2.28}$$

which has a root greater than 1 in general. However, as will be shown later, (2.26) is integrable under the condition (2.27).

The factor

$$(k - 1)(\lambda^2 + 1) + \lambda$$

in fact comes from gauge freedom and thus we must disregard it. Let a_n satisfy (2.27). Introducing $z_n = \gamma_n x_n$ with

$$\gamma_n = (\gamma_{n+1}\gamma_{n-1})^{1-k}, \tag{2.29}$$

we have

$$z_{n+1}z_{n-1} = \alpha_n - \alpha_n^k z_n,$$

where

$$\alpha_n = \gamma_n^{1/(k-1)} a_n.$$

Then there exists γ_n satisfying (2.29) such that this α_n satisfies (2.25).

It might seem that this example demonstrates a defect of the full-deautonomisation approach. However, we can find the gauge factor $(k-1)(\lambda^2+1)+\lambda$ and the condition (2.29) solely by singularity confinement. Key is to consider a late confinement of the same equation.

In order to explain this, we construct the space of initial conditions. Let us consider the equation

$$x_{n+1}x_{n-1} = a_n - b_n x_n, \tag{2.30}$$

where $a_n, b_n \neq 0$ for all n. It can be written as

$$\psi_n\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad (x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) = \left(y_n, \frac{a_n - b_n y_n}{x_n}\right). \tag{2.31}$$

Let $s_n = 1/x_n$ and $t_n = 1/y_n$. Then, ψ_n has two indeterminate points:

$$(s_n, t_n) = (0,0), \quad Q_n\colon (x_n, y_n) = \left(0, \frac{a_n}{b_n}\right),$$

and ψ_n^{-1} has two indeterminate points:

$$(s_{n+1}, t_{n+1}) = (0,0), \quad P_{n+1}\colon (x_{n+1}, y_{n+1}) = \left(\frac{a_n}{b_n}, 0\right).$$

By blowing up the n-th $\mathbb{P}^1 \times \mathbb{P}^1$ at $(s_n, t_n) = (0,0)$ and Q_n, and the $(n+1)$-st $\mathbb{P}^1 \times \mathbb{P}^1$ at $(s_{n+1}, t_{n+1}) = (0,0)$ and P_{n+1}, we obtain an isomorphism between surfaces as depicted in Figure 7.

The equation (2.31) can be regularized as a family of isomorphisms between surfaces if and only if there exists $\ell \geq 0$ such that

$$\psi_{n+5\ell} \circ \cdots \circ \psi_n(P_n) = Q_{n+5\ell+1}. \tag{2.32}$$

We will see that the case $\ell = 0$ corresponds to the early confinement, $\ell = 1$ to the standard confinement and $\ell \geq 2$ to late confinements.

Let us define the points

$$T_n^{(1)}(\alpha_1) : (s_n, y_n) = (\alpha_1, 0), \qquad T_n^{(2)}(\alpha_2) : (x_n, y_n) = (0, \alpha_2),$$

$$T_n^{(3)}(\alpha_3) : (x_n, t_n) = (\alpha_3, 0), \qquad T_n^{(4)}(\alpha_4) : \left(s_n, -\frac{t_n}{s_n}\right) = (0, \alpha_4),$$

$$T_n^{(5)}(\alpha_5) : (s_n, t_n) = (0, \alpha_5),$$

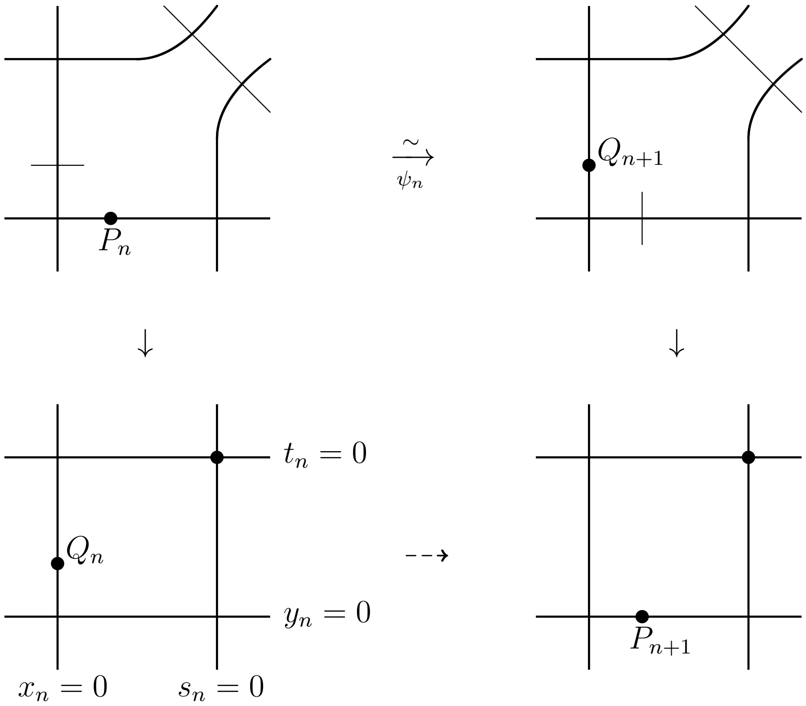


FIGURE 7. Diagram showing the blow-ups of the n-th $\mathbb{P}^1 \times \mathbb{P}^1$ at Q_n and $(s_n, t_n) = (0,0)$ and those of $(n+1)$-st $\mathbb{P}^1 \times \mathbb{P}^1$ at P_{n+1} and $(s_{n+1}, t_{n+1}) = (0,0)$. After these blow-ups, the mapping ψ_n becomes an isomorphism from the n-th surface to the $(n+1)$-st surface.

where $\alpha_1, \ldots, \alpha_5$ are general values. These points are mapped by ψ_n as follows:

$$\psi_n(T_n^{(1)}(\alpha_1)) = T_{n+1}^{(2)}(a_n\alpha_1), \qquad \psi_n(T_n^{(2)}(\alpha_2)) = T_{n+1}^{(3)}(\alpha_2),$$
$$\psi_n(T_n^{(3)}(\alpha_3)) = T_{n+1}^{(4)}\left(\frac{\alpha_3}{b_n}\right), \qquad \psi_n(T_n^{(4)}(\alpha_4)) = T_{n+1}^{(5)}\left(\frac{\alpha_4}{b_n}\right),$$
$$\psi_n(T_n^{(5)}(\alpha_5)) = T_{n+1}^{(1)}(\alpha_5).$$

Since

$$P_n = T_n^{(1)}\left(\frac{b_{n-1}}{a_{n-1}}\right), \quad Q_n = T_{n+1}^{(2)}\left(\frac{a_{n+1}}{b_{n+1}}\right)$$

and

$$T_n^{(1)}\left(\frac{b_{n-1}}{a_{n-1}}\right) \mapsto \cdots \mapsto T_{n+5\ell+1}^{(2)}\left(\frac{a_n b_{n-1}}{a_{n-1}} \prod_{j=1}^{\ell} \frac{a_{n+5j}}{b_{n+5j-2}b_{n+5j-3}}\right), \tag{2.33}$$

the condition (2.32) can be written by a_n and b_n as

$$\frac{a_n b_{n-1}}{a_{n-1}} \prod_{j=1}^{\ell} \frac{a_{n+5j}}{b_{n+5j-2}b_{n+5j-3}} = \frac{a_{n+5\ell+1}}{b_{n+5\ell+1}}. \tag{2.34}$$

If a_n and b_n satisfy this condition, we obtain a (family of) surface(s) as in Figure 8 with Picard number $5\ell + 5$.

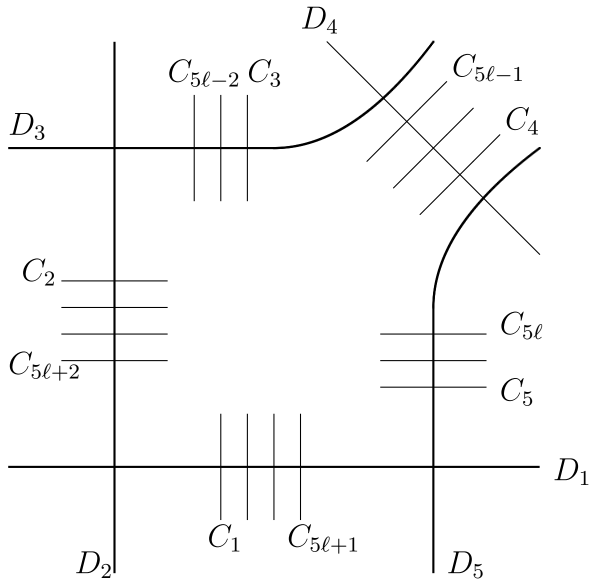


FIGURE 8. Family of surfaces on which the equation ψ_n acts as an isomorphism. The curves $D_1, \ldots, D_5$ are all $-(\ell+1)$-curves and compose the fundamental chain (2.35). The exceptional curves $C_1, \ldots, C_{5\ell+2}$ are obtained by the blow-ups at the points in (2.33).

The curves in Figure 8 move under ψ_n as follows:

$$D_1 \to D_2 \to \cdots \to D_5 \to D_1, \tag{2.35}$$

$$\left\{y = \frac{a}{b}\right\} \to C_1 \to \cdots \to C_{5\ell+2} \to \left\{x = \frac{a}{b}\right\}, \tag{2.36}$$

where we denote by $\{y = a/b\}$ and $\{x = a/b\}$ the strict transforms of the corresponding lines in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively. Since $C_{5\ell+1}$ is linearly equivalent to the divisor

$$-D_1 + D_3 + D_4 + \sum_{j=0}^{\ell-1}(-C_{5j+1} + C_{5j+3} + C_{5j+4}),$$

the matrix of ψ_* with respect to the basis

$$D_1, \ldots, D_5, C_1, \ldots, C_{5\ell}$$

is

$$\left(\begin{array}{c|c} \tau_5 & * \\ \hline 0 & A \end{array}\right),$$

where τ_5 is a permutation matrix corresponding to the motion of $D_1, \dots, D_5$ and where the matrix A is given by

$$A = \begin{pmatrix} 0 & & & & & & & & -1 \\ 1 & 0 & & & & & & & 0 \\ 0 & 1 & 0 & & & & & & 1 \\ & 0 & 1 & 0 & & & & & 1 \\ & & 0 & 1 & & & & & 0 \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \vdots \\ & & & & & & & & -1 \\ & & & & & & & & 0 \\ & & & & & & 0 & & 1 \\ & & & & & & 1 & 0 & 1 \\ & & & & & & 0 & 1 & 0 \end{pmatrix}. \tag{2.37}$$

The characteristic polynomial of A is

$$f_\ell(\lambda) = \lambda^{5\ell} + (-\lambda^3 - \lambda^2 + 1) \sum_{j=0}^{\ell-1} \lambda^{5j}.$$

While all roots of f_0 and f_1 have modulus 1, the polynomial f_ℓ ($\ell \geq 2$) has a real root greater than 1 since $f(1) < 0$. Thus, the equation (2.30) is integrable only for $\ell = 0, 1$.

Let us return to (2.26). Putting $b_n = a_n^k$ to (2.34), we obtain the condition

$$a_{n+5\ell+1}^{k-1} a_n a_{n-1}^{k-1} \prod_{j=1}^{\ell} \frac{a_{n+5j}}{a_{n+5j-2}^k a_{n+5j-3}^k} = 1,$$

and the characteristic polynomial of $\log a_n$

$$g_\ell(\lambda) = (k-1)\lambda^{5\ell+2} + \lambda^3(\lambda^3 - k\lambda - k) \sum_{j=0}^{\ell-1} \lambda^{5j} + \lambda + (k-1).$$

If $\ell = 1$, then these coincide with (2.27) and (2.28), respectively.

A direct calculation shows that

$$g_\ell(\lambda) = ((k-1)(\lambda^2+1) + \lambda) f_\ell(\lambda),$$

which implies that all g_ℓ have the same factor

$$(k-1)(\lambda^2+1) + \lambda. \tag{2.38}$$

This factor corresponds to the condition on a gauge function (2.29).

Therefore, we can find an appropriate gauge function by searching for the factor that appears in the characteristic polynomial of the coefficients in every late confinement.

Using this technique, the full-deautonomisation approach is refined as follows:

(1) We introduce sufficient nonautonomous coefficients to an autonomous equation.
(2) We calculate the characteristic polynomial f of the condition on the parameters such that the deautonomised equation has the same singularity patterns as of the original mapping.
(3) We also calculate the characteristic polynomial f' of the condition on the parameters in the case of the first late confinement.

(4) After disregarding the common factors in f and f', we calculate the root λ of f with maximum radius.
(5) Then, $\log \lambda$ is expected to be the entropy of the original equation.

Remark 2.16. Since a factor that comes from gauge freedom appears in all confinement cases, it also appears in the early confinement case. Thus, if an equation has an early confinement case, we can find the gauge freedom by calculating the early and standard confinement cases.

Remark 2.17. There exist other approaches to find gauge freedom.

One approach is to observe the form of the equation. For example, (2.26) looks strange for great k and thus it would be natural to suspect the existence of gauge freedom. However, this kind of observation sometimes leads us to the wrong conclusion.

Theorem 1.12 is sometimes also useful. For example, the roots of the gauge factor

$$(k-1)(\lambda^2+1)+\lambda$$

are neither reciprocal quadratic integers greater than 1 nor Salem numbers. Thus, we can presume that this factor comes from gauge freedom.

Constructing a space of initial conditions is the last resort. This is always valid and gives us an explicit value of the entropy but requires a lot of calculations.

2.5. **Family of late confinements.** As we have already seen before, the singularity pattern and the condition on the essential coefficients in the case of late confinements exhibit a periodic pattern. Thus, it is not difficult to consider a whole family of late confinements. In this subsection, we consider a family of late confinements for a nonintegrable equations. We will see through an example that a Pisot number is obtained as a limit of dynamical degrees of confining mappings. Several other examples are given in [15].

Example 2.18. Consider the equation

$$x_{n+1}+x_{n-1}=x_n+\frac{b_n}{x_n}+\frac{a_n}{x_n^2}, \tag{2.39}$$

where $a_n \neq 0$ for all n. We have already seen in Example 2.13 that, in the autonomous case, this mapping has the singularity pattern $\{\epsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon\}$, where ε is an infinitesimal quantity. The condition for the above equation to have this singularity pattern is

$$a_{n+3}=a_n, \quad b_{n+3}-2b_{n+2}-2b_{n+1}-b_n=0.$$

The characteristic polynomial of b_n is

$$P_0^0(\lambda)=\lambda^3-2\lambda^2-2\lambda+1.$$

Since (2.39) has two parameters, there are many possibilities of late confinements.

First we consider late confinements due to a_n. The first late confinement has the pattern

$$\{\varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon\}$$

and the confinement constraints

$$\begin{aligned}&a_{n+6}-a_{n+3}+a_n=0,\\&b_{n+6}-2b_{n+5}-2b_{n+4}+b_{n+3}-2b_{n+2}-2b_{n+1}+b_n=0.\end{aligned}$$

In this case, the characteristic polynomial of b_n is

$$P_1^0(\lambda)=\lambda^6-2\lambda^5-2\lambda^4+\lambda^3-2\lambda^2-2\lambda+1,$$

whose greatest root is $2.727167\cdots$. The m-th late confinement has the pattern

$$\{\varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \dots, \varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon\}.$$

In this case, the characteristic polynomial of b_n is

$$P^0_m(\lambda) = \sum_{i=0}^{m} \lambda^{3i}(\lambda^3 - 2\lambda^2 - 2\lambda) + 1$$
$$= \frac{\lambda^{3m+3} - 1}{\lambda^3 - 1}\left(P^0_0(\lambda) - 1\right) + 1.$$

Let λ_m be the greatest root of $P^0_m(\lambda)$ and let λ^* be that of $P^0_0(\lambda) - 1$. Then, the sequence $(\lambda_m)_{m\geq 1}$ is monotonically increasing and converges to λ^*. Thus, this λ^* gives the upper bound of the dynamical degree of all late confinements due to a_n.

Next, we consider late confinements due to b_n. We assume that a_n satisfies $a_{n+3} = a_n$ (for example, $a_n = 1$ for all n is sufficient). The k-th late confinement due to b_n has the pattern

$$\{\varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon, \varepsilon^{-1}, \varepsilon^{-1}, \ldots, \varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon, \varepsilon^{-2}, \varepsilon^{-2}, \varepsilon\}.$$

In this case, the characteristic polynomial of b_n is

$$P^k_0(\mu) = \sum_{i=0}^{k-1} \mu^{6i+4}(\mu^5 - 2\mu^4 - 2\mu^3 + \mu^2 - \mu - 1) + \mu^3 - 2\mu^2 - \mu + 1$$
$$= \frac{\mu^{6k} - 1}{\mu^6 - 1}\left(\mu^5 - 2\mu^4 - 2\mu^3 + \mu^2 - \mu - 1\right) + P^0_0(\mu).$$

Let μ_k be the greatest root of $P^k_0(\mu)$ and let μ^* be that of $\mu^5 - 2\mu^4 - 2\mu^3 + \mu^2 - \mu - 1$. As in the above case, $(\mu_k)_{k\geq 1}$ is monotonically increasing and converges to $\mu^* = 2.678712\cdots$. Since

$$\mu^5 - 2\mu^4 - 2\mu^3 + \mu^2 - \mu - 1 = (\mu - 1)(\mu^4 - 3\mu^3 + \mu^2 - 1)$$

and the second factor on the right hand side is irreducible as an integer-coefficient polynomial, this is the minimal polynomial of μ^*. Since the other roots of this polynomial have modulus less than 1, μ^* is a Pisot number but not a quadratic integer.

It is natural to presume that λ^* in the above example coincides with the dynamical degree of (2.39) for general a_n and that μ^* coincides with the dynamical degree in the case where $a_n = 1$ and b_n is general. However, it is not easy to calculate such dynamical degrees. An alternative approach is illustrated on the following example.

Example 2.19. Consider the equation

$$x_{n+1} + x_{n-1} = x_n + \frac{b_n}{x_n} + \frac{a_n}{x_n^k},$$

where $a_n \neq 0$ and $k \geq 3$ is odd.

The shortest singularity pattern of this equation is

$$\{\varepsilon, \varepsilon^{-k}, \varepsilon^{-k}, \varepsilon\}$$

and the corresponding condition on b_n is

$$b_{n+3} - kb_{n+2} - kb_{n+1} + b_n = 0.$$

Its characteristic polynomial is

$$P^0_0(\lambda) = \lambda^3 - k\lambda^2 - k\lambda + 1.$$

Let us consider a family of late confinements due to a_n. Let λ_m be the greatest root of the characteristic polynomial of b_n in the case of the m-th late confinement and let λ^* be the greatest root of $P^0_0(\lambda) - 1$. As in Example 2.18, $(\lambda_m)_{m\geq 1}$ is monotonically increasing and converges to λ^*.

On the other hand, in the case where $b_n = 0$ and $a_n = 1$ for all n, the equation does not pass the singularity confinement test. According to [25], the exact value of the dynamical degree is

$$\frac{k + \sqrt{k(k+4)}}{2},$$

which coincides with λ^*. It would be natural to naïvely think of this case as the "infinitely late" confinement due to a_n.

3. Space of initial conditions for nonautonomous systems

In this section, we define a space of initial conditions in nonautonomous cases.

As we have already seen in §2, when considering a space of initial conditions, it is most important for an equation to be regularized as a (family of) isomorphism(s) on surfaces. However, since there are a lot of artificial nonautonomous equations, this condition is so weak that we cannot say anything about general properties of such equations.

Let us first consider some several artificial examples. In the following examples, we fix the starting index as $n = 0$, i.e. by $\deg \psi^n$ we denote $\deg(\psi_{n-1} \circ \cdots \circ \psi_0)$ (Remark 1.2).

Example 3.1. Let φ be an arbitrary autonomous equation with unbounded degree growth and a space of initial condition X (for example, the φ in Example 1.5), and let $(d_n)_{n>0}$ be an arbitrary sequence of positive integers. Define sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n>0}$ by

$$p_0 = 0, \quad p_n = \max\{k \in \mathbb{Z}_{\geq 0} \mid \deg \varphi^k \leq d_n\}, \quad q_n = p_n - p_{n-1}.$$

Let

$$\psi_n = \varphi^{q_n} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

for all $n > 0$. Then, we have

$$\deg(\psi_n \circ \cdots \circ \psi_1) = \deg \varphi^{p_n} \approx d_n.$$

Since φ is an automorphism on X, so is ψ_n for all n. Therefore, by choosing $(d_n)_n$ appropriately, we can construct a lot of equations that can be reduced to families of isomorphisms (automorphisms) on surfaces but that have arbitrary degree growth.

Case 1

Let λ be an arbitrary real number greater than 1 and let d_n be the greatest integer not greater than λ^n. In this case, the entropy of the mapping ψ_n is $\log \lambda$.

Case 2

Let λ as in Case 1 and let

$$d_n = \begin{cases} \text{the greatest integer not greater than } \lambda^n & (n\colon \text{even}) \\ 1 & (n\colon \text{odd}). \end{cases}$$

In this case, the entropy of the mapping ψ_n does not exist. If we change the definition of the entropy to

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\deg \psi^n\right),$$

then the entropy exists and is $\log \lambda$.

Case 3

Let $d_n = n$. In this case, the degree of ψ^n grows linearly but the equation can be reduced to a family of automorphisms on X.

Case 4

Let d_n grow faster than any exponential function of n, for example $d_n = n^n$. In this case, the entropy of the mapping ψ_n is $+\infty$.

Example 3.2. The same technique as above is also valid in the case where the original φ does not have a space of initial conditions. Let φ be an autonomous equation with unbounded degree growth but no space of initial conditions (for example a linearizable mapping) and let $d_n = n^2$. Then, we obtain a mapping ψ_n that has a quadratic degree growth but cannot be regularized as a family of isomorphisms on surfaces.

Example 3.3. In the above two examples, the equations are quite artificial and practically impossible to write explicitly. Usually, the term "nonautonomous equation" refers to an equation with several nonautonomous coefficients such as examples in §2. However, even in this class of equations, there are strange mappings.

Consider the equation

$$x_{n+1} = a_n x_n^2 + (1 - a_n) x_n + b x_{n-1},$$

where b is a general constant and a_n is a nonautonomous coefficient. We are interested only in the case $a_n = 0, 1$.

In the case where a_n is always 0, this equation is a linear mapping and thus the degree growth is obviously bounded. On the other hand, in the case where a_n is always 1, this equation is a Hénon map [18] and its algebraic entropy is $\log 2$.

If a_n takes both 0 and 1, then these two cases are mixed. It is obvious that for any real number $\lambda \in [1, 2]$, there exist a sequence $(a_n)_n$ such that the dynamical degree of the above equation is λ.

It is always possible to mix two different equations by using one nonautonomous coefficient. For example, if we start with two autonomous equations that have the same space of initial conditions, then the mixed equation is reduced to a family of automorphisms on a surface but exhibits strange behavior.

What is important is that, even if the obtained surfaces and isomorphisms depend on n, their "fundamental structures" (for example, the intersection pattern of specific curves and the linear action induced on the Picard groups) are the same. When we consider a concrete equation such as in §2, it is (in principle) possible to check whether those structures do or do not depend on n. However, it is difficult to define mathematically what constitutes a fundamental structure for general equations. In this thesis we shall therefore define a space of initial conditions as follows:

Definition 3.4 (space of initial conditions for nonautonomous systems)**.** An equation $\varphi_n \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has a *space of initial conditions* if (after an appropriate coordinate change) the following three conditions are satisfied:

- There exists a composition of blow-ups $\pi_n = \pi_n^{(1)} \circ \cdots \circ \pi_n^{(r)} \colon X_n \to \mathbb{P}^2$ for each n such that the induced birational maps $\varphi_n \colon X_n \dashrightarrow X_{n+1}$ are all isomorphic:

$$\begin{array}{ccccccccc} \longrightarrow & X_{n-1} & \xrightarrow{\sim} & X_n & \xrightarrow{\sim} & X_{n+1} & \longrightarrow \\ & \downarrow{\scriptstyle \pi_{n-1}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} & \\ \dashrightarrow & \mathbb{P}^2 & \overset{\varphi_{n-1}}{\dashrightarrow} & \mathbb{P}^2 & \overset{\varphi_n}{\dashrightarrow} & \mathbb{P}^2 & \dashrightarrow . \end{array}$$

- Let $e_n = (e_n^{(0)}, \ldots, e_n^{(r)})$ be the geometric basis corresponding to π_n (Definition A.26). Then, the matrices of $\varphi_{n*} \colon \operatorname{Pic} X_n \to \operatorname{Pic} X_{n+1}$ with respect to these bases do not depend on n.
- The set of all effective classes in $\operatorname{Pic} X_n$ does not depend on n, i.e. if $\sum_i a^{(i)} e_n^{(i)} \in \operatorname{Pic} X_n$ is effective, then so is $\sum_i a^{(i)} e_k^{(i)} \in \operatorname{Pic} X_k$ for any k.

Note that in nonautonomous cases, a space of initial conditions does not consist of a single surface but of a family of surfaces. It also contains information about the centers and ordering of the blow-ups.

Remark 3.5. As in the autonomous case (Remark 1.7), blow-downs are necessary in general to construct a space of initial conditions. However, to avoid unnecessary complexity, we used the phrase "after an appropriate coordinate change" instead. We will see in Remark 3.21 the rigorous definition including blow-downs.

Usual nonconfining equations such as linearizable mappings and Hénon maps do not satisfy the first condition in Definition 3.4. On the other hand, Example 3.1 does satisfy the first and third conditions but does not satisfy the second. The third condition imposes some condition on the centers and ordering of blow-ups.

Unfortunately, it is not easy in general to check the third condition in Definition 3.4 for a concrete equation. However, if only the first and second conditions are satisfied, we can still calculate the degree growth by Proposition 4.2 since its proof does not need the third condition. One reason why we introduce the third condition is a correspondence to a space of initial conditions in Sakai's sense, which we shall introduce later.

Remark 3.6. Let us first have a closer look at the third condition. Let

$$\mathbb{Z}^{1,r} = \mathbb{Z}e^{(r)} \oplus \cdots \oplus \mathbb{Z}e^{(0)}$$

and define on $\mathbb{Z}^{1,r}$ a symmetric bilinear form $(-,-)$ by

$$(e^{(i)}, e^{(j)}) = \begin{cases} 1 & (i = j = 0) \\ -1 & (i = j \neq 0) \\ 0 & (i \neq j). \end{cases}$$

Let

$$\iota_n \colon \mathbb{Z}^{1,r} \to \operatorname{Pic} X_n, \quad e^{(i)} \mapsto e_n^{(i)}$$

and $\Phi_n = \iota_{n+1}^{-1} \varphi_{n*} \iota_n$:

$$\begin{array}{ccccccc} \longrightarrow & \mathbb{Z}^{1,r} & \xrightarrow{\Phi_{n-1}} & \mathbb{Z}^{1,r} & \xrightarrow{\Phi_n} & \mathbb{Z}^{1,r} & \longrightarrow \\ & \big\downarrow \iota_{n-1} & & \big\downarrow \iota_n & & \big\downarrow \iota_{n+1} & \\ \longrightarrow & \mathrm{PicX_{n-1}} & \xrightarrow{\varphi_{n-1*}} & \mathrm{PicX_n} & \xrightarrow{\varphi_{n*}} & \mathrm{PicX_{n+1}} & \longrightarrow . \end{array}$$

Then, the meaning of the second condition is that Φ_n does not depend on n. We then simply denote Φ_n by Φ.

We will use these notations in §4.

Lemma 3.7. *Let $K = \iota_n^{-1} K_{X_n} = -3e^{(0)} + e^{(1)} + \cdots + e^{(r)}$. Then Φ preserves K and $(-,-)$, i.e.*

$$\Phi K = K, \quad (v, w) = (\Phi v, \Phi w)$$

for all $v, w \in \mathbb{Z}^{1,r}$.

Proof. Immediate from the fact that φ_{n*} preserves the canonical class and the intersection number on the surface. □

Next, we review the notion of a space of initial conditions in Sakai's sense.

Let X be a rational surface, $e = (e^{(0)}, \ldots, e^{(r)}), \widetilde{e} = (\widetilde{e}^{(0)}, \ldots, \widetilde{e}^{(r)})$ geometric bases and let $\pi, \widetilde{\pi} \colon X \to \mathbb{P}^2$ be the corresponding birational morphisms. Then we obtain a birational automorphism $\widetilde{\pi} \circ \pi^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which will become a part of an equation.

Let σ be the $\mathbb{Z}$-linear map on $\operatorname{Pic} X$ defined by

$$e^{(0)} \mapsto \widetilde{e}^{(0)}, \dots, e^{(r)} \mapsto \widetilde{e}^{(r)}.$$

Suppose that $\sigma^n e = (\sigma^n e^{(0)}, \dots, \sigma^n e^{(r)})$ is a geometric basis for each n and let $\pi_n \colon X \to \mathbb{P}^2$ be the corresponding birational morphism. Then, we obtain the equation

$$\varphi_n = \pi_{n+1} \circ \pi_n^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2.$$

Example 3.8. Let us see the simplest example where $\sigma^n e$ is not a geometric basis.

We cover $\mathbb{P}^2$ by three copies of $\mathbb{C}^2$ as follows:

$$\mathbb{P}^2 = (x, y) \cup \left(\frac{x}{y}, \frac{1}{y}\right) \cup \left(\frac{y}{x}, \frac{1}{x}\right).$$

Let $\pi^{(1)}, \pi^{(2)}, \pi^{(3)}$ be the blow-ups at the following points:

(1) $\pi^{(1)}$: at $(x, y) = (0, 0)$,
(2) $\pi^{(2)}$: at $\left(\frac{1}{y}, \frac{x}{y}\right) = (0, 0)$,
(3) $\pi^{(3)}$: at $\left(\frac{1}{x}, \frac{x}{y}\right) = (0, 0)$.

Let X be the surface obtained by the blow-ups $\pi = \pi^{(1)} \circ \pi^{(3)} \circ \pi^{(3)}$ (Figure 9) and let $e = (e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)})$ be the corresponding geometric basis.

It is obvious that

$$\widetilde{e} = (\widetilde{e}^{(0)}, \widetilde{e}^{(1)}, \widetilde{e}^{(2)}, \widetilde{e}^{(3)}) = (e^{(0)}, e^{(2)}, e^{(3)}, e^{(1)})$$

is another geometric basis on $\operatorname{Pic} X$. Let σ be the $\mathbb{Z}$-linear transformation on $\operatorname{Pic} X$ defined by $e^{(i)} \mapsto \widetilde{e}^{(i)}$ for all i. While e and $\sigma e = \widetilde{e}$ are geometric, $\sigma^2 e = (e^{(0)}, e^{(3)}, e^{(1)}, e^{(2)})$ is not since $e^{(2)} - e^{(3)}$ is effective.

It is obvious that all problems in this case come from the ordering of $e^{(i)}$.

As in the above example, $\sigma^n e$ is not always geometric. Therefore, it is necessary to impose some condition on σ.

Definition 3.9 (Cremona isometry [29, 9, 37])**.** Let X be a rational surface and let σ be an invertible $\mathbb{Z}$-linear transformation on $\operatorname{Pic} X$. σ is said to be a *Cremona isometry* if it satisfies the following three conditions:

- σ preserves the intersection number on $\operatorname{Pic} X$, i.e. $F_1 \cdot F_2 = (\sigma F_1) \cdot (\sigma F_2)$ for all $F_1, F_2 \in \operatorname{Pic} X$,
- σ preserves K_X,
- σ preserves the set of effective classes, i.e. if F is effective, then so is σF (and $\sigma^{-1} F$).

Example 3.10. Let φ be an automorphism on a rational surface. Then the induced linear transformations φ^* and φ_* (Definition A.11) are Cremona isometries.

It is clear from the definition that the following holds.

Lemma 3.11. *Cremona isometries preserve the nef cone.*

It should be noted that, while an automorphism on a surface determines the motion of each curve, a Cremona isometry does *not*. It only determines the motion of the classes of curves. However, as shown in the following lemma, if an irreducible curve has a negative self-intersection, then its motion is completely determined.

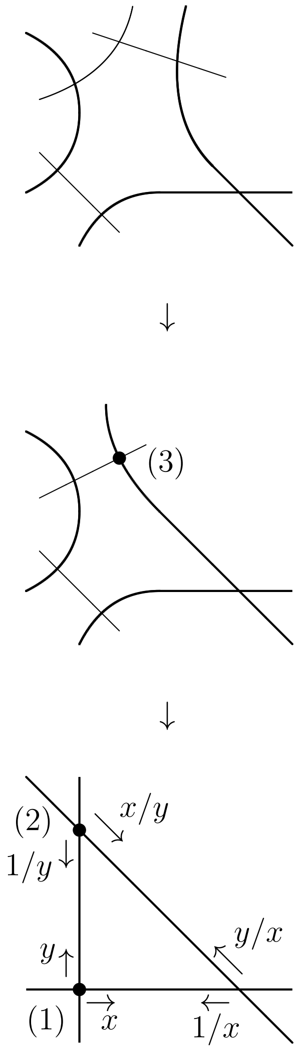


FIGURE 9. Diagram showing the blow-ups needed to obtain X in Example 3.8.

Lemma 3.12. *Let X be a rational surface and σ a Cremona isometry, and let C be an irreducible curve in X with a negative self-intersection. Then there exists only one effective divisor D such that $[D] = \sigma[C]$. Moreover, D is a prime divisor, i.e. an irreducible curve. In particular, σ acts as a permutation on the set of all exceptional curves of first kind.*

Proof. Let

$$\sigma[C] = \left[\sum_{i=1}^{k} m_i C_i\right],$$

where C_i are irreducible curves. Since

$$[C] = \sum_{i=1}^{k} m_i \sigma^{-1}[C_i]$$

and $\sigma^{-1}[C_i]$ are all effective, it follows from Proposition A.29 that $k = 1$ and $m_1 = 1$. □

Lemma 3.13. *Let σ be a Cremona isometry. If $e = (e^{(0)}, \ldots, e^{(r)})$ is a geometric basis on* Pic X*, then so is $\sigma e = (\sigma e^{(0)}, \ldots, \sigma e^{(r)})$.*

Proof. Let $\pi = \pi^{(1)} \circ \cdots \circ \pi^{(r)} \colon X \to \mathbb{P}^2$ be the composition of blow-ups corresponding to e and let $C_1, \ldots, C_r \subset X$ be the irreducible curves contracted by π. Since all these curves have negative self-intersection, by Lemma 3.12, their motions are determined by σ. Let us denote them by $C'_1, \ldots, C'_r$. Since $C_i \cdot C_j = C'_i \cdot C'_j$ for all i, j, it is possible to contract $C'_1, \ldots, C'_r$ in the same order as $C_1, \ldots, C_r$. It is clear that the geometric basis corresponding to this contraction is σe. □

Let us see how to obtain an equation from a Cremona isometry.

Definition 3.14. Let X be a basic rational surface (Definition A.6) and let σ be a Cremona isometry on $\operatorname{Pic} X$ and take an arbitrary geometric basis $e = (e^{(0)}, \ldots, e^{(r)})$. By Lemma 3.13, $\sigma^n e$ is a geometric basis for each n. Let π_n be the corresponding birational morphism to $\mathbb{P}^2$ and let $\varphi_n = \pi_{n+1} \circ \pi_n^{-1}$. Thus we obtain $(\varphi_n)_{n \in \mathbb{Z}}$, a family of birational automorphisms on $\mathbb{P}^2$:

$$\begin{array}{ccccccc} & & & X & & & \\ & & \swarrow^{\pi_{n-1}} & \downarrow{\scriptstyle \pi_n} & \searrow^{\pi_{n+1}} & & \\ \cdots\dashrightarrow & \mathbb{P}^2 & \overset{}{\underset{\varphi_{n-1}}{\dashrightarrow}} & \mathbb{P}^2 & \underset{\varphi_n}{\dashrightarrow} & \mathbb{P}^2 & \dashrightarrow \cdots \end{array}$$

This is the equation defined by X, σ and e, and we call X a space of initial conditions (in Sakai's sense). Since the choice of e only determines the specific coordinates, we sometimes think of (X, σ) as the equation itself.

Note that $(\varphi_n)_{n \in \mathbb{Z}}$ is determined by X, σ and e up to an automorphism of $\mathbb{P}^2$ for each n, i.e. if $(\varphi'_n)_{n \in \mathbb{Z}}$ is another family of birational automorphisms on $\mathbb{P}^2$ defined by X, σ and e, then there exist $f_n \in \mathrm{PGL}(3, \mathbb{C})$ such that $\varphi_n = f_{n+1} \circ \varphi'_n \circ f_n^{-1}$.

Definition 3.15 (generalized Halphen surface [37])**.** A rational surface X is called a *generalized Halphen surface* if it satisfies the following two conditions:

- $-K_X$ is effective.
- All components of $-K_X$ are orthogonal to $-K_X$, i.e. $D_i \cdot (-K_X) = 0$ for any $\sum_i m_i D_i \in |-K_X|$.

Lemma 3.16 (Proposition 2 in [37])**.** *Any generalized Halphen surface is a basic rational surface.*

Definition 3.17 (discrete Painlevé equation [37])**.** Let X be a generalized Halphen surface and σ a Cremona isometry on $\operatorname{Pic} X$ of infinite order. Then, the equation obtained by the above procedure is called a *discrete Painlevé equation*.

Remark 3.18. Note that according to this definition, autonomous mappings such as the QRT mappings are also labeled "discrete Painlevé."

Using generalized Halphen surfaces, Sakai has classified (and, in a sense, defined) all discrete Painlevé equations. Since we do not need such a detailed classification in this thesis, we only give a brief introduction.

${K_X}^{\perp}$, which is preserved under σ, is an affine root lattice of type $E_8^{(1)}$. If $\dim|-K_X| = 0$, then the expression $\sum_i m_i D_i \in |-K_X|$ is unique. Therefore, σ acts on the set $\{D_i\}_i$ as a permutation and preserves the lattice $\operatorname{span}_{\mathbb{Z}} D_i$ and its orthogonal compliment. These two lattices are both affine root sublattices of ${K_X}^{\perp}$ and play an important role in the classification of the discrete Painlevé equations.

Remark 3.19. While Cremona isometries can be defined for any rational surface, we only consider basic rational surfaces such as in Definition 3.14. Although it is possible to consider a family of blow-downs from a nonbasic rational surface to a Hirzebruch surface F_a ($a \geq 2$) instead of $\mathbb{P}^2$, Theorem A.38 implies that the degree growth of such an equation must be bounded. Hence, it is sufficient to consider only basic rational surfaces as long as we are interested in equations with unbounded degree growth.

Now let us clarify a correspondence between two definitions of a space of initial conditions.

Proposition 3.20. *The two definitions of a space of initial conditions, Definition 3.4 and Definition 3.14, are equivalent.*

Proof. First, consider the situation in Definition 3.4. Let $X = X_0$ and σ be the $\mathbb{Z}$-linear transformation on $\operatorname{Pic} X$ defined by

$$\sigma = \iota_0 \Phi^{-1} \iota_0^{-1}.$$

Then, a direct calculation shows that

$$\sigma^{\ell} e_0^{(i)} = \varphi_0^* \cdots \varphi_{\ell-1}^* e_{\ell}^{(i)}, \quad \sigma^{-\ell} e_0^{(i)} = \varphi_{-1*} \cdots \varphi_{-\ell *} e_{-\ell}^{(i)}$$

for all $\ell > 0$.

Let us show that σ is a Cremona isometry. It is clear, by construction, that σ satisfies the first and second conditions on a Cremona isometry. Let $F = \sum_i a^{(i)} e_0^{(i)} \in \operatorname{Pic} X$ be an effective class. Then we have

$$\begin{aligned} \sigma F &= \sum_i a^{(i)} \varphi_0^* e_1^{(i)} \\ &= \varphi_0^* \left(\sum_i a^{(i)} e_1^{(i)} \right). \end{aligned}$$

The third condition in Definition 3.4 implies that $\sum_i a^{(i)} e_1^{(i)}$ is effective. Since φ_0^* preserves the set of effective classes, σF is also effective. We can prove the effectiveness of $\sigma^{-1} F$ in the same way.

Next, consider the situation in Definition 3.14. That is, X is a basic rational surface and σ is a Cremona isometry on $\operatorname{Pic} X$. Take $e = (e^{(0)}, \ldots, e^{(r)})$ as a geometric basis on $\operatorname{Pic} X$ and consider the equation defined by X, σ and e. Let us recover the above situation from these data.

Let $X_n = X$ and $e_n = \sigma^n e$ for all $n \in \mathbb{Z}$. While the X_n themselves are all the same, the bases e_n vary depending on n. Then we have the following diagram:

$$\begin{array}{ccccccc}
 & X_{n-1} & & X_n & & X_{n+1} & \\
 & \| & & \| & & \| & \\
\longrightarrow & X & \xrightarrow{\mathrm{id}} & X & \xrightarrow{\mathrm{id}} & X & \longrightarrow \\
 & \downarrow{\scriptstyle \pi_{n-1}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} & \\
\dashrightarrow & \mathbb{P}^2 & \overset{\varphi_{n-1}}{\dashrightarrow} & \mathbb{P}^2 & \overset{\varphi_n}{\dashrightarrow} & \mathbb{P}^2 & \dashrightarrow .
\end{array}$$

It is important to note that, while the morphisms from X_n to X_{n+1} are all the identity map on $X_n = X_{n+1} = X$, the φ_n are not the identity map on $\mathbb{P}^2$ in general.

Let us check the second condition in Definition 3.4. Let $A_n = \left(a_n^{(i,j)} \right)_{i,j}$ be the matrix representation of φ_{n*} with respect to the bases e_n and e_{n+1}. Since $\varphi_{n*} = \mathrm{id}_{\operatorname{Pic} X}$, A_n are determined by

$$e_n^{(i)} = \sum_j a_n^{(j,i)} e_{n+1}^{(j)}.$$

Operating σ^k and using $\sigma^k e_n^{(i)} = e_{n+k}^{(i)}$, we have

$$e_{n+k}^{(i)} = \sum_j a_n^{(j,i)} e_{n+k+1}^{(j)},$$

which shows that A_n do not depend on n.

Finally we check the third condition in Definition 3.4. Let $F = \sum_i a^{(i)} e_n^{(i)} \in \operatorname{Pic} X$ be an effective class. Since σ^k preserves the effective classes,

$$\sigma^k F = \sum_i a^{(i)} e_{n+k}^{(i)}$$

are effective for all k. Hence the set of effective classes does not depend on n. □

We have seen that the two definitions of a space of initial conditions are equivalent. In this thesis, we will use both definitions depending on the situation.

Remark 3.21. If we consider blow-downs instead of an "appropriate coordinate change" in Definition 3.4, we must assume that all blow-downs do not depend on n. In this case, one possible rigorous definition is as follows:

An equation $(\varphi_n)_n$ has a space of initial conditions if there exist rational surfaces Y_n and X_n, blow-ups $\pi_n = \pi_n^{(1)} \circ \cdots \circ \pi_n^{(r)} \colon Y_n \to \mathbb{P}^2$ and blow-downs $\epsilon_n = \epsilon_n^{(1)} \circ \cdots \circ \epsilon_n^{(r')} \colon Y_n \to X_n$ for each n such that the following four conditions are satisfied:

- φ_n is an isomorphism from X_n to X_{n+1}.
- Let $\widetilde{e}_n = (\widetilde{e}_n^{(0)}, \ldots, \widetilde{e}_n^{(r)})$ be the geometric basis corresponding to π_n and we identify all $\operatorname{Pic} Y_n$ by these bases. Let $E_n^{(k)}$ be the total transform of the exceptional class of $\epsilon_n^{(k)}$. Then, $E_n^{(k)}$ does not depend n.
- Take a basis $e_n = (e_n^{(0)}, \ldots, e_n^{(r-r'+1)})$ of $\operatorname{Pic} X_n$ for each n such that $\epsilon_n^* e_n^{(i)}$ does not depend on n (under the above identification). We identify all $\operatorname{Pic} X_n$ by these bases. Then, φ_{n*} does not depend on n.
- The set of effective classes in $\operatorname{Pic} X_n$ (and in $\operatorname{Pic} Y_n$) does not depend on n (under the above identification).

As in the autonomous case (Remark 1.7), if the equation has unbounded degree growth, then it is possible to reduce the above situation to that in Definition 3.4 by taking new blow-downs $X_n \to \mathbb{P}^2$ (Figure 10). Needless to say, the new blow-downs must be such that the geometric basis on $\operatorname{Pic} X_n$ does not depend on n. As in the autonomous case, the existence of such blow-downs is guaranteed by Theorem A.38. Hence, as long as we are interested only in performing a classification, we may only consider the situation in Definition 3.4.

The reason why this kind of problem arises is that we start from a specific equation $(\varphi_n)_n$, whereas if we start from the situation in Definition 3.14, this kind of problem does not appear.

From now on, we shall assume that a space of initial conditions is obtained only by blow-ups, i.e. we shall simply consider the situation in Definition 3.4 or Definition 3.14.

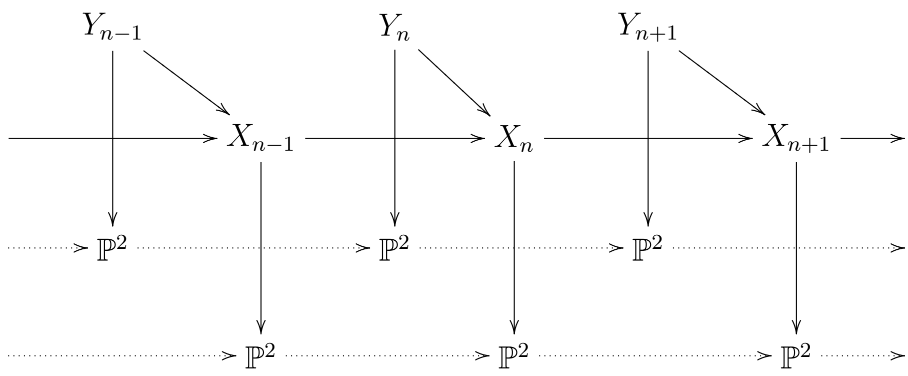


FIGURE 10. Diagram showing a space of initial conditions in the case where we consider blow-downs. The third row represents the original equation and the bottom row represents a new equation obtained by an appropriate coordinate change.

4. Basic properties of an equation with a space of initial conditions

In this section, we first recall Takenawa's result on the degree growth for an equation. Next we shall see that, as in the autonomous case, the degree growth of a nonautonomous equation with a space of initial conditions can be classified into three cases. Finally we show some relations between the degree growth of an equation and the Picard number of a space of initial conditions.

In this section, we consider the situation in Definition 3.4. We will also use the Φ and ι_n defined in Remark 3.6.

Since we will not use the third condition in Definition 3.4 in this section, the results still hold in the case where the third condition is not satisfied.

Lemma 4.1 (Takenawa [39])**.**

$$\deg \varphi^n = (\Phi^n e^{(0)}, e^{(0)}).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Z}^{1,r} & \xrightarrow{\Phi} & \cdots & \xrightarrow{\Phi} & \mathbb{Z}^{1,r} \\
\downarrow{\scriptstyle \iota_\ell} & & & & \downarrow{\scriptstyle \iota_{\ell+n}} \\
\operatorname{Pic} X_\ell & \xrightarrow[\varphi_{\ell *}]{} & \cdots & \xrightarrow[\varphi_{\ell+n-1 *}]{} & \operatorname{Pic} X_{\ell+n} \\
\uparrow{\scriptstyle \pi_\ell^*} & & & & \downarrow{\scriptstyle \pi_{\ell+n *}} \\
\operatorname{Pic}(\mathbb{P}^2) & & & & \operatorname{Pic}(\mathbb{P}^2).
\end{array}$$

Using Proposition A.24, we have

$$\begin{aligned}
\deg \varphi^n &= (\pi_{\ell+n*}(\varphi_{\ell+n-1} \circ \cdots \circ \varphi_\ell)_* \pi_\ell^* \mathcal{O}_{\mathbb{P}^2}(1)) \cdot \mathcal{O}_{\mathbb{P}^2}(1) \\
&= \left((\varphi_{\ell+n-1} \circ \cdots \circ \varphi_\ell)_* e_\ell^{(0)} \right) \cdot e_{\ell+n}^{(0)} \\
&= \left(\iota_{\ell+n-1} \Phi^n e^{(0)} \right) \cdot e_{\ell+n}^{(0)} \\
&= (\Phi^n e^{(0)}, e^{(0)}).
\end{aligned}$$

□

Proposition 4.2. *The Jordan normal form of Φ is one of the following three:*

- $$\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{r+1} \end{pmatrix},$$
  where μ_i are all roots of unity. In particular, there exists $\ell > 0$ such that $\Phi^\ell = \mathrm{id}$ and thus the degree growth of the equation is bounded.
- $$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & \mu_1 & & \\ & & & & \ddots & \\ & & & & & \mu_{r-2} \end{pmatrix},$$
  where μ_i are all roots of unity. In this case, the degree grows quadratically. The dominant eigenvector is isotropic.
- $$\begin{pmatrix} \lambda & & & & \\ & \frac{1}{\lambda} & & & \\ & & \mu_1 & & \\ & & & \ddots & \\ & & & & \mu_{r-1} \end{pmatrix},$$
  where λ is a reciprocal quadratic integer greater than 1 or a Salem number, and $|\mu_i| = 1$. In this case, the entropy of the equation is $\log \lambda > 0$. The two eigenvectors corresponding to λ and $1/\lambda$ are both isotropic.

These three cases correspond to the classes $1, 3$ and 4 in Theorem 1.8, respectively.

Corollary 4.3 (Takenawa [39])**.** *The dynamical degree of an equation is given by the maximum eigenvalue of Φ and the entropy by its logarithm.*

Corollary 4.4. *Theorem 1.12 still holds in nonautonomous cases.*

Remark 4.5. We have already seen in Example 3.2 that Theorem 1.11 does *not* hold in general nonautonomous cases. To extend Theorem 1.11 to nonautonomous cases, it is necessary to apply some conditions on the mapping φ_n itself. However, since there exist too many possible artificial equations in nonautonomous cases, it would be extremely difficult to describe such conditions in all generality.

It is easy to prove Proposition 4.2 if we admit the following two lemmas in linear algebra.

Lemma 4.6. *Let V be an $(r+1)$-dimensional $\mathbb{C}$-vector space with a Hermitian form $(-,-)$ of signature $(1, r)$. If $v \in V$ is isotropic, i.e. $(v, v) = 0$ and $v \neq 0$, then the signature of $(-,-)\big|_{v^\perp}$ is $(0, r-1)$ and its kernel is generated by v. In particular, if v_1, v_2 satisfy $(v_1, v_1) = (v_1, v_2) = (v_2, v_2) = 0$, then v_1 and v_2 are linearly dependent.*

Lemma 4.7. *Let V be an $(r+1)$-dimensional $\mathbb{R}$-vector space with a symmetric bilinear form $(-,-)$ of signature $(1, r)$, and let f be a linear transformation on V which preserves $(-,-)$.*

(1) *The Jordan normal form of f must be one of the following:*

$$\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{r+1} \end{pmatrix} \quad (|\mu_i| = 1). \tag{4.1}$$

$$\begin{pmatrix} \nu & 1 & & & & \\ & \nu & 1 & & & \\ & & \nu & & & \\ & & & \mu_1 & & \\ & & & & \ddots & \\ & & & & & \mu_{r-2} \end{pmatrix} \quad (\nu = \pm 1, |\mu_i| = 1), \tag{4.2}$$

$$\begin{pmatrix} \lambda & & & & \\ & \frac{1}{\lambda} & & & \\ & & \mu_1 & & \\ & & & \ddots & \\ & & & & \mu_{r-1} \end{pmatrix} \quad (\lambda \in \mathbb{R}, |\lambda| > 1, |\mu_i| = 1). \tag{4.3}$$

(2) *Consider the case where the Jordan normal form of f is (4.2). If $(v_1, v_2, v_3, u_1, \ldots, u_{r-2})$ is the corresponding Jordan basis on $V_{\mathbb{C}} = V \otimes \mathbb{C}$, then v_1 is isotropic and*

$$\lim_{n \to +\infty} \frac{1}{\nu^n n^2} f^n w = \frac{(w, v_1)}{2(v_3, v_1)} v_1$$

for any $w \in V_{\mathbb{C}}$.

(3) *Consider the case where the Jordan normal form of f is (4.3). If $(v_1, v_2, u_1, \ldots, u_{r-1})$ is the corresponding Jordan basis, then v_1 and v_2 are both isotropic and*

$$\lim_{n \to +\infty} \frac{1}{\lambda^n} f^n w = \frac{(w, v_2)}{(v_1, v_2)} v_1$$

for any $w \in V_{\mathbb{C}}$.

Although we use Lemma 4.6 throughout the thesis, we omit a proof since it is a well-known fact in linear algebra. The proof of Lemma 4.7 will be given in Appendix B since it is long and only of secondary importance in this thesis.

proof of Proposition 4.2. By Lemma 4.7, the Jordan normal form of Φ is (4.1), (4.2) or (4.3).

Case: (4.1)

It is sufficient to show that every eigenvalue of Φ is a root of unity. Since Φ preserves the lattice $\mathbb{Z}^{1,r}$, its characteristic polynomial has integer coefficients. Since all roots of this polynomial have modulus 1, they are all roots of unity by Kronecker's theorem [27].

Case: (4.2)

It is clear that the degree growth is at most quadratic, and the reason why μ_i are roots of unity is the same as above. Therefore, it is sufficient to show that $\nu = 1$ and that the degree growth is actually quadratic.

Using Lemma 4.1 and Lemma 4.7 (2), we have

$$\begin{aligned}\lim_{n\to+\infty}\frac{\deg\varphi^n}{\nu^n n^2} &= \left(\lim_{n\to+\infty}\frac{1}{\nu^n n^2}\Phi^n e^{(0)}, e^{(0)}\right)\\ &= \left(\frac{(e^{(0)}, v_1)}{2(v_3, v_1)}v_1, e^{(0)}\right)\\ &= \frac{(e^{(0)}, v_1)^2}{2(v_3, v_1)}.\end{aligned}$$

Since v_1 is isotropic, $(e^{(0)}, v_1)$ is not 0 and thus $\deg\varphi^n/\nu^n$ grows quadratically. Since $\deg\varphi^n$ is always positive, we have $\nu = 1$.

Case: (4.3)

Since λ has modulus greater than 1, as in the case of (4.2), we have

$$\lim_{n\to+\infty}\frac{\deg\varphi^n}{\lambda^n} = \frac{(e^{(0)}, v_1)(e^{(0)}, v_2)}{(v_1, v_2)}.$$

We can prove $(e^{(0)}, v_1) \neq 0$ and $\lambda > 1$ in the same way as above. □

The following proposition shows a relation between the Jordan normal form of Φ and the Picard number $\rho(X_n)$.

Proposition 4.8. (1) *If $\rho(X_n) < 10$, then the degree growth of the equation is bounded.*
(2) *If $\rho(X_n) \leq 10$, then the degree growth of the equation is bounded or quadratic.*

Proof. The key to the proof is that Φ preserves $K = 3e^{(0)} - e^{(1)} - \cdots - e^{(r)}$.

(1) Since

$$(K, K) = K_{X_n}^2 = 10 - \rho(X_n) > 0,$$

the bilinear form $(-,-)$ is negative definite on $K^{\perp}$. Since $\Phi\big|_{K^\perp}$ preserves the lattice $K^\perp$ with a negative definite bilinear form, there exists $\ell > 0$ such that $\left(\Phi\big|_{K^\perp}\right)^\ell = \mathrm{id}$.

(2) Let us assume that the dynamical degree λ is greater than 1 and show that $(K, K) < 0$. Let $v \in \mathbb{R}^{1,r}$ be the eigenvector corresponding to λ. Since

$$(v, K) = (\lambda v, K) = \lambda(v, K),$$

we have $K \in v^\perp$. Lemma 4.6 says that $(-,-)\big|_{v^\perp}$ is semi-negative definite and its kernel is generated by v. Since v and K are eigenvectors corresponding to different eigenvalues, we have $K \notin \mathbb{C}v$, and thus $(K, K) < 0$. □

Since all generalized Halphen surfaces have Picard number 10, one immediately obtains the following corollary, which was first shown by Takenawa on a case-by-case basis [39].

Corollary 4.9. *The degree growth of any discrete Painlevé equation is quadratic. In particular, all discrete Painlevé equations are integrable.*

As shown in the following example, the direct converse of Proposition 4.8 does not hold even in the autonomous case.

Example 4.10. Let φ be an automorphism on a rational surface X and let $P \in X$ be a fixed point of φ. Let $\epsilon\colon \widetilde{X} \to X$ be the blow-up at P. Then, φ is lifted to an automorphism on $\widetilde{X}$.

This procedure does not change the algebraic entropy of an equation but increases the Picard number of a space of initial conditions.

When we consider the classification of equations with a space of initial conditions, it is sometimes necessary to perform a minimization. A concrete approach to a minimization was considered by Carstea and Takenawa [7]. They gave an example of a minimization that contracts a curve passing through $\mathbb{C}^2$ (the finite region in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$). However, a general theory in nonautonomous cases was not yet known. In the following section, we will consider a minimization of a space of initial conditions in general cases, in order to classify all nonautonomous integrable equations with unbounded degree growth that possess a space of initial conditions.

5. Minimization of a space of initial conditions

Let us consider a minimization of a space of initial conditions for a nonautonomous mapping. In this section, we consider the situation in Definition 3.14 and think of (X, σ) as an equation itself.

We first recall the process of minimization in autonomous cases.

Definition 5.1. Let φ be an autonomous equation (automorphism) on a rational surface X. Then, φ is minimal on X if there do not exist a rational surface X', an automorphism φ' on X' and a birational morphism $\epsilon\colon X \to X'$ such that $\rho(X) > \rho(X')$ and $\epsilon \circ \varphi = \varphi' \circ \epsilon$:

$$\begin{array}{ccc} X & \xrightarrow[\varphi]{\sim} & X \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ X' & \xrightarrow[\varphi']{\sim} & X'. \end{array}$$

It is known that an automorphism φ on X is minimal if and only if there are no mutually disjoint exceptional curves of first kind in X that are permuted by φ. The proof is almost the same as that of Lemma 5.3.

Hence, what we call a minimization of an autonomous equation is first of all the process of finding such contractible curves and then to actually realize the contraction.

Definition 5.2. Let X be a rational surface and let σ be a Cremona isometry on X. A nonautonomous equation (X, σ) is minimal if there do not exist a rational surface X', a birational morphism $\epsilon\colon X \to X'$ and a Cremona isometry σ' on $\operatorname{Pic} X'$ such that $\rho(X) > \rho(X')$ and $\epsilon_* \sigma = \sigma' \epsilon_*$:

$$\begin{array}{ccc} \operatorname{Pic} X & \xrightarrow[\sigma]{} & \operatorname{Pic} X \\ \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} \\ \operatorname{Pic} X' & \xrightarrow[\sigma']{} & \operatorname{Pic} X'. \end{array}$$

As in autonomous cases, it is possible to verify the minimality with specific curves.

Lemma 5.3. *Let X be a rational surface and σ a Cremona isometry on $\operatorname{Pic} X$. The equation (X, σ) is minimal if and only if there are no mutually disjoint exceptional curves of first kind $C_1, \dots, C_N \subset X$ that are permuted by σ.*

Lemma 5.4. *Let X, X' be rational surfaces and $\epsilon\colon X \to X'$ a birational morphism. If a Cremona isometry σ on $\operatorname{Pic} X$ preserves the sublattice $\epsilon^*(\operatorname{Pic} X') \subset \operatorname{Pic} X$, then $\epsilon_* \sigma \epsilon^*$ is also a Cremona isometry on $\operatorname{Pic} X'$.*

Proof. Let $\epsilon = \epsilon^{(1)} \circ \cdots \circ \epsilon^{(L)}$ be a decomposition into blow-ups and let $E^{(i)}$ be the total transform of the class of the exceptional curve of $\epsilon^{(i)}$ for $i = 1, \dots, L$.

Let $F, F' \in \operatorname{Pic} X'$. Using $\sigma\epsilon^* F, \sigma\epsilon^* F' \in \epsilon^*(\operatorname{Pic} X')$, we have

$$\begin{aligned}(\epsilon_*\sigma\epsilon^* F) \cdot (\epsilon_*\sigma\epsilon^* F') &= (\sigma\epsilon^* F) \cdot (\sigma\epsilon^* F') \\ &= F \cdot F'.\end{aligned}$$

By Proposition A.17, we have

$$K_X = \epsilon^* K_{X'} + E^{(1)} + \cdots E^{(L)}.$$

Since $\epsilon_* E^{(i)} = 0$ and the $E^{(i)}$ are permuted by σ, we have

$$\begin{aligned}\epsilon_*\sigma\epsilon^* K_{X'} &= \epsilon_*\sigma(K_X - E^{(1)} - \cdots E^{(L)}) \\ &= \epsilon_*(K_X - E^{(1)} - \cdots E^{(L)}) \\ &= K_{X'}.\end{aligned}$$

The third condition in Definition 3.9 is trivial since $\epsilon^*, \sigma, \epsilon_*$ all preserve the effective class. □

Proof of Lemma 5.3. First let $C_1, \ldots, C_N$ be irreducible curves of first kind that are permuted by σ. It follows from Castelnuovo's contraction theorem that there exist a rational surface X' and a birational morphism $\epsilon\colon X \to X'$ such that ϵ contracts $C_1, \ldots, C_N$ and is isomorphic outside $C_1 \cup \cdots \cup C_N$. Let $\sigma' = \epsilon_*\sigma\epsilon^*$. By Lemma 5.4, σ' is a Cremona isometry on $\operatorname{Pic} X'$ and thus we obtain an equation (X', σ'). It is clear, by construction, that ϵ, X', σ' satisfy the conditions in Definition 5.2.

Next we show the converse. Let ϵ, X', σ' satisfy those conditions and take an exceptional curve of first kind C that is contracted by ϵ. Since

$$\epsilon_*\sigma^\ell[C] = \sigma'^\ell\epsilon_*[C] = 0,$$

$\sigma^\ell C$ is contracted by ϵ for all ℓ. However, ϵ contracts only a finite number of curves. Thus, there exists $N > 0$ such that $\sigma^N C = C$. Hence σ acts as a permutation on $\{C, \sigma C, \ldots, \sigma^{N-1} C\}$. Since these curves are exceptional curves of first kind and are contracted by σ, they are mutually disjoint. □

As in the autonomous case, one must first verify if there are such contractible curves. If so, then we obtain an equation (X', σ') by contracting these curves. It is clear that the degree growth of (X, σ) is the same as that of (X', σ'). Replacing (X, σ) with (X', σ') and repeating this procedure, we finally obtain a surface on which the equation is minimal.

As shown in the following example, a minimization is not unique in general.

Example 5.5. Let X be the surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at (∞, ∞), and let $\varphi(x, y) = (y, x)$. It is clear that φ is an automorphism on X.

X has three exceptional curves of first kind: $C, \{x = \infty\}$ and $\{y = \infty\}$ (Figure 11). This mapping has two minimizations.

The first possibility is $\mathbb{P}^1 \times \mathbb{P}^1$. Since C is fixed by φ, we can minimize φ from X to $\mathbb{P}^1 \times \mathbb{P}^1$, and it is trivial that φ is an automorphism on $\mathbb{P}^1 \times \mathbb{P}^1$.

The second possibility is $\mathbb{P}^2$. Since two curves $\{y = \infty\}$ and $\{x = \infty\}$ are permuted by φ, we can minimize φ from X to $\mathbb{P}^2$ by contracting these curves.

We will show in Proposition 5.12 (integrable case) and Proposition 5.18 (nonintegrable case) that if the degree growth is unbounded, i.e. σ is of infinite order, then the minimization is unique.

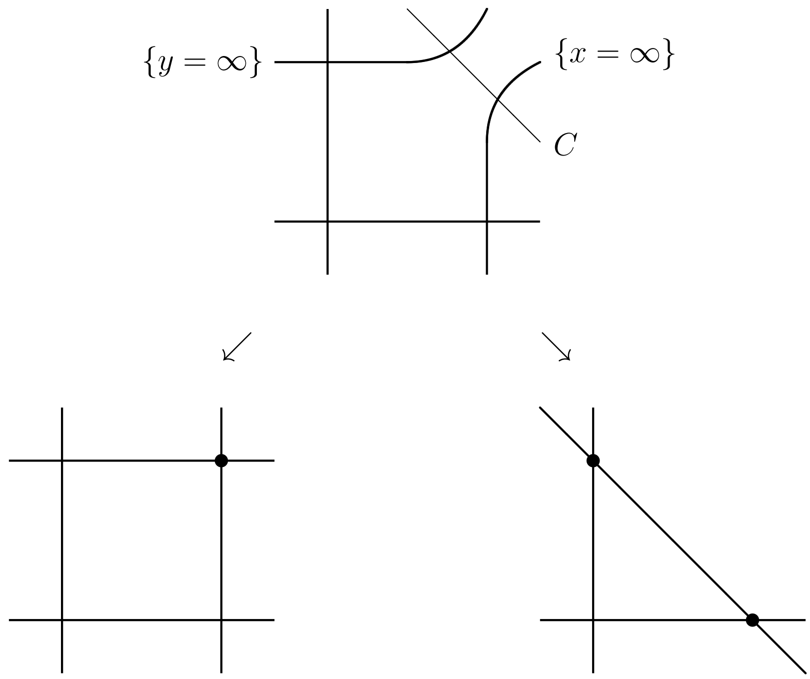


FIGURE 11. The mapping in Example 5.5 permutes two axes x and y. If we consider this mapping on the upper surface, it has two minimizations.

5.1. **Integrable case.** In this subsection, we consider a minimization in the case of integrable equations. The following is our main theorem in this thesis:

Theorem 5.6. *Consider an equation (X, σ) and assume that its degree growth is quadratic. Then, this equation can be minimized to a generalized Halphen surface.*

In particular, if a mapping of the plane with unbounded degree growth and zero algebraic entropy has a space of initial conditions, then it must be one of the discrete Painlevé equations.

Note that in this thesis, "discrete Painlevé equation" should be understood in Sakai's sense, as defined in Definition 3.17.

Lemma 5.7. *Let X be a rational surface with $\rho(X) = 10$. Then X is a generalized Halphen surface if and only if $-K_X$ is nef.*

Proof. Suppose X be a generalized Halphen surface. Let $C \subset X$ be an irreducible curve. If C is a component of $-K_X$, then $-K_X \cdot C = 0$ by definition. On the other hand, if C is not a component of $-K_X$, then $-K_X \cdot C \geq 0$ since $-K_X$ is effective. In both cases we have $-K_X \cdot C \geq 0$ and thus $-K_X$ is nef.

Let us prove the converse. Suppose $-K_X$ is nef. Since $\rho(X) = 10$, we have $(-K_X)^2 = 0$ and $-K_X$ is effective ([37], Proposition 2). Thus it is sufficient to show that every component of $-K_X$ is orthogonal to $-K_X$. Let $\sum_i a_i D_i \in |-K_X|$. Since $-K_X$ is nef, we have $a_i D_i \cdot (-K_X) \geq 0$. Summing them we obtain

$$\sum_i a_i D_i \cdot (-K_X) \geq 0.$$

Since the left hand side is equal to $(-K_X)^2$, $D_i \cdot (-K_X)$ must be 0 for all i. Hence X is a generalized Halphen surface. $\square$

Lemma 5.8. *Let X be a basic rational surface and let σ be a Cremona isometry on* $\operatorname{Pic} X$ *with quadratic growth. Let $v_1, v_2, v_3 \in \operatorname{Pic}_{\mathbb{Q}} X \setminus \{0\}$ satisfy*

$$\begin{aligned} \sigma v_1 &= v_1, \\ \sigma v_2 &= v_2 + v_1, \\ \sigma v_3 &= v_3 + v_2. \end{aligned}$$

Then, we have

- *v_1 is isotropic,*
- *either v_1 or $-v_1$ is nef,*
- *$v_1 \cdot K_X = 0$.*

Proof. That v_1 is isotropic follows immediately from Proposition 4.2 .

Let $e = (e^{(0)}, \dots, e^{(r)})$ be a geometric basis. Then, by Proposition 4.2 and Lemma 4.7, there exists $a \in \mathbb{Q}^\times$ such that

$$v_1 = a \lim_{n \to +\infty} \frac{1}{n^2} \sigma^n e^{(0)}.$$

Since $e^{(0)}$ is nef and σ preserves the nef cone (Lemma 3.11), $\frac{1}{n^2}\sigma^n e^{(0)}$ is nef for all n. Therefore, Proposition A.37 implies that $\frac{1}{a} v_1$ is nef.

Since

$$\begin{aligned} v_2 \cdot K_X &= (\sigma v_2) \cdot (\sigma K_X) \\ &= (v_2 + v_1) \cdot K_X \\ &= v_2 \cdot K_X + v_1 \cdot K_X, \end{aligned}$$

we have $v_1 \cdot K_X = 0$. □

Note that while v_2, v_3 above are not unique, v_1 is unique up to scaling. v_1 is determined by

$$\mathbb{Q} v_1 = \operatorname{Ker}(\sigma_{\mathbb{Q}} - \mathrm{id}) \cap \operatorname{Im}(\sigma_{\mathbb{Q}} - \mathrm{id})^2,$$

where $\sigma_{\mathbb{Q}}$ is the $\mathbb{Q}$-extension of σ to $\operatorname{Pic}_{\mathbb{Q}} X$.

Definition 5.9. Let us normalize v_1 so that

- v_1 is nef,
- $v_1 \in \operatorname{Pic} X$,
- v_1 is primitive in $\operatorname{Pic} X$, i.e. if a rational number a satisfies $a v_1 \in \operatorname{Pic} X$, then a is an integer.

We shall call this v_1 the *normalized dominant eigenvector* of σ.

Lemma 5.10. *Let X be a rational surface of Picard number* 10*. If X has a Cremona isometry which grows quadratically, then X must be a generalized Halphen surface and $-K_X$ coincides with the normalized dominant eigenvector.*

Proof. Let σ be a Cremona isometry on $\operatorname{Pic} X$ that grows quadratically and let v_1 be the normalized dominant eigenvector of σ. By Lemma 5.8, v_1 is isotropic and $v_1 \cdot K_X = 0$. However, K_X is also isotropic since $\rho(X) = 10$. Therefore, by Lemma 4.6, v_1 and K_X are linearly dependent. Since v_1 and K_X are both primitive in $\operatorname{Pic} X$, we have $v_1 = \pm K_X$. While v_1 is nef by Lemma 5.8, K_X cannot be nef since X is rational. Thus we have $v_1 = -K_X$ and Lemma 5.7 implies that X is a generalized Halphen surface. □

The following lemma is the key to the proof of Theorem 5.6

Lemma 5.11. *Let X be a rational surface with $\rho(X) > 10$ and let σ be a Cremona isometry on* $\operatorname{Pic} X$ *with quadratic growth. Then, the equation (X, σ) is not minimal.*

Proof. We will try to find mutually disjoint exceptional curves of first kind that are permuted by σ (Lemma 5.3).

Step 1

Let $v_1 \in \operatorname{Pic} X$ be the normalized dominant eigenvector of σ. We show that $v_1 + K_X$ is effective and nonzero.

By the Riemann-Roch inequality, we have

$$h^0(v_1 + K_X) + h^2(v_1 + K_X) \geq 1 + \frac{1}{2}(v_1 + K_X) \cdot v_1 = 1.$$

Using Serre duality we have

$$h^2(v_1 + K_X) = h^0(-v_1) = 0.$$

Hence, $h^0(v_1+K_X) \geq 1$ and v_1+K_X is effective. It immediately follows from $(v_1+K_X)^2 = 10 - \rho(X) < 0$ that $v_1 + K_X \neq 0$.

Step 2

Let

$$\mathcal{C} = \{C \subset X \colon \text{irreducible} \,|\, C \cdot (v_1 + K_X) < 0\}.$$

We show that $\mathcal{C}$ is a nonempty finite set.

By Step 1, we can express $v_1 + K_X$ as

$$v_1 + K_X = \left[\sum_{i=1}^{\ell} a_i C_i\right],$$

where the C_i are irreducible and $a_i > 0$. Since $(v_1 + K_X)^2 < 0$, at least one of $C_1, \ldots, C_\ell$ satisfy $C_i \cdot (v_1 + K_X) < 0$. Thus $\mathcal{C}$ is not empty.

On the other hand, if an irreducible curve C is different from $C_1, \ldots, C_\ell$, then it satisfies $C \cdot (v_1 + K_X) \geq 0$. Hence $\mathcal{C}$ is finite.

Step 3

We show that if $C \in \mathcal{C}$, then

$$C^2 = -1, \quad C \cdot K_X = -1, \quad C \cdot v_1 = 0, \quad C \cong \mathbb{P}^1.$$

By the genus formula, we have

$$\begin{aligned} g_a(C) &= 1 + \frac{1}{2} C \cdot (C + K_X) \\ &= 1 + \frac{1}{2} C^2 + \frac{1}{2} C \cdot (v_1 + K_X) - \frac{1}{2} C \cdot v_1. \end{aligned}$$

Since $g_a(C) \geq 0$, $C^2 < 0$, $C \cdot (v_1 + K_X) < 0$ and $C \cdot v_1 \geq 0$, the only possible case is

$$g_a(C) = 0, \quad C^2 = -1, \quad C \cdot (v_1 + K_X) = -1, \quad C \cdot v_1 = 0.$$

It follows from Proposition A.34 that $C \cong \mathbb{P}^1$.

Step 4

Since σ is a Cremona isometry, Lemma 3.12 implies that σ acts on $\mathcal{C}$ as a permutation.

Step 5

Let $C, C' \in \mathcal{C}$ satisfy $C \neq C$. We show that $C \cap C' = \emptyset$.

Let $m = C \cdot C'$. Since $(C + C') \cdot v_1 = 0$, Lemma 4.6 implies that

$$0 \geq (C + C')^2 = 2m - 2$$

and therefore $m = 0$ or $m = 1$. Assume that $m = 1$. In this case, v_1 and $C + C'$ are orthogonal and both isotropic. Thus, again by Lemma 4.6, there exists $a \in \mathbb{Q}^\times$ such that $[C + C'] = av_1$. Since v_1 and $[C + C']$ are both primitive and effective, we have $a = 1$. On the other hand, since C and C' are two different components of $v_1 + K_X$, there exists an effective class F such that $[C] + [C'] + F = v_1 + K_X$. Thus we have $F = K_X$, which is a contradiction since K_X cannot be effective when X is rational. Hence we have $C \cdot C' = 0$. □

proof of Theorem 5.6. Let X be a rational surface and let σ be a Cremona isometry on $\operatorname{Pic} X$ with quadratic growth. We show that one can minimize σ from X to a generalized Halphen surface.

It follows from Proposition 4.8 that $\rho(X) \geq 10$. If $\rho(X) = 10$, then Lemma 5.10 implies that X is a generalized Halphen surface, and thus (X, σ) is a discrete Painlevé equation.

Consider the case $\rho(X) > 10$. By Lemma 5.11, the equation (X, σ) is not minimal. Let $\epsilon \colon X \to X'$ be a minimization and let $\sigma' = \epsilon_* \sigma \epsilon^*$. The minimality of (X', σ') implies that $\rho(X') \leq 10$. However, it follows from Proposition 4.8 that $\rho(X') \geq 10$ since the degree grows quadratically. Thus Lemma 5.10 implies that X is a generalized Halphen surface and hence the equation (X', σ') is a discrete Painlevé equation. □

Although the proofs of Lemma 5.11 and Theorem 5.6 define a program to minimize (X, σ), it could be a little difficult to describe the $\mathcal{C}$ in Step 2 of the proof of Lemma 5.11 explicitly. The following proposition tells us how to find a minimization only by linear algebra and, at the same time, shows the uniqueness of the minimization.

Proposition 5.12. *Let X be a rational surface with $\rho(X) = r+1 > 10$ and σ a Cremona isometry on $\operatorname{Pic} X$ that grows quadratically. Let $\epsilon \colon X \to X'$ be a minimization of (X, σ). Decompose ϵ into a composition of blow-ups*

$$\epsilon = \epsilon^{(1)} \circ \cdots \circ \epsilon^{(r-9)}$$

and let $E^{(i)} \in \operatorname{Pic} X$ be the total transform of the exceptional class of $\epsilon^{(i)}$ for $i = 1, \ldots, r-9$. Let $v_1 \in \operatorname{Pic} X$ be the normalized dominant eigenvector and $e = (e^{(0)}, \ldots, e^{(r)})$ an arbitrary geometric basis on $\operatorname{Pic} X$. Then the set $\{E^{(1)}, \ldots E^{(r-9)}\}$ can be written as

$$\mathcal{E} = \{E \in \operatorname{Pic} X \mid E^2 = -1, E \cdot v_1 = 0, E \cdot K_X = -1, E \cdot e^{(0)} \geq 0, (v_1 - E) \cdot e^{(0)} \geq 3\}. \tag{5.1}$$

In particular, a minimization $\epsilon \colon X \to X'$ is unique.

Proof. Step 1

We show that $E^{(i)} \in \mathcal{E}$ for $i = 1, \ldots, r-9$. It is sufficient to show that $(v_1 - E^{(i)}) \cdot e^{(0)} \geq 3$ since the other conditions are trivial. Since $e^{(0)}$ is nef and

$$v_1 + K_X - E^{(i)} = E^{(1)} + \cdots E^{(i-1)} + E^{(i+1)} + \cdots E^{(r-9)}$$

is effective, we have

$$\begin{aligned} 0 &\leq (v_1 + K_X - E^{(i)}) \cdot e^{(0)} \\ &= -3 + (v_1 - E^{(i)}) \cdot e^{(0)}. \end{aligned}$$

Step 2

Let $E \in \mathcal{E}$. We show E and $K_X + v_1 - E$ are both effective.

By the Riemann-Roch inequality, we have

$$h^0(E) + h^2(E) \geq 1 + E \cdot (E - K_X) = 1.$$

Using Serre duality, we have

$$h^2(E) = h^0(K_X - E).$$

Since

$$(K_X - E) \cdot e^{(0)} \le -3,$$

$K_X - E$ is not effective and thus $h^0(K_X - E) = 0$. Therefore we have $h^0(E) > 0$.

By the Riemann-Roch inequality and Serre duality, we have

$$\begin{aligned} h^0(K_X + v_1 - E) &\ge 1 + (K_X + v_1 - E) \cdot (v_1 - E) - h^2(K_X + v_1 - E) \\ &= 1 - h^0(-v_1 + E). \end{aligned}$$

It follows from $(-v_1+E)\cdot e^{(0)} < 0$ that $h^0(-v_1+E) = 0$. Thus we have $h^0(K_X+v_1-E) > 0$.

Step 3

We show that if $E, E' \in \mathcal{E}$ and $E \neq E'$, then $E \cdot E' = 0$. It is important to note that $E, E' \in v_1^\perp$ and that the intersection is semi-negative definite on $v_1^\perp$ and its kernel is generated by v_1.

Let $m = E \cdot E'$. Since

$$0 \ge (E \pm E')^2 = -2 \pm 2m,$$

we have $m = 0, \pm 1$. We can exclude the cases $m = \pm 1$ as follows.

Assume that $m = 1$. In this case, $E + E'$ is isotropic and thus there exists α such that $E + E' = \alpha v_1$. However, this leads to the contradiction:

$$0 = \alpha v_1 \cdot K_X = (E + E') \cdot K_X = -2.$$

Assume that $m = -1$. As in the case of $m = 1$, there exists α such that $E - E' = \alpha v_1$. Since v_1 is primitive, α is a nonzero integer. We may assume $\alpha > 0$. Thus we have

$$E' + (K_X + v_1 - E) = (1 - \alpha) v_1 + K_X.$$

However, while the left hand side is effective, the right hand side is not. Hence we concluded that $E \cdot E' = 0$.

Step 4

We show that $\mathcal{E} \subset \{E^{(1)}, \dots, E^{(9-r)}\}$.

Assume that there exists $E \in \mathcal{E} \setminus \{E^{(1)}, \dots, E^{(9-r)}\}$. It follows from Steps 1 and 3 that $E \cdot E^{(i)} = 0$ for $i = 1, \dots, E^{(r-9)}$. However, this leads to the contradiction:

$$-1 = E \cdot K_X = E \cdot (-v_1 - E^{(1)} - \cdots - E^{(r-9)}) = 0.$$

Step 5

The uniqueness of a minimization follows from the fact that the set $\mathcal{E}$ does not depend on ϵ. □

The normalized dominant eigenvector v_1 is determined by

$$\mathbb{Z} v_1 = \operatorname{Ker}(\sigma_{\mathbb{Q}} - \mathrm{id}) \cap \operatorname{Im}(\sigma_{\mathbb{Q}} - \mathrm{id})^2 \cap \operatorname{Pic} X \quad \text{and} \quad v_1 \cdot e^{(0)} > 0.$$

Thus, in principle we can calculate v_1 and therefore $\mathcal{E}$ explicitly. Hence, this proposition allows us to obtain (X', σ') from (X, σ) only by linear algebra.

Example 5.13. Let k be an integer greater than 1 and consider the equation

$$\varphi_n \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) = \left(y_n, -x_n + \frac{b_n}{y_n} + \frac{a_n}{y_n^k} \right), \tag{5.2}$$

where (x_n, y_n) are inhomogeneous coordinates on $\mathbb{P}^2$ and a_n, b_n satisfy

$$a_{n+2} = (-1)^k a_n, \quad a_n \neq 0, \quad b_{n+2} - k b_{n+1} + b_n = 0$$

for all n. $\mathbb{P}^2$ is covered by three copies of $\mathbb{C}^2$ as follows:

$$\mathbb{P}^2 = (x_n, y_n) \cup \left(\frac{x_n}{y_n}, \frac{1}{y_n}\right) \cup \left(\frac{y_n}{x_n}, \frac{1}{x_n}\right).$$

Note that these three copies of $\mathbb{C}^2$ coincide with $\{z_3 \neq 0\}, \{z_2 \neq 0\}, \{z_1 \neq 0\}$, respectively, where $(z_1 : z_2 : z_3)$ are homogeneous coordinates on $\mathbb{P}^2$. We construct a space of initial conditions by blow-ups.

As a birational automorphism on $\mathbb{P}^2$, the mapping φ_n has only one indeterminate point:

$$T'_n \colon \left(\frac{y_n}{x_n}, \frac{1}{x_n}\right) = (0, 0).$$

Eliminating this indeterminacy requires $2k + 1$ blow-ups at the following points:

- T'_n,
- $R_n^{(\ell)} \colon \left(\frac{1}{x_n y_n^\ell}, y_n\right) = (0, 0) \quad (\ell = 0, 1, \ldots, k-1)$,
- $S_n^{(\ell)} \colon \left(\frac{1}{y_n^\ell}\left(\frac{1}{x_n y_n^k} - \frac{1}{a_n}\right), y_n\right) = (0, 0) \quad (\ell = 0, 1, 2, \ldots, k-2)$,
- $S_n^{(k-1)} \colon \left(\frac{1}{y_n^{k-1}}\left(\frac{1}{x_n y_n^k} - \frac{1}{a_n}\right), y_n\right) = \left(-\frac{b_n}{a_n^2}, 0\right)$.

On the other hand, as a birational automorphism on $\mathbb{P}^2$, the inversed mapping φ_{n-1}^{-1} also has only one indeterminate point:

$$T_n \colon \left(\frac{x_n}{y_n}, \frac{1}{y_n}\right) = (0, 0).$$

Eliminating this indeterminacy requires $2k + 1$ blow-ups at the following points:

- T_n,
- $P_n^{(\ell)} \colon \left(x_n, \frac{1}{x_n^\ell y_n}\right) = (0, 0) \quad (\ell = 0, 1, \ldots, k-1)$,
- $Q_n^{(\ell)} \colon \left(x_n, \frac{1}{x_n^\ell}\left(\frac{1}{x_n^k y_n} - \frac{1}{a_{n-1}}\right)\right) = (0, 0) \quad (\ell = 0, 1, 2, \ldots, k-2)$,
- $Q_n^{(k-1)} \colon \left(x_n, \frac{1}{x_n^\ell}\left(\frac{1}{x_n^k y_n} - \frac{1}{a_{n-1}}\right)\right) = \left(0, -\frac{b_{n-1}}{a_{n-1}^2}\right)$.

A straightforward calculation shows that

$$\varphi_n(T_n) = T'_n, \quad \varphi_n(P_n^{(\ell)}) = R_{n+1}^{(\ell)}, \quad \varphi_n(Q_n^{(\ell)}) = S_{n+1}^{(\ell)} \tag{5.3}$$

for $\ell = 0, \ldots, k-1$. Note that the condition $a_{n+1} = (-1)^k a_{n-1}$ is equivalent to $\varphi_n(Q_n^{(0)}) = S_{n+1}^{(0)}$ and, under this condition, $b_{n+1} - kb_n + b_{n-1} = 0$ is equivalent to $\varphi_n(Q_n^{(k-1)}) = S_{n+1}^{(k-1)}$.

Let X_n be the surface obtained by blowing up $\mathbb{P}^2$ at these $4k + 2$ points in the above order. That is, the part $(e_n^{(1)}, \ldots, e_n^{(4k+2)})$ of its geometric basis corresponds to

$$\left(T', R_n^{(0)}, \ldots, R_n^{(k-1)}, S_n^{(0)}, \ldots, S_n^{(k-1)}, T, P_n^{(0)}, \ldots, P_n^{(k-1)}, Q_n^{(0)}, \ldots, Q_n^{(k-1)}\right).$$

It follows from (5.3) that φ_n is an isomorphism from X_n to X_{n+1}.

Let us label specific curves in X_n as follows (Figure 12):

- E_n: the strict transform of the line at infinity in $\mathbb{P}^2$.
- $D_n^{(0)}, \overline{D}_n^{(0)}$: the strict transforms of the exceptional curves of the blow-ups at T_n and T'_n, respectively.

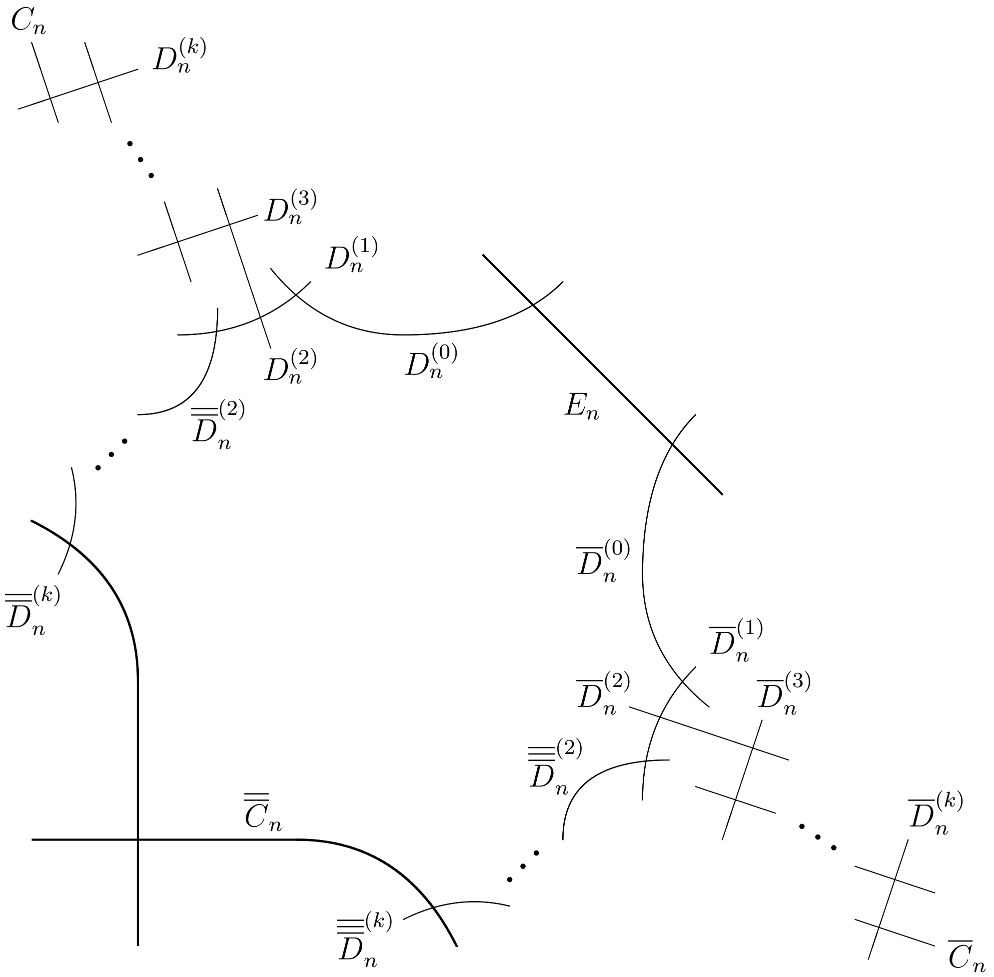


FIGURE 12. Diagram showing a space of initial conditions and specific curves. The thick lines represent the strict transforms of the three fundamental lines in $\mathbb{P}^2$.

- $D_n^{(1)}, \overline{D}_n^{(1)}$: the strict transforms of the exceptional curves of the blow-ups at $P_n^{(k-1)}$ and $R_n^{(k-1)}$, respectively.
- $D_n^{(\ell)}, \overline{D}_n^{(\ell)}, \overline{\overline{D}}_n^{(\ell)}, \overline{\overline{\overline{D}}}_n^{(\ell)}$: the strict transforms of the exceptional curves of the blow-ups at $Q_n^{(\ell-2)}, S_n^{(\ell-2)}, P_n^{(k-\ell)}, R_n^{(k-\ell)}$, respectively. ($\ell = 2, \dots, k$.)
- C_n: the exceptional curve of the blow-up at $Q_n^{(k-1)}$.
- $\overline{C}_n$: the exceptional curve of the blow-up at $S_n^{(k-1)}$.
- $\overline{\overline{C}}_n$: the strict transform of the curve $\{x_n = 0\}$ in $\mathbb{P}^2$.

Instead of introducing ι_n and Φ of Remark 3.6, we identify all $\operatorname{Pic} X_n$ by using the bases $e_n = (e^{(0)}, \dots, e^{(4k+2)})$ for all n. Clearly, the classes represented by the above curves do not depend on n and we shall omit the index n and simply denote $E, \overline{D}^{(\ell)}, \varphi_*$ and so on.

The above curves move under the equation as follows:

$$
\begin{aligned}
&E \to E \quad \text{(fixed)},\\
&D^{(0)} \to \overline{D}^{(0)} \to D^{(0)},\\
&D^{(1)} \to \overline{D}^{(1)} \to D^{(1)},\\
&D^{(\ell)} \to \overline{D}^{(\ell)} \to \overline{\overline{D}}^{(\ell)} \to \overline{\overline{\overline{D}}}^{(\ell)} \to D^{(\ell)} \quad (\ell = 2, \dots, k),\\
&C \to \overline{C} \to \overline{\overline{C}}.
\end{aligned}
$$

Since $\overline{\overline{C}}$ is linearly equivalent to the divisor

$$
E + D^{(0)} + kD^{(1)} - \overline{D}^{(1)} + \sum_{\ell=2}^{k} (kD^{(\ell)} - \overline{D}^{(\ell)} + (k+1-\ell)\overline{\overline{D}}^{(\ell)} - \overline{\overline{\overline{D}}}^{(\ell)}) + kC - \overline{C},
$$

the matrix of φ_* with respect to the basis

$$
E, D^{(0)}, \overline{D}^{(0)}, D^{(1)}, \overline{D}^{(1)}, D^{(2)}, \overline{D}^{(2)}, \overline{\overline{D}}^{(2)}, \overline{\overline{\overline{D}}}^{(2)}, \dots, D^{(k)}, \overline{D}^{(k)}, \overline{\overline{D}}^{(k)}, \overline{\overline{\overline{D}}}^{(k)}, C, \overline{C}
$$

is given by

$$
A = \begin{pmatrix}
1 & & & & & & \\
& \tau_2 & & & & & \\
& & \tau_2 & & & & \\
& & & \tau_4 & & & \\
& & & & \ddots & & \\
& & & & & \tau_4 & \\
& & & & & & B
\end{pmatrix},
$$

where the blank entries are all 0 except in the $(4k+3)$-rd column and where

$$
\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}.
$$

Therefore, this equation is integrable if and only if $k = 2$.

From now on, we assume that $k = 2$. We will consider the case $k \geq 3$ in Example 5.19.

Since $\rho(X_n) = 11$, this space of initial conditions is not minimal and thus there exists one exceptional curve of first kind fixed by the equation (under the identification of $\operatorname{Pic} X_n$). In this case it is obvious that E is such a curve.

Let Y_n be the surface obtained by contracting E_n. Then, Y_n is the minimal space of initial conditions since $\rho(Y_n) = 10$.

It is true that Proposition 5.12 tells us which curve we must contract in general. As in the above example, however, looking closely at the motion of curves in a space of initial conditions often tells us which curves we must contract.

5.2. **Nonintegrable case.** In this subsection, we consider a minimization of a space of initial conditions in nonintegrable cases. We will give in Proposition 5.17 a minimality criterion for a space of initial conditions and we show in Proposition 5.18 that the minimization of a space of initial conditions is unique.

In this subsection, we consider the following situation:

- X: a basic rational surface with $\rho(X) = r + 1 > 10$.

- σ: a Cremona isometry on $\operatorname{Pic} X$ with exponential growth.
- $\lambda > 1$: the maximum eigenvalue of σ.
- $v \in \operatorname{Pic}_{\mathbb{R}} X$: the dominant eigenvector of σ, which is isotropic.

Lemma 5.14. *v or $-v$ is nef.*

Proof. The proof is the same as that of Lemma 5.8. Take a geometric basis $e = (e^{(0)}, \ldots, e^{(r)})$ and consider the limit

$$\lim_{n \to +\infty} \frac{1}{\lambda^n} \sigma^n e^{(0)}.$$

□

As the sign can be changed at will, we may assume that v is nef.

We shall use the following lemma throughout this section.

Lemma 5.15. *The intersection number is negative definite on the lattice $v^{\perp} \cap \operatorname{Pic} X$.*

Proof. Lemma 4.6 says that the intersection number has signature $(0, r-1)$ on $v^{\perp}$ and its kernel is generated by v. However, any scalar multiplication of v does not belong to $\operatorname{Pic} X$ since λ is irrational. Thus the intersection number is negative definite on $v^{\perp} \cap \operatorname{Pic} X$. □

Lemma 5.16. *Two different exceptional curves of first kind that belong to $v^{\perp}$ are always orthogonal to each other.*

Proof. Let $C, C' \in v^{\perp}$ be two different exceptional curves of first kind. Using Lemma 5.15, we have

$$0 > (C + C')^2 = -2 + 2C \cdot C'$$

and thus $C \cdot C' = 0$. □

Proposition 5.17. *(X, σ) is minimal if and only if there exist no exceptional curves of first kind that are orthogonal to v.*

Proof. Suppose that (X, σ) is not minimal. Then there exist mutually disjoint exceptional curves of first kind $C_1, \ldots, C_N$ such that σ acts as a permutation on $\{C_1, \ldots, C_N\}$. It is therefore sufficient to show that $C_1 \cdot v = 0$.

Taking $\ell \in \mathbb{Z}$ that satisfies $\sigma^{\ell} C_1 = C_1$, we have

$$C_1 \cdot v = (\sigma^{\ell} C_1) \cdot (\sigma^{\ell} v) = \lambda^{\ell} C_1 \cdot v,$$

which shows that $C_1 \cdot v = 0$.

We now show the converse. Let $\mathcal{C}$ be the set of the exceptional curves of first kind that are orthogonal to v, and assume that $\mathcal{C}$ is nonempty. It is clear that σ acts on $\mathcal{C}$ as a permutation. Since all elements in $\mathcal{C}$ are mutually disjoint by Lemma 5.16, it follows from Lemma 5.3 that (X, σ) is not minimal. □

While the Picard numbers of the minimal spaces of initial conditions for integrable systems are always 10, those of nonintegrable systems depend on the detail of the equations. Therefore, it is impossible to check the minimality only by the Picard number. We can only say that the Picard numbers are greater than 10. However, Proposition 5.17 gives us a precise minimality criterion.

The following proposition is an analogue of Proposition 5.12.

Proposition 5.18. *Let $\epsilon \colon X \to X'$ be a minimization of (X, σ). Decompose ϵ into a composition of blow-ups*

$$\epsilon = \epsilon^{(1)} \circ \cdots \circ \epsilon^{(L)}$$

and let $E^{(i)} \in \operatorname{Pic} X$ *be the total transform of the exceptional class of* $\epsilon^{(i)}$ *for* $i = 1, \ldots, L$. *Let* $e = (e^{(0)}, \ldots, e^{(r)})$ *be an arbitrary geometric basis on* $\operatorname{Pic} X$. *Then the set* $\{E^{(1)}, \ldots E^{(L)}\}$ *can be written as*

$$\mathcal{E} = \{E \in \operatorname{Pic} X \mid E^2 = -1, E \cdot v = 0, E \cdot K_X = -1, E \cdot e^{(0)} \geq 0\}.$$

In particular, the minimization $\epsilon \colon X \to X'$ *is unique.*

Proof. It is clear that $E_i \in \mathcal{E}$ for $i = 1, \ldots, L$. We show that $\mathcal{E} \subset \{E_1, \ldots, E_L\}$.

Step 1

We show that every element $E \in \mathcal{E}$ is effective. Using the Riemann-Roch inequality and Serre duality, we have

$$h^0(E) \geq 1 - h^2(E) = 1 - h^0(K_X - E).$$

It follows from $e^{(0)} \cdot (K_X - E) < 0$ that $h^0(K_X - E) = 0$. Thus E is effective.

Step 2

We show that two different elements $E, E' \in \mathcal{E}$ are orthogonal to each other. Since the intersection number is negative definite on $v^\perp \cap \operatorname{Pic} X$, we have

$$0 > (E \pm E')^2 = -2 \pm 2E \cdot E'$$

and thus $E \cdot E' = 0$.

Step 3

Assume that there exists $E \in \mathcal{E} \setminus \{E_1, \ldots, E_L\}$. Let $E' = \epsilon_* E$ and $v' = \epsilon_* v$. Since $E \cdot E_i = 0$ by Step 2, we have

$$E'^2 = -1, \quad E' \cdot K_{X'} = -1, \quad E' \cdot v' = 0.$$

Since E is effective, so is E'. Let us express E' as a sum of irreducible curves:

$$E' = \sum_{j=1}^{\ell} a_i [C_i].$$

We show that there exists at least one j such that C_j is an exceptional curve of first kind. Since $E' \cdot v' = 0$ and v' is nef, we have $C_j \cdot v' = 0$ for $j = 1, \ldots, \ell$. Since the intersection number is negative definite on $v'^\perp \cap \operatorname{Pic} X'$, C_j^2 are all negative. By the genus formula, we have

$$g_a(C_j) = 1 + \frac{1}{2} C_j^2 + \frac{1}{2} C_j \cdot K_{X'}.$$

Multiplying with a_j and summing, we obtain

$$\sum_j a_j g_a(C_j) = \sum_j a_j + \frac{1}{2} \sum_j a_j C_j^2 + \frac{1}{2} \sum_j a_j C_j \cdot K_{X'}.$$

Using $\sum_j a_j C_j = E'$ and $E' \cdot K_{X'} = -1$, we have

$$\sum_j a_j (2 + C_j^2 - 2 g_a(C_j)) = 1.$$

If $C_j^2 - 2g_a(C_j) \leq -2$ for all j, then the left hand side is not positive, which is a contradiction. Therefore, there is at least one j such that $C_j^2 - 2g_a(C_j) \geq -1$. Since $C_j^2 < 0$, the only possible case is

$$C_j^2 = -1, \quad g_a(C_j) = 0.$$

Thus C_j is an exceptional curve of first kind.

Since $C_j \cdot v' = 0$, Proposition 5.17 implies that (X', σ') is not minimal, which is a contradiction. Hence, we have $\mathcal{E} = \{E_1, \ldots, E_L\}$. □

Example 5.19. Again we consider Example 5.13, but this time we assume that $k \geq 3$.

As in the case $k = 2$, it is possible to contract E since it is fixed by the equation (under the identification as explained in Example 5.13). Let $\epsilon_n \colon X_n \to Y_n$ be the blow-down that contracts E_n. We show that Y_n is minimal with respect to this equation.

Since the dynamical degree λ of this equation is greater than 1, we cannot verify the minimality of Y_n only by its Picard number. Instead, we use Proposition 5.17

Let $v \in \operatorname{Pic}_{\mathbb{R}} Y_n$ be the λ-eigenvector. The rank of the lattice $v^\perp \cap \operatorname{Pic} Y_n$ is at most $4k$ since the dimension of $v^\perp$ is $4k+1$, $v \in v^\perp$ and $v \notin (v^\perp \cap \operatorname{Pic} Y_n) \otimes \mathbb{R}$. We show that the rank of this lattice is in fact $4k$.

We denote the images in Y_n of the irreducible curves

$$D^{(0)}, \overline{D}^{(0)}, D^{(1)}, \overline{D}^{(1)}, D^{(2)}, \overline{D}^{(2)}, \overline{\overline{D}}^{(2)}, \overline{\overline{\overline{D}}}^{(2)}, \dots, D^{(k)}, \overline{D}^{(k)}, \overline{\overline{D}}^{(k)}, \overline{\overline{\overline{D}}}^{(k)}$$

by $G^{(1)}, \dots, G^{(4k)}$, respectively. These curves are all orthogonal to v since they are permuted by the equation. The classes of these curves are linearly independent in the Picard group and thus the rank of the lattice $v^\perp \cap \operatorname{Pic} Y_n$ is $4k$.

Since

$$\begin{aligned}\left(G^{(1)}\right)^2 &= \left(G^{(2)}\right)^2 = -k,\\ \left(G^{(3)}\right)^2 &= \cdots = \left(G^{(4k)}\right)^2 = -2,\end{aligned}$$

these curves are not contractible. Therefore, it is sufficient to show that there is no other irreducible curve C' such that $C' \cdot v = 0$.

Since $G^{(1)}, \dots, G^{(4k)}$ are linearly independent in $\operatorname{Pic} Y_n$ and the rank of the lattice $v^\perp \cap \operatorname{Pic} Y_n$ is $4k$, there exists $c_1, \dots, c_{4k}, d \in \mathbb{Z}$ such that $d > 0$ and

$$d[C'] = \sum_j c_j [G^{(j)}].$$

However, this leads to

$$\left[dC' + \sum_{c_j < 0} (-c_j) G^{(j)} \right] = \left[\sum_{c_j > 0} c_j G^{(j)} \right],$$

which contradicts Proposition A.29 since the matrix $\left(G^{(i)} \cdot G^{(j)}\right)_{ij}$ is negative definite (Lemma 5.15).

6. Conclusion

In this thesis, we studied nonautonomous mappings of the plane by means of singularity confinement and spaces of initial conditions.

In §2, we introduced the full-deautonomisation approach, which enables us to predict the value of the algebraic entropy of an equation with all singularities confined. First we performed a detailed algebro-geometric analysis of the late confinements through an example and observed the existence of a close relation between the conditions on the parameters and the linear action induced on the Picard group of the space of initial conditions. Using this relation, we proposed the full-deautonomisation approach. We also discussed how to discard gauge freedom in this approach. It is true that the whole analysis in §2 was performed on examples and that there is still no rigorous proof in the general case. However, it should be stressed that not a single counterexample to this method has been found to date. Therefore, we are certain that the full-deautonomisation approach is very useful in testing the integrability of concrete equations.

In §3–5, we studied nonautonomous equations with spaces of initial conditions and unbounded degree growth. Especially, Theorem 5.6 shows that if an integrable mapping of the plane with unbounded degree growth has a space of initial conditions, then it must be one of the discrete Painlevé equations. Since all discrete Painlevé equations have already been classified by Sakai [37], this means we have finished the classification of integrable mappings of the plane with a space of initial conditions and unbounded degree growth. Moreover, we have given a concrete procedure to minimize a space of initial condition to a generalized Halphen surface.

There are many future problems to address. The most important one is of course to prove Conjecture 2.10. If this conjecture is proven, then the singularity confinement approach combined with the full-deautonomisation method will become the most powerful integrability detector for mappings of the plane. It is also important to find an efficient way to introduce nonautonomous coefficients to an autonomous equation. We also wish to study the full-deautonomisation approach of equations of rank higher than 2. The reason, even now, why the main objects in the field of discrete integrable systems are mappings of the plane is the lack of appropriate examples of confining equations of higher rank. Very recently, however, a family of nonintegrable mappings of rank higher than 2 that are conjectured to have the coprimeness condition (an algebro-geometric reinterpretation of singularity confinement) has been discovered [26]. If the rank of an equation is higher than 2, then an algebro-geometric approach becomes very hard but singularity confinement still remains a valid approach. In the future, we intend to study the equations found in [26] by the full-deautonomisation approach.

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Appendix A. Algebraic surfaces

In this appendix we shall recall basic results on algebraic surfaces. We shall not give the proofs of propositions since they are given in many textbooks such as [17, 2, 1].

In this thesis, a *surface* means a smooth projective variety of dimension 2 over $\mathbb{C}$, which becomes a compact complex manifold of dimension 2.

Notation A.1.

- $\sim$: the linear equivalence of divisors, which is the same as the numerical equivalence if the surface is rational.
- $[D]$: the linear equivalence class of D.
- $\operatorname{Pic} X$: the Picard group of X.
- $\operatorname{Pic}_{\mathbb{Q}} X = \operatorname{Pic} X \otimes \mathbb{Q}$, $\operatorname{Pic}_{\mathbb{R}} X = \operatorname{Pic} X \otimes \mathbb{R}$, $\operatorname{Pic}_{\mathbb{C}} X = \operatorname{Pic} X \otimes \mathbb{C}$.
- $\rho(X)$: the Picard number of X, which is equal to $\dim_{\mathbb{Q}}(\operatorname{Pic}_{\mathbb{Q}} X)$ if X is rational.
- $H^i(X, D) = H^i(D)$: the i-th cohomology group of the divisor D (or its class).
- $h^i(D) = h^i(X, D) = \dim_{\mathbb{C}} H^i(X, D)$.
- $|F|$: the linear system of F.
- K_X: the canonical class of X.
- $D_1 \cdot D_2$: the intersection number of the divisors D_1 and D_2 (or their classes).
- $X \to Y$: a morphism from X to Y.

- $X \dashrightarrow Y$: a birational map from X to Y.
- $\mathcal{O}_{\mathbb{P}^2}(1)$: the class of lines in $\mathbb{P}^2$.
- $g_a(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$: the arithmetic genus of an irreducible curve C.

Definition A.2 (birational map)**.** A *rational map* is a morphism defined on a nonempty Zariski open subset. A *birational map* is a rational map that has as its inverse a rational map. A *birational morphism* is a morphism that has a rational inverse. A *birational automorphism* is a birational map from a surface to itself.

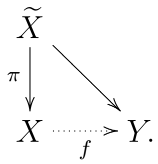
Definition A.3 (blow-up)**.** Let X be a surface and $P \in X$. There exists a surface $\widetilde{X}$ and a birational morphism $\pi\colon \widetilde{X} \to X$ such that $\pi^{-1}(P)$ is isomorphic to $\mathbb{P}^1$ and π is an isomorphism from $\widetilde{X} \setminus \pi^{-1}(P)$ to $X \setminus \{P\}$. This $\widetilde{X}$ (and π) is unique up to isomorphism and is called the *blow-up* of X at P. The fiber $\pi^{-1}(P)$ is called the *exceptional curve* of π and P the *center* of the blow-up.

Let (x, y) be a local coordinate around P, i.e. P is the point defined by $(x, y) = (0, 0)$. Then, $\widetilde{X}$ can be covered around $\pi^{-1}(P)$ with two coordinates $(x, y/x)$ and $(x/y, y)$.

Proposition A.4. *Let $f\colon Y \to X$ be a birational morphism.*

(1) *f can be written as a composition of a finite number of blow-ups.*
(2) *If f is bijective, then f is an isomorphism.*
(3) *If $\rho(Y) = \rho(X)$, then f is an isomorphism.*

Proposition A.5. *Let $f\colon X \dashrightarrow Y$ be a birational map. Then, there are only a finite number of indeterminate points. Moreover, it is always possible to eliminate these indeterminacies by blow-ups, i.e. there exists a composition of blow-ups $\pi\colon \widetilde{X} \to X$ such that $f \circ \pi$ is a birational morphism:*



In particular, f can be written as a composition of blow-ups and blow-downs.

Definition A.6 (rational surface)**.** A surface is *rational* if it is birational to $\mathbb{P}^2$. A rational surface that admits a birational morphism to $\mathbb{P}^2$ is called a *basic rational surface*. This means that a basic rational surface can always be obtained by a finite number of blow-ups of $\mathbb{P}^2$.

Definition A.7 (divisor)**.** A *divisor* on X is an integer-coefficient finite formal sum of irreducible curves in X. We denote by $\operatorname{Div} X$ the set of divisors on X. An irreducible curve is called a *prime divisor* when we think of it as a divisor.

If f is a nonzero rational function, then its zeros and poles define a divisor. This kind of divisor is called a *principal divisor*. Two divisors D, D' are *linearly equivalent* if $D - D'$ is principal. We denote by "$\sim$" the linear equivalence. The quotient group $\operatorname{Div} X/\sim$ is called the *Picard group* of X and denoted by $\operatorname{Pic} X$. We denote by $[D]$ the linear equivalence class of D. It is known that the Picard group is finitely generated.

A divisor is *effective* if its coefficients are all nonnegative. A divisor class is effective if it can be represented by an effective divisor.

Definition A.8 (intersection number)**.** There exists a unique symmetric $\mathbb{Z}$-bilinear form

$$\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}, \quad (F, F') \mapsto F \cdot F'$$

such that every two different irreducible curves C, C' satisfy

$$[C] \cdot [C'] = \sum_{P \in C \cap C'} m_P,$$

where m_P is the multiplicity of the intersection at P. This bilinear form is called the *intersection number*. We simply denote $F \cdot F$ by F^2, $[D] \cdot [D']$ by $D \cdot D'$ and so on.

Definition A.9 (numerical equivalence, Picard number)**.** Two divisors D_1, D_2 are *numerically equivalent* if

$$D_1 \cdot D' = D_2 \cdot D'$$

for every divisor D'. The *Picard number* is the rank of the quotient group of $\operatorname{Div} X$ by the numerical equivalence, and we denote it by $\rho(X)$.

It is known that for rational surfaces the numerical equivalence is the same as the linear equivalence. In this case the Picard group is a free abelian group whose rank is the Picard number. Since we are only interested in rational surfaces in this thesis, we shall not strictly distinguish these two equivalences.

Theorem A.10 (Hodge index theorem)**.** *The intersection number on a surface X has signature $(1, \rho(X) - 1)$.*

Definition A.11. Let $\pi\colon Y \to X$ be the blow-up at $P \in X$.

The *strict transform* (or proper transform) of an irreducible curve $C \subset X$ is

$$\overline{\pi^{-1}(C \setminus \{P\})},$$

where we denote by $\overline{Z}$ the closure of Z in Y. If a curve is irreducible, then so is its strict transform.

The *total transform* of a prime divisor C on X is the divisor

$$\pi^* C = \widetilde{C} + m_P(C) E \in \operatorname{Div} Y,$$

where $\widetilde{C}$ is the strict transform of C, $m_P(C)$ the multiplicity of C at P and E the exceptional curve of π. We define the total transform of a divisor by the $\mathbb{Z}$-linear extension.

For a divisor $D' = \sum_j a_j C'_j$ on Y, we define $\pi_* D' \in \operatorname{Div} X$ by

$$\pi_* D' = \sum_{C'_j \neq E} a_j \pi(C'_j).$$

These transforms for a birational morphism can be calculated by decomposing it into blow-ups.

It should be noted that, while π^* always preserves the intersection number, π_* does not do so in general.

Proposition A.12. *Let $\pi\colon Y \to X$ be the blow-up at $P \in X$ and let E be its exceptional curve. Let $C \subset X$ be an irreducible curve and let $\widetilde{C} \subset Y$ its strict transform. Then*

$$E^2 = -1, \quad E \cdot \widetilde{C} = m_P(C), \quad \widetilde{C}^2 = C^2 - m_P(C)^2,$$

where $m_P(C)$ is the multiplicity of C at P.

Let $C' \subset X$ be another irreducible curve and let $\widetilde{C}'$ its strict transform. Then

$$\widetilde{C} \cdot \widetilde{C}' = C \cdot C' - m_P(C) m_P(C').$$

Definition A.13. Let $\pi\colon Y \to X$ be a birational morphism. Since $\pi_*\colon \operatorname{Div} Y \to \operatorname{Div} X$ and $\pi^*\colon \operatorname{Div} X \to \operatorname{Div} Y$ preserve the linear equivalence, we have $\pi_*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$ and $\pi^*\colon \operatorname{Pic} X \to \operatorname{Pic} Y$.

Proposition A.14. *Let $\pi\colon Y \to X$ be a blow-up and $E \subset Y$ its exceptional curve. Then* $\operatorname{Pic} Y$ *is isomorphic to* $\operatorname{Pic} X \oplus \mathbb{Z}[E]$.

Proposition A.15. *If $f\colon Z \to Y$ and $g\colon Y \to X$ are birational morphisms, then*

$$(g \circ f)_* = g_* f_* \quad \text{and} \quad (g \circ f)^* = f^* g^*.$$

Definition A.16 (canonical class)**.** Let ω be a globally meromorphic 2-form on a surface X which is not identically zero. The *canonical class* of X, denoted by K_X, is the class of the divisor defined by the 2-form ω. It is known that the canonical class does not depend on the choice of ω.

Proposition A.17. *If $\pi\colon \widetilde{X} \to X$ is a blow-up, then we have*

$$K_{\widetilde{X}} = \pi^* K_X + [E],$$

where E is the exceptional curve of π.

Proposition A.18. *The Picard group of $\mathbb{P}^2$ is a free $\mathbb{Z}$-module of rank 1. Its generator is the class of lines, which is denoted by $\mathcal{O}_{\mathbb{P}^2}(1)$. The class of curves of degree d is $\mathcal{O}_{\mathbb{P}^2}(d) = d\mathcal{O}_{\mathbb{P}^2}(1)$, and the intersection number on $\mathbb{P}^2$ is determined by*

$$\mathcal{O}_{\mathbb{P}^2}(1) \cdot \mathcal{O}_{\mathbb{P}^2}(1) = 1.$$

The canonical class of $\mathbb{P}^2$ is $\mathcal{O}_{\mathbb{P}^2}(-3)$.

Definition A.19. Let A be an invertible square matrix of size 3. Then, the linear transformation $\mathbb{C}^3 \to \mathbb{C}^3$ corresponding to A induces an automorphism on $\mathbb{P}^2$. This kind of automorphism is called a projective linear transformation. The set of all these transformations is denoted by $\mathrm{PGL}(3)$.

It is known that any automorphism on $\mathbb{P}^2$ belongs to this class.

Definition A.20 (degree)**.** Using the homogeneous coordinate $(z_1 : z_2 : z_3) \in \mathbb{P}^2$, a birational automorphism φ on $\mathbb{P}^2$ can be written as

$$\varphi(z_1 : z_2 : z_3) = (\varphi_1(z_1, z_2, z_3) : \varphi_2(z_1, z_2, z_3) : \varphi_3(z_1, z_2, z_3)),$$

where $\varphi_1, \varphi_2, \varphi_3$ are homogeneous polynomials of z_1, z_2, z_3 of the same degree with no common factors. The *degree* of φ (as a birational automorphism on $\mathbb{P}^2$) is defined by $\deg \varphi_i$.

Definition A.21. Let $f\colon X \dashrightarrow Y$ be a birational map and let $\pi\colon \widetilde{X} \to X$ be a resolution of the indeterminacies of f:

$$\begin{array}{ccc} \widetilde{X} & & \\ {\scriptstyle \pi}\downarrow & \searrow^{g} & \\ X & \underset{f}{\dashrightarrow} & Y. \end{array}$$

Then, f_* and f^* are defined by

$$f_* = g_* \pi^*, \quad f^* = \pi_* g^*.$$

It is known that these linear maps do not depend on the choice of π.

These linear maps do not preserve the intersection number in general.

Proposition A.22. *The above f_* and f^* preserve the set of effective classes.*

Remark A.23. Proposition A.15 does *not* hold if f, g are simply birational maps.

Proposition A.24. *Let φ be a birational automorphism on $\mathbb{P}^2$. Then*

$$\deg \varphi = (\varphi_* \mathcal{O}_{\mathbb{P}^2}(1)) \cdot \mathcal{O}_{\mathbb{P}^2}(1) = (\varphi^* \mathcal{O}_{\mathbb{P}^2}(1)) \cdot \mathcal{O}_{\mathbb{P}^2}(1).$$

Example A.25. Let

$$\pi = \pi^{(1)} \circ \cdots \circ \pi^{(r)} \colon X \to \mathbb{P}^2$$

be a composition of blow-ups. Let $e^{(0)} \in \operatorname{Pic} X$ be the total transform of $\mathcal{O}_{\mathbb{P}^2}(1)$ and $e^{(i)} \in \operatorname{Pic} X$ the total transform of the class of the exceptional curve of $\pi^{(i)}$ for $i = 1, \dots, r$. Then we have

$$\operatorname{Pic} X = \mathbb{Z}e^{(0)} \oplus \cdots \oplus \mathbb{Z}e^{(r)}.$$

The intersection number can be calculated by

$$e^{(i)} \cdot e^{(j)} = \begin{cases} 1 & (i = j = 0) \\ -1 & (i = j \neq 0) \\ 0 & (i \neq j). \end{cases}$$

Definition A.26 (geometric basis [10])**.** Let X be a basic rational surface and $(e^{(0)}, \dots, e^{(r)})$ be a $\mathbb{Z}$-basis on $\operatorname{Pic} X$. Then $(e^{(0)}, \dots, e^{(r)})$ is said to be a *geometric basis* if there is a composition of blow-ups $\pi = \pi^{(1)} \circ \cdots \circ \pi^{(r)} \colon X \to \mathbb{P}^2$ such that $e^{(0)} = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $e^{(i)}$ is the total transform of the class of the exceptional curve of $\pi^{(i)}$ for $i = 1, \dots, r$.

Note that $e^{(i)} - e^{(j)}$ $(i, j > 0, i \neq j)$ is effective if and only if $i < j$ and the center of the j-th blow-up is infinitely near that of the i-th blow-up.

For any geometric basis $(e^{(0)}, \dots, e^{(r)})$ we have

$$K_X = -3e^{(0)} + e^{(1)} + \cdots + e^{(r)}, \quad e^{(i)} \cdot e^{(j)} = \begin{cases} 1 & (i = j = 0) \\ -1 & (i = j \neq 0) \\ 0 & (i \neq j). \end{cases}$$

A birational morphism to $\mathbb{P}^2$ is determined only by its geometric basis up to automorphism on $\mathbb{P}^2$, i.e. if two birational morphisms $\pi, \pi' \colon X \to \mathbb{P}^2$ give the same geometric basis on $\operatorname{Pic} X$, then there exists $f \in \mathrm{PGL}(3)$ such that $\pi' = f \circ \pi$. In fact, these birational morphisms are determined only by $e^{(0)}$, since the set $\{e^{(1)}, \dots, e^{(r)}\}$ is determined by

$$\left\{e^{(1)}, \dots, e^{(r)}\right\} = \left\{F \in \operatorname{Pic} X \mid F^2 = -1, F \cdot e^{(0)} = 0, F \colon \text{effective}\right\}.$$

Definition A.27 (cohomology group)**.** Let X be a surface and let D be a divisor on it. For a divisor D and $i = 0, 1, 2$, there is a finite dimensional $\mathbb{C}$-vector space $H^i(X, D)$ (or simply $H^i(D)$), called the *cohomology group* of D. We denote by $h^i(D)$ the dimension of $H^i(D)$.

It is known that if D and D' are linearly equivalent, then the corresponding cohomology groups are naturally isomorphic. Thus, $H^i(F)$ and $h^i(F)$ are well-defined for $F \in \operatorname{Pic} X$.

Definition A.28 (linear system)**.** The *linear system* of a divisor D (or its class) is

$$|D| = \{D' \colon \text{effective divisor} \mid D' \sim D\}.$$

It is known that there is a natural bijection from $|D|$ to $(H^0(D) \setminus 0)/\mathbb{C}^*$ and thus $|D|$ becomes a projective space. In particular, D is effective if and only if $h^0(D) \geq 1$. We define $\dim |D| = h^0(D) - 1$ where D is effective.

Proposition A.29. *Let $C_1, \ldots, C_m$ be irreducible curves in a surface X such that the matrix $(C_i \cdot C_j)_{ij}$ is negative definite. Then, for all nonnegative integers $a_1, \ldots, a_m$, we have*

$$h^0\left(\sum_i a_i C_i\right) = 1.$$

In particular, if an irreducible curve C has a negative self-intersection, then

$$h^0(mC) = 1$$

for $m \geq 0$ and thus the class $[C]$ cannot be written as a nontrivial sum of effective classes.

Proposition A.30. *If X is rational, then*

$$h^0(mK_X) = 0$$

for all $m \geq 1$.

Theorem A.31 (Riemann-Roch)**.** *Let $F \in \operatorname{Pic} X$. Then*

$$h^0(F) - h^1(F) + h^2(F) = \chi(\mathcal{O}_X) + \frac{1}{2} F \cdot (F - K_X),$$

where we denote by $\chi(\mathcal{O}_X)$ the Euler-Poincaré characteristic. Since $h^1(F)$ is not negative, we have

$$h^0(F) + h^2(F) \geq \chi(\mathcal{O}_X) + \frac{1}{2} F \cdot (F - K_X),$$

which is called the Riemann-Roch inequality*. If X is rational, then $\chi(\mathcal{O}_X) = 1$. Therefore we have*

$$h^0(F) - h^1(F) + h^2(F) = 1 + \frac{1}{2} F \cdot (F - K_X)$$

and

$$h^0(F) + h^2(F) \geq 1 + \frac{1}{2} F \cdot (F - K_X).$$

Theorem A.32 (Serre duality)**.** *Let $F \in \operatorname{Pic} X$. Then*

$$h^i(F) = h^{2-i}(K_X - F)$$

for $i = 0, 1, 2$.

Theorem A.33 (genus formula)**.** *Let $C \subset X$ be an irreducible curve. Then*

$$g_a(C) = 1 + \frac{1}{2} C \cdot (C + K_X),$$

where $g_a(C) = h^1(C, \mathcal{O}_C)$ is the arithmetic genus of C.

Proposition A.34. *Let C be a (possibly singular) irreducible curve. Then $g_a(C) = 0$ if and only if C is isomorphic to $\mathbb{P}^1$.*

Definition A.35 (exceptional curve of first kind)**.** An irreducible curve $C \subset X$ is called an *exceptional curve of first kind* if C is isomorphic to $\mathbb{P}^1$ and $C^2 = -1$. The genus formula and Proposition A.34 imply that these conditions are equivalent to $C^2 = C \cdot K_X = -1$.

Theorem A.36 (Castelnuovo's contraction theorem)**.** *Let X be a surface and $C \subset X$ an exceptional curve of first kind. Then there exist a surface X' and a birational morphism $\pi \colon X \to X'$ such that $\pi(C)$ is a point in X' and π is an isomorphism from $X \setminus C$ to $X' \setminus \pi(C)$. This procedure is called a* blow-down*. In other words, we can contract an exceptional curve of first kind by a blow-down.*

Definition A.37 (nef)**.** A divisor D on a surface X is *nef* if it satisfies

$$C \cdot D \geq 0$$

for every irreducible curve $C \subset X$. A class $F \in \operatorname{Pic} X$ (or $F \in \operatorname{Pic}_{\mathbb{R}} X$) is said to be nef if it satisfies

$$C \cdot F \geq 0$$

for every irreducible curve $C \subset X$.

It is known that the self-intersection of a nef class is always nonnegative. It is also known that K_X cannot be nef for a rational surface X.

The set

$$\{F \in \operatorname{Pic}_{\mathbb{R}} X \mid F\colon \text{ nef}\}$$

is a closed convex cone in $\operatorname{Pic}_{\mathbb{R}} X$, i.e.

- if F, F' are nef, then so is $F + F'$,
- if F is nef and $a > 0$, then aF is also nef,
- the above set is a closed set in $\operatorname{Pic}_{\mathbb{R}} X$.

The set of all nef classes in $\operatorname{Pic}_{\mathbb{R}} X$ is called the *nef cone* of X.

Theorem A.38 (Nagata [32])**.** *If a rational surface has infinitely many exceptional curves of first kind, then it is a basic rational surface.*

Appendix B. Proof of Lemma 4.7

Proof of Lemma 4.7. Let us extend $(-,-)$ to a Hermitian form on $V_{\mathbb{C}}$.

We show (1) in Steps 1–9.

Step 1

First let us consider the case where f has an eigenvalue whose modulus is not 1.

If λ is an eigenvalue with $|\lambda| \neq 1$ and v its corresponding eigenvector, then v is always isotropic since

$$\begin{aligned}(v, v) &= (fv, fv)\\ &= (\lambda v, \lambda v)\\ &= |\lambda|^2 (v, v).\end{aligned}$$

Step 2

Let us show that if λ is an eigenvalue whose modulus is not 1, then λ is simple.

Let v be the corresponding eigenvector and assume λ is not simple. Then there exists w, linearly independent of v, such that

$$fw = \lambda w \quad \text{or} \quad fw = \lambda w + v.$$

In the first case, v and w are orthogonal to each other since

$$\begin{aligned}(v, w) &= (\lambda v, \lambda w)\\ &= |\lambda|^2 (v, w),\end{aligned}$$

which, together with $(v, v) = (w, w) = 0$, contradicts Lemma 4.6. Let us consider the second case. Since

$$\begin{aligned}(v, w) &= (\lambda v, \lambda w + v)\\ &= |\lambda|^2 (v, w),\end{aligned}$$

we have

$$(v, w) = 0.$$

In the same way we find

$$\begin{aligned}(w,w) &= (\lambda w + v, \lambda w + v)\\ &= |\lambda|^2(w,w)\end{aligned}$$

and thus

$$(v,v) = (v,w) = (w,w) = 0,$$

which again contradicts Lemma 4.6.

Step 3

We show that if λ_1, λ_2 are different eigenvalues of f with $|\lambda_i| \neq 1$, then $\lambda_2 = 1/\overline{\lambda_1}$. In particular, λ_i must be real numbers.

Let v_1, v_2 be the corresponding eigenvectors. These vectors are both isotropic by Step 1, and linearly independent since they are eigenvectors corresponding to different eigenvalues. Thus it follows from Lemma 4.6 that $(v_1, v_2) \neq 0$. Since

$$\begin{aligned}(v_1,v_2) &= (\lambda_1 v_1, \lambda_2 v_2)\\ &= \lambda_1\overline{\lambda_2}(v_1,v_2),\end{aligned}$$

we have

$$\lambda_1\overline{\lambda_2} = 1.$$

Step 4

We show that if f has an eigenvalue whose modulus is not 1, then f is diagonalizable. We already know that such eigenvalues are simple.

We therefore consider an eigenvalue μ of modulus 1 and assume vectors u_1, u_2 satisfy

$$\begin{aligned}fu_1 &= \mu u_1,\\ fu_2 &= \mu u_2 + u_1.\end{aligned}$$

Since

$$\begin{aligned}(u_1,u_2) &= (\mu u_1, \mu u_2 + u_1)\\ &= (u_1,u_2) + \mu(u_1,u_1),\end{aligned}$$

u_1 is isotropic. In the same way we find

$$(v,u_1) = \lambda\overline{\mu}(v,u_1)$$

and thus

$$(v,v) = (v,u_1) = (u_1,u_1) = 0,$$

which contradicts Lemma 4.6. Hence, f is diagonalizable.

Step 5

We show that if $\lambda \in \mathbb{R} \setminus \{\pm 1\}$ is an eigenvalue of f, then so is $1/\lambda$.

Assume that $1/\lambda$ is not an eigenvalue. Then, since all eigenvalues except λ have modulus 1 (Step 3) and f is diagonalizable (Step 4), there exists a basis $(v, u_1, \ldots, u_r)$ of $V_{\mathbb{C}}$ such that

$$\begin{aligned}fv &= \lambda v,\\ fu_i &= \mu_i u_i,\end{aligned}$$

where $|\mu_i| = 1$. Since

$$\begin{aligned}(v,u_i) &= (\lambda v, \mu_i u_i)\\ &= \lambda\overline{\mu_i}(v,u_i),\end{aligned}$$

we have $(v, u_i) = 0$. Therefore, using Step 1, we obtain that v is orthogonal to all elements in $V_{\mathbb{C}}$. However, this contradicts the nondegeneratedness of $(-,-)$.

Steps 1–5 show that if f has an eigenvalue whose modulus is not 1, then the Jordan normal form of f is (4.3). From now on, let us consider the case where all eigenvalues of f have modulus 1.

Step 6

It is clear that if f is diagonalizable, then its Jordan normal form is (4.1). Thus it is sufficient to show that if f is not diagonalizable, then its Jordan normal form is (4.2).

Step 7

We show that the size of each Jordan block is at most 3.

Assume that linearly independent vectors v_1, v_2, v_3, v_4 satisfy

$$\begin{aligned} fv_1 &= \nu v_1, \\ fv_2 &= \nu v_2 + v_1, \\ fv_3 &= \nu v_3 + v_2, \\ fv_4 &= \nu v_4 + v_3. \end{aligned}$$

Using

$$\begin{aligned} (v_1, v_i) &= (\nu v_1, \nu v_i + v_{i-1}) \\ &= (v_1, v_i) + \nu(v_1, v_{i-1}), \end{aligned}$$

we have $(v_1, v_{i-1}) = 0$ for $i = 2, 3, 4$. Since

$$\begin{aligned} (v_2, v_3) &= (\nu v_2 + v_1, \nu v_3 + v_2) \\ &= (v_2, v_3) + \nu(v_2, v_2), \end{aligned}$$

v_2 is isotropic, which contradicts Lemma 4.6.

Step 8

We show that f has only one Jordan block whose size is greater than 1. In particular, the corresponding eigenvalue is ± 1.

Let μ, ν be eigenvalues of modulus 1 and let pairwise-linearly independent vectors v_1, v_2 and w_1, w_2 satisfy

$$\begin{aligned} fv_1 &= \mu v_1, & fw_1 &= \nu w_1, \\ fv_2 &= \mu v_2 + v_1, & fw_2 &= \nu w_2 + w_1. \end{aligned}$$

It is sufficient to show that v_1 and w_1 are linearly dependent.

The same calculation as in Step 4 implies that v_1 and w_1 are both isotropic. Therefore, since

$$\begin{aligned} (v_1, w_1) &= (\mu v_1, \nu w_1) \\ &= \frac{\mu}{\nu}(v_1, w_1), \end{aligned}$$

it follows from Lemma 4.6 that $\mu = \nu$. However, using

$$\begin{aligned} (v_1, w_2) &= (\mu v_1, \mu w_2 + w_1) \\ &= (v_1, w_2) + \mu(v_1, w_1), \end{aligned}$$

we have $(v_1, w_1) = 0$. Hence Lemma 4.6 shows that v_1 and w_1 are linearly dependent.

Step 9

Finally we show that the size of the Jordan block in Step 8 is exactly 3. It is sufficient to show that the size is not 2.

Assume that $(v_1, v_2, u_1, \ldots, u_{r-1})$ is a basis on $V_{\mathbb{C}}$ such that

$$\begin{aligned} fv_1 &= \nu v_1, \\ fv_2 &= \nu v_2 + v_1, \\ fu_i &= \mu_i u_i. \end{aligned}$$

Moreover, we can take v_1 and v_2 in V since $\nu = \pm 1$. As in Step 5, we deduce a contradiction by showing that v_1 is orthogonal to any of $v_1, v_2, u_1, \ldots, u_{r-1}$, baring in mind that a calculation similar to Step 4 shows that $(v_1, v_1) = 0$. Using

$$\begin{aligned} (v_2, v_2) &= (\nu v_2 + v_1, \nu v_2 + v_1) \\ &= (v_2, v_2) + 2\nu(v_1, v_2), \end{aligned}$$

we have $(v_1, v_2) = 0$. Finally we show that $(v_1, u_i) = 0$. If $\nu \neq \mu_i$, then

$$\begin{aligned} (v_1, u_i) &= (\nu v_1, \mu_i u_i) \\ &= \nu \overline{\mu_i}(v_1, u_i) \end{aligned}$$

and thus $(v_1, u_i) = 0$. If $\nu = \mu_i$, then

$$\begin{aligned} (v_2, u_i) &= (\nu v_2 + v_1, \nu u_i) \\ &= (v_2, u_i) + \nu(v_1, u_i) \end{aligned}$$

and thus $(v_1, u_i) = 0$.

We now show (2).

Step 10

Let us represent w as

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \sum_j b_j u_j.$$

Since $(v_1, \ldots, u_{r-2})$ is a Jordan basis corresponding to (4.2), we have

$$f^n w = \left(\nu^n a_1 + \nu^{n-1} n a_2 + \frac{n(n-1)\nu^n}{2} a_3\right) v_1 + (\nu^n a_2 + \nu^{n-1} n a_3) v_2 + \nu^n a_3 v_3 + \sum_j \mu_j^n b_j u_j.$$

Dividing both sides by $\nu^n n^2$ and taking the limit $n \to +\infty$, we have

$$\lim_{n \to +\infty} \frac{1}{\nu^n n^2} f^n w = \frac{a_3}{2} v_1.$$

Thus it is sufficient to show that $a_3 = (w, v_1)/(v_3, v_1)$. A calculation similar to that given in Step 9 leads to

$$(v_1, v_1) = (v_2, v_1) = (u_j, v_1) = 0.$$

Therefore, we have

$$(w, v_1) = a_3 (v_3, v_1)$$

and thus

$$a_3 = \frac{(w, v_1)}{(v_3, v_1)}.$$

We finally show (3) in a similar way as in Step 10.

Step 11

Let us represent w as

$$w = a_1 v_1 + a_2 v_2 + \sum_j b_j u_j.$$

Then we have

$$f^n w = \lambda^n a_1 v_1 + \frac{a_2}{\lambda^n} v_2 + \sum_j \mu_j^n b_j u_j$$

and

$$\lim_{n \to +\infty} \frac{1}{\lambda^n} w = a_1 v_1.$$

Step 1 and a calculation similar to that given in Step 5 lead to

$$(v_1, v_1) = (v_1, u_j) = 0$$

and thus we have

$$a_1 = \frac{(w, v_2)}{(v_1, v_2)}.$$

□

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